Marshallian Demand

LLE – Mathematics and Statistics Skills

1. Agent A's preferences over goods x and y can be represented by the utility function $U(x,y) = x^{0.4}y^{0.6}$. Suppose A's income is £300, the price of good x is £10, and the price of good y is £15. Use the Lagrangian method to find A's Marshallian demand for x and y (i.e., calculate the specific quantities consumed).

Solution to 1

The utility maximization problem is:

$$\max_{x,y} U(x,y) = x^{0.4}y^{0.6}$$
$$\text{subject to } 10x + 15y = 300$$

The Lagrangian function is: $L(x,y,\lambda) = x^{0.4}y^{0.6} + \lambda(300 - 10x - 15y)$

First-Order Conditions (FOCs):

$$\frac{\partial L}{\partial x} = 0.4x^{-0.6}y^{0.6} - 10\lambda = 0 \implies 0.4x^{-0.6}y^{0.6} = 10\lambda \quad (1)$$
$$\frac{\partial L}{\partial y} = 0.6x^{0.4}y^{-0.4} - 15\lambda = 0 \implies 0.6x^{0.4}y^{-0.4} = 15\lambda \quad (2)$$
$$\frac{\partial L}{\partial \lambda} = 300 - 10x - 15y = 0 \implies 10x + 15y = 300 \quad (3)$$

Divide equation (1) by equation (2) to eliminate λ:

$$\frac{0.4x^{-0.6}y^{0.6}}{0.6x^{0.4}y^{-0.4}} = \frac{10\lambda}{15\lambda}$$

Simplify the expression:

$$\frac{2}{3}\frac{y^{0.6}y^{0.4}}{x^{0.6}x^{0.4}} = \frac{2}{3}$$

$$\frac{2}{3}\frac{y}{x} = \frac{2}{3}$$

Multiply both sides by $\frac{3}{2}$ and x:

$$y = x$$

Substitute $y = x$ into the budget constraint (3):

$$10x + 15(x) = 300$$

$$25x = 300$$

$$x = \frac{300}{25}$$

$$x = 12$$

Since $y = x$, then $y = 12$.

Therefore, Agent A's Marshallian demand for x is 12 units and for y is 12 units.

2. Consumer B's preferences over goods x and y can be represented by the utility function $U(x, y) = x^{0.25}y^{0.75}$. Suppose B's income is $£240$, the price of good x is $£8$, and the price of good y is $£4$. Use the Lagrangian method to find B's Marshallian demand for x and y (i.e., calculate the specific quantities consumed).

Solution to 2

The utility maximization problem is:

$$\max_{x,y} U(x,y) = x^{0.25}y^{0.75}$$
$$\text{subject to } 8x + 4y = 240$$

The Lagrangian function is: $L(x,y,\lambda) = x^{0.25}y^{0.75} + \lambda(240 - 8x - 4y)$

First-Order Conditions (FOCs):

$$\frac{\partial L}{\partial x} = 0.25x^{-0.75}y^{0.75} - 8\lambda = 0 \implies 0.25x^{-0.75}y^{0.75} = 8\lambda \quad (1)$$
$$\frac{\partial L}{\partial y} = 0.75x^{0.25}y^{-0.25} - 4\lambda = 0 \implies 0.75x^{0.25}y^{-0.25} = 4\lambda \quad (2)$$
$$\frac{\partial L}{\partial \lambda} = 240 - 8x - 4y = 0 \implies 8x + 4y = 240 \quad (3)$$

Divide equation (1) by equation (2) to eliminate λ:

$$\frac{0.25x^{-0.75}y^{0.75}}{0.75x^{0.25}y^{-0.25}} = \frac{8\lambda}{4\lambda}$$

Simplify the expression:

$$\frac{1}{3}\frac{y^{0.75}y^{0.25}}{x^{0.75}x^{0.25}} = 2$$

$$\frac{1}{3}\frac{y}{x} = 2$$

Multiply both sides by $3x$:

$$y = 6x$$

Substitute $y = 6x$ into the budget constraint (3):

$$8x + 4(6x) = 240$$

$$8x + 24x = 240$$
$$32x = 240$$
$$x = \frac{240}{32}$$
$$x = 7.5$$

Since $y = 6x$, then $y = 6 \times 7.5 = 45$.

Therefore, Consumer B's Marshallian demand for x is 7.5 units and for y is 45 units.

3. Agent C's preferences over goods x and y can be represented by the utility function $U(x, y) = x^{0.9}y^{0.1}$. Let M denote C's income, and p_x and p_y denote the prices of the respective goods. Use the Lagrangian method to find C's Marshallian demand function for y.

Solution to 3

The utility maximization problem is:

$$\max_{x,y} U(x, y) = x^{0.9}y^{0.1}$$
$$\text{subject to } p_x x + p_y y = M$$

The Lagrangian function is: $L(x, y, \lambda) = x^{0.9}y^{0.1} + \lambda(M - p_x x - p_y y)$

First-Order Conditions (FOCs):

$$\frac{\partial L}{\partial x} = 0.9x^{-0.1}y^{0.1} - \lambda p_x = 0 \implies 0.9x^{-0.1}y^{0.1} = \lambda p_x \quad (1)$$
$$\frac{\partial L}{\partial y} = 0.1x^{0.9}y^{-0.9} - \lambda p_y = 0 \implies 0.1x^{0.9}y^{-0.9} = \lambda p_y \quad (2)$$
$$\frac{\partial L}{\partial \lambda} = M - p_x x - p_y y = 0 \implies p_x x + p_y y = M \quad (3)$$

Divide equation (1) by equation (2) to eliminate λ:

$$\frac{0.9x^{-0.1}y^{0.1}}{0.1x^{0.9}y^{-0.9}} = \frac{\lambda p_x}{\lambda p_y}$$

Simplify the expression:

$$\frac{0.9}{0.1}\frac{y^{0.1}y^{0.9}}{x^{0.1}x^{0.9}} = \frac{p_x}{p_y}$$

$$9\frac{y}{x} = \frac{p_x}{p_y}$$

Rearrange to express x in terms of y and prices (since we need demand for y):

$$9yp_y = xp_x$$

$$x = \frac{9yp_y}{p_x}$$

Substitute this expression for x into the budget constraint (3):

$$p_x\left(\frac{9yp_y}{p_x}\right) + p_y y = M$$

$$9yp_y + p_y y = M$$

$$10yp_y = M$$

$$y = \frac{M}{10p_y}$$

Therefore, Agent C's Marshallian demand function for y is $y = \frac{M}{10p_y}$.

4. Consumer D's preferences over goods x and y can be represented by the utility function $U(x, y) = x^a y^b$, where a and b are positive constants. Let M denote D's income, and p_x and p_y denote the prices of the respective goods. Use the Lagrangian method to find D's Marshallian demand function for x.

Solution to 4

The utility maximization problem is:

$$\max_{x,y} U(x,y) = x^a y^b$$
$$\text{subject to } p_x x + p_y y = M$$

The Lagrangian function is: $L(x,y,\lambda) = x^a y^b + \lambda(M - p_x x - p_y y)$

First-Order Conditions (FOCs):

$$\frac{\partial L}{\partial x} = a x^{a-1} y^b - \lambda p_x = 0 \implies a x^{a-1} y^b = \lambda p_x \quad (1)$$
$$\frac{\partial L}{\partial y} = b x^a y^{b-1} - \lambda p_y = 0 \implies b x^a y^{b-1} = \lambda p_y \quad (2)$$
$$\frac{\partial L}{\partial \lambda} = M - p_x x - p_y y = 0 \implies p_x x + p_y y = M \quad (3)$$

Divide equation (1) by equation (2) to eliminate λ:

$$\frac{a x^{a-1} y^b}{b x^a y^{b-1}} = \frac{\lambda p_x}{\lambda p_y}$$

Simplify the expression:

$$\frac{a}{b} \frac{y^b y^1}{x^a x^{1-a}} = \frac{p_x}{p_y}$$

$$\frac{a}{b} \frac{y}{x} = \frac{p_x}{p_y}$$

Rearrange to express y in terms of x and prices:

$$a y p_y = b x p_x$$

$$y = \frac{b p_x x}{a p_y}$$

Substitute this expression for y into the budget constraint (3):

$$p_x x + p_y \left(\frac{bp_x x}{ap_y} \right) = M$$

$$p_x x + \frac{bp_x x}{a} = M$$

Factor out $p_x x$:

$$p_x x \left(1 + \frac{b}{a} \right) = M$$

$$p_x x \left(\frac{a+b}{a} \right) = M$$

Solve for x:

$$x = \frac{Ma}{p_x(a+b)}$$

Therefore, Consumer D's Marshallian demand function for x is $x = \frac{aM}{(a+b)p_x}$.

5. Agent E's preferences over goods x and y can be represented by the utility function $U(x,y) = x + \ln(y)$. Let M denote E's income, and p_x and p_y denote the prices of the respective goods. Use the Lagrangian method to find E's Marshallian demand functions for x and y. Assume the consumer consumes positive amounts of both x and y.

Solution to 5

The utility maximization problem is:

$$\max_{x,y} U(x,y) = x + \ln(y)$$
$$\text{subject to } p_x x + p_y y = M$$

The Lagrangian function is: $L(x,y,\lambda) = x + \ln(y) + \lambda(M - p_x x - p_y y)$

First-Order Conditions (FOCs):

$$
\begin{aligned}
\frac{\partial L}{\partial x} &= 1 - \lambda p_x = 0 \implies 1 = \lambda p_x \quad (1) \\
\frac{\partial L}{\partial y} &= \frac{1}{y} - \lambda p_y = 0 \implies \frac{1}{y} = \lambda p_y \quad (2) \\
\frac{\partial L}{\partial \lambda} &= M - p_x x - p_y y = 0 \implies p_x x + p_y y = M \quad (3)
\end{aligned}
$$

From equation (1), solve for λ:

$$\lambda = \frac{1}{p_x}$$

Substitute this expression for λ into equation (2):

$$\frac{1}{y} = \left(\frac{1}{p_x}\right) p_y$$

$$\frac{1}{y} = \frac{p_y}{p_x}$$

Solve for y:

$$y = \frac{p_x}{p_y}$$

Substitute this expression for y into the budget constraint (3):

$$p_x x + p_y \left(\frac{p_x}{p_y}\right) = M$$

$$p_x x + p_x = M$$

Solve for x:

$$p_x x = M - p_x$$

$$x = \frac{M - p_x}{p_x}$$

Therefore, Agent E's Marshallian demand functions are:

$$x = \frac{M - p_x}{p_x}$$

$$y = \frac{p_x}{p_y}$$

(Note: For $x > 0$, it must be that $M > p_x$.)

6. Consumer F's preferences over goods x and y can be represented by the utility function $U(x, y) = xy + x$. Let M denote F's income, and p_x and p_y denote the prices of the respective goods. Use the Lagrangian method to find F's Marshallian demand functions for x and y. Assume the consumer consumes positive amounts of both x and y.

Solution to 6

The utility maximization problem is:

$$\max_{x,y} U(x, y) = xy + x = x(y + 1)$$
$$\text{subject to } p_x x + p_y y = M$$

The Lagrangian function is: $L(x, y, \lambda) = xy + x + \lambda(M - p_x x - p_y y)$

First-Order Conditions (FOCs):

$$\frac{\partial L}{\partial x} = y + 1 - \lambda p_x = 0 \implies y + 1 = \lambda p_x \quad (1)$$
$$\frac{\partial L}{\partial y} = x - \lambda p_y = 0 \implies x = \lambda p_y \quad (2)$$
$$\frac{\partial L}{\partial \lambda} = M - p_x x - p_y y = 0 \implies p_x x + p_y y = M \quad (3)$$

From equation (2), solve for λ:

$$\lambda = \frac{x}{p_y}$$

Substitute this expression for λ into equation (1):

$$y + 1 = \left(\frac{x}{p_y}\right) p_x$$

$$y + 1 = \frac{p_x x}{p_y}$$

Rearrange to express $p_x x$:

$$p_x x = p_y(y + 1)$$

$$p_x x = p_y y + p_y$$

Substitute this expression for $p_x x$ into the budget constraint (3):

$$(p_y y + p_y) + p_y y = M$$

$$2p_y y + p_y = M$$

Solve for y:

$$2p_y y = M - p_y$$

$$y = \frac{M - p_y}{2p_y}$$

Now, substitute this expression for y back into the equation for $p_x x$:

$$p_x x = p_y \left(\frac{M - p_y}{2p_y} + 1\right)$$

$$p_x x = p_y \left(\frac{M - p_y + 2p_y}{2p_y}\right)$$

$$p_x x = \frac{M + p_y}{2}$$

Solve for x:

$$x = \frac{M + p_y}{2p_x}$$

Therefore, Consumer F's Marshallian demand functions are:

$$x = \frac{M + p_y}{2p_x}$$

$$y = \frac{M - p_y}{2p_y}$$

(Note: For $y > 0$, it must be that $M > p_y$.)