Bordism Homology and Cohomology

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Definition 1.1. *If X, Y are spaces with base points x_0, y_0, we denote by $[X, Y]$ the* set of homotopy classes of maps $(X, x_0) \to (Y, y_0)$. *We have the suspension sequence*

$$[X, Y] \to [SX, SY] \to \cdots \to [S^n X, S^n Y] \to \ldots$$

in which all terms after the first two are abelian groups, the maps being then group homomorphisms. Moreover we have an isomorphism

$$[S^n X, S^n Y] \to [S^{n+1} X, S^{n+1} Y]$$

if $n + 2$(connectivity of Y) $\geq \dim X \ldots$

This is the beginning of the first definition in Atiyah, I don't understand it yet... Let's look in Bröcker now.

[atiyah]

Definition 1.2 (Manifold). *When we talk about a* ***manifold*** *M^n, we will mean an n-dimensional, paracompact, smooth manifold (with or without boundary). Its* ***boundary*** *∂M^n of M^n is a manifold without boundary and has dimension $n-1$. A map $f : M \to N$ between two manifolds will always be, unless specified otherwise, smooth.*

Beginning of Bröcker, chapter 1, Difftopo. From here, many things are known from AnaGeo

[brocker]

Definition 1.3 (Tangent space). *The* ***tangent vectors*** *of a manifold M^n form an n-dimensional smooth vector bundle $\pi : TM^n \to M^n$, the* ***tangent bundle*** *of M^n. The* ***fibre*** *$T_x M$ of TM over $x \in M$ is isomorphism to $\mathbb{R}^n$.*

[brocker]

Definition 1.4 (Differential). *A map on manifolds $F : M \to N$ induces a smooth linear map $df : TM \to TN$ on tangent bundles, the* ***differential of*** *f. So the tangent bundle is a functor between the categories* Man^∞ *and* vectbund

Bröcker writes Tf, but I'll stick to Côté's notation.

[brocker]

Definition 1.5 (Immersion). *A map $f : M \to N$ is called an* ***immersion****, if df is injective on every fibre, i.e. $T_p f$ is injective for every $p \in M$. If an immersion is a homeomorphism onto its image, it is called an* ***embedding****.*

[brocker]

We know by Whitney, that a manifold of dimension n can be embedded in $\mathbb{R}^{\geq 2n+1}$. Precisely:

Theorem 1.6 (Weak Whitney's embedding theorem). *Let $\varepsilon : M^n \to \mathbb{R}$ be a strictly positive map, and $f : M^n \to \mathbb{R}^p$ a map for $p > 2n$, which is an embedding in a neighbourhood of a closed subset $A \subset M^n$. Then there is a ε-approximation g of f, with $g_{|A} = f_{|A}$, which is an embedding. In particular, there is an embedding $g : M^n \to \mathbb{R}^p$, such that $g(M^n)$ is closed in $\mathbb{R}^p$*

g is an ε-approximation of f, if the distance of $f(x)$ and $g(x)$ is smaller than $\varepsilon(x)$ (given a metric). f only needs to be continuous

[brocker]

Proof. Unclear, if we need a proof here, we could copy one from AnaGeo... □ Still needed

Theorem 1.7. *Let $f : M \to N$ continuous, and on a closed subset $A \subset M$ smooth. Let $\varepsilon : M \to \mathbb{R}$ be strictly positive and N given a metric. Then there is a smooth ε-approximation $g : M \to N$ of f, with $g_{|A} = f_{|A}$.*

[brocker]

Proof. To be proven. □ Still needed

Definition 1.8 (Normal bundle). *Let now $f : M \to \mathbb{R}^p$ be an embedding. For f, there is a* ***normal bundle*** *$\nu_f : E(\nu_f) \to M$ of f.*

E is the normal space, we need f for the scalar product on $\mathbb{R}^p$. It probably suffices to be a Riemannian manifold

[brocker]

Considering M as a subset of $\mathbb{R}^p$ by f, the fibre of ν_f over $x \in M$ consists of the vectors $v \in \mathbb{R}^p$, which (w.r.t. the standard scalar product on $\mathbb{R}^p$), that are in x orthogonal to M.

The **Whitney sum** $\nu_f \oplus \pi$ of ν_f and the tangent bundle is trivial of dimension p over M, as it is the restriction of the trivial bundle $T\mathbb{R}$ on M^n, Definition needed!

$$\nu_f \oplus \pi_M \cong \mathrm{pr}_1 : M^n \times \mathbb{R}^p \to M^n$$

Theorem 1.9. *The inclusion*

$$(\nu_f : E(\nu_f) \to M) \to (\mathrm{pr}_1 : M \times \mathbb{R}^p \to M)$$

is a linear embedding of smooth vector bundles.

[**brocker**]

Proof. missing. $\square$ missing.