Expected Value and Utility Optimisation

LLE Mathematics and Statistics

1. Find the expected values for the following:

   a) A game pays 75 with probability 0.4, 25 with probability 0.3, and 10 with probability 0.3.

   b) A variable X takes values $5, 10, 15, 20$ with probabilities $0.1, 0.2, 0.4, 0.3$ respectively.

   c) A random process yields 100 with 20% chance, 20 with 50% chance, and -10 with 30% chance.

2. An investor has current wealth of $£500$ and utility function $U(w) = w^2$. They invest in a venture. Their total wealth will become $£600$ with 70% probability, or $£450$ with 30% probability. Calculate the expected utility of this investment.

3. For each of the options below, find the option that yields the greatest expected utility:

   a) An individual has current wealth $£1,000$ and utility function $U(w) = w$. They consider two investment choices:

      i) A secure investment where total wealth becomes $£1,100$ with 100% certainty.

      i) A risky investment where total wealth becomes $£1,300$ with 60% probability, or $£900$ with 40% probability.

   b) A decision-maker has $£500$ in current capital and utility function $U(c) = c^2$. They evaluate two project proposals:

      i) Project Alpha: Capital remains $£500$ with 100% certainty.

i) Project Beta: Capital becomes £700 with 25% probability, or £400 with 75% probability.

c) A firm with £10,000 in assets has utility function $U(a) = \sqrt{a}$. They face a 10% chance of a £2,000 loss due to an event. They can either:

  i) Accept the risk without protection.

  i) Purchase a protection plan for a premium of £150 that fully covers the £2,000 loss if the event occurs.

d) A policy maker with initial support score of 100 has utility function $U(s) = \ln(s)$. They are choosing between two policies:

  i) Policy X: Guarantees a support score of 110.

  i) Policy Y: Results in a support score of 130 with 50% probability, or 95 with 50% probability.

4. An individual has just purchased an asset worth £100,000. If a specific adverse event occurs, the asset's value would drop to £20,000. The probability of this event is 5%. An insurance company offers to pay £x in the event of the adverse event, if the individual pays an insurance premium of £$0.08x$. The premium must be paid regardless of whether the event occurs. The individual's utility function is $U(\text{wealth}) = (\text{wealth})^{0.5}$.

  a) Write down the individual's expected utility if they buy insurance coverage of £x.

  b) How much insurance (value of x) will the individual buy to maximize their expected utility, and what will their insurance premium be?

5. An investor has £10,000. They can invest an amount £x in a new project. With 60% probability, the project pays back $2.5x$ (a gain of $1.5x$). With 40% probability, the project results in a loss of $0.5x$ (meaning $0.5x$ is returned, $0.5x$ is lost). The money not invested

remains unchanged. The investor's utility function is
$U(\text{wealth}) = (\text{wealth})^{0.75}$.

a) Express the investor's expected utility as a function of the amount $£x$ invested.

b) Determine the optimal amount $£x$ that maximizes their expected utility.

6. A retired individual has $£50,000$ in savings. Their utility for total savings is $U(\text{savings}) = \ln(\text{savings})$. They are considering investing an amount $£x$ into a particular stock. If the stock performs well (with 70% probability), the invested amount yields a 60% return (i.e., $1.6x$). If it performs poorly (with 30% probability), they lose 40% of the invested amount (i.e., $0.6x$ remains). The rest of their savings are held securely and do not change.

a) Formulate the individual's expected utility as a function of the investment amount $£x$.

b) Calculate the amount $£x$ that the individual should invest to maximize their expected utility.

7. A fund manager has a current wealth of $£2,000,000$. Their utility for wealth is $U(W) = -e^{-0.0000005W}$. They can invest an amount $£x$ (where $x \geq 0$) into a highly volatile asset. This asset yields a return such that the invested x becomes $1.8x$ with 50% probability, or $0.5x$ with 50% probability. The rest of their wealth $(£2,000,000 - x)$ is kept in a risk-free account.

a) Write down the fund manager's expected utility as a function of the investment amount $£x$.

b) Determine the optimal investment amount $£x$ that maximizes their expected utility.