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MLE → principle → $L = f(x_1,\theta) \cdot f(x_2,\theta) \ldots f(x_n,\theta)$

likelihood function $= \prod_{i=1}^{n} f(x_i, \theta)$

Q

$x_1, x_2 \ldots x_n$ be a random sample taken from the,

$$P\{X = x\} = f(x, \lambda)$$

$$= \frac{e^{-\lambda} \lambda^{x}}{x!} \; ; x = 0, 1, 2 \ldots$$

$$= 0 \quad ; \text{otherwise}$$

find λ or, estimate λ. find its variance also. and show that it is sufficient estimator.

Q Given, $f(x,\lambda) = \frac{e^{-\lambda} \lambda^{x}}{x!} \; ; x = 0, 1, 2 \ldots$

find λ^{x}

step ① $\to$ find L

Soln: Let, $x_1, x_2, \dots x_n$ be a random sample of size n taken from the given pdf. Then the likelihood function denoted by $L(\lambda|x)$ is defined as:

$$\cancel{L(\theta/x)} = L = f(x_1, \lambda) . f(x_2, \lambda) \dots f(x_n \lambda)$$

$L(\lambda|x)$

$$= \frac{e^{-\lambda} \lambda^{x_1}}{x_1 !} . \frac{e^{-\lambda} \lambda^{x_2}}{x_2!} \dots \frac{e^{-\lambda} \lambda^{x_n}}{x_n !}$$

$$= \frac{e^{-(\lambda+\lambda+\dots \lambda)} \lambda^{x_1+x_2+\dots x_n}}{x_1 ! x_2 ! \dots x_n !}$$

$$= \frac{e^{-n\lambda} . \lambda^{\sum_{i=1}^{n} x_i}}{x_1! x_2! \dots x_n!}$$

$$\log L = -n\lambda + \sum_{i=1}^{n} x_i \log \lambda - \sum_{i=1}^{n} \log(x_i) \quad \cancel{\log \sum_{i=1}^{n} (x_i)}$$

$$\Rightarrow \frac{\partial \log L}{\partial \lambda} = 0$$

$$\Rightarrow 0 = \cancel{\frac{\partial L}{\partial \lambda}} \Rightarrow -n + \sum_{i=1}^{n} x_i \cdot \frac{1}{\lambda} = 0 \qquad \cancel{\frac{1}{\sum_{i=1} x_i}}$$

$$\Rightarrow \cancel{\frac{\partial L}{\partial \lambda}} \Rightarrow \left(-n + \sum_{i=1}^{n} x_i\left(\frac{1}{\hat{\lambda}}\right)\right) \longrightarrow \hat{\lambda} = 0$$

$$\Rightarrow \left(-n + \sum_{i=1}^{n} x_i \cdot \left(\frac{1}{\hat{\lambda}}\right)\right) = 0$$

$$\sum_{i=1}^{n} x_i \left(\frac{1}{\hat{\lambda}}\right) = n$$

$$\Rightarrow \hat{\lambda} = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$= \bar{X}$$

$\therefore$ The MLE of λ is $\bar{X}$.

The variance of the estimate is

$$\frac{1}{v(\hat{\lambda})} = E\left[-\frac{\partial^2}{\partial\lambda^2}(\log L)\right]$$

$$= E\left[-\frac{\partial}{\partial\lambda}\left(-n+\frac{n\bar{x}}{\lambda}\right)\right]$$

$$= E\left[0-\left(-\frac{n\bar{x}}{\lambda^2}\right)\right]$$

$$= \frac{n}{\lambda^2}E(\bar{x})$$

$$= \frac{n}{\lambda^2}\cdot\lambda$$

$$= \frac{n}{\lambda}$$

$$\therefore v(\hat{\lambda}) = \frac{\lambda}{n}$$

Note

$E(x)=\lambda$

$E(\bar{x})=\lambda$

$$\frac{\partial \log L}{\partial \lambda} = -n + \frac{n\bar{x}}{\lambda}$$

$$= n\left(\frac{\bar{x}}{\lambda} - 1\right)$$

$$= g_{\lambda}(t(x)) \cdot h(x)$$

$g_{\lambda}(t(x)) \cdot h(x)$ is a function of λ and x, where $h(x) = 1$ which is independent of λ.

Therefore by the factorization theorem it is $t(x) = \frac{\sum x_i}{n} = \bar{x}$ is a sufficient statistic of λ.

$$\lim_{n\to\infty} \frac{\sqrt{n+k}}{\sqrt{n}} = \lim_{n\to\infty} \frac{k}{n}$$

$$\lim_{n\to\infty}\left(1+\frac{\alpha}{n}\right)^{\alpha} = e^0 = 1$$

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n} = e^{x}$$

Point Estimation:

C-R: LR
Cramer

If t is an unbiased estimator for $\gamma(\theta)$, a function of θ, then,

[illegible]

$$v(t) \geq \boxed{\frac{\left\{\frac{\partial}{\partial\theta}\gamma(\theta)\right\}^2}{E\left[\frac{\partial}{\partial\theta}\log L\right]^2}} = \frac{\{\gamma'(\theta)\}^2}{I(\theta)}$$

where, $I(\theta)$ is information of θ, supplied by sample.

Example:

Show that, $\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$, in the random sampling from

$$f(x,\theta) = \begin{cases} \frac{1}{\theta} e^{-x/\theta}, & 0 < x < \infty;\ \theta > 0 \\ 0, & \text{otherwise} \end{cases}$$

$\boxed{\text{minimum variance bound}}$

is an MVB estimator of θ, and has variance $\frac{\theta^2}{n}$.

Sol^n:

Given, $f(x,\theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}}$, $x>0$, $\theta>0$

$\therefore L(\theta/x) = \frac{1}{\theta^n} e^{-\frac{1}{\theta}\sum x_i}$

Note: $L(\theta/x) = L$

$\Rightarrow \log L = -n\log\theta - \frac{1}{\theta}\sum x_i$

$\Rightarrow \frac{\partial \log L}{\partial\theta} = -\frac{n}{\theta} + \frac{1}{\theta^2}\sum x_i$

$\Rightarrow \frac{\partial^2 \log L}{\partial\theta^2} = \frac{n}{\theta^2} - \frac{2}{\theta^3}\sum x_i$

Now in sampling from exponential population,

$\therefore E(X) = \theta$ and $V(X) = \theta^2$

Then,

$I(\theta) = -E\left\{\frac{\partial^2 \log L}{\partial\theta^2}\right\}$

$$= -E\left\{\frac{n}{\theta^2} - \frac{2}{\theta^3}\sum x_i\right\}$$

$$= -\frac{n}{\theta^2} + \frac{2}{\theta^3}\sum E(X_i)$$

$$= -\frac{n}{\theta^2} + \frac{2}{\theta^3}\cdot n\theta$$

$$= \frac{n}{\theta^2}$$

And, $\gamma(\theta) = \theta$

$\therefore \gamma'(\theta) = 1$

Now, using CRLB

for θ,

$$\frac{\{\gamma'(\theta)\}^2}{I(\theta)}$$

$$= \frac{1}{\left[\frac{n}{\theta^2}\right]} = \frac{\theta^2}{n}$$

$$E(\bar{X}) = E\left\{\frac{\sum x_i}{n}\right\} = \frac{1}{n}\sum_{i=1}^{n} E(x_i)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\theta$$

$$= \frac{1}{n}\cdot n\theta$$

$$= \theta$$

$\therefore \bar{x} = \frac{\sum x_i}{n}$ is an unbiased estimator of θ.

$$v(\bar{x}) = v\left\{\frac{\sum x_i}{n}\right\}$$

$$= \frac{1}{n^2}\sum v(x_i)$$

$$= \frac{1}{n^2}\cdot\sum\theta^2$$

$$= \frac{1}{n^2}\cdot n\cdot\theta^2$$

$$= \left\{\frac{\theta^2}{n}\right\}$$