Expected Value and Utility Optimisation

LLE Mathematics and Statistics

Solutions

Solution to Question 1

a) The expected value (EV) is calculated as $EV = \sum x_i P(x_i)$.

$$
\begin{aligned}
EV &= (75 \times 0.4) + (25 \times 0.3) + (10 \times 0.3) \\
&= 30 + 7.5 + 3 \\
&= 40.5
\end{aligned}
$$

The expected value is 40.5.

b)

$$
\begin{aligned}
EV &= (5 \times 0.1) + (10 \times 0.2) + (15 \times 0.4) + (20 \times 0.3) \\
&= 0.5 + 2 + 6 + 6 \\
&= 14.5
\end{aligned}
$$

The expected value is 14.5.

c)

$$
\begin{aligned}
EV &= (100 \times 0.2) + (20 \times 0.5) + (-10 \times 0.3) \\
&= 20 + 10 - 3 \\
&= 27
\end{aligned}
$$

The expected value is 27.

Solution to Question 2

The investor's utility function is $U(w) = w^2$. Expected utility (EU) is calculated as $EU = \sum U(w_i)P(w_i)$.

$$\begin{aligned} EU &= U(600) \times 0.7 + U(450) \times 0.3 \\ &= (600^2 \times 0.7) + (450^2 \times 0.3) \\ &= (360000 \times 0.7) + (202500 \times 0.3) \\ &= 252000 + 60750 \\ &= 312750 \end{aligned}$$

The expected utility of this investment is 312750.

Solution to Question 3

a) Initial wealth is $£1,000$. Utility function is $U(w) = w$.

   i) Secure Investment: Wealth becomes $£1,100$ with 100% certainty.

$$\begin{aligned} EU_i &= U(1100) \times 1 \\ &= 1100 \times 1 \\ &= 1100 \end{aligned}$$

   i) Risky Investment: Wealth becomes $£1,300$ with 60% prob, or $£900$ with 40% prob.

$$\begin{aligned} EU_{ii} &= U(1300) \times 0.6 + U(900) \times 0.4 \\ &= 780 + 360 \\ &= 1140 \end{aligned}$$

Comparing $EU_i = 1100$ and $EU_{ii} = 1140$, the risky investment (ii) yields the greater expected utility.

b) Current capital is $£500$. Utility function is $U(c) = c^2$.

   i) Project Alpha: Capital remains $£500$ with 100% certainty.

$$\begin{aligned} EU_i &= U(500) \times 1 \\ &= 500^2 \times 1 \\ &= 250000 \end{aligned}$$

   i) Project Beta: Capital becomes $£700$ with 25% prob, or $£400$ with 75% prob.

$$\begin{aligned} EU_{ii} &= U(700) \times 0.25 + U(400) \times 0.75 \\ &= (700^2 \times 0.25) + (400^2 \times 0.75) \\ &= 122500 + 120000 \\ &= 242500 \end{aligned}$$

Comparing $EU_i = 250000$ and $EU_{ii} = 242500$, Project Alpha (i) yields the greater expected utility.

c) Current assets are $£10,000$. Utility function is $U(a) = (a)^{0.5}$. Probability of $£2,000$ loss is 10%.

i) Accept the risk without protection.

$$\begin{aligned}
EU_i &= U(10000) \times 0.9 + U(10000 - 2000) \times 0.1 \\
&= (10000)^{0.5} \times 0.9 + (8000)^{0.5} \times 0.1 \\
&= (100 \times 0.9) + (89.4427 \times 0.1) \\
&= 90 + 8.94427 \\
&= 98.94427
\end{aligned}$$

i) Purchase protection plan for £150: Loss is fully covered.

$$\begin{aligned}
EU_{ii} &= U(10000 - 150) \times 1 \\
&= (9850)^{0.5} \\
&= 99.24716
\end{aligned}$$

Comparing $EU_i = 98.94427$ and $EU_{ii} = 99.24716$, purchasing the protection plan (ii) yields the greater expected utility.

d) Initial support score is 100. Utility function is $U(s) = \text{ln}(s)$.

i) Policy X: Guarantees a support score of 110.

$$\begin{aligned}
EU_i &= U(110) \times 1 \\
&= \text{ln}(110) \\
&= 4.70048
\end{aligned}$$

i) Policy Y: Results in 130 with 50% prob, or 95 with 50% prob.

$$\begin{aligned}
EU_{ii} &= U(130) \times 0.5 + U(95) \times 0.5 \\
&= (\text{ln}(130) \times 0.5) + (\text{ln}(95) \times 0.5) \\
&= 2.433765 + 2.27694 \\
&= 4.710705
\end{aligned}$$

Comparing $EU_i = 4.70048$ and $EU_{ii} = 4.710705$, Policy Y (ii) yields the greater expected utility.

Solution to Question 4

Initial asset value: $£100,000$. Loss: $£80,000$ (to $£20,000$).
Probability of adverse event: $p_L = 0.05$. Coverage: $£x$. Premium: $£0.08x$.
Utility function: $U(\text{wealth}) = (\text{wealth})^{0.5}$.

a) Expected utility if they buy insurance coverage of $£x$:

- If no adverse event (prob 0.95):
  Wealth $= 100,000 - 0.08x$
- If adverse event (prob 0.05):
  Wealth $= 100,000 - 80,000 - 0.08x + x = 20,000 + 0.92x$

So, the expected utility $EU(x)$ is:

$$EU(x) = 0.95(100,000 - 0.08x)^{0.5} + 0.05(20,000 + 0.92x)^{0.5}$$

b) To maximize expected utility, we take the derivative of $EU(x)$ with respect to x and set it to zero.

$$\begin{aligned}\frac{dEU}{dx} &= 0.95 \times 0.5(100,000 - 0.08x)^{-0.5}(-0.08) \\ &\quad + 0.05 \times 0.5(20,000 + 0.92x)^{-0.5}(0.92) = 0\end{aligned}$$

$$\frac{-0.038}{(100,000 - 0.08x)^{0.5}} + \frac{0.023}{(20,000 + 0.92x)^{0.5}} = 0$$
$$\frac{0.023}{(20,000 + 0.92x)^{0.5}} = \frac{0.038}{(100,000 - 0.08x)^{0.5}}$$

Cross-multiply and square both sides:

$$\begin{aligned}
0.023(100,000 - 0.08x)^{0.5} &= 0.038(20,000 + 0.92x)^{0.5} \\
(0.023)^2(100,000 - 0.08x) &= (0.038)^2(20,000 + 0.92x) \\
0.000529(100,000 - 0.08x) &= 0.001444(20,000 + 0.92x) \\
52.9 - 0.00004232x &= 28.88 + 0.00132848x \\
52.9 - 28.88 &= 0.00132848x + 0.00004232x \\
24.02 &= 0.0013708x \\
x &= \frac{24.02}{0.0013708} \approx 17522.61
\end{aligned}$$

The optimal insurance coverage is $£17,522.61$.

The insurance premium will be $0.08x$:

$$\text{Premium} = 0.08 \times 17522.61 = £1,401.81$$

Solution to Question 5

Initial capital: $£10,000$. Investment amount: $£x$.
Utility function: $U(\text{capital}) = (\text{capital})^{0.75}$.

a) Express expected utility as a function of x: Final capital in states:

- Success (prob 0.6): Capital $= (10,000 - x) + 2.5x = 10,000 + 1.5x$
- Failure (prob 0.4): Capital $= (10,000 - x) + 0.5x = 10,000 - 0.5x$

So, the expected utility $EU(x)$ is:

$$EU(x) = 0.6(10,000 + 1.5x)^{0.75} + 0.4(10,000 - 0.5x)^{0.75}$$

b) To find the optimal x, take the derivative of $EU(x)$ with respect to x and set to zero.

$$\frac{dEU}{dx} = 0.6 \times 0.75(10,000 + 1.5x)^{-0.25}(1.5) + 0.4 \times 0.75(10,000 - 0.5x)^{-0.25}(-0.5) = 0$$

$$0.675(10,000 + 1.5x)^{-0.25} - 0.15(10,000 - 0.5x)^{-0.25} = 0$$
$$0.675(10,000 + 1.5x)^{-0.25} = 0.15(10,000 - 0.5x)^{-0.25}$$
$$\frac{0.675}{0.15} = \frac{(10,000 + 1.5x)^{0.25}}{(10,000 - 0.5x)^{0.25}}$$
$$4.5 = \left(\frac{10,000 + 1.5x}{10,000 - 0.5x}\right)^{0.25}$$

Raise both sides to the power of 4:

$$4.5^4 = \frac{10,000 + 1.5x}{10,000 - 0.5x}$$
$$410.0625 = \frac{10,000 + 1.5x}{10,000 - 0.5x}$$
$$410.0625(10,000 - 0.5x) = 10,000 + 1.5x$$
$$4,100,625 - 205.03125x = 10,000 + 1.5x$$
$$4,090,625 = 1.5x + 205.03125x$$
$$4,090,625 = 206.53125x$$
$$x = \frac{4,090,625}{206.53125} \approx 19806.31$$

The optimal investment amount is approximately £$19,806.31$.

Solution to Question 6

Initial savings: £$50,000$. Investment amount: £x.
Utility function: U(savings) $=$ ln(savings).

a) Formulate expected utility as a function of x: Final savings in states:

- Stock performs well (prob 0.7): Savings $= (50,000 - x) + 1.6x = 50,000 + 0.6x$
- Stock performs poorly (prob 0.3): Savings $= (50,000 - x) + 0.6x = 50,000 - 0.4x$

So, the expected utility $EU(x)$ is:

$$EU(x) = 0.7\ln(50,000 + 0.6x) + 0.3\ln(50,000 - 0.4x)$$

b) To find the optimal x, take the derivative of $EU(x)$ with respect to x and set to zero.

$$\frac{dEU}{dx} = 0.7 \times \frac{0.6}{50,000 + 0.6x} + 0.3 \times \frac{-0.4}{50,000 - 0.4x} = 0$$

$$\frac{0.42}{50,000 + 0.6x} - \frac{0.12}{50,000 - 0.4x} = 0$$

$$\frac{0.42}{50,000 + 0.6x} = \frac{0.12}{50,000 - 0.4x}$$

Cross-multiply:

$$\begin{aligned}
0.42(50,000 - 0.4x) &= 0.12(50,000 + 0.6x) \\
21,000 - 0.168x &= 6,000 + 0.072x \\
21,000 - 6,000 &= 0.072x + 0.168x \\
15,000 &= 0.24x \\
x &= \frac{15,000}{0.24} = 62,500
\end{aligned}$$

Since the initial savings are £$50,000$, and we cannot invest more than we have, the optimal investment amount is limited by the budget. Therefore, the optimal investment amount is £$50,000$.

Solution to Question 7

Initial wealth: $£2,000,000$. Investment amount: $£x$.
Utility function: $U(W) = -e^{-0.0000005W}$.

a) Write down expected utility as a function of x: Final wealth in states:

- Asset performs well (prob 0.5): Wealth $= (2,000,000 - x) + 1.8x = 2,000,000 + 0.8x$
- Asset performs poorly (prob 0.5): Wealth $= (2,000,000 - x) + 0.5x = 2,000,000 - 0.5x$

So, the expected utility $EU(x)$ is:

$$\begin{aligned} EU(x) &= 0.5(-e^{-0.0000005(2,000,000+0.8x)}) \\ &\quad + 0.5(-e^{-0.0000005(2,000,000-0.5x)}) \\ EU(x) &= -0.5e^{-1-0.0000004x} - 0.5e^{-1+0.00000025x} \end{aligned}$$

b) To find the optimal x, take the derivative of $EU(x)$ with respect to x and set to zero.

$$\begin{aligned} \frac{dEU}{dx} &= -0.5e^{-1-0.0000004x}(-0.0000004) \\ &\quad - 0.5e^{-1+0.00000025x}(0.00000025) = 0 \end{aligned}$$

$$\begin{aligned} 0.0000002e^{-1-0.0000004x} &= 0.000000125e^{-1+0.00000025x} \\ \frac{e^{-1-0.0000004x}}{e^{-1+0.00000025x}} &= \frac{0.000000125}{0.0000002} \\ e^{(-1-0.0000004x)-(-1+0.00000025x)} &= 0.625 \\ e^{-0.00000065x} &= 0.625 \end{aligned}$$

Take the natural logarithm of both sides:

$$
\begin{aligned}
-0.00000065x &= \ln(0.625) \\
-0.00000065x &= -0.4700036 \\
x &= \frac{-0.4700036}{-0.00000065} \approx 723082.46
\end{aligned}
$$

The optimal investment amount is approximately $\pounds 723,082.46$.