

Cross Product for Euclidean and Minkowskian spaces

Geometric Algebra

Geometric product: $ab = a \cdot b + a \wedge b$ $a \cdot b$: scalar part, $a \wedge b$: Bivector

Orthonormal base: $\{e_i\}$ Products: $e_{ii} = \pm 1$ (signature of e_i)

$$j \neq k \rightarrow e_j e_k = -e_k e_j \rightarrow e_{jk} = -e_{kj}$$

Euclidean space (0,3): $e_{ii} = 1$

$\{e_i\} = \{e_1, e_2, e_3\}$ Trivector (pseudoscalar) $i_{(3)} = e_{123} = e_1 e_2 e_3$

$a = a_1 e_1 + a_2 e_2 + a_3 e_3$, $b = b_1 e_1 + b_2 e_2 + b_3 e_3$

Euclidean Cross Product : $a \times b = -i_{(3)} (a \wedge b)$

$$\begin{aligned} a \wedge b &= a_1 b_2 e_1 e_2 + a_1 b_3 e_1 e_3 + a_2 b_1 e_2 e_1 + a_2 b_3 e_2 e_3 + a_3 b_1 e_3 e_1 + a_3 b_2 e_3 e_2 = \\ &= a_1 b_2 e_{12} - a_1 b_3 e_{31} - a_2 b_1 e_{12} + a_2 b_3 e_{23} + a_3 b_1 e_{31} - a_3 b_2 e_{23} = \\ &= (a_1 b_2 - a_2 b_1) e_{12} + (a_2 b_3 - a_3 b_2) e_{23} + (a_3 b_1 - a_1 b_3) e_{31} \\ a \times b &= -e_{123} [(a_1 b_2 - a_2 b_1) e_{12} + (a_2 b_3 - a_3 b_2) e_{23} + (a_3 b_1 - a_1 b_3) e_{31}] = \\ &= -[(a_1 b_2 - a_2 b_1) e_{123} e_{12} + (a_2 b_3 - a_3 b_2) e_{123} e_{23} + (a_3 b_1 - a_1 b_3) e_{123} e_{31}] = \\ &= -[(a_1 b_2 - a_2 b_1) e_{12312} + (a_2 b_3 - a_3 b_2) e_{12323} + (a_3 b_1 - a_1 b_3) e_{12331}] = \\ &= -[(a_1 b_2 - a_2 b_1) e_{11232} + (a_2 b_3 - a_3 b_2) e_{12323} + (a_3 b_1 - a_1 b_3) e_{12133}] = \\ &= -[(a_1 b_2 - a_2 b_1)(-e_{11223}) + (a_2 b_3 - a_3 b_2)(-e_{12233}) + (a_3 b_1 - a_1 b_3)(-e_{11233})] = \\ &= -[(a_1 b_2 - a_2 b_1)(-e_3) + (a_2 b_3 - a_3 b_2)(-e_1) + (a_3 b_1 - a_1 b_3)(-e_2)] = \\ &= -[(a_1 b_2 - a_2 b_1)(-e_3) + (a_3 b_1 - a_1 b_3)(-e_2) + (a_2 b_3 - a_3 b_2)(-e_1)] = \\ &= (a_1 b_2 - a_2 b_1) e_3 + (a_3 b_1 - a_1 b_3) e_2 + (a_2 b_3 - a_3 b_2) e_1 \end{aligned}$$

$$\vec{a} \times \vec{b} = \begin{bmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

Minkowskian (1,2) space: $e_{00} = -1$ (timelike direction)

$\{e_i\} = \{e_1, e_2, e_0\}$ Trivector (pseudoscalar) $i_{(2,1)} = e_{120} = e_1 e_2 e_0$

$a = a_1 e_1 + a_2 e_2 + a_0 e_0$, $b = b_1 e_1 + b_2 e_2 + b_0 e_0$

Minkowskian Cross Product: $a \times b = -i_{(2,1)} (a \wedge b)$

$$\begin{aligned} a \wedge b &= a_1 b_2 e_1 e_2 + a_1 b_0 e_1 e_0 + a_2 b_1 e_2 e_1 + a_2 b_0 e_2 e_0 + a_0 b_1 e_0 e_1 + a_0 b_2 e_0 e_2 = \\ &= a_1 b_2 e_{12} - a_1 b_0 e_{01} - a_2 b_1 e_{12} + a_2 b_0 e_{20} + a_0 b_1 e_{01} - a_0 b_2 e_{20} = \\ &= (a_1 b_2 - a_2 b_1) e_{12} + (a_0 b_1 - a_1 b_0) e_{01} + (a_2 b_0 - a_0 b_2) e_{20} \\ a \times b &= -e_{120} [(a_1 b_2 - a_2 b_1) e_{12} + (a_0 b_1 - a_1 b_0) e_{01} + (a_2 b_0 - a_0 b_2) e_{20}] = \\ &= -[(a_1 b_2 - a_2 b_1) e_{12012} + (a_0 b_1 - a_1 b_0) e_{12001} + (a_2 b_0 - a_0 b_2) e_{12020}] = \\ &= -[(a_1 b_2 - a_2 b_1)(-e_{11220}) + (a_0 b_1 - a_1 b_0)(-e_{11200}) + (a_2 b_0 - a_0 b_2)(-e_{12200})] = \\ &= -[(a_1 b_2 - a_2 b_1)(-e_0) + (a_0 b_1 - a_1 b_0)(-e_{200}) + (a_2 b_0 - a_0 b_2)(-e_{100})] = \\ &= -[(a_1 b_2 - a_2 b_1)(-e_0) + (a_0 b_1 - a_1 b_0) e_2 + (a_2 b_0 - a_0 b_2) e_1] = \\ &= (a_1 b_2 - a_2 b_1) e_0 + (a_0 b_1 - a_1 b_0)(-e_2) + (a_2 b_0 - a_0 b_2)(-e_1) \end{aligned}$$

$$\vec{a} \times \vec{b} = \begin{bmatrix} -e_1 & -e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$