Bordism Homology and Cohomology

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Contents

0 Introduction/Motivation

Recall the definition of homotopy groups $\pi_n(X)$ as the set of homotopy classes of maps from the n-sphere S^n to a space X. The problem with these groups is that they are generally hard to compute, even for simple spaces. For example, the homotopy groups of spheres are not known in general. The general idea of bordism is to replace the n-sphere with a manifold of dimension n and to consider the homotopy classes of maps from this manifold to a space X.

1 Basic Definitions

Definition 1.1 (Manifold). *A* ***smooth manifold with boundary*** $M = (M, [\mathcal{A}])$ *is the data of a topological space* M *and an equivalence class* $[\mathcal{A}]$ *of smooth atlases such that:*

- M *is Hausdorff,*
- M *is second countable,*
- $\mathcal{A}$ *is locally finite.*

Remark. *In this thesis, with* manifold *we will always mean a smooth manifold with boundary.*

Definition 1.2 (Boundary). *The* ***boundary*** *of a manifold* M *is the set of points* $x \in M$ *such that there exists a chart* (U, φ) *such that* $\varphi(U) \subseteq \mathbb{R}^n_+$. *The boundary is denoted by* ∂M.

Example (Standard n-disk). *The* ***standard n-disk*** *is the set of points in* $\mathbb{R}^n$ *such that* $|x| \leq 1$. *The boundary of the standard* n*-disk is the standard* $(n-1)$*-sphere. The standard* n*-disk is denoted by* D^n *and the standard* $(n-1)$*-sphere is denoted by* S^{n-1}.

Non-example. *Line with two origins*

The problem with working with manifolds is that they are hard to classify up to homeomorphism or diffeomorphism.

2 Bordism

Definition 2.1 (Singular manifold).

Definition 2.2 (nullbordant).

Example.

Definition 2.3 (bordant).

Remark.

$$(M, f) + (\emptyset, g) \text{ are bordant} \iff (M, f) \text{ is nullbordant}$$

Example.

Non-example.

Proposition 2.4. *Being bordant is an equivalence relation on the set of singular manifolds.*

Proof.

- Reflexivity: $(M, f) \sim (M, f)$ is trivial.
- Symmetry: If $(M, f) \sim (N, g)$, then $(N, g) \sim (M, f)$.
- Transitivity: If $(M, f) \sim (N, g)$ and $(N, g) \sim (P, h)$, then $(M, f) \sim (P, h)$.

□

Definition 2.5 (bordism group).

Observe the similarity with the definition of singular homology groups.

Theorem 2.6. *The bordism groups are abelian groups.*

Definition 2.7 (graded bordism ring).

3 Cobordism

classifying spaces, Thom spaces, Thom isomorphism, Thom class, Thom isomorphism theorem

4 Pontryagin-Thom Construction