# Gradient, Divergence & Curl

The vector differential operator $\nabla$is generally called delta and is defined by

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

It is useful in defining three quantities which are known as the gradient, the divergence and the curl. The operator $\nabla$is also known as nabla.

## The Gradient

The gradient of a scalar function $\phi(x, y, z)$ is written by grad $\phi$ or $\nabla\phi$ is defined by

$$\nabla\phi = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

Note that $\nabla\phi$ defines a vector field.

The component of $\nabla\phi$ in the direction of a unit vector a s given by $\nabla\phi.\mathbf{a}$and is called the directional derivative of $\phi$ in the direction **a**. Physically, this is the rate of change of $\phi$ at (x,y,z) in the direction **a**.

## The Divergence

The divergence of a vector $\mathbf{V}(x, y, z) = V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}$ is written by div **V** or $\nabla.\mathbf{V}$ is defined by

$$\nabla\cdot\mathbf{V} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k})$$

$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

Note that Divergence of a vector **V** is a scalar quantity ($\mathbf{A.B} = A_1B_1 + A_2B_2 + A_3B_3$) and $\mathbf{A.B} \neq \mathbf{B.A}$.A vector **F** is solenoidal if div **F** =0.

A fluid moves so that its velocity at any point is v(x,y,z). div v represents the loss or gain of fluid per unit volume per unit time in a small parallelepiped having a center at P(x,y,z) and edges parallel to the coordinate axes and having magnitude $\Delta_x$, $\Delta_y$, $\Delta_z$ respectively.

## The Curl

If **V**(z,y,z) is a differentiable vector field then the curl or rotation of **V**, written $\nabla\times\mathbf{V}$, curl **V**, is defined by

$$\nabla \times \mathbf{V} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \times (V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

## THE GRADIENT

1. If $\phi(x,y,z) = 3x^2y - y^3z^2$, find $\nabla\phi$ (or grad $\phi$) at the point $(1,-2,-1)$.

$$\nabla\phi = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)(3x^2y - y^3z^2)$$

$$= \mathbf{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \mathbf{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \mathbf{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= 6xy\,\mathbf{i} + (3x^2 - 3y^2z^2)\mathbf{j} - 2y^3z\,\mathbf{k}$$

$$= 6(1)(-2)\mathbf{i} + \{3(1)^2 - 3(-2)^2(-1)^2\}\mathbf{j} - 2(-2)^3(-1)\mathbf{k}$$

$$= -12\mathbf{i} - 9\mathbf{j} - 16\mathbf{k}$$

2.

Show that $\nabla r^n = nr^{n-2}\mathbf{r}$.

$$\nabla r^n = \nabla(\sqrt{x^2+y^2+z^2})^n = \nabla(x^2+y^2+z^2)^{n/2}$$

$$= \mathbf{i}\frac{\partial}{\partial x}\{(x^2+y^2+z^2)^{n/2}\} + \mathbf{j}\frac{\partial}{\partial y}\{(x^2+y^2+z^2)^{n/2}\} + \mathbf{k}\frac{\partial}{\partial z}\{(x^2+y^2+z^2)^{n/2}\}$$

$$= \mathbf{i}\left\{\frac{n}{2}(x^2+y^2+z^2)^{n/2-1}2x\right\} + \mathbf{j}\left\{\frac{n}{2}(x^2+y^2+z^2)^{n/2-1}2y\right\} + \mathbf{k}\left\{\frac{n}{2}(x^2+y^2+z^2)^{n/2-1}2z\right\}$$

$$= n(x^2+y^2+z^2)^{n/2-1}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$= n(r^2)^{n/2-1}\mathbf{r} = nr^{n-2}\mathbf{r}$$

3.

Show that $\nabla\phi$ is a vector perpendicular to the surface $\phi(x,y,z) = c$ where $c$ is a constant.

Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the position vector to any point $P(x, y, z)$ on the surface.

Then $d\,\mathbf{r} = dx\,\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$ lies in the tangent plane to the surface at P.

But $d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = 0$

or $\left(\frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}\right)\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}) = 0$

i.e. $\nabla\phi \cdot d\mathbf{r} = 0$ so that $\nabla\phi$ is perpendicular to $d\mathbf{r}$ and therefore to the surface.

**4.** Find a unit normal to the surface $x^2y + 2xz = 4$ at the point (2,-2,3).

The normal vector to the surface $x^2y + 2xz = 4$ is

$\nabla(x^2y + 2xz) = (2xy + 2z)\mathbf{i} + x^2\mathbf{j} + 2x\,\mathbf{k}$

at the point (2,-2,3) the normal vector is

$-2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$

Then a unit normal to the surface $= \dfrac{-2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}}{\sqrt{(-2)^2 + (4)^2 + (4)^2}} = -\frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}.$

Another unit normal is $\frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$ having direction opposite to that above.

**5.** Find the directional derivative of $0 = x^2yz + 4xz^2$ at (1,-2,-1) in the direction $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ .

The normal vector to the surface is

$$\nabla\phi = \nabla(x^2yz + 4xz^2) = (2xyz + 4z^2)\mathbf{i} + x^2z\,\mathbf{j} + (x^2y + 8xz)\mathbf{k}$$
$$= 8\mathbf{i} - \mathbf{j} - 10\mathbf{k} \quad \text{at} \quad (1,-2,-1).$$

The unit vector in the direction of $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ is

$$\mathbf{a} = \frac{2\mathbf{i} - \mathbf{j} - 2\mathbf{k}}{\sqrt{(2)^2 + (-1)^2 + (-2)^2}} = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$$

Then the required directional derivative is

$$\nabla\phi\cdot\mathbf{a} = (8\mathbf{i} - \mathbf{j} - 10\mathbf{k})\cdot\left(\frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right) = \frac{16}{3} + \frac{1}{3} + \frac{20}{3} = \frac{37}{3}$$

**6.** Find the angle between the surfaces $x^2+y^2+z^2 = 9$ and $z = x^2+y^2-3$ at the point (2,-1,2).

The angle between the surfaces at the point is the angle between the normals to the surfaces at the

point.

A normal to $x^2+y^2+z^2 = 9$ at $(2,-1,2)$ is

$$\nabla\phi_1 = \nabla(x^2+y^2+z^2) = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k} = 4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$$

A normal to $z = x^2+y^2-3$ or $x^2+y^2-z = 3$ at $(2,-1,2)$ is

$$\nabla\phi_2 = \nabla(x^2+y^2-z) = 2x\,\mathbf{i} + 2y\,\mathbf{j} - \mathbf{k} = 4\mathbf{i} - 2\mathbf{j} - \mathbf{k}$$

$(\nabla\phi_1)\cdot(\nabla\phi_2) = |\nabla\phi_1|\,|\nabla\phi_2|\cos\theta$, where $\theta$ is the required angle. Then

$$(4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})\cdot(4\mathbf{i} - 2\mathbf{j} - \mathbf{k}) = |4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}|\,|4\mathbf{i} - 2\mathbf{j} - \mathbf{k}|\cos\theta$$

$$16 + 4 - 4 = \sqrt{(4)^2+(-2)^2+(4)^2}\,\sqrt{(4)^2+(-2)^2+(-1)^2}\,\cos\theta$$

and $\cos\theta = \dfrac{16}{6\sqrt{21}} = \dfrac{8\sqrt{21}}{63} = 0.5819$; thus the acute angle is $\theta = \arccos 0.5819 = 54^\circ 25'$.

**Gradient Theorem**

Let $C$ be a Differentiable curve given by the vector $\boldsymbol{r}(t), a \le t \le b$.

Let $f$ be a differentiable function on three variables whose gradient vector $\nabla f$ is continuous on $C$.

then

$$\int_C \nabla f \cdot d\boldsymbol{r} = f(b) - f(a).$$

Proof:

$$\int_C \nabla f \cdot d\boldsymbol{r} = \int_a^b \nabla f \cdot \frac{d\boldsymbol{r}}{dt}\,dt$$

$$= \int_a^b \nabla f \cdot \boldsymbol{r}'(t)\,dt$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial y}\boldsymbol{k}\right)\cdot\left(\frac{dx}{dt}\boldsymbol{i} + \frac{dy}{dt}\boldsymbol{j} + \frac{dz}{dt}\boldsymbol{k}\right)dt$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\right)dt$$

$$= \int_a^b \frac{df}{dt}\,dt$$

$$= \int_a^b df$$

$$= f(b) - f(a)$$

7. If $A = x^2z\,\mathbf{i} - 2y^3z^2\mathbf{j} + xy^2z\,\mathbf{k}$, find $\nabla.\mathbf{A}$ (or div A) at the point (1,-1,1).

$$\nabla\cdot\mathbf{A} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot\left(x^2z\,\mathbf{i} - 2y^3z^2\,\mathbf{j} + xy^2z\,\mathbf{k}\right)$$

$$= \frac{\partial}{\partial x}(x^2z) + \frac{\partial}{\partial y}(-2y^3z^2) + \frac{\partial}{\partial z}(xy^2z)$$

$$= 2xz - 6y^2z^2 + xy^2 = 2(1)(1) - 6(-1)^2(1)^2 + (1)(-1)^2 = -3 \quad \text{at } (1,-1,1).$$

8. Prove that $\nabla.\nabla\phi = \nabla^2\phi$

Proof:

$$\nabla\cdot\nabla\phi = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot\left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = \nabla^2\phi$$

The operator $\nabla.\nabla = \nabla^2$ is called the Laplacian operator.

9. Prove that $\nabla^2(\frac{1}{r^2})$ where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

Proof: we have

$$\nabla^2\left(\frac{1}{r}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

Now

$$\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} = -x(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{\partial}{\partial x}\left[-x(x^2+y^2+z^2)^{-3/2}\right]$$

$$= 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$$

Similarly, we have

$$\frac{\partial^2}{\partial y^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{2y^2-z^2-x^2}{(x^2+y^2+z^2)^{5/2}}$$

$$\frac{\partial^2}{\partial z^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$$

Then by addition, $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = 0.$

10. Determine the constant a so that the vector **V** = (x +3y) **i** + (y- 2z) **j** + (x +az) **k** is solenoidal.

A vector **V** is solenoidal if its divergence is zero, then $\nabla.\mathbf{V} = 0$

Now

$$\nabla\cdot\mathbf{v} = \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+az) = 1 + 1 + a$$

Then $a + 2 = 0$, that is $a = -1$.

## THE CURL

11. If A = $xz^3$**i** - $2x^2yz$ **j** + $2yz^4$**k** , find ∇xA (or curl A) at the point (1,-1,1).

$$\nabla\times\mathbf{A} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\times(xz^3\,\mathbf{i} - 2x^2yz\,\mathbf{j} + 2yz^4\,\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(2yz^4) - \frac{\partial}{\partial z}(-2x^2yz)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(xz^3) - \frac{\partial}{\partial x}(2yz^4)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(-2x^2yz) - \frac{\partial}{\partial y}(xz^3)\right]\mathbf{k}$$

$$= (2z^4 + 2x^2y)\,\mathbf{i} + 3xz^2\,\mathbf{j} - 4xyz\,\mathbf{k} = 3\mathbf{j} + 4\mathbf{k} \quad \text{at } (1,-1,1).$$

12. Evaluate ∇. (**A** x **r**) if ∇ x **A** = 0 where **A** = $A_1$**i** + $A_2$**j** + $A_3$**k**, r = xi + yj + zk.

Now

$$\mathbf{A}\times\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_1 & A_2 & A_3 \\ x & y & z \end{vmatrix}$$

or, $\mathbf{A}\times\mathbf{r} = (zA_2 - yA_3)\mathbf{i} + (xA_3 - zA_1)\mathbf{j} + (yA_1 - xA_2)\mathbf{k}$

13. Prove $\nabla \mathrm{x}(\nabla \mathrm{x}\ \mathbf{A}) = -\nabla^2 \mathbf{A}+\nabla(\nabla.\mathbf{A})$.

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \nabla \times \left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\mathbf{k}\right]$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} & \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} & \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\right]\mathbf{i}$$

$$+ \left[\frac{\partial}{\partial z}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\right]\mathbf{j}$$

$$+ \left[\frac{\partial}{\partial x}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\right]\mathbf{k}$$

$$= \left(-\frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2}\right)\mathbf{i} + \left(-\frac{\partial^2 A_2}{\partial z^2} - \frac{\partial^2 A_2}{\partial x^2}\right)\mathbf{j} + \left(-\frac{\partial^2 A_3}{\partial x^2} - \frac{\partial^2 A_3}{\partial y^2}\right)\mathbf{k}$$

$$+ \left(\frac{\partial^2 A_2}{\partial y\,\partial x} + \frac{\partial^2 A_3}{\partial z\,\partial x}\right)\mathbf{i} + \left(\frac{\partial^2 A_3}{\partial z\,\partial y} + \frac{\partial^2 A_1}{\partial x\,\partial y}\right)\mathbf{j} + \left(\frac{\partial^2 A_1}{\partial x\,\partial z} + \frac{\partial^2 A_2}{\partial y\,\partial z}\right)\mathbf{k}$$

$$= \left(-\frac{\partial^2 A_1}{\partial x^2} - \frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2}\right)\mathbf{i} + \left(-\frac{\partial^2 A_2}{\partial x^2} - \frac{\partial^2 A_2}{\partial y^2} - \frac{\partial^2 A_2}{\partial z^2}\right)\mathbf{j} + \left(-\frac{\partial^2 A_3}{\partial x^2} - \frac{\partial^2 A_3}{\partial y^2} - \frac{\partial^2 A_3}{\partial z^2}\right)\mathbf{k}$$

$$+ \left(\frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_2}{\partial y\,\partial x} + \frac{\partial^2 A_3}{\partial z\,\partial x}\right)\mathbf{i} + \left(\frac{\partial^2 A_1}{\partial x\,\partial y} + \frac{\partial^2 A_2}{\partial y^2} + \frac{\partial^2 A_3}{\partial z\,\partial y}\right)\mathbf{j} + \left(\frac{\partial^2 A_1}{\partial x\,\partial z} + \frac{\partial^2 A_2}{\partial y\,\partial z} + \frac{\partial^2 A_3}{\partial z^2}\right)\mathbf{k}$$

$$= -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})$$

$$+ \mathbf{i}\frac{\partial}{\partial x}\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) + \mathbf{j}\frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) + \mathbf{k}\frac{\partial}{\partial z}\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)$$

$$= -\nabla^2 \mathbf{A} + \nabla\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)$$

$$= -\nabla^2 \mathbf{A}+\nabla(\nabla.\mathbf{A}).$$

$$\text{and} \quad \nabla\cdot(\mathbf{A}\times\mathbf{r}) = \frac{\partial}{\partial x}(zA_2 - yA_3) + \frac{\partial}{\partial y}(xA_3 - zA_1) + \frac{\partial}{\partial z}(yA_1 - xA_2)$$

$$= z\frac{\partial A_2}{\partial x} - y\frac{\partial A_3}{\partial x} + x\frac{\partial A_3}{\partial y} - z\frac{\partial A_1}{\partial y} + y\frac{\partial A_1}{\partial z} - x\frac{\partial A_2}{\partial z}$$

$$= x\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) + y\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) + z\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

$$= [x\mathbf{i} + y\mathbf{j} + z\mathbf{k}]\cdot\left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\mathbf{k}\right]$$

$$= \mathbf{r}\cdot(\nabla\times\mathbf{A})$$

$= 0$ where $\nabla\mathrm{x}\mathbf{A}=0$.

14. A vector **V** is called irrotational if curl **V**=0. Find constants a,b,c so that **V** = (x+2y+az)**i** + (bx-3y+z)**j**+(4x+cy+2z)**k** is irrotational. Show that **V** can be expressed as the gradient of a scalar function.

Now

$$\text{curl } \mathbf{V} = \nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

or, $(c+1)\boldsymbol{i} + (a-4)\boldsymbol{j} + (b-2)\boldsymbol{k} = 0$

Thus $C+1=0$, $a-4=0$, and $b-2=0$. Hence $a=4, b=2, c=-1$ and the irrotational vector becomes

$\mathbf{V} = (x+2y+4z)\boldsymbol{i} + (2x-3y-z)\boldsymbol{j} + (4x-y+2z)\boldsymbol{k}.$

Now assume $\mathbf{V}=\nabla\phi$

$$\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} = (x+2y+4z)\mathbf{i} + (2x-3y-z)\mathbf{j} + (4x-y+2z)\mathbf{k}$$

Then

$$\frac{\partial\phi}{\partial x} = x+2y+4z, \quad (1)$$

$$\frac{\partial\phi}{\partial y} = 2x-3y-z \quad (2)$$

$$\frac{\partial\phi}{\partial z} = 4x-y+2z \quad (3)$$

Integrating (1) partially with respect to x, keeping y and z constant, we get

$$\phi = \frac{x^2}{2} + 2xy + 4xz + f(y,z) \quad (4)$$

where f (y z) is an arbitrary function of y and z. Similarly, from (2) and (3),

$$\phi = 2xy - \frac{3y^2}{2} - yz + g(x,z) \quad (5)$$

$$\phi = 4xz - yz + z^2 + h(x,y). \quad (6)$$

A comparison of (4), (5) and (6) shows that there will be a common value of $\phi$ if we choose

$$f(y,z) = -\frac{3y^2}{2} + z^2 - yz$$

$$g(x,z) = \frac{x^2}{2} + z^2 + 4xz$$

$$h(x,y) = -\frac{3y^2}{2} + \frac{x^2}{2} + 2xy$$

so that

$$\phi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4xz - yz$$

15. Show that $\nabla^2 f(r) = (2/r)\, f'(r) + f''(r)$.

Soln. We Know $\nabla^2 f(r) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f(r)$

Now $\frac{\partial^2}{\partial x^2} f(r) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f(r)\right)$

$$= \frac{\partial}{\partial x}\left( f'(r)\frac{\partial r}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left( f'(r)\frac{\partial\sqrt{x^2+y^2+z^2}}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left( f'(r)\frac{x}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \frac{\partial}{\partial x}\left( f'(r)\frac{x}{r}\right)$$

$$= \frac{\partial}{\partial x}\left( f'(r)\frac{x}{r}\right)$$

$$= f''(r)\frac{\partial r}{\partial x}\frac{x}{r} + f'(r)\frac{\partial}{\partial x}\left(\frac{x}{r}\right)$$

$$= f''(r)\frac{x}{r}\frac{x}{r} + f'(r)\left[\frac{1}{r} - \frac{x}{r^2}\frac{\partial r}{\partial x}\right]$$

$$= f''(r)\frac{x^2}{r^2} + f'(r)\left[\frac{1}{r} - \frac{x}{r^2}\frac{x}{r}\right]$$

$$= f''(r)\frac{x^2}{r^2} + f'(r)\left[\frac{1}{r} - \frac{x^2}{r^3}\right]$$

Thus $\frac{\partial^2}{\partial x^2} f(r) = f''(r)\frac{x^2}{r^2} + f'(r)\left[\frac{1}{r} - \frac{x^2}{r^3}\right]$ (1)

Similarly, we get

$\frac{\partial^2}{\partial y^2} f(r) = f''(r)\frac{y^2}{r^2} + f'(r)\left[\frac{1}{r} - \frac{y^2}{r^3}\right]$ (2)

$$\frac{\partial^2}{\partial z^2} f(r) = f''(r)\frac{z^2}{r^2} + f'(r)\left[\frac{1}{r} - \frac{z^2}{r^3}\right] \qquad (3)$$

Now adding (1), (2) and (3), we get

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f(r) = f''(r)\left\{\frac{x^2 + y^2 + z^2}{r^2}\right\} + f'(r)\left\{\frac{3}{r} - \frac{x^2 + y^2 + z^2}{r^3}\right\}$$

or, $\nabla^2 f(r) = f''(r) + f'(r)\left\{\frac{3}{r} - \frac{r^2}{r^3}\right\}$

or, $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$

16. If $\mathbf{v} = \boldsymbol{\omega} \times \boldsymbol{r}$, prove $\boldsymbol{\omega} = \frac{1}{2}\text{curl}\mathbf{v}$ where $\boldsymbol{\omega} = \omega_1 \boldsymbol{i} + \omega_2 \boldsymbol{j} + \omega_3 \boldsymbol{k}$ is a constant vector

Now $curl\mathbf{v} = \nabla \times \mathbf{v}$

$= \nabla \times (\boldsymbol{\omega} \times \mathbf{r})$

$$= \nabla \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$= \nabla \times [(\omega_2 z - \omega_3 y)\mathbf{i} + (\omega_3 x - \omega_1 z)\mathbf{j} + (\omega_1 y - \omega_2 x)\mathbf{k}]$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$= 2(\omega_1 \mathbf{i} + \omega_2 \mathbf{j} + \omega_3 \mathbf{k})$

$= 2\boldsymbol{\omega}$ .

Thus $\text{curl}\mathbf{v} = 2\boldsymbol{\omega}$

or, $\boldsymbol{\omega} = \frac{1}{2}\text{curl}\mathbf{v}$

This problem indicates that the curl of a vector field has something to do with rotational properties of the field. For a vector field F. If curl$\mathbf{F} \neq 0$in theregion, then the field F is then called rotational. If curl$\mathbf{F} = 0$ in theregion, then the field F is then called irrotational.