Cross Product for Euclidean and Minkowskian spaces

Geometric Algebra

Geometric product: $ab = a\cdot b + a\wedge b$ $a\cdot b$: scalar part, $a\wedge b$: Bivector

Orthonormal base: $\{e_i\}$ Products: e_{ii}: ± 1 (signature of e_i)

$j \neq k \rightarrow \quad e_j e_k = - e_k e_j \quad \rightarrow \quad e_{jk} = - e_{kj}$

Euclidean space (0,3): $e_{ii} = 1$

$\{e_i\} = \{ e_1 , e_2 , e_3 \}$ Trivector (pseudoscalar) $i_{(3)} = e_{123} = e_1 e_2 e_3$

$a = a_1e_1 + a_2e_2 + a_3e_3$, $b = b_1e_1 + b_2e_2 + b_3e_3$

Euclidean Cross Product : $a \times b = - i_{(3)} (a \wedge b)$

$$
\begin{aligned}
a \wedge b &= a_1b_2e_1e_2 + a_1b_3e_1e_3 + a_2b_1e_2e_1 + a_2b_3e_2e_3 + a_3b_1e_3e_1 + a_3b_2e_3e_2 = \\
&= a_1b_2e_{12} - a_1b_3e_{31} - a_2b_1e_{12} + a_2b_3e_{23} + a_3b_1e_{31} - a_3b_2e_{23} = \\
&= (a_1b_2 - a_2b_1) e_{12} + (a_2b_3 - a_3b_2) e_{23} + (a_3b_1 - a_1b_3) e_{31}
\end{aligned}
$$

$$
\begin{aligned}
a \times b &= - e_{123} \left[(a_1b_2 - a_2b_1) e_{12} + (a_2b_3 - a_3b_2) e_{23} + (a_3b_1 - a_1b_3) e_{31} \right] = \\
&= -\left[(a_1b_2 - a_2b_1) e_{123}e_{12} + (a_2b_3 - a_3b_2) e_{123} e_{23} + (a_3b_1 - a_1b_3) e_{123}e_{31} \right] = \\
&= -\left[(a_1b_2 - a_2b_1) e_{12312} + (a_2b_3 - a_3b_2) e_{12323} + (a_3b_1 - a_1b_3) e_{12331} \right] = \\
&= -\left[(a_1b_2 - a_2b_1) e_{11232} + (a_2b_3 - a_3b_2) e_{12323} + (a_3b_1 - a_1b_3) e_{12133} \right] = \\
&= -\left[(a_1b_2 - a_2b_1)(- e_{11223}) + (a_2b_3 - a_3b_2)(- e_{12233}) + (a_3b_1 - a_1b_3)(-e_{11233}) \right] = \\
&= -\left[(a_1b_2 - a_2b_1) (- e_3) + (a_2b_3 - a_3b_2) (- e_1) + (a_3b_1 - a_1b_3) (- e_2) \right] = \\
&= -\left[(a_1b_2 - a_2b_1) (- e_3) + (a_3b_1 - a_1b_3) (- e_2) + (a_2b_3 - a_3b_2) (- e_1) \right] = \\
&= (a_1b_2 - a_2b_1) (e_3) + (a_3b_1 - a_1b_3) (e_2) + (a_2b_3 - a_3b_2) (e_1)
\end{aligned}
$$

$$\vec{a} \times \vec{b} = \begin{bmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

Minkowskian (1,2) space: $e_{00} = -1$ *(timelike direction)*

$\{e_i\} = \{ e_1 , e_2 , e_0 \}$ Trivector (pseudoscalar) $i_{(2,1)} = e_{102} = e_1e_0e_2$

$a = a_1e_1 + a_2e_2 + a_0e_0$, $b = b_1e_1 + b_2e_2 + b_0e_0$

Minkowskian Cross Product: $a \times b = - i_{(2,1)} (a \wedge b) = e_{120}(a \wedge b)$

$$
\begin{aligned}
a \wedge b &= a_1b_2e_1e_2 + a_1b_0e_1e_0 + a_2b_1e_2e_1 + a_2b_0e_2e_0 + a_0b_1e_0e_1 + a_0b_2e_0e_2 = \\
&= a_1b_2e_{12} - a_1b_0e_{01} - a_2b_1e_{12} + a_2b_0e_{20} + a_0b_1e_{01} - a_0b_2e_{20} = \\
&= (a_1b_2 - a_2b_1) e_{12} + (a_0b_1 - a_1b_0) e_{01} + (a_2b_0 - a_0b_2) e_{20}
\end{aligned}
$$

$$
\begin{aligned}
a \times b &= e_{120} \left[(a_1b_2 - a_2b_1) e_{12} + (a_0b_1 - a_1b_0) e_{01} + (a_2b_0 - a_0b_2) e_{20} \right] = \\
&= (a_1b_2 - a_2b_1) e_{12012} + (a_0b_1 - a_1b_0) e_{12001} + (a_2b_0 - a_0b_2) e_{12020} = \\
&= (a_1b_2 - a_2b_1)(- e_{11220}) + (a_0b_1 - a_1b_0)(- e_{11200}) + (a_2b_0 - a_0b_2)(- e_{12200}) = \\
&= (a_1b_2 - a_2b_1) (- e_0) + (a_0b_1 - a_1b_0)(- e_{200}) + (a_2b_0 - a_0b_2) (- e_{100}) = \\
&= (a_1b_2 - a_2b_1) (- e_0) + (a_0b_1 - a_1b_0) e_2 + (a_2b_0 - a_0b_2) e_1
\end{aligned}
$$

$$\vec{a} \times \vec{b} = \begin{bmatrix} e_1 & e_2 & -e_0 \\ a_1 & a_2 & a_0 \\ b_1 & b_2 & b_0 \end{bmatrix}$$