# Constrained Optimisation Solutions

LLE – Mathematics and Statistics Skills

1. Given the function:

$$z = x^2 + 2xy + 4y + 10$$

with constraint:

$$x + y = 100$$

Lagrangian:

$$L(x, y, \lambda) = x^2 + 2xy + 4y + 10 + \lambda(100 - x - y)$$

FOC:

$$\begin{aligned} L_x &= 2x + 2y - \lambda = 0 \\ L_y &= 2x + 4 - \lambda = 0 \\ L_\lambda &= 100 - x - y = 0 \end{aligned}$$

Make $\lambda$ the subject of $L_x$ and $L_y$ and divide:

$$\begin{aligned} 2x + 2y &= \lambda \\ 2x + 4 &= \lambda \\ \frac{2x + 2y}{2x + 4} &= \frac{\lambda}{\lambda} \end{aligned}$$

Rearranging gives:

$$2x + 2y = 2x + 4 \implies 2y = 4 \implies y = 2$$

Substitute into constraint:

$$x + 2 = 100 \implies x = 98$$

Substitute into $z$:

$$z = 98^2 + 2 \times 98 + 4 \times 2 + 10 = 9818$$

Optimal solution $x = 98$, $y = 2$, $z = 9818$

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2. Given the function:

$$z = 12x + 4y$$

with constraint:

$$x^{0.3}y^{0.7} = 1000$$

Optimise the function $z$ subject to the constraint.

$$L = 12x + 4y + \lambda(1000 - x^{0.3}y^{0.7})$$

$$\begin{aligned} L_x &= 12 - 0.3\lambda x^{-0.7}y^{0.7} = 0 \\ L_y &= 4 - 0.7\lambda x^{0.3}y^{-0.3} = 0 \\ L_\lambda &= 1000 - x^{0.3}y^{0.7} = 0 \end{aligned}$$

$$\begin{aligned} 12 &= 0.3\lambda x^{-0.7}y^{0.7} \\ 4 &= 0.7\lambda x^{0.3}y^{-0.3} \end{aligned}$$

$$\begin{aligned} \frac{12}{4} &= \frac{3y}{7x} \\ y &= 7x \end{aligned}$$

Substitute $y = 7x$ into constraint and solve:

$$\begin{aligned} x^{0.3}(7x)^{0.7} &= 1000 \\ 7^{0.7}x &= 1000 \\ x &= \frac{1000}{7^{0.7}} = 256.1 \\ y &= 7x = 1792.8 \\ z &= 12x + 4y = 10244 \end{aligned}$$

Optimal solution $x = 256.1$, $y = 1792.8$, $z = 10244$

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3. Given the function:

$$z = x^{0.6}y^{0.4}$$

with constraint:

$$5x + 8y = 120$$

Optimise the function $z$ subject to the constraint.

$$L = x^{0.6}y^{0.4} + \lambda(120 - 5x - 8y)$$

$$\begin{aligned} L_x &= 0.6x^{-0.4}y^{0.4} - 5\lambda = 0 \\ L_y &= 0.4x^{0.6}y^{-0.6} - 8\lambda = 0 \\ L_\lambda &= 120 - 5x - 8y = 0 \end{aligned}$$

$$\begin{aligned} 0.6x^{-0.4}y^{0.4} &= 5\lambda \\ 0.4x^{0.6}y^{-0.6} &= 8\lambda \end{aligned}$$

$$\begin{aligned} \frac{3y}{2x} &= \frac{5}{8} \\ y &= \frac{5}{12}x \end{aligned}$$

Substitute into constraint:

$$\begin{aligned} 5x + 8\left(\frac{5}{12}x\right) &= 120 \\ 5x + \frac{10}{3}x &= 120 \\ \frac{25}{3}x &= 120 \\ x &= 14.4 \\ y &= 6 \\ z &= 10.1 \end{aligned}$$

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4. A consumer has the following utility function for two goods $x$ and $y$:

$$U(x, y) = x^{0.5}y^{0.5}$$

The consumer has an income of 100. The price of good $x$ is $p_x = 2$ and the price of good $y$ is $p_y = 4$.
Constraint: $2x + 4y = 100$

$$L = x^{0.5}y^{0.5} + \lambda(100 - 2x - 4y)$$

$$\begin{aligned} L_x &= 0.5x^{-0.5}y^{0.5} - 2\lambda = 0 \\ L_y &= 0.5x^{0.5}y^{-0.5} - 4\lambda = 0 \\ L_\lambda &= 100 - 2x - 4y = 0 \end{aligned}$$

$$\begin{aligned} \frac{0.5x^{-0.5}y^{0.5}}{0.5x^{0.5}y^{-0.5}} &= \frac{2\lambda}{4\lambda} \\ \frac{y}{x} &= \frac{1}{2} \\ x &= 2y \end{aligned}$$

Substitute into constraint and solve:

$$\begin{aligned}
2(2y) + 4y &= 100 \\
8y &= 100 \\
y &= 12.5 \\
x &= 25 \\
U &= 17.7
\end{aligned}$$

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5. The production function for output ($Q$), for two inputs of labour ($L$) and capital ($K$) is given by:

$$Q = L^{\alpha} K^{\beta}$$

The cost of labour is 10 and the cost of capital is 20.

(a) Production constraint of an output of 100 units
Minimize $C = 10L + 20K$ subject to $L^{\alpha}K^{\beta} = 100$

$$L = 10L + 20K + \lambda(100 - L^{\alpha}K^{\beta})$$

$$\begin{aligned}
L_L &= 10 - \lambda\alpha L^{\alpha-1}K^{\beta} = 0 \\
L_K &= 20 - \lambda\beta L^{\alpha}K^{\beta-1} = 0 \\
L_{\lambda} &= 100 - L^{\alpha}K^{\beta} = 0
\end{aligned}$$

$$\begin{aligned}
\frac{10}{20} &= \frac{\lambda\alpha L^{\alpha-1}K^{\beta}}{\lambda\beta L^{\alpha}K^{\beta-1}} \\
\frac{1}{2} &= \frac{\alpha K}{\beta L} \\
L &= \frac{2\alpha K}{\beta}
\end{aligned}$$

Substitute into constraint and solve:

$$
\begin{aligned}
\left(\frac{2\alpha K}{\beta}\right)^{\alpha} K^{\beta} &= 100 \\
\left(\frac{2\alpha}{\beta}\right)^{\alpha} K^{\alpha+\beta} &= 100 \\
K^{\alpha+\beta} &= 100\left(\frac{2\alpha}{\beta}\right)^{-\alpha} \\
K = K^{*} &= \left(100\left(\frac{2\alpha}{\beta}\right)^{-\alpha}\right)^{\frac{1}{\alpha+\beta}}
\end{aligned}
$$

$$
\begin{aligned}
L &= \frac{2\alpha K}{\beta} \\
L &= \left(\frac{2\alpha}{\beta}\right)^{\frac{\alpha+\beta}{\alpha+\beta}}\left(100\left(\frac{2\alpha}{\beta}\right)^{-\alpha}\right)^{\frac{1}{\alpha+\beta}} \\
L &= \left(100\left(\frac{2\alpha}{\beta}\right)^{\alpha+\beta}\left(\frac{2\alpha}{\beta}\right)^{-\alpha}\right)^{\frac{1}{\alpha+\beta}} \\
L = L^{*} &= \left(100\left(\frac{2\alpha}{\beta}\right)^{\beta}\right)^{\frac{1}{\alpha+\beta}}
\end{aligned}
$$

(b) Find the optimal values and the minimum cost when:

i. $\alpha = \beta = 0.5$

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\begin{aligned}
K^{*} &= \left(100\left(\frac{2\times 0.5}{0.5}\right)^{-0.5}\right)^{\frac{1}{0.5+0.5}} = 70.7 \\
L^{*} &= \left(100\left(\frac{2\times 0.5}{0.5}\right)^{0.5}\right)^{\frac{1}{0.5+0.5}} = 141.4 \\
C^{8} &= 10\times 141.4 + 20\times 70.7 = 2828
\end{aligned}
$$

ii. $\alpha = 0.3$, $\beta = 0.7$

$$K^* = 104.7$$
$$L^* = 89.8$$
$$C^* = 2992$$

6. A single good is produced by a monopoly. The demand function for price ($P$) from quantity ($Q$) for this good is given by:

$$P = 50 - Q$$

The cost function for the company producing the good is given by:

$$C = 10 + 2Q$$

Maximise $\Pi = PQ - C$ subject to $P = 50 - Q$

$$\Pi = PQ - (10 + 2Q) = PQ - 10 - 2Q$$

$$L = PQ - 10 - 2Q + \lambda(50 - Q - P)$$

$$L_P = Q - \lambda = 0$$
$$L_Q = P - 2 - \lambda = 0$$
$$L_\lambda = 50 - Q - P = 0$$

$$Q = \lambda$$
$$P - 2 = \lambda$$
$$Q = P - 2$$

Substitute into constraint and solve:

$$\begin{aligned} P &= = 50 - (P - 2) \\ 2P &= 52 \\ P &= 26 \\ Q &= 24 \\ \Pi &= 566 \end{aligned}$$