Optimisation (Single Variable Functions)

LLE Mathematics and Statistics

Differentiation Practice

For each function, we find the first and second derivatives.

1. $y = 5x^3 + 2x^2 - 8x + 3$

$$\begin{aligned}\frac{dy}{dx} &= 5(3x^2) + 2(2x) - 8(1) + 0\\ &= 15x^2 + 4x - 8\end{aligned}$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= 15(2x) + 4(1) - 0\\ &= 30x + 4\end{aligned}$$

2. $m = 4x - 3 + \frac{2}{x} - \frac{4}{x^2}$

$$\begin{aligned}m &= 4x - 3 + 2x^{-1} - 4x^{-2}\\ \frac{dm}{dx} &= 4(1) - 0 + 2(-1x^{-2}) - 4(-2x^{-3})\\ &= 4 - 2x^{-2} + 8x^{-3}\\ &= 4 - \frac{2}{x^2} + \frac{8}{x^3}\end{aligned}$$

$$
\begin{aligned}
\frac{d^2m}{dx^2} &= 0 - 2(-2x^{-3}) + 8(-3x^{-4}) \\
&= 4x^{-3} - 24x^{-4} \\
&= \frac{4}{x^3} - \frac{24}{x^4}
\end{aligned}
$$

3. $f(t) = 24 + 4\sqrt{t}$

$$
\begin{aligned}
f(t) &= 24 + 4t^{1/2} \\
f'(t) &= 0 + 4\left(\frac{1}{2}t^{-1/2}\right) \\
&= 2t^{-1/2} \\
&= \frac{2}{\sqrt{t}}
\end{aligned}
$$

$$
\begin{aligned}
f''(t) &= 2\left(-\frac{1}{2}t^{-3/2}\right) \\
&= -t^{-3/2} \\
&= -\frac{1}{t^{3/2}}
\end{aligned}
$$

4. $P = t^3(5t^3 - 4t^{-3/2})$

$$
\begin{aligned}
P &= 5t^{3+3} - 4t^{3-3/2} \\
&= 5t^6 - 4t^{3/2} \\
\frac{dP}{dt} &= 5(6t^5) - 4\left(\frac{3}{2}t^{1/2}\right) \\
&= 30t^5 - 6t^{1/2} \\
&= 30t^5 - 6\sqrt{t}
\end{aligned}
$$

$$\begin{aligned}
\frac{d^2P}{dt^2} &= 30(5t^4) - 6\left(\frac{1}{2}t^{-1/2}\right) \\
&= 150t^4 - 3t^{-1/2} \\
&= 150t^4 - \frac{3}{\sqrt{t}}
\end{aligned}$$

5. $g(x) = 3x^5 - 4x^2 + \frac{7}{x}$ (domain: $x > 0$)

$$\begin{aligned}
g(x) &= 3x^5 - 4x^2 + 7x^{-1} \\
\frac{dg}{dx} &= 3(5x^4) - 4(2x) + 7(-1x^{-2}) \\
&= 15x^4 - 8x - 7x^{-2} \\
&= 15x^4 - 8x - \frac{7}{x^2}
\end{aligned}$$

$$\begin{aligned}
\frac{d^2g}{dx^2} &= 15(4x^3) - 8(1) - 7(-2x^{-3}) \\
&= 60x^3 - 8 + 14x^{-3} \\
&= 60x^3 - 8 + \frac{14}{x^3}
\end{aligned}$$

6. $y = 10x^4 + 6x^{1/3} - \frac{3}{x^2}$

$$\begin{aligned}
y &= 10x^4 + 6x^{1/3} - 3x^{-2} \\
\frac{dy}{dx} &= 10(4x^3) + 6\left(\frac{1}{3}x^{-2/3}\right) - 3(-2x^{-3}) \\
&= 40x^3 + 2x^{-2/3} + 6x^{-3} \\
&= 40x^3 + \frac{2}{x^{2/3}} + \frac{6}{x^3}
\end{aligned}$$

$$
\begin{aligned}
\frac{d^2y}{dx^2} &= 40(3x^2) + 2\left(-\frac{2}{3}x^{-5/3}\right) + 6(-3x^{-4}) \\
&= 120x^2 - \frac{4}{3}x^{-5/3} - 18x^{-4} \\
&= 120x^2 - \frac{4}{3x^{5/3}} - \frac{18}{x^4}
\end{aligned}
$$

Stationary Points

For each function, we will find the first derivative, stationary points, second derivative, and determine the nature of the stationary points.

1. $y = x^2 + 8x - 9$

   - **First Derivative:** $\frac{dy}{dx} = 2x + 8$
   - **Stationary Point(s):** Set $\frac{dy}{dx} = 0$.

     $$\begin{aligned} 2x + 8 &= 0 \\ 2x &= -8 \\ x &= -4 \end{aligned}$$

     Substitute $x = -4$ into y:

     $$\begin{aligned} y &= (-4)^2 + 8(-4) - 9 \\ &= 16 - 32 - 9 \\ &= -25 \end{aligned}$$

     Stationary point: $(-4, -25)$.
   - **Second Derivative:** $\frac{d^2y}{dx^2} = 2$
   - **Nature of Stationary Point(s):** At $x = -4$:

     $$\frac{d^2y}{dx^2} = 2$$

     Since $\frac{d^2y}{dx^2} = 2 > 0$, the stationary point $(-4, -25)$ is a **local minimum**.

2. $y = 2x^3 + 6x^2 - 90x - 2$

   - **First Derivative:** $\frac{dy}{dx} = 6x^2 + 12x - 90$

- **Stationary Point(s):** Set $\frac{dy}{dx} = 0$.

$$\begin{aligned} 6x^2 + 12x - 90 &= 0 \\ x^2 + 2x - 15 &= 0 \qquad \text{(Divide by 6)} \\ (x+5)(x-3) &= 0 \end{aligned}$$

  $x = -5$ or $x = 3$. For $x = -5$:

$$\begin{aligned} y &= 2(-5)^3 + 6(-5)^2 - 90(-5) - 2 \\ &= 2(-125) + 6(25) + 450 - 2 \\ &= -250 + 150 + 450 - 2 \\ &= 348 \end{aligned}$$

  Stationary point: $(-5, 348)$. For $x = 3$:

$$\begin{aligned} y &= 2(3)^3 + 6(3)^2 - 90(3) - 2 \\ &= 2(27) + 6(9) - 270 - 2 \\ &= 54 + 54 - 270 - 2 \\ &= -164 \end{aligned}$$

  Stationary point: $(3, -164)$.

- **Second Derivative:** $\frac{d^2y}{dx^2} = 12x + 12$
- **Nature of Stationary Point(s):** At $x = -5$:

$$\begin{aligned} \frac{d^2y}{dx^2} &= 12(-5) + 12 \\ &= -60 + 12 \\ &= -48 \end{aligned}$$

  Since $\frac{d^2y}{dx^2} = -48 < 0$, the stationary point $(-5, 348)$ is a **local**

maximum. At $x = 3$:

$$\begin{aligned} \frac{d^2y}{dx^2} &= 12(3) + 12 \\ &= 36 + 12 \\ &= 48 \end{aligned}$$

Since $\frac{d^2y}{dx^2} = 48 > 0$, the stationary point $(3, -164)$ is a **local minimum**.

3. $y = 4x^{3/2} - x^2 \quad$ (domain: $x > 0$)

   - **First Derivative:**

$$\begin{aligned} \frac{dy}{dx} &= 4\left(\frac{3}{2}x^{1/2}\right) - 2x \\ &= 6x^{1/2} - 2x \\ &= 6\sqrt{x} - 2x \end{aligned}$$

   - **Stationary Point(s):** Set $\frac{dy}{dx} = 0$.

$$\begin{aligned} 6\sqrt{x} - 2x &= 0 \\ 2\sqrt{x}(3 - \sqrt{x}) &= 0 \end{aligned}$$

Given domain $x > 0$, we consider only $3 - \sqrt{x} = 0 \Rightarrow \sqrt{x} = 3 \Rightarrow x = 9$. (Note: $x = 0$ is excluded by the domain $x > 0$). For $x = 9$:

$$\begin{aligned} y &= 4(9)^{3/2} - (9)^2 \\ &= 4(27) - 81 \\ &= 108 - 81 \\ &= 27 \end{aligned}$$

Stationary point: $(9, 27)$.

- **Second Derivative:**

$$\begin{aligned}\frac{d^2y}{dx^2} &= 6\left(\frac{1}{2}x^{-1/2}\right) - 2 \\ &= 3x^{-1/2} - 2 \\ &= \frac{3}{\sqrt{x}} - 2\end{aligned}$$

- **Nature of Stationary Point(s):** At $x = 9$:

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{3}{\sqrt{9}} - 2 \\ &= \frac{3}{3} - 2 \\ &= 1 - 2 \\ &= -1\end{aligned}$$

  Since $\frac{d^2y}{dx^2} = -1 < 0$, the stationary point $(9, 27)$ is a **local maximum**.

4. $y = x \ln x$ (use the product rule)

   - **First Derivative:** Use product rule $\frac{d}{dx}(uv) = u'v + uv'$. Let $u = x, v = \ln x$.

$$\begin{aligned}\frac{dy}{dx} &= (1)(\ln x) + (x)\left(\frac{1}{x}\right) \\ &= \ln x + 1\end{aligned}$$

   - **Stationary Point(s):** Set $\frac{dy}{dx} = 0$.

$$\begin{aligned}\ln x + 1 &= 0 \\ \ln x &= -1 \\ x &= e^{-1} = \frac{1}{e}\end{aligned}$$

Substitute $x = \frac{1}{e}$ into y:

$$
\begin{aligned}
y &= \frac{1}{e} \ln \left(\frac{1}{e} \right) \\
&= \frac{1}{e}(-1) \\
&= -\frac{1}{e}
\end{aligned}
$$

Stationary point: $\left(\frac{1}{e}, -\frac{1}{e} \right)$.

- **Second Derivative:** $\frac{d^2y}{dx^2} = \frac{1}{x}$
- **Nature of Stationary Point(s):** At $x = \frac{1}{e}$:

$$
\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{1}{1/e} \\
&= e
\end{aligned}
$$

Since $\frac{d^2y}{dx^2} = e > 0$, the stationary point $\left(\frac{1}{e}, -\frac{1}{e} \right)$ is a **local minimum**.

5. $y = 6x^{2/3} - 4x$ (domain: $x > 0$)

- **First Derivative:**

$$
\begin{aligned}
\frac{dy}{dx} &= 6 \left(\frac{2}{3} x^{-1/3} \right) - 4 \\
&= 4x^{-1/3} - 4 \\
&= \frac{4}{\sqrt[3]{x}} - 4
\end{aligned}
$$

- **Stationary Point(s):** Set $\frac{dy}{dx} = 0$.

$$\begin{aligned}
\frac{4}{\sqrt[3]{x}} - 4 &= 0 \\
\frac{4}{\sqrt[3]{x}} &= 4 \\
4 &= 4\sqrt[3]{x} \\
1 &= \sqrt[3]{x} \\
x &= 1^3 \\
x &= 1
\end{aligned}$$

Substitute $x = 1$ into y:

$$\begin{aligned}
y &= 6(1)^{2/3} - 4(1) \\
&= 6 - 4 \\
&= 2
\end{aligned}$$

Stationary point: $(1, 2)$.

- **Second Derivative:**

$$\begin{aligned}
\frac{d^2y}{dx^2} &= 4\left(-\frac{1}{3}x^{-4/3}\right) - 0 \\
&= -\frac{4}{3}x^{-4/3} \\
&= -\frac{4}{3x^{4/3}}
\end{aligned}$$

- **Nature of Stationary Point(s):** At $x = 1$:

$$\begin{aligned}
\frac{d^2y}{dx^2} &= -\frac{4}{3(1)^{4/3}} \\
&= -\frac{4}{3}
\end{aligned}$$

Since $\frac{d^2y}{dx^2} = -\frac{4}{3} < 0$, the stationary point $(1, 2)$ is a **local maximum**.

6. $y = x^3 - 3x$

- **First Derivative:** $\frac{dy}{dx} = 3x^2 - 3$
- **Stationary Point(s):** Set $\frac{dy}{dx} = 0$.

$$\begin{aligned} 3x^2 - 3 &= 0 \\ 3x^2 &= 3 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

  For $x = 1$: $y = (1)^3 - 3(1) = 1 - 3 = -2$. Stationary point: $(1, -2)$. For $x = -1$: $y = (-1)^3 - 3(-1) = -1 + 3 = 2$. Stationary point: $(-1, 2)$.

- **Second Derivative:** $\frac{d^2y}{dx^2} = 6x$
- **Nature of Stationary Point(s):** At $x = 1$:

$$\begin{aligned} \frac{d^2y}{dx^2} &= 6(1) \\ &= 6 \end{aligned}$$

  Since $\frac{d^2y}{dx^2} = 6 > 0$, the stationary point $(1, -2)$ is a **local minimum**. At $x = -1$:

$$\begin{aligned} \frac{d^2y}{dx^2} &= 6(-1) \\ &= -6 \end{aligned}$$

  Since $\frac{d^2y}{dx^2} = -6 < 0$, the stationary point $(-1, 2)$ is a **local maximum**.

7. $y = x + \frac{3}{x} + 2$

- **First Derivative:**

$$\begin{aligned}
y &= x + 3x^{-1} + 2 \\
\frac{dy}{dx} &= 1 + 3(-1x^{-2}) + 0 \\
&= 1 - 3x^{-2} \\
&= 1 - \frac{3}{x^2}
\end{aligned}$$

- **Stationary Point(s):** Set $\frac{dy}{dx} = 0$.

$$\begin{aligned}
1 - \frac{3}{x^2} &= 0 \\
1 &= \frac{3}{x^2} \\
x^2 &= 3 \\
x &= \pm\sqrt{3}
\end{aligned}$$

For $x = \sqrt{3}$:

$$\begin{aligned}
y &= \sqrt{3} + \frac{3}{\sqrt{3}} + 2 \\
&= \sqrt{3} + \sqrt{3} + 2 \\
&= 2\sqrt{3} + 2
\end{aligned}$$

Stationary point: $(\sqrt{3}, 2\sqrt{3} + 2)$. For $x = -\sqrt{3}$:

$$\begin{aligned}
y &= -\sqrt{3} + \frac{3}{-\sqrt{3}} + 2 \\
&= -\sqrt{3} - \sqrt{3} + 2 \\
&= -2\sqrt{3} + 2
\end{aligned}$$

Stationary point: $(-\sqrt{3}, -2\sqrt{3} + 2)$.

- **Second Derivative:**

$$\begin{aligned}\frac{d^2y}{dx^2} &= 0 - 3(-2x^{-3}) \\ &= 6x^{-3} \\ &= \frac{6}{x^3}\end{aligned}$$

- **Nature of Stationary Point(s):** At $x = \sqrt{3}$:

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{6}{(\sqrt{3})^3} \\ &= \frac{6}{3\sqrt{3}} \\ &= \frac{2}{\sqrt{3}}\end{aligned}$$

  Since $\frac{d^2y}{dx^2} = \frac{2}{\sqrt{3}} > 0$, the stationary point $(\sqrt{3}, 2\sqrt{3} + 2)$ is a **local minimum**. At $x = -\sqrt{3}$:

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{6}{(-\sqrt{3})^3} \\ &= \frac{6}{-3\sqrt{3}} \\ &= -\frac{2}{\sqrt{3}}\end{aligned}$$

  Since $\frac{d^2y}{dx^2} = -\frac{2}{\sqrt{3}} < 0$, the stationary point $(-\sqrt{3}, -2\sqrt{3} + 2)$ is a **local maximum**.

8. $y = x^3 - 6x^2 + 9x + 5$

   - **First Derivative:** $\frac{dy}{dx} = 3x^2 - 12x + 9$

- **Stationary Point(s):** Set $\frac{dy}{dx} = 0$.

$$\begin{aligned} 3x^2 - 12x + 9 &= 0 \\ x^2 - 4x + 3 &= 0 && \text{(Divide by 3)} \\ (x-1)(x-3) &= 0 \end{aligned}$$

  $x = 1$ or $x = 3$. For $x = 1$:

$$\begin{aligned} y &= (1)^3 - 6(1)^2 + 9(1) + 5 \\ &= 1 - 6 + 9 + 5 \\ &= 9 \end{aligned}$$

  Stationary point: $(1, 9)$. For $x = 3$:

$$\begin{aligned} y &= (3)^3 - 6(3)^2 + 9(3) + 5 \\ &= 27 - 54 + 27 + 5 \\ &= 5 \end{aligned}$$

  Stationary point: $(3, 5)$.

- **Second Derivative:** $\frac{d^2y}{dx^2} = 6x - 12$
- **Nature of Stationary Point(s):** At $x = 1$:

$$\begin{aligned} \frac{d^2y}{dx^2} &= 6(1) - 12 \\ &= 6 - 12 \\ &= -6 \end{aligned}$$

  Since $\frac{d^2y}{dx^2} = -6 < 0$, the stationary point $(1, 9)$ is a **local maximum**. At $x = 3$:

$$\begin{aligned} \frac{d^2y}{dx^2} &= 6(3) - 12 \\ &= 18 - 12 \\ &= 6 \end{aligned}$$

Since $\frac{d^2y}{dx^2} = 6 > 0$, the stationary point $(3, 5)$ is a **local minimum**.

9. $y = x^4 - 8x^2 + 5$

- **First Derivative:** $\frac{dy}{dx} = 4x^3 - 16x$
- **Stationary Point(s):** Set $\frac{dy}{dx} = 0$.

$$\begin{aligned} 4x^3 - 16x &= 0 \\ 4x(x^2 - 4) &= 0 \\ 4x(x-2)(x+2) &= 0 \end{aligned}$$

  $x = 0$, $x = 2$, or $x = -2$. For $x = 0$: $y = (0)^4 - 8(0)^2 + 5 = 5$. Stationary point: $(0, 5)$. For $x = 2$:

$$\begin{aligned} y &= (2)^4 - 8(2)^2 + 5 \\ &= 16 - 32 + 5 \\ &= -11 \end{aligned}$$

  Stationary point: $(2, -11)$. For $x = -2$:

$$\begin{aligned} y &= (-2)^4 - 8(-2)^2 + 5 \\ &= 16 - 32 + 5 \\ &= -11 \end{aligned}$$

  Stationary point: $(-2, -11)$.

- **Second Derivative:** $\frac{d^2y}{dx^2} = 12x^2 - 16$
- **Nature of Stationary Point(s):** At $x = 0$:

$$\begin{aligned} \frac{d^2y}{dx^2} &= 12(0)^2 - 16 \\ &= -16 \end{aligned}$$

  Since $\frac{d^2y}{dx^2} = -16 < 0$, the stationary point $(0, 5)$ is a **local**

maximum. At $x = 2$:

$$\begin{aligned}\frac{d^2y}{dx^2} &= 12(2)^2 - 16 \\ &= 48 - 16 \\ &= 32\end{aligned}$$

Since $\frac{d^2y}{dx^2} = 32 > 0$, the stationary point $(2, -11)$ is a **local minimum**. At $x = -2$:

$$\begin{aligned}\frac{d^2y}{dx^2} &= 12(-2)^2 - 16 \\ &= 48 - 16 \\ &= 32\end{aligned}$$

Since $\frac{d^2y}{dx^2} = 32 > 0$, the stationary point $(-2, -11)$ is a **local minimum**.

Optimisation: Open-Top Box Problem

A box has a square base of dimensions $x \times x$ and height y. The box is open at the top.

1. Write a formula for the surface area of the open-top box.

$$
\begin{aligned}
A &= (\text{Area of base}) + (\text{Area of 4 sides}) \\
A &= (x \cdot x) + 4(x \cdot y) \\
A &= x^2 + 4xy
\end{aligned}
$$

2. Suppose the surface area is fixed at 300 cm^2. Show that this constraint implies:

$$y = \frac{300}{4x} - \frac{x}{4}$$

Given $A = 300$:

$$
\begin{aligned}
300 &= x^2 + 4xy \\
300 - x^2 &= 4xy \\
\frac{300 - x^2}{4x} &= y \\
y &= \frac{300}{4x} - \frac{x^2}{4x} \\
y &= \frac{300}{4x} - \frac{x}{4}
\end{aligned}
$$

3. Write a formula for the volume of the box in terms of x and y.

$$
\begin{aligned}
V &= \text{length} \times \text{width} \times \text{height} \\
V &= x \cdot x \cdot y \\
V &= x^2 y
\end{aligned}
$$

4. Substitute the expression for y into the volume formula so the volume

is a function of x only. Substitute $y = \frac{300}{4x} - \frac{x}{4}$ into $V = x^2 y$:

$$
\begin{aligned}
V(x) &= x^2 \left(\frac{300}{4x} - \frac{x}{4} \right) \\
V(x) &= \frac{300x^2}{4x} - \frac{x^3}{4} \\
V(x) &= 75x - \frac{1}{4}x^3
\end{aligned}
$$

5. Find the value of x that maximises the volume. To maximise volume, find $\frac{dV}{dx}$ and set it to zero.

$$
\begin{aligned}
\frac{dV}{dx} &= \frac{d}{dx}\left(75x - \frac{1}{4}x^3\right) \\
&= 75 - \frac{3}{4}x^2
\end{aligned}
$$

Set $\frac{dV}{dx} = 0$:

$$
\begin{aligned}
75 - \frac{3}{4}x^2 &= 0 \\
75 &= \frac{3}{4}x^2 \\
300 &= 3x^2 \\
100 &= x^2 \\
x &= \pm 10
\end{aligned}
$$

Since x is a dimension, $x > 0$, so $x = 10$ cm.

To confirm this is a maximum, find the second derivative $\frac{d^2V}{dx^2}$.

$$
\begin{aligned}
\frac{d^2V}{dx^2} &= \frac{d}{dx}\left(75 - \frac{3}{4}x^2\right) \\
&= 0 - \frac{3}{4}(2x) \\
&= -\frac{3}{2}x
\end{aligned}
$$

At $x = 10$:

$$\left.\frac{d^2V}{dx^2}\right|_{x=10} = -\frac{3}{2}(10)$$
$$= -15$$

Since $\frac{d^2V}{dx^2} = -15 < 0$, the volume is maximised at $x = 10$ cm.

6. Calculate the corresponding height y and determine the maximum volume. Using $x = 10$ cm and $y = \frac{300}{4x} - \frac{x}{4}$:

$$\begin{aligned} y &= \frac{300}{4(10)} - \frac{10}{4} \\ &= \frac{300}{40} - 2.5 \\ &= 7.5 - 2.5 \\ &= 5 \text{ cm} \end{aligned}$$

The corresponding height is **5** cm.

Calculate the maximum volume using $V = x^2y$:

$$\begin{aligned} V_{\text{max}} &= (10 \text{ cm})^2 \cdot (5 \text{ cm}) \\ &= 100 \text{ cm}^2 \cdot 5 \text{ cm} \\ &= 500 \text{ cm}^3 \end{aligned}$$

The maximum volume is **500** cm**³**.