Bordism Homology and Cohomology

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Vorwort

Diese Bachelorarbei beschäftigt sich mit der Entwicklung der Bordismus-Homologie und -Kohomologie. Dafür werden in Kapitel 1.1zunächst die benötigten Grundlagen der Differentialtopologie gelegt. In Kapitel 1.2 wird dann die Bordismus-Relation für Mannigfaltigkeiten definiert. Dabei heißen zwei Mannigfaltigkeiten bordant, wenn ihre disjunkte Vereinigung der Rand einer anderen Mannigfaltigkeit ist. Wir werden zeigen, dass dies eine Äquivalenzrelation ist und dass die Bordismus-Klassen eine Gruppe bilden. In Abschnitt 1.2.2 werden Homologietheorien axiomatisch definiert und dann für die in Abschnitt 1.2.3 zu relativen Bordismus erweiterten Bordismus Klassen in Abschnitt 1.2.4 nachgeprüft.
In Kapitel 1.3 wird der Begriff der Orientierung auf glatten Mannigfaltigkeiten eingeführt, um in Kapitel 1.4 eine orientierte Version der Bordismus-Relation zu definieren. Wieder werden in Abschnitt 1.4.3 die Axiome der Homologie-Theorie nachgeprüft.
Im zweiten Teil der Arbeit geht es um die Bordismus-Kohomologie. Diese werden wir über Spektren definieren. Dazu werden in Abschnitt 2.1.1 die benötigten Grundlagen der Spektrentheorie. Hier werden auch die (orthogonalen) Thom-Spektren definiert, deren assoziierte Kohomologietheorie die Bordismus-Kohomologie ist. Wie in Teil 1 werden dann die Axiome der Kohomologietheorie in Abschnitt 2.1.3 für die Bordismus-Kohomologie überprüft.
Zuletzt wird in Kapitel 2.2 die Verbindung zwischen der Bordismus-Homologie und -Kohomologie hergestellt.

0 Motivation

Bordism theory starts with an equivalence relation on manifolds, where two manifolds are called bordant if their disjoint union is the boundary of a higher-dimensional manifold. First, this seems like a very abstract definition, but it turns out to be a very useful one for algebraic topology.
Obviously, this is a far weaker equivalence relation than diffeomorphism or homeomorphism, but while classifying manifolds up to diffeomorphism or homeomorphism is known to be difficult in higher dimensions, classifying manifolds up to bordism has been fully understood.
Bordism theory is a powerful generalization of homotopy theory. Homotopy groups $\pi_n(X)$ classify maps from spheres to a space X, but are often difficult to compute, even for simple spaces.
Bordism groups take a slightly different approach: Instead of spheres, one considers arbitrary compact manifolds and two maps are considered equivalent if there exists a bordism between the manifolds making the maps compatible. This makes computation a lot easier.
Additionally, bordism gives rise to a extraordinary homology theory, which we will define and prove to be a homology theory in later chapters.

1 Bordism

Before we can define bordism, we need some preliminary definitions. We will start with the recalling the definition of manifolds. We will also introduce some useful tools from differential topology.

1.1 Manifolds

Definition 1.1 (Topological manifold [Lee13, pp.2-3])**.** An n-dimensional **topological manifold** is a topological space M such that:

- M is Hausdorff, i.e. any two distinct points can be separated by disjoint open sets,
- M is second countable, i.e. there exists a countable basis for the topology of M and
- M is locally Euclidean i.e. every point in M has a neighbourhood homeomorphic to an open subspace of $\mathbb{R}^n$.

We will often write M^n for an n-dimensional manifold. n-dimensional manifolds are also called **n-manifolds**.

Definition 1.2 (Chart)**.** For a manifold M^n, a **chart** is a pair (U, φ) of an open subset $U \subset M$ and a map $\varphi : U \to \mathbb{R}^n$ such that $\varphi : U \to \varphi(U)$ is a homeomorphism.

Example 1. $\mathbb{R}^n$ is an n-dimensional topological manifold.

Example 2. The n-dimensional sphere $S^n \subset \mathbb{R}^{n+1}$ with subspace topology is an n-dimensional topological manifold. Hausdorffness and second countability follow from $S^n \subset \mathbb{R}^{n+1}$. For local Euclideanness, we can use charts onto the open ball $B_1^n(0)$

$$\varphi_i^{\pm} : U_i := \{(x_0, \ldots, x_n) \in S^n \mid \pm x_i > 0\} \to B_1^n(0)$$

by

$$(x_0, \ldots, x_i, \ldots, x_n) \mapsto (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

Non-examples 3.

- A “cross” in $\mathbb{R}^2$ ($\{(x_1, x_2 \mid x_1 = 0 \vee x_2 = 0)\}$) is not a topological manifold, because it is not locally Euclidean at the crossing point.
- The line with two origins $((\mathbb{R} \times \{0\} \sqcup \mathbb{R} \times \{1\})/((x, 0) \sim (y, 1) \Leftrightarrow (x = y \wedge x \neq 0)))$ is not a topological manifold. It is not Hausdorff, as the two origins cannot be separated by disjoint open sets. However, it is locally Euclidean.
- Let $\{\mathrm{pt}\}$ denote the point space. $\coprod_{i \in \mathbb{R}} \{\mathrm{pt}\}$ is not a topological manifold, because it is not second countable, as it has uncountably many connected components.
- $S^1 \sqcup \{\mathrm{pt}\}$ is not a topological manifold, because it is, locally homeomorphic to $\mathbb{R}^1$ for any point in S^1 or $\mathbb{R}^0$, if the chosen point is not in S^1, and the dimension of a manifold needs to be constant.

Remark. One could replace the condition of being second countable with the condition of being paracompact (i.e. every open cover of M admits a locally finite refinement). The following equivalence holds for any topological space:

$$\text{second countable} \iff \text{paracompact and countably many connected components}$$

Definition 1.3 ((Smooth) Atlas [Lee13, p.12])**.** Let M be a topological manifold. A **(smooth) atlas** $\mathcal{A}$ on M is a collection of charts $(U_\alpha, \varphi_\alpha)$ such that:

- the $\{U_\alpha\}$ cover M,
- the charts are pairwise smoothly compatible, i.e. the transition functions $\varphi_j \circ \varphi_i^{-1} : \mathbb{R}^n \supset \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j) \subset \mathbb{R}^n$ are smooth as maps from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Example 4. The charts we chose for the n-sphere S^n in example 2 form a smooth atlas on S^n.

Definition 1.4 (Equivalence of atlases)**.** Two atlases $\mathcal{A}$ and $\mathcal{A}'$ (on a fixed topological manifold) are said to be **equivalent**, if their union is still an atlas.

Remark. This is indeed an equivalence relation, we only have to check transitivity, but this is also straightforward since we can just take the union of the three atlases.

Definition 1.5 (Smooth manifold [Lee13, p.13])**.** A **smooth manifold** $M = (M, [\mathcal{A}])$ consists of

- a topological manifold M,
- an equivalence class $[\mathcal{A}]$ of smooth atlases on M.

Remark. An alternative, equivalent definition would take a maximal atlas instead of an equivalence class of atlases.

We already know what it means for a map $f : \mathbb{R}^n \to \mathbb{R}^m$ to be smooth and what it means for charts to be smoothly compatible. We can now extend the definition of smoothness to maps between manifolds.

Definition 1.6 (Smoothness of maps)**.** A map $f : (M^m, \mathcal{A}) \to (N^n, \mathcal{B})$ between two smooth manifolds is said to be **smooth**, if for all $p \in M$, there exists a chart $(U, \varphi) \in \mathcal{A}$ such that $p \in U$ and $(V, \psi) \in \mathcal{B}$ such that

- $f(U \subset V)$
- $\psi \circ f \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^m \to R^n$ is smooth as a map from $\mathbb{R}^m$ to $\mathbb{R}^n$.

Notation: For two smooth manifolds M, N, when writing $M = N$, we will mean that they are diffeomorphic, i.e. there exists a bijective map $f : M \to N$ such that f and f^{-1} are smooth.

Example 5 (Spheres)**.** The n-sphere S^n with the atlas given by the charts in example 2 is a smooth manifold.

Example 6 (Subset of manifolds)**.** For a smooth manifold $M, [\mathcal{A}]$ any open subset U of M is a smooth manifold. The atlas is given by restriction of the charts in $[\mathcal{A}]$ to U. We call U an **open submanifold** of M.

Example 7 (Product of manifolds)**.** For smooth manifolds M^m, N^n, the product $M \times N$ is a smooth $(m \cdot n)$-manifold with the charts

$$\{(U \times V, (\varphi, \psi)) \mid (U, \varphi), (V, \psi) \text{ charts of } M, N\}$$

While being a topological manifold is just a property of the topological space M, being a smooth manifold gives the manifold *extra structure*.

Example 8. $(\mathbb{R}, [(\mathbb{R}, \mathrm{id})])$ and $(\mathbb{R}, [\mathbb{R}, x \mapsto x^3])$ are *different* smooth manifolds, because the transition functions between the charts are not smooth: $\mathrm{id} \circ (x \mapsto x^3)^{-1} = \varphi$, where

$$\varphi(x) = \begin{cases} x^{\frac{1}{3}} & x \geq 0 \\ -|x|^{\frac{1}{3}} & \end{cases}.$$

φ is not differentiable at 0, hence the atlases are not equivalent.
However, the manifolds are diffeomorphic, as the map φ is a diffeomorphism between the two manifolds: $(x \mapsto x^3) \circ \varphi \circ \mathrm{id}^{-1} = \mathrm{id}$. Its inverse is given by $x \mapsto x^3$, which is also smooth: $\mathrm{id} \circ (x \mapsto x^3) \circ (x \mapsto x^3)^{-1} = \mathrm{id}$

Example 9 (Exotic spheres)**.** There exist 15 pairwise non-diffeomorphic smooth structures on S^7. See [KM63] for a construction of these exotic spheres.

Definition 1.7 (Manifold with boundary [Lee13, p.25])**.** To define a (smooth or topological) **manifold with boundary**, replace the condition of the manifold being locally Euclidean with the condition that every point has a neighbourhood homeomorphic to an open subspace of the half space $\mathbb{H}^n := \{(x_1, \ldots, x_n) \in \mathbb{R} \mid x_1 \geq 0\}$. Naturally, the charts now map into $\mathbb{H}^n$ instead of $\mathbb{R}^n$.

Remark. In this thesis, by **manifold** we will always mean a smooth manifold with boundary, unless specified otherwise.

Definition 1.8 (Boundary [Lee13, p.25])**.** Let M^n be a manifold. A point $x \in M^n$ is called an **interior point** if it admits a neighbourhood homeomorphic to $\mathbb{R}^n$. Otherwise, it is called a **boundary point**.
The set of interior points is denoted by $\mathrm{int}(M)$ and is called the **interior** of M. The set of boundary points is denoted by ∂M and is called the **boundary** of M.
If M is compact and has empty boundary, M is called a **closed manifold**.

Example 10. $([0,1], [([0,1], i)])$, where i is the inclusion into $\mathbb{R}$, is a 1-manifold. Its boundary is $\partial[0,1] = \{0,1\}$.

Example 11. $\mathbb{H}^n$ is a manifold with boundary $\mathbb{R}^{n-1}$

Example 12. The n-disk D^n is a manifold with boundary S^{n-1}

Theorem 1.9 (Boundaries are submanifolds)**.** *The boundary of an n-manifold is a closed $(n-1)$-dimensional (embedded) submanifold.*

Proof. Let $(M^n, [\mathcal{A}])$ be a manifold. A smooth structure on ∂M is given by the restriction of the charts in $[\mathcal{A}]$ to ∂M:

$$\{(U \cap \partial M, \varphi_{|U\cap\partial M}) \mid (U, \varphi) \in [\mathcal{A}]\}$$

Smooth compatibility follows from the smooth compatibility of the charts in $[\mathcal{A}]$. This makes ∂M a submanifold of M. It remains to show that it is closed and of codimension 1. The charts in $[\mathcal{A}]$ map every point in ∂M to $\partial\mathbb{H}^n = \mathbb{R}^{n-1}$. If they didn't, we could find a Euclidean neighbourhood of that point, contradicting the fact that it was a boundary point in M. As the charts restricted to ∂M now map to $\mathbb{R}^{n-1}$, we have an $(n-1)$-dimensional submanifold without boundary.

□

Theorem 1.10 (Collar theorem [BD70, I, Satz 1.5])**.** *Let M be a manifold. Then there exists a neighbourhood U of ∂M with a diffeomorphism $s : \partial M \times [0, 1) \to U$ with $s(\partial M \times \{0\}) = \partial M$. U is called a* ***collar*** *of ∂M in M.*

Proof. omitted. A proof can be found in [Lee13, p.223]. □

Observation. *If ∂M has multiple path components, then we can find a collar for each path component.*

Theorem 1.11 (Smooth Urysohn lemma [Lee13, Exercise 2-14])**.** *Let M be a manifold, $A, B \subset M$ disjoint closed sets. Then, there is a smooth function $f : M \to \mathbb{R}$ such that $f_{|A} = 0$ and $f_{|B} = 1$.*

Proof. This follows from the normality of manifolds and the existence of bump functions [Lee13, Proposition 2.25], i.e. functions $\psi : M \to \mathbb{R}$ that are smooth and take the value 1 on a closed subset S, its support is contained in a chosen neighbourhood of S and never takes values outside of $[0, 1]$.
So we just take a bump function for B supported in a neighbourhood of B which we choose to be disjoint from A. □

Definition 1.12 (Tangent space [Lee13, p.72])**.** The **tangent space** of a manifold M^n at a point $p \in M$, denoted by T_pM is the set of equivalence classes of smooth curves $\gamma : [-\varepsilon, \varepsilon] \to M$, $\gamma(0) = p$ under the equivalence relation $\gamma_1 \sim \gamma_2 :\Leftrightarrow$ for any smooth function $f : M \to \mathbb{R}^n$ defined in a neighbourhood of p, we have $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ ($\varepsilon > 0$ depends on γ)

Remark. [Lee13, Proposition 3.10] For an n-manifold, T_pM is an n-dimensional vector space over $\mathbb{R}$ for every point $p \in M$.

Definition 1.13 (Differential [Lee13, p.55])**.** Let $f : M \to N$ be a smooth map between two manifolds. For $p \in M$, the **differential of** f **at** p is given by

$$df_p : T_pM \to T_{f(p)}N$$
$$[\gamma] \mapsto [f \circ \gamma]$$

Definition 1.14 (critical value [Lee13, p.105])**.** Let $f : M \to N$ be a smooth map between two manifolds. A point $p \in M$ is a **critical point** of f, if the differential $df_p : T_pM \to T_{f(p)}N$ fails to be surjective. Otherwise, it is called a **regular point**.
A point $q \in N$ is a **critical value** of f, if $f^{-1}(q)$ contains a critical point of f. Otherwise, it is called a **regular value**.

Proposition 1.15 ([Lee13, Proposition 5.47a, Corollary 5.14])**.** *For a smooth map $f : M^n \to \mathbb{R}$, r a regular value of f, $\{p \in M : f(p) \leq r\} = f^{-1}(-\infty, r]$ is an n-dimensional submanifold of M. $f^{-1}(r)$ is an $(n-1)$-dimensional submanifold of M.*

Proof. omitted. A proof can be found in [Lee13]. □

Definition 1.16 (Measure zero [Lee13, p.126])**.** A subset $S \subset \mathbb{R}^n$ has **measure zero** if, for any $\varepsilon > 0$, there exists a countable collection of rectangles

$$\mathbb{R}^n \supset C_i = (a_1^i - \varepsilon_1^i, a_1^i + \varepsilon_1^i) \times \cdots \times (a_n^i - \varepsilon_n^i, a_n^i + \varepsilon_n^i)$$

where $(a_1^i, \ldots, a_n^i) \in \mathbb{R}^n$ and $(\varepsilon + 1^i, \ldots, \varepsilon_n^i) \in \mathbb{R}^n_{>0}$ such that

$$S \subset \bigcup_{i=1}^{\infty} C_i$$

$$\sum_{i=1}^{\infty} vol(C_i) < \varepsilon.$$

A subset S of a manifold M is said to have **measure zero**, if for every chart (U, φ) in its atlas, the set $\varphi(S \cap U)$ has measure zero in $\mathbb{R}^n$.

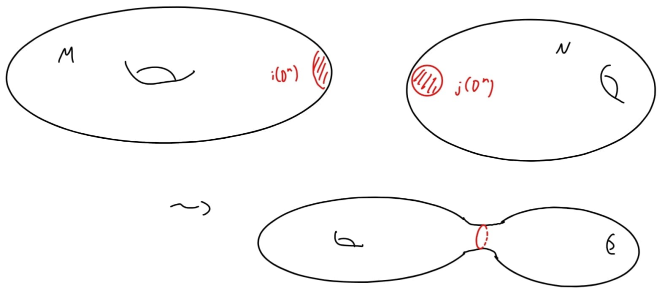
Theorem 1.17 (Sard's theorem [Lee13, Theorem 6.10])**.** *For a smooth map $f : M \to N$ between two manifolds, the set of critical values of f has measure zero in N.*

Proof. See [Lee13] for a proof. □

Remark. We will not need that the set of critical values has measure zero, but we will use that the set of regular values is dense in N, which follows from Sard's theorem.

Definition 1.18 (Connected sum [Lee13, Example 9.31])**.** For two connected manifolds M^n, N^n, choose smooth embeddings $D^n \overset{i}{\hookrightarrow} \text{int}(M)$, $D^n \overset{j}{\hookrightarrow} \text{int}(N)$. Then the connected sum of M and N is defined as the pushout

$$\begin{tikzcd}
\partial D^n \arrow[r, "i_{|\partial D^n}"] \arrow[d, "j_{|\partial D^n}"'] & M \setminus i(\mathring{D}^n) \arrow[d] \\
N \setminus j(\mathring{D}^n) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & M\#N
\end{tikzcd}$$



Remark. This construction is a priori only unique up to homeomorphism. It can be shown that $M\#N$ is a smooth manifold if M and N are smooth and that it is unique up to diffeomorphism, see [Kos93, VI, Theorem 1.1.] for details.

1.2 Unoriented Bordism

1.2.1 Definitions

Definition 1.19 (Singular manifold [BD70, II, Definition 1.1])**.** Let X be a topological space. An n-dimensional **singular manifold** in X is a pair (M, f) of a compact manifold M and a continuous map $f : M \to X$.
The **boundary** of a singular manifold is $\partial(M, f) := (\partial M, \partial f) := (\partial M, f_{|\partial M})$.

Definition 1.20 (Nullbordant [BD70, II, Definition 1.2])**.** Let (M, f) be a singular n-manifold in X. We say that (M, f) is **nullbordant**, if there exists a singular $(n + 1)$-manifold (B, F) in X, such that $\partial(B, F) = (M, f)$.
B, F is then called a **nullbordism** of (M, f).

In the following, we will sometimes omit f from the notation, if it is clear, what f is, e.g. if $X = \mathrm{pt}$ or $M = \emptyset$

Example 13. For $X = \mathrm{pt}$, and $M = S^n$, we have a nullbordism $B = D^{n+1}$, the disk.

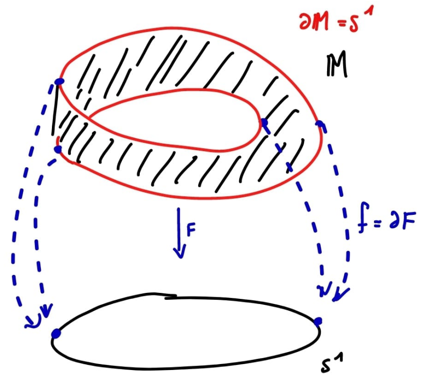
Example 14. For $X = \mathrm{pt}$, and $M = \mathbb{T}^2$, the torus, a nullbordism is $S^1 \times D^2$, the filled torus.

Remark. These examples can be generalized to f just being a constant map, and B any manifold such that $\partial B = M$.

Example 15. For $M = \emptyset$, any closed manifold is a nullbordism of M, no matter what the space X is.

Observation. *A singular manifold (M, f) can be nullbordant even if f is not nullhomotopic.*

Example 16. $X = S^1$, $M = S^1$, and f is given by wrapping around the circle twice, a nullbordism is given by the Möbius strip $\mathbb{M}$ with F as projection onto the circle.



Definition 1.21 (Bordant [BD70, II, Definition 1.3])**.** Let (M, f) and (N, g) be singular manifolds in X. We say that (M, f) and (N, g) are **bordant**, if their sum $(M, f) + (N, g) := (M + N, (f, g)) := (M \sqcup N, f \sqcup g)$ is nullbordant.

A nullbordism of $(M, f) + (N, g)$ is called a **bordism** between (M, f) and (N, g).

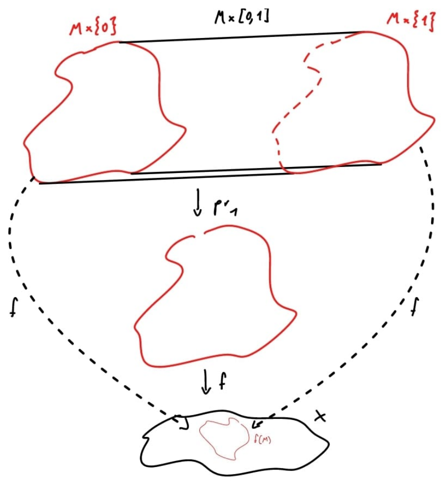
We will refer to this relation as **bordism relation**.

Remark.

$$(M, f) + (\emptyset, g) \text{ are bordant } \iff (M, f) \text{ is nullbordant}$$

Remark. It follows that (M, f) and (N, g) can only be nullbordant if M and N are of the same dimension.

Example 17 (Cylinder)**.** For an arbitrary X, and $(M, f) = (N, g)$, we always get the cylinder as a bordism: $(M \times [0, 1], f \circ \mathrm{pr}_1)$, where pr_1 is the projection onto the first factor.



Example 18 (Pair of pants)**.** For $X = \mathrm{pt}$, $M_1 = M^n \sqcup N^n$ and $M_2 = M\#N$, we get the "pair of pants" as a bordism between them. We can construct the pants as follows. Take the two cylinders $M \times [0,1]$ and $N \times [0,1]$. Now

$$f : (M \times [0,1]) \sqcup (N \times [0,1]) \to [0,1]$$

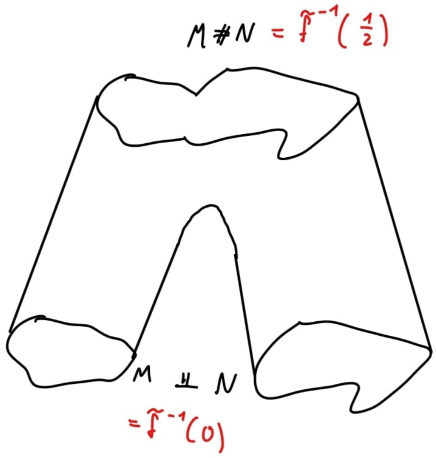
given by

$$(p,t) \mapsto t$$

regardless of $p \in M$ or $p \in N$ is a smooth map between manifolds. By theorem 1.17, we may assume that $\frac{1}{2}$ is a regular value. We embed D^{n+1} by i and j into the cylinders such that $f \circ i(0) = f \circ j(0) = \frac{1}{2}$ and $f \circ i(x) = f \circ j(x)$ for all $x \in D^{n+1}$. We may also assume that $(f \circ i)^{-1}(\frac{1}{2})$ and is connected, else we can just consider smaller disks inside the disks. Taking the connected sum now with respect to i and j leaves

$$\tilde{f} : (M \times [0,1])\#(N \times [0,1]) \to [0,1],$$

which is induced by f, well-defined. Furthermore, $\frac{1}{2}$ is still a regular value of $\tilde{f}$. Note that $\tilde{f}^{-1} = M\#N$, as the embeddings i, j induce embeddings of D^n into M, N. Now we define the pair of pants to be $\tilde{f}^{-1}([0, \frac{1}{2}])$, which is an $(n+1)$-dimensional manifold by with boundary $\tilde{f}^{-1}(0) \sqcup \tilde{f}^{-1}(\frac{1}{2}) = (M \sqcup N) \sqcup (M\#N)$ by construction. Thus, it is a bordism between M_1 and M_2.



Example 19. $\{\mathrm{pt}\}$ and $\{\mathrm{pt}\} \sqcup \{\mathrm{pt}\}$ are not bordant, as 1-manifolds either have 2 or 0 boundary points. Compact 1-manifolds can be classified up to homeomorphism as the circle S^1 (no boundary) and the interval $[0,1]$ (two boundary points).

Remark. It follows that an odd number of points is never nullbordant as a singular manifold in $\{\mathrm{pt}\}$, but an even number of points is. (This gives us $\mathfrak{N}_0(\mathrm{pt}) \cong \mathbb{F}_2$, as we will see later.)

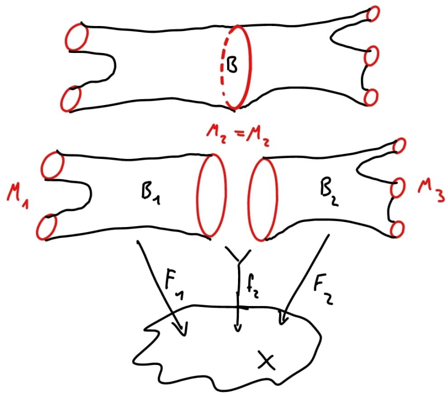
Proposition 1.22. *[BD70, II, Satz 1.4] Being bordant is an equivalence relation on the set of closed singular manifolds.*

Proof. [BD70], [CF64, p.8]

- Symmetry: Follows from the symmetry of the disjoint union. If (M, f) and (N, g) are bordant, then there exists a nullbordism of $(M, f) + (N, g) = (M \sqcup N, f \sqcup g) = (N \sqcup M, g \sqcup f) = (N, g) + (M, f)$. So, a bordism between (M, f) and (N, g) is also bordism between (N, g) and (M, f).

- Reflexivity: We have already constructed a bordism between (M, f) and itself in example 17.

- Transitivity: Let (B_1, F_1) be the bordism between (M_1, f_1) and (M_2, f_2), and let (B_2, F_2) be the bordism between (M_2, f_2) and (M_3, f_3). Then we can "glue" the two bordisms together at the common boundary (M_2, f_2). We only need to check that the gluing is smooth, i.e. $(B_1, F_1) \cup_{(M_2, f_2)} (B_2, F_2)$ is a smooth manifold. By theorem 1.10, we can find a collar of M_2 in both B_1 and B_2. Both these collars have the induced smooth structure of $M_2 \times [0, 1)$. Gluing the bordisms along M_2 glues the collars to something diffeomorphic to $M_2 \times (-1, 1)$, as the smooth structures align in the collars. Since smoothness is local and we now have that the gluing is smooth for a open neighbourhood of M_2, the whole gluing is smooth.



□

Remark. We required the singular manifolds to be closed, because on manifolds with non-empty boundary the bordism relation does not make sense, as they cannot be the boundary of another manifold (see theorem 1.9).

Definition 1.23 (bordism group [BD70, II, Definition 1.5])**.** The equivalence classes of the bordism relation are called **bordism classes**. The bordism class of (M, f) is denoted by $[M, f]$. The set of bordism classes of n-dimensional singular manifolds in X is denoted by $\mathfrak{N}_n(X)$ and is called the ***n*-th bordism group of** X.

$$\mathfrak{N}_n(X) = \{\text{closed singular } n\text{-manifolds in } X\}/\text{bordism}$$

For negative n, we set $\mathfrak{N}_n(X)$ to be the trivial group.

Observation. *This might seem similar to the definition of singular homology groups. They were also defined by the quotient of the kernel by the image of a boundary map. In our case, the boundary map is ∂, defined in definition 1.19. Let us introduce an index for this map to denote the dimension: $\partial_n : \{n\text{-manifolds}\} \to \{\text{closed } (n-1)\text{-manifolds}\}$. Then:*

$$\mathfrak{N}_n = \ker(\partial_n)/\mathrm{im}(\partial_{n+1})$$

This notion only makes sense if we have an neutral element, so we can speak of a kernel. This is given by an abelian structure given in the next theorem.

Theorem 1.24 ([BD70, II, Satz 2.1])**.** *The bordism groups are abelian groups with the operation $+$ defined in definition 1.21:*

$$[M_1, f_1] + [M_2, f_2] = [M_1 + M_2, (f_1, f_2)]$$

Every element in this group has order at most 2, making $\mathfrak{N}_n(X)$ a $\mathbb{F}_2$-vector space.

Proof. [BD70] The neutral element is the bordism class of the empty manifold (i.e. the class of all nullbordant manifolds).
"+" is associative and commutative, because the disjoint union is associative and commutative.
It is well-defined: Let $(M_1, f_1), (M_1', f_1') \in [M_1, f_1]$, and $(M_2, f_2), (M_2', f_2') \in [M_2, f_2]$. Moreover, let (B_1, F_1) and (B_2, F_2) be bordisms between (M_1, f_1) and (M_1', f_2'), respectively between (M_2, f_2) and (M_2', f_2'). Then, a bordism between $(M_1, f_1) + (M_2, f_2)$ and $(M_1', f_1') + (M_2', f_2')$ is given by $(B_1, F_1) + (B_2, F_2)$.
Since being bordant is reflexive, every element is its own inverse. $\square$

Definition 1.25 (Product map [BD70, pp.12-13])**.** Let X, Y be two topological spaces. Then we define a product map

$$\cdot : \mathfrak{N}_p(X) \times \mathfrak{N}_q(Y) \to \mathfrak{N}_{p+q}(X \times Y)$$

as

$$([M, f], [N, g]) \mapsto [M \times N, f \times g]$$

Observation. *This map is well defined, as for two bordant singular manifolds (M, f) and (M', f') with bordism (B, F), we get a bordism between $(M \times N, f \times g)$ and $(M' \times N, f' \times g)$ by $(B \times N, F \times g)$. This map is also bilinear (as a map between $\mathbb{F}_2$-vector spaces). Scalar multiplication is defined as $0 \cdot [M, f] = [\emptyset, \emptyset \to X]$, $1 \cdot [M, f] = [M, f]$. As $[\emptyset] \cdot [N, g] = [\emptyset] = [M, f] \cdot [\emptyset]$, we conclude for any $a \in \mathbb{F}_2$: $[M, f] \cdot (a \cdot [N, g]) = a \cdot ([M, f] \cdot [N, g]) = (a \cdot [M, f]) \cdot [N, g]$. Additivity follows, as addition is defined as disjoint union:*

$$\begin{aligned}
([M, f] + [M', f']) \cdot [N, g] &= [M + M', (f, f')] \cdot [N, g] \\
&= [((M + M') \times N), (f, f') \times g] \\
&= [(M \times N) + (M' \times N), (f \times g, f' \times g)] \\
&= [M, f] \cdot [N, g] + [M', f'] \cdot [N, g]
\end{aligned}$$

Notation. For $X = \{\mathrm{pt}\}$, we will write $\mathfrak{N}_n$ for $\mathfrak{N}_n(X)$ and for elements of $\mathfrak{N}_n$, we will omit the map from the notation ($[M] = [M, f]$).

Definition 1.26 (graded bordism ring [BD70, Satz 2.2])**.**

$$\mathfrak{N}_* := \bigoplus_{n\in\mathbb{Z}} \mathfrak{N}_n$$

is a $\mathbb{Z}$-graded ring over $\mathbb{F}_2$ via $+$ and $\cdot$ and is called the **bordism ring**.

Remark. As $\mathbb{F}_2$ is a field, $\mathfrak{N}_*$ is a graded vector space over $\mathbb{F}_2$.

Definition 1.27 (graded bordism module [BD70, Satz 2.3])**.**

$$\mathfrak{N}_*(X) := \bigoplus_{n\in\mathbb{Z}} \mathfrak{N}_n(X)$$

is a $\mathbb{Z}$-graded module over $\mathfrak{N}_*$. It is called the **bordism module**. Explicitly, the product map acts as follows:

$$[M] \cdot [N, f] = [M \times N, f \circ \mathrm{pr}_2]$$

where $\mathrm{pr}_2 : M \times N \to N$ is the projection onto the second factor.
This is well-defined by the above observation.

Remark. Notice that in this definition, this is a left module, but since we could just project to the first factor instead, we also get a right module structure.

1.2.2 The Eilenberg-Steenrod Axioms

The Eilenberg-Steenrod axioms are a set of axioms that characterize homology and cohomology theories.

Definition 1.28 (Homology theory [ES52, I.3], [Lüc05, Definition 1.1.])**.** An **ordinary homology theory** $\mathcal{H}_* = (\mathcal{H}_*, \partial_*)$ with coefficients in R-modules is a covariant functor

$$\mathcal{H}_* : \mathrm{TOP}^2 \to \mathbb{Z}\text{-graded } R\text{-modules}$$

together with a natural boundary operator

$$\partial_* : \mathcal{H}_* \to \mathcal{H}_{*-1} \circ I$$

where I is a forgetful functor $I : \mathrm{TOP}^2 \to \mathrm{TOP}^2$, given by $I(X, A) = (A, \emptyset)$. We will often write X for $(X, \emptyset)$ for any space X.
$\mathcal{H}_*$ has to satisfy the following **Eilenberg-Steenrod axioms for homology theories**:

- **Homotopy invariance**
  Let $f, g : (X, A) \to (Y, B)$ be homotopic maps. Then for all $n \in \mathbb{Z}$, we have

$$\mathcal{H}_n(f) = \mathcal{H}_n(g) : \mathcal{H}_n(X, A) \to \mathcal{H}_n(Y, B)$$

- **Long exact sequence**
  Let (X, A) be a pair of spaces. Then for all $n \in \mathbb{Z}$, we have the long exact sequence of homology groups:
  $$\dots \xrightarrow{\partial_{n+1}(X,A)} \mathcal{H}_n(A) \xrightarrow{\mathcal{H}_n(i)} \mathcal{H}_n(X) \xrightarrow{\mathcal{H}_n(j)} \mathcal{H}_n(X, A) \xrightarrow{\partial_n(X,A)} \mathcal{H}_{n-1}(A) \to \dots$$

- **Excision**
  Let $A \subset B \subset X$ be subspaces of X such that $\overline{A} \subset B^\circ$. Then the inclusion $i : (X \setminus B, A \setminus B) \to (X, A)$ induces an isomorphism of homology groups for all $n \in \mathbb{Z}$:
  $$\mathcal{H}_n(i) : \mathcal{H}_n(X \setminus A, B \setminus A) \xrightarrow{\cong} \mathcal{H}_n(X, B)$$

- **Dimension axiom**
  For the point space pt, we have
  $$\mathcal{H}_n(\mathrm{pt}) \cong \begin{cases} R & n = 0 \\ \{0\} & n \neq 0 \end{cases}$$

The following axiom is often considered part of the Eilenberg-Steenrod axioms, although it was formulated by Milnor 10 years later

- **Disjoint union axiom** [Mil62, p.337, Additivity axiom],[Lüc05, Definition 1.1.]
  Let $\{X_i\}_{i \in I}$ be a family of topological spaces. Let $j_i : X_i \to \coprod_{i \in I} X_i$ be the inclusion. Then for all $n \in \mathbb{Z}$, we have a bijection:
  $$\bigoplus_{i \in I} \mathcal{H}_n(j_i) : \bigoplus_{i \in I} \mathcal{H}_n(X_i) \xrightarrow{\cong} \mathcal{H}_n\left(\coprod_{i \in I} X_i\right)$$

The motivation behind the disjoint union axiom is the following uniqueness theorem.

Theorem 1.29 ([Mil62, p.340, Uniqueness Theorem]). *Let $\mathcal{H}_*$ be an ordinary homology theory satisfying the disjoint union axiom on CW-complexes with values in R. Then there is a natural isomorphism $\mathcal{H}_* \xrightarrow{\cong} H_*$, where H_* denotes singular homology.*

A proof can be found in [Mil62], but we will not need this here.

Remark. If a functor only fails to satisfy the dimension axiom from definition 1.28, it is called an **extraordinary homology theory**.

1.2.3 Relative Bordism

As homology theories are defined as functors from TOP^2, we will extend the definition of bordism to pairs of topological spaces.

Definition 1.30 (relative bordism [Die08, pp.524-525])**.** For a pair of topological spaces (X, A), we call $(M, f) = (M, \partial M, f)$ a **singular manifold in** (X, A), if $f : (M, \partial M) \to (X, A)$ is a continuous map of pairs.
Two n-dimensional singular manifolds (M_0, f_0) and (M_1, f_1) in (X, A) are called **bordant**, if there exists an $(n + 1)$-dimensional singular manifold (B, F) in X, such that:

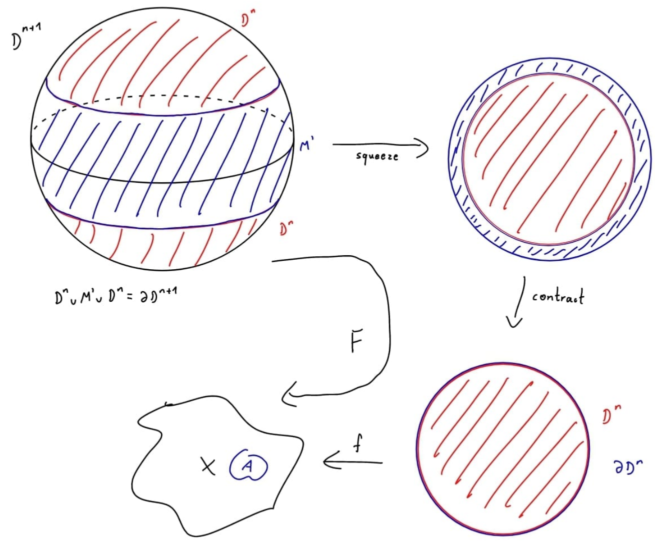
- ∂B can be decomposed as $\partial B = M_0 \cup M_1 \cup M'$, where M_0, M_1, M' are considered as embedded submanifolds of B with boundary, such that $\partial M' = \partial M_0 \sqcup \partial M_1$, $M_i \cap M' = \partial M_i$ and $\partial M_0 \cap \partial M_1 = \emptyset$, for $i \in \{0, 1\}$.
- $\partial F_{|M_i} = f_i$, for $i \in \{0, 1\}$
- $F(M') \subseteq A$

As before, (B, F) will be called a **bordism** between (M_0, f_0) and (M_1, f_1) and a **null-bordism** of (M_0, f_0) if $M_1 = \emptyset$.

Remark. If we take $A = \emptyset$, we get the definition of (absolute) bordism as we had before. In particular, if (M, f) and (N, g) are bordant in the absolute sense, then they are also bordant in the relative sense with the same bordism (but there may be more possible bordisms in the relative sense).

Now, the bordism relation also makes sense for manifolds with non-empty boundary.

Example 20. Consider the disk (D^n, f) as a singular manifold in (X, A). We have a bordism B, F between (D^n, f) and itself by $B = D^{n+1}$. We have to cover $\partial D^{n+1} = S^n$ by two D^ns and M', which can be done by taking almost all of each the upper and the lower hemisphere as D^n – in this example, the chosen embeddings $i, j : D^n \to S^n$ heavily matters. We will embed them such that $\mathrm{im}(i) = -\mathrm{im}(j)$ but $i(x) = -j(x) \Rightarrow x = 0$ – and the remaining cylinder around the equator as M'. Now F just has to map M' to A (and along with it, also ∂M_0 and ∂M_1, as desired). To do this, F "squeezes" the disk such that the D^ns are identified together and then maps the resulting disk with the remains of M' to $(D^n, \partial D^n)$ via the identity on D^n and contracting M' to its boundary. Lastly, apply f.



Remark. The above example is just the same as the cylinder bordism in example 17, if we had constructed it as a cylinder we would not have had to worry about the embedding. If cylinder bordism works for manifolds with non-empty boundary, we can take its lateral surface ($= \partial M_0 \times [0,1]$) as M'. For this to work, it has to be shown that $M \times [0,1]$ admits a smooth structure; a priori it is a manifold with corners. This is done by applying a technique called **straightening the angle** [CF64, p. I.3]. We will not go into detail here, but the main idea is to find an submanifold of codimension 2 containing all corners and a open neighbourhood of it which can be charted into $\mathbb{R}^{n-2} \times \mathbb{R}^2_{\geq 0}$ and then getting new charts by composing with a diffeomorphism $\mathbb{R}^2_{\geq 0} \xrightarrow{\cong} \mathbb{R} \times \mathbb{R}_{\geq 0}$.

Theorem 1.31 ([Die08, p.525])**.** *Relative bordism is an equivalence relation on the set of singular manifolds in (X, A).*

Proof. The proof is similar to the proof of proposition 1.22. Symmetry is clear, for reflexivity, we now also have the cylinder as stated in the above remark. For transitivity, we again glue the bordisms along their common boundary M_2, but as M_2 may have boundary now, we glue the bordism along an open neighbourhood U (in ∂B_1 and ∂B_1) of M_2. To see that this is smooth, we again look at the collars of ∂B_i. This time we restrict the collars to U and see that the collars are diffeomorphic to $M_2 \times (-1, 1)$. The rest of the proof is the same as in proposition 1.22. □

Relative bordism groups are also defined to be the set of relative bordism classes, analogously to absolute bordism groups, but as manifolds with boundary are also allowed now, we have

$$\mathfrak{N}_n(X, A) = \{\textit{compact} \text{ singular } n\text{-manifolds in } (X, A)\}\big/\text{relative bordism}.$$

These also are abelian by the same proof (theorem 1.24). The product map is defined as

$$\cdot : \mathfrak{N}_p(X, A) \times \mathfrak{N}_q(Y, B) \to \mathfrak{N}_{p+q}(X \times Y, (A \times Y) \cup (X \times B))$$

on elements, it does the same as before. A product of manifolds is again a smoothable manifold by straightening the angle as in the remark after example 20. Again, we get a graded module structure

$$\mathfrak{N}_*(X, A) := \bigoplus_{n \in \mathbb{Z}} \mathfrak{N}_n(X, A)$$

over $\mathfrak{N}_*$, the relative bordism module.

Lemma 1.32 ([Die91, VIII, Lemma 13.10])**.** *Let $[M, f] \in \mathfrak{N}_n(X, A)$ and N an n-dimensional submanifold of M. Suppose that $[N, f_{|N}] \in \mathfrak{N}_n(X, A)$ and $f(M \setminus N) \subseteq A$. Then $[M, f] = [N, f_{|N}]$ in $\mathfrak{N}_n(X, A)$.*

Proof. [Die91] We need to show that (M, f) and $(N, f_{|N})$ are bordant.
Let $B = M \times [0, 1]$ the cylinder. $\partial B = M \times \{0\} \cup M \times \{1\} \cup \partial M \times I$. Define $F : B \to X$ as $F(p, t) = f(p)$.
Claim: (B, F) is a bordism between $(M \times \{0\}, f \circ \mathrm{pr}_1)$ and $(N \times \{1\}, f_{|N} \circ \mathrm{pr}_1)$.
As both are embedded submanifolds of B, we only need to check that $\partial B \setminus ((M \times \{0\}) \cup (N \times \{1\}))$ is mapped into A by F.

$$\begin{aligned} F(\partial B \setminus ((M \times \{0\}) \cup (N \times \{1\}))) &\subseteq F((\partial M \times I) \cup ((M \times \{1\}) \setminus (N \times \{1\}))) \\ &= f(\partial M) \cup f(M \setminus N) \subseteq A \end{aligned}$$

□

1.2.4 Bordism Homology

Lemma 1.33. *[BD70, II, Satz 3.2] Relative bordism is a covariant functor*

$$\mathfrak{N}_* : \mathrm{TOP}^2 \to \textit{graded } \mathfrak{N}_* \textit{modules}$$

Proof. [BD70] Let $(X, A) \in \mathrm{Ob}(\mathrm{TOP}^2)$, we already saw

$$(X, A) \overset{\mathfrak{N}_*}{\longmapsto} \mathfrak{N}_*(X, A)$$

For a map $\mathrm{Mor}(\mathrm{TOP}^2) \ni f : (X, A) \to (Y, B)$, we take the induced map on the bordism groups:

$$f_* := \mathfrak{N}_*(f) : \mathfrak{N}_*(X, A) \to \mathfrak{N}_*(Y, B)$$

given by

$$f_*[M, g] = [M, f \circ g]$$

Then we get that $\mathfrak{N}_*(\mathrm{id}_{(X,A)}) = \mathrm{id}_{\mathfrak{N}_*(X,A)}$ and for $f : (X, A) \to (Y, B), g : (Y, B) \to (Z, C)$, we have for any $[M, h] \in \mathfrak{N}_*(X, A)$:

$$(g \circ f)_*[M, h] = [M, g \circ f \circ h] = g_*[M, f \circ h] = g_* \circ f_*[M, h]$$

□

The boundary map is defined similarly as before.

$$\partial_n : \mathfrak{N}_n(X, A) \to \mathfrak{N}_{n-1}(A, \emptyset)$$

$$\partial_n(M, f) = (\partial M, f_{|\partial M})$$

Lemma 1.34 (Naturality of the boundary map)**.** *The following diagram commutes for any $f : (X, A) \to (Y, B)$ and $n \in \mathbb{Z}$:*

$$\begin{array}{ccc} \mathfrak{N}_n(X, A) & \xrightarrow{\partial_n} & \mathfrak{N}_{n-1}(A) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle (f_{|A})_*} \\ \mathfrak{N}_n(Y, B) & \xrightarrow{\partial_n} & \mathfrak{N}_{n-1}(B) \end{array}$$

Proof. Let $[M, g] \in \mathfrak{N}_n(X, A)$. $\partial_n[M, g] = [\partial M, g_{|\partial M}] \in \mathfrak{N}_{n-1}(A)$, as ∂M is closed and g maps ∂M to A. Now $f_{|A*}([\partial M, g_{|\partial M}]) = [\partial M, f \circ g_{|\partial M}] \in \mathfrak{N}_{n-1}(B)$.
On the other side, $(\partial_n \circ f_*)[M, g] = \partial_n[M, f_* \circ g] = [\partial M, (f \circ g)_{|\partial M}] \in \mathfrak{N}_{n-1}(B)$, which is the same as the previous composition, hence,

$$\partial_n \circ f_* = f_{|A*} \circ \partial_n$$

and the diagram commutes. □

Lemma 1.35 (Homotopy invariance)**.** *[BD70, II, Satz 3.1] $\mathfrak{N}_*$ is homotopy invariant.*

Proof. [BD70][CF64, Chapter I, 5.5] Let $f, g : (X, A) \to (Y, B)$ be homotopic maps of pairs. Let $h : (X \times I, A \times I) \to (Y, B)$ be a homotopy between f and g. Then, for $[M, F] \in \mathfrak{N}_*(X, A)$, we have a bordism between $f_*[M, F]$ and $g_*[M, F]$ by the cylinder $(M \times I, h \circ (F \times \mathrm{id}_I))$. □

Remark. For a closed manifold M with $[M, f] = 0 \in \mathfrak{N}_n(X, A)$ and a nullbordism (B, F) of (M, f). Then $\partial B \setminus M$ is a closed n-dimensional submanifold of B. For a proof, see [Zha23, Lemma 5.4].

Lemma 1.36 (Long exact sequence [Die08, Proposition 21.1.9])**.** *$\mathfrak{N}_*$ satisfies the long exact sequence axiom.*

Proof. [Die08] Let i, j be the inclusion maps $i : A \to X, j : X = (X, \emptyset) \to (X, A)$.
Claim: The sequence

$$\dots \xrightarrow{\partial_{n+1}} \mathfrak{N}_n(A) \xrightarrow{i_*} \mathfrak{N}_n(X) \xrightarrow{j_*} \mathfrak{N}_n(X, A) \xrightarrow{\partial_n} \mathfrak{N}_{n-1}(A) \xrightarrow{i_*} \dots$$

is exact.

- **Exactness at $\mathfrak{N}_n(A)$:** $i_* \circ \partial = 0$, as for $[M, f] \in \mathfrak{N}_{n+1}(X, A)$, $\partial_{n+1}[M, f] = [\partial M, f_{|\partial M}]$, which, considered as an element in $\mathfrak{N}_n(X)$ is nullbordant via the nullbordism (M, f).

  Let $(M, f) \in \mathfrak{N}_n(A)$ with nullbordism (B, F) in X. Then, $\partial_{n+1}[B, F] = [M, f]$.

- **Exactness at $\mathfrak{N}_n(X)$:** Let $[M, f] \in \mathfrak{N}_n(A)$. Choose $N = \emptyset$ and use lemma 1.32 to get $[M, f] = 0$ in $\mathfrak{N}_n(X, A)$, so $j_* \circ i_* = 0$.

  Now let $[M, f] \in \ker(j_*) \subseteq \mathfrak{N}_n(X)$. Then there exists a singular manifold (B^{n+1}, F) in (X, A) of $[M, f]$. (B, F) is a bordism between M, f and $\partial B \backslash M, F_{|_{\partial B \backslash M}}$, as $\partial B \backslash M$ is a closed submanifold of B by the previous remark. Since $F(\partial B \setminus M) \subseteq A$, $[\partial B \setminus M, F_{|_{\partial B \setminus M}}] \in \mathfrak{N}_n(A)$. So, $i_*[\partial B \setminus M, F_{|_{\partial B \setminus M}}] = [M, f] \in \mathfrak{N}_n(X)$.

- **Exactness at $\mathfrak{N}_n(X, A)$:** $\partial \circ j_* = 0$ holds because every element in $\mathfrak{N}_n(X)$ is a closed manifold, so it has empty boundary, hence applying the boundary map gets us to $[\emptyset, \emptyset \to X]$ which is the zero element in $\mathfrak{N}_{n-1}(A)$.

  Let $\partial[M, f] = 0$, and $[B, F]$ be a nullbordism of $[\partial M, f_{|_{\partial M}}]$. Identify (M, f) and (B, F) along ∂M. This is smooth by the same argument as in proposition 1.22. Call the resulting singular manifold (C, g). This has no boundary, so $[C, g] \in \mathfrak{N}_n(X)$ Now with lemma 1.32, get $j_*[C, g] = [M, f]$.

□

Only the excision axiom is left to check. To see the excision property, we need some preliminary lemmas.

Lemma 1.37 ([CF64, Chapter I, 3.1]). *Let M^n be a compact manifold, let $K, L \subseteq M$ closed, such that $K \cap L = \emptyset$. Then there exists a closed n-dimensional submanifold $N \subset M$ with boundary ∂N such that $K \subseteq N$ and $L \cap N = \emptyset$.*

Proof. [CF64]
$K \cap \partial M$ and $L \cap \partial M$ are still disjoint, closed in ∂M, so we can find disjoint closed subsets $K', L' \subseteq \partial M$ such that $K \subseteq \overset{\circ}{K'}$, $L \subseteq \overset{\circ}{L'}$. We find a collar C of ∂M by theorem 1.10 and identify it with $\partial M \times [0, 1)$. As M is compact, there exists a $t \in (0, 1)$ such that $L \cap (\partial M \times [0, t)) \subseteq L'$ and $K \cap (\partial M \times [0, t)) \subseteq K'$. Now $M' := M \setminus (\partial M \times [0, t))$ is a closed n-dimensional submanifold of M. By Urysohn's lemma 1.11, there exists a smooth function $\alpha : M' \to [0, 1]$ such that $\alpha_{|_{(K' \times \{t\}) \cup (K \cap M')}} = 0$ and $\alpha_{|_{(L' \times \{t\}) \cup (L \cap M')}} = 1$. We can extend α to M by $\alpha(p, s) := \alpha(p, t)$ for $p \in \partial M$ and $s \in [0, t)$. By theorem 1.17, there is a regular value $r \in (0, 1)$ of α. Then $N := \alpha^{-1}([0, r]) = \alpha^{-1}(-\infty, r]$ is a n-dimensional closed submanifold of M. By construction, $K \subseteq N$, $L \cap N = \emptyset$.
It remains to show that N is smoothable, but we will omit the proof here. □

Lemma 1.38 ([Zha23, Lemma 5.8]). *Let K, L, M, N, α, r be as in lemma 1.37. Then,*

$$\partial N \subseteq \partial M \cup \alpha^{-1}(r) \subseteq \partial M \cup ((M \setminus K) \cap (M \setminus L))$$

Proof. [Zha23] *First inclusion:* Assume $p \in \partial N \setminus \partial M$. We need to show that $\alpha(p) = r$. By assumption, p is an interior point of M, so we can find a euclidean neighbourhood U of p in M. As $p \in N$, $\alpha(p) \leq r$. Suppose $\alpha(p) < r$. Then, for a $r' \in (\alpha(p), r)$, we have $p \in \alpha^{-1}([0, r')) \subseteq N$. Since $\alpha^{-1}([0, r'))$ is open in M, it is open in N and $U \cap V$ is an open euclidean neighbourhood of p in N. But then, p is an interior point of N,

contradicting the assumption that $p \in \partial N$. So, $\alpha(p) = r$.
Second inclusion: Since $\alpha_{|_K} = 0$ and $\alpha_{|_L} = 1$, we have $\alpha^{-1}(r) \subseteq M \setminus (K \cup L)$. $\square$

Lemma 1.39 (Excision axiom [Zha23, Theorem 5.10])**.** *Let (X, A, Z) be a triple of topological spaces satisfying $\overline{Z} \subseteq \mathring{A}$. Then the inclusion map $i : (X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces an isomorphism of bordism groups:*

$$i_* : \mathfrak{N}_n(X \setminus Z, A \setminus Z) \xrightarrow{\cong} \mathfrak{N}_n(X, A)$$

Proof. [Zha23] **Surjectivity**: Let $[M, f] \in \mathfrak{N}_n(X, A)$. Then the preimages $K = f^{-1}(X \setminus \mathring{A})$ and $L = f^{-1}(\overline{Z})$ are disjoint and closed subsets of M. By lemma 1.37, there exists a closed submanifold with boundary $N \subset M$ such that $K \subseteq N$ and $L \cap N = \emptyset$.
From $L \cap N = \emptyset$, it follows that $f(N) \subseteq X \setminus \overline{Z}$. By lemma 1.38, we have $\partial N \subseteq \partial M \cup ((M \setminus K) \cap (M \setminus L))$. So, for any $p \in \partial N$, we have either $p \in \partial M$, implying $f(p) \in A$, or $p \in (M \setminus K)$, implying $f(p) \in \mathring{A}$. In any case, we get $f(\partial N) \subseteq A \setminus \overline{Z}$, so $[N, f_{|_N}] \in \mathfrak{N}_n(X \setminus Z, A \setminus Z)$.
As $f^{-1}(X \setminus \mathring{A})$, we get $f(M \setminus N) \subseteq \mathring{A}$. By lemma 1.32, we get $i_*[N, f_{|_N}] = [M, f]$.
Injectivity: Take $[M, f] \in \mathfrak{N}_n(X \setminus Z, A \setminus Z)$ such that $i_*[M, f] = 0$ in $\mathfrak{N}_n(X, A)$. Then there exists a nullbordism (B, F) with $F(\partial B \setminus M) \subseteq A$.
Again, let $K = F^{-1}(X \setminus \mathring{A})$, $L = F^{-1}(\overline{Z})$. By lemma 1.37, we have a submanifold $N^{n+1} \subseteq B$ such that $K \subseteq N$, $N \cap L = \emptyset$. Then, $[\partial N, F_{|_{\partial N}}] \in \mathfrak{N}_n(X \setminus Z, A \setminus Z)$ nullbordant by $(N, F_{|_N})$.
Claim: $M \cap \partial N = M \cap N$.
"$\subseteq$" is clear. For "$\supseteq$", take $p \in M \cap N$. Then $p \in \partial B$, so there exists a chart $\varphi : U \to \mathbb{H}^{n+1}$ of B, such that $\varphi(p) \in \partial \mathbb{H}^{n+1}$. So, $p \in \partial N$.
Then $M \cap \partial N$ is an submanifold of M, because

$$M \cap \partial N = M \cap N = (\alpha_{|_M})^{-1}([0, r]), \tag{$\star$}$$

where α is again such that $N = \alpha^{-1}([0, r])$. By theorem 1.17, we assume that r is also a regular value of $\alpha_{|_M}$. Now we see that $[M \cap \partial N, f_{|_{M \cap \partial N}}] \in \mathfrak{N}_n(X \setminus Z, A \setminus Z)$: $f(M \cap \partial N) \subseteq f(M) \subseteq X \setminus Z$ and $f(\partial(M \cap \partial N)) \subseteq A \setminus Z$ because of ($\star$) and lemma 1.38, as $\partial(M \cap \partial N) \subseteq \partial M \cup (B \setminus (K \cup L))$.
Claim: $[M, f] = [M \cap N, f_{|_{M \cap N}}] \in \mathfrak{N}_n(X \setminus Z, A \setminus Z)$.
By lemma 1.32, it is enough to show that $f(M \setminus (M \cap N)) = f(M \setminus M \cap \partial N) \subseteq A \setminus Z$. Let $p \in M \setminus \partial N = M \setminus N$, then $f(p) \in X \setminus Z$ because $p \in M$, and $f(p) \in A$, because $p \notin N$.
By exactly the same argument, $[M \cap N, f_{|_{M \cap N}}] = [\partial N, F_{|_{\partial N}}] \in \mathfrak{N}_n(X \setminus Z, A \setminus Z)$. Since $M \cap N$ is an submanifold of M, it is also an submanifold of ∂N. Now, we only need to show that $F(\partial N \setminus (M \cap N)) = F(\partial N \setminus M) \subseteq A \setminus Z$. We know $\partial N \subseteq \partial B \cup (B \setminus (K \cup L))$ (lemma 1.38). Let $p \in \partial N \setminus M$, then either $p \in (B \setminus (K \cup L)) \setminus M$ or $p \in \partial B \setminus M$. In the first case, $F(p) \in A \setminus Z$, so we are done. In the second case, we know $p \notin L$, because $B \cap L = \emptyset$. So, $F(p) \notin \overline{Z}$. By construction, we have $F(\partial B \setminus M) \subseteq A$, so $F(p) \in A$.

$\Rightarrow g(p) \in A \setminus Z$.
In total, we have shown that $(M, f), (M \cap N, f_{|_{M\cap N}}), (\partial N, F_{|_{\partial N}})$ are bordant. All three are nullbordant, since $(\partial N, F_{|_{\partial N}})$ is nullbordant in $\mathfrak{N}_n(X \setminus Z, A \setminus Z)$ by $(N, F_{|_N})$. Thus, $[M, f] = 0$ in $\mathfrak{N}_n(X \setminus Z, A \setminus Z)$. □

Lemma 1.40 (Disjoint union axiom [Zha23, Theorem 5.3])**.** *The disjoint union axiom holds for bordism.*

Proof. [Zha23] We need to show that

$$\bigoplus_{i\in I} \mathfrak{N}_n(j_i) : \bigoplus_{i\in I} \mathfrak{N}_n(X_i) \to \mathfrak{N}_n \left(\coprod_{i\in I} X_i \right)$$

is an isomorphism.
Claim:

$$\iota : \bigoplus_{i\in I} [M_i, f_i] \mapsto \left[\coprod_{i\in I} M_i, \coprod_{i\in I} f_i \right]$$

gives us the desired isomorphism.
Well-definedness: All but finitely many $[M_i, f_i]$ are 0. Thus, $\coprod M_i$ is a finite disjoint union of compact manifolds, so it is compact.
Injectivity: Suppose $\iota\left(\bigoplus_i [M_i, f_i]\right) = 0$ in $\mathfrak{N}_n(\coprod_i X_i)$. Then there exists a nullbordism (B, F) of it.B as a space is the disjoint union $\coprod_i B_i := \coprod_i F^{-1}(X_i)$, all of the B_i being open and closed in B. Thus, the B_i are compact $(n+1)$-manifolds. Also, $\partial(B_i, F_{|_{B_i}}) = (M_i, f_i)$, so all the (M_i, f_i) are nullbordant and the sum $\bigoplus [M_i, f_i] = 0$.
Surjectivity: Suppose $[M, f] \in \mathfrak{N}_n(\coprod_i X_i)$. As in the proof of injectivity, we can write $M = \coprod_i M_i := f^{-1}(X_i)$, with all M_i compact manifolds. Then a preimage of $[M, f]$ under ι is $\bigoplus [M_i, f_{|_{M_i}}]$. □

Observation. *Bordism does not satisfy the dimension axiom.*
Check: Consider $[\mathbb{RP}^2] \in \mathfrak{N}_2$. There is no 3-manifold that has $\mathbb{RP}^2$ as its boundary. We can see this by the euler characteristic: $\chi(\mathbb{RP}^2) = 1$. However boundaries of manifolds always have even euler characteristic [Die08, Proposition 18.6.2]. So, $\mathfrak{N}_2 \ncong \{0\}$.

Now we can finally conclude:

Theorem 1.41. *Bordism defines a homology theory satisfying the disjoint union axiom.*

Proof. This follows directly from the lemmas 1.33, 1.35, 1.36, 1.39 and 1.40.
As it is not an ordinary homology theory (i.e. the dimension axiom does not hold), we call bordism an **extraordinary** or a **generalized** homology theory. □

1.2.5 Calculations

As we have already noted in example 19, we have $\mathfrak{N}_0 \cong \mathbb{F}_2$. An even number of points is nullbordant, an odd number is not. Let us try to argue for higher dimensions.

The only closed 1-dimensional manifold (up to diffeomorphism) is S^1. As $S^1 = \partial D^2$, we conclude $\mathfrak{N}_1 \cong \{0\}$.
Closed 2-manifolds are classified to be $S^2, \#_i \mathbb{T}^2, \#_i \mathbb{RP}^2$. I.e. they are classified by euler characteristic and orientability. We know $S^2 = \partial D^3$ and $\mathbb{T}^2 = \partial(S^1 \times D^2)$. So $\#_i \mathbb{T}^2 = \partial(\#_i D^3)$. From the above observation, we know $\mathbb{RP}^2$ is not nullbordant. By example 18, we have $[\mathbb{RP}^2 \# \mathbb{RP}^2] = [\mathbb{RP}^2] + [\mathbb{RP}^2]$, and by proposition 1.22, this is $[\mathbb{RP}^2] + [\mathbb{RP}^2] = 0$. Inductively, any even number of connected sums of $\mathbb{RP}^2$s is nullbordant, and any odd number of connected sums of $\mathbb{RP}^2$s is not nullbordant, but in the same equivalence class as $\mathbb{RP}^2$. So we have $\mathfrak{N}_2 \cong \mathbb{F}_2$ with $\mathbb{RP}^2$ as generator.
Closed 3-manifolds are harder to classify, but there still are geometric arguments that show that every closed 3-manifold is nullbordant, so $\mathfrak{N}_3 \cong \{0\}$.
But for dimension 4, already for $\mathbb{R}^4$, there are uncountably many pairwise non-diffeomorphic smooth structures, so we cannot hope to continue this argumentation further. We know that $\mathfrak{N}_4 \not\cong \{0\}$, as $\mathbb{RP}^4$ is not nullbordant, by the same argument as for $\mathbb{RP}^2$.
For dimensions 5 and above, we would need surgery theory, which we will not discuss here.

But while classifications of manifolds up to diffeomorphism are hard, the bordism ring $\mathfrak{N}_*$ has been completely determined in [Tho54]. It is isomorphic to the graded polynomial ring over $\mathbb{F}_2$ with one generator in each dimension that cannot be written as $2^k - 1$.

$$\mathfrak{N}_* \cong \mathbb{F}_2[x_i \mid i \neq 2^k - 1]$$

We get

	$\mathfrak{N}_0$	$\mathfrak{N}_1$	$\mathfrak{N}_2$	$\mathfrak{N}_3$	$\mathfrak{N}_4$	$\mathfrak{N}_5$	$\mathfrak{N}_6$	$\mathfrak{N}_7$	$\mathfrak{N}_8$	...
	$\mathbb{F}_2$	0	$\mathbb{F}_2$	0	$\mathbb{F}_2 \oplus \mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2^2$	0	$\mathbb{F}_2^3$	...
New generator	pt	–	$\mathbb{RP}^2$	–	$\mathbb{RP}^4$	$V_{2,4}$	$\mathbb{RP}^6$	–	$\mathbb{RP}^8$	...

where $V_{k,n} = \{\text{orthonormal } k\text{-frames in } \mathbb{R}^n\}$ denotes the Stiefel manifold. The generators in even dimension are always $\mathbb{RP}^n$, as shown in [Tho54], and in odd dimensions, the generators have been determined in [Dol56].

Definition 1.42 (Reduced bordism homology group [Zha23, p.13])**.** The **reduced bordism homology group** is defined as

$$\tilde{\mathfrak{N}}_n(X) := \ker(\mathfrak{N}_n(X) \xrightarrow{\varepsilon} \mathfrak{N}_n)$$

where ε is the induced map of $X \to \mathrm{pt}$.

Remark.

$$\mathfrak{N}_n(X) \cong \tilde{\mathfrak{N}}_n(X) \oplus \mathfrak{N}_n$$

Remark. For any reduced homology theory, we have the suspension isomorphism

$$\tilde{\mathfrak{N}}_n(X) \cong \tilde{\mathfrak{N}}_{n+1}(\Sigma X)$$

where ΣX denotes the suspension of X.

Using the above two remarks, we can calculate the bordism groups of spheres [Zha23, Proposition 6.1]. We first notice that $S^k = \Sigma(S^{k-1})$. So,

$$\mathfrak{N}_n(S^k) \cong \tilde{\mathfrak{N}}_n(S^k) \oplus \mathfrak{N}_n \cong \tilde{\mathfrak{N}}_{n-1}(S^{k-1}) \oplus \mathfrak{N}_n \cong \ldots \cong \tilde{\mathfrak{N}}_{n-k}(S^0) \oplus \mathfrak{N}_n$$

It suffices to calculate $\tilde{\mathfrak{N}}_n(S^0) = \ker(\mathfrak{N}_n(S^0)) \xrightarrow{\varepsilon} \mathfrak{N}_n$. As $S^0 = \mathrm{pt} \sqcup \mathrm{pt}$, we have by lemma 1.40 $\mathfrak{N}_n(S^0) \xrightarrow{\iota} \mathfrak{N}_n \oplus \mathfrak{N}_n$ is an isomorphism. Consider $\varepsilon \circ \iota$. $\varepsilon \circ \iota([M] \oplus [M']) = [M] + [M']$. This is zero if and only if $[M] = [M']$, as elements are self-inverse. It follows that $\ker(\varepsilon \cong \mathfrak{N}_n)$. Altogether, we have

$$\mathfrak{N}_n(S^k) \cong \mathfrak{N}_{n-k} \oplus \mathfrak{N}_n$$

.

1.3 Orientation

We will now define additional structure on manifolds.

Definition 1.43 (Orientation on vector spaces [Lee13, p.379])**.** An **orientation** on a real vector space V with $\dim V \geq 1$ is an equivalence class of ordered bases $(e_1, \ldots, e_{\dim V})$. Two bases are equivalent if the basis transformation has positive determinant. For $\dim V = 0$ an orientation is the choice of $\pm$.
This gives us exactly two orientations for any vector space.

Definition 1.44 (Orientation preserving [Die91, p. VIII.8])**.** A linear isomorphism $f : U \to V$ between two oriented vector spaces is called **orientation preserving** if for the orientation $(u_1, \ldots, u_n)$ of U, the images $(f(u_1), \ldots, f(u_n))$ give us the chosen orientation of V.

Definition 1.45 (Orienting atlas [Die91, p. VIII.8])**.** Let M be a topological n-manifold with or without boundary. An **orienting atlas** on M is a atlas $\mathcal{O}$ such that the transition maps are not only smooth but also **oriented-related**, i.e. for any $\varphi, \psi \in \mathcal{O}$, the Jacobian $D(\psi \circ \varphi^{-1})$ has positive determinant.
A manifold is called **orientable** if it admits an orienting atlas.

Remark. For smooth manifolds this definition of being orientable coincides with the homological definition.

Definition 1.46 (Oriented manifold [Die91, p. VIII.8])**.** An **oriented manifold** is the pair $(M, [\mathcal{O}])$, of an orientable manifold M and a choice of an equivalence class of orientating atlases $[\mathcal{O}]$ of M. Two orienting atlases are equivalent, if, analogous to definition 1.4, their union is again an orienting atlas. The choice of $[\mathcal{O}]$ is called an **orientation** on M. We will often write just M for an oriented manifold.

Observation. *For a connected manifold M, every equivalence class of atlases with the relation as in definition 1.4 ($\sim_{smooth}$) is split into two equivalence classes with the relation as in 1.46 ($\sim_{oriented}$). I.e. for any atlas $\mathcal{A}$, there exists, if $\mathcal{A}$ is orientable, an atlas $-\mathcal{A} \in$*

$[\mathcal{A}]_{smooth}$ and two equivalence classes $[\mathcal{O}]_{oriented}$ and $[\mathcal{O}]_{oriented}$ such that $\mathcal{A} \in [\mathcal{O}]_{oriented}$ and $-\mathcal{A} \in [-\mathcal{O}]_{oriented}$.
Moreover, $[\mathcal{O}]_{oriented} \cup [-\mathcal{O}]_{oriented} = [\mathcal{A}]_{smooth} \cap \{orientable\ atlases\ on\ M\}$
We call $[-\mathcal{O}]$ the ***opposite orientation*** *to $[\mathcal{O}]$.*

Remark. Clearly, $[\mathcal{O}] \cap [-\mathcal{O}] = \emptyset$. If $[\mathcal{O}] = \emptyset$, the opposite orientation is also empty. This happens if the manifold is not orientable. Typical examples for non-orientable manifolds are $\mathbb{RP}^{2n}$ and the Klein bottle.

An orientation on a manifold M induces an orientation on the tangent spaces T_pM via the differential of charts $\varphi : U \to \mathbb{R}^n$

$$d\varphi_p : T_pU \to T_{\varphi(p)}\mathbb{R}^n \cong \mathbb{R}^n$$

and the standard orientation on $\mathbb{R}^n$.

Remark. Orientation on manifolds can also be defined via pointwise orientation (orientation on the tangent spaces), but as not every pointwise orientation gives a orientation on the manifold, one has to be careful here. The pointwise orientation has to admit an orienting atlas in the sense that the differentials of the charts are orientation preserving.

For manifolds with boundary, the **boundary orientation** is induced by the orientation of $T_p\partial M$, which is, in turn, induced by the orientation of T_pM by forgetting the first basis vector.

1.4 Oriented Bordism

1.4.1 Definitions

We will adapt our definition of bordism to respect orientations.
The Definition of singular manifolds stays the same, but we additionally require M to be oriented now.

Definition 1.47 (bordant [Die08, p.526], [Ati61, p.202])**.** Two closed singular oriented n-manifolds $(M, f), (N, g)$ are called **bordant**, if there exists a singular oriented $n+1$-manifold (B, F) with oriented boundary such that $\partial(B, F) = (N, g) - (M, f)$. $(N, g) - (M, f)$ is defined as

$$(N, g) - (M, f) = (N, g) + (M^-, f),$$

where $M^- = (M, -[\mathcal{O}])$, M with opposite orientation. (B, F) is then called a **oriented bordism** between (M, f) and (N, g).

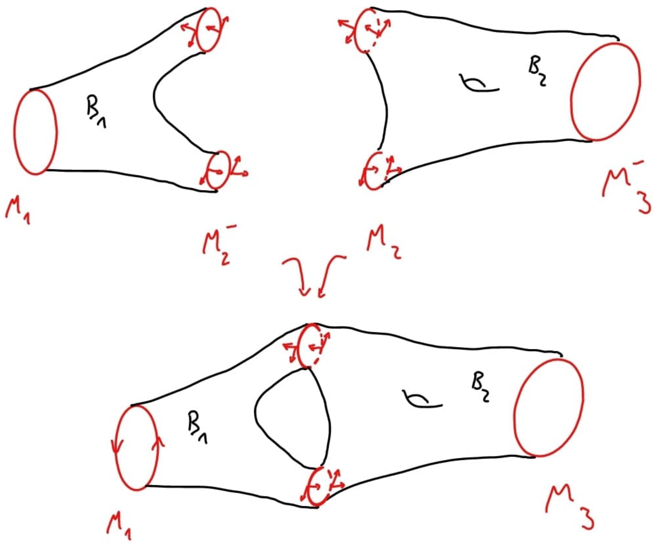
Remark. The definition of being nullbordant follows if we take one of the singular oriented manifolds to be empty.

Proposition 1.48. *Being bordant is an equivalence relation on the set of closed singular oriented manifolds.*

The proof can be copied from the proof of proposition 1.22, but we have to check a few more things.

Proof.

- **Symmetry**: If (B, F) is an oriented bordism between (M, f) and (N, g), then (B^-, F) (again, the negative sign is denoting the opposite orientation) is an oriented bordism between (N, g) and (M, f).
- **Reflexivity**: The cylinder still works as the proof of reflexivity here. We have defined oriented bordism in this way with giving one side the opposite orientation, so the cylinder is still a bordism. $\partial(M \times [0,1]) = M \sqcup M^-$.
- **Transitivity**: The gluing of the manifolds has to be orientation preserving now. Let there be bordisms (B_1, F_1) between (M_1, f_1) and (M_2, f_2), and (B_2, F_2) between (M_2, f_2) and M_3, f_3. Focus on (M_2, f_2), where the gluing happens. Since $M_2 \subset B_1$ has negative orientation and $M_2 \subset B_2$ has positive orientation, so the gluing we had before respects the orientation.



□

Definition 1.49 (oriented bordism group [Ati61, p.202, 2])**.** The equivalence classes of the oriented bordism relation called **oriented bordism classes** and the set of oriented bordism classes of n-dimensional singular oriented manifolds in X is denoted by $\Omega_n(X)$. $\Omega_n(X)$ is called the **n-th oriented bordism group** of X

$$\Omega_n(X) = \{\text{closed singular oriented } n\text{-manifolds in } X\}/\text{oriented bordism}$$

Up until now, everything seems to work out the same as in the unoriented case, but we will now see a critical difference.

Observation. *As in the oriented case, any odd number of points is not nullbordant, but now, an even number does not guarantee* $\coprod_i\{\text{pt}\}$ *being nullbordant. For instance,* $(\text{pt}, +)$ *and* $(\text{pt}, -)$ *are not bordant.*

Theorem 1.50. *The oriented bordism groups are abelian groups via the operation +*

Proof. See proof of theorem 1.24. The only thing that changes is the inverse element. Note that the elements are not self-inverse anymore, as (M, f) and (M, f) being bordant only gives us that $(M, f) + (M^-, f)$ is nullbordant, and in general, $M \neq M^-$. So the inverse for $[M, f]$ exists, but is $[M^-, f]$ rather than $[M, f]$. □

So, $\Omega_n(X)$ is not a $\mathbb{F}_2$-vector space anymore! This makes $\Omega_n(X)$ harder to compute.

The graded ring Ω_* and the graded module $\Omega_*(X)$ are defined in the same way as in the unoriented case (definitions 1.26, 1.27).

1.4.2 Relative Oriented Bordism

The path from absolute oriented bordism to relative oriented bordism is exactly the same as in the unoriented case. We just always need to remember that we reverse the orientations for the second singular oriented manifold. Relative oriented bordism extends the equivalence relation to not necessarily closed singular oriented manifolds.

Again, $\Omega_n(X, A)$ is not a $\mathbb{F}_2$-vector space. So we get a more complicated homology theory now.

1.4.3 Oriented Bordism Homology

Lemma 1.51. *Relative oriented bordism is a covariant functor*

$$\Omega_* : \mathrm{TOP}^2 \to \textit{graded } \Omega_* \textit{ modules}$$

Proof. See proof of lemma 1.33. □

Lemma 1.52 (Naturality of the boundary map)**.** *The following diagram commutes for $f : (X, A) \to (Y, A)$, $n \in \mathbb{Z}$:*

$$\begin{array}{ccc} \Omega_n(X,A) & \xrightarrow{\partial_n} & \Omega_{n-1}(A) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle (f_{|A})_*} \\ \Omega_n(Y,B) & \xrightarrow{\partial_n} & \Omega_{n-1}(B) \end{array}$$

Proof. See proof of lemma 1.34 □

Lemma 1.53 (Homotopy invariance [Ati61, Lemma $2 \cdot 1$])**.** *Let $f, g : (X, A) \to (Y, B)$ be homotopic maps. Then $f_*, g_* : \Omega_n(X, A) \to \Omega_n(Y, B)$ are the same homomorphisms.*

Proof. [Ati61], [CF64] The proof is exactly the same as the proof of lemma 1.35; noting that the cylinder is now oriented and $\partial M \times I$ is now $\partial I \times (M \cup M^-) = M \sqcup M^-$ □

Lemma 1.54 (Long exact sequence, [CF64])**.** *The sequence*

$$\cdots \xrightarrow{\partial} \Omega_n(A) \xrightarrow{i_*} \Omega_n(X) \xrightarrow{j_*} \Omega_n(X, A) \xrightarrow{\partial} \Omega_{n-1}(A) \xrightarrow{i_*} \cdots$$

is exact.

Proof. [CF64] The proof is exactly the same as in the unoriented case, up to one minor adjustment. For exactness at $\Omega_n(X, A)$, we identify the boundaries of (M, f) and (B^-, F); B with the opposite orientation. $\square$

Remark. We used lemma 1.32 in the proof. The oriented version of this lemma also holds, by the same proof, just keep in mind that $M \times \{1\}$ and $N \times \{1\}$ have the opposite orientation.

Lemma 1.55 (Excision axiom, [CF64, p. 5.7])**.** *If $\overline{U} \subset \overset{\circ}{A}$, then $i : (X \setminus U, A \setminus U) \subset (X, A)$ induces an isomorphism of relative oriented bordism groups:*

$$i_* : \Omega_n(X \setminus U, A \setminus U) \xrightarrow{\cong} \Omega_n(X, A)$$

Proof. [CF64] Again, everything stays the same as the proof of lemma 1.39. The used lemmas 1.37 and 1.38 are left unbothered by the orientations. $\square$

Lemma 1.56 (Disjoint union axiom)**.** *Relative oriented bordism satisfies the disjoint union axiom.*

Proof. See proof of lemma 1.40 $\square$

Remark. The dimension axiom does not hold. Seeing this is harder than in the unoriented case, but for dimension 4 the signature map gives an isomorphism

$$\text{sign} : \Omega_4 \xrightarrow{\cong} \mathbb{Z}$$

This map is well-defined, because the signature is a bordism invariant in every dimension, but we will not show this here.

We conclude:

Theorem 1.57. *Relative oriented bordism is an extraordinary homology theory satisfying the disjoint union axiom.*

1.4.4 Calculations

Let us try to calculate Ω_n for low n.
For $n = 0$, we need to see if a disjoint union of oriented points is nullbordant. An orientation of such spaces is a choice of a sign for every point. We know that $(\text{pt}, +)$ is bordant to itself, by symmetry of the bordism relation, as $(\text{pt}, +)$ and $(\text{pt}, -)$ are the boundary of the interval. $(\text{pt}, +)$ and $(\text{pt}, -)$ are not bordant, again by the classification of compact 1-manifolds. Let us write $\text{pt}_{n,k}$ for the disjoint union of n positively oriented points and k negatively oriented points. Then $\text{pt}_{n,k}$ is bordant to $\text{pt}_{n-1,k-1}$, as we can take $n + k - 1$ intervals. So we can assume $n = 0$ or $k = 0$ and can write $[\text{pt}_{n-k}]$ for the equivalence class of $\text{pt}_{n,k}$. We notice that pt_n and $\text{pt}_{n'}$ are bordant if and only if $n = n'$. Thus, we get an isomorphism $\Omega_0 \cong \mathbb{Z}$ by $[\text{pt}_n] \mapsto n$
For $n = 1$, we see $\Omega_1 \cong \{0\}$ by seeing that S^1 with any orientation has D^2 with corresponding orientation as nullbordism. S^1 is the only connected closed 1-manifold.

For $n = 2$, our problem from the unoriented case has been solved, as singular manifolds are oriented now and any connected sum of $\mathbb{RP}^2$s is not orientable. So, by the argumentation of the unoriented case, we get $\Omega_2 \cong \{0\}$.
Again, from dimension 3 onwards, calculations are difficult, so we will just state the results of [MS74, §17]: We only have $\Omega_n \cong 0$ for $n \in \{1, 2, 3, 5, 6\}$ (and, of course, negative n). Else, we have:

$$\Omega_n \cong \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z} & n = 4 \\ \mathbb{F}_2 & n = 5 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 8 \\ \mathbb{F}_2 \oplus \mathbb{F}_2 & n = 9 \\ \dots \end{cases}$$

Note that we got rid of the real even-dimensional projective spaces, as they are not orientable, but we still have the complex projective spaces $\mathbb{CP}^n$ as generators in some dimensions, notably in dimension 4, where $\mathbb{CP}^2$ is a generator of Ω_4. Just like in the unoriented case, the oriented bordism groups have been completely determined.

We can define the **reduced oriented bordism homology** analogously to reduced unoriented bordism homology.

$$\tilde{\Omega}_n(X) := \ker(\Omega_n(X) \to \Omega_n)$$

By the same arguments as in the unoriented case, we get

$$\Omega_n(S^k) \cong \Omega_n \oplus \Omega_{n-k}$$

2 Cobordism

Now that we have seen that bordism is a homology theory, we can ask the question wether there is a dual to this, giving rise to a cohomology theory. The answer is yes, as we will see now.

2.1 Cobordism

Cobordism is defined in a less intuitive way than bordism, we will define them over spectra.

2.1.1 (Thom) Spectra

Definition 2.1 (Principal bundle [Die91, I.11])**.** Let G be a topological group, $r : E \times G \to E$, $r(x, g) = xg$ be a free right action on E, and $p : E \to B$ a continuous map. Then (p, r) is called a **G-principal bundle**, if

- $\forall g \in G, x \in E : p(xg) = p(x)$

- $\forall b \in B \exists$ a neighbourhood U and a G-homeomorphism $\varphi : p^{-1}(U) \to U \times G$ which is a local trivialization, i.e. $p \circ \varphi^{-1}(x, g) = x$ for all $g \in G$ and $g \mapsto \varphi^{-1}(x, g)$ is an isomorphism between G and $p^{-1}(x)$. φ is given by $((u, x), g) \mapsto (u, xg)$.

Definition 2.2 (Numerable cover)**.** Let X be a topological space, and $\{U_i\}_{i \in I}$ an open cover of X. $\{U_{i\_}i \in I\}$ is called **numerable** if it admits a partition of unity subordinate to this cover.

Remark. For X a manifold, every cover is numerable, as manifolds are paracompact.

Definition 2.3 (Numerable bundle [Die91, IX.4])**.** A vector bundle $\xi : E \to B$ is called **numerable** if there exists a numerable cover $\{U_i\}_{i \in I}$ of B such that ξ is trivial over each U_i.

Definition 2.4 (Universal bundle [Die91, IX.4])**.** A G-principal bundle $\xi : EG \to BG$ is called **universal**, if it is numerable and for every numerable G-principal bundle $\eta : E \to B$ has a unique bundle map to ξ (up to homotopy).

Remark. The universal G-bundle exists for every topological group G and is unique up to homotopy equivalence, in particular, EG and BG are unique up to homotopy equivalence.

Definition 2.5 (Classifying space [Die91, IX.4])**.** Let G be a topological group. The **classifying space** BG of G is the base space of the universal principal G-bundle.

Remark. [Tho54, p.32]

$$BO(n) = \mathbb{G}_{n,\infty} := \lim_{k \to \infty} \mathbb{G}_{n,k}$$

$$BSO(n) = \widetilde{\mathbb{G}}_{n,\infty} := \lim_{k \to \infty} \widetilde{\mathbb{G}_{n,k}}$$

Where $\mathbb{G}_{n,k} = \{n - \text{dim. linear subspaces of } \mathbb{R}^k\}$ is the Grassmannian, and $\widetilde{\mathbb{G}_{n,k}}$ the space of the oriented linear subspaces. The direct limit is taken with respect to the inclusions $\mathbb{G}_{n,k} \hookrightarrow \mathbb{G}_{n,k+1}$, resp. $\widetilde{\mathbb{G}_{n,k}} \hookrightarrow \widetilde{\mathbb{G}_{n,k+1}}$.

Definition 2.6 (Vector bundle)**.** A (real) **vector bundle** is the triple (π, E, B) such that $\pi : E \to B$ is a continuous map between topological spaces and every fibre $E_n := \pi^{-1}(b)$ carries the structure of a finite dimensional vector space. Additionally, for every point $b \in B$, there has to be an open neighbourhood U and a homeomorphism $\varphi : U \times \mathbb{R}^k \to \pi^{-1}(U)$, which is a local trivialization for some k. k is called the dimension of the vector bundle.

Definition 2.7 (Thom space [Tho54, p.29])**.** Let $\xi : E \to B$ be a real vector bundle with Riemannian metric over a manifold B. Its **disk bundle** is defined by $D(\xi) : DE \to X$, $DE = \{v \in E \mid \|v\| \leq 1\}$ and similarly, the **sphere bundle** is defined by $S(\xi) : SE \to X$, $SE = \{v \in E \mid \|v\| = 1\}$. Then the Thom space of ξ is defined as

$$M(\xi) = D(\xi) / S(\xi)$$

where the sphere bundle is collapsed to a point. We can also get the Thom space without a choice of a Riemannian metric, but I will omit this here.
For ξ the universal principal G-bundle, we can write $M(G)$ instead of $M(\xi)$.

Example 21 ($MO(n), MSO(n)$ [BD70])**.**

$$MO(n) = M(\gamma_{n,\infty})$$

with $\gamma_{n,\infty}$ being the universal real vector bundle over $\mathbb{G}_{k,\infty}$. Similarly,

$$MSO(n) = M(\tilde{\gamma}_{n,\infty})$$

is the Thom space of the universal real oriented vector bundle.

Definition 2.8 (Spectrum [BD70, Definition IV.1.1.])**.** A **spectrum** $\underline{E} = \{(E_n, \sigma_n) \mid n \in \mathbb{Z}\}$ is a sequence of pointed spaces E_n with pointed **structure maps**

$$\sigma_n : \Sigma E_n \to E_{n+1}$$

where Σ denotes the reduced suspension functor.

Example 22 (Suspension spectrum [Die08, 7.7.2])**.** Given a pointed space X, we can define its **suspension spectrum** $\Sigma^\infty X$ by $(\Sigma^\infty X)_n = \Sigma^n X$ and the structure maps as the identities for $n \geq 0$. For negative n, we take $(\Sigma^\infty X)_n = \{\mathrm{pt}\}$ and the structure maps as the identity map $\sigma_n : \Sigma(\{\mathrm{pt}\}) = \{\mathrm{pt}\} \to \{\mathrm{pt}\}$ except for $n = -1$, where we take the inclusion map to the basepoint. Note that $(\Sigma^\infty X)_0 = X$

Example 23 (Sphere spectrum [Die08, 7.7.1])**.** Taking $X = S^0$ if example 22 gives us the **sphere spectrum** $\underline{S}$. Then, $S_n = S^n$ for non-negative n.

Example 24 (Thom spectrum [BD70, Beispiel IV.1.2(b)])**.** Let γ_n be a vector bundle. The trivial map $\gamma_n \oplus \varepsilon \to \gamma_{n+1}$ induces a pointed map

$$\sigma_n : \Sigma M(\gamma_n) \cong M(\gamma_n \oplus \varepsilon) \to M(\gamma_{n+1})$$

where $\varepsilon : B \times \mathbb{R} \to B$ denotes the trivial line bundle. The homeomorphism comes from $M(\gamma \oplus \varepsilon) \cong M(\gamma) \wedge M(\varepsilon) \cong M(\gamma) \wedge S^1 \cong \Sigma M(\gamma_n)$.
The (orthogonal) **Thom spectra MO** $= (MO(n), e_n)$ and **MSO** $= (MSO(n), \widetilde{e_n})$ with the structure maps being the induced maps of the classifying maps of the universal real (oriented) vector bundles

Theorem/Definition 2.9 (Cohomology theory associated to a spectrum [Whi62])**.** *Let $\underline{E}$ be a spectrum. Then the following is an extraordinary cohomology theory.*

$$\mathfrak{H}^k(X, A; \underline{E}) := \lim_{n \to \infty} [\Sigma^{n-k}(X/A), E_n]$$

This is shown in [Whi62].

Example 25 (Stable cohomotopy groups [Die08, 7.7.1]). For the sphere spectrum $\underline{S}$, we get the **stable cohomotopy groups** via the above definition:

$$\pi_s^k(X, A) := \mathfrak{H}^k(X, A; \underline{S})$$

We have

$$\pi_x^k(X, A) = \lim_{n\to\infty} \pi^n(\Sigma^{n-k}(X/A)),$$

the colimit of the cohomotopy groups with respect to the suspension maps.

Definition 2.10 (Cobordism group [Ati61, p.201]). Let (X, Y) be a pair of spaces, then for $k \in \mathbb{Z}$, the k**-th oriented cobordism group** is

$$\Omega^k(X, Y) := \mathfrak{H}^k(X, Y; MSO) = \lim_{n\to\infty} [\Sigma^{n-k}(X/Y), MSO(n)]$$

with respect to the above map. Analogously, we define the **relative unoriented cobordism group** as

$$\mathfrak{N}^k(X, Y) := \mathfrak{H}^k(X/Y; MO) = \lim_{n\to\infty} [\Sigma^{n-k}(X/Y), MO(n)]$$

2.1.2 The Eilenberg-Steenrod Axioms

To get a the axioms for cohomology theories, intuitively, we "reverse all arrows" in the previously defined axioms for homology theories in 1.2.2.

Definition 2.11 (Cohomology theory [ES52, I.3c][Lüc05, Definition 5.2]). An **ordinary cohomology theory** $\mathcal{H}^* = (\mathcal{H}^*, \partial^*)$ with coefficients in R-modules is a contravariant functor

$$\mathcal{H}^* : \mathrm{TOP}^2 \to \mathbb{Z}\text{-graded } R\text{-modules}$$

together with a natural boundary map

$$\partial^* : \mathcal{H}^* \circ I \to \mathcal{H}^{*+1}$$

satisfying the following axioms:

- **Homotopy invariance** For homotopic maps of pairs $f, g : (X, A \to Y, B)$, we have
$$\mathcal{H}^n(f) = \mathcal{H}^n(g) : \mathcal{H}^n(Y, B) \to \mathcal{H}^n(X, A)$$

- **Long exact sequence** For a pair of spaces (X, A), the sequence
$$\dots \xrightarrow{\partial^{n-1}} \mathcal{H}^n(X, A) \xrightarrow{\mathcal{H}^n(j)} \mathcal{H}^n(X) \xrightarrow{\mathcal{H}^n(i)} \mathcal{H}^n(A) \xrightarrow{\partial^n(X,A)} \mathcal{H}^{n+1}(X, A) \to \dots$$
is exact, where $i : A \hookrightarrow X$ and $j : (X, \emptyset) \hookrightarrow (X, A)$ are the inclusions.

- **Excision** Let (X, B, A) be a triple of spaces such that $\overline{A} \subseteq \overset{\circ}{B}$. Then the inclusion map $i : (X \setminus A, B \setminus A) \hookrightarrow (X, B)$ induces an isomorphism of cohomology groups:

$$\mathcal{H}^n(i) : \mathcal{H}^n(X, B) \xrightarrow{\cong} \mathcal{H}^n(X \setminus A, B \setminus A)$$

- **Dimension axiom** For all $n \in \mathbb{Z}$, we have

$$\mathcal{H}^n(\{\mathrm{pt}\}) \cong \begin{cases} R, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

Again, one may add Milnor's disjoint union axiom [Mil62, p.337, Additivity Axiom]:

- **Disjoint union axiom** For a disjoint union of spaces $\coprod_{i \in I} X_i$ over any index set I and the inclusions $j_i : X_i \to \coprod_{i \in I} X_i$, the map

$$\prod_{i \in I} \mathcal{H}^n(j_i) : \mathcal{H}^n\left(\coprod_{i \in I X_i}\right) \xrightarrow{\cong} \prod_{i \in I} \mathcal{H}^n(X_i)$$

Again, if only the dimension axiom doesn't not hold, we have a **extraordinary cohomology theory**.

2.1.3 Bordism Cohomology

To prove theorem 2.9, we will check the axioms one by one. For this section, we will fix a spectrum $\underline{E}$, and $\mathfrak{H}^*$ will be the associated functor, as defined in theorem 2.9.

Theorem 2.12 ([Tho54, Théorème II.7])**.** *For n large enough, we have*

$$\pi_{n+k}(\Sigma MO(n)) \cong \pi_{n+k}(MO(n+1))$$

$$\pi_{n+k}(\Sigma MSO(n)) \cong \pi_{n+k}(MSO(n+1))$$

Corollary 2.13 ([Ati61, p.201])**.** *For X a pointed finite CW-complex, n large enough, we have bijections*

$$[X, \Sigma MO(n)] \to [X, MO(n+1)]$$

$$[X, \Sigma MSO(n)] \to [X, MSO(n+1)]$$

Theorem 2.14 (Freudenthal suspension theorem [Ati61, p.201])**.** *Let X, Y a pointed CW-complexes. Then for large enough n, the suspension map*

$$[\Sigma^n X, \Sigma^n Y] \to [\Sigma^{n+1} X, \Sigma^{n+1} Y]$$

is an isomorphism.

Lemma 2.15 (Functoriality). *$\mathfrak{H}^*$ defines a contravariant functor*

$$\mathfrak{H}^* : \mathrm{TOP}^2 \to \textit{graded abelian groups}$$

Remark. Every abelian group is a $\mathbb{Z}$-module.

Proof. Let $(X, A) \in \mathrm{Ob}(\mathrm{TOP}^2)$. Then $\mathfrak{H}^*(X, A) = \lim_{n\to\infty}[\Sigma^{n-k}(X/A), E_n]$ is an abelian group, as for every n, k, $[\Sigma^{n-k}(X/A), E_n]$ is an abelian group and the limit of a sequence of abelian groups is an abelian group.
Now, for a map $f \in \mathrm{Mor}(\mathrm{TOP}^2)$, $f : (X, A) \to (Y, B)$, we get a map

$$f^* := \mathfrak{H}^*(f) : \lim_{n\to\infty}[\Sigma^{n-k}(Y/B), E_n] \to \lim_{n\to\infty}[\Sigma^{n-k}(X/A, E_n)]$$

by

$$f^*[x] = [x \circ \Sigma^{n-k}(\tilde{f})]$$

for $[x] \in [\Sigma^{n-k}(Y/B), E_n]$. Here, $\tilde{f} : X/A \to Y/B$ is the induced map on the quotients. Taking the colimit, we get the desired map. Then we have that $\mathfrak{H}^*(\mathrm{id}_{(X,A)}) = \mathrm{id}_{\mathfrak{H}^*(X,A)}$, as all $\Sigma^{n-k}(\tilde{f}) = \mathrm{id}$ for every n.
For $f : (X, A) \to (Y, B)$, $g : (Y, B) \to (Z, C)$, we have, for $[x] \in [\Sigma^{n-k}(Y/B), E_n]$

$$(g \circ f)^*([x]) = [x \circ \Sigma^{n-k}(\widetilde{g \circ f})] = [x \circ \Sigma^{n-k}(\tilde{g}) \circ \Sigma^{n-k}(\tilde{f})] = f^*(g^*([x]))$$

. Since the equality holds for every n, it holds for the limit, so we have $(g \circ f)^* = f^* \circ g^*$. □

Lemma 2.16 (Naturality). *The boundary operator*

$$\partial^* : \mathfrak{H}^{*-1}(A) \to \mathfrak{H}^*(X, A)$$

is natural.

Remark. Let $[x] \in [\Sigma^{n-k+1}(A/\emptyset), E_n]$ for n large enough. This x induces a map $\Sigma x : \Sigma^{n-k+2}(A/\emptyset) \to \Sigma E_n$. We can extend this map to a map $y : \Sigma^{n-k+2}(X/\emptyset) \to \Sigma E_n$ via the inclusion $i : A/\emptyset \hookrightarrow X/\emptyset$ such that y sends the complement of $A/\emptyset$ to the basepoint of ΣE_n. Now we collapse $\Sigma^{n+k+2}A/\emptyset \subset \Sigma^{n+k+2}X/\emptyset$ and get a map $\tilde{y}$ from $\Sigma^{n+k+2}X/A$. By suspending twice afterwards, we get a map

$$\tilde{y} \circ \Sigma^2 : \Sigma^{n-k+2}X/A \to \Sigma^2 E_n$$

Thus, $[\tilde{y} \circ \Sigma^2] \in [\Sigma^{n-k+2}X/A, \Sigma^2 E_n] \cong [\Sigma^{n-k}X/A, E_n]$ by the Freudenthal theorem 2.14.
We define the boundary map as $\partial([x]) = \tilde{y}$. Since the construction works for every n large enough, the boundary map works in the colimit.

Proof. omitted. □

Lemma 2.17. *The homotopy invariance axiom is satisfied.*

Proof. This is immediate by the definition of the cohomology groups as sets of homotopy classes of maps. □

Lemma 2.18 (Long exact sequence [Ati61, Proposition 1·1])**.** *For a finite CW-complex X, and a subcomplex A, we have the long exact sequences*

$$\cdots \to \mathfrak{N}^n(X,A) \xrightarrow{i^*} \mathfrak{N}^n(X) \xrightarrow{j^*} \mathfrak{N}^n(A) \xrightarrow{\partial} \mathfrak{N}^{n+1}(X,Y) \to \cdots$$

$$\cdots \to \Omega^n(X,A) \xrightarrow{i^*} \Omega^n(X) \xrightarrow{j^*} \Omega^n(A) \xrightarrow{\partial} \Omega^{n+1}(X,Y) \to \cdots$$

where i^* and j^* are the induced maps on cohomology groups by the inclusions $i : A \hookrightarrow X$ and $j : (X, \emptyset) \hookrightarrow (X, A)$.

Proof. This is a consequence of the exact sequence given in [Spa58, p.261]. □

Lemma 2.19 (Excision)**.** *Let (X, A) be a pair and $U \subset X$ open such that $\overline{U} \subset \mathring{A}$. Then*

$$\iota : X \setminus U, A \setminus U \hookrightarrow (X, A)$$

induces an isomorphisms

$$\mathfrak{N}^*(X,A) \xrightarrow{\cong} \mathfrak{N}^*(X \setminus U, A \setminus U)$$

$$\Omega^*(X,A) \xrightarrow{\cong} \Omega^*(X \setminus U, A \setminus U)$$

Proof Sketch for CW-complexes. [Ada95, Chapter 6, p.250] Let (X, A) be a CW-pair. By cellular approximation, we can assume that $\overline{U}$ lies in a subcomplex of A. In particular, X is a union of two subcomplexes $X = A \cup (X \setminus U)$ with intersection $A \cap (X \setminus U) = A \setminus U$. Then

$$A/(A \setminus U) \to X/(X \setminus U)$$

is a homeomorphism. Because of the $\overline{U} \subset \mathring{A}$ assumption, we also have $A/(A \setminus U) \cong (X \setminus U)/(A \setminus U)$ and $X/(X \setminus U) \cong X/A$. So we have

$$(X \setminus U)/(A \setminus U) \cong X/A$$

Applying the the functor $\mathfrak{N}^*$ resp. Ω^* concludes the proof. □

Remark. This argument also works for any other spectrum-cohomology theory.

Lemma 2.20 (Disjoint union axiom)**.** *The disjoint union axiom holds for $\mathfrak{H}^*$*

Remark/ Proof Sketch. [Ada95, Chapter 6] For this (for CW-complexes), one would show the Milnor-Brown wedge axiom for the associated reduced cohomology, i.e.

$$\tilde{\mathfrak{H}}^n\left(\left(\bigvee_i X_i\right); \underline{E}\right) \to \prod_i \tilde{\mathfrak{H}}^n(X_i; \underline{E})$$

Then, we can conclude that the disjoint union axiom holds for $\mathfrak{H}^*$ by the following argument:

$$\mathfrak{H}^n(X \sqcup Y; \underline{E}) = \tilde{\mathfrak{H}}^n((X \sqcup Y)_+; \underline{E}) \cong \tilde{\mathfrak{H}}^n(X_+ \vee Y_+; \underline{E})$$
$$\cong \tilde{\mathfrak{H}}^n(X_+; \underline{E}) \times \tilde{\mathfrak{H}}^n(Y_+, \underline{E}) = \mathfrak{H}^n(X; \underline{E}) \times \mathfrak{H}^n(Y; \underline{E}).$$

□

Remark (Dimension axiom). [Tho54] The dimension axiom does not hold, in particular, we have $\mathfrak{N}^{-2} \cong \mathbb{F}_2$, generated by $\mathbb{RP}^2$, and $\Omega^{-4} \cong \mathbb{Z}$, generated by $\mathbb{CP}^2$.

We finally conclude:

Theorem 2.21 (Bordism cohomology theory)**.** *$\mathfrak{N}^*$ and Ω^* are extraordinary cohomology theories satisfying the disjoint union axiom.*

Proof. This can be seen as a corollary to theorem 2.9, or as a consequence of the above lemmas 2.15, 2.16, 2.17, 2.18, 2.19, 2.20. □

2.2 Duality to Bordism Homology

We still need to satisfy the name "cobordism" and "bordism cohomology". It is not at all clear that the cohomology theory we defined above is the dual to the bordism homology theory.

Theorem 2.22 (Brown's representation theorem [Swi02, Theorem 9.27])**.** *Let $\mathcal{H}^*$ be a extraordinary cohomology theory with values in $\mathbb{Z}$-modules satisfying the disjoint union axiom defined on CW-pairs. Then there exists a spectrum $\underline{E}$ and a natural equivalence of cohomology theories*

$$\mathfrak{H}(--, \underline{E}) \to \mathcal{H}^*$$

Remark. There is also a version of this theorem for homology theories, at least over finite CW-complexes.

But the theorem we really need is dual to theorem 2.9:

Theorem/Definition 2.23 (Homology theory assigned to a spectrum [BD70, IV, Definition 1.5])**.** *Let $\underline{E}$ be a spectrum. Then the following is an extraordinary homology theory:*

$$\mathfrak{H}_*(X, A; \underline{E}) := \mathfrak{H}^{-*}(S^0; (X/A) \wedge \underline{E})$$

Remark. This is well-defined, as $X \wedge \underline{E}$ again defines a spectrum for any pointed space X.

Example 26 (Stable homotopy groups [Die08, 20.9])**.** Taking the sphere spectrum $\underline{S}$, we get the **stable homotopy groups**:

$$\pi_k^s(X) := \mathfrak{H}_k(X; \underline{S}).$$

Then

$$\pi_k^s(X) = \lim_{n\to\infty} \pi_n(\Sigma^{n-k}X),$$

is the colimit of the homotopy groups with respect to the suspension maps. This is dual to the stable cohomotopy groups defined in example 25.

Now, we get the duality between bordism and cobordism by the fact that they are both defined by the same spectra, namely **MO** (unoriented) resp. **MSO** (oriented).

Theorem 2.24 ([BD70, III])**.** *There are natural equivalences of homology theories*

$$\mathfrak{N}_*(--) \xrightarrow{\cong} \mathfrak{H}_*(--;\mathbf{MO})$$

$$\Omega_*(--) \xrightarrow{\cong} \mathfrak{H}_*(--;\mathbf{MSO})$$

Proof. omitted. □

Outlook. There also exists a stronger connection between the bordism and stable homotopy groups.

Theorem 2.25 ([Tho54, Théorème IV.8])**.** *The absolute bordism groups $\mathfrak{N}_k$ and Ω_k are isomorphic to the stable homotopy groups of the respective Thom spectra:*

$$\mathfrak{N}_k \cong \lim_{n\to\infty} \pi_{n+k}(MO(k))$$

$$\Omega_k \cong \lim_{n\to\infty} \pi_{n+k}(MSO(k))$$

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