Vector Analysis

The dot or scalar product of two vectors **A** and **B**:

The dot product of two vectors **A** and **B**, denoted by **A.B** (read **A** dot **B**), is defined as the product of the magnitudes of **A** and **B** and the cosine of the angle θ between them. Mathematically, $\mathbf{A}\cdot\mathbf{B} = \mathrm{AB}\cos,\ \ 0 \leq \theta \leq \pi$

$\mathbf{A}\cdot\mathbf{B}$ is a scalar and not a vector.

Problem: Prove that $\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$

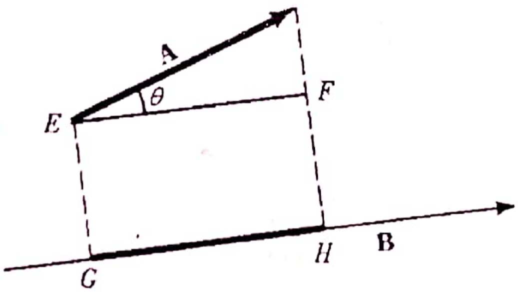
Sol. Let **A** and **B** two vectors, then from definition we get

$$\mathbf{A}\cdot\mathbf{B} = \mathrm{AB}\cos$$
$$= \mathrm{BA}\cos$$
$$= \mathbf{B}\cdot\mathbf{A}$$

Thus, $\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$

Problem: Prove that the projection of **A** on **B** is equal to $\mathbf{A}\cdot\mathbf{b}$, where **b** is a unit vector in the direction of **B**.

Sol. Draw perpendiculars from initial and terminal points of **A** to **B** and cut the vector **B** at G and H, respectively. Draw EF parallel to GH at initial point E.



The projection of **A** on **B** is $GH = EF = A\cos\theta$ where θ be the angle between **A** and **B**.

From the definition, we get

$$\mathbf{A}\cdot\mathbf{B} = \mathrm{AB}\cos\theta$$
$$\Rightarrow \mathrm{A}\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{B} = \boldsymbol{A}\cdot\boldsymbol{b}$$

where $\boldsymbol{b}$ is unit vector of **B**.

Then, the projection of **A** on **B** is $\boldsymbol{A}\cdot\boldsymbol{b}$.

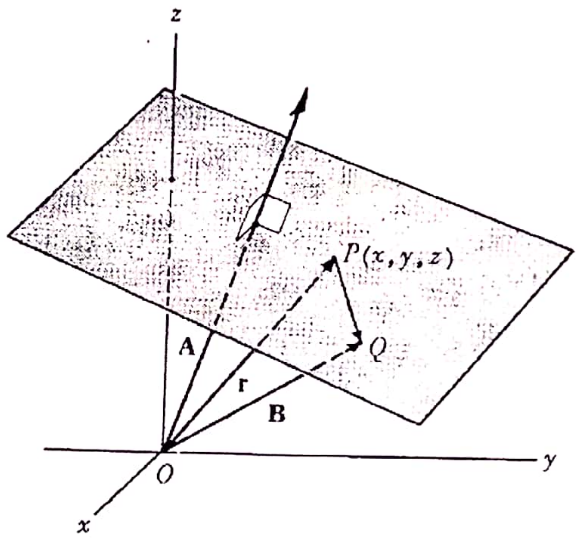
Problem: Find the angle between **A**=2**i**+2**j**-**k** and **B**=6**i**-3**j**+2**k**.

Problem: Determine the value of a so that **A**=2**i**+a**j**+**k** and **B**=4**i**-2**j**-2**k** are perpendicular.

Problem: Find the projection of the vector **A**=**i**-2**j**+**k** on the vector **B**=4**i**-4**j**+7**k**.

Problem: Find an equation for the plane perpendicular to the vector A=2**i**+3**j**+6**k** and passing through the terminal point of the vector **B**=**i**+5**j**+3**k**.

Sol: Let **r** be the position vector of the point P (x,y,z), that is, **OP**= x**i**+y**j**+z**k** and Q be the terminal point of **B**.



From the figure, **OP+PQ=OQ**

or, **OP=B-r** which is perpendicular to **A**, then $\mathbf{OP} \cdot \mathbf{A} = 0$

$$(\mathbf{B} - \mathbf{r}) \cdot \mathbf{A} = 0$$

$$\mathbf{B} \cdot \mathbf{A} = \mathbf{r} \cdot \mathbf{A}$$

or, $(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot (2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}) = (\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}) \cdot (2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k})$

or, $2x + 3y + 6z = (1)(2) + (5)(3) + (3)(6) = 35$

The Cross product

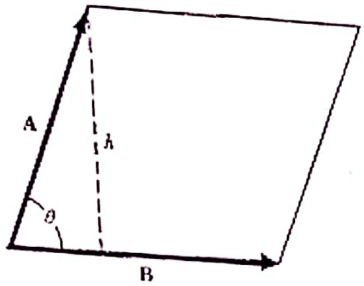
The cross product of two vectos a A and B is a vector $\boldsymbol{C} = \boldsymbol{C} = \boldsymbol{A} \times \boldsymbol{B}$. The magnitude of $\boldsymbol{A} \times \boldsymbol{B}$ is defined as the product of the magnitudes of **A** and **B** and the sine of the angle θ between them. The direction of the vector $\boldsymbol{C} = \boldsymbol{A} \times \boldsymbol{B}$ is perpendicular to the plane of **A** and **B** and such that A, **B** and **C** from a right-handed system.

$$\boldsymbol{A} \times \boldsymbol{B} = AB \sin\theta \ \mathbf{u} \qquad 0 \leq \theta \leq \pi$$

u is a unit vector indicating the direction of $\boldsymbol{A} \times \boldsymbol{B}$.

Problem: Prove that the area of a parallelogram with sides **A** and **B** is $|\boldsymbol{A} \times \boldsymbol{B}|$.

Sol. Let **A** and **B** be two vectors, θ be the angle between them and draw a parallelogram with sides **A** and **B** as shown in the figure.



From the figure, Area of the parallelogram $= h|B|$

$$= |A| \sin\theta \, |B|$$

$$= |A \times B|$$

Note that the area of the triangle with sides **A** and **B** is $\frac{1}{2}|A \times B|$.

Problem: Determine a unit vector perpendicular to the plane of **A=2i-6j-2k** and **B=4i+3j-k.**

Problem: Find the area of the triangle having vertices at $P(1,3,2)$, $Q(2,-1,1)$, $R(-1,2,3)$.

Multiple products of vectors.

Let A, **B** and **C** be any three vectors. Dot and cross multiplication of three vectors **A**, **B** and **C** may produce meaningful products of the form **(A . B)C**, **A. (BxC)** and **Ax(BxC).**

The scalar triple product

The scalar triple product of the three vectors **A**, **B** and **C** is defined by **A. (BxC)**

If $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$, $\mathbf{C} = C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k}$ show that

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

Proof: Now

$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$

$$= \mathbf{A}\cdot\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$= (A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}) \cdot [(B_2C_3 - B_3C_2)\mathbf{i} + (B_3C_1 - B_1C_3)\mathbf{j} + (B_1C_2 - B_2C_1)\mathbf{k}]$$

$$= A_1(B_2C_3 - B_3C_2) + A_2(B_3C_1 - B_1C_3) + A_3(B_1C_2 - B_2C_1)$$

$$= \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

Prove that $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$.

Proof: We know that

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

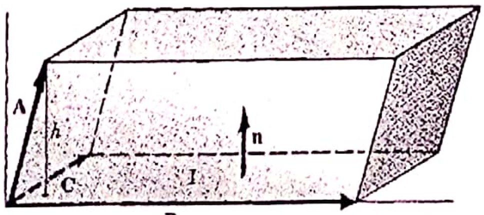
By a theorem of determinants which states that interchange of two rows of a determinant changes its sign, we have

$$\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = -\begin{vmatrix} B_1 & B_2 & B_3 \\ A_1 & A_2 & A_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \begin{vmatrix} B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \end{vmatrix} = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})$$

$$\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = -\begin{vmatrix} C_1 & C_2 & C_3 \\ B_1 & B_2 & B_3 \\ A_1 & A_2 & A_3 \end{vmatrix} = \begin{vmatrix} C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$

Show that the absolute value of $\mathbf{A}.(\mathbf{B}\mathrm{x}\mathbf{C})$ is the volume of a parallelepiped with sides **A, B** and **C.**

Let **n** be a unit normal to parallelogram I, having the direction of **B**x**C**, and let h be the height of the terminal point of **A** above the parallelogram I.



Volume of parallelepiped = (height h)(area of parallelogram I)

Let θ is the angle between **A** and **n** then $A.n = A\cos\theta = A\frac{h}{A} = h$

Then, volume of parallelepiped= (height h) (area of the parallelogram)

$$= A.n(|B \times C|)$$

$$= \mathbf{A}\cdot\{|\mathbf{B}\times\mathbf{C}|\,\mathbf{n}\} = \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$$

The product **A.** (**B**x**C**) is sometimes called the scalar triple product or box product and may be denoted by [**ABC**].

The Vector triple product

The vector triple product of the three vectors **A**, **B** and **C** is defined by **A**x (**B**x**C**).

Prove: (*a*) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$,

(*b*) $(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{A}(\mathbf{B}\cdot\mathbf{C})$.

Solution:

(*a*) Let $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$, $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$, $\mathbf{C} = C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k}$.

Then
$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}) \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}.$$

$$= (A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}) \times ([B_2C_3 - B_3C_2]\mathbf{i} + [B_3C_1 - B_1C_3]\mathbf{j} + [B_1C_2 - B_2C_1]\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_1 & A_2 & A_3 \\ B_2C_3 - B_3C_2 & B_3C_1 - B_1C_3 & B_1C_2 - B_2C_1 \end{vmatrix}$$

$$= (A_2B_1C_2 - A_2B_2C_1 - A_3B_3C_1 + A_3B_1C_3)\mathbf{i} + (A_3B_2C_3 - A_3B_3C_2 - A_1B_1C_2 + A_1B_2C_1)\mathbf{j}$$
$$+ (A_1B_3C_1 - A_1B_1C_3 - A_2B_2C_3 + A_2B_3C_2)\mathbf{k}$$

Also $\mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

$$= (B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k})(A_1C_1 + A_2C_2 + A_3C_3) - (C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k})(A_1B_1 + A_2B_2 + A_3B_3)$$

$$= (A_2B_1C_2 + A_3B_1C_3 - A_2C_1B_2 - A_3C_1B_3)\mathbf{i} + (B_2A_1C_1 + B_2A_3C_3 - C_2A_1B_1 - C_2A_3B_3)\mathbf{j}$$
$$+ (B_3A_1C_1 + B_3A_2C_2 - C_3A_1B_1 - C_3A_2B_2)\mathbf{k}$$

Thus,

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

Again

$(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$

$= -\mathbf{C}\times(\mathbf{A}\times\mathbf{B})$

$= -\{\mathbf{A}(\mathbf{C}\cdot\mathbf{B}) - \mathbf{B}(\mathbf{C}\cdot\mathbf{A})\}$

$= \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{A}(\mathbf{B}\cdot\mathbf{C})$

Prove: $(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}\times\mathbf{D}) - \mathbf{A}(\mathbf{B}\cdot\mathbf{C}\times\mathbf{D}) = \mathbf{C}(\mathbf{A}\cdot\mathbf{B}\times\mathbf{D}) - \mathbf{D}(\mathbf{A}\cdot\mathbf{B}\times\mathbf{C})$.

Proof:

Let $\mathbf{X} = \mathbf{A}\times\mathbf{B}$; then

$(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = \mathbf{X}\times(\mathbf{C}\times\mathbf{D})$

$= \mathbf{C}(\mathbf{X}\cdot\mathbf{D}) - \mathbf{D}(\mathbf{X}\cdot\mathbf{C}).$

Thus,

$$\begin{aligned}(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) &= \mathbf{C}(\mathbf{A}\times\mathbf{B}\cdot\mathbf{D}) - \mathbf{D}(\mathbf{A}\times\mathbf{B}\cdot\mathbf{C})\\ &= \mathbf{C}(\mathbf{A}\cdot\mathbf{B}\times\mathbf{D}) - \mathbf{D}(\mathbf{A}\cdot\mathbf{B}\times\mathbf{C})\end{aligned}$$

Again,

Let $\mathbf{Y} = \mathbf{C}\times\mathbf{D}$; then

$(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\times\mathbf{B})\times\mathbf{Y}$

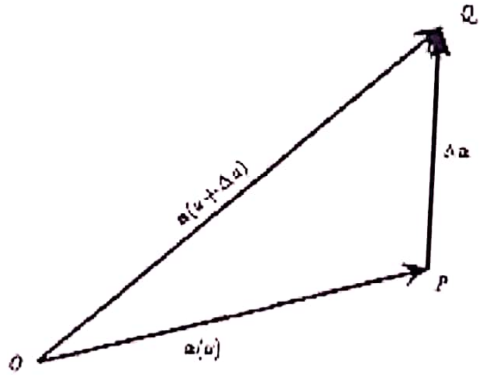
$= \mathbf{B}(\mathbf{A}\cdot\mathbf{Y}) - \mathbf{A}(\mathbf{B}\cdot\mathbf{Y}).$

Thus,

$$(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}\times\mathbf{D}) - \mathbf{A}(\mathbf{B}\cdot\mathbf{C}\times\mathbf{D})$$

Vector Differentiation

Differentiation with respect to a scalar variable. Let u be a scalar variable. If there is a value of a vector a corresponding to each value of the scalar u, $\vec{a}$ is said to be a function of u. We write $\vec{a}(u)$. Let us consider a general value of the scalar u and the corresponding vector $\vec{a}(u)$. Let the vector $\overrightarrow{OP}$ in Figure denote this vector. We now increase the scalar u by an amount Δu.



The vector corresponding to the scalar $u + \Delta u$ is $\vec{a}(u + \Delta u)$. Let the vector OQ in the Figure denote this vector $\vec{a}(u + \Delta u)$. The change in $\vec{a}(u)$ corresponding to the change Δu in u is then $\vec{a}(u + \Delta u) - \vec{a}(u)$. In the usual notation of calculus, we denote it by $\Delta\vec{a}$, so that

$$\Delta\vec{a} = \vec{a}(u + \Delta u) - \vec{a}(u).$$

From the figure it is seen that $\Delta\vec{a} = \overrightarrow{OP}$. Since Δu is a scalar, the vector $\dfrac{\Delta\vec{a}}{\Delta u}$ has the same direction as $\overrightarrow{PQ}$. The vector

$$\lim_{\Delta u \to 0} \frac{\Delta\vec{a}}{\Delta u} = \frac{\vec{a}(u+\Delta u) - \vec{a}(u)}{\Delta u}$$

is the rate of change of $\vec{a}$ with respect to u. It is also called the derivative of a with respect to u, and is denoted by the symbol $\dfrac{d\bar{a}}{dt}$ so that, $\dfrac{d\bar{a}}{dt} = \lim\limits_{\Delta u \to 0} \dfrac{\Delta\bar{a}}{\Delta u}$

In precisely the same way, we define the derivative with respect to u of the vector $\dfrac{d\bar{a}}{dt}$. This vector is denoted by $\dfrac{d^2\bar{a}}{dt^2}$.

Let **a**(u) and **b**(u) be any two vectors which are functions of a scalar u, and let $m$be a scalar function of u. We shall now derive the following

$$\frac{d}{du}(\mathbf{a}+\mathbf{b}) = \frac{d\mathbf{a}}{du} + \frac{d\mathbf{b}}{du},$$

$$\frac{d}{du}(m\mathbf{a}) = m\frac{d\mathbf{a}}{du} + \frac{dm}{du}\mathbf{a},$$

$$\frac{d}{du}(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot\frac{d\mathbf{b}}{du} + \frac{d\mathbf{a}}{du}\cdot\mathbf{b},$$

$$\frac{d}{du}(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times\frac{d\mathbf{b}}{du} + \frac{d\mathbf{a}}{du}\times\mathbf{b}.$$

Ex.2 If $\vec{a}=t^2\vec{i}-t\vec{j}+(2t+1)\vec{k}$ then find $\frac{d\vec{a}}{dt},\frac{d^2\vec{a}}{dt^2},\left|\frac{d\vec{a}}{dt}\right|,\left|\frac{d^2\vec{a}}{dt^2}\right| at\ t=0$

Soln. Given $\vec{a}=t^2\vec{i}-t\vec{j}+(2t+1)\vec{k}$

therefore $\frac{da}{dt}=2t\vec{i}-\vec{j}+2\vec{k}$ ....(i)

now $\frac{d^2\vec{a}}{dt^2}=2\vec{i}-0\vec{j}+0\vec{k}=2\vec{i}$ ....(ii)

also from (i) & (ii)

$$\left|\frac{da}{dt}\right|=\sqrt{\{4t^2+1+4\}}=\sqrt{(4t^2+5)}$$ ....(iii)

$$\left|\frac{d^2\vec{a}}{dt^2}\right|=\sqrt{(2)^2}=2$$ ....(iv)

Hence from (i) (ii) (iii) &(iv) at t=0 , we have

$$\frac{d\vec{a}}{dt}=-\vec{j}+2\vec{k},\frac{d^2a}{dt^2}=2\vec{i}\left|\frac{da}{dt}\right|=\sqrt{5}\text{ and }\left|\frac{d^2a}{dt^2}\right|=2.$$ Ans

Ex.3 Show that $\hat{r}\times d\hat{r}=\frac{\vec{r}\times d\vec{r}}{r^2}$, where $\vec{r}=r.\hat{r}$

Soln. Since $\hat{r}=\frac{\vec{r}}{r}$ ....(i)

$$\therefore d\hat{r}=\frac{1}{r}d\vec{r}+\vec{r}d\left(\frac{1}{r}\right)$$

$$=\frac{1}{r}d\vec{r}+\vec{r}\left(-\frac{1}{r^2}dr\right)$$

$$\therefore \hat{r}\times d\hat{r}=\hat{r}\times\left(\frac{1}{r}d\vec{r}-\frac{\vec{r}}{r^2}dr\right)$$

$$=\frac{\vec{r}}{r}\times\left(\frac{1}{r}d\vec{r}-\frac{\vec{r}}{r^2}dr\right)$$ [by (i)]

$$=\frac{1}{r^2}(\vec{r}\times d\vec{r})-\frac{1}{r^3}(\vec{r}\times d\vec{r})dr$$

$$=\frac{1}{r^2}(\vec{r}\times d\vec{r})[\because \vec{r}\times\vec{r}=0] hence proved$$

Ex.5 If $\vec{a} = a(t)$ has a constant magnitude, then show that $\vec{a} \cdot \frac{d\vec{a}}{dt} = 0$.

Soln. Given $\vec{a}$ has a constant magnitude ,therefore $|a|^2 = \vec{a}.\vec{a} =$ constant ......(i)

On differentiating (i) both side w.r.t. t , we get

$$\frac{d(\vec{a}.\vec{a})}{dt} = 0$$

$$\Rightarrow \vec{a}.\frac{d\vec{a}}{dt} + \frac{d\vec{a}}{dt}.\vec{a} = 0$$

$$\Rightarrow 2\,\vec{a}.\frac{d\vec{a}}{dt} = 0$$

$$\Rightarrow \vec{a}.\frac{d\vec{a}}{dt} = 0.$$

Thus, $\bar{a}$ and $\frac{d\bar{a}}{dt}$ are perpendicular to each other.

If u is the time t and $\bar{a}$ is the position vector at any point (x, y, z), $\frac{d\bar{a}}{dt}$ represents the velocity with which the terminal point of $\bar{a}$ describes the curves. Similarly, $\frac{d\bar{v}}{dt} = \frac{d^2\bar{a}}{dt^2}$ represents its acceleration along the curve.

Ex. 6

A particle moves along a curve whose parametric equations are $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$, where t is the time.

(a) Determine its velocity and acceleration at any time.

(b) Find the magnitudes of the velocity and acceleration at $t = 0$.

(a) The position vector $\mathbf{r}$ of the particle is $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = e^{-t}\mathbf{i} + 2\cos 3t\,\mathbf{j} + 2\sin 3t\,\mathbf{k}$.

Then the velocity is $\mathbf{v} = \frac{d\mathbf{r}}{dt} = -e^{-t}\mathbf{i} - 6\sin 3t\,\mathbf{j} + 6\cos 3t\,\mathbf{k}$

and the acceleration is $\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = e^{-t}\mathbf{i} - 18\cos 3t\,\mathbf{j} - 18\sin 3t\,\mathbf{k}$

(b) At $t = 0$, $\frac{d\mathbf{r}}{dt} = -\mathbf{i} + 6\mathbf{k}$ and $\frac{d^2\mathbf{r}}{dt^2} = \mathbf{i} - 18\mathbf{j}$. Then

magnitude of velocity at $t = 0$ is $\sqrt{(-1)^2 + (6)^2} = \sqrt{37}$

magnitude of acceleration at $t = 0$ is $\sqrt{(1)^2 + (-18)^2} = \sqrt{325}$.

Ex. 7

A particle moves along the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$, where t is the time. Find the components of its velocity and acceleration at time $t = 1$ in the direction $\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$.

$$\text{Velocity} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}\left[2t^2\mathbf{i} + (t^2 - 4t)\mathbf{j} + (3t - 5)\mathbf{k}\right]$$

$$= 4t\mathbf{i} + (2t - 4)\mathbf{j} + 3\mathbf{k} = 4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} \quad \text{at } t = 1.$$

Unit vector in direction $\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ is $\dfrac{\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}}{\sqrt{(1)^2 + (-3)^2 + (2)^2}} = \dfrac{\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}}{\sqrt{14}}$.

Then the component of the velocity in the given direction is

$$\frac{(4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})\cdot(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k})}{\sqrt{14}} = \frac{(4)(1) + (-2)(-3) + (3)(2)}{\sqrt{14}} = \frac{16}{\sqrt{14}} = \frac{8\sqrt{14}}{7}.$$

$$\text{Acceleration} = \frac{d^2\mathbf{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right) = \frac{d}{dt}\left[4t\mathbf{i} + (2t - 4)\mathbf{j} + 3\mathbf{k}\right] = 4\mathbf{i} + 2\mathbf{j} + 0\mathbf{k}.$$

Then the component of the acceleration in the given direction is

$$\frac{(4\mathbf{i} + 2\mathbf{j} + 0\mathbf{k})\cdot(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k})}{\sqrt{14}} = \frac{(4)(1) + (2)(-3) + (0)(2)}{\sqrt{14}} = \frac{-2}{\sqrt{14}} = \frac{-\sqrt{14}}{7}$$

If C is a space curve defined by the function $r(u)$, then we have seen that $\dfrac{d\bar{r}}{du}$ is a vector in the direction of the tangent to C. If the scalar u is taken as the arc length s measured from some fixed point on C, then $\dfrac{d\bar{r}}{ds}$ is a unit tangent vector to C and is denoted by T.

Ex. 8

(a) Find the unit tangent vector to any point on the curve $x = t^2 + 1$, $y = 4t - 3$, $z = 2t^2 - 6t$.
(b) Determine the unit tangent at the point where $t = 2$.

(a) A tangent vector to the curve at any point is

$$\frac{d\mathbf{r}}{dt} = \frac{d}{dt}\left[(t^2 + 1)\mathbf{i} + (4t - 3)\mathbf{j} + (2t^2 - 6t)\mathbf{k}\right] = 2t\mathbf{i} + 4\mathbf{j} + (4t - 6)\mathbf{k}$$

The magnitude of the vector is $\left|\dfrac{d\mathbf{r}}{dt}\right| = \sqrt{(2t)^2 + (4)^2 + (4t - 6)^2}$.

Then the required unit tangent vector is $\mathbf{T} = \dfrac{2t\mathbf{i} + 4\mathbf{j} + (4t - 6)\mathbf{k}}{\sqrt{(2t)^2 + (4)^2 + (4t - 6)^2}}$

Note that since $\left|\dfrac{d\mathbf{r}}{dt}\right| = \dfrac{ds}{dt}$, $\mathbf{T} = \dfrac{d\mathbf{r}/dt}{ds/dt} = \dfrac{d\mathbf{r}}{ds}$.

(b) At $t = 2$, the unit tangent vector is $\mathbf{T} = \dfrac{4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}}{\sqrt{(4)^2 + (4)^2 + (2)^2}} = \dfrac{2}{3}\mathbf{i} + \dfrac{2}{3}\mathbf{j} + \dfrac{1}{3}\mathbf{k}$.

INTEGATION OF VECTORS

Let be a $\boldsymbol{a}(t)$vector function and there exists a function $f(t)$ such that $\frac{df}{dt} = \boldsymbol{a}(t)$ then $f(t)$ is defined as integral of $\boldsymbol{a}(t)$ and written as.

$$\int \boldsymbol{a}(t)dt = f(t) + C \qquad \text{... (1)}$$

C is arbitrary constant vector.

In general if $\boldsymbol{a}(t) = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}$then$\int \boldsymbol{a}(t)dt = \boldsymbol{i}\int a_x dt + \boldsymbol{j}\int a_y dt + \boldsymbol{k}\int a_z dt.$

Ex.1 If $\vec{f}(t) = t\vec{i} + (t^2 - 2t)\vec{j} + (3t^2 + 3t^3)\vec{k}$ then find that $\int_0^1 [\vec{f}(t)]dt$

Soln. $\int_0^1 f(t)dt = \int_0^1 [ti + (t^2 - 2t)j + (3t^2 + 3t^3)k]dt$

$$= i\int_0^1 t\,dt + j\int_0^1 (t^2 - 2t)dt + k\int_0^1 (3t^2 + 3t^3)dt + c$$

$$= i\left[\frac{t^2}{2}\right]_0^1 + j\left[\frac{t^3}{3} - t^2\right]_0^1 + k\left[t^3 + \frac{3}{4}t^4\right]_0^1 + c$$

$$= i\left(\frac{1}{2}\right) + j\left[\frac{1}{3} - 1\right] + k\left[1 + \frac{3}{4}\right]$$

$$= \frac{1}{2}i - \frac{2}{3}j + \frac{7}{4}k$$

Ex.2 If $\vec{a}(t) = 5t^2\vec{i} + t\vec{j} - t^3\vec{k}$, show that $\int_1^2 \left(\vec{a} \times \frac{d^2\vec{a}}{dt}\right)dt = -14\vec{i} + 75\vec{j} - 15\vec{k}$

Soln. Given $\vec{a} = 5t^2\vec{i} + t\vec{j} - t^3\vec{k}$

$$\therefore \quad \frac{d\vec{a}}{dt} = 10t\,\vec{i} + \vec{j} - 3t^2\vec{k}$$

and $\frac{d^2\vec{a}}{dt^2} = 10\vec{i} - 6t\vec{k}$

$$\text{Now } \vec{a} \times \frac{d^2\vec{a}}{dt^2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5t^2 & t & -t^3 \\ 10 & 0 & -6t \end{vmatrix}$$

$$= \quad \vec{i}\left[-6t^2 - 0\right] - \vec{j}\left[-30t^3 + 10t^3\right] + \vec{k}\left[0 - 10t\right]$$

$$= \quad -6t^2\vec{i} + 20t^3\vec{j} - 10t\vec{k}$$

$$\therefore \quad \int_1^2 \left(\vec{a} \times \frac{d^2a}{dt^2}\right) dt = \int_1^2 \left[-6t^2\vec{i} + 20t^3\vec{j} - 10t\vec{k}\right] dt$$

$$= \quad \left[-6\frac{t^3}{3}\vec{i}\right]_1^2 + \left[20\frac{t^4}{4}\vec{j}\right]_1^2 + \left[-10\frac{t^2}{2}\vec{k}\right]_1^2$$

$$= \quad \left[-2t^3\right]_1^2\vec{i} + \left[5t^4\right]_1^2\vec{j} - \left[5t^2\right]_1^2\vec{k}$$

$$= \quad -14\vec{i} + 75\vec{j} - 15\vec{k}$$

Ex. 3

The acceleration of a particle at any time $t \geqq 0$ is given by

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = 12 \cos 2t \, \mathbf{i} - 8 \sin 2t \, \mathbf{j} + 16t \, \mathbf{k}$$

If the velocity $\mathbf{v}$ and displacement $\mathbf{r}$ are zero at $t = 0$, find $\mathbf{v}$ and $\mathbf{r}$ at any time.

Integrating, $\mathbf{v} = \mathbf{i} \int 12 \cos 2t \, dt + \mathbf{j} \int -8 \sin 2t \, dt + \mathbf{k} \int 16t \, dt$

$= 6 \sin 2t \, \mathbf{i} + 4 \cos 2t \, \mathbf{j} + 8t^2 \, \mathbf{k} + \mathbf{c}_1$

Putting $\mathbf{v} = \mathbf{0}$ when $t = 0$, we find $\mathbf{0} = 0\mathbf{i} + 4\mathbf{j} + 0\mathbf{k} + \mathbf{c}_1$ and $\mathbf{c}_1 = -4\mathbf{j}$.

Then $\mathbf{v} = 6 \sin 2t \, \mathbf{i} + (4 \cos 2t - 4) \, \mathbf{j} + 8t^2 \, \mathbf{k}$

so that $\frac{d\mathbf{r}}{dt} = 6 \sin 2t \, \mathbf{i} + (4 \cos 2t - 4) \, \mathbf{j} + 8t^2 \, \mathbf{k}$.

Integrating, $\mathbf{r} = \mathbf{i} \int 6 \sin 2t \, dt + \mathbf{j} \int (4 \cos 2t - 4) \, dt + \mathbf{k} \int 8t^2 \, dt$

$= -3 \cos 2t \, \mathbf{i} + (2 \sin 2t - 4t) \, \mathbf{j} + \frac{8}{3}t^3 \, \mathbf{k} + \mathbf{c}_2$

Putting $\mathbf{r} = \mathbf{0}$ when $t = 0$, $\mathbf{0} = -3\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} + \mathbf{c}_2$ and $\mathbf{c}_2 = 3\mathbf{i}$.

Then $\mathbf{r} = (3 - 3 \cos 2t)\mathbf{i} + (2 \sin 2t - 4t)\mathbf{j} + \frac{8}{3}t^3 \, \mathbf{k}$.