Bordism Homology and Cohomology

Paul Jin Robaschik

Geboren am 25. Juni 2004 in Köln

7. Mai 2025

Bachelorarbeit Mathematik

Betreuer: Prof. Dr. Markus Hausmann

Zweitgutachterin: Dr. Elizabeth Tatum

Mathematisches Institut

Mathematisch-Naturwissenschaftliche Fakultät der

Rheinischen Friedrich-Wilhelms-Universität Bonn

Contents

0 Introduction/Motivation

Recall the definition of homotopy groups $\pi_n(X)$ as the set of homotopy classes of maps from the n-sphere S^n to a space X. The problem with these groups is that they are generally hard to compute, even for simple spaces. For example, the homotopy groups of spheres are not known in general. The general idea of bordism is to replace the n-sphere with a manifold of dimension n and to consider the homotopy classes of maps from this manifold to a space X.

Testing citations:[Ati61],[BD70],[Tho54],[Lee13],[Hat02],[Die08],[Lüc05],[**luck**]

1 Basic Definitions

Here, Examples still need to be added.

Definition 1.1 (Topological manifold). *An n-dimensional **topological manifold** is a topological space M such that:*

- *M is Hausdorff, (i.e. any two distinct points can be separated by disjoint open sets),*
- *M is second countable, (i.e. there exists a countable basis for the topology of M),*
- *M is locally Euclidean (i.e. every point in M has a neighbourhood homeomorphic to an open subspace of $\mathbb{R}^n$).*

We will often write M^n for an n-dimensional manifold.

[Lee13]

Remark. *One could replace the condition of being second countable with the condition of being paracompact (i.e. every open cover of M admits a locally finite refinement). The following equivalence holds:*

$$M \text{ is second countable} \iff M \text{ is paracompact and countably many connected components}$$

This is shown in [Lee13].

Definition 1.2 ((Smooth) Atlas). *Let M be a topological manifold. A **(smooth) atlas** $\mathcal{A}$ on M is a collection of smooth charts $(U_\alpha, \varphi_\alpha)$ such that:*

- *the $\{U_\alpha\}$ cover M,*
- *the charts are pairwise smoothly compatible (i.e. the transition functions $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ are smooth)*

[Lee13]

Definition 1.3. *Two atlases $\mathcal{A}$ and $\mathcal{A}'$ (on a fixed topological manifold) are said to be **equivalent**, if their union is still on atlas.*

[Lee13]
This is an equivalence relation. [Lee13]

Definition 1.4 (Smooth manifold). *A **smooth manifold** $M = (M, [\mathcal{A}])$ consists of*

- *a topological manifold M,*
- *an equivalence class $[\mathcal{A}]$ of smooth atlases on M.*

[Lee13]

Remark. *While being a topological manifold is just a property of the topological space M, begin a smooth manifold gives the manifold extra structure.*

Definition 1.5 (Manifold with boundary)**.** *To get a definition of a (smooth or topological)* ***manifold with boundary****, replace the condition of the manifold being locally Euclidean with the condition that every point has a neighbourhood homeomorphic to an open subspace of* $\mathbb{H}^n := \{(x_1, \dots, x_n) \in \mathbb{R} \mid x_1 \geq 0\}$ *(the half space).*

[Lee13]

Remark. *In this thesis, with* manifold *we will always mean a smooth manifold with boundary.*

Definition 1.6. *A point $x \in M^n$ is called a* ***interior point*** *if it admits a neighbourhood homeomorphic to $\mathbb{R}^n$. Otherwise, it is called a* ***boundary point****. The set of boundary points is denoted by ∂M and is called the* ***boundary*** *of M.*
If a manifold is compact and has empty boundary, it is called a ***closed manifold****.*

[Lee13]

Example (Standard n-disk)**.** *The* ***standard n-disk*** *is the set of points in $\mathbb{R}^n$ such that $|x| \leq 1$. The boundary of the standard n-disk is the standard $(n-1)$-sphere. The standard n-disk is denoted by D^n and the standard $(n-1)$-sphere is denoted by S^{n-1}.*

Remark. *The boundary of an n-dimensional manifold is an $(n-1)$-dimensional submanifold.*

Maybe I will make a theorem out of this and add a proof.

Non-example. *Line with two origins*

2 Bordism

The problem with working with manifolds is that they are hard to classify up to homeomorphism or diffeomorphism. Bordism is a way to classify manifolds up to a weaker equivalence relation, which is easier to work with.

2.1 unoriented bordism

Definition 2.1 (Singular manifold)**.** *Let X be a topological space. An n-dimensional* ***singular manifold*** *in X is a pair M, f of a compact manifold M^n and a continuous map $f : M \to X$.*
The ***boundary*** *of a singular manifold is $\partial(M, f) := (\partial M, f_{|\partial M})$.*

[BD70]

Definition 2.2 (Nullbordant)**.** *Let (M, f) be a singular manifold in X. We say that (M, f) is* ***nullbordant****, if there exists a singular manifold (B, F), such that $\partial(B, F) = (M, f)$.*
B, F is then called a ***nullbordism*** *of (M, f).*

[BD70]

Example.

Definition 2.3 (bordant)**.** *Let (M, f) and (N, g) be singular manifolds in X. We say that (M, f) and (N, g) are* ***bordant****, if their sum $(M, f) + (N, g) = (M + N, (f, g)) := (M \amalg N, f \amalg g)$ is nullbordant.*
A nullbordism of $(M, f) + (N, g)$ is called a ***bordism*** *between (M, f) and (N, g).*

[BD70]

Remark.

$$(M, f) + (\emptyset, g) \text{ are bordant} \iff (M, f) \text{ is nullbordant}$$

Example.

Non-example.

Proposition 2.4. *Being bordant is an equivalence relation on the set of singular manifolds.*

Proof. [BD70]

- Symmetry: Follows from the symmetry of the disjoint union. If (M,f) and (N,g) are bordant, then there exists a nullbordism of $(M,f)+(N,g)=(M \amalg N, f \amalg g)=(N \amalg M, g \amalg f)=(N,g)+(M,f)$. So, a bordism between (M,f) and (N,g) is also bordism between (N,g) and (M,f).
- Reflexivity: Cylinder
- Transitivity: Draw a picture

□

Check smooth structure!

Definition 2.5 (bordism group).

Observe the similarity with the definition of singular homology groups.

Theorem 2.6. *The bordism groups are abelian groups.*

Definition 2.7 (graded bordism ring).

Definition 2.8 (relative bordism).

2.2 The Eilenberg-Steenrod Axioms

The Eilenberg-Steenrod axioms are a set of axioms that characterize the homology and cohomology theories.

Definition 2.9 (Homology theory [Lüc05]). *A* ***homology theory*** $\mathcal{H}_* = (\mathcal{H}_*, \partial_*)$ *with coefficients in R-modules is a covariant functor*

$$\mathcal{H}_* : \mathrm{TOP}^2 \to \mathbb{Z}\text{-graded } R\text{-modules}$$

together with a natural transformation

$$\partial_* : \mathcal{H}_* \to \mathcal{H}_{*-1} \circ I$$

- ***Homotopy invariance***
  Let $f,g : (X,A) \to (Y,B)$ be homotopic maps. Then for all $n \in \mathbb{Z}$, we have

  $$\mathcal{H}_n(f) = \mathcal{H}_n(g) : \mathcal{H}_n(X,A) \to \mathcal{H}_n(Y,B)$$

- ***Long exact sequence***
  Let (X,A) be a pair of spaces. Then for all $n \in \mathbb{Z}$, we have the long exact sequence of homology groups:

  $$\ldots \xrightarrow{\partial_{n+1}(X,A)} \mathcal{H}_n(A) \xrightarrow{\mathcal{H}_n(i)} \mathcal{H}_n(X) \xrightarrow{\mathcal{H}_n(j)} \mathcal{H}_n(X,A) \xrightarrow{\partial_n(X,A)} \mathcal{H}_{n-1}(A) \to \ldots$$

- ***Excision***
  Let $A \subset B \subset X$ be subspaces of X such that $\overline{A} \subset B^\circ$. Then the inclusion $i : (X \setminus B, A \setminus B) \to (X,A)$ induces an isomorphism of homology groups for all $n \in \mathbb{Z}$:

  $$\mathcal{H}_n(i) : \mathcal{H}_n(X \setminus A, B \setminus A) \xrightarrow{\cong} \mathcal{H}_n(X,B)$$

Sometimes one adds the following axioms:

- ***Disjoin union axiom***
  Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $j_i : X_i \to \coprod_{i \in I} X_i$ be the inclusion. Then for all $n \in \mathbb{Z}$, we have a bijection:

  $$\bigoplus_{i \in I} \mathcal{H}_n(j_i) : \bigoplus_{i \in I} \mathcal{H}_n(X_i) \xrightarrow{\cong} \mathcal{H}_n\left(\coprod_{i \in I} X_i\right)$$

- ***Dimension axiom***
  For the point space pt*, we have*

$$\mathcal{H}_n(\mathrm{pt}) \cong \begin{cases} R & n = 0 \\ \{0\} & n \neq 0 \end{cases}$$

Theorem 2.10. *Bordism defines a homology theory satisfying the disjoint union axiom.*

Proof. We will check the axioms one by one. □

Observation. *Bordism does not satisfy the dimension axiom.*

We will now calculate the bordism groups.

2.3 oriented bordism

Definition 2.11 (vector bundle)**.**

Definition 2.12 (orientable)**.**

Definition 2.13 (bordant)**.**

Definition 2.14 (oriented bordism group)**.**

3 Cobordism

classifying spaces, Thom spaces, Thom isomorphism, Thom class, Thom isomorphism theorem

4 Pontryagin-Thom Construction

References

[Ati61] M. F. Atiyah. „Bordism and cobordism“. In: *Proc. Cambridge Philos. Soc.* 57 (1961), pp. 200–208. ISSN: 0008-1981. DOI: 10.1017/s0305004100035064. URL: https://doi.org/10.1017/s0305004100035064.

[BD70] Theodor Bröcker and Tammo tom Dieck. *Kobordismentheorie*. Vol. Vol. 178. Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1970, pp. xvi+191.

[Die08] Tammo tom Dieck. *Algebraic topology*. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2008, pp. xii+567. ISBN: 978-3-03719-048-7. DOI: 10.4171/048. URL: https://doi.org/10.4171/048.

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002, pp. xii+544. ISBN: 0-521-79160-X; 0-521-79540-0.

[Lee13] John M. Lee. *Introduction to smooth manifolds*. Second. Vol. 218. Graduate Texts in Mathematics. Springer, New York, 2013, pp. xvi+708. ISBN: 978-1-4419-9981-8.

[Lüc05] Wolfgang Lück. *Algebraische Topologie Homologie und Mannigfaltigkeiten*. vieweg, 2005. ISBN: 978-3-528-03218-0.

[Tho54] René Thom. „Quelques propriétés globales des variétés différentiables“. In: *Comment. Math. Helv.* 28 (1954), pp. 17–86. ISSN: 0010-2571,1420-8946. DOI: 10.1007/BF02566923. URL: https://doi.org/10.1007/BF02566923.