Bordism Homology and Cohomology

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0 Introduction/Motivation

Recall the definition of homotopy groups $\pi_n(X)$ as the set of homotopy classes of maps from the n-sphere S^n to a space X. The problem with these groups is that they are generally hard to compute, even for simple spaces. For example, the homotopy groups of spheres are not known in general. The general idea of bordism is to replace the n-sphere with a manifold of dimension n and to consider the homotopy classes of maps from this manifold to a space X.

Testing citations:[Ati61],[BD70], [Tho54], [Lee13], [Hat02], [Die08]

1 Basic Definitions

Maybe I will split this definition into topological manifold, top. manifold with boundary, smooth manifold, smooth manifold with boundary..

Remark. *One could equivalently replace the condition of being second countable with the condition of being paracompact.*

$$\textit{second countable} \iff \textit{paracompact and countably many connected components}$$

Maybe I will add a definition of paracompactness later.

Definition 1.1 (Manifold). *A **smooth manifold with boundary** $M = (M, [\mathcal{A}])$ is the data of a topological space M and an equivalence class $[\mathcal{A}]$ of smooth atlases such that:*

- *M is Hausdorff,*
- *M is second countable,*
- *$\mathcal{A}$ is locally finite.*

Remark. *While being a topological manifold is just a property of the topological space M, begin a smooth manifold gives the manifold extra structure.*

Remark. *In this thesis, with* manifold *we will always mean a smooth manifold with boundary.*

Definition 1.2 (Boundary). *The **boundary** of a manifold M is the set of points $x \in M$ such that there exists a chart (U, φ) such that $\varphi(U) \subseteq \mathbb{R}^n_+$. The boundary is denoted by ∂M.*

Example (Standard n-disk). *The **standard n-disk** is the set of points in $\mathbb{R}^n$ such that $|x| \leq 1$. The boundary of the standard n-disk is the standard $(n-1)$-sphere. The standard n-disk is denoted by D^n and the standard $(n-1)$-sphere is denoted by S^{n-1}.*

Remark. *The boundary of an n-dimensional manifold is an $(n-1)$-dimensional submanifold.*

Non-example. *Line with two origins*

The problem with working with manifolds is that they are hard to classify up to homeomorphism or diffeomorphism.

2 Bordism

2.1 unoriented bordism

Definition 2.1 (Singular manifold).

Definition 2.2 (nullbordant).

Example.

Definition 2.3 (bordant).

Remark.

$$(M, f) + (\emptyset, g) \text{ are bordant} \iff (M, f) \text{ is nullbordant}$$

Example.

Non-example.

Proposition 2.4. *Being bordant is an equivalence relation on the set of singular manifolds.*

Proof.

- Reflexivity: $(M, f) \sim (M, f)$ is trivial.
- Symmetry: If $(M, f) \sim (N, g)$, then $(N, g) \sim (M, f)$.
- Transitivity: If $(M, f) \sim (N, g)$ and $(N, g) \sim (P, h)$, then $(M, f) \sim (P, h)$.

□

Definition 2.5 (bordism group)**.**

Observe the similarity with the definition of singular homology groups.

Theorem 2.6. *The bordism groups are abelian groups.*

Definition 2.7 (graded bordism ring)**.**

2.2 The Eilenberg-Steenrod Axioms

The Eilenberg-Steenrod axioms are a set of axioms that characterize the homology and cohomology theories.

We will now calculate the bordism groups.

2.3 oriented bordism

Definition 2.8 (vector bundle)**.**

Definition 2.9 (orientable)**.**

Definition 2.10 (bordant)**.**

Definition 2.11 (oriented bordism group)**.**

3 Cobordism

classifying spaces, Thom spaces, Thom isomorphism, Thom class, Thom isomorphism theorem

4 Pontryagin-Thom Construction

References

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