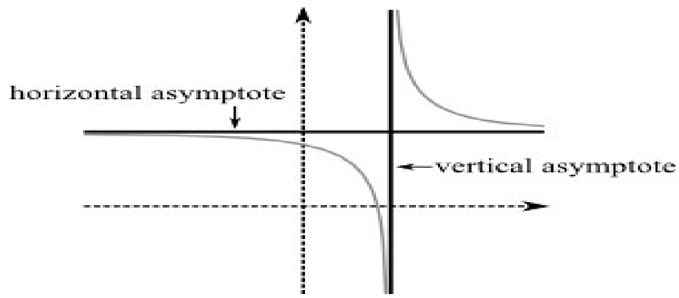
ASYMPTOTES

4.1 Introduction:

An asymptote is a line that approaches closer to a given curve as one or both of x or y coordinates tend to infinity but never intersects or crosses the curve. There are two types of asymptotes viz. Rectangular asymptotes and Oblique asymptotes

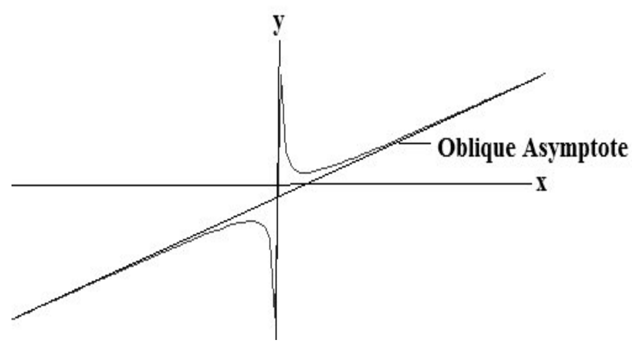
Rectangular Asymptote:

If an asymptote is parallel to x-axis or to y-axis, then it is called rectangular asymptote. An asymptote parallel to x-axis is called horizontal asymptote and the asymptote parallel to y-axis is called vertical asymptote.



Oblique Asymptote:

If an asymptote is neither parallel to x-axis nor to y-axis then it is called an oblique asymptote. An oblique asymptote occurs when the degree of polynomial in the numerator is greater than that of polynomial in the denominator. To find the oblique asymptote, numerator must be divided by the denominator by using either long division or synthetic division.



4.2 Method of finding rectangular asymptote:

- To find an asymptote parallel to x-axis equate to zero the coefficient of highest power of x in the equation of the curve.
- To find an asymptote parallel to y-axis equate to zero the coefficient of highest power of y in the equation of the curve.

Example1 Find the asymptotes parallel to coordinate axes of the curve

$$4x^2 + 9y^2 = x^2y^2$$

Solution: The equation of the given curve is $4x^2 + 9y^2 - x^2y^2 = 0$

Equating it to zero, coefficient of x^2 (which is highest power of x) we get,

$4 - y^2 = 0 \Rightarrow y = \pm 2$

$\therefore y = 2, y = -2$ are the two asymptotes parallel to x-axis

Equating to zero the coefficient of highest power of y, we get

$$9 - x^2 = 0 \Longrightarrow x = \pm 3$$

$\therefore x = 3, x = -3$ are the asymptotes parallel to y – axis

4.3 Method of finding oblique asymptote:

Let the asymptote be $y = mx + c$

Let the equation of the curve be

$$\phi_n(x,y) + \phi_{n-1}(x,y) + \ldots\ldots\ldots + \phi_1(x,y) + k = 0 \quad -(1)$$

where $\phi_n(x,y)$ denotes the term of highest degree of the curve.

Step-1 Put x = 1, y = m in ϕ_n(x, y), ϕ_{n-1} (x, y), ……, $\phi_1(x,y)$

Step-2 Find all the real roots of $\phi_n(\mathrm{m}) = 0$

Step-3 If m is a non-repeated root, then corresponding value of c is given by

$$c\ \phi_n'(\mathrm{m}) + \phi_{n-1}(\mathrm{m}) = 0, \quad (\phi_n'(\mathrm{m}) \neq 0)$$

If ϕ_n' (m) = 0 then there is no asymptote to the curve corresponding to this value of m.

Step-4 If m is a repeated root occurring twice, then the two values of c are given by

$$\frac{c^2}{2!}\phi_n''(\mathrm{m}) + \frac{c}{1!}\phi_{n-1}'(\mathrm{m}) + \phi_{n-2}(\mathrm{m}) = 0 \quad (\phi_n''(\mathrm{m}) \neq 0)$$

Step-5 The asymptote of the curve is y = m x + c

Example2 Find all the asymptotes of the curve

$$x^3 - x^2y - xy^2 + y^3 + 2x^2 - 4y^2 + 2xy + x + y + 1 = 0$$

Solution: In the curve the highest degree term of x is x^3 and its coefficient is 1.Equating it to 0 we get 1= 0 which is absurd, thus the curve has no asymptote parallel to x – axis. Also the coefficient of highest degree term in y is 1, thus the curve has no asymptote parallel to y-axis.

Now finding oblique asymptote

Here ϕ_3 (x, y) $= x^3 - x^2y + y^3 - xy^2$

ϕ_2 (x, y) $= 2x^2 - 4y^2 + 2xy$, ϕ_1 (x, y) $= x + y$

Let the asymptote be given by $y = mx + c$

Step-1 Putting x = 1 & y = m in ϕ_3 (x, y), ϕ_2 (x, y), ϕ_1 (x, y)

$$\phi_3(\mathrm{m}) = m^3 - m^2 - m + 1$$

$\phi_2(\mathrm{m}) = 2 - 4m^2 + 2m$ and ϕ_1 (m) = 1 + m

Step-2 The values of m are obtained by solving ϕ_3 (m) = 0

$$\therefore m^3 - m^2 - m + 1 = 0 \Rightarrow (m^2 - 1)(m - 1) = 0$$

$\Rightarrow$ m = 1, 1, – 1

Here m = –1 is a non repeated root & m = 1 is a repeated root.

Step-3: For m = – 1 (non-repeated root),the corresponding value of c is given by

$$c\phi_3'(\mathrm{m}) + \phi_2(\mathrm{m}) = 0$$

Now $\phi_3(\mathrm{m}) = m^3 - m^2 - m + 1$

$$\Rightarrow \phi_3' = 3m^2 - 2m - 1$$

$$\therefore c(3m^2 - 2m - 1) + (2 + 2m - 4m^2) = 0$$

$$\Rightarrow c(3m^2 - 2m - 1) = 4m^2 - 2m - 2 \quad \Rightarrow c = \frac{4m^2 - 2m - 2}{(3m^2 - 2m - 1)} = \frac{4}{4} = 1$$

Thus for $m = -1$, c= 1

Now the asymptote is $y = mx + c$ i.e. $y = -x + 1$ or $x + y = 1$

Step-4 For m = 1 (repeated root), the value of c is given by

$$\frac{c^2}{2!}\phi_3''(m) + \frac{c}{1!}\phi_2'(m) + \phi_1(m) = 0 \text{........(2)}$$

Now $\phi_3'(m) = 3m^2 - 2m - 1$

$\Rightarrow \phi_3''(m) = 6m - 2$

$\phi_2(m) = 2 + 2m - 4m^2$

$\Rightarrow \phi_2''(m) = 2 - 8m$

$\phi_1(m) = 1 + m$

Putting these values in (2) we get

$$\frac{c^2}{2}(6m-2) + c(-8m + 2) + 1 + m = 0$$

$For\ m = 1$

$$\frac{c^2}{2}(4) + c(-6) + 2 = 0 \Rightarrow c^2 - 3c + 1 = 0$$

$$\Rightarrow c = \frac{3 \pm \sqrt{5}}{2} = \frac{3+\sqrt{5}}{2}, \quad \frac{3-\sqrt{5}}{2}$$

For m = 1, c= $\frac{3+\sqrt{5}}{2}$, ∴ the asymptote is y=mx + c i.e. $y = x + \frac{3+\sqrt{5}}{2}$

$$\Rightarrow 2y = 2x + (3 + \sqrt{5})$$

For m = 1, c= $\frac{3-\sqrt{5}}{2}$ ∴ the asymptote is $2y = 2x + (3 - \sqrt{5})$

∴ Three asymptotes of the given curve is

$$x + y = 1,\ 2(y-x) = 3 + \sqrt{5},\ 2(y-x) = 3 - \sqrt{5}$$

Example3 Find the asymptotes of the curve $y^3 + x^2y + 2xy^2 - y + 1 = 0$

Solution: The highest degree term of y is y^3 whose coefficient is 1. Equating it to 0 we get 1 = 0 which is not possible ∴ there is no asymptote parallel to y–axis The coefficient of highest degree term in x is y thus y = 0 is the asymptote parallel to x – axis.

To find oblique asymptote:

$\phi_3(x,y) = y^3 + x^2y + 2xy^2$

$\phi_2(x,y) = 0,\ \phi_1(x,y) = -y,$

Step –1 Putting x = 1, y = m in the above functions, we get

$\phi_3(m) = m^3 + m + 2m^2$, $\phi_2(m) = 0$ and $\phi_1(m) = -m$

$\therefore \phi_3'(m) = 3m^2 + 4m + 1$

Step – 2 The values of m are obtained by solving the equation $\phi_3(m) = 0$

i.e. $m^3 + 2m^2 + m = 0$

$\Rightarrow m(m^2 + 2m + 1) = 0 \Rightarrow m = 0, -1, -1$

Step – 3 For m = 0 (non–repeating value), the value of c is given by

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = \frac{0}{3m^2 + 4m + 1} = 0$$

∴y = 0 is the asymptote

Step – 4 For m = – 1, – 1 (repeated root)

$\frac{c^2}{2!}\phi_3''(\text{m}) + c\phi_2''(\text{m}) + \phi_1(\text{m}) = 0$

$\frac{c^2}{2}(6m+4) + c(0) - m = 0$

$\Rightarrow (3m+2)\,c^2 - m = 0$

Putting m = – 1 in this equation, we get

$-c^2 + 1 = 0 \Rightarrow c^2 = 1 \Rightarrow \text{C} = \pm 1$

∴ Asymptote corresponding to m = – 1, c = 1 is

$y = -x + 1 \Rightarrow y + x - 1 = 0$

And asymptote corresponding to m = – 1, c = –1 is

y = –x – 1 $\Rightarrow y + x - 1 = 0$

Thus the asymptotes of the given curve are

$y = 0, y + x - 1 = 0$ & $y + x + 1 = 0$

Note: If m is a repeated root occurring thrice then the values of c are given by

$\frac{c^3}{3!}\phi_n'''(\text{m}) + \frac{c^2}{2!}\phi_{n-1}''(\text{m}) + \frac{c}{1!}\phi_{n-2}'(\text{m}) + \phi_{n-3}(\text{m}) = 0$

Example 4 Find the asymptotes of the curve

$y^4 - 2xy^3 + 2x^3y - x^4 - 3x^3 + 3x^2y + 3xy^2 - 3y^3 - 2x^2 + 2y^2 - 1 = 0$

Solution: In this case, vertical and horizontal asymptotes do not exist.

To find oblique asymptotes:

$$\left.\begin{aligned} &\text{Here } \phi_4(\text{x,y}) = y^4 - 2xy^3 + 2x^3y - x^4 \\ &\phi_3(\text{x,y}) = -3x^3 + 3x^2y + 3xy^2 - 3y^3 \\ &\phi_2(\text{x,y}) = -2\text{x}^2 + 2\text{x}^2 \\ &\phi_1(\text{x,y}) = 0 \end{aligned}\right] - (1)$$

Step -1 Putting x = 1, y = m in (1), we get

$\phi_4(\text{m}) = m^4 - 2m^3 + 2m - 1$

$\phi_3(\text{m}) = -3 + 3m + 3m^2 - 3m^3$

$\phi_2(\text{m}) = -2 + 2m^2$

$\phi_1(\text{m}) = 0$

Step -2 The values of m are obtained by solving ϕ_4 (m) = 0

$\Rightarrow m^4 - 2m^3 + 2m - 1 = 0$

$\Rightarrow (m-1)^3(m+1) = 0$

⇒ m = 1, 1, 1, –1

Step -3 For m = –1 (non – repeated root), the value of c is given by

$\text{c} = -\frac{\phi_3(\text{m})}{\phi_4'(\text{m})} = \frac{-3(-\text{m}^3+\text{m}^2+\text{m}-1)}{3(\text{m}-1)^2(\text{m}+1)+(\text{m}-1).^3 1}$

= 0 (for m= -1)

∴ y = – x is the asymptote corresponding to m = – 1

Step -4 For m = 1 (repeated root), we have

$\frac{c^3}{3!}\ \phi_4''\ (\mathrm{m})\ +\frac{c^2}{2!}\ \phi_3''\ (\mathrm{m}) + \mathrm{c}\ \phi_2'\ (\mathrm{m}) + \phi_1\ (\mathrm{m})\ =\ 0$

$\Rightarrow \frac{c^3}{6}\ (24\,m - 12)\ + \frac{c^2}{2}\big(-18m\ +\ 6\big) +\ c.4m\ +\ 0\ =\ 0$

$\Rightarrow c\ [\ (4m - 2)\ c^2\ +\ (-\ 9m\ +\ 3)\ c\ +\ 4\,m\ ]\ =\ 0$

For m = 1, we get c = 0 and $2c^2 - 6c\ +\ 4\ =\ 0$

$\Rightarrow (c - 1)\ (c - 2)\ =\ 0$

∴ Values of c are 0, 1, 2

∴ Asymptotes are $y\ =\ x, y\ =\ x\ +\ 1\ ,\ y\ =\ x\ +\ 2$ and $y\ = -\ x$

Example 5 Find the asymptotes of the curve

$$(x^2 - y^2)\ (x\ +\ 2y)\ + 5\ (x^2\ +\ y^2)\ +\ x\ +\ y\ =\ 0$$

Solution: The curve has neither horizontal asymptote nor vertical asymptote.

$$\left.\begin{aligned} \text{Here } \phi_3\ (\mathrm{x,y})\ &=\ (x^2 - y^2)\ (x\ +\ 2y) \\ \phi_2(x,y)\ &=\ 5\ (x^2\ +\ y^2) \\ \phi_1(\mathrm{x,y})\ &=\ \mathrm{x}\ +\ \mathrm{y} \end{aligned}\right] \quad -(1)$$

Step -1 Putting x = 1 and y = m in (1), we get

$\phi_3\ (\mathrm{m})\ =\ (1 - m^2)\ (1\ +\ 2m)$

$\phi_2\ (\mathrm{m})\ =\ 5\ (1\ +\ m^2)$

$\phi_1\ (\mathrm{m})\ =\ 1\ +\ \mathrm{m}$

Step -2 The values of m are obtained by ϕ_3 (m) = 0

$\Rightarrow (1 - m^2)\ (1\ +\ 2m)\ =\ 0$

$\Rightarrow\ \mathrm{m}\ =\ \pm 1, \frac{-1}{2}$

Step -3 For m = $\pm 1, \frac{-1}{2}$ we have

c = $-\frac{\phi_2(\mathrm{m})}{\phi_3'(\mathrm{m})}$ (Since all are non repeated roots)

Now m = 1 gives $c = \frac{5}{3}$ ⇒ y = $\frac{x+5}{3}$ is the asymptote

m = – 1 gives $c = \frac{-5}{3}$ ⇒ y = $-x - \frac{5}{3}$ is the asymptote

m = $-\frac{1}{2}$ gives c = $-\frac{25}{6}$ ⇒ y = $-\frac{x}{2} - \frac{25}{6}$ is the asymptote

∴ All the asymptotes are

$3y - 3\ y\ +\ 5\ =\ 0,\ \ 3x\ +\ 3y\ +\ 5\ =\ 0$ and $\ 3x\ +\ 6y\ +\ 25\ =\ 0$

Exercise 4A

1. Find the asymptotes of the following curves:

(i) $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1$

Ans. x = $\pm$ b, y = $\pm$ a

(ii) $x^3 + 3x^2y - 4y^3 - x + y + 3 = 0$

Ans. $y = x,\ x + 2y = 1$ and $x + 2y + 1 = 0$

(iii) $y\,(x - y)^2 = x + y$

Ans. $y = 0, y = x + \sqrt{2}$ and $y = x - \sqrt{2}$

(iv) $x^2y + xy^2 - xy + y^2 + 3x = 0$

Ans. y = 0 and y = –x

(v) $y^3 - 2xy^2 - x^2y + 2x^3 + 2x^2 - 3xy + x - 2y + 1 = 0$

Ans. $2\,(x - y) = 1,\ 6\,(x + y) + 5 = 0$ and $6x - 3\,y + 4 = 0$

(vi) $(x^2 - y^2)^2 - 4x^2 + x = 0$

Ans. y = $x + 1,\ y = x - 1,\ y = -x + 1, y = -x - 1$

(vii) $x^3 + 4x^2y + 5xy^2 + 2y^2 + 2x^2 + 4xy + 2y^2 - x - 9y + 1 = 0$

Ans. $x + 2y + 2 = 0,\ x + y \pm 2\sqrt{2} = 0$

(viii) $(x - y)^2\,(x - 2y)\,(x - 3y) - 2a\,(x^3 - y^3) - (x - 2y)\,(x + y) = 0$

Ans. $2y = x + 14a,\ 3y = x - 13a,\ y = x - a,\ y = x - 2a$