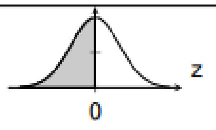
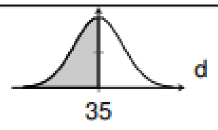
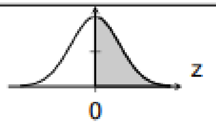
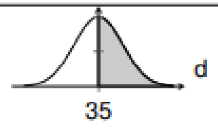
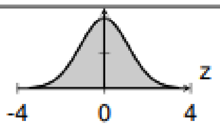
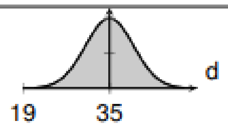
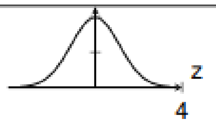
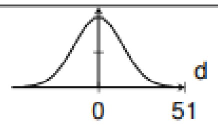
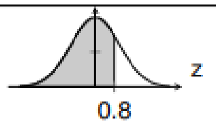
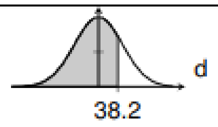
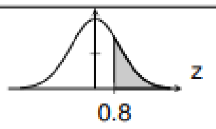
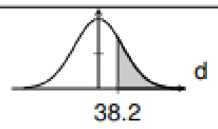
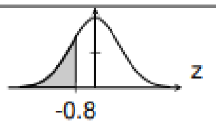
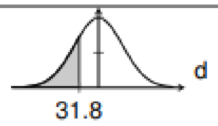
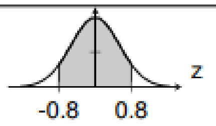
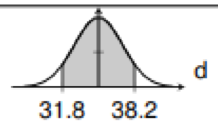
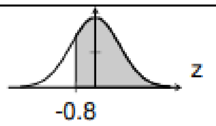
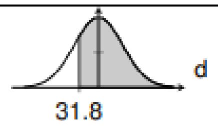
The Normal Distribution

LLE – Mathematics and Statistics

1. These questions concern the standard normal distribution, Z, which has a mean $\mu = 0$ and a variance $\sigma^2 = 1$, so you can write $Z \sim N(0,1)$. You will also consider the distribution of the length of donkey's tails, $D \sim N(35,16)$, that is the normal distribution with mean $\mu = 35$ cm and variance $\sigma^2 = 16$ cm^2. Fill in the blanks using the information throughout the table below. No tables or calculators are required.

$P(Z<0)=$ 0.5		$P(D<35cm)=$ 0.5	
$P(Z>0)=$ 0.5		$P(D>35cm)=$ 0.5	
$P(Z>-4)\approx$ 1		$P(D>19)\approx$ 1	
$P(Z>4)\approx$ 0		$P(D>51)\approx$ 0	
$P(Z<0.8)=$ 0.788		$P(D<38.2)=$ $=P(Z<0.8)=0.788$	
$P(Z>0.8)=$ 0.212		$P(D>38.2)=$ 0.212	
$P(Z<-0.8)=$ 0.212		$P(D<31.8)=$ 0.212	
$P(-0.8<Z<0.8)=$ $0.788-0.212=0.576$		$P(31.8<D<38.2)=$ 0.576	
$P(Z>-0.8)=$ 0.788		$P(D>31.8)=$ 0.788	

2. When raw scores are turned into z-scores, it is known as standardising. The standardised normal distribution will have a mean of 0 and a standard deviation of 1, and can therefore be written:

$$Z \sim N(0, 1)$$

You can convert normal distributions into standard normal distributions by standardising. This allows you to calculate proportions of the population that are more than, less than, or between values. You can calculate these probabilities using tables, software, or websites such as at this link or this one.

   (a) Calculate the following. Make a sketch of the curve with labeled axes and the area being found.

      i. $P(Z < 1) = 0.8413$
      ii. $P(Z < -0.45) = 0.3264$
      iii. $P(Z > 0.45) = 0.6736$
      iv. $P(-1.2 < Z < 1.6) = 0.8301$

   (b) Calculate the following z-scores, correct to 2 decimal places, making a sketch of the situation.

      i. $P(Z < z) = 0.975, z = 1.96$
      ii. $P(Z < z) = 0.2, z = -0.842$
      iii. $P(Z > z) = 0.8, z = -0.842$
      iv. $P(Z > z) = 0.15, z = 1.036$

3. Given the information about the population distribution, find the probabilities requested by first standardising the raw scores.

   (a) $X \sim N(80, 25)$

      i. $P(X < 85) = P(Z < \frac{85-80}{5}) = P(Z < 1) = 0.8413$
      ii. $P(X > 70) = P(Z > -2) = 0.9773$
      iii. $P(78 < X < 82) = P(-0.4 < Z < 0.4) = 0.3108$

(b) $X \sim N(150, 156.25)$

i. $P(X < 143.75) = P(Z < \frac{143.75-150}{12.5}) = P(Z < -0.5) = 0.3085$

ii. $P(X > 180) = P(Z > 2.4) = 0.0082$

iii. $P(145 < X < 170) = P(-0.4 < Z < 1.6) =$
$P(Z < 1.6) - P(Z < -0.4) = 0.6096$

4. Given the information about the population distribution, find the values that are represented by the following probabilities/proportions. Give your answers correct to 1 decimal place. You will need to find the z-score from the probability given and then transform this back to a raw score.

(a) $X \sim N(100, 16)$

i. $P(X < r) = 0.9772, z = 2, r = 100 + 2 \times 4 = 108$

ii. $P(X > s) = 0.9772, z = -2, s = 92$

iii. $P(X < t) = 0.35, z = -0.385, t = 98.5$

(b) $X \sim N(20, 4)$

i. $P(X < r) = 0.9772, z = 2, r - 20 + 2 \times 2 = 24$

ii. $P(X > t) = 65\%, z = -0.385, t = 19.2$

5. A hot drinks machine is known to dispense liquid in a volume that is normally distributed with a mean of 250 ml and a standard deviation of 5 ml, $V \sim N(250, 25)$

(a) What proportion of drinks are expected to be dispensed that are over 254 ml?
$z = \frac{254-250}{5} = 0.8$
$P(V > 254) = P(Z > 0.8) = 0.2119 = 21.2\%$

(b) What proportion of drinks that are expected to be under 246 ml in volume.
$P(V < 246) = P(Z < -0.8) = 0.2119 = 21.2\%$

(c) Use your answers and sketches to parts b and c to work out the probability that a drink is dispensed that is between 246 ml and 254 ml.

Over 254 ml and under 246 ml is 21.2 + 21.2 = 42.4%

Therefore between these amounts is 100 - 42.4 = 57.6 %

(d) What volume are 75% of drinks less than?

$P(V < v) = 0.75$

$P(Z < 0.674) = 0.75 \implies v = 250 + 0.674 \times 5 = 253.4$ ml

(e) You find that your cup overflows 2% of the time. How big is your cup?

$P(V > c) = 0.02$

$P(Z > 2.054) = 0.02 \implies c = 250 + 2.054 \times 5 = 260.3$ ml