Constrained Optimisation

LLE – Mathematics and Statistics Skills

1. Given the function:

$$z = x^2 + 2xy + 4y + 10$$

   with constraint:

$$x + y = 100$$

   Optimise the function z subject to the constraint by:

   (a) Writing down the Langrangian function

   (b) Writing down the three first order conditions

   (c) Solving the first order condictions to find the optimal x, y, and the value of z

2. Given the function:

$$z = 12x + 4y$$

   with constraint:

$$x^{0.3}y^{0.7} = 1000$$

   Optimise the function z subject to the constraint.

3. Given the function:

$$z = x^{0.6}y^{0.4}$$

   with constraint:

$$5x + 8y = 120$$

   Optimise the function z subject to the constraint.

4. A consumer has the following utility function for two goods x and y:

$$U(x, y) = x^{0.5}y^{0.5}$$

The consumer has an income of 100. The price of good x is $p_x = 2$ and the price of good y is $p_y = 4$.

(a) Write down the budget constraint for the consumer.

(b) Maximise the customer's utility, subject to their budget constraint.

5. The production function for output (Q), for two inputs of labour (L) and capital (K) is given by:

$$Q = L^{\alpha} K^{\beta}$$

The cost of labour is 10 and the cost of capital is 20.

(a) Given a production constraint of an output of 100 units, use the Langrangian method to show that minimum cost occurs when:

$$L = \frac{2\alpha K}{\beta}$$

to give:

$$K^* = \left(100\left(\frac{2\alpha}{\beta}\right)^{-\alpha}\right)^{\frac{1}{\alpha+\beta}}$$

$$L^* = \left(100\left(\frac{2\alpha}{\beta}\right)^{\beta}\right)^{\frac{1}{\alpha+\beta}}$$

(b) Find the optimal values and the minimum cost when:

i. $\alpha = \beta = 0.5$

ii. $\alpha = 0.3$, $\beta = 0.7$

6. A single good is produced by a monopoly. The demand function for price (P) from quantity (Q) for this good is given by:

$$P = 50 - Q$$

The cost function for the company producing the good is given by:

$$C = 10 + 2Q$$

Find the maximum profit, Π, and the values of P and Q that give this profit, subject to the constraint of the demand function.