Indeterminate Form

MM

INDETERMINATE FORM

Q. State L' Hospital's theorem. Evaluate the following limits:

(i) $\lim_{x\to 0} x^2 \ln x$ (ii) $\lim_{x\to 0} x^{2x}$ (iii) $\lim_{x\to 0} (\cos x)^{\cot^2 x}$

(iv) $\lim_{x\to 0} \left(\frac{1}{x^2}\right)^{\tan x}$ (v) $\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x}}$ (vi) $\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$

(vii) $\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$ (viii) $\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$

Sol^n:

L' Hospital's theorem :

statement : If $\phi(x)$, $\psi(x)$ and their derivatives $\phi'(x)$, $\psi'(x)$ are continuous at $x=a$, and if $\phi(a)=\psi(a)=0$ [i.e. $\lim_{x\to a} \phi(x) = \lim_{x\to a} \psi(x) = 0$], then

$$\cancel{\lim_{x\to a}} \quad \cancel{\frac{\phi(x)}{\psi}} \qquad \lim_{x\to a} \frac{\phi(x)}{\psi(x)} = \lim_{x\to a} \frac{\phi'(x)}{\psi'(x)} = \frac{\phi'(a)}{\psi'(a)}$$

provided $\psi'(a) \neq 0$.

Note: L' Hospital's theorem $\frac{0}{0}$ Form

$$\frac{\infty}{\infty} = \frac{\frac{1}{\infty}}{\frac{1}{\infty}} = \frac{0}{0}$$

$$0\times\infty = \frac{\infty}{\frac{1}{0}} = \frac{\infty}{\infty} \quad \text{or} \quad 0\times\infty = \frac{0}{\frac{1}{\infty}} = \frac{0}{0}$$

Sol^n: (i) $\lim_{x \to 0} x^2 \ln x = \lim_{x \to 0} \dfrac{\ln x}{\frac{1}{x^2}}$ $\quad \frac{\infty}{\infty}$ Form

~~Let~~

$= \lim_{x \to 0} \dfrac{\frac{1}{x}}{-\frac{2}{x^3}}$

$= \lim_{x \to 0} \dfrac{1}{x} \times \dfrac{-x^3}{2}$

$= \lim_{x \to 0} \dfrac{-x^2}{2}$

$= 0$ Answer

Sol^n (ii) $\lim_{x \to 0} x^{2x}$

Let, $y = x^{2x}$

$\therefore \log y = 2x \log x$

$\Rightarrow \lim_{x \to 0} (\log y) = \lim_{x \to 0} \dfrac{2 \log x}{\frac{1}{x}}$ $\quad \frac{\infty}{\infty}$ Form

$\Rightarrow \log \left(\lim_{x \to 0} y \right) = \lim_{x \to 0} \dfrac{\frac{2}{x}}{-\frac{1}{x^2}}$

$= \lim_{x \to 0} \dfrac{2}{x} \times \dfrac{-x^2}{1}$

$= \lim_{x \to 0} -2x$

$= 0$ $\quad$ [$e^{\log x} = x$

$\therefore \lim_{x \to 0} x^{2x} = e^0 = 1$ Answer

$\underline{\text{Sol}^n\text{(iii)}}$ $\lim_{x\to 0} (\cos x)^{\cot^2 x}$

$\underline{\text{Sol}^n:}$ Let $y = (\cos x)^{\cot^2 x}$

$\therefore \log y = \cot^2 x \log \cos x$

$\therefore \lim_{x\to 0} \log y = \lim_{x\to 0} \frac{\log \cos x}{\tan^2 x}$ $\frac{0}{0}$ form

$\Rightarrow \log\left(\lim_{x\to 0} y\right) = \lim_{x\to 0} \frac{\frac{1}{\cos x}\cdot(-\sin x)}{2\tan x \cdot \sec^2 x}$

$= \lim_{x\to 0} \frac{-\tan x}{2\tan x \cdot \sec^2 x}$

$= \lim_{x\to 0} \frac{-\cos^2 x}{2}$

$= -\frac{1}{2}$

$\therefore \lim_{x\to 0} (\cos x)^{\cot^2 x} = e^{-\frac{1}{2}}$ Answer

$\underline{\text{Sol}^n\text{(iv)}}$ $\lim_{x\to 0} \left(\frac{1}{x^2}\right)^{\tan x}$

Let, $y = \left(\frac{1}{x^2}\right)^{\tan x}$

$\therefore \log y = \tan x\,(\log 1 - 2\log x)$

$\therefore \lim_{x\to 0} \log y = \lim_{x\to 0} \frac{-2\log x}{\cot x}$ $\frac{\infty}{\infty}$ form

$\Rightarrow \log\left(\lim_{x\to 0} y\right) = \lim_{x\to 0} \frac{\frac{-2}{x}}{-\operatorname{cosec}^2 x}$

$$\therefore \log\left(\lim_{x\to 0} y\right) = \lim_{x\to 0} \frac{2\sin^2 x}{x} \quad \frac{0}{0} \text{ form}$$

$$= \lim_{x\to 0} \frac{2\sin x \cdot \cos x}{1}$$

$$= 0$$

$$\therefore \lim_{x\to 0} \left(\frac{1}{x^2}\right)^{\tan x} = e^0 = 1$$

Answer

Sol^n (V) $\displaystyle \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x}}$

Let, $y = \left(\frac{\sin x}{x}\right)^{\frac{1}{x}}$

$$\therefore \log y = \frac{1}{x} \log\left(\frac{\sin x}{x}\right)$$

$$\therefore \lim_{x\to 0} \log y = \lim_{x\to 0} \frac{\log\left(\frac{\sin x}{x}\right)}{x} \quad \frac{0}{0} \text{ form}$$

$$\Rightarrow \log\left(\lim_{x\to 0} y\right) = \lim_{x\to 0} \frac{\frac{x}{\sin x}\left(\frac{x\cos x - \sin x}{x^2}\right)}{1}$$

$$= \lim_{x\to 0} \frac{x\cos x - \sin x}{x \sin x} \quad \frac{0}{0} \text{ form}$$

$$= \lim_{x\to 0} \frac{-x\sin x + \cos x - \cos x}{x\cos x + \sin x}$$

$$= \lim_{x\to 0} \frac{-x\sin x}{x\cos x + \sin x} \quad \frac{0}{0} \text{ form}$$

$$= \lim_{x\to 0} \frac{-x\cos x - \sin x}{-x\sin x + \cos x + \cos x}$$

$$\therefore \log\left(\lim_{x\to 0} y\right) = \frac{-0-0}{0+1+1} = 0$$

$$\therefore \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x}} = e^0 = 1 \quad \text{Answer}$$

Sol^n (vi) $\displaystyle \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$

Sol^n: Let $A = \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$

$$\therefore \log A = \frac{1}{x^2}\log\left(\frac{\sin x}{x}\right)$$

$$\therefore \lim_{x\to 0}\log A = \lim_{x\to 0}\frac{\log\left(\frac{\sin x}{x}\right)}{x^2} \quad \frac{0}{0} \text{ form}$$

$$\Rightarrow \log\left(\lim_{x\to 0} A\right) = \lim_{x\to 0}\frac{\frac{x}{\sin x}\left(\frac{x\cos x - \sin x}{x^2}\right)}{2x}$$

$$= \lim_{x\to 0}\frac{x\cos x - \sin x}{2x^2\sin x} \quad \frac{0}{0} \text{ form}$$

$$= \lim_{x\to 0}\frac{-x\sin x + \cancel{\cos x} - \cancel{\cos x}}{2x^2\cos x + 4x\sin x} \quad \frac{0}{0} \text{ form}$$

$$= \lim_{x\to 0}\frac{-x\cos x - \sin x}{-2x^2\sin x + 4x\cos x + 4x\cos x + 4\sin x}$$

$$= \lim_{x\to 0}\frac{-x\cos x - \sin x}{-2x^2\sin x + 8x\cos x + 4\sin x} \quad \frac{0}{0} \text{ form}$$

$$= \lim_{x\to 0}\frac{x\sin x - \cos x - \cos x}{-2x^2\cos x - 4x\sin x + 8\cos x - 8x\sin x + 4\cos x}$$

$$\therefore \log\left(\lim_{x\to 0} A\right) = \frac{0 - 1 - 1}{-0 - 0 + 8 - 0 + 4}$$

$$= \frac{-2}{12}$$

$$= -\frac{1}{6}$$

$$\therefore \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}} = e^{-\frac{1}{6}}$$

Sol^n (vii) $\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$

Let $A = \left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$

$$\therefore \log A = \frac{1}{x} \log\left(\frac{\tan x}{x}\right)$$

$$\therefore \lim_{x\to 0} \log A = \lim_{x\to 0} \frac{\log\left(\frac{\tan x}{x}\right)}{x} \quad \frac{0}{0} \text{ form}$$

$$\Rightarrow \log\left(\lim_{x\to 0} A\right) = \lim_{x\to 0} \frac{\frac{x}{\tan x}\left(\frac{x\sec^2 x - \tan x}{x^2}\right)}{1}$$

$$= \lim_{x\to 0} \frac{x\sec^2 x - \tan x}{x\tan x}$$

$$= \lim_{x\to 0} \frac{\frac{x}{\cos^2 x} - \frac{\sin x}{\cos x}}{\frac{x\sin x}{\cos x}}$$

$$= \lim_{x\to 0} \frac{x - \sin x\cos x}{\cos^2 x} \times \frac{\cos x}{x\sin x}$$

$$\therefore \log\left(\lim_{x\to 0} A\right) = \lim_{x\to 0} \frac{x - \sin x \cos x}{x \sin x \cos x}$$

$$= \lim_{x\to 0} \frac{2x - 2\sin x \cos x}{x.2\sin x \cos x}$$

$$= \lim_{x\to 0} \frac{2x - \sin 2x}{x \sin 2x} \quad \frac{0}{0} \text{ form}$$

$$= \lim_{x\to 0} \frac{2 - 2\cos 2x}{2x \cos 2x + \sin 2x} \quad \frac{0}{0} \text{ form}$$

$$= \lim_{x\to 0} \frac{0 + 4\sin 2x}{-4x \sin 2x + 2\cos 2x + 2\cos 2x}$$

$$= \frac{0 + 0}{0 + 2 + 2}$$

$$= 0$$

$$\therefore \lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x}} = e^0 = 1$$

(viii) H.W. $\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}} = e^{\frac{1}{3}}$