Hicksian Demand and Expenditure Functions

LLE – Mathematics and Statistics Skills

Worked Solutions

1. Consider a consumer with the utility function $U(x, y) = xy$, where x and y are two goods. Let p_x and p_y denote their respective prices, and let $\bar{U}$ be the target level of utility.

   (a) **Derive the Hicksian demand functions:**

   The expenditure minimization problem is to minimize $E = p_x x + p_y y$ subject to the utility constraint $U(x, y) = xy = \bar{U}$.

   The Lagrangian is:

   $$L(x, y, \lambda) = p_x x + p_y y + \lambda(\bar{U} - xy)$$

   The first-order conditions (FOCs) are:

   $$\begin{aligned}
   \frac{\partial L}{\partial x} &= p_x - \lambda y = 0 \quad &\implies p_x = \lambda y \quad (1) \\
   \frac{\partial L}{\partial y} &= p_y - \lambda x = 0 \quad &\implies p_y = \lambda x \quad (2) \\
   \frac{\partial L}{\partial \lambda} &= \bar{U} - xy = 0 \quad &\implies xy = \bar{U} \quad (3)
   \end{aligned}$$

   Divide equation (1) by equation (2) to find the optimal ratio of goods:

   $$\frac{p_x}{p_y} = \frac{\lambda y}{\lambda x} = \frac{y}{x}$$

   This implies $y = \frac{p_x}{p_y} x$.

Substitute this expression for y into the utility constraint (3):

$$x\left(\frac{p_x}{p_y}x\right) = \bar{U}$$
$$\frac{p_x}{p_y}x^2 = \bar{U}$$

Solving for x, we get the Hicksian demand for x:

$$x_H(p_x, p_y, \bar{U}) = \sqrt{\frac{p_y\bar{U}}{p_x}}$$

Substitute x_H back into $y = \frac{p_x}{p_y}x$ to find the Hicksian demand for y:

$$\begin{aligned} y_H(p_x, p_y, \bar{U}) &= \frac{p_x}{p_y}\sqrt{\frac{p_y\bar{U}}{p_x}} \\ &= \sqrt{\frac{p_x^2 p_y \bar{U}}{p_y^2 p_x}} \\ &= \sqrt{\frac{p_x\bar{U}}{p_y}} \end{aligned}$$

(b) **Derive the general expenditure function:** The expenditure function $E(p_x, p_y, \bar{U})$ is obtained by substituting the Hicksian demand functions (x_H and y_H) back into the objective function $E = p_x x + p_y y$:

$$\begin{aligned} E(p_x, p_y, \bar{U}) &= p_x\left(\sqrt{\frac{p_y\bar{U}}{p_x}}\right) + p_y\left(\sqrt{\frac{p_x\bar{U}}{p_y}}\right) \\ &= \sqrt{p_x^2\frac{p_y\bar{U}}{p_x}} + \sqrt{p_y^2\frac{p_x\bar{U}}{p_y}} \\ &= \sqrt{p_x p_y \bar{U}} + \sqrt{p_x p_y \bar{U}} \\ &= 2\sqrt{p_x p_y \bar{U}} \end{aligned}$$

(c) **Numerical Applications (using solutions from parts (a) and (b)):**

(i) **Scenario:** $\bar{U} = 100$. **Initial** $p_x = 1, p_y = 1$; **New** $p_x = 4$:

Initial Hicksian demands and minimum expenditure:

$$x_H = \sqrt{\frac{1 \cdot 100}{1}} = \sqrt{100} = 10$$
$$y_H = \sqrt{\frac{1 \cdot 100}{1}} = \sqrt{100} = 10$$
$$E_0 = 2\sqrt{1 \cdot 1 \cdot 100} = 2\sqrt{100} = 2 \cdot 10 = 20$$

New Hicksian demands and minimum expenditure:

$$x'_H = \sqrt{\frac{1 \cdot 100}{4}} = \sqrt{25} = 5$$
$$y'_H = \sqrt{\frac{4 \cdot 100}{1}} = \sqrt{400} = 20$$
$$E_1 = 2\sqrt{4 \cdot 1 \cdot 100} = 2\sqrt{400} = 2 \cdot 20 = 40$$

Compensating Variation (CV):

$$CV = E_1 - E_0 = 40 - 20 = 20$$

(ii) **Scenario: $\bar{U} = 1000$. Initial $p_x = 1, p_y = 1$; New $p_x = 5$:**
Initial Hicksian demands and minimum expenditure:

$$x_H = \sqrt{\frac{1 \cdot 1000}{1}} = \sqrt{1000} = 10\sqrt{10} \approx 31.62$$
$$y_H = \sqrt{\frac{1 \cdot 1000}{1}} = \sqrt{1000} = 10\sqrt{10} \approx 31.62$$
$$E_0 = 2\sqrt{1 \cdot 1 \cdot 1000} = 2\sqrt{1000} = 20\sqrt{10} \approx 63.25$$

New Hicksian demands and minimum expenditure:

$$x'_H = \sqrt{\frac{1 \cdot 1000}{5}} = \sqrt{200} = 10\sqrt{2} \approx 14.14$$
$$y'_H = \sqrt{\frac{5 \cdot 1000}{1}} = \sqrt{5000} = 50\sqrt{2} \approx 70.71$$
$$E_1 = 2\sqrt{5 \cdot 1 \cdot 1000} = 2\sqrt{400} = 2 \cdot 20 = 40$$

Compensating Variation (CV):

$$CV = E_1 - E_0 = 40 - 20 = 20$$

(iii) **Scenario:** $\bar{U} = 36$**. Initial** $p_x = 1, p_y = 2$**; New** $p_x = 3$**:**
Initial Hicksian demands and minimum expenditure:

$$x_H = \sqrt{\frac{2 \cdot 36}{1}} = \sqrt{72} = 6\sqrt{2} \approx 8.49$$
$$y_H = \sqrt{\frac{1 \cdot 36}{2}} = \sqrt{18} = 3\sqrt{2} \approx 4.24$$
$$E_0 = 2\sqrt{1 \cdot 2 \cdot 36} = 2\sqrt{72} = 12\sqrt{2} \approx 16.97$$

New Hicksian demands and minimum expenditure:

$$x'_H = \sqrt{\frac{2 \cdot 36}{3}} = \sqrt{24} = 2\sqrt{6} \approx 4.90$$
$$y'_H = \sqrt{\frac{3 \cdot 36}{2}} = \sqrt{54} = 3\sqrt{6} \approx 7.35$$
$$E_1 = 2\sqrt{3 \cdot 2 \cdot 36} = 2\sqrt{216} = 12\sqrt{6} \approx 29.39$$

Compensating Variation (CV):

$$CV = E_1 - E_0 = 12\sqrt{6} - 12\sqrt{2} = 12(\sqrt{6} - \sqrt{2}) \approx 29.39 - 16.97 = 12.42$$

(iv) **Scenario:** $\bar{U} = 256$**. Initial** $p_x = 1, p_y = 1$**; New** $p_y = 4$**:**
Initial Hicksian demands and minimum expenditure:

$$x_H = \sqrt{\frac{1 \cdot 256}{1}} = \sqrt{256} = 16$$
$$y_H = \sqrt{\frac{1 \cdot 256}{1}} = \sqrt{256} = 16$$
$$E_0 = 2\sqrt{1 \cdot 1 \cdot 256} = 2\sqrt{256} = 2 \cdot 16 = 32$$

New Hicksian demands and minimum expenditure:

$$\begin{aligned}
x'_H &= \sqrt{\frac{4 \cdot 256}{1}} = \sqrt{1024} = 32 \\
y'_H &= \sqrt{\frac{1 \cdot 256}{4}} = \sqrt{64} = 8 \\
E_1 &= 2\sqrt{1 \cdot 4 \cdot 256} = 2\sqrt{1024} = 2 \cdot 32 = 64
\end{aligned}$$

Compensating Variation (CV):

$$CV = E_1 - E_0 = 64 - 32 = 32$$

2. Consider a consumer with the utility function $U(x, y) = x^{1/2}y^{1/2}$. Let p_x and p_y denote the prices of x and y, and let $\bar{U}$ be the target level of utility.

   (a) **Derive the Hicksian demand functions:** The expenditure minimization problem is to minimize $E = p_x x + p_y y$ subject to $x^{1/2}y^{1/2} = \bar{U}$.

   The Lagrangian is:

$$L(x, y, \lambda) = p_x x + p_y y + \lambda(\bar{U} - x^{1/2}y^{1/2})$$

   The first-order conditions (FOCs) are:

$$\begin{aligned}
\frac{\partial L}{\partial x} &= p_x - \lambda \cdot \frac{1}{2}x^{-1/2}y^{1/2} = 0 \quad \Longrightarrow \; p_x = \lambda \frac{1}{2}\sqrt{\frac{y}{x}} \quad (1) \\
\frac{\partial L}{\partial y} &= p_y - \lambda \cdot \frac{1}{2}x^{1/2}y^{-1/2} = 0 \quad \Longrightarrow \; p_y = \lambda \frac{1}{2}\sqrt{\frac{x}{y}} \quad (2) \\
\frac{\partial L}{\partial \lambda} &= \bar{U} - x^{1/2}y^{1/2} = 0 \quad \Longrightarrow \; x^{1/2}y^{1/2} = \bar{U} \quad (3)
\end{aligned}$$

   Divide equation (1) by equation (2):

$$\begin{aligned}
\frac{p_x}{p_y} &= \frac{\lambda \frac{1}{2}\sqrt{y/x}}{\lambda \frac{1}{2}\sqrt{x/y}} \\
&= \frac{\sqrt{y/x}}{\sqrt{x/y}} \\
&= \sqrt{\frac{y}{x} \cdot \frac{y}{x}} = \sqrt{\left(\frac{y}{x}\right)^2} = \frac{y}{x}
\end{aligned}$$

This implies $y = \frac{p_x}{p_y} x$.
Substitute this relationship into the utility constraint (3):

$$
\begin{aligned}
x^{1/2} \left(\frac{p_x}{p_y} x \right)^{1/2} &= \bar{U} \\
x^{1/2} \left(\frac{p_x}{p_y} \right)^{1/2} x^{1/2} &= \bar{U} \\
x^{(1/2+1/2)} \left(\frac{p_x}{p_y} \right)^{1/2} &= \bar{U} \\
x \left(\frac{p_x}{p_y} \right)^{1/2} &= \bar{U}
\end{aligned}
$$

Solving for x, we get the Hicksian demand for x:

$$x_H(p_x, p_y, \bar{U}) = \bar{U} \left(\frac{p_y}{p_x} \right)^{1/2} = \bar{U} \sqrt{\frac{p_y}{p_x}}$$

Substitute x_H back into $y = \frac{p_x}{p_y} x$ to find the Hicksian demand for y:

$$
\begin{aligned}
y_H(p_x, p_y, \bar{U}) &= \frac{p_x}{p_y} \left(\bar{U} \sqrt{\frac{p_y}{p_x}} \right) \\
&= \bar{U} \sqrt{\frac{p_x^2 p_y}{p_y^2 p_x}} \\
&= \bar{U} \sqrt{\frac{p_x}{p_y}}
\end{aligned}
$$

(b) **Derive the general expenditure function:** The expenditure function $E(p_x, p_y, \bar{U})$ is obtained by substituting the Hicksian demand functions into $E = p_x x_H + p_y y_H$:

$$\begin{aligned}
E(p_x, p_y, \bar{U}) &= p_x \left(\bar{U} \sqrt{\frac{p_y}{p_x}} \right) + p_y \left(\bar{U} \sqrt{\frac{p_x}{p_y}} \right) \\
&= \bar{U} \sqrt{p_x^2 \frac{p_y}{p_x}} + \bar{U} \sqrt{p_y^2 \frac{p_x}{p_y}} \\
&= \bar{U} \sqrt{p_x p_y} + \bar{U} \sqrt{p_x p_y} \\
&= 2\bar{U} \sqrt{p_x p_y}
\end{aligned}$$

(c) **Numerical Applications (using solutions from parts (a) and (b)):**

(i) **Scenario:** $\bar{U} = 16$**. Initial** $p_x = 1, p_y = 1$**; New** $p_x = 4$**:**

Initial Hicksian demands and minimum expenditure:

$$\begin{aligned}
x_H &= 16\sqrt{\frac{1}{1}} = 16 \\
y_H &= 16\sqrt{\frac{1}{1}} = 16 \\
E_0 &= 2 \cdot 16\sqrt{1 \cdot 1} = 32 \cdot 1 = 32
\end{aligned}$$

New Hicksian demands and minimum expenditure:

$$\begin{aligned}
x'_H &= 16\sqrt{\frac{1}{4}} = 16 \cdot \frac{1}{2} = 8 \\
y'_H &= 16\sqrt{\frac{4}{1}} = 16 \cdot 2 = 32 \\
E_1 &= 2 \cdot 16\sqrt{4 \cdot 1} = 32\sqrt{4} = 32 \cdot 2 = 64
\end{aligned}$$

Compensating Variation (CV):

$$CV = E_1 - E_0 = 64 - 32 = 32$$

(ii) **Scenario:** $\bar{U} = 25$**. Initial** $p_x = 2, p_y = 2$**; New** $p_y = 8$**:**

Initial Hicksian demands and minimum expenditure:

$$
\begin{aligned}
x_H &= 25\sqrt{\frac{2}{2}} = 25 \\
y_H &= 25\sqrt{\frac{2}{2}} = 25 \\
E_0 &= 2 \cdot 25\sqrt{2 \cdot 2} = 50\sqrt{4} = 50 \cdot 2 = 100
\end{aligned}
$$

New Hicksian demands and minimum expenditure:

$$
\begin{aligned}
x'_H &= 25\sqrt{\frac{8}{2}} = 25\sqrt{4} = 25 \cdot 2 = 50 \\
y'_H &= 25\sqrt{\frac{2}{8}} = 25\sqrt{\frac{1}{4}} = 25 \cdot \frac{1}{2} = 12.5 \\
E_1 &= 2 \cdot 25\sqrt{2 \cdot 8} = 50\sqrt{16} = 50 \cdot 4 = 200
\end{aligned}
$$

Compensating Variation (CV):

$$
CV = E_1 - E_0 = 200 - 100 = 100
$$

(iii) **Scenario:** $\bar{U} = 81$**. Initial** $p_x = 3, p_y = 1$**; New** $p_y = 3$**:**
Initial Hicksian demands and minimum expenditure:

$$
\begin{aligned}
x_H &= 81\sqrt{\frac{1}{3}} = \frac{81}{\sqrt{3}} = 27\sqrt{3} \approx 46.77 \\
y_H &= 81\sqrt{\frac{3}{1}} = 81\sqrt{3} \approx 140.29 \\
E_0 &= 2 \cdot 81\sqrt{3 \cdot 1} = 162\sqrt{3} \approx 281.04
\end{aligned}
$$

New Hicksian demands and minimum expenditure:

$$
\begin{aligned}
x'_H &= 81\sqrt{\frac{3}{3}} = 81 \\
y'_H &= 81\sqrt{\frac{3}{3}} = 81 \\
E_1 &= 2 \cdot 81\sqrt{3 \cdot 3} = 162\sqrt{9} = 162 \cdot 3 = 486
\end{aligned}
$$

Compensating Variation (CV):

$$CV = E_1 - E_0 = 486 - 162\sqrt{3} \approx 486 - 281.04 = 204.96$$

3. Consider a consumer with the utility function $U(x, y) = x^{0.4}y^{0.6}$. Let p_x and p_y denote the prices of x and y, and let $\bar{U}$ be the target level of utility.

   (a) **Derive the Hicksian demand functions:** The expenditure minimization problem is to minimize $E = p_x x + p_y y$ subject to $x^{0.4}y^{0.6} = \bar{U}$.

   The Lagrangian is:

$$L(x, y, \lambda) = p_x x + p_y y + \lambda(\bar{U} - x^{0.4}y^{0.6})$$

   The first-order conditions (FOCs) are:

$$\frac{\partial L}{\partial x} = p_x - \lambda(0.4)x^{-0.6}y^{0.6} = 0 \quad \Longrightarrow \; p_x = \lambda(0.4)\left(\frac{y}{x}\right)^{0.6} \quad (1)$$
$$\frac{\partial L}{\partial y} = p_y - \lambda(0.6)x^{0.4}y^{-0.4} = 0 \quad \Longrightarrow \; p_y = \lambda(0.6)\left(\frac{x}{y}\right)^{0.4} \quad (2)$$
$$\frac{\partial L}{\partial \lambda} = \bar{U} - x^{0.4}y^{0.6} = 0 \quad \Longrightarrow \; x^{0.4}y^{0.6} = \bar{U} \quad (3)$$

   Divide equation (1) by equation (2) to find the optimal ratio of goods:

$$\begin{aligned}\frac{p_x}{p_y} &= \frac{0.4y^{0.6}x^{-0.6}}{0.6x^{0.4}y^{-0.4}} \\ &= \frac{0.4}{0.6}\left(\frac{y^{0.6}y^{0.4}}{x^{0.6}x^{0.4}}\right) \\ &= \frac{2}{3}\frac{y}{x}\end{aligned}$$

   This implies $y = \frac{3p_x}{2p_y}x$.

Substitute this expression for y into the utility constraint (3):

$$\begin{aligned} x^{0.4}\left(\frac{3p_x}{2p_y}x\right)^{0.6} &= \bar{U} \\ x^{0.4}x^{0.6}\left(\frac{3p_x}{2p_y}\right)^{0.6} &= \bar{U} \\ x\left(\frac{3p_x}{2p_y}\right)^{0.6} &= \bar{U} \end{aligned}$$

Solving for x, we get the Hicksian demand for x:

$$x_H(p_x, p_y, \bar{U}) = \bar{U}\left(\frac{2p_y}{3p_x}\right)^{0.6}$$

Substitute $x = \frac{2p_y}{3p_x}y$ (derived from $y = \frac{3p_x}{2p_y}x$) into the utility constraint:

$$\begin{aligned} \left(\frac{2p_y}{3p_x}y\right)^{0.4} y^{0.6} &= \bar{U} \\ y^{0.4}y^{0.6}\left(\frac{2p_y}{3p_x}\right)^{0.4} &= \bar{U} \\ y\left(\frac{2p_y}{3p_x}\right)^{0.4} &= \bar{U} \end{aligned}$$

Solving for y, we get the Hicksian demand for y:

$$y_H(p_x, p_y, \bar{U}) = \bar{U}\left(\frac{3p_x}{2p_y}\right)^{0.4}$$

(b) **Derive the general expenditure function:** The expenditure function $E(p_x, p_y, \bar{U})$ is found by substituting these Hicksian demand functions into $E = p_x x_H + p_y y_H$:

$$E(p_x, p_y, \bar{U}) = p_x \left(\bar{U} \left(\frac{2p_y}{3p_x} \right)^{0.6} \right) + p_y \left(\bar{U} \left(\frac{3p_x}{2p_y} \right)^{0.4} \right)$$
$$= \bar{U} \left[p_x^1 \frac{2^{0.6} p_y^{0.6}}{3^{0.6} p_x^{0.6}} + p_y^1 \frac{3^{0.4} p_x^{0.4}}{2^{0.4} p_y^{0.4}} \right]$$
$$= \bar{U} \left[p_x^{0.4} p_y^{0.6} \left(\frac{2}{3} \right)^{0.6} + p_x^{0.4} p_y^{0.6} \left(\frac{3}{2} \right)^{0.4} \right]$$
$$= \bar{U} p_x^{0.4} p_y^{0.6} \left[\left(\frac{2}{3} \right)^{0.6} + \left(\frac{3}{2} \right)^{0.4} \right]$$

(c) **Numerical Applications (using solutions from parts (a) and (b)):**

(i) **Scenario:** $\bar{U} = 20$**. Initial** $p_x = 1, p_y = 2$**; New** $p_x = 2$**:**
Initial Hicksian demands and minimum expenditure:

$$x_H = 20 \left(\frac{2 \cdot 2}{3 \cdot 1} \right)^{0.6} = 20 \left(\frac{4}{3} \right)^{0.6} \approx 23.156$$
$$y_H = 20 \left(\frac{3 \cdot 1}{2 \cdot 2} \right)^{0.4} = 20 \left(\frac{3}{4} \right)^{0.4} \approx 17.758$$
$$E_0 = 20 \cdot (1)^{0.4} (2)^{0.6} \left[\left(\frac{2}{3} \right)^{0.6} + \left(\frac{3}{2} \right)^{0.4} \right] \approx 65.885$$

New Hicksian demands and minimum expenditure:

$$x'_H = 20 \left(\frac{2 \cdot 2}{3 \cdot 2} \right)^{0.6} = 20 \left(\frac{2}{3} \right)^{0.6} \approx 15.202$$
$$y'_H = 20 \left(\frac{3 \cdot 2}{2 \cdot 2} \right)^{0.4} = 20 \left(\frac{3}{2} \right)^{0.4} \approx 23.036$$
$$E_1 = 20 \cdot (2)^{0.4} (2)^{0.6} \left[\left(\frac{2}{3} \right)^{0.6} + \left(\frac{3}{2} \right)^{0.4} \right]$$
$$\approx 131.77$$

Compensating Variation (CV):

$$CV = E_1 - E_0 \approx 131.77 - 65.885 = 65.885$$

(ii) **Scenario:** $\bar{U} = 50$**. Initial** $p_x = 4, p_y = 2$**; New** $p_x = 1$**:**
Initial Hicksian demands and minimum expenditure:

$$x_H = 50\left(\frac{2 \cdot 2}{3 \cdot 4}\right)^{0.6} = 50\left(\frac{1}{3}\right)^{0.6} \approx 24.035$$
$$y_H = 50\left(\frac{3 \cdot 4}{2 \cdot 2}\right)^{0.4} = 50(3)^{0.4} \approx 77.59$$
$$E_0 = 50 \cdot (4)^{0.4}(2)^{0.6}\left[\left(\frac{2}{3}\right)^{0.6} + \left(\frac{3}{2}\right)^{0.4}\right] \approx 286.79$$

New Hicksian demands and minimum expenditure:

$$x'_H = 50\left(\frac{2 \cdot 2}{3 \cdot 1}\right)^{0.6} = 50\left(\frac{4}{3}\right)^{0.6} \approx 57.89$$
$$y'_H = 50\left(\frac{3 \cdot 1}{2 \cdot 2}\right)^{0.4} = 50\left(\frac{3}{4}\right)^{0.4} \approx 44.395$$
$$E_1 = 50 \cdot (1)^{0.4}(2)^{0.6}\left[\left(\frac{2}{3}\right)^{0.6} + \left(\frac{3}{2}\right)^{0.4}\right] \approx 164.71$$

Compensating Variation (CV):

$$CV = E_1 - E_0 \approx 164.71 - 286.79 = -122.08$$

(iii) **Scenario:** $\bar{U} = 100$**. Initial** $p_x = 2, p_y = 1$**; New** $p_y = 3$**:**
Initial Hicksian demands and minimum expenditure:

$$x_H = 100\left(\frac{2 \cdot 1}{3 \cdot 2}\right)^{0.6} = 100\left(\frac{1}{3}\right)^{0.6} \approx 48.07$$
$$y_H = 100\left(\frac{3 \cdot 2}{2 \cdot 1}\right)^{0.4} = 100(3)^{0.4} \approx 155.18$$
$$E_0 = 100 \cdot (2)^{0.4}(1)^{0.6}\left[\left(\frac{2}{3}\right)^{0.6} + \left(\frac{3}{2}\right)^{0.4}\right] \approx 286.79$$

New Hicksian demands and minimum expenditure:

$$\begin{aligned}
x'_H &= 100\left(\frac{2\cdot 3}{3\cdot 2}\right)^{0.6} = 100(1)^{0.6} = 100 \\
y'_H &= 100\left(\frac{3\cdot 2}{2\cdot 3}\right)^{0.4} = 100(1)^{0.4} = 100 \\
E_1 &= 100\cdot(2)^{0.4}(3)^{0.6}\left[\left(\frac{2}{3}\right)^{0.6} + \left(\frac{3}{2}\right)^{0.4}\right] \approx 500
\end{aligned}$$

Compensating Variation (CV):

$$CV = E_1 - E_0 \approx 500 - 286.79 = 213.21$$