Optimisation (Single Variable Functions)

LLE Mathematics and Statistics

Differentiation Practice

For each of the functions below, find the first and second derivative:

1. $y = 5x^3 + 2x^2 - 8x + 3$

   Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

2. $m = 4x - 3 + \frac{2}{x} - \frac{4}{x^2}$

   Find $\frac{dm}{dx}$ and $\frac{d^2m}{dx^2}$

3. $f(t) = 24 + 4\sqrt{t}$

   Find $f'(t)$ and $f''(t)$

4. $P = t^3(5t^3 - 4t^{-3/2})$

   Find $\frac{dP}{dt}$ and $\frac{d^2P}{dt^2}$

5. $g(x) = 3x^5 - 4x^2 + \frac{7}{x}$ (domain: $x > 0$)

   Find $\frac{dg}{dx}$ and $\frac{d^2g}{dx^2}$

6. $y = 10x^4 + 6x^{1/3} - \frac{3}{x^2}$

   Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

Stationary Points

A point is stationary if its gradient is equal to zero, i.e. $\frac{dy}{dx} = 0$.

Substituting the stationary point into the second derivative of the function determines whether the stationary point is a local minimum, local maximum, or a point of inflection:

- $\frac{d^2y}{dx^2} < 0 \quad \Rightarrow$ **Local maximum**
- $\frac{d^2y}{dx^2} > 0 \quad \Rightarrow$ **Local minimum**
- $\frac{d^2y}{dx^2} = 0$ **and** $\frac{d^3y}{dx^3} \neq 0 \quad \Rightarrow$ **Point of inflection**

For each function below:

- Find the first derivative
- Solve where the first derivative equals zero, and find the coordinates of the stationary point(s)
- Find the second derivative
- Substitute result(s) into the second derivative to determine the nature of the stationary point(s)
- Plot the functions using an online graphing tool (e.g. Desmos)

Functions:

1. $y = x^2 + 8x - 9$
2. $y = 2x^3 + 6x^2 - 90x - 2$
3. $y = 4x^{3/2} - x^2$

4. $y = x \ln x$ (use the product rule)
5. $y = 6x^{2/3} - 4x$ (domain: $x > 0$)
6. $y = x^3 - 3x$
7. $y = x + \frac{3}{x} + 2$
8. $y = x^3 - 6x^2 + 9x + 5$
9. $y = x^4 - 8x^2 + 5$

Optimisation: Open-Top Box Problem

A box has a square base of dimensions $x \times x$ and height y. The box is open at the top.

1. Write a formula for the surface area of the open-top box.
2. Suppose the surface area is fixed at 300 cm^2. Show that this constraint implies:
$$y = \frac{300}{4x} - \frac{x}{4}$$
3. Write a formula for the volume of the box in terms of x and y.
4. Substitute the expression for y into the volume formula so the volume is a function of x only.
5. Find the value of x that maximises the volume.
6. Calculate the corresponding height y and determine the maximum volume.