# Partial Differentiation Solutions

LLE – Mathematics and Statistics Skills

## Partial Derivatives

1. For solutions we shall use:

$$z_x \equiv \frac{\partial z}{\partial x}, \quad z_y \equiv \frac{\partial z}{\partial y}, \quad z_{xx} \equiv \frac{\partial^2 z}{\partial x^2},$$

$$z_{yy} \equiv \frac{\partial^2 z}{\partial y^2}, \quad z_{yx} \equiv \frac{\partial^2 z}{\partial x \partial y}, \quad z_{xy} \equiv \frac{\partial^2 z}{\partial y \partial x}$$

(a) $z = 5x + 2y$

$$\begin{aligned} z_x &= 5 & z_y &= 2 \\ z_{xx} &= 0 & z_{yy} &= 0 \\ z_{yx} &= z_{xy} = 0 \end{aligned}$$

(b) $z = 5x^3 + 3y^2 + 4$

$$\begin{aligned} z_x &= 15x^2 & z_y &= 6y \\ z_{xx} &= 30x & z_{yy} &= 6 \\ z_{yx} &= z_{xy} = 0 \end{aligned}$$

(c) $z = 8xy$

$$\begin{aligned} z_x &= 8y & z_y &= 8x \\ z_{xx} &= 0 & z_{yy} &= 0 \\ z_{yx} &= z_{xy} = 8 \end{aligned}$$

(d) $z = 2xy^{0.5}$

$$z_x = 2y^{0.5} \qquad z_y = 0.5 \cdot 2xy^{-0.5} = xy^{-0.5}$$
$$z_{xx} = 0 \qquad z_{yy} = -0.5xy^{-1.5}$$
$$z_{yx} = z_{xy} = y^{-0.5}$$

(e) $z = e^{2x} + 4e^y$

$$z_x = 2e^{2x} \qquad z_y = 4e^y$$
$$z_{xx} = 4e^{2x} \qquad z_{yy} = 4e^y$$
$$z_{yx} = z_{xy} = 0$$

(f) $z = x^2 + 2xy + y^2$

$$z_x = 2x + 2y \qquad z_y = 2x + 2y$$
$$z_{xx} = 2 \qquad z_{yy} = 2$$
$$z_{yx} = z_{xy} = 2$$

(g) $z = 10x^2y^3 - 2x - 4y$

$$z_x = 20xy^3 - 2 \qquad z_y = 30x^2y^2 - 4$$
$$z_{xx} = 20y^3 \qquad z_{yy} = 60x^2y$$
$$z_{yx} = z_{xy} = 60xy^2$$

(h) $z = 5x^2 + 2x + 5y^3 - 2y^2$

$$z_x = 10x + 2 \qquad z_y = 15y^2 - 4y$$
$$z_{xx} = 10 \qquad z_{yy} = 30y - 4$$
$$z_{yx} = z_{xy} = 0$$

(i) $z = 10x^4y^3$

$$z_x = 40x^3y^3 \qquad z_y = 30x^4y^2$$
$$z_{xx} = 120x^2y^3 \qquad z_{yy} = 60x^4y$$
$$z_{yx} = z_{xy} = 120x^3y^2$$

(j) $z = 2e^{5x} - 4x^2y^2 + \ln(3xy)$

$$z_x = 10e^{5x} - 8xy^2 + \frac{1}{x} \qquad z_y = -8x^2y + \frac{1}{y}$$
$$z_{xx} = 50e^{5x} - 8y^2 - \frac{1}{x^2} \qquad z_{yy} = -8x^2 - \frac{1}{y^2}$$
$$z_{yx} = z_{xy} = -16xy$$

(k) $z = x^3 \ln y$

$$z_x = 3x^2 \ln y \qquad z_y = \frac{x^3}{y}$$
$$z_{xx} = 6x \ln y \qquad z_{yy} = -\frac{x^3}{y^2}$$
$$z_{yx} = z_{xy} = \frac{3x^2}{y}$$

(l) $z = y(5 - 2x)$

$$z_x = -2y \qquad z_y = 5 - 2x$$
$$z_{xx} = 0 \qquad z_{yy} = 0$$
$$z_{yx} = z_{xy} = -2$$

(m) $z = y(5 - 2x)^2$

$$z_x = -4y(5 - 2x) \qquad z_y = (5 - 2x)^2$$
$$z_{xx} = 8y \qquad z_{yy} = 0$$
$$z_{yx} = z_{xy} = -4(5 - 2x)$$

2. Demand function: $Q = 50 - 2P + 0.5I$, where $Q$ is quantity demanded, $P$ price, $I$ income.

   (a) Partial derivatives:

   $$\frac{\partial Q}{\partial P} = -2, \quad \frac{\partial Q}{\partial I} = 0.5$$

   (b) Interpretation:
   $\frac{\partial Q}{\partial P} = -2$ means that for every 1 unit increase in price $P$, quantity demanded $Q$ decreases by 2 units, holding income constant.
   $\frac{\partial Q}{\partial I} = 0.5$ means that for every 1 unit increase in income $I$, quantity demanded $Q$ increases by 0.5 units, holding price constant.

   (c) Change in demand if income increases by 100:

   $$\Delta Q = \frac{\partial Q}{\partial I} \times 100 = 0.5 \times 100 = 50$$

3. Production function: $Q(L, K) = AL^{\alpha}K^{1-\alpha}$

   (a) Partial derivatives:

   $$\frac{\partial Q}{\partial L} = A\alpha L^{\alpha-1}K^{1-\alpha}, \quad \frac{\partial Q}{\partial K} = A(1-\alpha)L^{\alpha}K^{-\alpha}$$

   (b) With $A = 1$, $\alpha = 0.3$, at $L = 10$, $K = 5$:

   $$\frac{\partial Q}{\partial L} = 1 \times 0.3 \times 10^{0.3-1} \times 5^{0.7} \approx 0.3 \times 10^{-0.7} \times 5^{0.7}$$

   Calculate numerically:

   $$10^{-0.7} \approx 0.1995, \quad 5^{0.7} \approx 3.084$$

   Thus,

   $$\frac{\partial Q}{\partial L} \approx 0.3 \times 0.1995 \times 3.084 \approx 0.185$$

Similarly,

$$\frac{\partial Q}{\partial K} = 1 \times 0.7 \times 10^{0.3} \times 5^{-0.3}$$

$$10^{0.3} \approx 1.995, \quad 5^{-0.3} \approx 0.617$$

$$\frac{\partial Q}{\partial K} \approx 0.7 \times 1.995 \times 0.617 \approx 0.862$$

(c) At $L = 20$, $K = 5$:

$$\frac{\partial Q}{\partial L} = 0.3 \times 20^{-0.7} \times 5^{0.7}$$

$$20^{-0.7} \approx 0.122, \quad 5^{0.7} \approx 3.084$$

$$\frac{\partial Q}{\partial L} \approx 0.3 \times 0.122 \times 3.084 \approx 0.113$$

**Explanation:** Increasing $L$ reduces the marginal product of labour because the exponent $\alpha - 1$ is negative, meaning the derivative decreases as $L$ increases.

# First Order Conditions

4. For each function $f(x, y)$, find:

(i) Partial derivatives $f_x$ and $f_y$

(ii) Points where $f_x = 0$ and $f_y = 0$, and find $f(x, y)$ there

(a) $f(x, y) = x^2 + y^2$

$$f_x = 2x, \quad f_y = 2y$$

Set equal to zero:

$$f_x = 0 \Rightarrow x = 0$$

$$f_y = 0 \Rightarrow y = 0$$

At $(0,0)$,

$$f(0,0) = 0^2 + 0^2 = 0$$

(b) $f(x,y) = x^2 - 2x - y^2 - 4y - 3$

$$f_x = 2x - 2, \quad f_y = -2y - 4$$

Set equal to zero:

$$2x - 2 = 0 \Rightarrow x = 1$$

$$-2y - 4 = 0 \Rightarrow y = -2$$

At $(1,-2)$,

$$f(1,-2) = 1 - 2 - 4 + 8 - 3 = 0$$

(c) For $f(x,y) = x^2y - 2xy^2 + 3xy + 4$:

**Step 1:** Compute the partial derivatives

$$f_x = 2xy - 2y^2 + 3y, \quad f_y = x^2 - 4xy + 3x$$

**Step 2:** Set the partial derivatives equal to zero

$$2xy - 2y^2 + 3y = 0 \tag{1}$$

$$x^2 - 4xy + 3x = 0 \tag{2}$$

**Step 3:** Factor both equations

From equation (1), factor out $y$:

$$y(2x - 2y + 3) = 0$$

This means either

$$y = 0 \quad \text{or} \quad 2x - 2y + 3 = 0.$$

From equation (2), factor out $x$:

$$x(x - 4y + 3) = 0$$

This means either

$$x = 0 \quad \text{or} \quad x - 4y + 3 = 0.$$

**Case 1:** $y = 0$
Substitute $y = 0$ into equation (2):

$$x(x - 4 \cdot 0 + 3) = x(x + 3) = 0$$

So,

$$x = 0 \quad \text{or} \quad x = -3$$

Points:

$$(0, 0), \quad (-3, 0)$$

**Case 2:** $x = 0$
Substitute $x = 0$ into equation (1):

$$y(2 \cdot 0 - 2y + 3) = y(-2y + 3) = 0$$

So,

$$y = 0 \quad \text{or} \quad y = \frac{3}{2}$$

Points:

$$(0, 0), \quad \left(0, \frac{3}{2}\right)$$

**Case 3:** $2x - 2y + 3 = 0$
Rearranged as

$$x - y = -\frac{3}{2} \implies x = y - \frac{3}{2}$$

Substitute $x = y - \frac{3}{2}$ into equation (2):

$$(y - \frac{3}{2})\left((y - \frac{3}{2}) - 4y + 3\right) = 0$$

Simplify the second factor:

$$(y - \frac{3}{2})(-3y + \frac{3}{2}) = 0$$

Solve:

$$y - \frac{3}{2} = 0 \implies y = \frac{3}{2}$$

$$-3y + \frac{3}{2} = 0 \implies y = \frac{1}{2}$$

Corresponding $x$ values:

$$\text{If } y = \frac{3}{2}, \quad x = \frac{3}{2} - \frac{3}{2} = 0$$

$$\text{If } y = \frac{1}{2}, \quad x = \frac{1}{2} - \frac{3}{2} = -1$$

Points:

$$\left(0, \frac{3}{2}\right), \quad \left(-1, \frac{1}{2}\right)$$

**Final points where $f_x = 0$ and $f_y = 0$:**

$$(0, 0), \quad (-3, 0), \quad \left(0, \frac{3}{2}\right), \quad \left(-1, \frac{1}{2}\right)$$

**Evaluate $f(x, y)$ at these points:**

$$f(0, 0) = 4$$

$$f(-3, 0) = 4$$

$$f\left(0, \frac{3}{2}\right) = 4$$

$$f\left(-1, \frac{1}{2}\right) = (-1)^2 \cdot \frac{1}{2} - 2 \cdot (-1) \cdot \left(\frac{1}{2}\right)^2 + 3 \cdot (-1) \cdot \frac{1}{2} + 4 = \frac{1}{2} + \frac{1}{2} - \frac{3}{2} + 4 = 3.5$$