Profit and Profit Maximisation for Monopolies

LLE Mathematics and Statistics

1. You are given the demand function and the total cost function for a monopolist. Use this information to calculate the profit function, π, and the price, quantity, and profit that maximises the profit.

   (a) Market demand: $Q = 100 - P$
   Re-arrange: $P = 100 - Q$
   $TR = PQ = (100 - Q)Q = 100Q - Q^2$

      i. $TC = 250 + 20Q$

$$\begin{aligned} \pi &= TR - TC = 100Q - Q^2 - 250 - 20Q \\ &= 80Q - Q^2 - 250 \\ \frac{\mathrm{d}\pi}{\mathrm{d}Q} &= 80 - 2Q \end{aligned}$$

For max profit, set $\frac{\mathrm{d}\pi}{\mathrm{d}Q} = 0$

$$\begin{aligned} 80 - 2Q &= 0 \\ 2Q &= 80 \\ Q &= 40 \\ \implies P &= 100 - Q = 60 \\ \implies \pi &= 80(40) - 40^2 - 250 = 1350 \end{aligned}$$

ii. $TC = Q^2 - 4Q + 200$

$$\begin{aligned}
\pi &= 100Q - Q^2 - Q^2 + 4Q - 200 \\
&= -2Q^2 + 104Q - 200 \\
\frac{\mathsf{d}\pi}{\mathsf{d}Q} &= 0 \\
\implies -4Q + 104 &= 0 \\
\implies 4Q &= 104 \\
\implies Q &= 26 \\
\implies P &= 74 \\
\implies \pi &= -2(26)^2 + 104(26) - 200 = 1152
\end{aligned}$$

iii. $TC = 100 + 2\,\mathsf{ln}(Q+1)$

$$\begin{aligned}
\pi &= 100Q - Q^2 - 100 - 2\,\mathsf{ln}(Q+1) \\
\frac{\mathsf{d}\pi}{\mathsf{d}Q} &= 100 - 2Q - \frac{2}{Q+1} = 0 \\
100(Q+1) - 2Q(Q+1) - 2 &= 0 \\
100Q + 100 - 2Q^2 - 2Q - 2 &= 0 \\
-2Q^2 + 98Q + 98 &= 0 \\
Q^2 - 49Q - 49 &= 0 \\
Q &= \frac{49 \pm \sqrt{49^2 - 4(1)(-49)}}{2(1)}
\end{aligned}$$

Use the answer $Q \approx 50$, $P \approx 50$

$$\pi = 100(50) - 50^2 - 100 - 2\,\mathsf{ln}(51) = 2392$$

(b) Market Demand: $Q = 500e^{-0.2P}$

$$\begin{aligned}
\frac{Q}{500} &= e^{-0.2P} \\
\ln\left(\frac{Q}{500}\right) &= -0.2P \\
-5\ln\left(\frac{Q}{500}\right) &= P
\end{aligned}$$

$$TR = -5Q\ln\left(\frac{Q}{500}\right)$$

i. $TC = 100$

$$\begin{aligned}
\pi &= -5Q\ln\left(\frac{Q}{500}\right) - 100 \\
\frac{d\pi}{dQ} &= -5\ln\left(\frac{Q}{500}\right) - 5Q\frac{1}{Q} \qquad \text{:product rule} \\
&= -5\ln\left(\frac{Q}{500}\right) - 5
\end{aligned}$$

$$\begin{aligned}
\frac{d\pi}{dQ} &= 0 \\
5\ln\left(\frac{Q}{500}\right) &= -5 \\
\ln\left(\frac{Q}{500}\right) &= -1 \\
\frac{Q}{500} &= e^{-1} \\
Q &= 500e^{-1} \approx 184
\end{aligned}$$

$$\begin{aligned}
Q &= 500e^{-0.2P} \\
500e^{-1} &= 500e^{-0.2P} \\
-1 &= -0.2P \\
P &= 5
\end{aligned}$$

$$\pi \approx -5(184) \ln \frac{184}{500} - 100$$
$$\approx 820$$

ii. $TC = 10 + 15Q$

$$\begin{aligned}
\pi &= -5Q \ln\left(\frac{Q}{500}\right) - 10 - 15Q \\
\frac{\mathrm{d}\pi}{\mathrm{d}Q} &= -5 \ln\left(\frac{Q}{500}\right) - 5Q\frac{1}{Q} - 15 \\
&= -5 \ln\left(\frac{Q}{500}\right) - 20
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}\pi}{\mathrm{d}Q} &= 0 \\
5 \ln\left(\frac{Q}{500}\right) &= -20 \\
\ln\left(\frac{Q}{500}\right) &= -4 \\
\frac{Q}{500} &= e^{-4} \\
Q &= 500e^{-4} \approx 9
\end{aligned}$$

$$\begin{aligned}
Q &= 500e^{-0.2P} \\
500e^{-4} &= 500e^{-0.2P} \\
-4 &= -0.2P \\
P &= 20
\end{aligned}$$

$$\pi \approx -5(9) \ln \frac{9}{500} - 10 - 15(9)$$
$$\approx 36$$

2. A fish a chip shop knows demand for cod and chips is different for pensioners compared to those who are not pensioners.

Demand for pensioners: $Q_P = 200 - 4P_P$ where P_P is the price charged to pensioners.
Demand for others: $Q_N = 160 - 2P_N$ where P_N is the price charged to non-pensioners.
The shop has fixed costs of 100 and variable costs of $0.5Q$

Find the maximum profits if:

(a) The shop charges the same price to both groups, $P_P = P_N$

$$\begin{aligned}
P &= P_P = P_N \\
Q &= Q_P + Q_N \\
&= 200 - 4P + 160 - 2P \\
&= 360 - 6P
\end{aligned}$$

$$\begin{aligned}
TR &= PQ = 360P - 6P^2 \\
TC &= 100 + 0.5Q = 280 - 3P \\
\pi &= TR - TC = -6P^2 + 363P - 280
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}\pi}{\mathrm{d}P} &= -12P + 363 \\
\frac{\mathrm{d}\pi}{\mathrm{d}P} &= 0 \implies P = \frac{363}{12} = 30.25 \\
Q &= 360 - 6(30.25) = 178.5 \approx 179 \\
\pi &= -6(30.25)^2 + 363(30.25) - 280 \approx 5210
\end{aligned}$$

(b) The shop charges different prices to both groups.

$$\begin{aligned}
TR_P &= P_P Q_P \\
&= P_P(200 - 4P_P) \\
&= 200P_P - 4P_P^2 \\
TR_N &= P_N Q_N \\
&= P_N(160 - 2P_N) \\
&= 160P_N - 2P_N^2 \\
TR &= 200P_P - 4P_P^2 + 160P_N - 2P_N^2 \\
TC &= 100 + 0.5(Q_P + Q_N) \\
&= 100 + 100 - 2P_P + 80 - P_N \\
&= 280 - 2P_P - P_N
\end{aligned}$$

$$\begin{aligned}
\pi &= 200P_P - 4P_P^2 + 160P_N - 2P_N^2 - 280 + 2P_P + P_N \\
&= 202P_P - 4P_P^2 + 161P_N - 2P_N^2 - 280
\end{aligned}$$

Solve $\frac{\partial \pi}{\partial P_P} = 0$ and $\frac{\partial \pi}{\partial P_N} = 0$

$$\frac{\partial \pi}{\partial P_P} = 202 - 8P_P = 0 \tag{1}$$

$$\frac{\partial \pi}{\partial P_N} = 161 - 4P_N = 0 \tag{2}$$

$$\begin{aligned}
(1) &\implies P_P = \frac{202}{8} = 25.25 \\
(2) &\implies P_N = \frac{161}{4} = 40.25
\end{aligned}$$

$$\begin{aligned}
Q_P &= 200 - 4(25.25) = 99 \\
Q_N &= 160 - 2(40.25) = 79.5 \approx 80
\end{aligned}$$

$$
\begin{aligned}
\pi &= 202(25.25) - 4(25.25)^2 + 161(40.25) - 2(40.25)^2 - 280 \\
&\approx 5510
\end{aligned}
$$

There is more maximum profit to be had by having different prices. If they charge 30.25 to both groups, they will sell 179 at a profit of 5210. If they charge 20.25 to pensioners and 40.25 to non-pensioners, they will sell 99 to pensioners and 80 to non-pensioners (a total of 179) at a profit of 5510.