Nesting Probabilistic Programs

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With help from Rob Cornish, Hongseok Yang, Andrew Warrington, and Frank Wood

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Probabilistic programming presents and opportunity to nest queries

```
(defquery outer [D]
 (let [y (sample (beta 2 3))
       dist (conditional inner)
       z (sample (dist y D))]
  (* y z)))
```

```
(defquery inner [y D]
 (let [z (sample (gamma y 1))]
  (observe (normal y z) D)
  z))
```

Nested Expectations

$$\gamma_0 = \mathbb{E}\left[f_0\left(y, \mathbb{E}\left[f_1\left(y, z\right)|y\right]\right)\right]$$

$$\gamma_0 = \mathbb{E}[f_0(y, \mathbb{E}[f_1(y, z, \mathbb{E}[f_2(y, z, u)|y, z])|y])]$$

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There are problems we cannot encode without nesting

- E.g. entropy of KL divergence of a marginal distribution:

$$H\left(p(y)\right) = -\mathbb{E}\left[\log\left(\mathbb{E}\left[p(y|\theta)|y\right]\right)\right]$$

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Agents reasoning about other agents

Bad cards, should probably fold

Can I bluff him though? I doubt he has good cards

Cards are ok, but if he bets I'll fold

He might be bluffing though…

Aims

- When does nesting probabilistic programs no longer give a standard Bayesian model?
- What are the statistical implications?
- How can we construct valid inference engines?

Three Ways We Might Nest

- Nested conditioning
- Nested inference
- Estimates as first class variables

Nested Conditioning

```
(defquery outer [D]
 (let [y (sample (beta 2 3))]
  (observe (nest inner [y D] :smc 100) nil)
  y))
```

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Formalizing Nested Conditioning

$$\pi_o(y) = \boxed{\psi(y)}\,\boxed{p_i(y)} \approx \psi(y)\ \hat{p}_i(y)$$

Density terms from outer

Marginal likelihood for inner

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Formalizing Nested Conditioning

$$\mathbb{E}_{q(y)}\left[\frac{\psi(y)\hat{p}_i(y)}{q(y)}\right] = \mathbb{E}_{q(y)}\left[\frac{\pi_o(y)}{q(y)}\right]$$

Nested Inference

```
(defquery inner [y D]
 (let [z (sample (gamma y 1))]
  (observe (normal y z) D)
  z))


(defquery outer [D]
 (let [y (sample (beta 2 3))
       dist (conditional inner)
       z (sample (dist y D))]
  (* y z)))
```

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Formalizing Nested Inference

$$\pi_o(y,z) = \psi(y,z)p_i(z|y)$$

$$p_i(z|y) = \frac{\pi_i(y,z)}{\int \pi_i(y,z')dz'}$$

$$\pi_o(y,z) = \frac{\psi(y,z)\pi_i(y,z)}{\int \pi_i(y,z')dz'}$$

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Formalizing Nested Inference

$$\pi_o(y,z) \approx \psi(y,z)\hat{p}_i(z|y)$$

$$\mathbb{E}_{q(y,z)}\left[\frac{\pi_o(y,z)}{q(y,z)}\right] = \mathbb{E}_{q(y,z)}\left[\frac{\psi(y,z)\pi_i(y,z)}{q(y,z)\mathbb{E}_{z'\sim q(z|y)}\left[\pi_i(y,z')/q(z'|y)\right]}\right]$$

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Estimates as First Class Variables

```
(defm prior [] (normal 0 1))
(defm lik [theta d] (normal theta d))

(defquery inner-q [y d]
 (let [theta (sample (prior))]
  (observe (lik theta d) y)))

(defn inner-E [y d M]
 (->> (doquery :importance
        inner-q [y d])
      (take M)
      log-marginal))

(with-primitive-procedures [inner-E]
```

```
 (defquery outer-q [d M]
  (let [theta (sample (prior))
        y (sample (lik theta d))
        log-lik (observe*
                  (lik theta d) y)
        log-marg (inner-E y d M)]
   (- log-lik log-marg))))

(defn outer-E [d M N]
 (->> (doquery :importance
        outer-q [d M])
      (take N)
      collect-results
      empirical-mean))
```

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Estimates as First Class Variables

$$\pi_0(y) = \psi(y, z(y)) \approx \psi(y, \hat{z}(y))$$

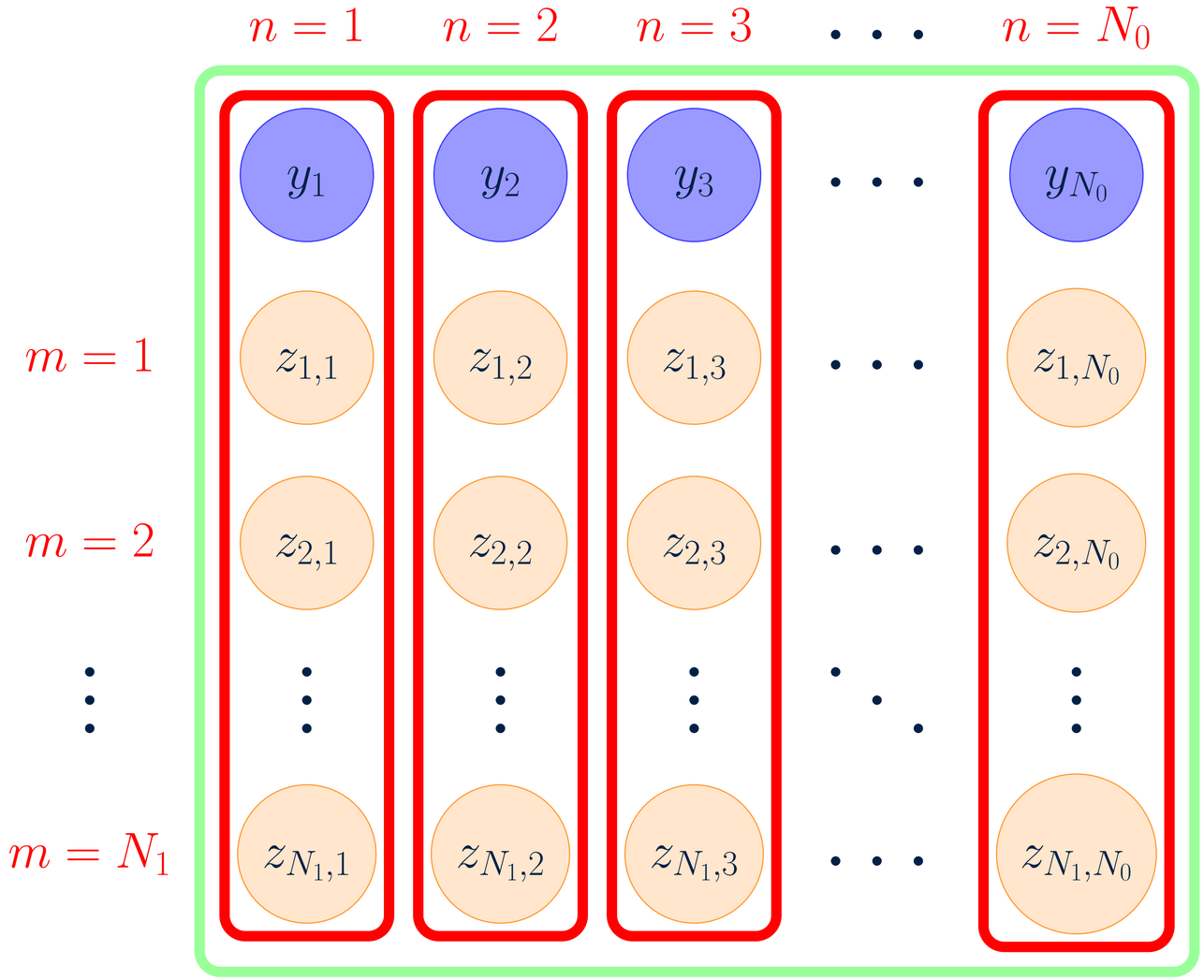
$$\mathbb{E}_{q(y)}\left[\frac{\pi_0(y)}{q(y)}\right] = \mathbb{E}_{q(y)}\left[\frac{\psi(y, \mathbb{E}\left[\hat{z}(y)|y\right])}{q(y)}\right]$$

Nested Monte Carlo

$$I_0 = \frac{1}{N_0} \sum_{n=1}^{N_0} f_0\left(y_n, I_1^n(y_n)\right) \quad \text{where} \quad y_n \sim p(y)$$

$$I_1^n(y_n) = \frac{1}{N_1} \sum_{m=1}^{N_1} f_1\left(y_n, z_{m,n}\right) \quad \text{where} \quad z_{m,n} \sim p(z|y = y_n)$$

[Rainforth, Cornish, Yang, Warrington, and Wood. On the Opportunities and Pitfalls of Nesting Monte Carlo Estimators. ICML 2018]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$z_{N_1,2}$
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Convergence Rate

- If each f_k is continuously differentiable, the MSE converges at a rate

$$O\left(\frac{1}{N_0} + \left(\sum_{k=1}^{D} \frac{1}{N_k}\right)^2\right)$$

Variance

Bias Squared

- We need each $N_k \to \infty$ for convergence

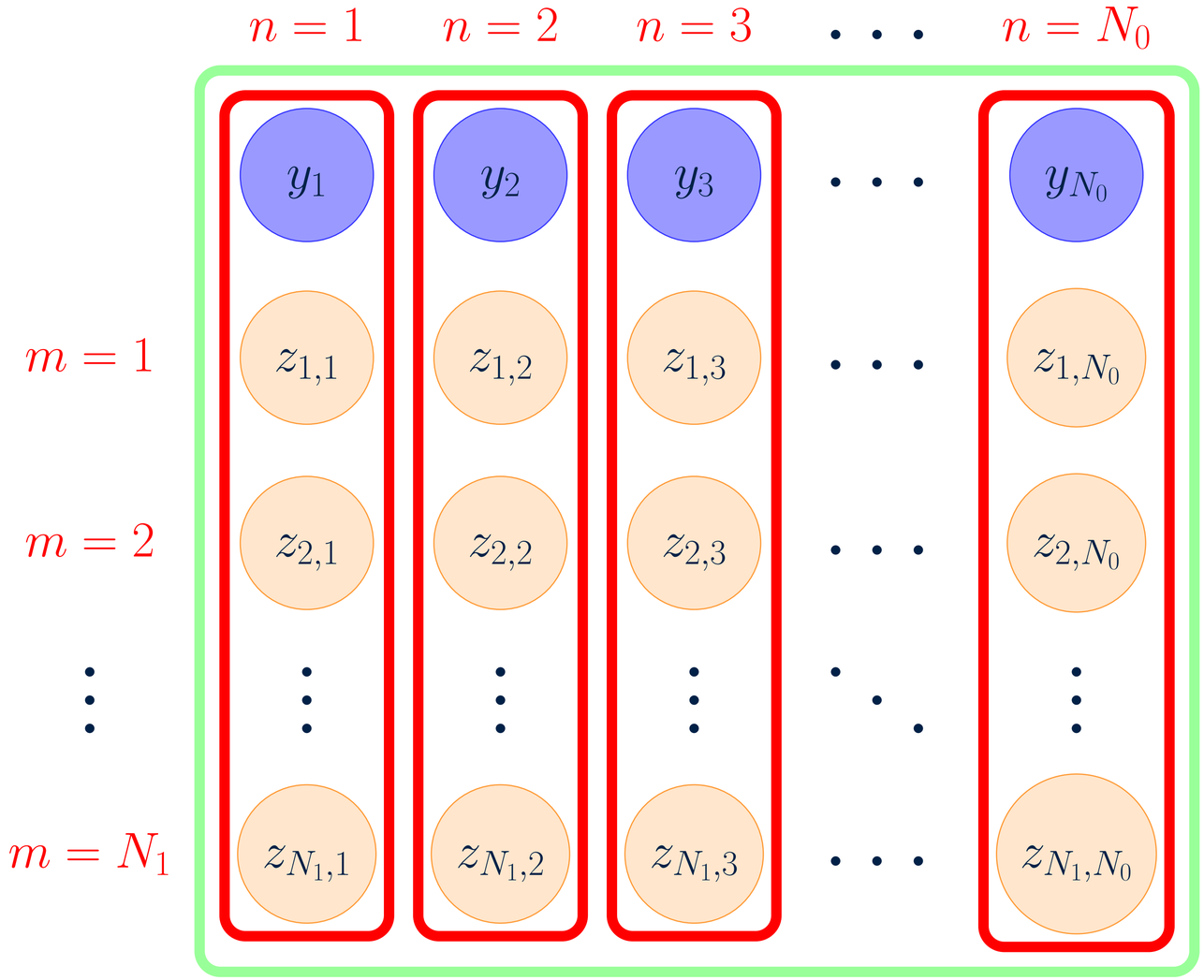
Optimal Setup

- The asymptotic mean squared error is minimised when

$$\sqrt{N_0} \propto N_1 \propto N_2 \propto \cdots \propto N_D$$

- This yields an overall convergence rate of $O\left(\frac{1}{T^{\frac{2}{2+D}}}\right)$

$$D = 0,\ O\left(\frac{1}{T}\right); \quad D = 1,\ O\left(\frac{1}{T^{\frac{2}{3}}}\right); \quad D = 2,\ O\left(\frac{1}{T^{\frac{1}{2}}}\right)$$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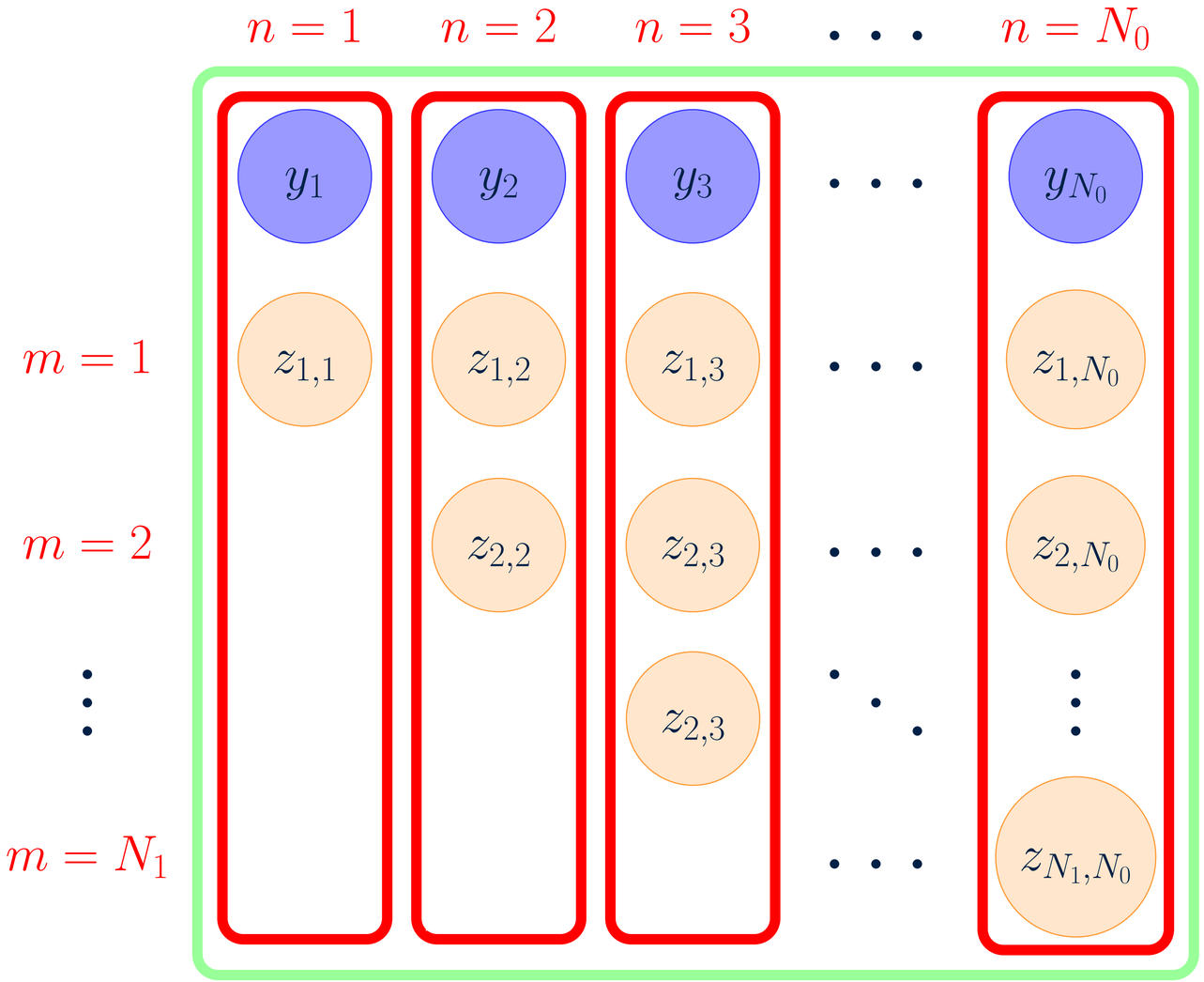
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Online NMC

$$\mathrm{Var}[\mathrm{ONMC}] \to c\,\mathrm{Var}[\mathrm{NMC}] \quad \text{where} \quad c < 1$$
$$\mathrm{Bias}[\mathrm{ONMC}] \to d\,\mathrm{Bias}[\mathrm{NMC}] \quad \text{where} \quad d < 2$$

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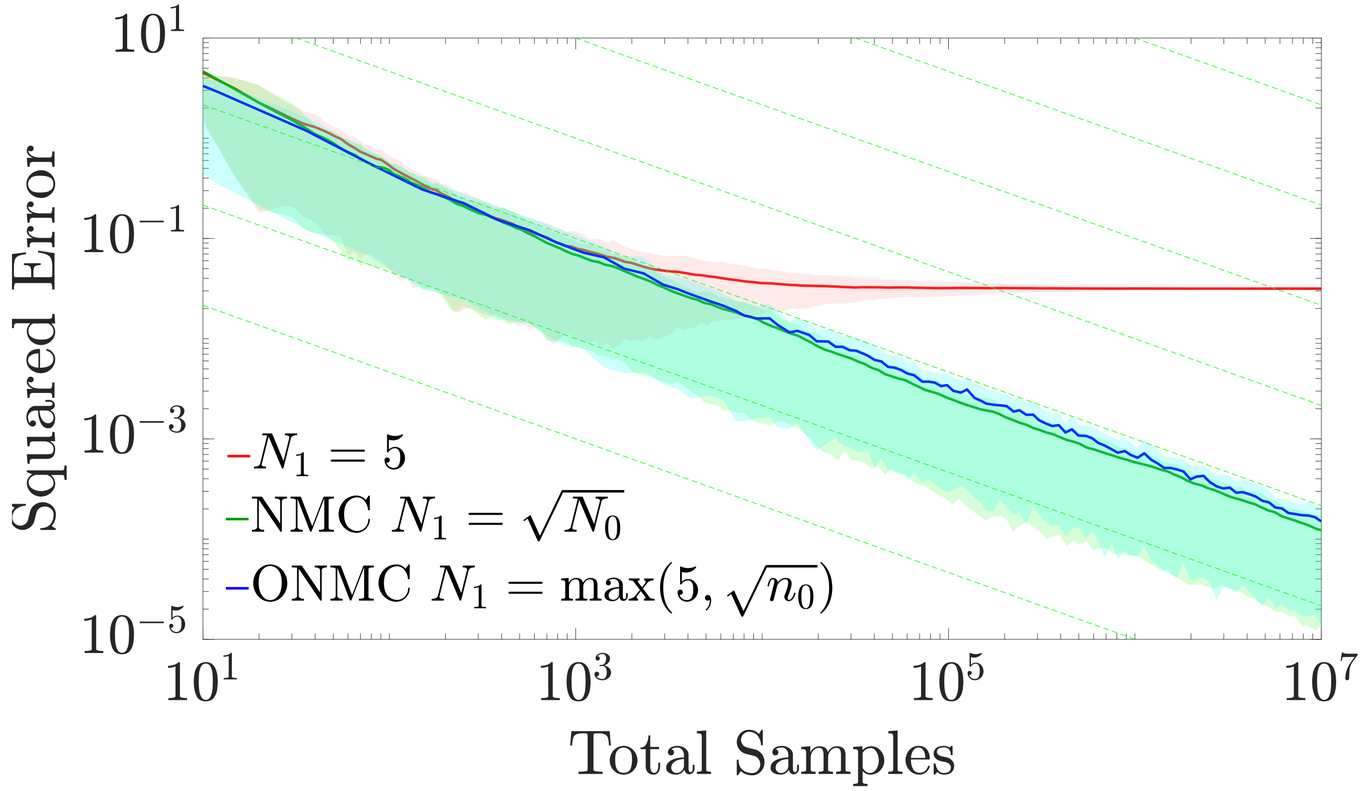
Online NMC



Take Homes

- Nesting probabilistic programs allows encoding of models we could not otherwise express
  - Allows encoding of nested expectations
- We need to be careful in the design of our inference engines
- Online NMC gives an easy way of making sure consistency is maintained

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Thanks!

Rob Cornish

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[Rainforth, Cornish, Yang, Warrington, and Wood. On the Opportunities and Pitfalls of Nesting Monte Carlo Estimators. ICML 2018]

[Rainforth. Nesting Probabilistic Programs. UAI 2018]

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