Complex stAngles

We can define a more general (complex) measure for a stAngle
using [Euler´s Formula] $e^{ix} = \cos(x) + i\cdot\sin(x)$
and extending the definitions $\cosh(x) = ½(e^{x} + e^{-x})$, $\sinh(x) = ½(e^{x} - e^{-x})$
to the complex numbers in order to get $\cosh(ix) = \cos(x)$, $\sinh(ix) = i\cdot\sin(x)$

We can now recall the cosine and sine formulas for the angle between two vectors

$$cos(\alpha) = \frac{a \bullet b}{|a|\cdot|b|} \quad , \quad sin(\alpha) = \frac{|a \times b|}{|a|\cdot|b|}$$

where the dot (a·b) and cross (axb) products in two (spatial) dimensions

$$a \bullet b = a_1 b_1 + a_1 b_1 \qquad a \times b = a_1 b_2 - a_2 b_1$$

and defining $|a| = \sqrt{a \bullet a}$

and extend them to the hyperbolic functions for the stAngle (or hyperbolic angle) between two spacetime vectors in the following way:

$$cosh(\alpha) = \frac{a \bullet b}{|a|\cdot|b|} \quad , \quad sinh(\alpha) = \frac{|a \times b|}{|a|\cdot|b|}$$

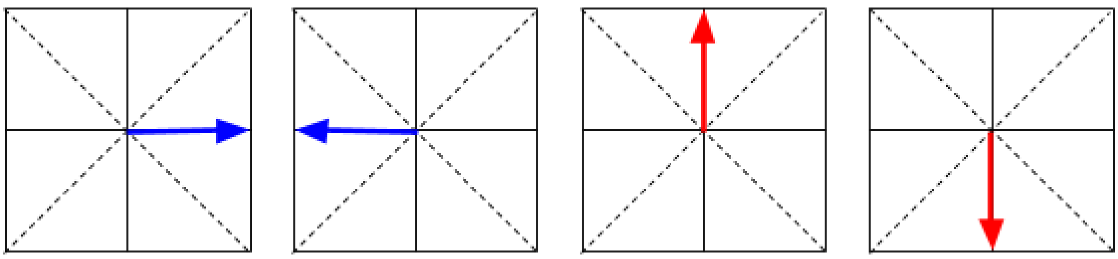
The formulas are apparently the same, but the operations they imply are not, because we now use **stDot** and **stCross** products instead of their euclidean counterparts:

$$a \bullet b = a_1 b_1 - a_0 b_0 \qquad a \times b = a_1 b_0 - a_0 b_1$$

The product of two timelike vectors is a negative number ($a_1 b_1 < a_0 b_0$), for this reason we must introduce the imaginary unit $i = \sqrt{-1}$ at the expression for the modulus of timelike vectors:

$$|a| = \sqrt{a \bullet a} = \sqrt{a_1^2 - a_0^2} = \sqrt{-(a_1^2 - a_0^2)} = i\sqrt{a_0^2 - a_1^2}$$

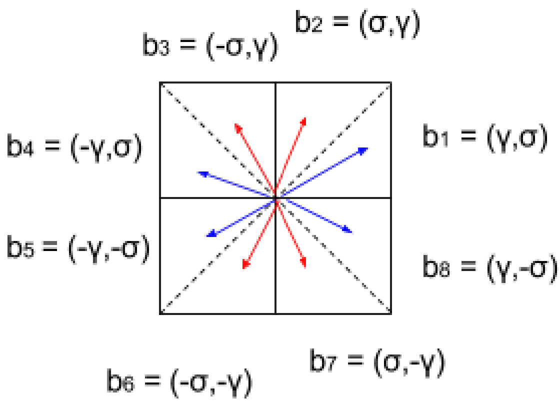
To classify the different stAngles between two spacetime Vectors, we first make a reduction to the stAngle between their stUnit vectors (which is the same), acknowledging further that the stAngle does not change under a stRotation (boost). Any (non-Lightlike) two-dimensional spacetime vector can be transformed in this way to one of those four stUnit vectors: $e_1 = (1,0)$, $-e_1 = (-1,0)$, $e_0 = (0,1)$, $-e_0 = (0,-1)$
These four stUnit vectors are represented in the following figure, where we use blue color to identify the spacelike vectors (whose modulus is equal to 1) and red color for the timelike vectors, whose modulus, as we have seen, is equal to the imaginary unit i.



$a = e_1 = (1,0)$ $\|a\| = 1$	$a = -e_1 = (-1,0)$ $\|a\| = 1$	$a = e_0 = (0,1)$ $\|a\| = i$	$a = -e_0 = (0,-1)$ $\|a\| = i$

We can reduce every pair of (not-Lightlike) vectors (or lines) first to their stUnitvectors and then making a boost in order to stRotate the first of them (a) into one of those four possibilities.

The other stUnit vector (b), then, will be in one of those eight possible orientations:



To express the coordinates of these stUnit vectors, we use the symbols γ, σ to represent the hyperbolic sine and cosine functions of a general (positive) stAngle α:

$\gamma = \cosh(\alpha)$, $\sigma = \sinh(\alpha)$

with the following properties:

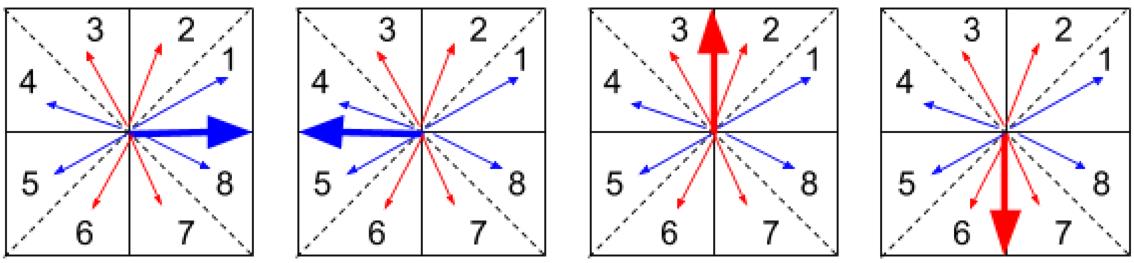
$\gamma^2 = 1 + \sigma^2$ and $\gamma > \sigma$

It is straightforward to write

$$|b_1| = |b_4| = |b_5| = |b_8| = \sqrt{\gamma^2 - \sigma^2} = 1$$

and $|b_2| = |b_3| = |b_6| = |b_7| = \sqrt{\sigma^2 - \gamma^2} = \sqrt{-1} = i$

Combining vectors a and b, we can get their stDot products $a \bullet b = a_1 b_1 - a_0 b_0$:



$a{\cdot}b_1 = \gamma$	$a{\cdot}b_1 = -\gamma$	$a{\cdot}b_1 = -\sigma$	$a{\cdot}b_1 = \sigma$
$a{\cdot}b_2 = \sigma$	$a{\cdot}b_2 = -\sigma$	$a{\cdot}b_2 = -\gamma$	$a{\cdot}b_2 = \gamma$
$a{\cdot}b_3 = -\sigma$	$a{\cdot}b_3 = \sigma$	$a{\cdot}b_3 = -\gamma$	$a{\cdot}b_3 = \gamma$
$a{\cdot}b_4 = -\gamma$	$a{\cdot}b_4 = \gamma$	$a{\cdot}b_4 = -\sigma$	$a{\cdot}b_4 = \sigma$
$a{\cdot}b_5 = -\gamma$	$a{\cdot}b_5 = \gamma$	$a{\cdot}b_5 = \sigma$	$a{\cdot}b_5 = -\sigma$
$a{\cdot}b_6 = -\sigma$	$a{\cdot}b_6 = \sigma$	$a{\cdot}b_6 = \gamma$	$a{\cdot}b_6 = -\gamma$
$a{\cdot}b_7 = \sigma$	$a{\cdot}b_7 = -\sigma$	$a{\cdot}b_7 = \gamma$	$a{\cdot}b_7 = -\gamma$
$a{\cdot}b_8 = \gamma$	$a{\cdot}b_8 = -\gamma$	$a{\cdot}b_8 = \sigma$	$a{\cdot}b_8 = -\sigma$

Applying $cosh(\alpha) = \frac{a \bullet b}{|a| \cdot |b|}$, and using

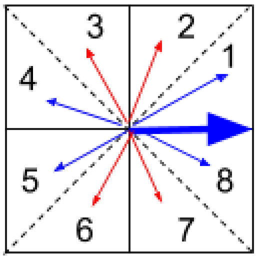
$1/|b_1| = 1/|b_4| = 1/|b_5| = 1/|b_8| = 1$

$1/|b_2| = 1/|b_3| = 1/|b_6| = 1/|b_7| = -i$

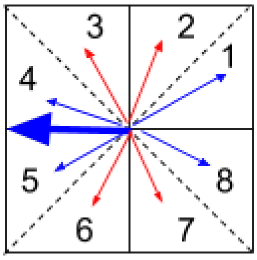
we get

$\lvert a\rvert=1$	$\lvert a\rvert=1$	$\lvert a\rvert=i$	$\lvert a\rvert=i$
$cosh(a,b_1)= \gamma$ $cosh(a,b_2)= -i\sigma$ $cosh(a,b_3)= i\sigma$ $cosh(a,b_4)= -\gamma$ $cosh(a,b_5)= -\gamma$ $cosh(a,b_6)= i\sigma$ $cosh(a,b_7)= -i\sigma$ $cosh(a,b_8)= \gamma$	$cosh(a,b_1)= -\gamma$ $cosh(a,b_2)= i\sigma$ $cosh(a,b_3)= -i\sigma$ $cosh(a,b_4)= \gamma$ $cosh(a,b_5)= \gamma$ $cosh(a,b_6)= -i\sigma$ $cosh(a,b_7)= i\sigma$ $cosh(a,b_8)= -\gamma$	$cosh(a,b_1)= -i\sigma$ $cosh(a,b_2)= -\gamma$ $cosh(a,b_3)= -\gamma$ $cosh(a,b_4)= -i\sigma$ $cosh(a,b_5)= i\sigma$ $cosh(a,b_6)= \gamma$ $cosh(a,b_7)= \gamma$ $cosh(a,b_8)= i\sigma$	$cosh(a,b_1)= i\sigma$ $cosh(a,b_2)= \gamma$ $cosh(a,b_3)= \gamma$ $cosh(a,b_4)= i\sigma$ $cosh(a,b_5)= -i\sigma$ $cosh(a,b_6)= -\gamma$ $cosh(a,b_7)= -\gamma$ $cosh(a,b_8)= -i\sigma$

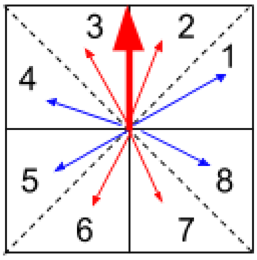
In the same way, we get their stCross products $a \times b = a_1 b_0 - a_0 b_1$:



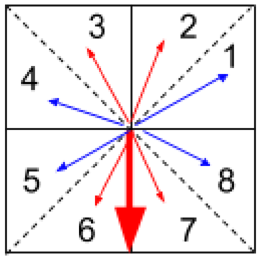
$axb_1 = \sigma$
$axb_2 = \gamma$
$axb_3 = \gamma$
$axb_4 = \sigma$
$axb_5 = -\sigma$
$axb_6 = -\gamma$
$axb_7 = -\gamma$
$axb_8 = -\sigma$



$axb_1 = -\sigma$
$axb_2 = -\gamma$
$axb_3 = -\gamma$
$axb_4 = -\sigma$
$axb_5 = \sigma$
$axb_6 = \gamma$
$axb_7 = \gamma$
$axb_8 = \sigma$



$axb_1 = -\gamma$
$axb_2 = -\sigma$
$axb_3 = \sigma$
$axb_4 = \gamma$
$axb_5 = \gamma$
$axb_6 = \sigma$
$axb_7 = -\sigma$
$axb_8 = \gamma$



$axb_1 = \gamma$
$axb_2 = \sigma$
$axb_3 = -\sigma$
$axb_4 = -\gamma$
$axb_5 = -\gamma$
$axb_6 = -\sigma$
$axb_7 = \sigma$
$axb_8 = \gamma$

Applying $sinh(\alpha) = \frac{|a \times b|}{|a| \cdot |b|}$, and using

$1/|b_1| = 1/|b_4| = 1/|b_5| = 1/|b_8| = 1$

$1/|b_2| = 1/|b_3| = 1/|b_6| = 1/|b_7| = - i$

we get

$\|a\|=1$	$\|a\|=1$	$\|a\|=i$	$\|a\|=i$
$sinh(a,b_1)= \sigma$ $sinh(a,b_2)= -i\gamma$ $sinh(a,b_3)= -i\gamma$ $sinh(a,b_4)= \sigma$ $sinh(a,b_5)= -\sigma$ $sinh(a,b_6)= i\gamma$ $sinh(a,b_7)= i\gamma$ $sinh(a,b_8)= -\sigma$	$sinh(a,b_1)= -\sigma$ $sinh(a,b_2)= i\gamma$ $sinh(a,b_3)= i\gamma$ $sinh(a,b_4)= -\sigma$ $sinh(a,b_5)= \sigma$ $sinh(a,b_6)= -i\gamma$ $sinh(a,b_7)= -i\gamma$ $sinh(a,b_8)= \sigma$	$sinh(a,b_1)= -i\gamma$ $sinh(a,b_2)= -\sigma$ $sinh(a,b_3)= \sigma$ $sinh(a,b_4)= i\gamma$ $sinh(a,b_5)= i\gamma$ $sinh(a,b_6)= \sigma$ $sinh(a,b_7)= -\sigma$ $sinh(a,b_8)= -i\gamma$	$sinh(a,b_1)= i\gamma$ $sinh(a,b_2)= \sigma$ $sinh(a,b_3)= -\sigma$ $sinh(a,b_4)= -i\gamma$ $sinh(a,b_5)= -i\gamma$ $sinh(a,b_6)= -\sigma$ $sinh(a,b_7)= \sigma$ $sinh(a,b_8)= i\gamma$

Combining these results into the general formula

$e^x = exp(x) = cosh(x) + sinh(x)$,

we get

$exp(a,b_1)= \gamma+\sigma$ $exp(a,b_2)= -i\sigma-i\gamma$ $exp(a,b_3)= i\sigma-i\gamma$ $exp(a,b_4)= -\gamma+\sigma$ $exp(a,b_5)= -\gamma-\sigma$ $exp(a,b_6)= i\sigma+i\gamma$ $exp(a,b_7)= -i\sigma+i\gamma$ $exp(a,b_8)= \gamma-\sigma$	$exp(a,b_1)= -\gamma-\sigma$ $exp(a,b_2)= i\sigma+i\gamma$ $exp(a,b_3)= -i\sigma+i\gamma$ $exp(a,b_4)= \gamma-\sigma$ $exp(a,b_5)= \gamma+\sigma$ $exp(a,b_6)= -i\sigma-i\gamma$ $exp(a,b_7)= i\sigma-i\gamma$ $exp(a,b_8)= -\gamma+\sigma$	$exp(a,b_1)= -i\sigma-i\gamma$ $exp(a,b_2)= -\gamma-\sigma$ $exp(a,b_3)= -\gamma+\sigma$ $exp(a,b_4)= -i\sigma+i\gamma$ $exp(a,b_5)= i\sigma+i\gamma$ $exp(a,b_6)= \gamma+\sigma$ $exp(a,b_7)= \gamma-\sigma$ $exp(a,b_8)= i\sigma-i\gamma$	$exp(a,b_1)= i\sigma+i\gamma$ $exp(a,b_2)= \gamma+\sigma$ $exp(a,b_3)= \gamma-\sigma$ $exp(a,b_4)= i\sigma-i\gamma$ $exp(a,b_5)= -i\sigma-i\gamma$ $exp(a,b_6)= -\gamma-\sigma$ $exp(a,b_7)= -\gamma+\sigma$ $exp(a,b_8)= -i\sigma+i\gamma$

Recalling that $\gamma = cosh(\alpha)$, $\sigma = sinh(\alpha)$

we can write $\gamma+\sigma = exp(\alpha)$ $\gamma-\sigma = exp(-\alpha)$, $-\gamma-\sigma = - exp(\alpha)$ $-\gamma+\sigma = - exp(-\alpha)$

and $i\gamma+i\sigma = iexp(\alpha)$ $i\gamma-i\sigma = iexp(-\alpha)$, $-i\gamma-i\sigma = -i\, exp(\alpha)$ $-i\gamma+i\sigma = -i\, exp(-\alpha)$

to get

$exp(a,b_1)= exp(\alpha)$ $exp(a,b_2)= -iexp(\alpha)$ $exp(a,b_3)= -iexp(-\alpha)$ $exp(a,b_4)= -exp(-\alpha)$ $exp(a,b_5)= -exp(\alpha)$ $exp(a,b_6)= iexp(\alpha)$ $exp(a,b_7)= -iexp(-\alpha)$ $exp(a,b_8)= exp(-\alpha)$	$exp(a,b_1)= -exp(\alpha)$ $exp(a,b_2)= iexp(\alpha)$ $exp(a,b_3)= -iexp(-\alpha)$ $exp(a,b_4)= exp(-\alpha)$ $exp(a,b_5)= exp(\alpha)$ $exp(a,b_6)= -iexp(\alpha)$ $exp(a,b_7)= -iexp(\alpha)$ $exp(a,b_8)= -exp(-\alpha)$	$exp(a,b_1)= -iexp(\alpha)$ $exp(a,b_2)= -exp(\alpha)$ $exp(a,b_3)= -exp(-\alpha)$ $exp(a,b_4)= -iexp(-\alpha)$ $exp(a,b_5)= iexp(\alpha)$ $exp(a,b_6)= exp(\alpha)$ $exp(a,b_7)= exp(-\alpha)$ $exp(a,b_8)= -iexp(\alpha)$	$exp(a,b_1)= iexp(\alpha)$ $exp(a,b_2)= exp(\alpha)$ $exp(a,b_3)= exp(-\alpha)$ $exp(a,b_4)= -iexp(\alpha)$ $exp(a,b_5)= -iexp(\alpha)$ $exp(a,b_6)= -exp(\alpha)$ $exp(a,b_7)= -exp(-\alpha)$ $exp(a,b_8)= -iexp(-\alpha)$

Applying logarithms, and taking into account that

$i\exp(\alpha) = \exp(\alpha+i\pi/2)$, $-i\exp(\alpha) = \exp(\alpha-i\pi/2)$,

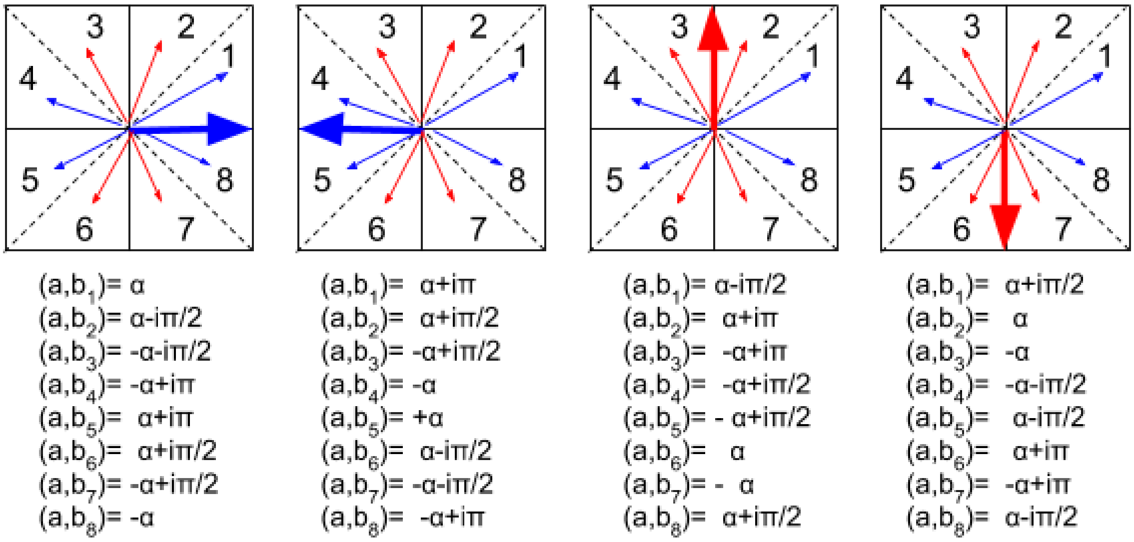
$i\exp(-\alpha) = \exp(-\alpha+i\pi/2)$, $-i\exp(-\alpha) = \exp(-\alpha-i\pi/2)$,

$-\exp(\alpha) = \exp(\alpha+i\pi)$, $-\exp(-\alpha) = \exp(-\alpha+i\pi)$

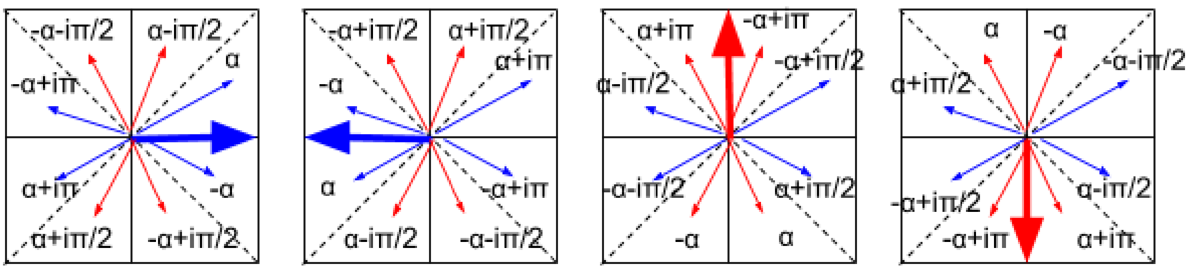
we get

$(a,b_1)= \alpha$	$(a,b_1)= \alpha+i\pi$	$(a,b_1)= \alpha -i\pi/2$	$(a,b_1)= \alpha+i\pi/2$
$(a,b_2)= \alpha-i\pi/2$	$(a,b_2)= \alpha+i\pi/2$	$(a,b_2)= \alpha+i\pi$	$(a,b_2)= \alpha$
$(a,b_3)= -\alpha-i\pi/2$	$(a,b_3)= -\alpha+i\pi/2$	$(a,b_3)= -\alpha+i\pi$	$(a,b_3)= -\alpha$
$(a,b_4)= -\alpha+i\pi$	$(a,b_4)= -\alpha$	$(a,b_4)= \alpha+i\pi/2$	$(a,b_4)= -\alpha-i\pi/2$
$(a,b_5)= \alpha+i\pi$	$(a,b_5)= \alpha$	$(a,b_5)= -\alpha+i\pi/2$	$(a,b_5)= \alpha-i\pi/2$
$(a,b_6)= \alpha+i\pi/2$	$(a,b_6)= \alpha-i\pi/2$	$(a,b_6)= \alpha$	$(a,b_6)= \alpha+i\pi$
$(a,b_7)= -\alpha+i\pi/2$	$(a,b_7)= -\alpha-i\pi/2$	$(a,b_7)= -\alpha$	$(a,b_7)= -\alpha+i\pi$
$(a,b_8)= -\alpha$	$(a,b_8)= -\alpha+i\pi$	$(a,b_8)= \alpha+i\pi/2$	$(a,b_8)= \alpha-i\pi/2$

Putting these results on the figure:

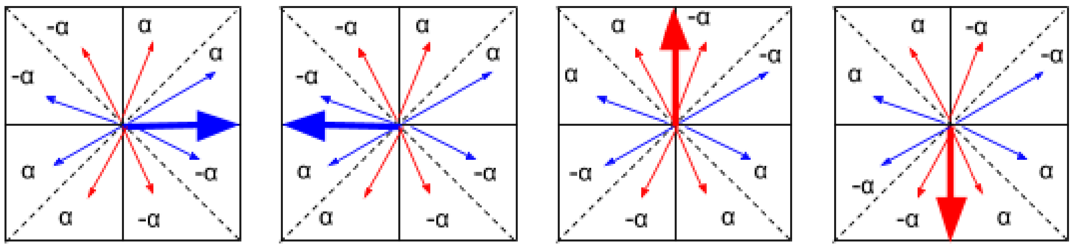


The resulting complex stAngles can be seen in the next figure:



Separating real and imaginary parts

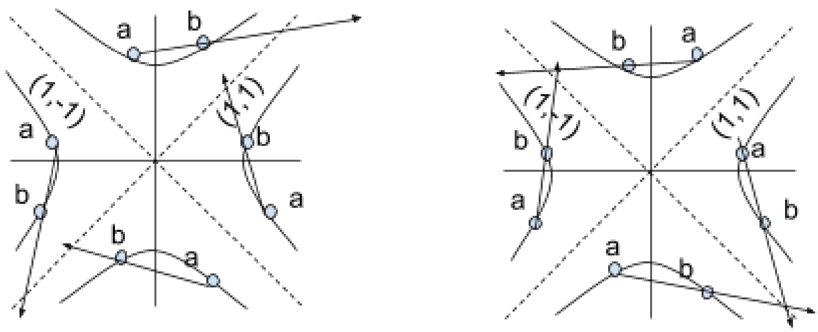
Real Parts:



The real part of any of those stAngles is a measure of the (stOriented and double) area of the stUnit stCircular sector between the direction and the stAxis with the same stType and direction.

We consider that the stOrientation of a vector b with respect to another vector a is POSITIVE when the ray that goes from a to b intersects the (1,1) diagonal line and NEGATIVE when the intersection is with the (1,-1) diagonal.

stOrientation of **b** with respect to **a**

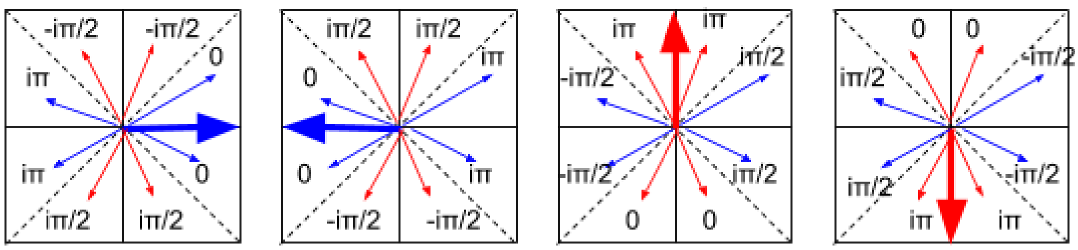


Positive stOrientation

Negative stOrientation

In the general case of any two stVectors, their stAngle is a measure of the (stOriented and double) area of the stUnit stCircular sector between one of the directions (b) and the direction from the tetrad {a, -a, stPerp(a), -stPerp(a)} which has the same stType and direction than b.

Imaginary parts:



The imaginary part of a complex stAngle shows the relationship between the stTypes and orientations of both vectors, as shown in the following table:

a / b	+s	-s	+t	-t
+s	0	iπ	-iπ/2	iπ/2
-s	iπ	0	iπ/2	-iπ/2
+t	iπ/2	-iπ/2	iπ	0
-t	-iπ/2	iπ/2	0	iπ

dividing sinh(φ)/cosh(φ)=tanh(φ) we get the hyperbolic tangents:

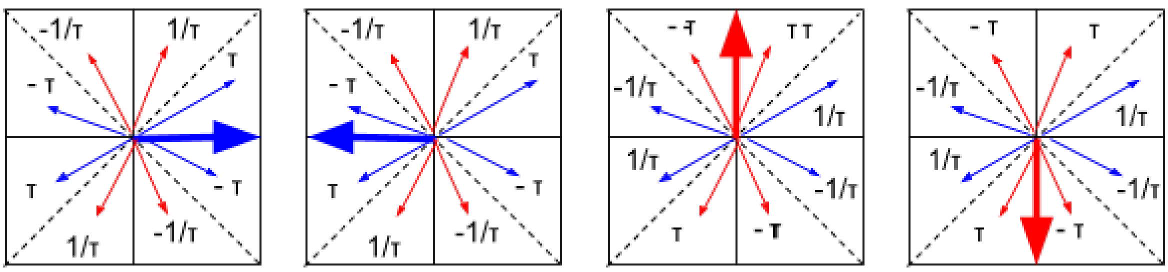
$\tanh(a,b_1)= \sigma/\gamma$ $\tanh(a,b_2)= \gamma/\sigma$ $\tanh(a,b_3)= -\gamma/\sigma$ $\tanh(a,b_4)= -\sigma/\gamma$ $\tanh(a,b_5)= \sigma/\gamma$ $\tanh(a,b_6)= \gamma/\sigma$ $\tanh(a,b_7)= -\gamma/\sigma$ $\tanh(a,b_8)= -\sigma/\gamma$	$\tanh(a,b_1)= \sigma/\gamma$ $\tanh(a,b_2)= \gamma/\sigma$ $\tanh(a,b_3)= -\gamma/\sigma$ $\tanh(a,b_4)= -\sigma/\gamma$ $\tanh(a,b_5)= \sigma/\gamma$ $\tanh(a,b_6)= \gamma/\sigma$ $\tanh(a,b_7)= -\gamma/\sigma$ $\tanh(a,b_8)= -\sigma/\gamma$	$\tanh(a,b_1)= \gamma/\sigma$ $\tanh(a,b_2)= \sigma/\gamma$ $\tanh(a,b_3)= -\sigma/\gamma$ $\tanh(a,b_4)= -\gamma/\sigma$ $\tanh(a,b_5)= \gamma/\sigma$ $\tanh(a,b_6)= \sigma/\gamma$ $\tanh(a,b_7)= -\sigma/\gamma$ $\tanh(a,b_8)= -\gamma/\sigma$	$\tanh(a,b_1)= \gamma/\sigma$ $\tanh(a,b_2)= \sigma/\gamma$ $\tanh(a,b_3)= -\sigma/\gamma$ $\tanh(a,b_4)= -\gamma/\sigma$ $\tanh(a,b_5)= \gamma/\sigma$ $\tanh(a,b_6)= \sigma/\gamma$ $\tanh(a,b_7)= -\sigma/\gamma$ $\tanh(a,b_8)= -\gamma/\sigma$

We can observe that all hyperbolic tangents of stAngles are real-valued.

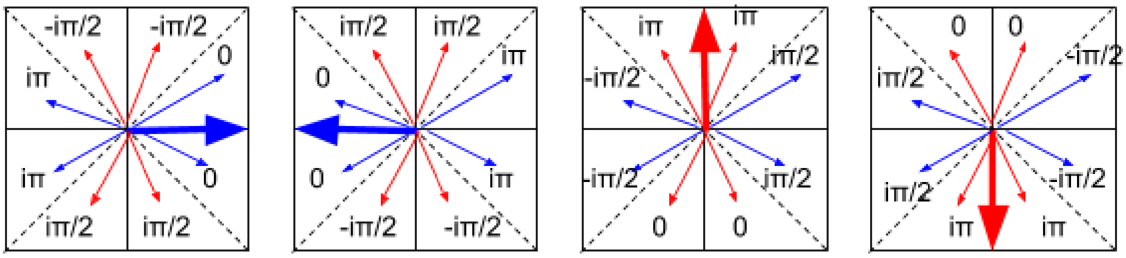
If we write $\sigma/\gamma = \tau$, we have

$\tanh(a,b_1)= \tau$ $\tanh(a,b_2)= \tau^{-1}$ $\tanh(a,b_3)= -\tau^{-1}$ $\tanh(a,b_4)= -\tau$ $\tanh(a,b_5)= \tau$ $\tanh(a,b_6)= \tau^{-1}$ $\tanh(a,b_7)= -\tau^{-1}$ $\tanh(a,b_8)= -\tau$	$\tanh(a,b_1)= \tau$ $\tanh(a,b_2)= \tau^{-1}$ $\tanh(a,b_3)= -\tau^{-1}$ $\tanh(a,b_4)= -\tau$ $\tanh(a,b_5)= \tau$ $\tanh(a,b_6)= \tau^{-1}$ $\tanh(a,b_7)= -\tau^{-1}$ $\tanh(a,b_8)= -\tau$	$\tanh(a,b_1)= \tau^{-1}$ $\tanh(a,b_2)= \tau$ $\tanh(a,b_3)= -\tau$ $\tanh(a,b_4)= -\tau^{-1}$ $\tanh(a,b_5)= \tau^{-1}$ $\tanh(a,b_6)= \tau$ $\tanh(a,b_7)= -\tau$ $\tanh(a,b_8)= -\tau^{-1}$	$\tanh(a,b_1)= \tau^{-1}$ $\tanh(a,b_2)= \tau$ $\tanh(a,b_3)= -\tau$ $\tanh(a,b_4)= -\tau^{-1}$ $\tanh(a,b_5)= \tau^{-1}$ $\tanh(a,b_6)= \tau$ $\tanh(a,b_7)= -\tau$ $\tanh(a,b_8)= -\tau^{-1}$

tanh(φ) = tanh (α + iβ)



Im(φ) = iβ



A complementary approach to complex stAngles implies the use of the expression for the hyperbolic tangent: $tanh\,(\varphi) = \dfrac{e^{\varphi} - e^{-\varphi}}{e^{\varphi} + e^{-\varphi}}$ and making φ = α + βi

We make the substitutions cosh(α)= γ , sinh(α)= σ , cos(β) = c, sin(β) = s

in e^{α} = cosh(α)+sinh(α) = γ + σ and $e^{-i\beta}$ = cos(β) - i·sin(β) = c - is

and $e^{-\alpha}$ = cosh(α)-sinh(α) = γ - σ and $e^{i\beta}$ = cos(β)+ i·sin(β) = c + is

$e^{\varphi} = e^{\alpha+\beta i} = e^{\alpha} e^{\beta i} = (\gamma+\sigma)(c+is) = \gamma c + \sigma c + i\gamma s + i\sigma s$

$e^{-\varphi} = e^{-\alpha-\beta i} = e^{-\alpha} e^{-\beta i} = (\gamma-\sigma)(c-is) = \gamma c - \sigma c - i\gamma s + i\sigma s$

$e^{\varphi} + e^{-\varphi} = 2\gamma c + 2i\sigma s$ $\qquad e^{\varphi} - e^{-\varphi} = 2\sigma c + 2i\gamma s$

$$tanh(\varphi) = \frac{\sigma c + i\gamma s}{\gamma c + i\sigma s} = \frac{(\sigma c + i\gamma s)(\gamma c - i\sigma s)}{(\gamma c + i\sigma s)(\gamma c - i\sigma s)} = \frac{\gamma\sigma c^2 + \gamma\sigma s^2 + isc\gamma^2 - isc\sigma^2}{(\gamma c)^2 + (\sigma s)^2} =$$

$$= \frac{\gamma\sigma(c^2+s^2) + isc(\gamma^2-s^2)}{(\gamma c)^2 + (\sigma s)^2} = \frac{\gamma\sigma + isc}{(\gamma c)^2 + (\sigma s)^2}$$

If we want to impose the condition for tanh of being real-valued, this implies that isc=0
The condition can be satisfied either by s = 0 or by c = 0

$s=0 \rightarrow \beta = 0,\ \beta = \pi \rightarrow c^2 = 1 \rightarrow tanh(\varphi) = \frac{\gamma\sigma}{\gamma 2} = \frac{\sigma}{\gamma} = tanh(\alpha) = \tau$

$c=0 \rightarrow \beta = \pm\pi/2 \rightarrow s^2 = 1 \rightarrow tanh(\varphi) = \frac{\gamma\sigma}{-\sigma 2} = \frac{\gamma}{-\sigma} = \frac{-1}{tanh(\alpha)} = \frac{-1}{\tau}$

This additional approach confirms the result obtained previously, and it adds the fact that we have taken into account all the possibilities for a real hyperbolic tangent.
In other words: All possible complex angles with a real hyperbolic tangent are stAngles.