Bordism Homology and Cohomology

Paul Jin Robaschik

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Betreuer: Prof. Dr. Markus Hausmann

Zweitgutachterin: Dr. Elizabeth Tatum

Mathematisches Institut

Mathematisch-Naturwissenschaftliche Fakultät der

Rheinischen Friedrich-Wilhelms-Universität Bonn

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0 Introduction/Motivation

Recall the definition of homotopy groups $\pi_n(X)$ as the set of homotopy classes of maps from the n-sphere S^n to a space X. The problem with homotopy groups is that they are, in general, hard to compute, even for simple spaces such as spheres.

The general idea of bordism is to replace the n-sphere with a manifold of dimension n and to consider the homotopy classes of maps from this manifold to a space X.

1 Bordism

1.1 Manifolds

Definition 1.1 (Topological manifold [Lee13, pp.2-3])**.** An n-dimensional **topological manifold** is a topological space M such that:

- M is Hausdorff, (i.e. any two distinct points can be separated by disjoint open sets),
- M is second countable, (i.e. there exists a countable basis for the topology of M) and
- M is locally Euclidean (i.e. every point in M has a neighbourhood homeomorphic to an open subspace of $\mathbb{R}^n$).

We will often write M^n for an n-dimensional manifold. n-dimensional manifolds are also called **n-manifolds**.

Example 1. *$\mathbb{R}^n$ is an n-dimensional topological manifold.*

Example 2. *The n-dimensional sphere S^n is an n-dimensional topological manifold. Hausdorffness and second countability follow from $S^n \subset \mathbb{R}^{n+1}$. For local Euclideanness, we can use the local charts*

$$\varphi_i^{\pm} : U_i := \{(x_0, \ldots, x_n) \in S^n \mid \pm x_i > 0\} \to B_1^n(0)$$

by

$$(x_0, \ldots, x_i, \ldots, x_n) \mapsto (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

Non-examples.

- *A "cross" in $\mathbb{R}^2$ ($\{(x_1, x_2 \mid x_1 = 0 \vee x_2 = 0)\}$) is not a topological manifold, because it is not locally Euclidean at the crossing point.*
- *The line with two origins ($(\mathbb{R} \times \{0\} \sqcup \mathbb{R} \times \{1\})/((x, 0) \sim (y, 1) \Leftrightarrow (x = y \wedge x \neq 0))$) is not a topological manifold. It is not Hausdorff, as the two origins cannot be separated by disjoint open sets.*
- *Let $\{\mathrm{pt}\}$ denote the point space. $\coprod_{i \in \mathbb{R}} \{\mathrm{pt}\}$ is not a topological manifold, because it is not second countable, as it has uncountably many connected components.*

- $S^1 \sqcup \{\mathrm{pt}\}$ *is not a topological manifold, because it is, depending on the point, locally homeomorphic to* $\mathbb{R}^2$ *or* $\mathbb{R}$*, and the dimension needs of a manifold needs to be constant.*

Remark. One could replace the condition of being second countable with the condition of being paracompact (i.e. every open cover of M admits a locally finite refinement). The following equivalence holds:

$$\text{second countable} \iff \text{paracompact and countably many connected components}$$

Definition 1.2 ((Smooth) Atlas [Lee13, p.12])**.** Let M be a topological manifold. A **(smooth) atlas** $\mathcal{A}$ on M is a collection of smooth charts $(U_\alpha, \varphi_\alpha)$ such that:

- the $\{U_\alpha\}$ cover M,
- the charts are pairwise smoothly compatible (i.e. the transition functions $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ are smooth)

Example 3. *The charts we chose for the n-sphere* S^n *in Example 2 form a smooth atlas on* S^n*.*

Definition 1.3 (Equivalence of atlases)**.** Two atlases $\mathcal{A}$ and $\mathcal{A}'$ (on a fixed topological manifold) are said to be **equivalent**, if their union is still on atlas.

Definition 1.4 (Smooth manifold [Lee13, p.13])**.** A **smooth manifold** $M = (M, [\mathcal{A}])$ consists of

- a topological manifold M,
- an equivalence class $[\mathcal{A}]$ of smooth atlases on M.

Notation: For two smooth manifolds M, N, when writing $M = N$, we will mean that they are diffeomorphic, i.e. there exists a bijective map $f : M \to N$ such that f and f^{-1} are smooth. (We say a map f between two manifolds M^m, N^n is smooth, if for all $p \in M$, there exists a charts $(U \ni p, \varphi)$ and $(V \subset M, \psi)$, such that (i) $f(U) \subset V$ and (ii) $\psi \circ f \circ \varphi : \varphi(U) \subset \mathbb{R}^m \to \mathbb{R}^n$ is smooth.)

Example 4 (Spheres)**.** *The n-sphere* S^n *with the atlas given by the charts in 2 is a smooth manifold.*

Example 5 (Subset of manifolds)**.** *For a manifold* $M, [\mathcal{A}]$ *any open subset* U *of* M *is a manifold. The atlas is given by restriction of the charts in* $[\mathcal{A}]$ *to* U*. We call* U *an* ***open submanifold*** *of* M*.*

Example 6 (Product of manifolds)**.** *For manifolds* M, N*,* $M \times N$ *is a manifold with the charts*

$$\{(U \times V, (\varphi, \psi)) \mid (U, \varphi), (V, \psi) \text{ charts of } M, N\}$$

Remark. While being a topological manifold is just a property of the topological space M, being a smooth manifold gives the manifold extra structure.

Example 7. *$\mathbb{R}, [(\mathbb{R}, \mathrm{id})]$ and $\mathbb{R}, [\mathbb{R}, x \mapsto x^3]$ are different smooth manifolds, because the transition functions between them are not smooth: $\mathrm{id} \circ (x \mapsto x^3)^{-1} = (x \mapsto x^{\frac{1}{3}})$, hence the atlases are not equivalent.*
But they are diffeomorphic, as the map $x \mapsto x^{\frac{1}{3}}$ is a diffeomorphism between the two manifolds: $(x \mapsto x^3) \circ (x \mapsto x^{\frac{1}{3}}) \circ \mathrm{id}^{-1} = \mathrm{id}$

I am confused a bit, doesn't the diffeomorphism have to be smooth?? Check again.

Example 8. *(Exotic spheres) There exists 15 pairwise non-diffeomorphic smooth structures on S^7. See [KM63] for a construction of these exotic spheres.*

Definition 1.5 (Manifold with boundary [Lee13, p.25])**.** To get a definition of a (smooth or topological) **manifold with boundary**, replace the condition of the manifold being locally Euclidean with the condition that every point has a neighbourhood homeomorphic to an open subspace of $\mathbb{H}^n := \{(x_1, \ldots, x_n) \in \mathbb{R} \mid x_1 \geq 0\}$ (the half space). Naturally, the charts now map into $\mathbb{H}^n$ instead of $\mathbb{R}^n$.

Remark. In this thesis, with **manifold** we will always mean a smooth manifold with boundary, unless specified otherwise.

Definition 1.6 (Boundary [Lee13, p.25])**.** Let M^n be a manifold. A point $x \in M^n$ is called a **interior point** if it admits a neighbourhood homeomorphic to $\mathbb{R}^n$. Otherwise, it is called a **boundary point**. The set of boundary points is denoted by ∂M and is called the **boundary** of M.
If a M is compact and has empty boundary, M is called a **closed manifold**.

Example 9. $([0,1], [([0,1], i)])$*, where i is the inclusion into $\mathbb{R}$, is a 1-manifold. Its boundary is $\partial[0,1] = \{0,1\}$.*

Example 10. *$\mathbb{H}^n$ is a manifold with boundary $\mathbb{R}^{n-1}$*

Example 11. *The n-disk D^n is a manifold with boundary S^{n-1}*

Theorem 1.7 (Boundaries are submanifolds)**.** *The boundary of an n-manifold is a closed $(n-1)$-dimensional (embedded) submanifold.*

Proof. Let $(M^n, [\mathcal{A}])$ be a manifold. A smooth structure of ∂M is given by the restriction of the charts in $[\mathcal{A}]$ to ∂M:

$$\{(U \cap \partial M, \varphi_{|_{U \cap \partial M}}) \mid (U, \varphi) \in [\mathcal{A}]\}$$

Smooth compatability follows from the smooth compatibility of the charts in $[\mathcal{A}]$. This makes ∂M a submanifold of M. It remains to show that it is closed and of codimension 1. The charts of $[\mathcal{A}]$ map every point in ∂M to $\partial \mathbb{H}^n = \mathbb{R}^{n-1}$. If they didn't, we could find a euclidean neighbourhood of that point, contradicting the fact that it was a boundary point in M. As the charts restricted to ∂M no map to $\mathbb{R}^{n-1}$, we have an $(n-1)$-dimensional submanifold without boundary. $\square$

Theorem 1.8 (Collar theorem [BD70, I, Satz 1.5]). *Let M be a manifold. Then there exists a neighbourhood U of ∂M with a diffeomorphism $s : \partial M \times [0,1) \to U$ with $s(\partial M \times \{0\}) = \partial M$. U is called a* ***collar*** *of ∂M in M.*

Proof. See [Lee13, p.223] for a proof. □

Observation. *If ∂M has multiple path components, then we can find a collar for each path component.*

Definition 1.9 (Tangent space [Lee13, p.72]). The **tangent space** of a manifold M at a point $p \in M$, denoted by T_pM is the set of equivalence classes of smooth curves $\gamma : [-\varepsilon, \varepsilon] \to M$, $\gamma(0) = p$ with the equivalence relation $\gamma_1 \sim \gamma_2 :\Leftrightarrow$ for any smooth function defined in a neighbourhood of p, we have $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ ($\varepsilon > 0$ depends on γ)

Remark. For an n-manifold, T_pM is an n-dimensional vector space over $\mathbb{R}$ for every point $p \in M$.

Definition 1.10 (Differential [Lee13, p.55]). Let $f : M \to N$ be a smooth map between two manifolds. For $p \in M$, the **differential of f at p** is given by

$$df_p : T_pM \to T_{f(p)}N$$

$$[\gamma] \mapsto [F \circ \gamma]$$

Definition 1.11 (critical value [Lee13, p.105]). Let $f : M \to N$ be a smooth map between two manifolds. A point $p \in M$ is a **critical point** of f, if the differential $df_p : T_pM \to T_{f(p)}N$ fails to be surjective. Otherwise, it is called a **regular point**.
A point $q \in N$ is a **critical value** of f, if $f^{-1}(q)$ containa a critical point of f. Otherwise, it is called a **regular value**.

For a smooth map $f : M^n \to \mathbb{R}$ between two manifolds, r a regular value of f, $\{p \in M : f(p) \leq r\} = f^{-1}(-\infty, r]$ is an n-dimensional submanifold of M.

Theorem 1.12 (Sard's theorem [Lee13]). *For a smooth map $f : M \to N$ between two manifolds, the set of critical values of f has measure zero in N.*

Proof. See [Lee13] for a proof. □

Definition 1.13 (Connected sum). to be done (maybe). Maybe define it in an example.

Definition 1.14 (Separating function). to be done. I need this in an example that still needs to be done (pair of pants).

The problem with working with manifolds is that they are, in higher dimensions, hard to classify up to homeomorphism or diffeomorphism. Bordism is a way to classify manifolds up to a weaker equivalence relation, which is easier to work with.

1.2 Unoriented bordism

1.2.1 Definitions

Definition 1.15 (Singular manifold [BD70, II, Definition 1.1])**.** Let X be a topological space. An n-dimensional **singular manifold** in X is a pair (M, f) of a compact manifold M and a continuous map $f : M \to X$.
The **boundary** of a singular manifold is $\partial(M, f) := (\partial M, \partial f) := (\partial M, f_{|\partial M})$.

Definition 1.16 (Nullbordant [BD70, II, Definition 1.2])**.** Let (M, f) be a singular n-manifold in X. We say that (M, f) is **nullbordant**, if there exists a singular $(n+1)$-manifold (B, F) in X, such that $\partial(B, F) = (M, f)$.

B, F is then called a **nullbordism** of (M, f).

Example 12. *For $X = \mathrm{pt}$, and $M = S^n$, we have a nullbordism $B = D^{n+1}$, the disk. (f can be omitted, as it is constant.)*

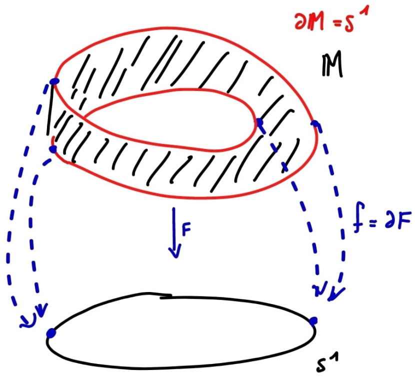
Example 13. *For $X = \mathrm{pt}$, and $M = \mathbb{T}^2$, the torus, $S^1 \times D^2$, the filled torus, is a nullbordism.*

Remark. These examples can be generalized to just f being a constant map, and B any manifold such that $\partial B = M$.

Example 14. *For $M = \emptyset$, any closed manifold is a nullbordism of M, no matter what the space X is. (Again, f can be omitted, as it is the empty map.)*

Observation. *A singular manifold (M, f) can be nullbordant even if f is not nullhomotopic.*

Example 15. *$X = S^1$, $M = S^1$, and f is given by wrapping around the circle twice, a nullbordism is given by the Möbius strip $\mathbb{M}$ with F as projection onto the circle.*



Definition 1.17 (Bordant [BD70, II, Definition 1.3])**.** Let (M, f) and (N, g) be singular manifolds in X. We say that (M, f) and (N, g) are **bordant**, if their sum $(M, f) + (N, g) := (M + N, (f, g)) := (M \sqcup N, f \sqcup g)$ is nullbordant.

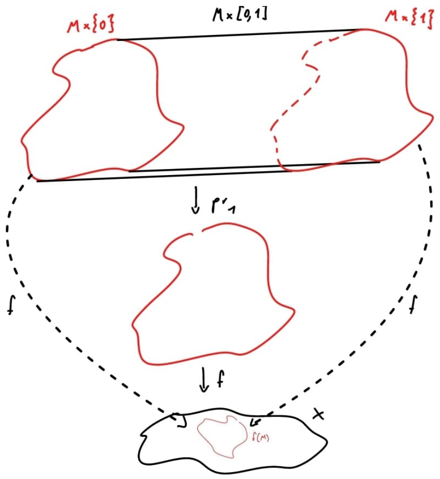
A nullbordism of $(M, f) + (N, g)$ is called a **bordism** between (M, f) and (N, g).

We will refer to this relation as **bordism relation**.

Remark.

$$(M, f) + (\emptyset, g) \text{ are bordant } \iff (M, f) \text{ is nullbordant}$$

Example 16 (Cylinder)**.** *For an arbitrary X, and $(M_0, f_0) = (M_1, f_1)$, we always get the cylinder as a bordism: $(M \times [0,1], f \circ \mathrm{pr}_1)$, where pr_1 is the projection onto the first factor.*



Example 17 (TODO: Pair of pants)**.** *For $X = \mathrm{pt}$, $(M_1, f_1) = (M, f) \sqcup (M, f)$ and $M_2 = M \# M$.*

Look this up again. Maybe it suffices to take X path-connected. Draw a picture. Probably to do: separating function

Example 18. *$\{\mathrm{pt}\}$ and $\{\mathrm{pt}\} \sqcup \{\mathrm{pt}\}$ are not bordant, as 1-manifolds either have 2 or 0 boundary points. (Compact 1-manifolds can be classified up to homeomorphism as the circle S^1 (no boundary) and the interval $[0,1]$ (two boundary points).)*

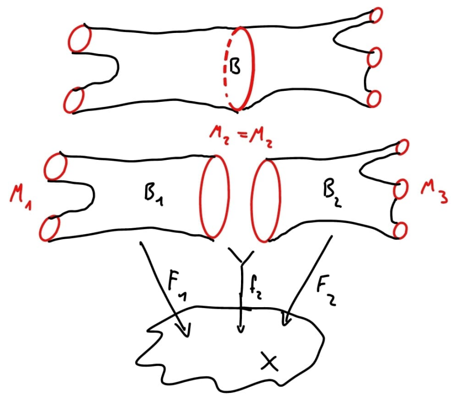
Remark. It follows that an odd number of points is never nullbordant as a singular manifold in $\{\mathrm{pt}\}$, but an even number of points is. (This gives us $\mathfrak{N}_0(\mathrm{pt}) = \mathbb{F}_2$, as we will see later.)

Proposition 1.18. *[BD70, II, Satz 1.4] Being bordant is an equivalence relation on the set of closed singular manifolds.*

Proof. [BD70], [CF64, p.8]

- Symmetry: Follows from the symmetry of the disjoint union. If (M, f) and (N, g) are bordant, then there exists a nullbordism of $(M, f) + (N, g) = (M \sqcup N, f \sqcup g) = (N \sqcup M, g \sqcup f) = (N, g) + (M, f)$. So, a bordism between (M, f) and (N, g) is also bordism between (N, g) and (M, f).

- Reflexivity: We have already constructed a bordism between (M, f) and itself in example 16.
- Transitivity: Let (B_1, F_1) be the bordism between (M_1, f_1) and (M_2, f_2), and let (B_2, F_2) be the bordism between (M_2, f_2) and (M_3, f_3). Then we can "glue" the two bordisms together at the common boundary (M_2, f_2). We only need to check that the gluing is smooth, i.e. $(B_1, F_1) \cup_{(M_2,f_2)} (B_2, F_2)$ is a smooth manifold. By theorem 1.8, we can find a collar of M_2 in both B_1 and B_2. Both these collars have the induced smooth structure of $M_2 \times [0, 1)$. Glueing the bordisms along M_2 glues the collars to something diffeomorphic to $M_2 \times (-1, 1)$, as the smooth structures align in the collars. Since smoothness is local and we now have that the glueing is smooth for a open neighbourhood or M_2, the whole glueing is smooth.



$\square$

Remark. We required the singular manifolds to be closed, because on manifolds with non-empty boundary, the bordism relation does not make any sense, as they cannot be the boundary of another manifold (see theorem 1.7).

Definition 1.19 (bordism group [BD70, II, Definition 1.5]). The equivalence classes of the bordism relation are called **bordism classes**. The bordism class of (M, f) is denoted by $[M, f]$. The set of bordism classes of n-dimensional singular manifolds in X is denoted by $\mathfrak{N}_n(X)$ and is called the n**-th bordism group of** X.

$$\mathfrak{N}_n(X) = \{\text{closed singular } n\text{-manifolds in } X\}/\text{bordism}$$

Observation. *This might seem similar to the definition of singular homology groups. They were also defined by the quotient of the kernel and the image of a boundary map. In our case, the boundary map is ∂, defined in definition 1.15. Let us introduce an index for this map to denote the dimension: $\partial_n : \{n\text{-manifolds}\} \to \{\text{closed } (n-1)\text{-manifolds}\}$. Then:*

$$\mathfrak{N}_n = \ker(\partial_n)/\operatorname{im}(\partial_{n+1})$$

Theorem 1.20. *[BD70, II, Satz 2.1] The bordism groups are abelian groups with the operation + defined in definition 1.17:*

$$[M_1, f_1] + [M_2, f_2] = [M_1 + M_2, (f_1, f_2)]$$

Every element in this group has order at most 2*, making* $\mathfrak{N}_n(X)$ *a* $\mathbb{F}_2$*-vector space.*

Proof. [BD70] The neutral element is the bordism class of the empty manifold (i.e. the class of all nullbordant manifolds).
"+" is associative and commutative, because the disjoint union is associative and commutative.
It is well-defined: Let $(M_1, f_1), (M_1', f_1') \in [M_1, f_1]$, and $(M_2, f_2), (M_2', f_2') \in [M_2, f_2]$. Moreover, let (B_1, F_1) and (B_2, F_2) be bordisms between (M_1, f_1) and (M_1', f_2'), respectively between (M_2, f_2) and (M_2', f_2'). Then, a bordism between $(M_1, f_1) + (M_2, f_2)$ and $(M_1', f_1') + (M_2', f_2')$ is given by $(B_1, F_1) + (B_2, F_2)$.
Since being bordant is a reflexive, every element is its own inverse. $\square$

Definition 1.21 (Product map [BD70, pp.12-13])**.** Let X, Y be two topological spaces. Then we define a product map

$$\cdot : \mathfrak{N}_p(X) \times \mathfrak{N}_q(Y) \to \mathfrak{N}_{p+q}(X \times Y)$$

as

$$([M, f], [N, g]) \mapsto [M \times N, f \times g]$$

Observation. *This map is well defined, as for two bordant singular manifolds* (M, f) *and* (M', f') *with bordism* (B, F)*, we get a bordism between* $(M \times N, f \times g)$ *and* $(M' \times N, f' \times g)$ *by* $(B \times N, F \times g)$*. This map is also bilinear (as a map between* $\mathbb{F}_2$*-vector spaces). Scalar multiplication is defined as* $0 \cdot [M, f] = [\emptyset, \emptyset \to X]$*,* $1 \cdot [M, f] = [M, f]$*. As* $[\emptyset] \cdot [N, g] = [\emptyset] = [M, f] \cdot [\emptyset]$*, we conclude for any* $a \in \mathbb{F}_2$*:* $[M, f] \cdot (a \cdot [N, g]) = a \cdot ([M, f] \cdot [N, g]) = (a \cdot [M, f]) \cdot [N, g]$*. Additivity follows, as addition is defined as disjoint union:*

$$\begin{aligned}([M, f] + [M', f']) \cdot [N, g] &= [M + M', (f, f')] \cdot [N, g] \\ &= [((M + M') \times N), (f, f') \times g] \\ &= [(M \times N) + (M' \times N), (f \times g, f' \times g)] \\ &= [M, f] \cdot [N, g] + [M', f'] \cdot [N, g]\end{aligned}$$

Notation. For $X = \{\mathrm{pt}\}$, we will write $\mathfrak{N}_n$ for $\mathfrak{N}_n(X)$ and for elements of $\mathfrak{N}_n$, we will omit the map from the notation ($[M] = [M, f]$).

Definition 1.22 (graded bordism ring [BD70, Satz 2.2])**.**

$$\mathfrak{N}_* := \bigoplus_{n \in \mathbb{Z}} \mathfrak{N}_n$$

is a $\mathbb{Z}$-graded ring over $\mathbb{F}_2$ via + and $\cdot$ and is called the **bordism ring**.

Remark. As $\mathbb{F}_2$ is a field, $\mathfrak{N}_*$ is a graded vector space over $\mathbb{F}_2$.

Definition 1.23 (graded bordism module [BD70, Satz 2.3])**.**

$$\mathfrak{N}_*(X) := \bigoplus_{n\in\mathbb{Z}} \mathfrak{N}_n(X)$$

is a $\mathbb{Z}$-graded module over $\mathfrak{N}_*$. It is called the **bordism module**. Explicitly, the product map acts as follows:

$$[M] \cdot [N, f] = [M \times N, f \circ \mathrm{pr}_2]$$

where $\mathrm{pr}_2 : M \times N \to N$ is the projection onto the second factor.

Remark. Notice that in this definition, this is a left module, but since we could just project to the first factor instead, we also get a right module structure.

1.2.2 The Eilenberg-Steenrod axioms

The Eilenberg-Steenrod axioms are a set of axioms that characterize the homology and cohomology theories.

Definition 1.24 (Homology theory [Lüc05, Definition 1.1])**.** A **homology theory** $\mathcal{H}_* = (\mathcal{H}_*, \partial_*)$ with coefficients in R-modules is a covariant functor

$$\mathcal{H}_* : \mathrm{TOP}^2 \to \mathbb{Z}\text{-graded } R\text{-modules}$$

together with a natural transformation

$$\partial_* : \mathcal{H}_* \to \mathcal{H}_{*-1} \circ I$$

where I is a forgetful functor $I : \mathrm{TOP}^2 \to \mathrm{TOP}^2$, given by $I(X, A) = (A, \emptyset)$. We will often write X for $(X, \emptyset)$ for any space X.
$\mathcal{H}_*$ has to satisfy the following **Eilenberg-Steenrod axioms for homology theories**:

- **Homotopy invariance**
  Let $f, g : (X, A) \to (Y, B)$ be homotopic maps. Then for all $n \in \mathbb{Z}$, we have
  $$\mathcal{H}_n(f) = \mathcal{H}_n(g) : \mathcal{H}_n(X, A) \to \mathcal{H}_n(Y, B)$$
- **Long exact sequence**
  Let (X, A) be a pair of spaces. Then for all $n \in \mathbb{Z}$, we have the long exact sequence of homology groups:
  $$\dots \xrightarrow{\partial_{n+1}(X,A)} \mathcal{H}_n(A) \xrightarrow{\mathcal{H}_n(i)} \mathcal{H}_n(X) \xrightarrow{\mathcal{H}_n(j)} \mathcal{H}_n(X, A) \xrightarrow{\partial_n(X,A)} \mathcal{H}_{n-1}(A) \to \dots$$
- **Excision**
  Let $A \subset B \subset X$ be subspaces of X such that $\overline{A} \subset B^\circ$. Then the inclusion $i : (X \setminus B, A \setminus B) \to (X, A)$ induces an isomorphism of homology groups for all $n \in \mathbb{Z}$:
  $$\mathcal{H}_n(i) : \mathcal{H}_n(X \setminus A, B \setminus A) \xrightarrow{\cong} \mathcal{H}_n(X, B)$$

Sometimes one adds the following axioms:

- **Disjoint union axiom**
  Let $\{X_i\}_{i\in I}$ be a family of topological spaces. Let $j_i : X_i \to \coprod_{i\in I} X_i$ be the inclusion. Then for all $n \in \mathbb{Z}$, we have a bijection:

$$\bigoplus_{i\in I} \mathcal{H}_n(j_i) : \bigoplus_{i\in I} \mathcal{H}_n(X_i) \xrightarrow{\cong} \mathcal{H}_n\left(\coprod_{i\in I} X_i\right)$$

- **Dimension axiom**
  For the point space pt, we have

$$\mathcal{H}_n(\mathrm{pt}) \cong \begin{cases} R & n = 0 \\ \{0\} & n \neq 0 \end{cases}$$

For homology theories, the abbreviation $\mathcal{H}_* = \mathcal{H}_*(\{\mathrm{pt}\}) = \mathcal{H}_*(\{\mathrm{pt}\}, \emptyset)$ is often used.

Remark. If a homology theory satisfies the dimension axiom, it is called an **ordinary homology theory**.

As homology theories are defined as functors from TOP^2, we will extend the definition of bordism to pairs of topological spaces.

1.2.3 Relative bordism

Definition 1.25 (relative bordism [Die08, pp.524-525])**.** For a pair of topological spaces (X, A), we call $(M, f) = (M, \partial M, f)$ a **singular manifold in** (X, A), if $f : (M, \partial M) \to (X, A)$ is a continuous map of pairs.
Two n-dimensional singular manifolds (M_0^n, f_0) and (M_1^n, f_1) in X are called **bordant**, if there exists an $(n+1)$-dimensional singular manifold (B^{n+1}, F) in X, such that:

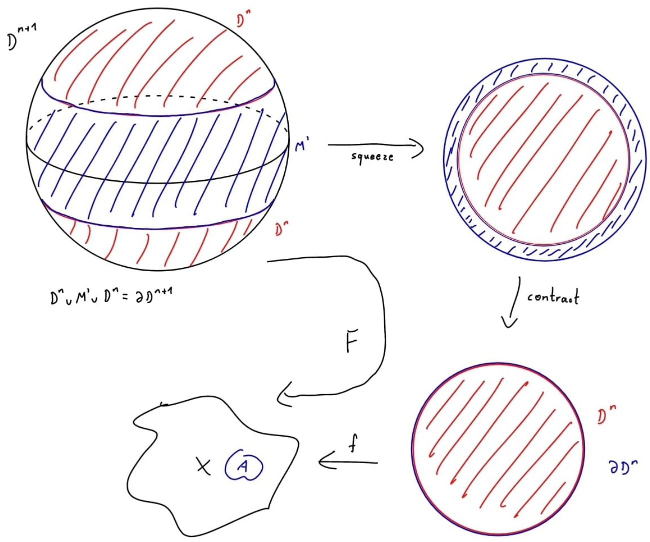
- ∂B can be decomposed as $\partial B = M_0 \cup M_1 \cup M'$, where M_0, M_1, M' are considerd as embedded submanifolds of B with boundary, such that $\partial M' = \partial M_0 \sqcup M_1$, for $i \in \{0,1\}$, $M_i \cap M' = \partial M_i$ and $\partial M_0 \cap \partial M_1 = \emptyset$.
- $\partial F_{|M_i} = f_i$, for $i \in \{0,1\}$
- $F(M') \subseteq A$

As before, (B, F) will be called a **bordism** between (M_0, f_0) and (M_1, f_1) and a **null-bordism** of (M_0, f_0) if $M_1 = \emptyset$.

Remark. If we take $A = \emptyset$, we get the definition of (absolute) bordism as we had before. In particular, if (M, f) and N, g are bordant in the absolute sense, then they are also bordant in the relative sense with the same bordism (but there may be more possible bordisms in the relative sense).

Observation. *Now, the bordism relation also makes sense fo manifolds with non-empty boundary.*

Example 19. *Consider the disk (D^n, f) as a singular manifold in (X, A). We have a bordism B, F between (D^n, f) and itself by $B = D^{n+1}$. We have to cover $\partial D^{n+1} = S^n$ by two D^ns and M', which can be done by taking almost all of each the upper and the lower hemisphere as D^n (In this example, the chosen embedding heavily matters. We will embed them such that there is only one pair of points such that the antipodal points in the embedding correspond to each other.) and the remaining cylinder around the equator as M'. Now F just has to map M' to A (and along with it, also ∂M_0 and ∂M_1, as desired). To do this, F will "squeeze" the disk such that the D^ns are identified together and will then map the resulting disk with the remains of M' to $D^n, \partial D^n$ via the identity on D^n and contracting M' to its boundary. Lastly, apply f.*



Remark. The above example is just the same as the cylinder bordism in example 16 (if we had constructed the cylinder we would not have had to worry about the embedding). This cylinder bordism now also works for manifolds with non-empty boundary, we can take its lateral surface ($= \partial M_0 \times [0, 1]$) as M'.

Theorem 1.26. *[Die08, p.525] Relative bordism is an equivalence relation on the set of singular manifolds in X.*

Proof. The proof is similar to the proof of proposition 1.18. Symmetry is clear, for reflexivity, we now also have the cylinder as stated in the above remark. For transitivity, we again glue the bordisms along their common boundary M_2, but as M_2 may have boundary now, we glue the bordism along a open neighbourhood U (in ∂B_1 and ∂B_1) of M_2. To see this is smooth, we again look at the collars of ∂B_i. This time we restrict the collars to U and see that the collars are diffeomorphic to $M_2 \times (-1, 1)$. The rest of the proof is the same as in Proposition 1.18. □

Relative bordism groups are defined in the same way as absolute bordism groups, they also are abelian by the same proof. The product map is defined as

$$\cdot : \mathfrak{N}_p(X, A) \times \mathfrak{N}_q(Y, B) \to \mathfrak{N}_{p+q}(X \times Y, A \times B)$$

on elements, it does the same as before. Again, we get a graded module structure

$$\mathfrak{N}_*(X, A) := \bigoplus_{n \in \mathbb{Z}} N_n(X, A)$$

over $\mathfrak{N}_*$, the relative bordism module.

Lemma 1.27. *[Die91, VIII, Lemma 13.10] Let $[M, f] \in \mathfrak{N}_n(X, A)$ and N an n-dimensional submanifold of M. Suppose that $[N, f_{|N}] \in \mathfrak{N}_n(X, A)$ and $f(M \setminus N) \subseteq A$. Then $[M, f] = [N, f_{|N}]$ in $\mathfrak{N}_n(X, A)$.*

Proof. [Die91] We need to show that (M, f) and $(N, f_{|N})$ are bordant.
Let $B = M \times [0, 1]$ the cylinder. $\partial B = M \times \{0\} \cup M \times \{1\} \cup \partial M \times I$. Define $F : B \to X$ as $F(p, t) = f(p)$.
Claim: (B, F) is a bordism between $(M \times \{0\}, f \circ \mathrm{pr}_1)$ and $(N \times \{1\}, f_{|N} \circ \mathrm{pr}_1)$.
As both are embedded submanifolds of B, we only need to check that $\partial B \setminus ((M \times \{0\}) \cup (N \times \{1\}))$ is mapped into A by F.

$$\begin{aligned} F(\partial B \setminus ((M \times \{0\}) \cup (N \times \{1\}))) &\subseteq F((\partial M \times I) \cup ((M \times \{1\}) \setminus (N \times \{1\}))) \\ &= f(\partial M) \cup f(M \setminus N) \subseteq A \end{aligned}$$

□

1.2.4 Bordism homology

Lemma 1.28. *[BD70, II, Satz 3.2] Relative bordism is a covariant functor*

$$\mathfrak{N}_* : \mathrm{TOP}^2 \to \textit{graded } \mathfrak{N}_* \textit{modules}$$

Proof. [BD70] Let $(X, A) \in \mathrm{Ob}(\mathrm{TOP}^2)$, we already saw

$$(X, A) \xmapsto{\mathfrak{N}_*} \mathfrak{N}_*(X, A)$$

For a map $\mathrm{Mor}(\mathrm{TOP}^2) \ni f : (X, A) \to (Y, B)$, we take the induced map on the bordism groups:

$$f_* := \mathfrak{N}_*(f) : \mathfrak{N}_*(X, A) \to \mathfrak{N}_*(Y, B)$$

given by

$$f_*[M, g] = [M, f \circ g]$$

Then we get that $\mathfrak{N}_*(\mathrm{id}_{(X,A)}) = \mathrm{id}_{\mathfrak{N}_*(X,A)}$ and for $f : (X, A) \to (Y, B), g : (Y, B) \to (Z, C)$, we have for any $[M, h] \in \mathfrak{N}_*(X, A)$:

$$(g \circ f)_*[M, h] = [M, g \circ f \circ h] = g_*[M, f \circ h] = g_* \circ f_*[M, h]$$

□

Lemma 1.29 (Naturality of the boundary map). *The following diagram commutes for any $f : (X, A) \to (Y, B)$ and $n \in \mathbb{Z}$:*

$$\begin{array}{ccc} \mathfrak{N}_n(X, A) & \xrightarrow{\partial_n} & \mathfrak{N}_{n-1}(A) \\ {\scriptstyle f_*}\big\downarrow & & \big\downarrow{\scriptstyle (f_{|A})_*} \\ \mathfrak{N}_n(Y, B) & \xrightarrow{\partial_n} & \mathfrak{N}_{n-1}(B) \end{array}$$

Proof. Let $[M, g] \in \mathfrak{N}_n(X, A)$. $\partial_n[M, g] = [\partial M, g_{|\partial M}] \in \mathfrak{N}_{n-1}(A)$, as ∂M is closed and g maps ∂M to A. Now $f_{|A*}([\partial M, g_{|\partial M}]) = [\partial M, f \circ g_{|\partial M}] \in \mathfrak{N}_{n-1}(B)$.
On the other side, $(\partial_n \circ f_*)[M, g] = \partial_n[M, f_* \circ g] = [\partial M, (f \circ g)_{|\partial M}] \in \mathfrak{N}_{n-1}(B)$, which is the same as the previous composition, hence,

$$\partial_n \circ f_* = f_{|A*} \circ \partial_n$$

and the diagram commutes. □

Lemma 1.30 (Homotopy invariance). *[BD70, II, Satz 3.1] $\mathfrak{N}_*$ is homotopy invariant.*

Proof. [BD70][CF64, Chapter I, 5.5] Let $f, g : (X, A) \to (Y, B)$ be homotopic maps. Let $h : (X \times I, A \times I) \to (Y, B)$ be a homotopy between f and g. Then, for $[M, f] \in \mathfrak{N}_*(X, A)$, we have a bordism between $f_*[M, F]$ and $g_*[M, F]$ by the cylinder $(M \times I, h \circ (F \times \mathrm{id}_I))$. □

Remark. For a closed manifold M with $[M, f] = 0 \in \mathfrak{N}_n(X, A)$ and a nullbordism (B, F) of (M, f). Then $\partial B \setminus M$ is a closed n-dimensional submanifold of B. For a proof, see [Zha23, Lemma 5.4].

Lemma 1.31 (Long exact sequence [Die08, Proposition 21.1.9]). *$\mathfrak{N}_*$ satisfies the long exact sequence axiom.*

Proof. [Die08] Let i, j be the inclusion maps $i : A \to X, j : X = (X, \emptyset) \to (X, A)$.
Claim: The sequence

$$\dots \xrightarrow{\partial_{n+1}} \mathfrak{N}_n(A) \xrightarrow{i_*} \mathfrak{N}_n(X) \xrightarrow{j_*} \mathfrak{N}_n(X, A) \xrightarrow{\partial_n} \mathfrak{N}_{n-1}(A) \xrightarrow{i_*} \dots$$

is exact.

- **Exactness at $\mathfrak{N}_n(A)$** $i_* \circ \partial = 0$, as for $[M, f] \in \mathfrak{N}_{n+1}(X, A)$, $\partial_{n+1}[M, f] = [\partial M, f_{|\partial M}]$, which, considered as an element in $\mathfrak{N}_n(X)$ is nullbordant via the nullbordism (M, f).

  Let $(M, f) \in \mathfrak{N}_n(A)$ with nullbordism (B, F) in X. Then, $\partial_{n+1}[B, F] = [M, f]$.

- **Exactness at $\mathfrak{N}_n(X)$** Let $[M, f] \in \mathfrak{N}_n(A)$. Choose $V = \emptyset$ and use lemma 1.27 to get $[M, f] = 0$ in $\mathfrak{N}_n(X, A)$, so $j_* \circ i_* = 0$.

  Now let $[M, f] \in \ker(j_*) \subseteq \mathfrak{N}_n(X)$. Then there exists a singular manifold (B^{n+1}, F) in X, A of $[M, f]$. (B, F) is a bordism between M, f and $\partial B \setminus M, F_{|\partial B \setminus M}$, as $\partial B \setminus M$ is a closed submanifold of B by the previous remark. Since $F(\partial B \setminus M) \subseteq A$, $[\partial B \setminus M, F_{|\partial B \setminus M}] \in \mathfrak{N}_n(A)$. So, $i_*[\partial B \setminus M, F_{|\partial B \setminus M}] = [M, f] \in \mathfrak{N}_n(X)$.

- **Exactness at $\mathfrak{N}_n(X, A)$** $\partial \circ j_* = 0$ holds because every element in $\mathfrak{N}_n(X)$ is a closed manifold, so it has empty boundary, hence applying the boundary map gets us to $[\emptyset, \emptyset \to X]$ which is the zero element in $\mathfrak{N}_{n-1}(A)$.

  Let $\partial[M, f] = 0$, and $[B, F]$ be a nullbordism of $[\partial M, f_{|\partial M}]$. Identify (M, f) and (B, F) along ∂M. This is smooth by the same argument as in theorem 1.18. Call the resulting singular manifold (C, g). This has no boundary, so $[C, g] \in \mathfrak{N}_n(X)$ Now with lemma 1.27, get $j_*[C, g] = [M, f]$.

$\square$

Only the excision axiom is left to check. But to see the excision property, we need some preliminary lemmas.

Lemma 1.32 ([CF64, Chapter I, 3.1]). *Let M^n be a closed manifold, let $K, L \subseteq M$ closed, such that $K \cap L = \emptyset$. Then there exists a closed n-dimensional submanifold with boundary N of M such that $K \subseteq N$ and $L \cap N = \emptyset$.*

Proof. [CF64] $K \cap \partial M$ and $L \cap \partial M$ are still disjoint, closed in ∂M, so we can find disjoint closed subsets $K', L' \subseteq \partial M$ such that $K \subseteq \mathring{K}'$, $L \subseteq \mathring{L}'$. We find a collar C of ∂M by theorem 1.8 and identify it by $\partial M \times [0, 1)$. As M is compact, there exists a $t \in (0, 1)$ such that $L \cap (\partial M \times [0, t)) \subseteq L'$ and $K \cap (\partial M \times [0, t)) \subseteq K'$. Now $M' := M \setminus (\partial M \times [0, t))$ is a closed n-dimensional submanifold of M. By Urysohn's lemma, there exists a smooth function $\alpha : M' \to [0, 1]$ such that $\alpha_{|(K' \times \{t\}) \cup (K \cap M')} = 0$ and $\alpha_{|(L' \times \{t\}) \cup (L \cap M')} = 1$. We can extend α to M by $\alpha(p, s) := \alpha(p, t)$ for $p \in \partial M$ and $s \in [0, t)$. By theorem 1.12, there is a regular value $r \in (0, 1)$ of α. Then $N := \alpha^{-1}([0, r]) = \alpha^{-1}(-\infty, r]$ is a n-dimensional closed submanifold of M. By construction, $K \subseteq N$, $L \cap N = \emptyset$.
It remains to show that N is smoothable, but we will omit the proof here. $\square$

Lemma 1.33 ([Zha23, Lemma 5.8]). *Let $K, L, M, N\alpha, r$ be as in lemma 1.32. Then,*

$$\partial N \subseteq \partial M \cup \alpha^{-1}(r) \subseteq \partial M \cup ((M \setminus K) \cap (M \setminus L))$$

Proof. [Zha23] *First inclusion:* Assume $p \in \partial N \setminus \partial M$. We need to show that $\alpha(p) = r$. By assumption, p is an interior point of M, so we can find a euclidean neighbourhood U of p in M. As $p \in N$, $\alpha(p) \leq r$. Suppose $\alpha(p) < r$. Then, for a $r' \in (\alpha(p), r)$, we have $p \in \alpha^{-1}([0, r')) \subseteq N$. Since $\alpha^{-1}([0, r'))$ is open in M, it is open in N and $U \cap V$ is an open euclidean neighbourhood of p in N. But then, p is an interior point of N, contradicting the assumption that $p \in \partial N$. So, $\alpha(p) = r$.
Second inclusion: Since $\alpha_{|K} = 0$ and $\alpha_{|L} = 1$, we have $\alpha^{-1}(r) \subseteq M \setminus (K \cup L)$. $\square$

Lemma 1.34 (Excision axiom [Zha23, Theorem 5.10]). *Let (X, A, Z) be a triple of topological spaces satisfying $\overline{Z} \subseteq \mathring{A}$. Then the inclusion map $i : (X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces an isomorphism of bordism groups:*

$$i_* : \mathfrak{N}_n(X \setminus Z, A \setminus Z) \xrightarrow{\cong} \mathfrak{N}_n(X, A)$$

Proof. [Zha23] **Surjectivity**: Let $[M, f] \in \mathfrak{N}_n(X, A)$. Then the preimages $K = f^{-1}(X \setminus \mathring{A})$ and $L = f^{-1}(\overline{Z})$ are disjoint and closed subsets of M. By lemma 1.32, there exists a closed submanifold with boundary $N \subset M$ such that $K \subseteq N$ and $L \cap N = \emptyset$.
From $L \cap N = \emptyset$, it follows that $f(N) \subseteq X \setminus \overline{Z}$. By 1.33, we have $\partial N \subseteq \partial M \cup ((M \setminus K) \cap (M \setminus L))$. So, for any $p \in \partial N$, we have either $p \in \partial M$, implying $f(p) \in A$, or $p \in (M \setminus K)$, implying $f(p) \in \mathring{A}$. In any case, we get $f(\partial N) \subseteq A \setminus \overline{Z}$, so $[N, f_{|_N}] \in \mathfrak{N}_n(X \setminus Z, A \setminus Z)$. As $f^{-1}(X \setminus \mathring{A})$, we get $f(M \setminus N) \subseteq \mathring{A}$. By lemma 1.27, we get $i_*[N, f_{|_N}] = [M, f]$.
Injectivity: Take $[M, f] \in \mathfrak{N}_n(X \setminus Z, A \setminus Z)$ such that $i_*[M, f] = 0$ in $\mathfrak{N}_n(X, A)$. Then there exists a nullbordism (B, F) with $F(\partial B \setminus M) \subseteq A$.
Again, let $K = F^{-1}(X \setminus \mathring{A})$, $L = F^{-1}(\overline{Z})$. By lemma 1.32, we have a submanifold $N^{n+1} \subseteq B$ such that $K \subseteq N$, $N \cap L = \emptyset$. Then, $[\partial N, F_{|_{\partial N}}] \in \mathfrak{N}_n(X \setminus Z, A \setminus Z)$ nullbordant by $(N, F_{|_N})$.
Claim: $M \cap \partial N = M \cap N$.
"$\subseteq$" is clear. For "$\supseteq$", take $p \in M \cap N$. Then $p \in \partial B$, so there exists a chart $\varphi : U \to \mathbb{H}^{n+1}$ of B, such that $\varphi(p) \in \partial \mathbb{H}^{n+1}$. So, $p \in \partial N$.
Then $M \cap \partial N$ is an submanifold of M, because

$$M \cap \partial N = M \cap N = (\alpha_{|_M})^{-1}([0, r]), \tag{$\star$}$$

where α is again such that $N = \alpha^{-1}([0, r])$. By theorem 1.12, we assume that r is also a regular value of $\alpha_{|_M}$. Now we see that $[M \cap \partial N, f_{|_{M \cap \partial N}}] \in \mathfrak{N}_n(X \setminus Z, A \setminus Z)$: $f(M \cap \partial N) \subseteq f(M) \subseteq X \setminus Z$ and $f(\partial(M \cap \partial N)) \subseteq A \setminus Z$ because of ($\star$) and lemma 1.33, as $\partial(M \cap \partial N) \subseteq \partial M \cup (B \setminus (K \cup L))$.
Claim: $[M, f] = [M \cap N, f_{|_{M \cap N}}] \in \mathfrak{N}_n(X \setminus Z, A \setminus Z)$.
By lemma 1.27, it is enough to show that $f(M \setminus (M \cap N)) = f(M \setminus M \cap \partial N) \subseteq A \setminus Z$. Let $p \in M \setminus \partial N = M \setminus N$, then $f(p) \in X \setminus Z$ because $p \in M$, and $f(p) \in A$, because $p \notin N$.
By exactly the same argument, $[M \cap N, f_{|_{M \cap N}}] = [\partial N, F_{|_{\partial N}}] \in \mathfrak{N}_n(X \setminus Z, A \setminus Z)$. Since $M \cap N$ is an submanifold of M, it is also an submanifold of ∂N. Now, we only need to show that $F(\partial N \setminus (M \cap N)) = F(\partial N \setminus M) \subseteq A \setminus Z$. We know $\partial N \subseteq \partial B \cup (B \setminus (K \cup L))$ (lemma 1.33). Let $p \in \partial N \setminus M$, then either $p \in (B \setminus (K \cup L)) \setminus M$ or $p \in \partial B \setminus M$. In the first case, $F(p) \in A \setminus Z$, so we are done. In the second case, we know $p \notin L$, because $B \cap L = \emptyset$. So, $F(p) \notin \overline{Z}$. By construction, we have $F(\partial B \setminus M) \subseteq A$, so $F(p) \in A$. $\Rightarrow g(p) \in A \setminus Z$.
In total, we have shown that $(M, f), (M \cap N, f_{|_{M \cap N}}), (\partial N, F_{|_{\partial N}})$ are bordant, so nullbordant, since the last one is nullbordant in $\mathfrak{N}_n(X \setminus Z, A \setminus Z)$ by $(N, F_{|_N})$, so $[M, f] = 0$ in $\mathfrak{N}_n(X \setminus Z, A \setminus Z)$. □

Now it already follows that bordism is a homology theory. Let let's take a look at the other axioms too.

Lemma 1.35 (Disjoint union axiom [Zha23, Theorem 5.3]). *The disjoint union axiom holds for bordism.*

Proof. [Zha23] We need to show that

$$\bigoplus_{i\in I} \mathfrak{N}_n(j_i) : \bigoplus_{i\in I} \mathfrak{N}_n(X_i) \to \mathfrak{N}_n \left(\coprod_{i\in I} X_i \right)$$

is an isomorphism.
Claim:

$$\iota : \bigoplus_{i\in I} [M_i, f_i] \mapsto \left[\coprod_{i\in I} M_i, \coprod_{i\in I} f_i \right]$$

gives us the desired isomorphism.
Well-definedness: All but finitely many $[M_i, f_i]$ are 0. So $\coprod M_i$ is a finite disjoint union of compact manifolds, so it is compact.
Injectivity: Suppose $\iota\left(\bigoplus_i [M_i, f_i]\right) = 0$ in $\mathfrak{N}_n(\coprod_i X_i)$. Then there exists a nullbordism (B, F) of it. B as a space is the disjoint union $\coprod_i B_i := \coprod_i F^{-1}(X_i)$, all of the B_i being open and closed in B. So, the B_i are compact $(n+1)$-manifolds. Also, $\partial(B_i, F_{|B_i}) = (M_i, f_i)$, so all the (M_i, f_i) are nullbordant and the sum $\bigoplus [M_i, f_i] = 0$.
Surjectivity: Suppose $[M, f] \in \mathfrak{N}_n(\coprod_i X_i)$. As in the proof of injectivity, we can write $M = \coprod_i M_i := f^{-1}(X_i)$, with all M_i compact manifolds. Then a preimage of $[M, f]$ under ι is $\bigoplus [M_i, f_{|M_i}]$. □

Observation. *Bordism does not satisfy the dimension axiom.*
Check: Consider $[\mathbb{RP}^2] \in \mathfrak{N}_2$. *There is no* 3*-manifold that has* $\mathbb{RP}^2$ *as its boundary. We can see this by the eulercharacteristic:* $\chi(\mathbb{RP}^2) = 1$. *But boundaries of manifolds always have even euler characteristic [Die08, Proposition 18.6.2]. So,* $\mathfrak{N}_2 \ncong \{0\}$.

Now we can finally conclude:

Theorem 1.36. *Bordism defines a homology theory satisfying the disjoint union axiom.*

Proof. This follows directly from the lemmas 1.28, 1.30, 1.31, 1.34 and 1.35.
As it is not an ordinary homology theory (i.e. the dimension axiom does not hold), we call bordism an **extraordinary** of a **generalized** homology theory. □

1.2.5 Calculations

As we have already noted in example 18, we have $\mathfrak{N}_0 \cong \mathbb{F}_2$. An even number of points is nullbordant, an odd number is not. Let us try to argue for higher dimensions.
The only closed 1-dimensional manifold (up to diffeomorphism) is S^1. As $S^1 = \partial D^2$, we conclude $\mathfrak{N}_1 \cong \{0\}$.
Closed 2-manifolds are classified to be $S^2, \#_i\mathbb{T}^2, \#_i\mathbb{RP}^2$. I.e. they are classified by genus and orientability. We know $S^2 = \partial D^3$ and $\mathbb{T}^2 = \partial(S^1 \times D^2)$. So $\#_i\mathbb{T}^2 = \partial(\#_i D^3)$.

$$
\begin{aligned}
\mathfrak{N}_0 &\cong \mathbb{F}_2\\
\mathfrak{N}_1 &\cong \{0\}\\
\mathfrak{N}_2 &\cong \mathbb{F}_2\\
\mathfrak{N}_3 &\cong \{0\}\\
\mathfrak{N}_4 &\cong \mathbb{F}_2 \oplus \mathbb{F}_2\\
\mathfrak{N}_5 &\cong \mathbb{F}_2
\end{aligned}
$$

Of spheres

1.3 Orientation

Definition 1.37 (Tangent space)**.**

I probably need this already before. (Along with tangent bundle, vector fields)

Definition 1.38 (Orientation on vector spaces [Lee13, p.379])**.** An **orientation** on a vector space V with $\dim V \geq 1$ is an equivalence class of ordered bases $(e_1, \ldots, e_{\dim V})$. Two bases are equivalent, if the basis transformation has positive determinant. For $\dim V = 0$, an orientation is the choice of $\pm$.
This gives us exactly two orientations for any vector space.

Definition 1.39 (Pointwise orientation on manifolds [Lee13, p.380])**.** For each point on a manifold, we have an associated vector space: the tangent space. A **pointwise orientation** on a manifold is a choice of orientation on each tangent space.

Definition 1.40 (Local frame)**.**

Maybe also need global frame.
Check again

Definition 1.41 (Continuous orientation)**.** A pointwise orientation on a manifold M is called **continuous**, if for every point $p \in M$, there exists a neighborhood U of p such that the orientations on the tangent spaces of all points in U are equivalent.

Definition 1.42 (Orientation on manifolds [Lee13, p.380])**.** An **orientation** of a manifold M is a continuous pointwise orientation of M. If there exists an orientation on M, M is called **orientable**.

Definition 1.43 (Oriented manifold [Lee13, p.380])**.** An **oriented manifold** is the pair $(M, \mathcal{O})$, of an orientable manifold M and a choice of orientation $\mathcal{O}$ on M. We will often write just M for an oriented manifold.

Definition 1.44 (vector bundle)**.** To be put somewhere else

Maybe I should say something about orientations covers...

1.4 Oriented bordism

1.4.1 Definitions

Probably I should use less confusing notations (e.g. M^- instead of M).

Now, we have defined additional structure on manifolds. We will adapt our definition of bordism to respect the additional structure.
The Definition of singular manifolds stays the same, but we additionally require M to be oriented now.

Definition 1.45 (bordant [Die08, p.526], [Ati61, p.202]). Two closed singular oriented n-manifolds $(M_0, f_0), (M_1, f_1)$ are called **bordant**, if there exists a singular oriented $n+1$-manifold (B, g) with oriented boundary such that $\partial(B, g) = (M_1, f_1) - (M_0, f_0)$. $(M_1, f_1) - (M_0, f_0)$ is defined as

$$(M_1, f_1) - (M_0, f_0) = (M_1, f_1) + ((M_0, -\mathcal{O}_0), f_0),$$

where $\mathcal{O}_0$ is the orientation on M_0. (B, g) is then called a **oriented bordism** between (M_0, f_0) and (M_1, f_1).

Remark. The definition of being nullbordant follows if we take one of the singular oriented manifolds to be empty.

Proposition 1.46. *Being bordant is an equivalence relation on the set of singular oriented manifolds.*

The proof can be copied from the proof in the oriented case, but we have to check a few more things.

Proof.

- **Symmetry**: If (B, g) is an oriented bordism between (M_0, f_0) and M_1, f_1, then $-(B, g)$ (again, the negative sign is denoting the opposite orientation) is an oriented bordism between (M_1, f_1) and (M_0, f_0).
- **Reflexivity**: The cylinder still works as the proof of reflexivity here. We have defined oriented bordism in this way with giving M_1, f_1 the opposite orientation, so that the cylinder is still a bordism.
- **Transitivity**: Again, here the reversed orientation is important. We have flipped the orientation of both M_1 now once, giving them the same orientation now and can glue them together without any problems on orientations.

□

This should probably be done in more detail.

Definition 1.47 (oriented bordism group). The equivalence classes of the oriented bordism relation called **oriented bordism classes** and the set of oriented bordism classes of n-dimensional singular oriented manifolds in X is denoted by $\Omega_n(X)$. $\Omega_n(X)$ is called the ***n*-th oriented bordism group** of X

$$\Omega_n(X) = \{\text{singular oriented } n\text{-manifolds in } X\}/\text{oriented bordism}$$

Up until now, everything seems to be the same as in the unoriented case, but we will now see a critical difference.

Theorem 1.48. *The oriented bordism groups are abelian groups via the operation +*

Proof. For this proof, the only thing we need to change is for the existence of the inverse. We have seen in an example (to be added) that the elements are not self-inverse anymore. The inverse of $[M, f]$ is now given by $[-M, f]$. We have seen in the proof of 1.46 that $[M, f] + [-M, f] = 0$. $\square$

So, $\Omega_n(X)$ is not a $\mathbb{F}_2$-vector space anymore! This makes $\Omega_n(X)$ harder to compute.

The graded ring Ω_* and the graded module $\Omega_*(X)$ are defined in the same way as in the unoriented case.

I still need to see that this is ok with the orientation.

1.4.2 Relative oriented bordism

The path from absolute oriented bordism to relative oriented bordism is exactly the same as in the unoriented case. We just always need to remember that we reverse the orientations for the second singular oriented manifold.

Again, $\Omega_n(X, A)$ is not a $\mathbb{F}_2$-vector space. So we get a more complicated homology theory now.

1.4.3 Oriented bordism homology

Lemma 1.49. *Relative oriented bordism is a covariant functor*

$$\Omega_* : \mathrm{TOP}^2 \to \textit{graded } \Omega_* \textit{ modules}$$

Proof. $\square$

Lemma 1.50 (Homotopy invariance [Ati61, Lemma $2 \cdot 1$])**.** *Let $f_0, f_1 : (X, A) \to (Y, B)$ be homotopic maps. Then $f_*, g_* : \Omega_n(X, A) \to \Omega_n(Y, B)$ are the same homomorphisms.*

Proof. [Ati61], [CF64] Let $h : (X, A) \times I \to (Y, B)$ be a homotopy between f_0 and f_1. The proof is exactly the same as in the unoriented case; noting that the cylinder is now oriented and $\partial M \times I$ is now $\partial I \times (M \cup M^-) = M \sqcup M^-$ $\square$

Maybe do this more precisely.

Lemma 1.51 (Long exact sequence, [CF64])**.** *The sequence*

$$\cdots \xrightarrow{\partial} \Omega_n(A) \xrightarrow{i_*} \Omega_n(X) \xrightarrow{j_*} \Omega_n(X, A) \xrightarrow{\partial} \Omega_{n-1}(A) \xrightarrow{i_*} \cdots$$

is exact.

Proof. [CF64] The proof is exactly the same as in the unoriented case, one minor adjustment. For exactness at $\Omega_n(X, A)$, we identify the boundaries of (M, f) and (B^-, F); B with the opposite orientation. $\square$

The used lemma might have to be adjusted.

Lemma 1.52 (Excision axiom, [CF64, p. 5.7])**.** *If* $\overline{U} \subset \overset{\circ}{A}$*, then* $i : (X \setminus U, A \setminus U) \subset (X, A)$ *induces an isomorphism of relative oriented bordism groups:*

$$i_* : \Omega_n(X \setminus U, A \setminus U) \xrightarrow{\cong} \Omega_n(X, A)$$

Proof. [CF64] Again, everything stays the same as in the unoriented case. □

Lemma 1.53 (Disjoint union axiom)**.** *Relative oriented bordism satisfies the disjoint union axiom.*

Proof. The proof can be copied from the unoriented case. □

We can finally conclude:

Theorem 1.54. *Relative oriented bordism is a homology theory satisfying the disjoint union axiom.*

Observation. *Relative oriented bordism does not satisfy the dimension axiom.*

2 Cobordism

Now that we have seen that bordism is a homology theory, we can ask the question wether there is a dual theory, giving rise to a cohomology theory. The answer is yes, as we will see now.
I should give a explanation of how $O(n)$ (real bundles) and $SO(n)$ (oriented real bundles) come into play.

2.1 Cobordism

2.1.1 Definitions

Definition 2.1 (Local trivialization [Tu17, p.242])**.** Let E, B, F be manifolds. A **local trivialization** for a smooth surjective map $\pi : E \to B$ is a collection of charts $\{(U_i, \varphi_i)\}_{i \in I}$ (for $\{U_i\}$ an open cover of M) such that $\pi^{-1}(U_i)$ is diffeomorphic to $U_i \times F$ via φ_i for all $i \in I$. The charts are called **trivializing charts**.

Definition 2.2 (Fibre bundle [Tu17, p.242])**.** Let E, B, F be manifolds. A **fibre bundle** is a smooth surjective map $\pi : E \to B$ with a local trivialization with fibre F. E is called the **total space**, B the **base space** and F the **fibre** of the fibre bundle.

Maybe I will not need this

Definition 2.3 (Classifying space)**.** Let G be a topological group. The **classifying space** BG of G is the base space of the universal principal G-bundle

Observation.

$$BO(n) = \mathbb{G}_{k,\infty}$$

$$BSO(n) = \widetilde{\mathbb{G}}_{k,\infty}$$

I might adjust this to be more like [Tho54]'s definition.

Definition 2.4 (Thom space[BD70], [Tho54, p.29], [Ati61, p.201])**.** Let $\xi : E \to B$ be a real k-dimensional vector bundle over a compact manifold B. Then the Thom space of ξ is defined as

$$M(\xi) = E^c$$

the one-point compactification of the total space of ξ, the added point serving as the base point.
Alternatively (without compact assumption):
Let $\xi : E \to B$ be a real vector bundle with Riemannian metric over a manifold B. Its **disk bundle** is defined by $D(\xi) : DE \to X$, $DE = \{v \in E \mid \|v\| \leq 1\}$ and similarly, the **sphere bundle** is defined by $S(\xi) : SE \to X$, $SE = \{v \in E \mid \|v\| = 1\}$. Then the Thom space of ξ is defined as

$$M(\xi) = D(\xi)/S(\xi)$$

where the sphere bundle is collapsed to a point. We can also get the Thom space without a choice of a Riemannian metric, but I will omit this here.
For ξ the universal principal G-bundle, we can write $M(G)$ instead of $M(\xi)$.

Definition 2.5 ($MO(n), MSO(n)$ [BD70])**.**

$$MO(n) := M(\xi_{n,\infty})$$

with $\xi_{n,\infty}$ being the universal real vector bundle over $\mathbb{G}_{k,\infty}$.

$$MSO(n) :=$$

Maybe it is enough to just say that these are called this way because the Grassmannians are the classifying spaces of $O(n), SO(n)$ instead of defining classifying spaces.

Definition 2.6 (Spectrum [BD70, Definition IV.1.1.])**.** A **spectrum** $\underline{E} = \{(E_n, \sigma_n) \mid n \in \mathbb{Z}\}$ is a sequence of pointed spaces E_n with pointed **structure maps**

$$\sigma_n : E_n \wedge S^1 \to E_{n+1}$$

Definition 2.7 (Thom spectrum [BD70, Beispiele IV.1.2(b)])**.** Let $\gamma_{n,\infty}$ be the universal real vector bundle over $BO(n) = \mathbb{G}_{n,\infty}$

Theorem 2.8 (Suspension sequence)**.** *Let X, Y be pointed spaces. We denote by $[X,Y]^\circ$ the set of homotopy classes of pointed maps $X \to Y$. Then we have the **suspension sequence**:*

$$[X,Y] \to [\Sigma X, \Sigma Y] \to \cdots \to [\Sigma^n X, \Sigma^n Y]$$

Theorem 2.9 (Freudenthal suspension theorem). *For large enough n, the suspension map*

$$[\Sigma^n X, \Sigma^n Y] \to [\Sigma^{n+1} X m \Sigma^{n+1} Y]$$

is an isomorphism.

Lemma 2.10. *[Tho54] The natural map*

$$\Sigma\{MSO(n)\} \to MSO(n+1)$$

induces isomorphisms of homotopy groups π_{n+1} for n large.

Corollary 2.11 ([Ati61, p.201]). *For a finite CW-complex X with basepoint,*

$$[X, \Sigma\{MSO(n)\}] \to [X, MSO(n+1)]$$

is bijective for n large.

Lemma 2.12 ([Ati61, p.201]). *Let Y be a subcomplex of a CW-complex X. Then from 2.8 and 2.10, we get a map*

$$[\Sigma^{n-k}(X/Y), MSO(n)] \to [\Sigma^{n+1-k}(X/Y, MSO(n+1))]$$

Definition 2.13 (relative oriented cobordism group [Ati61, p.201]). Let (X, Y) be a pair of spaces, then for $k \in \mathbb{Z}$, the **k-th oriented cobordism group** is

$$\Omega^k(X, Y) := MSO^k(X, Y) = \lim_{n\to\infty} [\Sigma^{n-k}(X/Y), MSO(n)]$$

with respect to the above map. Analogously, we define the **relative unoriented cobordism group** as

$$\mathfrak{N}^k(X, Y) := MO^k(X, Y) = \lim_{n\to\infty} [\Sigma^{n-k}(X/Y), MO(n)]$$

2.1.2 The Eilenberg-Steenrod axioms

To get a definition for cohomology theories, intuitively, we "reverse all arrows" in the previously defined axioms for homology theories in 1.2.2.

Definition 2.14 (Cohomology theory [Lüc05, Definition 5.2]). A **cohomology theory** $\mathcal{H}^* = (\mathcal{H}^*, \partial^*)$ with coefficients in R-modules is a contravariant functor

$$\mathcal{H}^* : \mathrm{TOP}^2 \to \mathbb{Z}\text{-graded } R\text{-modules}$$

together with a natural transformation

$$\partial^* : \mathcal{H}^* \circ I \to \mathcal{H}^{*+1}$$

satisfying the following axioms:

- **Homotopy invariance** For homotopic maps of pairs $f, g : (X, A \to Y, B)$, we have
$$\mathcal{H}^n(f) = \mathcal{H}^n(g) : \mathcal{H}^n(Y, B) \to \mathcal{H}^n(X, A)$$

- **Long exact sequence** For a pair of spaces (X, A), the sequence
$$\dots \xrightarrow{\partial^{n-1}} \mathcal{H}^n(X, A) \xrightarrow{\mathcal{H}^n(j)} \mathcal{H}^n(X) \xrightarrow{\mathcal{H}^n(i)} \mathcal{H}^n(A) \xrightarrow{\partial^n(X,A)} \mathcal{H}^{n+1}(X, A) \to \dots$$
is exact, where $i : A \hookrightarrow X$ and $j : (X, \emptyset) \hookrightarrow (X, A)$ are the inclusions.

- **Excision** Let (X, B, A) be a triple of spaces such that $\overline{A} \subseteq \mathring{B}$. Then the inclusion map $i : (X \setminus A, B \setminus A) \hookrightarrow (X, B)$ induces an isomorphism of cohomology groups:
$$\mathcal{H}^n(i) : \mathcal{H}^n(X, B) \xrightarrow{\cong} \mathcal{H}^n(X \setminus A, B \setminus A)$$

Sometimes one adds the following axioms:

- **Disjoint union axiom** For a disjoint union of spaces $\coprod_{i \in I} X_i$ over any index set I and the inclusions $j_i : X_i \to \coprod_{i \in I} X_i$, the map
$$\prod_{i \in I} \mathcal{H}^n(j_i) : \mathcal{H}^n \left(\coprod_{i \in I X_i} \right) \xrightarrow{\cong} \prod_{i \in I} \mathcal{H}^n(X_i)$$

- **Dimension axiom** For all $n \in \mathbb{Z}$, we have
$$\mathcal{H}^n(\{\mathrm{pt}\}) \cong \begin{cases} R, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

2.1.3 Bordism cohomology

Again, we will check the axioms one by one.

Lemma 2.15 (Functoriality)**.**

Lemma 2.16 (Naturality)**.**

Lemma 2.17 (Homotopy invariance)**.**

Lemma 2.18 (Long exact sequence)**.**

Lemma 2.19 (Excision)**.**

Lemma 2.20 (Disjoint union axiom)**.**

Observation (Dimension axiom)**.**

Conclude:

Theorem 2.21 (Bordism cohomology theory)**.**

2.1.4 Calculations

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