$$\hat{Z} = \prod_{n=1}^{N} \hat{Z}_n = \prod_{n=1}^{N} \frac{1}{K} \sum_{k=1}^{K} w(\tilde{\mathbf{x}}_{1:n}^k)$$