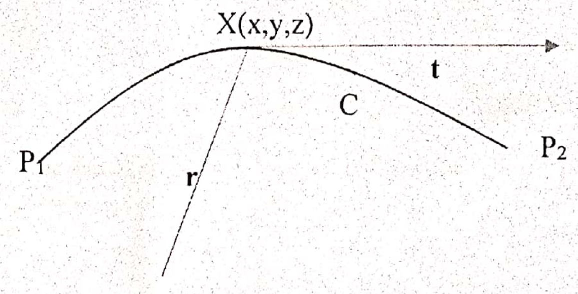
Line Integration

Let $\boldsymbol{r}(u) = x(u)\boldsymbol{i} + y(u)\boldsymbol{j} + z(u)\boldsymbol{k}$, where r(u) is the position vector of $X(x,y,z)$, define a curve C joining points P1 and P2. $\boldsymbol{A}\,(x,y,z) = A_1\boldsymbol{i} + A_2\boldsymbol{j} + A_3\boldsymbol{k}$ be a vector function continuous along C.



Let $X(x,y,z)$ be a general point on the curve C joining two points P1 and P2. Let s denote the arc length of C measured from the end P1 and P2. The vector *dr/ds* be a unit tangent vector to C at $X(x,y,z)$ in the direction of s increasing. Denoting this unit tangent vector by $\boldsymbol{t}$, we have

$$\mathbf{t} = \frac{d\boldsymbol{r}}{ds} \quad \dots (1)$$

Let $\boldsymbol{A}(x,y,z)$ be a vector field defined over C. The orthogonal projection of $\boldsymbol{A}$ on the unit tangent vector $\boldsymbol{t}$ is called the tangential component of $\boldsymbol{A}$. If we denote it by b_1 we have

$$b_1 = \boldsymbol{A}.\boldsymbol{t}$$

The line integral of $\boldsymbol{A}$ over C is defined to be

$$\int_{p_1}^{p_2} b_1 ds = \int_{p_1}^{p_2} \boldsymbol{A}.\boldsymbol{t}\, ds$$

$$\int_{p_1}^{p_2} b_1 ds = \int_{p_1}^{p_2} \boldsymbol{A}.d\boldsymbol{r} \text{ using (1)}$$

$$\int_{p_1}^{p_2} b_1 ds = \int_{p_1}^{p_2} (\mathrm{A}_1\mathbf{i} + \mathrm{A}_2\mathbf{j} + \mathrm{A}_3\mathbf{k}).(dx\mathbf{i} + dy\boldsymbol{j} + dz\boldsymbol{k})$$

$$\int_{p_1}^{p_2} b_1 ds = \int_{p_1}^{p_2} (\mathrm{A}_1 dx + \mathrm{A}_2 dy + A_3 dz)$$

Then the integral of the tangential component of $\boldsymbol{A}$ along C from P1 to P2 is

$$\int_{p_1}^{p_2} \boldsymbol{A}.d\boldsymbol{r} = \int_{p_1}^{p_2} (A_1 dx + A_2 dy + A_3 dz)$$

Definition: Let $\boldsymbol{r}(u) = x(u)\boldsymbol{i} + y(u)\boldsymbol{j} + z(u)\boldsymbol{k}$, where $\boldsymbol{r}(u)$ is the position vector of $X(x, y, z)$, define a curve C joining points P1 and P2. $\boldsymbol{A}(x, y, z) = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ be a vector function of position defined and continuous along C. Then the integral of the tangential component of $\boldsymbol{A}$ along C from P1 to P2, written as

$$\int_{P_1}^{P_2} \mathbf{A}\cdot d\mathbf{r} = \int_C \mathbf{A}\cdot d\mathbf{r} = \int_C A_1\,dx + A_2\,dy + A_3\,dz$$

Ex 1. Evaluate $\int_c \boldsymbol{F}.d\boldsymbol{r}$ where $F=x^2i+y^3j$ and the curve C is the arc of the parabola $y=x^2$ in xy-plane from (0,0) to (1,1).

Soln. Given curve is $y = x^2 \Rightarrow dy = 2xdx$ .....(1)

$$\therefore \quad \mathrm{F} = x^2\vec{i} + y^3\vec{j} = x^2\vec{i} + (x^2)^3\vec{j}$$

and $\boldsymbol{r}=x\boldsymbol{i}+y\boldsymbol{j}$

or,$\boldsymbol{r}=x\boldsymbol{i}+x^2\boldsymbol{j}$

or,$d\boldsymbol{r}= dx\ \boldsymbol{i} +2xdx\ \boldsymbol{j}$

Required line integral

$$\int_C \boldsymbol{F}.d\boldsymbol{r} = \int_0^1 (x^2\boldsymbol{i} + x^6\boldsymbol{j}).(dx\,\boldsymbol{i} + 2xdx\,\boldsymbol{j})$$

$$= \int_0^1 (x^2dx + 2x^7).$$

$$= \left[\frac{x^3}{3} + 2\frac{x^8}{8}\right]_0^1$$

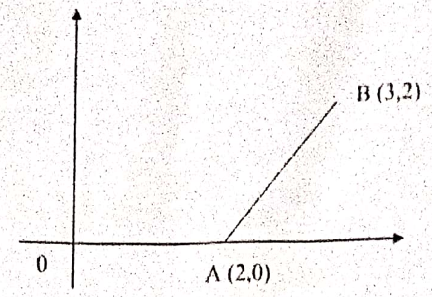
$$= \frac{7}{12}$$

Ex2. If **F=(2x+y)i+(3y-x)j** then evaluate$\int_c \boldsymbol{F}.d\boldsymbol{r}$ where C is the curve in xy-plane consisting the straight line from (0,0) to (2,0) and then to (3,2).

Sol. In the xy-plane $\boldsymbol{r}=x\boldsymbol{i}+y\boldsymbol{j}$

i. e. $d\boldsymbol{r}= dx\ \boldsymbol{i} +dy\boldsymbol{j}$

then $\boldsymbol{F}.d\boldsymbol{r} = ((2x+y)\boldsymbol{i} + (3y-x)\boldsymbol{j}).(dx\,\boldsymbol{i} + dy\boldsymbol{j}) = (2x+y)dx + (3y-x)dy$ ...(i)



Path of integration C consist line OA and AB as shown in the figure. The required integral is

$$\int_C F.dr = \int_{c_1=OA} F.dr + \int_{c_1=AB} F.dr \qquad \text{(ii)}$$

Along the line OA, $y = 0$ i.e., $dy = 0$ and x varies from 0 to 2.

Now $\int_{C_1}(F.dr) = \int_{OA}(2x+y)dx+(3y-x)dy$

$$= \int_{x=0}^{x=2} 2x dx$$

$$= \left[x^2\right]_{x=0}^{x=2} = 4 \quad \text{...(iii)}$$

Now the equation of a straight line passing through two given point A(2,0) and B(3,2) is

$$(y-0) = \frac{2-0}{3-2}(x-2) \Rightarrow y = 2x-4$$

Along the line AB, $y = 2x - 4$ i.e. $dy = 2dx$ and x varies from 2 to 3.

Now $\int_{C_2} F.dr = \int_{AB}(2x+y)dx+(3y-x)dy$

$$= \int_{AB} (2x + (2x-4))dx + (3(2x-4) - x)2dx$$

$$= \int_{AB} (4x-4)dx + ((5x-12))2dx$$

$$= \int_{AB} (14x-28)dx$$

$$= \left[7(x^2) - 28x\right]_{x=2}^{x=3}$$

$$= 7(9-4) - 28(3-2)$$

$$= 35 - 28 = 7 \qquad \text{...(iv)}$$

Putting the values from (iii) and (iv) into (ii), we get

$$\int_c F.dr = 4 + 7 = 11$$

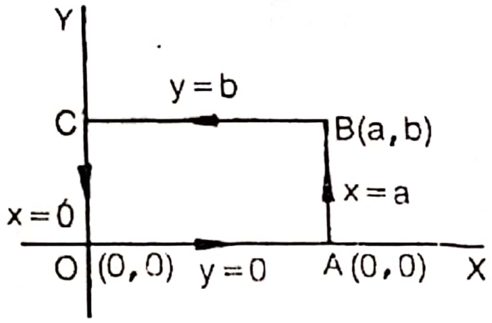
Ex.3 Evaluate $\int_c F.dr$ where $F = (x^2 + y^2)i - 2xyj$ curve C is the rectangle in xy-plane bounded by y = 0, x = a, y = b, x = 0

Sol. In the xy plane we have, $r = xi + yj$

i. e. $dr = dx\, i + dyj$

Then $F.dr = ((x^2 + y^2)i - 2xyj).(dx\, i + dyj)$

that is, $F.dr = ((x^2 + y^2)dx - 2xydy)$ ...(i)



Clearly the path of integration C consists four straight line OA, AB, BC and CO as shown in the diagram. Required line integral is

$$\int_c F.dr = \int_{OA} F.dr + \int_{AB} F.dr + \int_{BC} Fdr + \int_{CO} Fdr \quad .....(ii)$$

Clearly on OA $y = 0 \Rightarrow dy = 0$ and x varies from 0 to a.

On AB $x = a \Rightarrow dx = 0$ and y varies from 0 to b.

On BC $y = b \Rightarrow dy = 0$ and x varies from a to 0

and on CO $x = 0 \Rightarrow dx = 0$ and y varies from b to 0

$$\therefore \int_{OA} F.dr = \int_{OA} (x^2 + y^2)dx - 2xydy$$

$$= \int_{x=0}^{x=a} x^2 dx \qquad \text{as } y = 0 \;\&\; dy = 0$$

$$= \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3} \qquad ....(iii)$$

$$\int_{AB} F.dr = \int_{y=0}^{y=b} -2aydy \qquad [\because x = a \;\&\; dx = 0]$$

$$= -2a\left[\frac{y^2}{2}\right]_0^b = -ab^2 \qquad ....(iv)$$

$$\int_{BC} F.dr = \int_{x=a}^{x=0} (x^2 + b^2)dx$$

$$= \left[\frac{x^3}{3} + b^2 x\right]_a^0 \qquad [\because y = b \;\&\; dy = 0]$$

$$= \frac{-a^3}{3} - ab^2 \qquad ...(v)$$

and $\int_{Co} F.dr = \int_{y=a}^{y=0} 0\,dy = 0$ .....(vi)

On putting the values from (iii) (iv) (v) and (vi) in (ii), we get required result i.e.

$$\int_c F.dr = \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 + 0$$

$$= -2ab^2$$

Ex 4. Find the work done in moving a particle once around a circle C in the xy plane, if the circle has center at the origin and radius 3 and if the force field is given by

$$\mathbf{F} = (2x - y + z)\mathbf{i} + (x + y - z^2)\mathbf{j} + (3x - 2y + 4z)\mathbf{k}$$

Sol. In the xy plane $z = 0$.

In the plane $z=0$, $\mathbf{F} = (2x - y)\mathbf{i} + (x + y)\mathbf{j} + (3x - 2y)\mathbf{k}$ and $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$ so that the work done is

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C [(2x - y)\mathbf{i} + (x + y)\mathbf{j} + (3x - 2y)\mathbf{k}]\cdot[dx\,\mathbf{i} + dy\,\mathbf{j}]$$

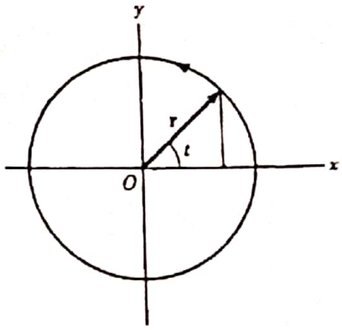
$$= \int_C (2x - y)\,dx + (x + y)\,dy$$

We know polar equation of a circle of radius 3 and center at origin is $x = 3cost$, $y = 3sint$.

Choose the parametric equations of the circle as $x = 3\cos t$, $y = 3\sin t$ where t varies from 0 to 2π (see adjoining figure). Then the line integral equals

$$\int_{t=0}^{2\pi} [2(3\cos t) - 3\sin t][-3\sin t]\,dt + [3\cos t + 3\sin t][3\cos t]\,dt$$

$$= \int_{0}^{2\pi} (9 - 9\sin t\cos t)\,dt = 9t - \frac{9}{2}\sin^2 t\Big|_{0}^{2\pi} = 18\pi$$



$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$
$= 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}$

In traversing C we have chosen the counterclockwise direction indicated in the adjoining figure. We call this the *positive* direction, or say that C has been traversed in the positive sense. If C were traversed in the clockwise (negative) direction the value of the integral would be -18π.

Ex. 5: If F is a conservative field, prove that curl $\boldsymbol{F} = \nabla \times \boldsymbol{F} = 0$ (i.e. F is irrotational). Conversely, if $\nabla \times \boldsymbol{F} = 0$ (i.e. $\boldsymbol{F}$ is irrotational), prove that $\boldsymbol{F}$ is conservative.

(See Problem 11, M.R. Spiegel)