Cross Product for Euclidean and Minkowskian spaces

Geometric Algebra

Geometric product: $\vec{a}\vec{b} = \vec{a}\cdot\vec{b} + \vec{a}\wedge\vec{b}$ $\quad\vec{a}\cdot\vec{b}$: scalar part, $\vec{a}\wedge\vec{b}$: bivector.

Orthonormal base: $\{e_i\}$ $\quad$ Products: e_{ii}: ± 1 (signature of e_i)

$$j \neq k \quad \rightarrow \quad e_j e_k = -e_k e_j \quad \rightarrow \quad e_{jk} = -e_{kj}$$

Euclidean space (0,3): $e_{ii} = 1$

$\{e_i\} = \{e_1, e_2, e_3\}$ $\quad$ Trivector (pseudoscalar) $\quad i_{(3)} = e_{123} = e_1 e_2 e_3$

$\vec{a} = a_1 e_1 + a_2 e_2 + a_3 e_3,\qquad \vec{b} = b_1 e_1 + b_2 e_2 + b_3 e_3$

Euclidean Cross Product: $\vec{a}\times\vec{b} = i_{(3)}(\vec{a}\wedge\vec{b})$

$$\begin{aligned}
\vec{a}\wedge\vec{b} &= a_1 b_2 e_1 e_2 + a_1 b_3 e_1 e_3 + a_2 b_1 e_2 e_1 + a_2 b_3 e_2 e_3 + a_3 b_1 e_3 e_1 + a_3 b_2 e_3 e_2 &&= \\
&= a_1 b_2 e_{12} - a_1 b_3 e_{31} - a_2 b_1 e_{12} - a_2 b_3 e_{23} - a_3 b_1 e_{31} - a_3 b_2 e_{23} &&= \\
&= (a_1 b_2 - a_2 b_1) e_{12} + (a_2 b_3 - a_3 b_2) e_{23} + (a_3 b_1 - a_1 b_3) e_{31}
\end{aligned}$$

$$\begin{aligned}
\vec{a}\times\vec{b} &= -e_{123}\ [\,(a_1 b_2 - a_2 b_1)\, e_{12} + (a_2 b_3 - a_3 b_2)\, e_{23} + (a_3 b_1 - a_1 b_3)\, e_{31}\,] &&= \\
&= -[\,(a_1 b_2 - a_2 b_1)\, e_{123} e_{12} + (a_2 b_3 - a_3 b_2)\, e_{123} e_{23} + (a_3 b_1 - a_1 b_3)\, e_{123} e_{31}\,] &&= \\
&= -[\,(a_1 b_2 - a_2 b_1)\, e_{12312} + (a_2 b_3 - a_3 b_2)\, e_{12323} + (a_3 b_1 - a_1 b_3)\, e_{12331}\,] &&= \\
&= -[\,(a_1 b_2 - a_2 b_1)\, e_{11232} + (a_2 b_3 - a_3 b_2)\, e_{12323} + (a_3 b_1 - a_1 b_3)\, e_{12133}\,] &&= \\
&= -[(a_1 b_2 - a_2 b_1)(-e_{11223}) + (a_2 b_3 - a_3 b_2)(-e_{12233}) + (a_3 b_1 - a_1 b_3)(-e_{11233})] &&= \\
&= -[\,(a_1 b_2 - a_2 b_1)\,(-e_3) + (a_2 b_3 - a_3 b_2)\,(-e_1) + (a_3 b_1 - a_1 b_3)\,(-e_2)\,] &&= \\
&= -[\,(a_1 b_2 - a_2 b_1)\,(-e_3) + (a_3 b_1 - a_1 b_3)\,(-e_2) + (a_2 b_3 - a_3 b_2)\,(-e_1)\,] &&= \\
&= (a_1 b_2 - a_2 b_1)\, e_3 + (a_3 b_1 - a_1 b_3)\, e_2 + (a_2 b_3 - a_3 b_2)\, e_1 \\
&= \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}
\end{aligned}$$

Minkowskian (1,2) space: $e_{00} = -1$ *(timelike direction)*

$\{e_i\} = \{e_1, e_2, e_0\}$ $\quad$ Trivector (pseudoscalar)

$i_{(2,1)} = e_{120} = e_1 e_2 e_0$

$$\vec{a} = a_1 e_1 + a_2 e_2 + a_0 e_0, \qquad \vec{b} = b_1 e_1 + b_2 e_2 + b_0 e_0$$

Minkowskian Cross Product: $\vec{a}\times\vec{b} = -i_{(2,1)}(\vec{a}\wedge\vec{b})$

$$\begin{aligned}
\vec{a}\wedge\vec{b} &= a_1 b_2 e_1 e_2 + a_1 b_0 e_1 e_0 + a_2 b_1 e_2 e_1 + a_2 b_0 e_2 e_0 + a_0 b_1 e_0 e_1 + a_0 b_2 e_0 e_2 &=\\
&= a_1 b_2 e_{12} - a_1 b_0 e_{01} - a_2 b_1 e_{12} + a_2 b_0 e_{20} + a_0 b_1 e_{01} - a_0 b_2 e_{20} &=\\
&= (a_1 b_2 - a_2 b_1) e_{12} + (a_0 b_1 - a_1 b_0) e_{01} + (a_2 b_0 - a_0 b_2) e_{20}
\end{aligned}$$

$$\begin{aligned}
\vec{a}\times\vec{b} &= -e_{120}\ [\,(a_1 b_2 - a_2 b_1)\, e_{12} + (a_0 b_1 - a_1 b_0)\, e_{01} + (a_2 b_0 - a_0 b_2)\, e_{20}\,] &=\\
&= -[\,(a_1 b_2 - a_2 b_1)\, e_{12012} + (a_0 b_1 - a_1 b_0)\, e_{12001} + (a_2 b_0 - a_0 b_2)\, e_{12020}\,] &=\\
&= -[(a_1 b_2 - a_2 b_1)(-e_{11220}) + (a_0 b_1 - a_1 b_0)(-e_{11200}) + (a_2 b_0 - a_0 b_2)(-e_{12200})] &=\\
&= -[\,(a_1 b_2 - a_2 b_1)\,(-e_0) + (a_0 b_1 - a_1 b_0)\,(-e_{200}) + (a_2 b_0 - a_0 b_2)(-e_{100})] &=\\
&= -[\,(a_1 b_2 - a_2 b_1)\,(-e_0) + (a_0 b_1 - a_1 b_0)\, e_2 + (a_2 b_0 - a_0 b_2)\, e_1\,] &=\\
&= (a_1 b_2 - a_2 b_1)\, e_0 + (a_0 b_1 - a_1 b_0)\,(-e_2) + (a_2 b_0 - a_0 b_2)\,(-e_1)\\
&= \begin{vmatrix} -e_1 & -e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}
\end{aligned}$$