Chapter 8

Asymptotes

8.1 Introduction

A curve in a plane is either closed or open. Examples of closed curves are circle, ellipse, whose lengths are limited. Open curves are those whose graphs extend to infinity, such as parabola and hyperbola.

Thus a straight line touching a curve at infinity is called its *asymptotes*. Let a curve be given and a tangent be drawn at some point of the curve. If the point of contact of the tangent goes further away from the origin, then the distance of the tangent from the origin will also go on changing; somestimes it will increase continuously and sometimes it will decrease continuously. But it may be possible that when the point of contact tends to infinity, then the tangent takes up a definite position of a straight line. This is called 'asmyptotes'.

In other words, if P be a point on a branch of curves which extends to infinite and a straight line exists at a finite distance from the origin, from which the distance of P gradually diminishes and ultimately tends to zero as P tends to infinity, moving along the area, then such a straight line is called an 'asymptote' to the curve.

In a simple language, an asymptote is a straight line, which cuts a curve in two points at infinity (i.e. touches at infinity) but is not itself at infinity. In other words, an asymptote is a tangent whose points of contact are $x = \infty$, $y = \infty$.

This can be understood as follows: Let $P(x, y)$ be any point on the curve $y = f(x)$. Then a straight line will be the asymptote of the curve if the perpendicular distance of the point $P(x, y)$ from the straight line tends to zero as $x \to \infty$ or $y \to \infty$ or both $x, y \to \infty$.

For example, the straight line $x = 2a$ is an asymptote of the cissoid $y^2(2a - x) = x^3$. We find that as $P(x, y)$ moves to infinity, its distance from the line $x = 2a$ tends to zero (Fig. 8.1).

Asymptote of a curve can be obtained in a number of ways and we shall discuss them one by one.

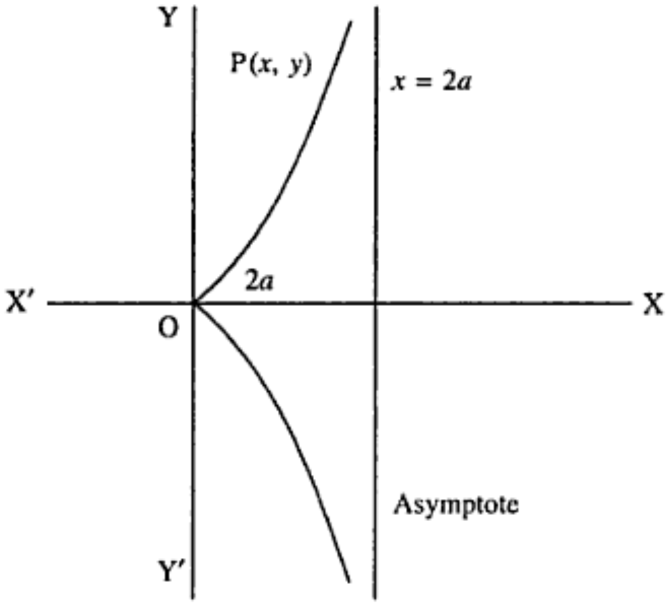

Fig. 8.1 An asymptote.

Solution to an equation with two infinite roots

If $y = mx + c$ be an asymptote of the curve $\phi(x, y) = 0$, we solve these equations for at least two infinite roots, such roots help find sets of suitable values of m and c for the asymptotes. This is the basic method to find the asymptotes.

Asymptotes of an algebraic curve

Let the equation of the curve be

$$(a_0x^n + a_1x^{n-1}y + \cdots + a_ny^n) + (b_1x^{n-1} + b_2x^{n-2}y + b_3x^{n-3}y^2 + \cdots b_ny^{n-1}) +$$
$$(c_2x^{n-2} + c_3x^{n-3}y + c_4x^{n-4}y^2 + \cdots + c_ny^{n-2}) + \cdots = 0,$$

where a's, b's and c's, are constants. In the first parentheses, we put all those terms in which the sum of the indices of x and y is n, i.e. the term in the first parentheses is a homogeneous function of x and y of degree n. Similarly, the term in the second parentheses is a homogeneous function of degree $(n - 1)$, and so on. Therefore,

$$x^n\phi_n\left(\frac{y}{x}\right) + x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right) + x^{n-2}\phi_{n-2}\left(\frac{y}{x}\right) + \cdots = 0 \tag{8.1}$$

where $x^r\phi_r(y/x)$ is a homogeneous function of degree r in x and y.

Let the equation of the asymptote be

$$y = mx + c \tag{8.2}$$

Now we want to find out the point of intersection of the line $y = mx + c$ with the given curve after simplification, we get

$$\frac{y}{x} = m + \frac{c}{x}$$

Putting this in Eq. (8.1), we get

$$x^n\phi_n\left(m+\frac{c}{x}\right)+x^{n-1}\phi_{n-1}\left(m+\frac{c}{x}\right)+x^{n-2}\phi_{n-2}\left(m+\frac{c}{x}\right)+\cdots=0 \qquad (8.3)$$

which gives the abscissa of point of intersection of the line and the curve. Expanding Eq. (8.3) by Taylor's theorem, we get

$$x^n\left[\phi_n(m)+\frac{c}{x}\phi_n'(m)+\frac{c^2}{x^2}\frac{1}{2!}\phi_n''(m)+\cdots\right]+x^{n-1}\left[\phi_{n-1}(m)+\frac{c}{x}\phi_{n-1}'(m)+\cdots\right]$$

$$+\,x^{n-2}\left[\phi_{n-2}(m)+\cdots\right]+\cdots=0$$

or

$$x^n\phi_n(m)+x^{n-1}\left[c\phi_n'(m)+\phi_{n-1}(m)\right]+x^{n-2}\left[\frac{c^2}{2!}\phi_n''(m)+c\phi_{n-1}'(m)+\phi_{n-2}(m)\right]+\cdots=0$$

(8.4)

This is the equation of the nth degree in x showing that the straight line cuts a curve of the nth degree in n points, in general (real or imaginary). If the straight line (8.2) is an asymptote to the curve, it cuts the curve at infinity. Therefore this equation has two infinite roots for which the coefficients of two highest-degree terms should be zero:

$$\phi_n(m)=0 \qquad (8.5)$$

$$c\phi_n'(m)+\phi_{n-1}(m)=0 \qquad (8.6)$$

If the roots of (8.5) be $m_1, m_2, m_3, \cdots, m_n$, the corresponding values of c, i.e. $(c_1, c_2, c_3, \cdots, c_n)$ are given by Eq. (8.6). Therefore,

$$c=-\frac{\phi_{n-1}(m)}{\phi_n'(m)}$$

Hence the n asymptotes are

$$y=m_1x+c_1, \qquad y=m_2x+c_2, \qquad y=m_3x+c_3 \quad \cdots, \qquad y=m_nx+c_n.$$

Working rules:

(i) In the highest-degree terms, put $x = 1$ and $y = m$; then we get $\phi_n(m)$. Equating it to zero and solving it, we get $m = m_1, m_2, m_3, \cdots$

(ii) In the next lower-degree terms, put $x = 1$, $y = m$, then we get $\phi_{n-1}(m)$.

(iii) To get c put the values of m in the formula $c = -\phi_{n-1}(m)/\phi'_n(m)$.

(iv) If this formula takes the form 0/0 by the substitution of the value of m, then use

$$\frac{c^2}{2!}\phi_n''(m)+c\phi_{n-1}'(m)+\phi_{n-2}(m)=0$$

to get c.

Example 8.1 Find the asymptotes to the curve $y^3 - 3x^2y + xy^2 - 3x^3 + 2y^2 + 2xy + 4x - 5y + 6 = 0$.

Solution Let the asymptote be $y = mx + c$. Here we put $x = 1$, $y = m$ in the highest-degree terms, i.e. in the third-degree terms of the given equation of the curve and equate it to zero, we get,

$$\phi_3(m) = m^3 - 3m + m^2 - 3 \tag{1}$$

Therefore, $\phi_3(m) = 0$ gives

$$m^3 - 3m + m^2 - 3 = 0 \quad \text{or} \quad (m + 1)(m^2 - 3) = 0.$$

We get

$$m = -1, \pm \sqrt{3}. \tag{2}$$

Again, put $x = 1$ and $y = m$ in the second-degree terms of the given equation of the curve, we get

$$\phi_2(m) = 2m^2 + 2m \tag{3}$$

Differentiating (1) with respect to m, we get $\phi_3' = 3m^2 - 3 + 2m$. Therefore,

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{2m^2 + 2m}{3m^2 + 2m - 3}$$

Here, we put m from (2), we get $c = 0$, when $m = -1$, $c = 1$, when $m = \sqrt{3}$ and $c = 1$, when $m = -\sqrt{3}$. Hence the required asymptotes are $y = mx + 1$, that is

$$x + y = 0 \quad \text{and} \quad y = \pm\sqrt{3}x + 1$$

Oblique asymptotes of algebraic curve

Let the rational algebraic expression containing terms of the rth and lower, but of no higher degrees be denoted by P_r, F_r.

(a) Let the equation of the curve of nth degree can be put into the form

$$(ax + by + c)\, P_{n-r} + F_{n-1} = 0. \tag{8.7}$$

Then the straight line parallel to $ax + by = 0$, obviously cuts the curve (8.7) in one point at infinity. We are now to find out the particular member of this family of parallel straight lines which cuts the curve (8.7) in a second point at infinity. We will now examine the ultimate linear form to which the curve reaches infinity. We make x and y in the equation of the curve in the ratio by

$$\frac{x}{y} = -\frac{b}{a}.$$

Therefore, (8.7) becomes

$$ax + by + c \lim_{x\to\infty,\, y=-a/b} \frac{F_{n-1}}{P_{n-1}} = 0$$

$a_1y + b_1 = 0$, $y = -b_1/a_1$ is the asymptote parallel to x-axis. Similarly, rearranging the terms of the equation of the curve in descending powers of y, we get

$$a_n y^n + (a_{n-1}x + b_n)y^{n-1} + (a_{n-2}x^2 + b_{n-1}x + c_n)y^{n-2} + \cdots = 0 \qquad (8.12)$$

Hence if $a_n = 0$, and x be so chosen that $a_{n-1}x + b_n = 0$, the coefficient of the two highest powers of y in (8.12) vanish, and therefore two of its root are infinite.

Hence the straight line $a_{n-1}x + b_n = 0$ or $x = -b_n/a_{n-1}$ is an asymptote parallel to the axis of y.

Again if $a_0 = 0$, $a_1 = 0$, $b_1 = 0$ and if y be so chosen that $a_2y^2 + b_2y + c_2 = 0$, three roots of Eq. (8.11) be infinite and the lines represented by $a_2y^2 + b_2y + c_2 = 0$ represent a pair of asymptotes (real or imaginary) parallel to the x-axis.

Similarly, if $a_n = 0$, $a_{n-1} = 0$, $b_n = 0$ and x be so chosen that $a_{n-2}x^2 + b_{n-1}x + c_n = 0$, three roots of Eq. (8.12) be infinity and the lines represented by $a_{n-2}x^2 + b_{n-1}x + c_n = 0$ represent a pair of asymptotes (real or imaginary) parallel to the y-axis.

Working rules:

(i) In order to obtain the asymptotes parallel to the axis of x, equate to zero, the coefficient of the highest power of x. For example, if the curve be of the nth degree and term containing x^n be absent, the coefficient of x^{n-1} equated to zero will give the asymptotes parallel to the axis of x.

(ii) If both the terms containing x_n and x^{n-1} be absent, then the coefficient of x^{n-2} equated to zero will give two asymptotes parallel to the axis of x.

(iii) To get the asymptotes parallel to the axis of y, equated to zero, the coefficient of the highest power of y. For example, if the curve be of nth degree and the term containing y_n be absent then the coefficient of y_{n-1} equated to zero will give the asymptotes parallel to the axis of y.

(iv) If both the terms containing y^n and y^{n-1} be absent then the coefficients of y^{n-2} equated to zero will give two asymptotes parallel to the axis of y.

Corollary If the equation of the curve be the nth degree, and the coefficient of x^n is not zero, then there will be no asymptote parallel to the axis of x. Similarly, if the coefficient of y^n is not zero, then there will be no asymptote parallel to the axis of y. For example, the curve $x^3 + y^3$, $3axy$ will have no asymptote parallel either to x-axis or y-axis, as the coefficients of x^3 and y^3, the highest-degree terms, are not zero.

Example 8.2 Find the asymptotes to the curve $x^2y^2 = a^2y^2 + b^2x^2$.

Solution Here $x^2y^2 = a^2y^2 + b^2x^2$ be the equation of the curve. This is the 4th degree equation. The terms containing, x^4, x^3, y^4 and y^3 are absent. Hence equating to zero the coefficient of x^2 and y^2, will give the asymptotes parallel to the axis of x and the axis of y.

Here the coefficient of $x^2 = y^2 - b^2$ or $y = \pm b$. Hence the asymptotes to the axis of x are $y = b$ and $y = -b$. Again, the coefficient of $y^2 = x^2 - a^2$ to zero. Therefore $x = \pm a$.

Hence the asymptotes parallel to the axis of y are $x = a$ and $x = -a$. Thus the required asymptotes are $y - b = 0$, $y + b = 0$, $x - a = 0$ and $x + a = 0$.

8.3 Asymptotes by Inspection

(a) If the equation of an algebraic curve be put in the form $F_n + F_{n-2} = 0$, where F_n consists of nth degree and lower degree terms which cannot be expressed as a product of n linear factors, none of which is repeated; and F_{n-2} consists of terms of degree $(n - 2)$ or lower degree terms. Then all the asymptotes to the given curve will be given by $F_n = 0$.

(b) If in the equation $F_n + F_{n-2} = 0$ of the curve, F_n consists of real linear factors (some repeated and some non-repeated factors). Then, the non-repeated factors equated to zero will definitely be the asymptotes to the curve. But the asymptotes corresponding to the repeated factors will however have to be obtained as in the general case.

Total number of asymptotes to a curve

Let $y = mx + c$ be the equation of an asymptote. We know that the value of m is found out by solving the equation $\phi_n(m) = 0$. Since the equations of nth degree has n roots, we shall get n values of m by solving $\phi_n(m) = 0$. We shall get an asymptote corresponding to each value of m. Hence the curve of nth degree has generally n asymptotes, real or imaginary.

Theorem 8.1 If $y = mx + c$ is an asymptote to a curve then

$$m = \lim_{x\to\infty} \frac{y}{x} \quad \text{and} \quad c = \lim_{x\to\infty}(y - mx)$$

Proof Let $y = mx + c$ be an asymptote to the curve, where m and c are to be obtained. Let P(x, y) be any point on the curve (Fig. 8.2). From P, draw a perpendicular on $y = mx + c$, whose length is p. Then

$$p = \frac{y - mx - c}{\sqrt{1 + m^2}} \tag{8.15}$$

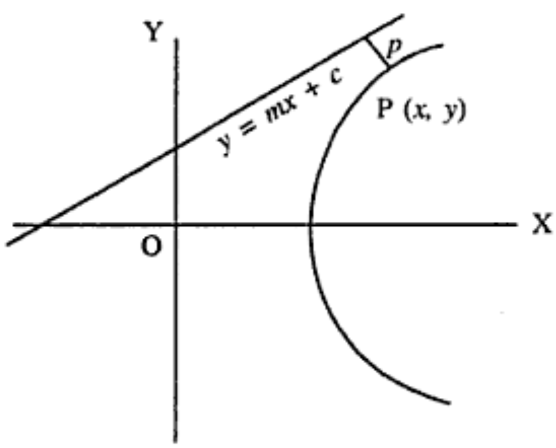

Fig. 8.2 Perpendicular from an asymptote to a curve.

From the figure, it is obvious that as the point tends to infinity along the curve, the distance between the curve and the line becomes lesser and lesser, i.e.

$p \to 0$. When $y = mx + c$ touches the curve at infinity, then $p \to 0$. Thus when $x \to \infty$, $p \to 0$. From this we get the values of m and c. Now taking the limit of Eq. (8.15), when $x \to \infty$,

$$\lim_{p\to 0} = \lim_{x\to\infty} \frac{y - mx - c}{\sqrt{1+m^2}} \to 0 = \lim_{x\to\infty}(y - mx - c)$$

Therefore,

$$c = \lim_{x\to\infty} (y - mx) \tag{8.16}$$

Again,

$$y = mx + c + p\sqrt{1+m^2}$$

or

$$\frac{y}{x} = m + \frac{c}{x} + \frac{p}{x}\sqrt{1+m^2}$$

Therefore,

$$\lim_{x\to\infty} \frac{y}{x} = m + \lim_{x\to\infty}\frac{c}{x} + \lim_{x\to\infty}\frac{p}{x}\sqrt{1+m^2}$$

Hence

$$m = \lim_{x\to\infty} \frac{y}{x} \tag{8.17}$$

Thus, we find out the values of c and m from Eqs. (8.16) and (8.17) and thereby the equation of the asymptotes $y = mx + c$ is found out.

Example 8.4 Find the asympotes of the curve $x^3 + y^3 = 3axy$.

Solution Let $y = mx + c$ be the equation of the asymptote. Here we divide the equation of the curve by x_1^3, we get

$$1 + \left(\frac{y}{x}\right)^3 = 3a\frac{y}{x^2} = 3a\frac{y}{x}\frac{1}{x}$$

Therefore,

$$\lim_{x\to\infty}\left[1 + \left(\frac{y}{x}\right)^3\right]^x = 3a \lim_{x\to\infty}\frac{y}{x} \lim_{x\to\infty}\frac{1}{x}$$

Then

$$1 + m^3 = 0 \quad \text{or} \quad (1 + m)(m^2 - m + 1) = 0.$$

Therefore, the real value of $m = -1$. Now

$$C = \lim_{x\to\infty} (y - mx) = \lim_{x\to\infty} (y + x)$$

Now put $(c - x)$ for y and take the limit $x \to \infty$. Therefore,

$$x^3 + (c - x)^3 = 3ax\,(c - x)$$

or

$$(3c+3a)-3c(c+a)\frac{1}{x}+\frac{c^3}{x^2}=0$$

Taking the limit when $x \to \infty$, we have

$$3c + 3a = 0 \quad \text{or} \quad c = -a.$$

Hence, the required equation of the asymptote is

$$y = -x - a \quad \text{or} \quad x + y + a = 0.$$

Example 8.5 Find asymptotes of the curve $x^3 - 2y^3 + 2x^2y - xy^2 + xy - y^2 + 1 = 0$.

Solution Putting $x = 1$, $y = m$ in the third-degree terms and equate to zero, we get

$$\phi_3(m) = 1 - 2m^3 + 2m - m^2 = 0,$$

or

$$(1 + 2m) - m^2(1 + 2m) = 0$$

or

$$(1 + 2m)(1 + m)(1 - m) = 0$$

We get $m = 1, -1, -1/2$. Again,

$$\phi_2(m) = m - m^2 = m(1 - m)$$

Therefore,

$$c=-\frac{\phi_2(m)}{\phi_3'(m)}=-\frac{m(1-m)}{-6m^2+2-2m}$$

$$\text{when } m = 1, \qquad c = 0$$

$$\text{when } m = 1, \qquad c = -\frac{(-1)(1+1)}{-6.1+2+2}=-\frac{-2}{-2}=-1$$

$$\text{when } m = -\frac{1}{2}, \qquad c=-\frac{(-1/2)(1+1/2)}{-6(1/4)+2+2(1/2)}=\frac{1}{2}.$$

Therefore, putting the corresponding values of m and c in $y = mx + c$, we get the asymptotes as: $y = x$, $x + y + 1 = 0$, $x + 2y - 1 = 0$.

Example 8.6 Find the asymptotes of $x^3 + x^2y - xy^2 - y^3 + 2xy + 2y^2 - 3x + y = 0$.

Solution Putting $x = 1$ and $y = m$ in the third-degree terms equating to zero $\phi_3(m) = 1 + m - m^2 - m^3 = 0$. Therefore, $m = -1, -1, 1$.

Again $\phi_2(m) = 2m + 2m^2 = 2m(m + 1)$. Then

$$c=-\frac{\phi_2(m)}{\phi_3'(m)}=-\frac{2m(m+1)}{1-2m-3m^2}$$

$$\text{when } m = 1, \qquad c = -\frac{2.2}{1-2-3} = 1$$

$$\text{when } m = -1, \qquad c = \frac{0}{0}.$$

Now to find the value of c, we choose first-degree terms to zero

$$\frac{c^2}{2!}\phi_3''(m) + c\phi_2'(m) + \phi_1(m) = 0$$

or

$$\frac{c^2}{2}(-2-6m) + c(2+4m) + (-3+m) = 0$$

or

$$-c^2(1 + 3m) + c(2 - 4) + (-3 - 1) = 0$$

Solving, we get $c = -1, 2$.

When $m = 1$, then $c = 1$, when $m = -1$, then $c = -1$ or 2. Therefore, the equation to the asymptotes are: $y = x + 1$, $y + x + 1 = 0$, $y + x - 2 = 0$.

Example 8.7 Find the asymptotes to $4x^3 - 3xy^2 - y^3 + 2x^2 - xy - y^2 - 1 = 0$.

Solution Putting $x = 1$ and $y = m$ in the third-degree terms and equating to zero, we get $\phi_3(m) = 4 - 3m^2 - m^3 = 0$. Solving, we get $m = 1, -2, -2$.

Again,

$$\phi_2(m) = 2 - m - m^2 = -(m^2 + m - 2) = -(m - 1)(m + 2)$$

Therefore,

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{-(m-1)(m+2)}{-6m-3m^2} = -\frac{m-1}{3m}$$

$$\text{when } m = 1, \qquad c = -\frac{1-1}{3} = 0$$

$$\text{when } m = -2, \qquad c = \frac{0}{0}$$

Therefore, using

$$\frac{c^2}{2!}\phi_3''(m) + c\phi_2'(m) + \phi_1(m) = 0.$$

we get

$$\frac{c^2}{2}(-6-6m) + c(-1-2m) + 0 = 0$$

Solving, we get

$$c = 0 \text{ or } 3c\,(1 + m) + (1 + 2m) = 0.$$

Thus when $m = 1$, $c = 0$; when $m = -2$, $c = -1$. Therefore, the equations of asymptotes are: $y - x = 0$, $y + 2x = 0$ and $y + 2x + 1 = 0$.

Example 8.8 Find the asymptotes to the curve

$$x^3 - 2x^2y + xy^2 + x^2 - xy + 2 = 0.$$

Solution Here the highest power term in x is x^3 and the coefficient of x^3 is $1(\neq 0)$. Therefore, there is no asymptote parallel to x-axis. Again, the highest power term in y is $xy^2 = 0$. We shall get one asymptote $x = 0$ from the coefficient x of y^2. For finding out the equation of the other two remaining systems, we follow the previous method.

Here the highest degree of the equation is 3. Therefore, putting $x = 1$ and $y = m$ in the third degree terms and equating to zero, we get

$$\phi_3(m) = 1 - 2m + m^2 = 0 \quad \text{or} \quad m = 1, 1$$

Again $\phi_2(m) = 1 - m$. Therefore,

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{1-m}{-2+2m} = -\frac{1-m}{2(m-1)}.$$

When $m = 1$, then $c = 0/0$. Therefore, we get

$$\frac{c^2}{2!}\phi_3''(m) + c\phi_2'(m) + \phi_1(m) = 0$$

or

$$\frac{c^2}{2}(2) + c(-1) + 0 = 0.$$

Solving, we get $c = 0$ or $c = 1$. Thus when $m = 1$, $c = 0$ or $c = 1$, the equations of the asymptotes are: $y = x$, $y = x + 1$ and $x = 0$.

Alternative method: The equation of the curve can be written as

$$x(x^2 - 2xy + y^2) + x(x - y) + 2 = 0$$

or

$$x(x - y)^2 + x(x - y) + 2 = 0$$

or

$$x(x - y)(x - y + 1) + 2 = 0,$$

which is of the form of $F_n + F_{n-2} = 0$. Therefore, the three asymptotes are:

$$x = 0, x - y = 0 \quad \text{and} \quad x - y + 1 = 0.$$

Example 8.9 Find the asymptotes to the curve $y^2(x - 2a) = x^3 - a^3$.

Solution The equation of the given curve is of third order. Here the coefficient of x^3 is $1(\neq 0)$. Therefore, there is no asymptote parallel to x-axis.

Again the coefficient of $y^3 = 0$. Then equating to zero the coefficient of y^2, we get $x - 2a = 0$. Therefore, the equation of the asymptote parallel to y-axis is $x - 2a = 0$. Now, putting $x = 1$, $y = m$ in the third-degree terms and equating to zero, we get

$$\phi_3(m) = m^2 - 1 = 0 \quad \text{or} \quad m = 1, -1.$$

Now, by putting $x = 1/t$, $y = -1/t$ so that $t \to 0$, we have

$$x + y = \pm \lim_{t \to 0} \sqrt{\frac{1/t - 9/t + 2}{1/t - 2/t + 2}} = \lim_{t \to 0} \pm \sqrt{\frac{-8 + 2t}{-1 + 2t}} = \pm\sqrt{8} = \pm 2\sqrt{2}.$$

Hence the three asymptotes are: $x + 2y + 2 = 0$, $x + y = 2\sqrt{2}$ and $x + y = -2\sqrt{2}$.

[redacted]mple 8.11 Find the asymptotes to the curve $(x^2 - y^2)y - 2ay^2 + 5x - 7 = 0$, and prove that the asymptotes form a triangle of area a^2.

Solution Here the given equation of the curve can be written as $x^2y - y^3 - 2ay^2 + 5x - 7 = 0$. Here, equating the coefficient y of x^2, the highest power term of x, to zero, we get the equation of the asymptote parallel to x-axis as $y = 0$.

Now, putting $x = 1$, $y = m$ in the highest power, i.e. in the third-degree term in the equation of the curve and equating to zero, we get

$$\phi_3(m) = m - m^3 = 0 \quad \text{or} \quad m = 0, 1, -1.$$

Again, $\phi_2(m) = -2am^2$. Then

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{-2am^2}{1 - 3m^2} = \frac{2am^2}{1 - 3m^2}.$$

Therefore, when $m = 0$, $c = 0$; when $m = 1$, $c = -a$; and when $m = -1$, $c = -a$.

Therefore, the equation of the asymptotes are

$$y = 0, \qquad y = x - a, \qquad y = -x - a.$$

Solving the three asymptotes, we get

$$x = a,\ \ y = 0; \qquad x = -a,\ \ y = 0; \qquad x = 0,\ \ y = -a.$$

Hence the coordinates of the three vertices of a triangle are: (a, 0), (–a, 0), and $(0, -a)$. Hence

$$\text{Area of } \Delta = \frac{1}{2}\,[a\,(0 + a) + (-a)(-a - 0) + 0(0 - 0)] = a^2.$$

Example 8.12 Find the asymptotes to the cubic $x^2y - xy^2 + xy + y^2 + x - y = 0$, and show that they cut the curve again in three points lying on the straight line.

Solution Putting $x = 1$, $y = m$, we get, $\phi_3(m) = m - m^2 = 0$. Therefore, $m = 0, 1$. Again $\phi_2(m) = m + m^2$. Hence.

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{m(1 + m)}{1 - 2m}.$$

When $m = 0$, $c = 0$; also when $m = 1$, $c = 2$. Therefore, the equation of two asymptotes are: $y = 0$ and $y = x + 2$.

Again equating to zero the higher powers in x and y, the equations to the asymptote parallel to the coordinate axes are: $y = 0$ and $1 - x = 0$, i.e. $x = 1$.

Thus the equations of three asymptotes are

$$y = 0, \quad x - 1 = 0, \quad x - y + 2 = 0.$$

Now the joint asymptotes of the three equations will be

$$y(x - 1)(x - y + z) = 0$$

or

$$x^2y - xy^2 + xy + y^2 - 2y = 0 \tag{1}$$

which we write as $P_3 = 0$.

But the equations to the curve is $x^2y - xy^2 + xy + y^2 - 2y + (x + y) = 0$, i.e. $P_3 + (x + y) = 0$. Hence the point of intersection of the curve and asymptotes satisfies $x + y = 0$, which represents a straight line.

But a straight line cuts a curve of third-degree in 3 points. But each asymptote passes through two points of infinity. Therefore, it will cut the given curve at $(3 - 2)$ or 1 point more. Since the number of asymptotes = 3, the number of points of interesction of the curve and asymptotes = $3 \times 1 = 3$.

Thus all the three points of intersection of the curve and asymptotes lie on a straight line.

Example 8.13 Determine the asymptotes to the curve.

$$4(x^4 + y^4) - 17x^2y^2 - 4x(4y^2 - x^2) + 2(x^2 - 2) = 0$$

and show that they pass through the point if intersection of the curve with the ellipse $x^2 + 4y^2 = 4$.

Solution The equation of the curve can be written as

$$(4x^4 + 4y^2 - 17x^2y^2) - 4x(4y^2 - x^2) + 2(x^2 - 2) = 0$$

or

$$(x^2 - 4y^2)(4x^2 - y^2) + 4x(x^2 - 4y^2) + 2(x^2 - 2) = 0.$$

Simplifying, we get

$$(x + 2y)(x - 2y)(2x + 1 + y)(2x + 1 - y) + (x^2 + 4y^2 - 4) = 0$$

Hence the required asymptotes are

$$(x - 2y) = 0, \quad x + 2y = 0, \quad 2x + y + 1 = 0, \quad 2x - y + 1 = 0. \tag{1}$$

From the equation of the curve and (1), it is evident that the points of intersection of the curve and the ellipse $x^2 + 4y^2 = 4$ satisfy the equations of the asymptotes.

Example 8.14 Find the asymptotes to the curve

$$x^3 + 3x^2y - 4y^3 - x + y + 3 = 0.$$

Solution Here the equation of the curve is written as

$$x^3 - x^2y + 4x^2y - 4y^2 - x + y + 3 = 0$$

or

$$(x - y)(x + 2y)^2 = x - y - 3.$$

or

$$y - 2x + 4 = 0.$$

Thus the asymptotes are: $y + x + 1 = 0$, $y + x + z = 0$ and $y - 2x + 4 = 0$.

Example 8.16 Find the asymptotes of the curve $y^3 + x^2y + 2xy^2 - y + 1 = 0$.

Solution The equation of the curve can be written as

$$y(y^2 + x^2 + 2xy) - y + 1 = 0,$$

or

$$y(y + x)^2 - y + 1 = 0.$$

Therefore, the curve has a pair of asymptotes

$$x + y = \pm\sqrt{\lim_{\substack{x,y\to\infty \\ x=-y}} \frac{y-1}{y}} = \pm 1$$

Thus the two asymptotes are $x + y = 1$ and $x + y = -1$.

The third asymptote is obtained by equating to zero the coefficient of x^2 which is $y = 0$. Thus the asympototes are $x + y = 1$, $x + y + 1 = 0$ and $y = 0$.

Example 8.17 Find the asymptotes of $x^3 + 2x^2y - 4xy^2 - 8y^3 - 4x + 8y - 1 = 0$.

Solution The given equation can be written as

$$x^2(x + 2y) - 4y^2(x + 2y) - 4(x - 2y) - 1 = 0$$

or

$$(x + 2y)(x^2 - 4y^2) - 4(x - 2y) - 1 = 0$$

or

$$(x - 2y)(x + 2y + 2)(x + 2y - 2) - 1 = 0$$

which is in the from of $F_n + F_{n-2} = 0$. Therefore, the asymptotes are: $x - 2y = 0$, $x + 2y + 2 = 0$, and $x + 2y - 2 = 0$.

Example 8.18 Find the asymptotes of $y^3 - xy^2 - x^2y + x^3 + x^2 - y^2 = 1$.

Solution The given equation can be written as

$$y^2(y - x) - x^2(y - x) - (y - x)(y + x) - 1 = 0.$$

or

$$(y - x)[y^2 - x^2 - (y + x)] - 1 = 0$$

or

$$(y - x)(y + x)(y - x - 1) - 1 = 0$$

which is in the form of $F_n + F_{n-2} = 0$. Hence the asymptotes are: $y - x = 0$, $y + x = 0$ and $y - x - 1 = 0$.

Example 8.19 Show that there is an infinite series of parrallel asymptotes to the curve

$$r = \frac{a}{\theta \sin\theta} + b$$

and show that their distance from the pole are in harmonic progression.

Solution Here

$$r = \frac{a}{\theta \sin \theta} + b \quad \text{or} \quad r\,\theta \sin \theta - (a + b\theta \sin \theta) = 0.$$

It is of the form of $rf_1(\theta) + f_0(\theta) = 0$. The directions for asymptotes are

$$f_1(\theta) = 0 \quad \text{or} \quad \theta \sin \theta = 0 \quad \text{or} \quad \theta = n\pi,$$

where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$ Thus the asymptotes are

$$r \sin (n\pi + \theta) = \frac{f_0(n\pi)}{f_1'(n\pi)} = -\frac{a + bn\pi \sin n\pi}{\sin n\pi + n\pi \cos n\pi} \tag{1}$$

If $n \neq 0$, by (1) the asymptotes are

$$r(-1)^{n+1} \sin \theta = \frac{-a}{n\pi(-1)^n} \quad \text{or} \quad r \sin \theta = \frac{a}{n\pi} \tag{2}$$

When $n = 0$, the right-hand side of (1) is infinity so we get no asymptote in this case. These are infinite number of asymptotes corresponding to $n = \pm 1, \pm 2, \pm 3, \cdots$ and the perpendicular distance of these from the pole are

$$\frac{a}{\pi}, \frac{a}{2\pi}, \frac{a}{3\pi}, \cdots$$

which are clearly in harmonic progression.

Exercises 8.1

Find the asymptotes of the following:

1. $x^3 + y^3 = a^3$

2. $y^3 = x(a^2 - x^2)$

3. $x^2y + xy^2 = a^2$

4. $x^2y = x^3 + x + y$

5. $y^2(a - x) = x^2(x + a)$

6. $y^2(x^2 - y^2) = x^2(x^2 - 4a^2)$

7. $xy(x - y) + bx^2 - ay^2 = 0$

8. $y^2(a^2 + x^2) = x^2(a - x)^2$

9. $(y - a)^2(x^2 - a^2) = x^4 + a^4$

10. $9x^4 - 4x^2y^2 + x^2 + y^2 - 1 = 0$

11. $(a + x)^2(b^2 + x^2) = x^2y^2$,

12. $y^3 - xy^2 - x^2y + x^3 - x^2 - y^2 = 1$

13. $x^2y^2 - x^2y - xy^2 + x + y + 1 = 0$

14. $2y^3 + 3y^2x + 3yx^2 + x^3 - y^2 - x^2 - x = 0$

15. $x^3 + 2x^2y - xy^2 - 2y^3 + x^2 - y^2 - 2x - 3y = 0$

16. $x^3 - 4xy^2 - 3x^2 + 12xy - 12y^2 + 8x + 2y + 4 = 0$

17. $x^3 + 2x^2y - xy^2 - 2y^3 + 4y^2 + 2xy + y - 1 = 0$

18. $(x - y + 2)(2x - 3y + 4)(4x - 5y + 6) + 5x - 6y + 7 = 0$

58. The asymptotes to the curve $x^2y^2 - x^2 - y^2 - x - y + 1 = 0$ form a square through two of whose angular points the curve passes.

59. The asymptotes to the curve $x^2y^2 - a^2(x^2 + y^2) - a^2(x + y) + a^4 = 0$ form a square two of whose angular points lie on the curve.

60. The four asymptotes to the curve

$$(x^2 - y^2)(y^2 - 4x^2) - 6x^3 + 5x^2y + 3xy^2 - 2y^3 - x^2 + 3xy - 1 = 0$$

cut the curve again in eight points which lie on a circle $x^2 + y^2 = 1$. [*Hint:* The given equation is $(x - y)(y - 2x)(x + y + 1)(y + 2x + 1) + x^2 + y^2 - 1 = 0$.]

61. All the asymptotes of the curve, $r \tan n\theta = a$ touch the circle $r = a/n$.