Title: Functions, Limits and Continuity

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Time: 10:00 am-12:00 am

Outline

- **Functions and its graphs.**
- **One-one, Onto and inverse functions.**
- **Transcendental functions.**
- **Bounded and monotonic functions.**
- **Limits of functions.**
- **Right and left hand limits.**
- **Special limits.**
- **Continuity.**
- **Right and left hand continuity.**
- **Sectional continuity.**
- **Uniform continuity, Lipschitz continuity.**

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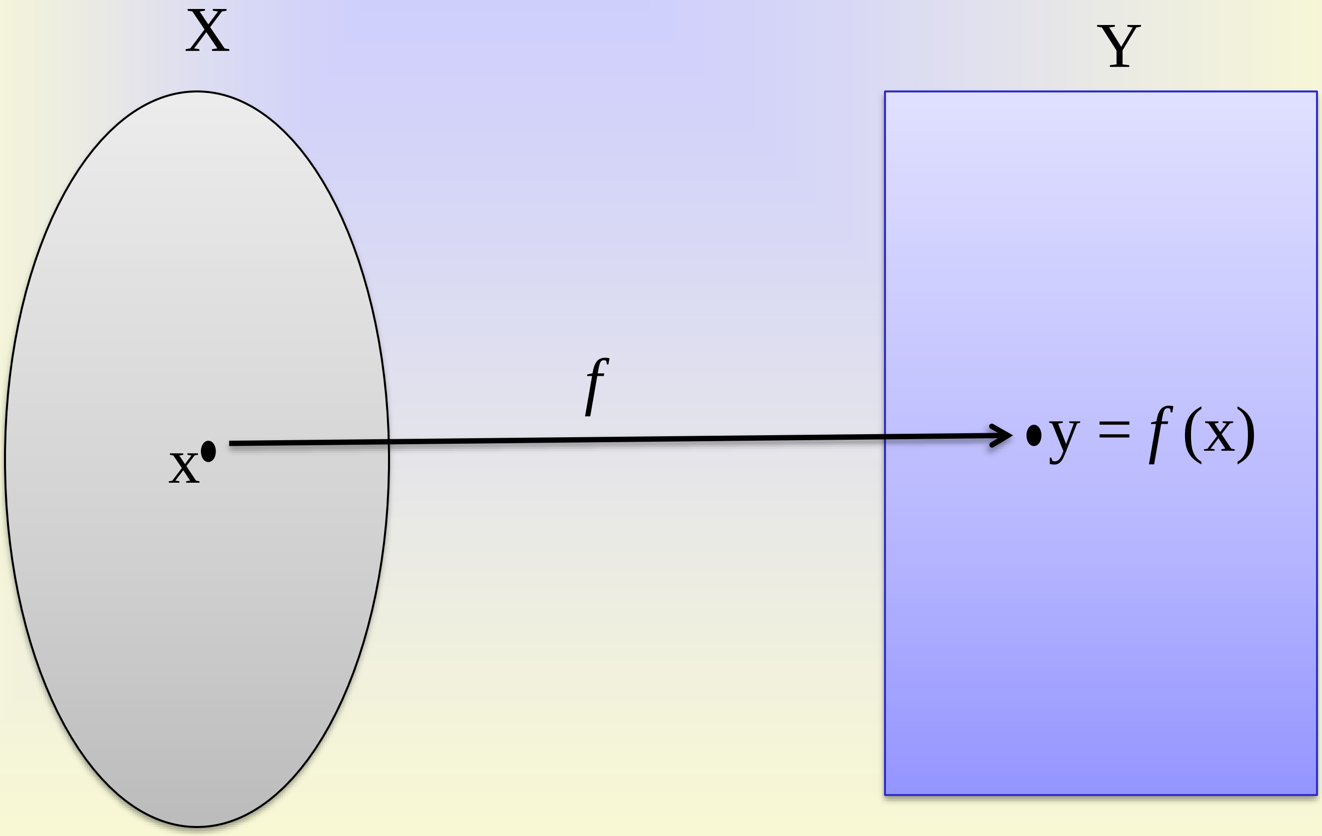
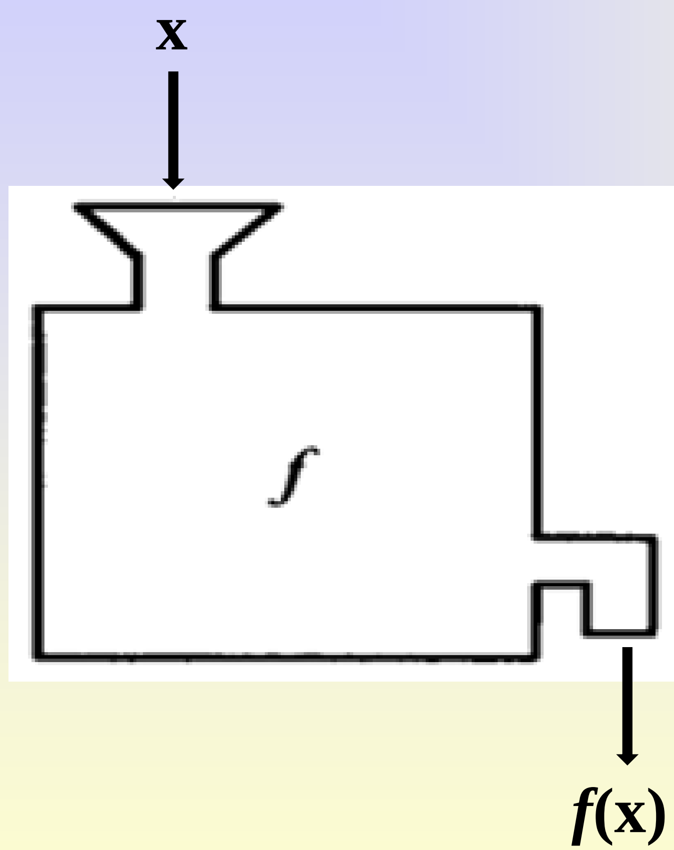
Functions and its Graphs



Functions and its Graphs



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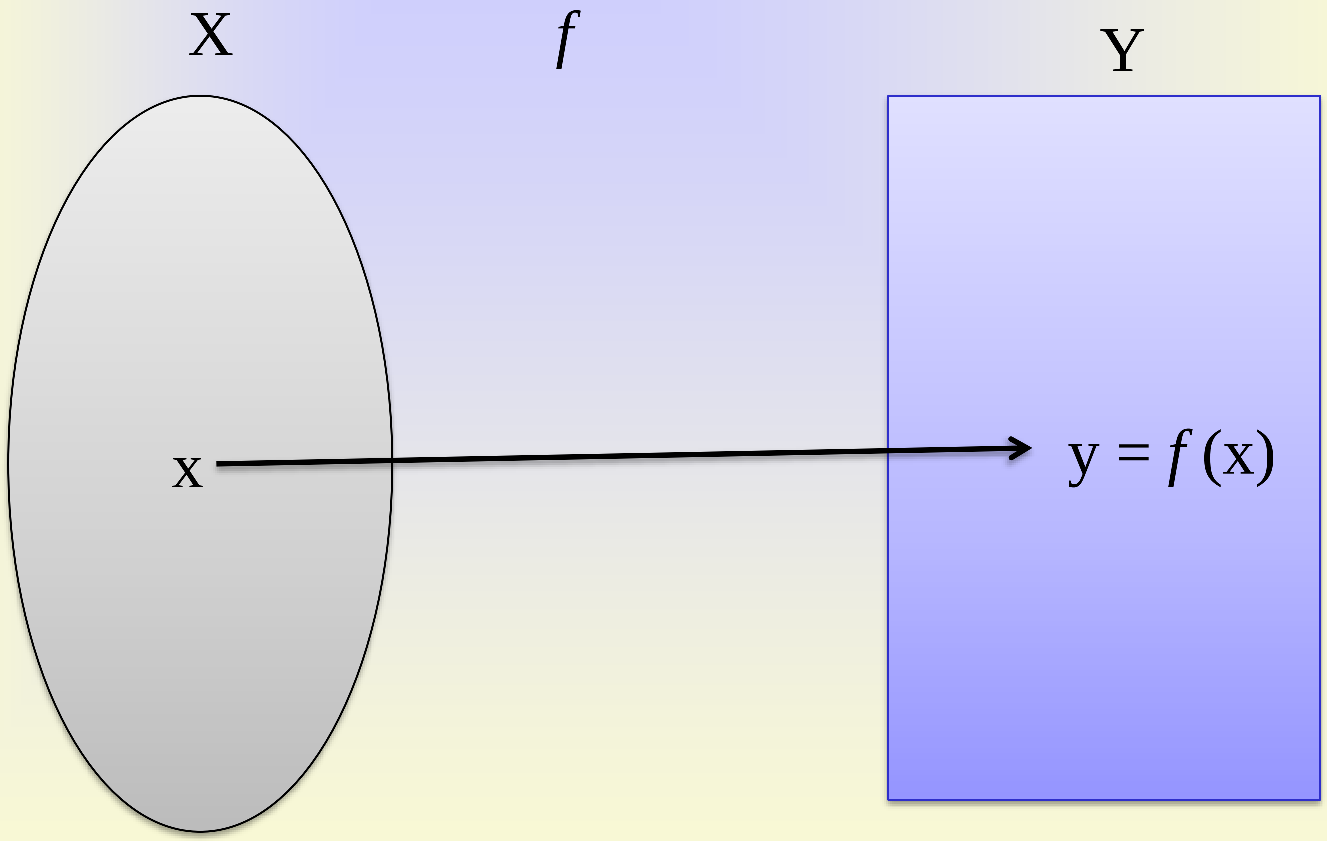
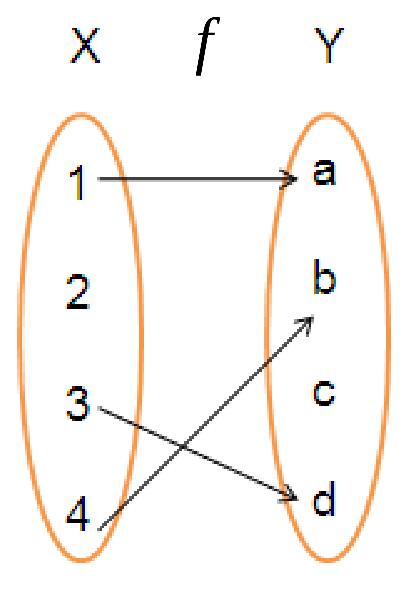
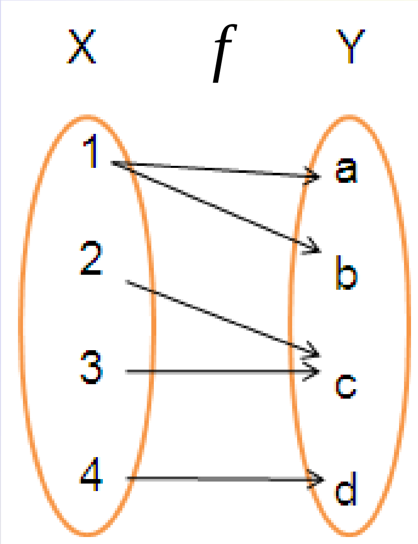


$f : X \to Y$ if for each $x \in X$ $\exists$ a unique $y \in Y$ such that $y = f(x)$.

Functions and its Graphs



f is not a function



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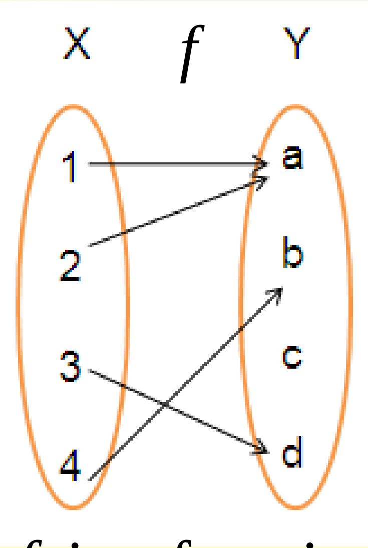

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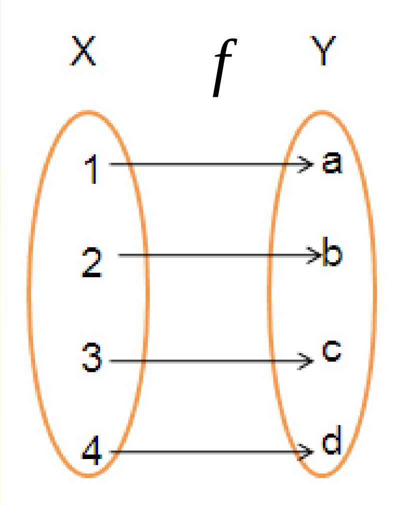

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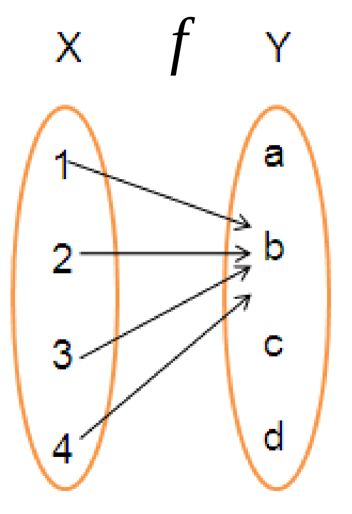
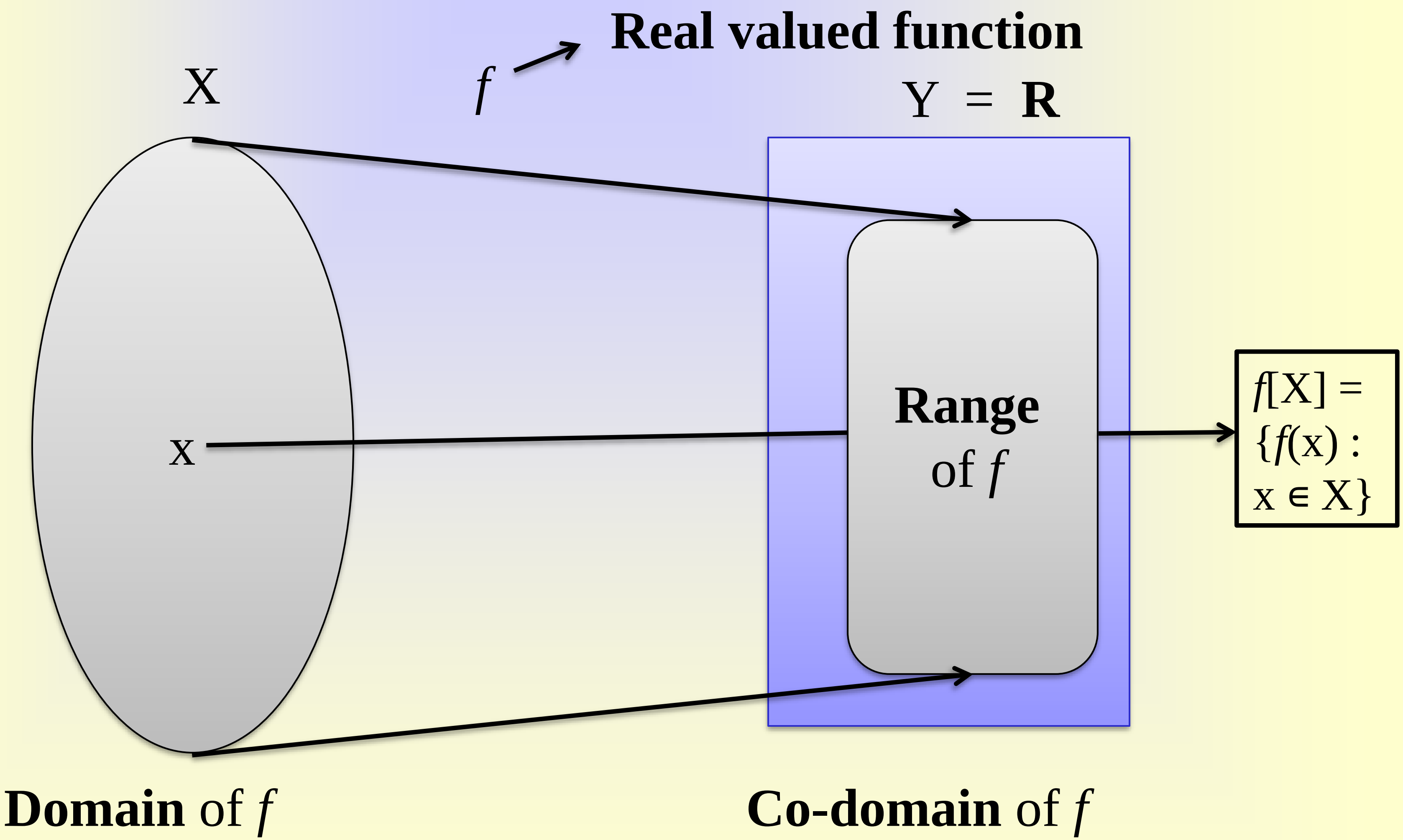


f is a function

Functions and its Graphs



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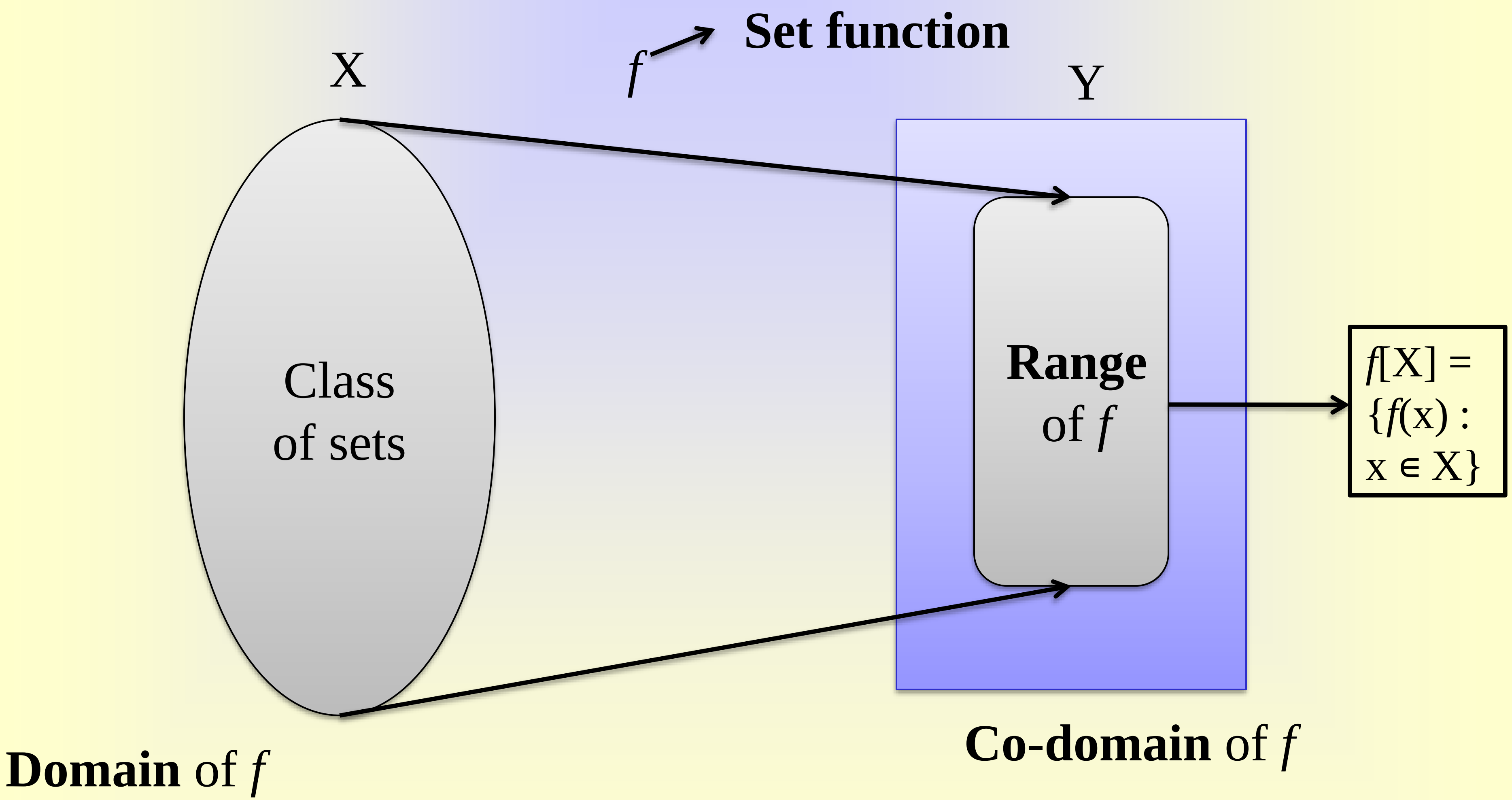
Functions and its Graphs



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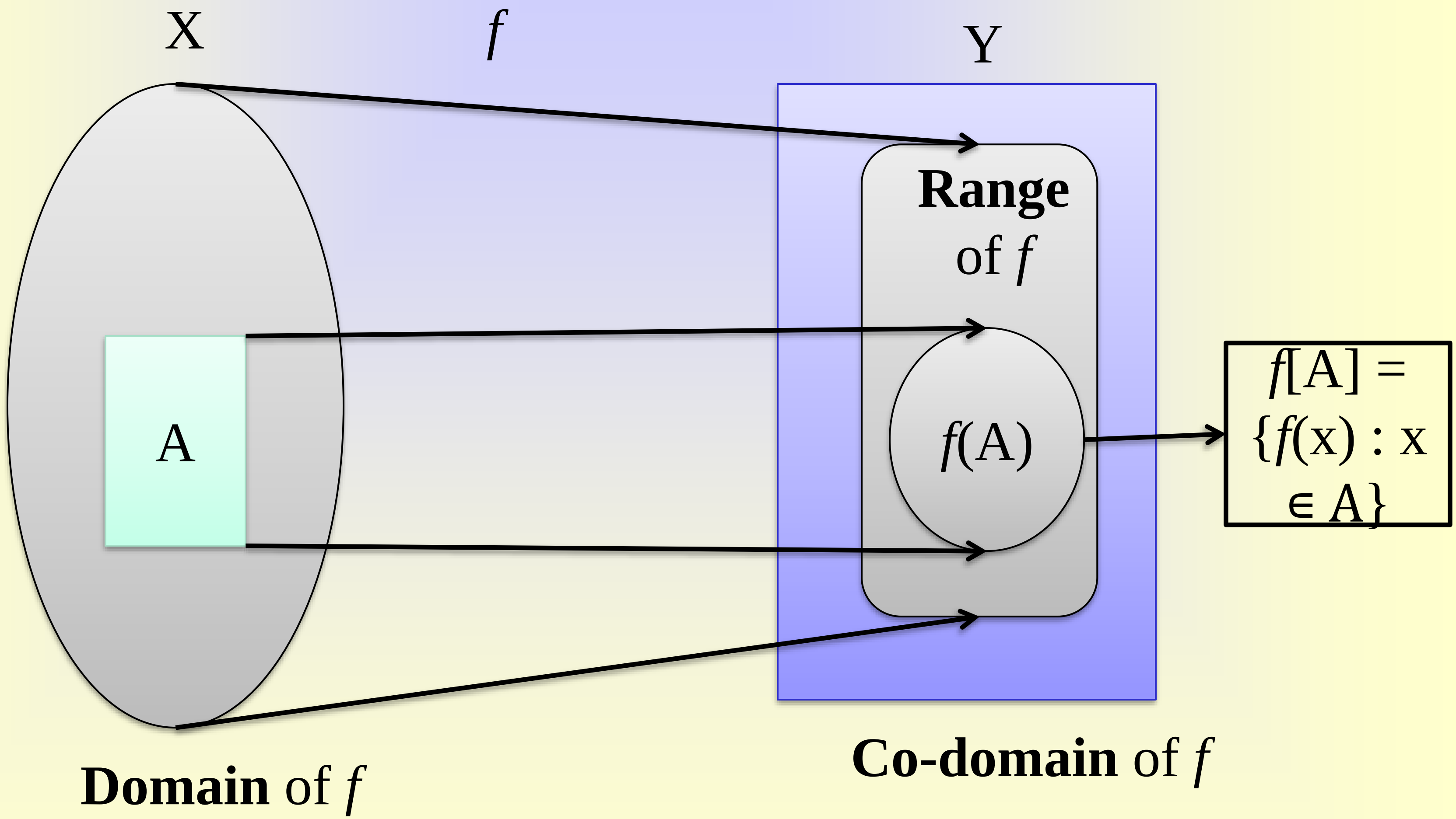
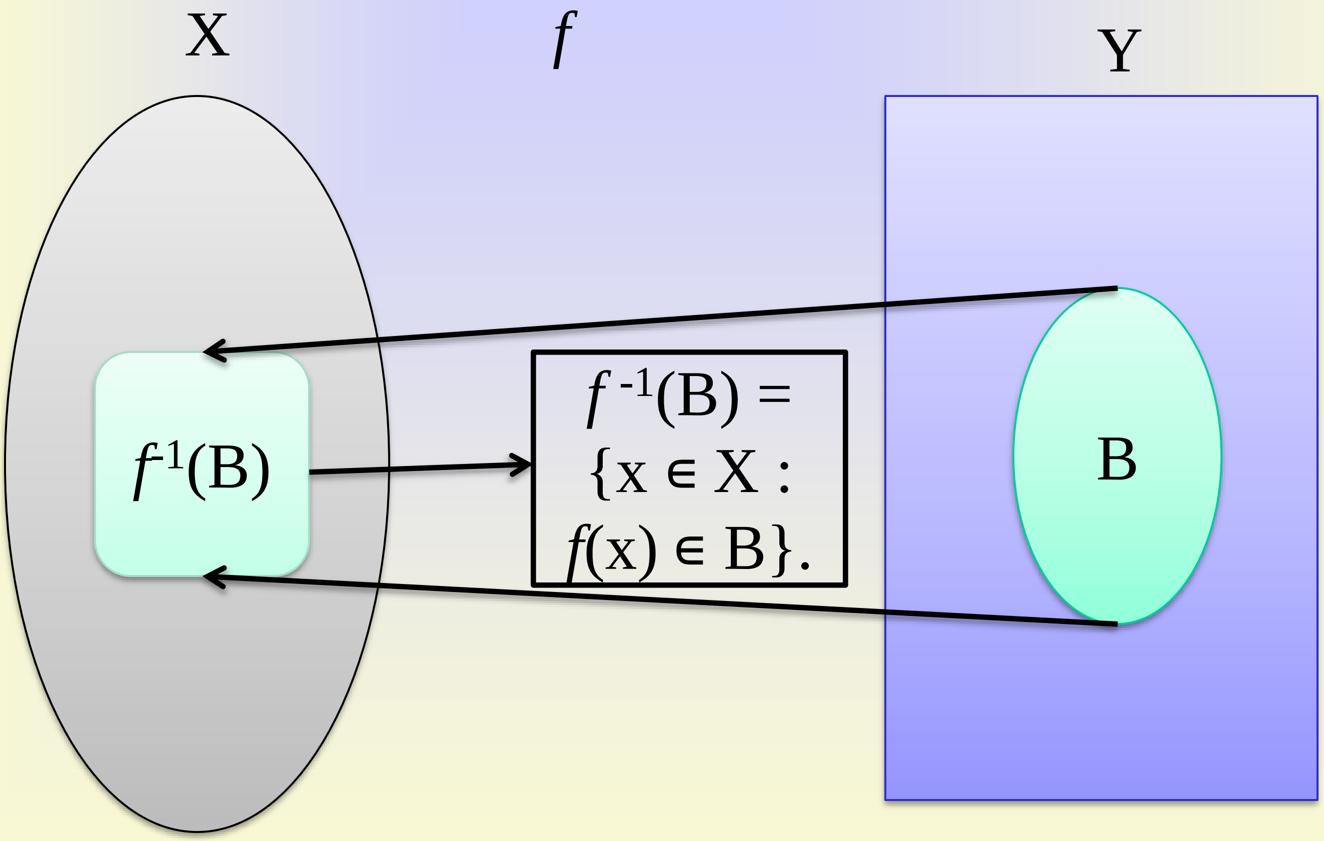
Functions and its Graphs



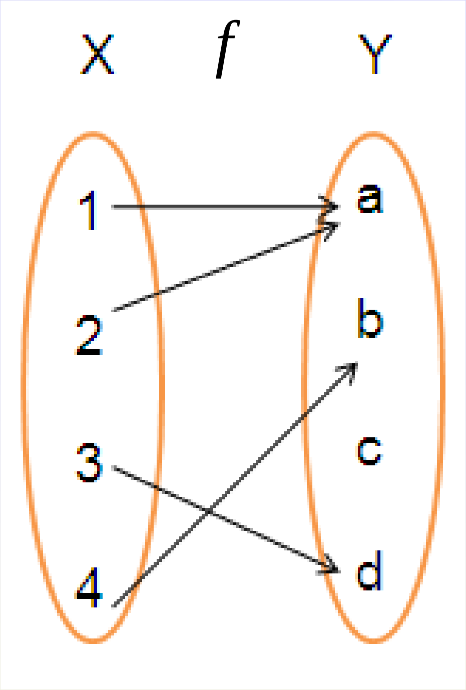
If $f : X \rightarrow Y$ then $\exists$ two set functions $f : 2^X \rightarrow 2^Y$ and $f^{-1} : 2^Y \rightarrow 2^X$

Functions and its Graphs



If $f : X \to Y$ then $\exists$ two set functions $f : 2^X \to 2^Y$ and $f^{-1} : 2^Y \to 2^X$

Functions and its Graphs



$f(\{1, 2\}) = \{a\}$, $f(\{1, 3, 4\}) = \{a, b, d\}$, $f(\{2, 3\}) = \{a, d\}$

also

$f^{-1}(\{a\}) = \{1, 2\}$, $f^{-1}(\{b, c\}) = \{4\}$, $f^{-1}(\{a, b, d\}) = \{1, 2, 3, 4\}$,

$f^{-1}(\{c\}) = \varnothing$,

Functions and its Graphs

Between these two set functions f^{-1} plays very important role in **topology** and **measure**. Since f^{-1} preserves arbitrary union, arbitrary intersection, difference, monotonicity, complementation etc. i.e.

Theorem

If $f : X \to Y$ then for any subset A and B of Y,

(i) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

(ii) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

(iii) $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$.

(iv) If $A \subseteq B$ then $f^{-1}(A) \subseteq f^{-1}(B)$.

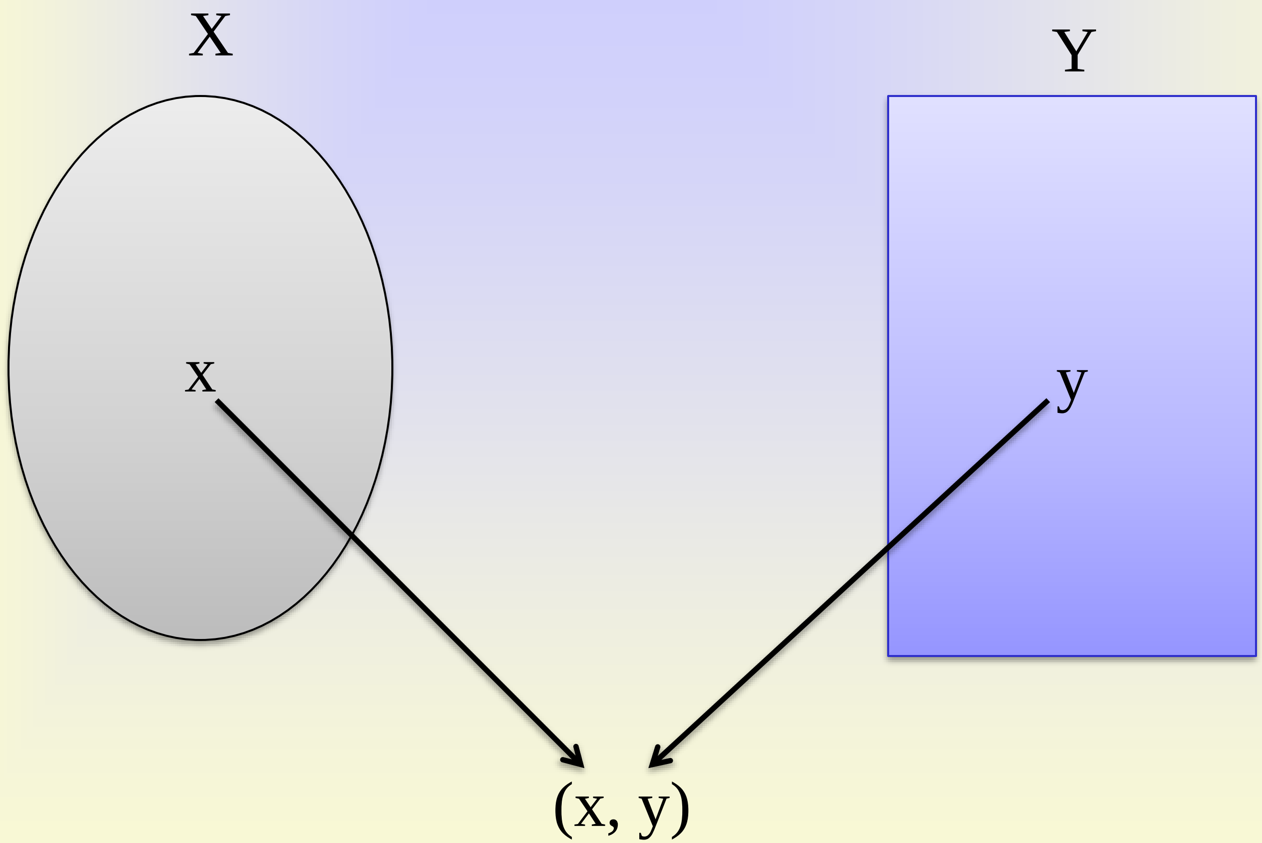
(v) $f^{-1}(A^c) = (f^{-1}(A))^c$.

And, more generally, for any indexed $\{A_i\}$ of subsets of Y,

(vi) $f^{-1}(\cup_i A_i) = \cup_i f^{-1}(A_i)$.

(vii) $f^{-1}(\cap_i A_i) = \cap_i f^{-1}(A_i)$.

Functions and its Graphs



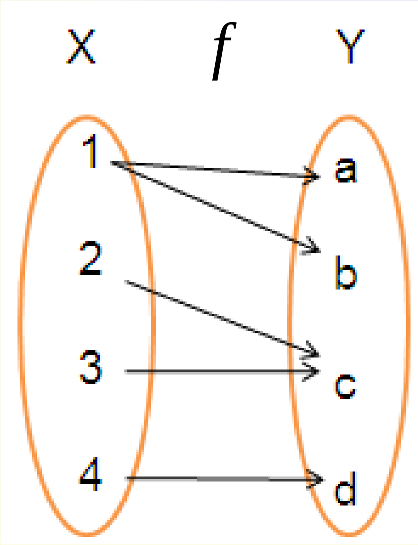
$$X \times Y = \{(x, y) : x \in X \,,\, y \in Y\}$$

A **relation** from X to Y is a subset of $X \times Y$.

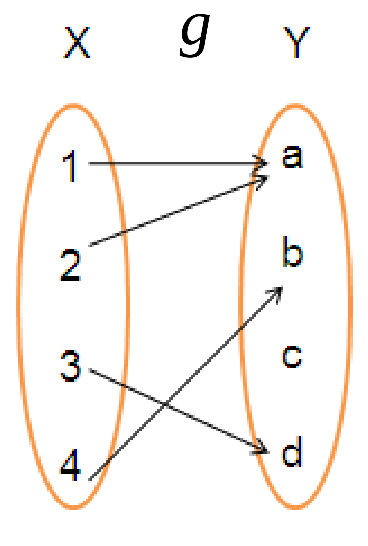
Functions and its Graphs

A **function** can also be described as a set of **ordered pairs** (x, y) such that for any x-value in the set, there is only one y-value. This means that there cannot be any repeated x-values with different y-values.

A **relation** is called a **function** if for any x-value in the set, there is only one y-value. This means that there cannot be any repeated x-values with different y-values.



f is not a function

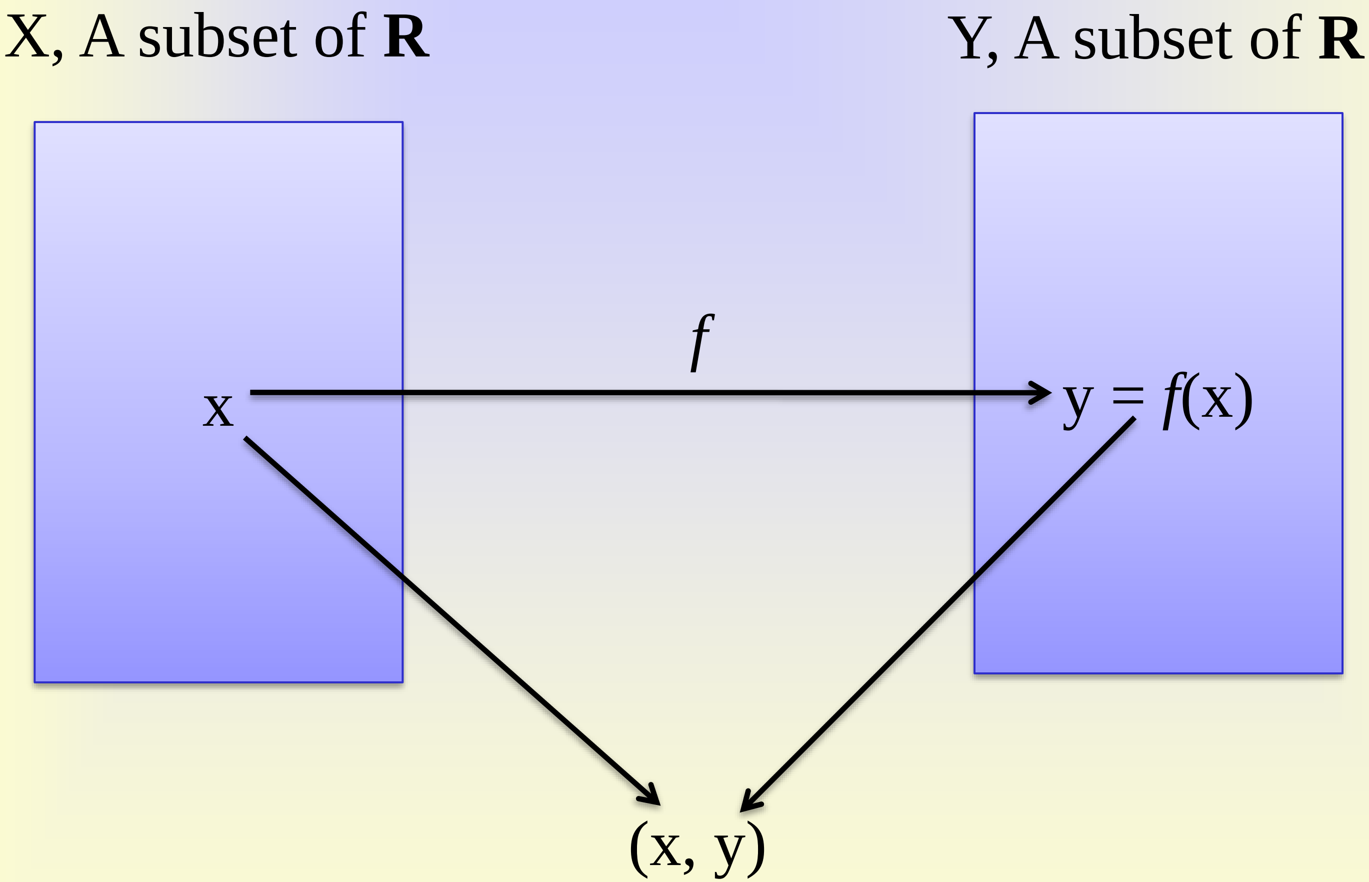


g is a function

f = {(1, a), (1, b), (2, c), (3, c), (4, d)} is not a function.

g = {(1, a), (2, a), (3, d), (4, b)} is a function.

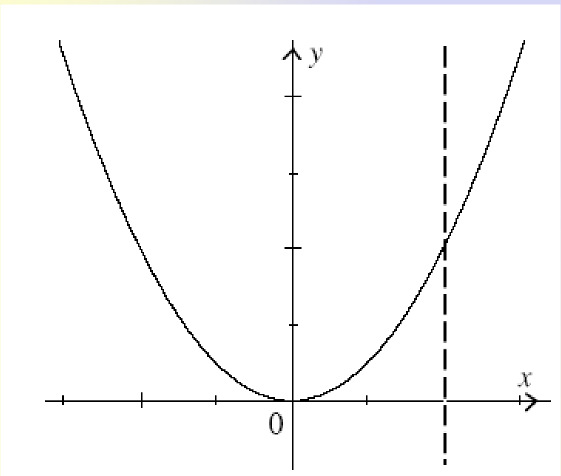
Functions and its Graphs

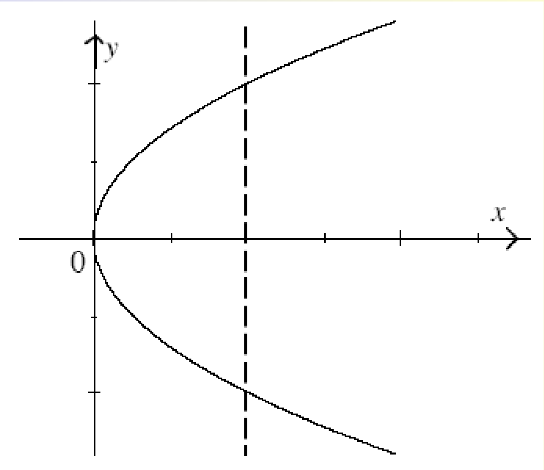


We plot the **domain** X on the x-axis, and the **co-domain** Y on the y-axis. Then for each point x in X we plot the point (x, y), where y = f(x). The totality of such points (x, y) is called the **graph of the function.**

Functions and its Graphs

The Vertical Line Test





This is the **graph of a function**. All possible vertical lines will cut this graph only **once**.

This is not the **graph of a function**. The vertical line we have drawn cuts the graph **twice**.

Functions and its Graphs

Now we consider some **examples** of **real functions**.

Example 1.

The identity function.

Let the function $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x$ for all real x. This function is often called the **identity function** on **R** and it is denoted by $1_{\mathbf{R}}$. Its **domain** is the **real line**, that is, the set of all **real numbers**. Here x = y for each point (x, y) on the **graph** of *f*. The **graph** is a straight line making equal angles with the coordinates axes (see Figure-1). The **range** of *f* is the set of all **real numbers**.

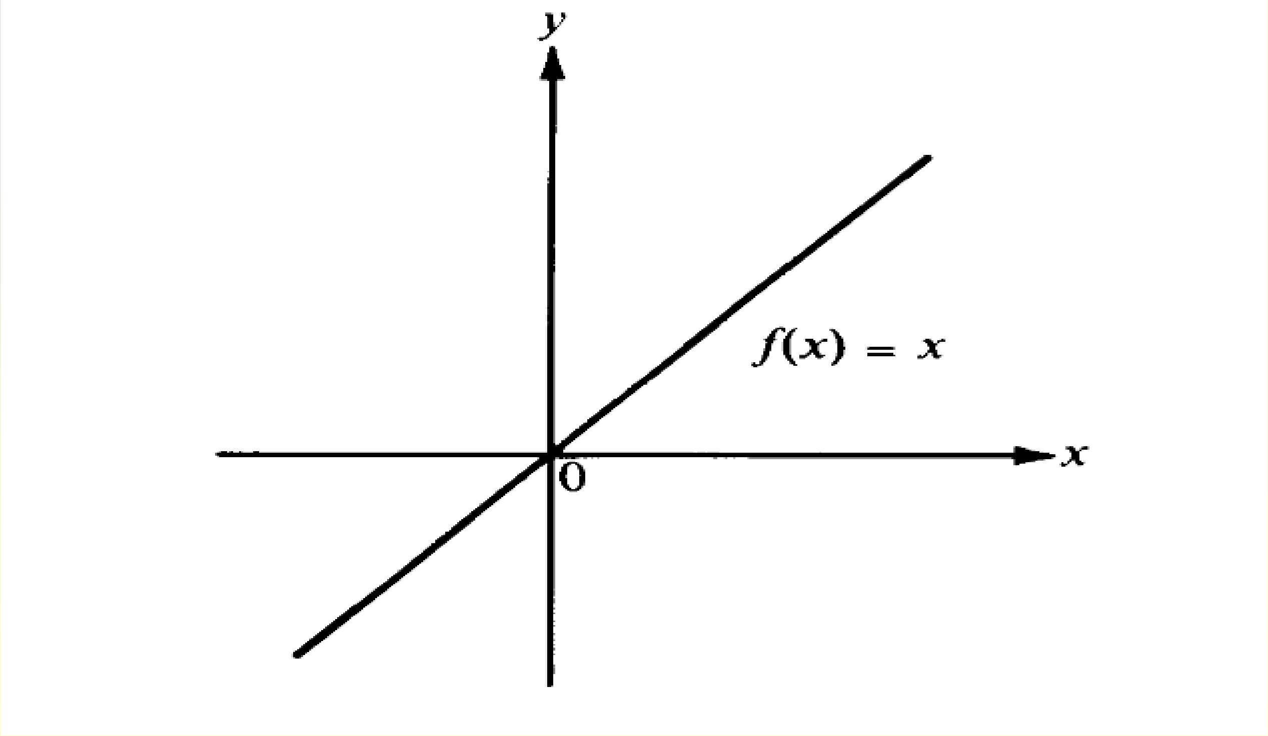


Figure-1 Graph of the identity function *f*(x) = x.

Functions and its Graphs

Example 2.

The absolute-value function.

Consider the function which assigns to each **real number** x the **nonnegative** number |x|. We define the function y = |x| as

$$y = \begin{cases} +x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

From this definition we can graph the function by taking each part separately. The graph of y = |x| is given below.

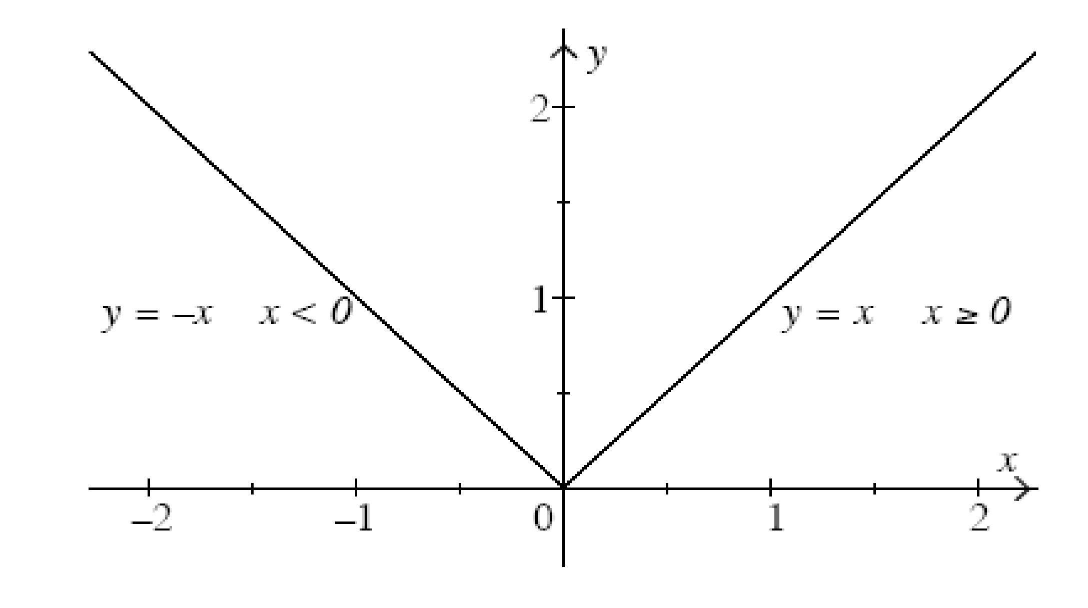


Figure- 2 Graph of the absolute-value function y = *f*(x) = |x|.

Functions and its Graphs

Example 3.

Constant functions.

A **function** whose **range** consists of a single number is called a **constant function**. An example is shown in Figure-3, where $f(x) = 3$ for every real x. The **graph** is a horizontal line cutting the Y-axis at the point (0, 3).

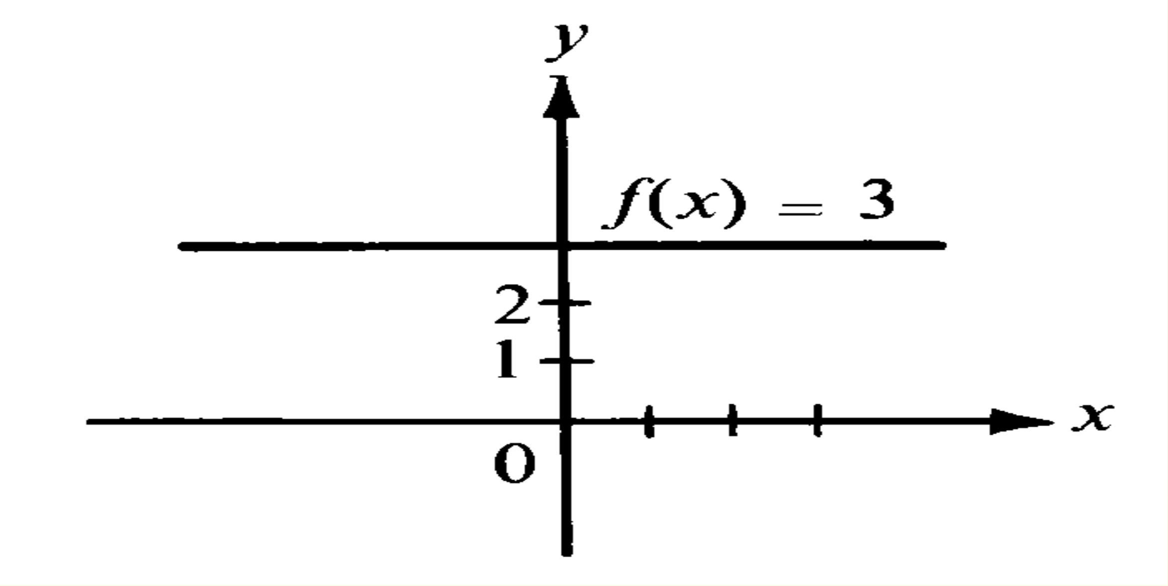


Figure- 3 Graph of the constant function *f*(x) = 3.

Functions and its Graphs

Example 4.

Linear functions and **affine linear function.**

Let the function g be defined for all real x by a formula of the form g(x) = ax + b, where a and b are real numbers, then *g* is called a **linear function** if b = 0. Otherwise, *g* is called a **affine linear function.** The example, *f*(x) = x, shown in Figure-1 is a **linear function**. And, g(x) = 2x - 1, shown in Figure-4 is a **affine linear function**.

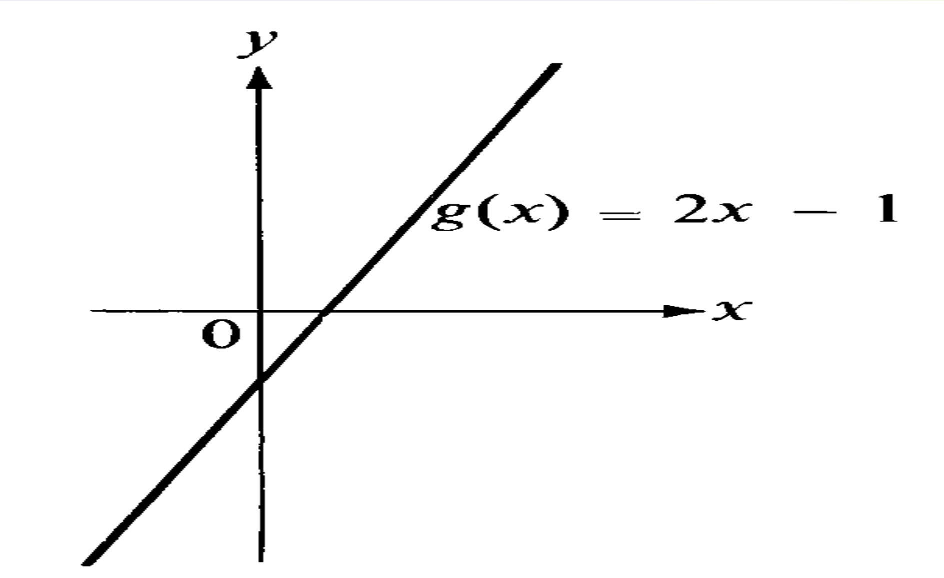


Figure- 4 Graph of the affine linear function g(x) = 2x - 1.

Functions and its Graphs

Example 5.

The greatest integer function

The greatest integer function is defined by $f(x) = [x]$ = The greatest integer less than or equal to x. **Figure- 5** shows the graph of $f(x) = [x]$.

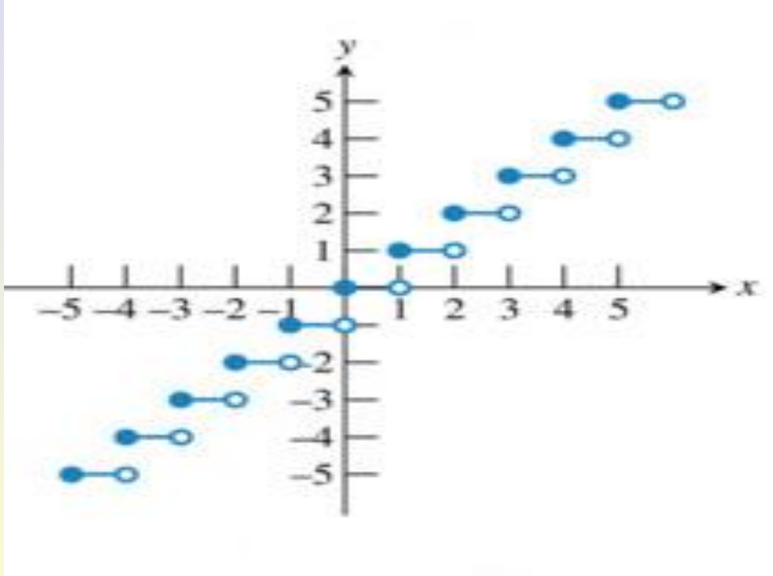


Figure- 5 Graph of the greatest integer function is defined by $f(x) = [x]$.

Functions and its Graphs

Example 6.

Polynomial functions.

A polynomial function P is one defined for all real a by an equation of the form

$P(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots\ldots\ldots\ldots\ldots + c_nx^n$

The numbers c_0 , c_1, c_2, c_3,……………,c_n are called the coefficients of the **polynomial**, and the **nonnegative integer** n is called its **degree** (if $c_n \neq 0$). They include the **constant functions** and the **power functions** as special cases. Polynomials of degree 2, 3, and 4 are called quadratic, cubic, and quartic polynomials, respectively. **Figure-6** shows a portion of the graph of a quartic polynomial P given by $P(x) = ½ x^4 – 2x^2$.

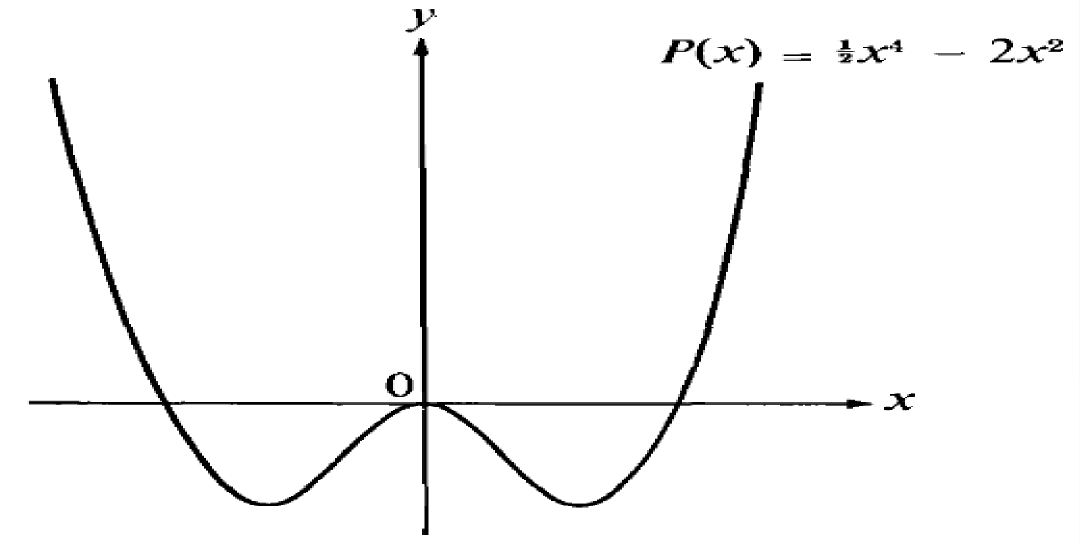


Figure- 6 Graph of a quartic polynomial function $p(x) = ½ x^4 – 2x^2$.

Functions and its Graphs

f $\qquad\qquad$ g

are functions

k = constant

$h_1 = f + g$

$h_2 = f \, . g$

$h_3 = f / g$

$h_4 = k + f$

$h_5 = k \, . \, f$

$h_6 = f \circ g$

$h_7 = g \circ f$

Functions and its Graphs

We can define

$$h_1 = f + g, \quad h_2 = f\,.g\,, \quad h_3 = f\,/g, \quad h_4 = \mathrm{k} + f \textit{ and } \quad h_5 = \mathrm{k}\,.\,f$$

by

$h_1(\mathrm{x}) = (f+g)(\mathrm{x}) = f(\mathrm{x}) + g(\mathrm{x}),$

$h_2(\mathrm{x}) = (f\,.g)(\mathrm{x}) = f(\mathrm{x})\,.\,g(\mathrm{x}),$

$h_3(\mathrm{x}) = (f\,/g)(\mathrm{x}) = f(\mathrm{x})\,/\,g(\mathrm{x}),$

$h_4(\mathrm{x}) = (\mathrm{k} + f)(\mathrm{x}) = \mathrm{k} + f(\mathrm{x})\,,$

$h_5(\mathrm{x}) = (\mathrm{k}\,.\,f)(\mathrm{x}) = \mathrm{k}\,.\,f(\mathrm{x})\,,$

if

f and *g* are **real valued** and defined on the same domain X and k is a real number.

Also we can define

$$h_6 = f \circ g \text{ and } h_7 = g \circ f$$

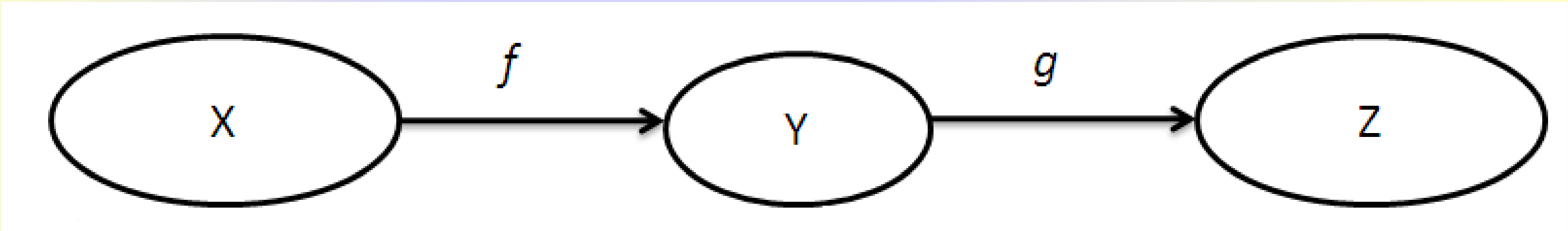
by

$h_6(\mathrm{x}) = (f \circ g)(\mathrm{x}) = f(g(\mathrm{x}))$ if **co-domain** of g = **domain** of f

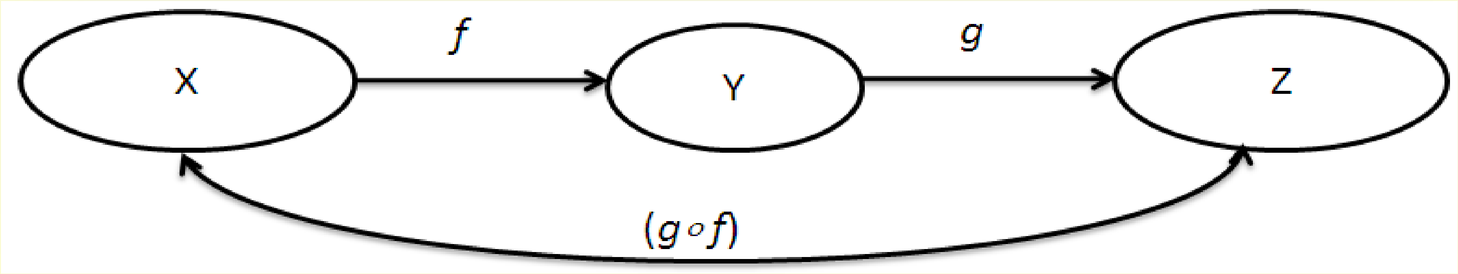
and $h_7(\mathrm{x}) = (g \circ f)(\mathrm{x}) = g\,(f(\mathrm{x}))$ if **co-domain** of f = **domain** of g

Generally, $f \circ g \neq g \circ f$

Composition Function



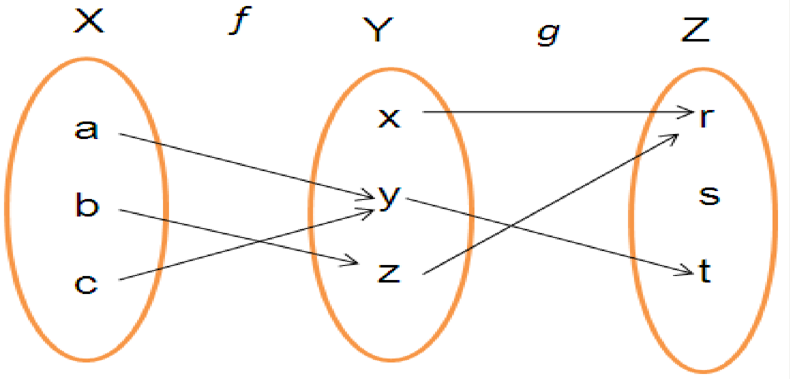
If $f : X \to Y$ and $g : Y \to Z$ then we define a function $(g \circ f) : X \to Z$ by $(g \circ f)(\text{x}) \equiv g(f(\text{x}))$.



Composition Function

Example

Let $f : X \to Y$ and $g : Y \to Z$ be define by the following diagrams



Then we can compute $(g \circ f) : X \to Z$ by its definition:

$$(g \circ f)(\text{a}) \equiv g(f(\text{a})) = g(\text{y}) = \text{t}$$

$$(g \circ f)(\text{b}) \equiv g(f(\text{b})) = g(\text{z}) = \text{r}$$

$$(g \circ f)(\text{c}) \equiv g(f(\text{c})) = g(\text{y}) = \text{t}$$

Remark:

Let $f : X \to Y$. Then $1_Y \circ f = f$ and $f \circ 1_X = f$ where 1_X and 1_Y are **identity functions** on X and Y respectively. That is the product of any function and the **identity function** is the function itself.

Functions and its Graphs

Theorem:

Let $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$. Then $(h \circ g) \circ f = h \circ (g \circ f)$.

Theorem

Two functions *f* and g are equal if and only if

(a) *f* and g have the **same domain**, and

(b) *f*(x) = g(x) for every x in the domain of *f*.

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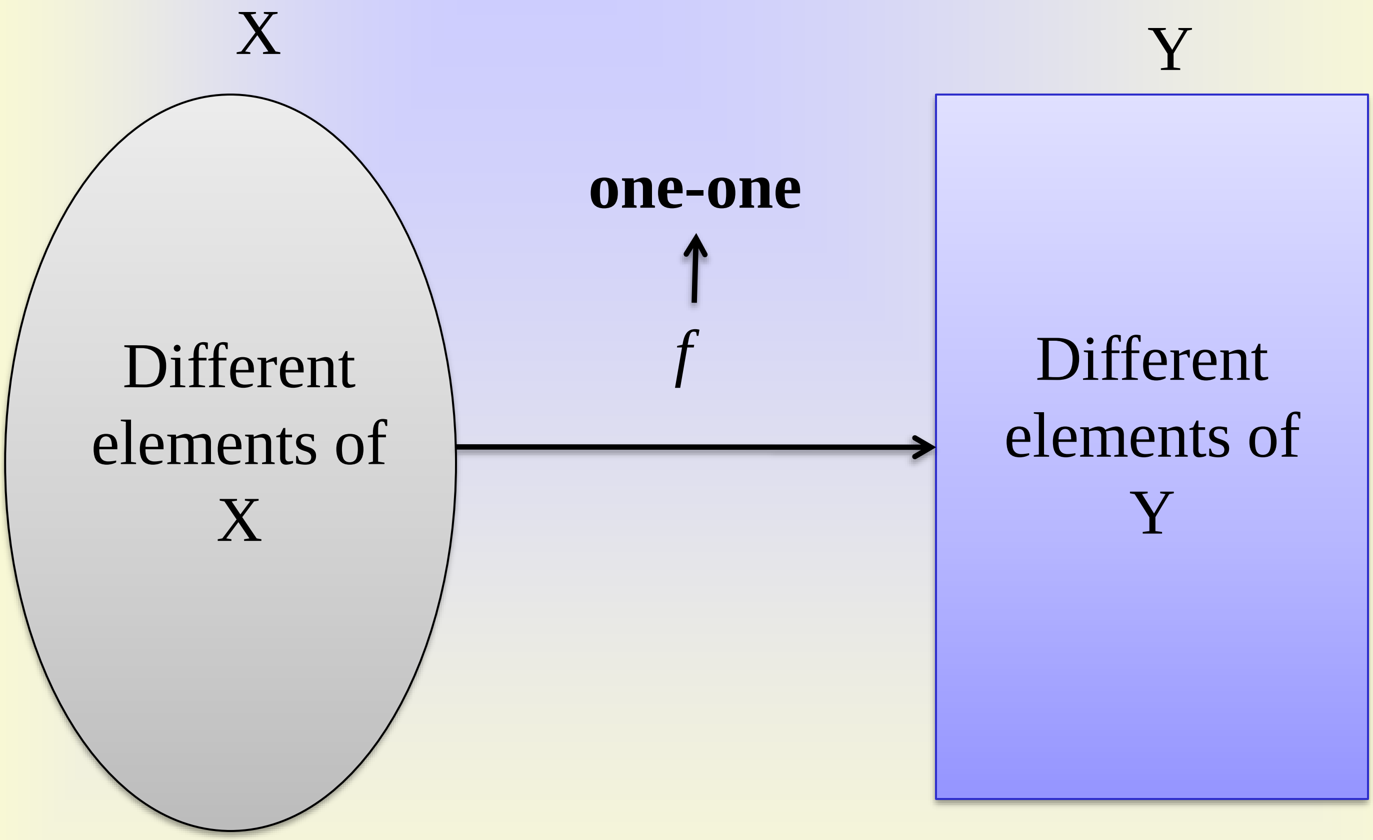
One-one Function

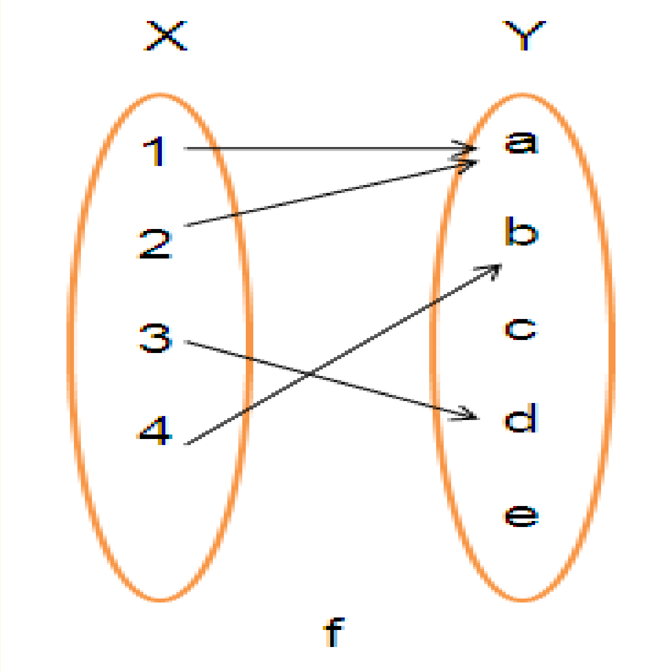


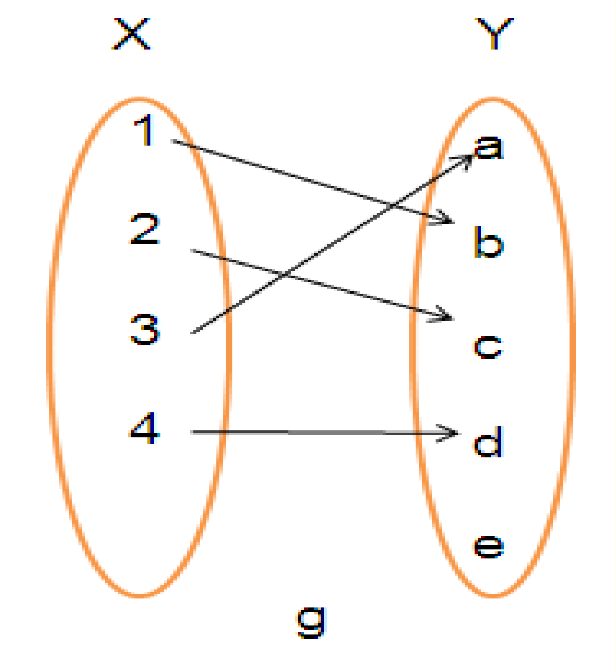
$f : X \rightarrow Y$ is said to be **one-one** if f(x) = f(y) implies x = y or, equivalently, x ≠ y implies f(x) ≠ f(y).

One-one Function

Examples

(vi) Let X = {1, 2, 3, 4} and Y = {a, b, c, d, e} and f and g be two functions from X into Y given by the following diagrams



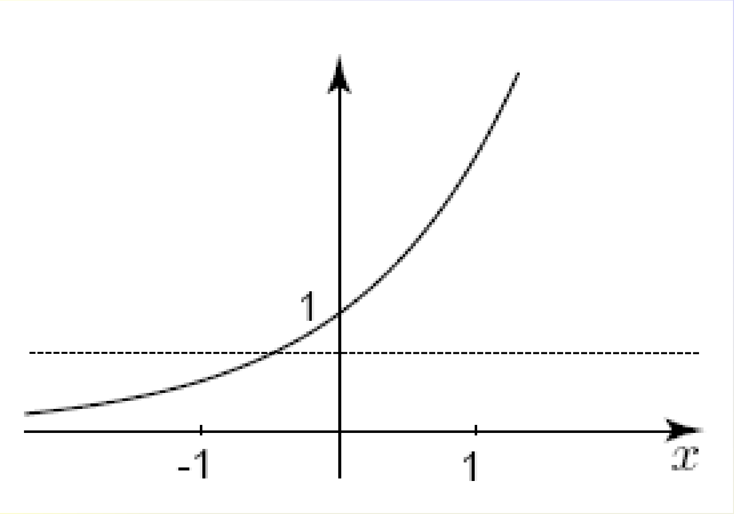


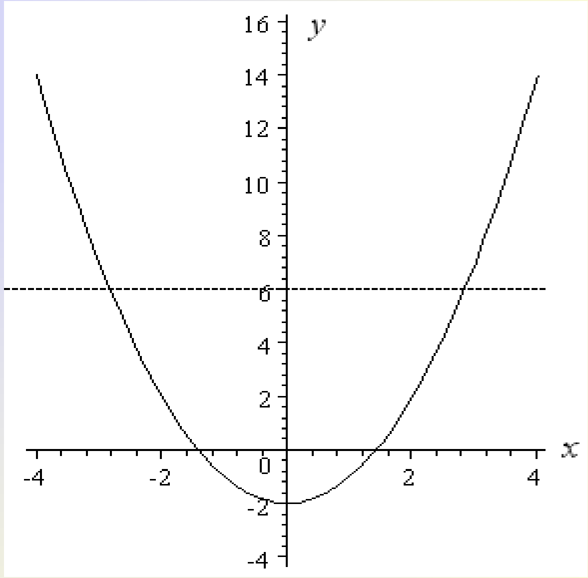
Here f is not a **one-one function** since a is the image of two different elements 1 and 2 of X.

Here g is a **one-one function** since different elements of X have different images.

Functions and its Graphs

The Horizontal Line Test





This is the **graph of a one-one function**. All possible horizontal lines will cut this graph only **once**.

This is not the **graph of a one-one function**. The horizontal line we have drawn cuts the graph **twice**.

One-one Function

Examples

The function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^2$.

Then f is not a **one-one function** since $f(2) = f(-2) = 4$.

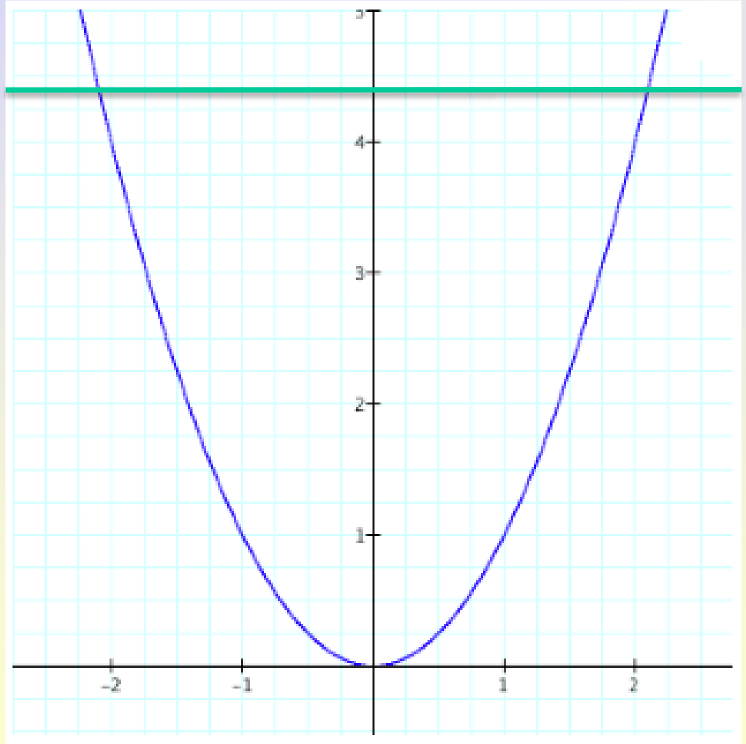


Figure- 7 Graph of the function $f(x) = x^2$.

One-one Function

Examples

The function $f : \mathbf{R} \rightarrow \mathbf{R}$ defined by $f(x) = e^x$ is a **one-one function.**

Proof:

Let $f(x) = f(y)$

then $e^x = e^y$

i.e. $x = y$

Hence by definition f is a **one-one function.**

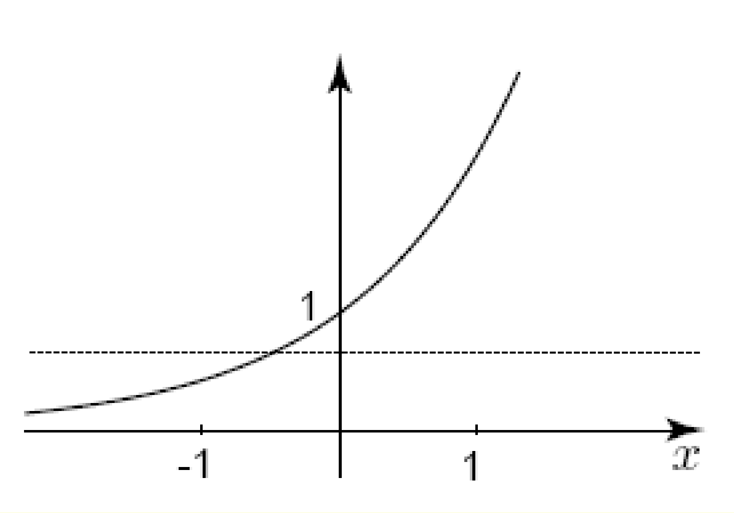


Figure- 8 Graph of the function $f(x) = e^x$.

One-one Function

Examples

The absolute-value function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = |x|$ is not a **one-one function** since $f(2) = f(-2) = 2$.

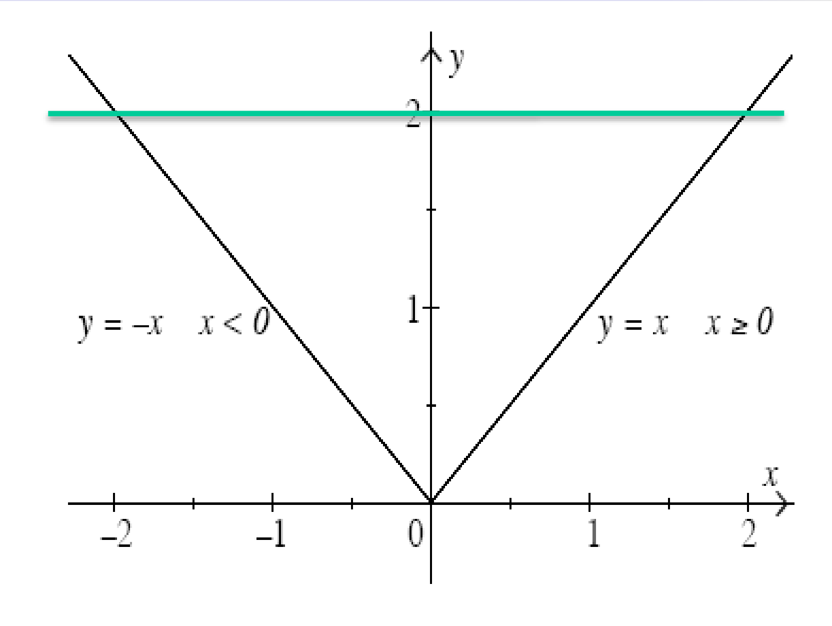


Figure- 9 Graph of the absolute-value function y = *f*(x) = |x|.

One-one Function

Examples

The **identity function** $f : \mathbf{R} \rightarrow \mathbf{R}$ defined by $f(x) = x$ is a **one-one function.**

Proof:

Let $f(x) = f(y)$

then $x = y$

Hence by definition f is a **one-one function.**

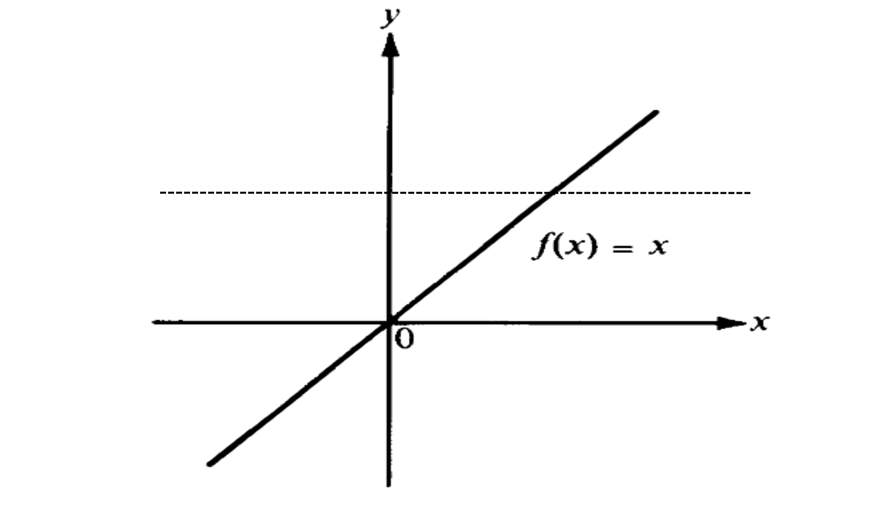
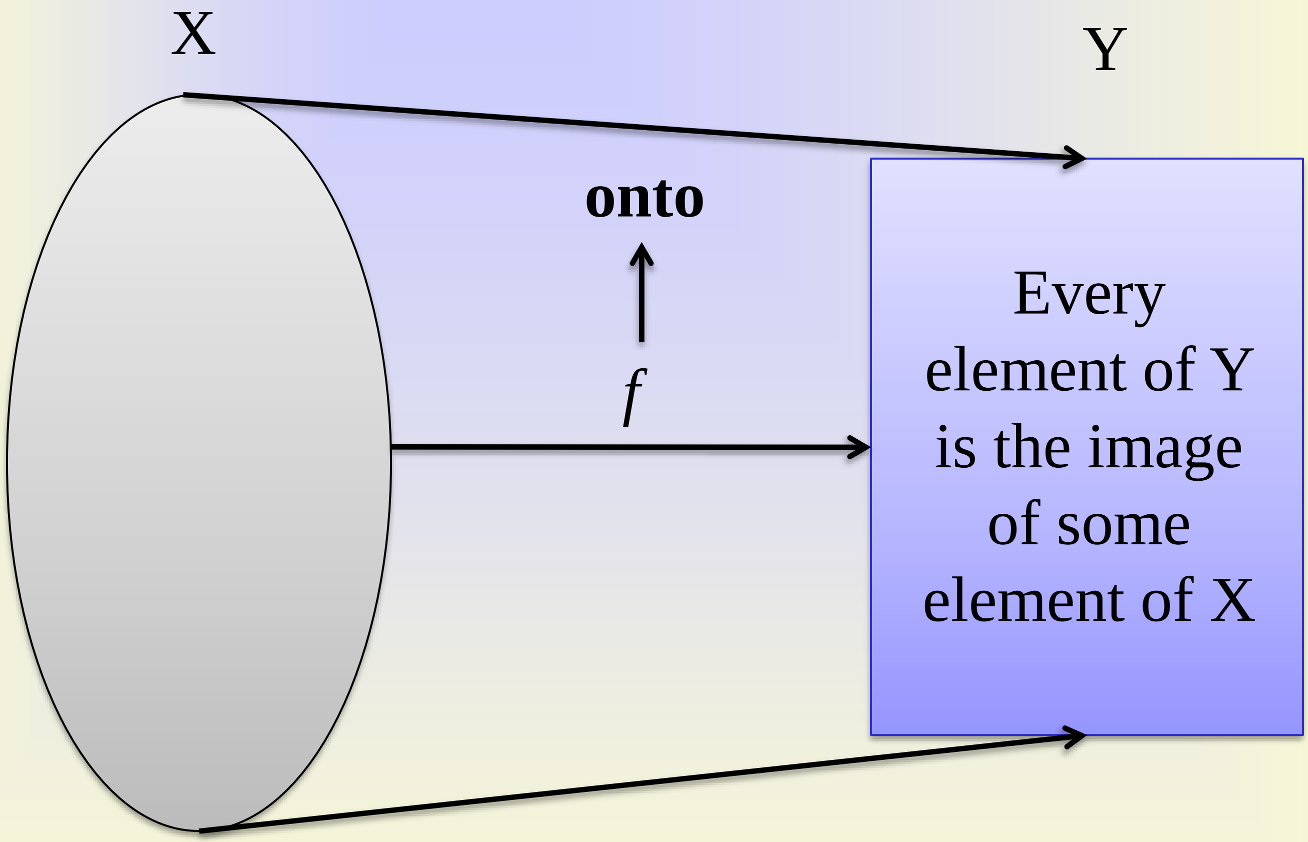


Figure- 10 Graph of the identity function $f(x) = x$.

Onto Function

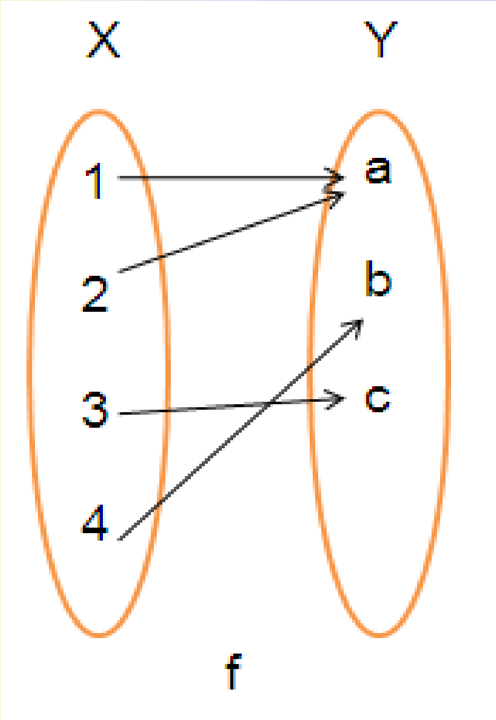


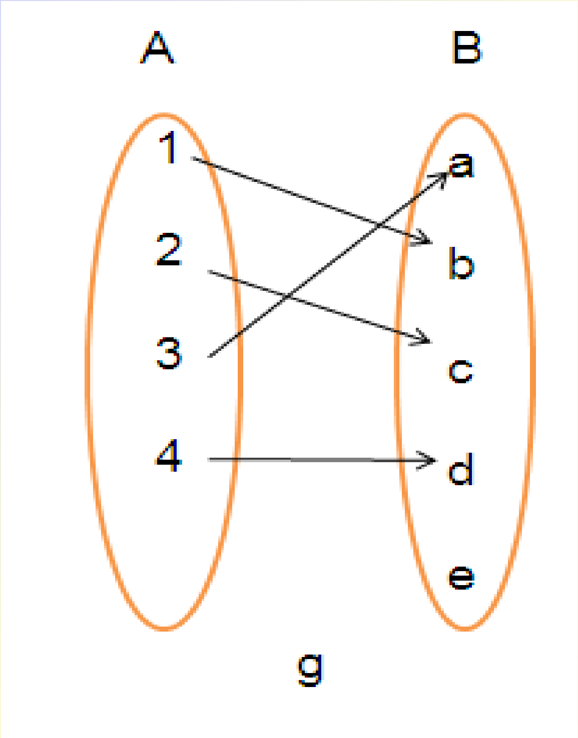
$f : X \rightarrow Y$ is said to be **onto** if for every $y \in Y$, $\exists$ an element $x \in X$ such that $y = f(x)$, i.e. $f(X) = Y$.

Onto Function

Examples

(vi) Let X = A = {1, 2, 3, 4}, Y = {a, b, c} and B= {a, b, c, d, e} and f and g be two functions from X into Y and A into B respectively given by the following diagrams



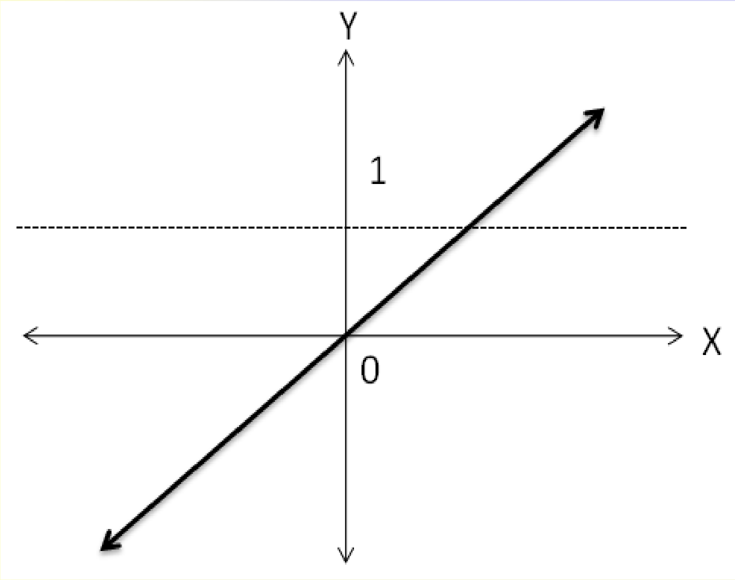


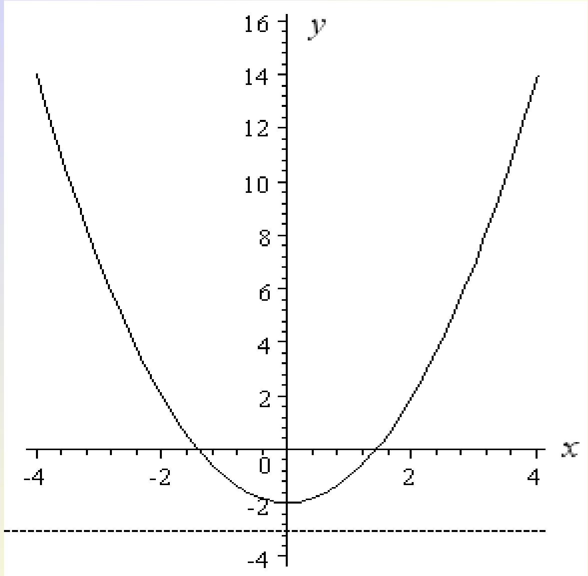
Here f is a **onto function** since every element of Y appears in the **range** of f

Here g is not a **onto function** since e ∊ B is not an image of any element of A.

Onto Function

The Horizontal Line Test





This is the **graph of a onto function**. All possible horizontal lines will cut this graph at least **once**.

This is not the **graph of a onto function**. the horizontal line drawn above does not cut the graph.

Onto Function

Examples

The function $f : \mathbf{R} \rightarrow \mathbf{R}$ defined by $f(x) = e^x$ is a not an **onto function.**

Proof:
Since -2 is an element of the **co-domain R** then there does not exist any element x in the **domain R** such that $-2 = e^x = f(x)$. Hence by definition f is not a **onto function.**

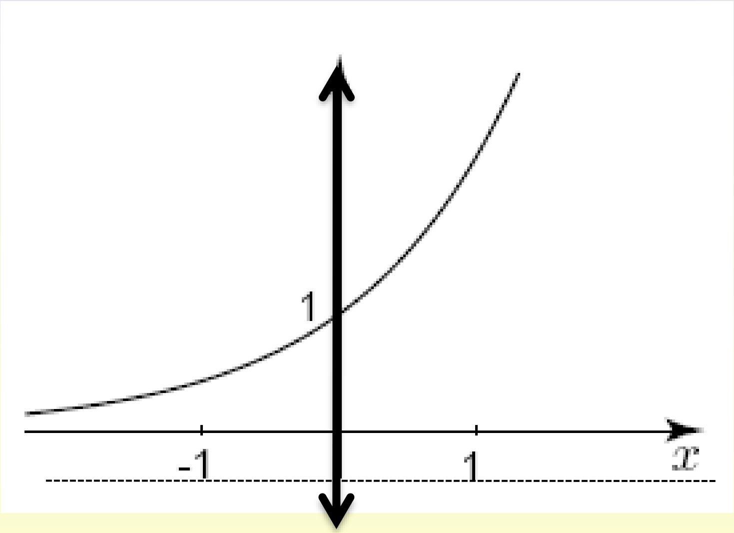


Figure- 11 Graph of the function $f(x) = e^x$.

Onto Function

Examples

The **identity function** f : **R** → **R** defined by $f(x) = x$ is a **onto function.**

Proof:
Since for every y in the **co-domain R,** ∃ an element x in the **domain R** such that $y = f(x)$. Hence by definition f is a **onto function.**

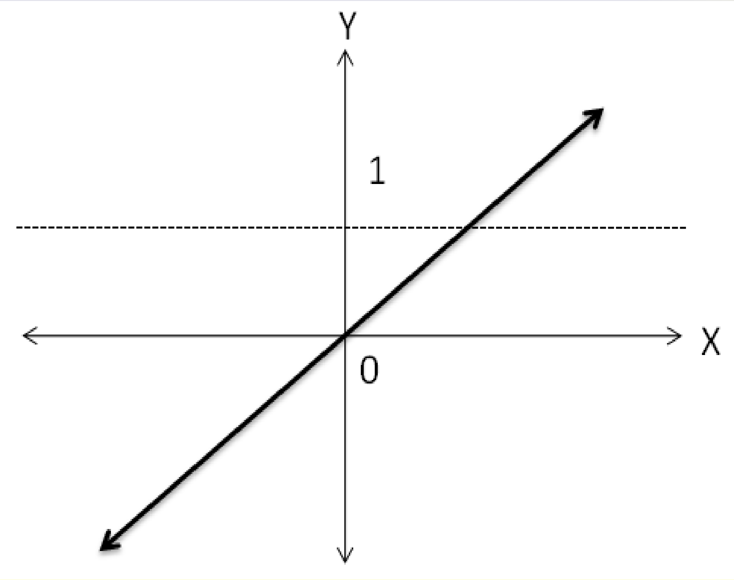


Figure- 12 Graph of the identity function $f(x) = x$.

Onto Function

Examples

The function f : **R** → **R** defined by $f(x) = x^2$ is not an **onto function.**

Proof:

Since -2 is an element of the **co-domain R** then there does not exist any element x in the **domain R** such that $-2 = x^2 = f(x)$. Hence by definition f is not a **onto function.**

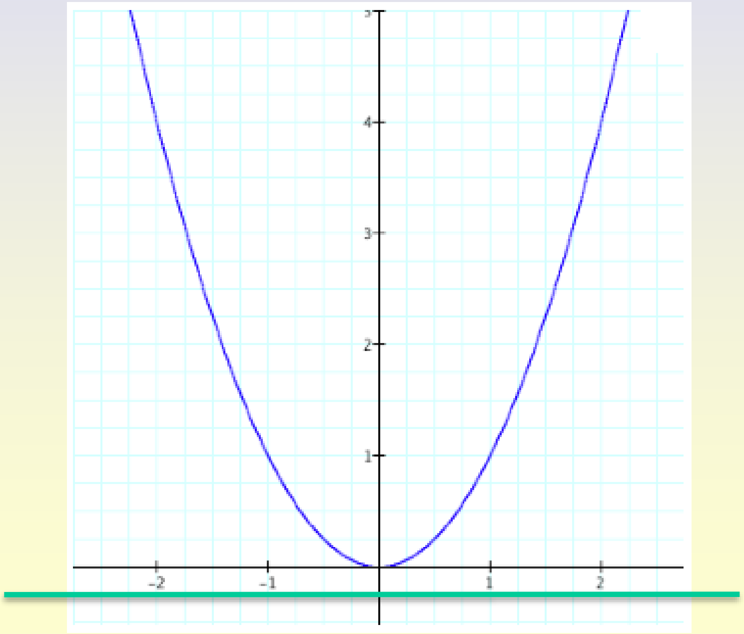
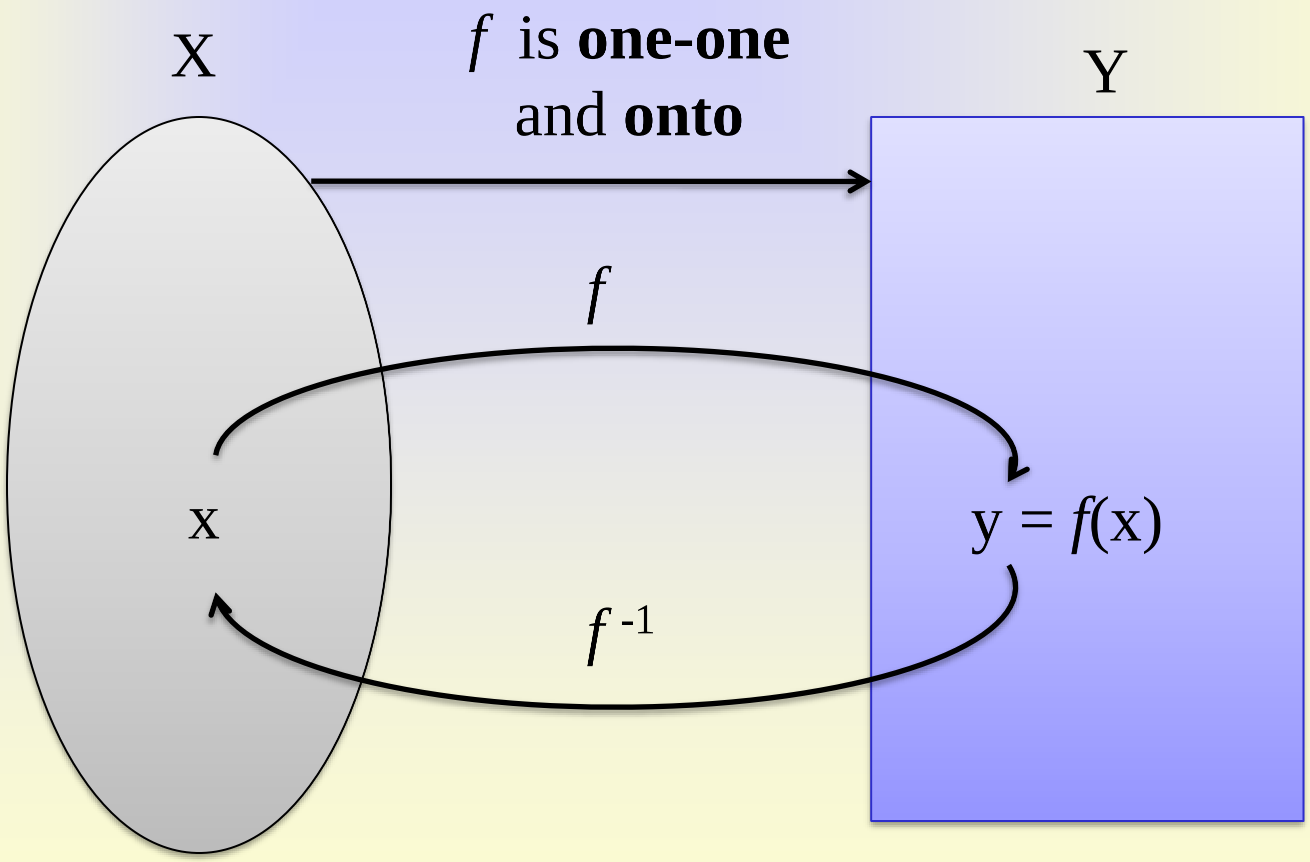


Figure- 13 Graph of the function $f(x) = x^2$.

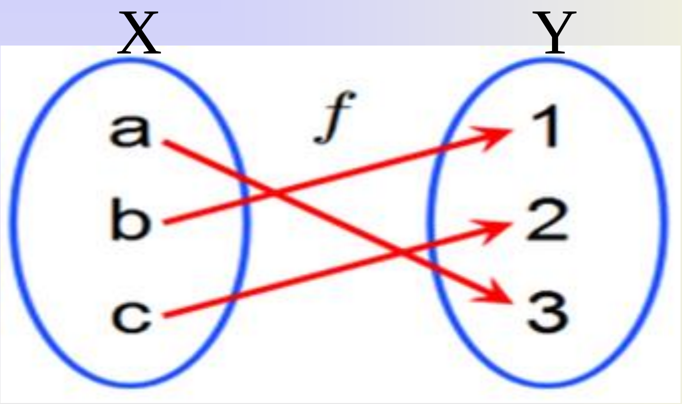
Inverse Function



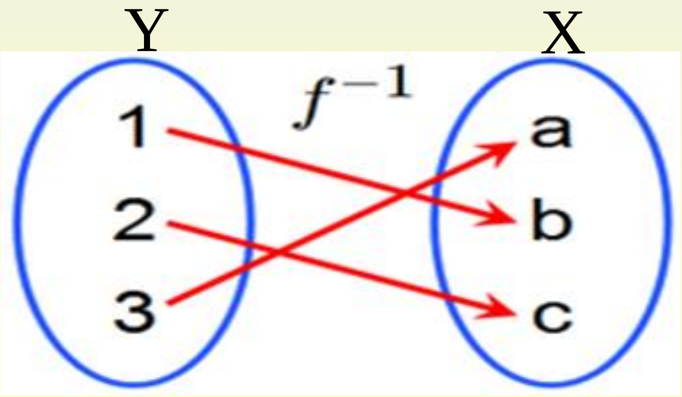
Inverse Function

Example

Let $f : X \to Y$ be define by the following diagram



Here f is **one-one** and **onto**. Therefore f^{-1}, the **inverse function**, exists. We describe $f^{-1} : Y \to X$ by the diagram



Inverse Function

Theorem on the inverse function.

Theorem 1

Let the function $f : X \to Y$ be one-one and onto; i.e. the inverse function $f^{-1} : Y \to X$ exists. Then the composition function

$$(f^{-1} \circ f) : X \to X$$

is the identity function on X, and the composition function

$$(f \circ f^{-1}) : Y \to \mathrm{Y}$$

is the identity function on Y, i.e. $(f^{-1} \circ f) = 1_{\mathrm{X}}$ and $(f \circ f^{-1}) = 1_{\mathrm{Y}}$.

Theorem 2

Let $f : X \to Y$ and $g : Y \to X$. Then g is the inverse of f, i.e. $g = f^{-1}$, if the composition function

$$(g \circ f) : X \to X$$

is the identity function on X, and the composition function

$$(f \circ g) : Y \to \mathrm{Y}$$

is the identity function on Y, i.e. $(g \circ f) = 1_{\mathrm{X}}$ and $(f \circ g) = 1_{\mathrm{Y}}$.

Outline

- **Functions and its graphs.**
- **One-one, Onto and inverse functions.**
- **Transcendental functions.**
- **Bounded and monotonic functions.**
- **Limits of functions.**
- **Right and left hand limits.**
- **Special limits.**
- **Continuity.**
- **Right and left hand continuity.**
- **Sectional continuity.**
- **Uniform continuity, Lipschitz continuity.**

Algebraic and Transcendental Functions

Algebraic functions

Algebraic functions are functions *y = f (x) satisfying an equation of the form*

$p_0(x)y^n + p_1(x)y^{n-1} + . . . + p_{n-1}(x)y + p_n(x) = 0$ ……………………………… (1)

where $p_0(x)$, . . . , $p_n(x)$ are **polynomials** in x.

If the function can be expressed as the **quotient** of two **polynomials**, i.e., P(x)/Q(x) where P(x) and Q(x) are **polynomials**, it is called **a rational algebraic function**; otherwise, it is **an irrational algebraic function**.

Example $$y = \frac{\sqrt{x} + x}{x - 1}$$

is an algebraic function since it satisfies the equation

$(x^2 – 2x + 1)y^2 + (2x^2 – 2x)y + (x^2 – x) = 0.$

Transcendental functions

Transcendental functions are **functions** which are not **algebraic;** i.e., they do not satisfy equations of the form of **Equation (1)**.

Algebraic and Transcendental Functions

The following are sometimes called ***elementary transcendental functions***.

1. **Exponential function**: *$f(x) = a^x$, $a \neq 0, 1$.*

2. **Logarithmic function**: *$f(x) = \log_a x$, $a \neq 0, 1$. This and **the exponential function** are **inverse functions**.*
If a = e = 2.71828 . . . , called the **natural base** of **logarithms**,
we write $f(x) = \log_e x = \ln x$, called the **natural logarithm** of x.

3. **Trigonometric functions:**

$$\sin x, \cos x, \tan x = \frac{\sin x}{\cos x}, \csc x = \frac{1}{\sin x}, \sec x = \frac{1}{\cos x}, \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

The variable x is generally expressed in **radians** (π radians = 180°). For real values of x, *sin x* and *cos x* lie between –1 and 1 inclusive.

4. **Inverse trigonometric functions.**
The following is a list of the **inverse trigonometric functions** and their **principal values**:

(a) $y = \sin^{-1} x$, $(-\pi/2 \leqq y \leqq \pi/2)$

(b) $y = \cos^{-1} x$, $(0 \leqq y \leqq \pi)$

(c) $y = \tan^{-1} x$, $(-\pi/2 < y < \pi/2)$

(d) $y = \csc^{-1} x = \sin^{-1} 1/x$, $(-\pi/2 \leqq y \leqq \pi/2)$

(e) $y = \sec^{-1} x = \cos^{-1} 1/x$, $(0 \leqq y \leqq \pi)$

(f) $y = \cot^{-1} x = \pi/2 - \tan^{-1} x$, $(0 < y < \pi)$

Outline

- Functions and its graphs.
- One-one, Onto and inverse functions.
- Transcendental functions.
- Bounded and monotonic functions.
- Limits of functions.
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- Uniform continuity, Lipschitz continuity.

Bounded Function

Bounded function

A function f defined on some set X with real values is called **bounded**, if the set of its values is bounded. In other words, there exists a real number $M < \infty$ such that $|f(x)| \leq M$ or $-M \leq f(x) \leq M$ for all x in X.

Geometrically, the graph of such a function lies between the graphs of two constant step functions s and t having the values -M and +M, respectively.

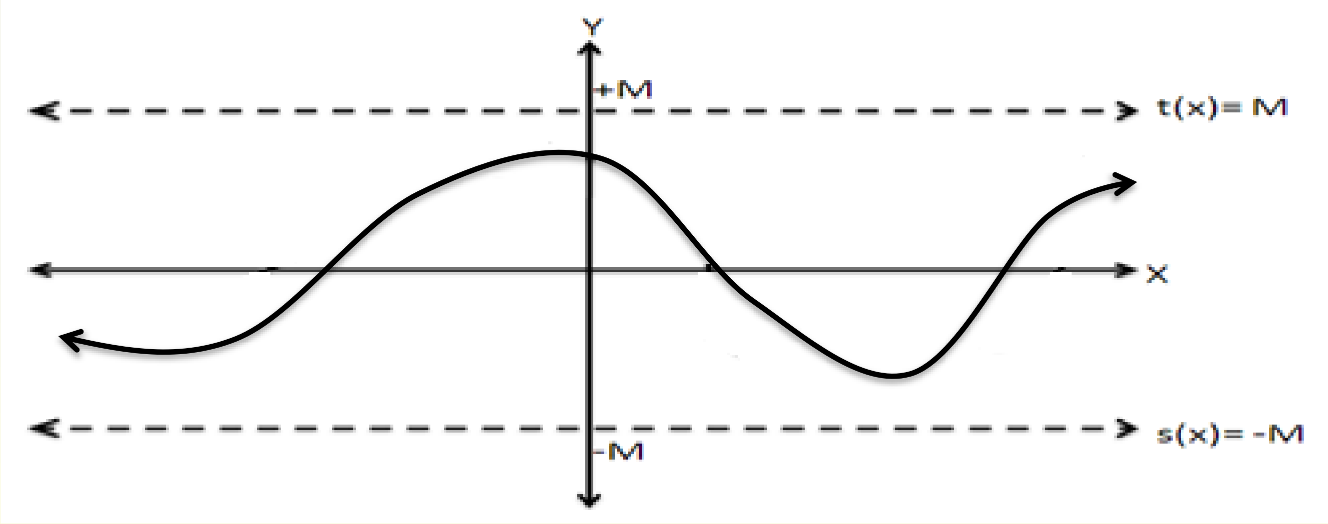

Figure- 14 Graph of a bounded function.

Intuitively, the **graph** of a **bounded function** stays within a horizontal band.

Bounded Function

Bounded above function and Bounded below function

Sometimes, if $f(x) \le A$ for all x in X, then the function is said to be **bounded above** by A. On the other hand, if $f(x) \ge B$ for all x in X, then the function is said to be **bounded below** by B.

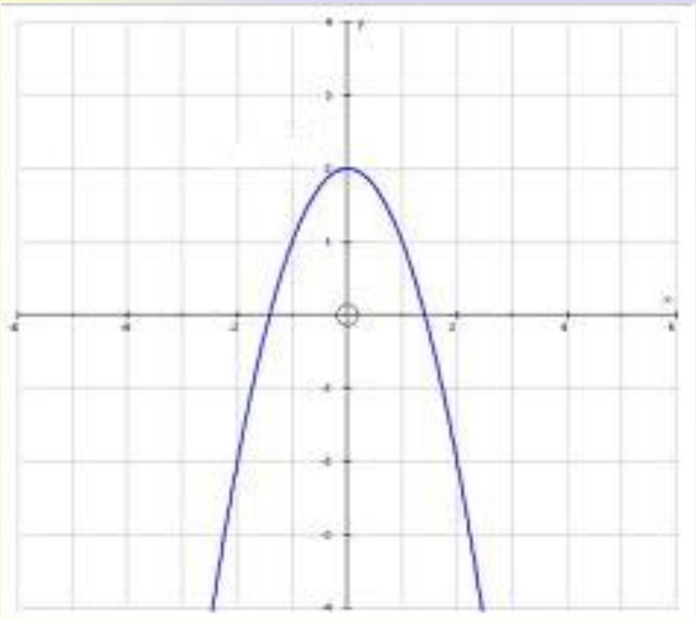

Figure- 15 Graph of a bounded above function.

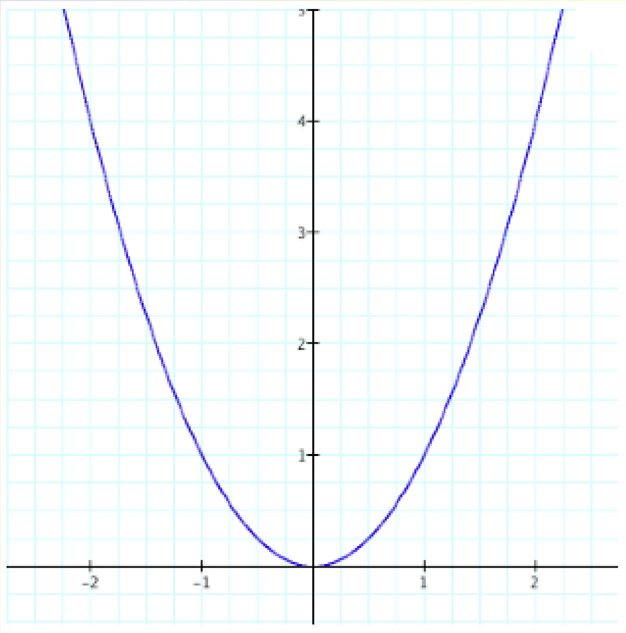

Figure- 16 Graph of a bounded below function.

Bounded Function

Unbounded function

The function which is not bounded is called **unbounded function**.

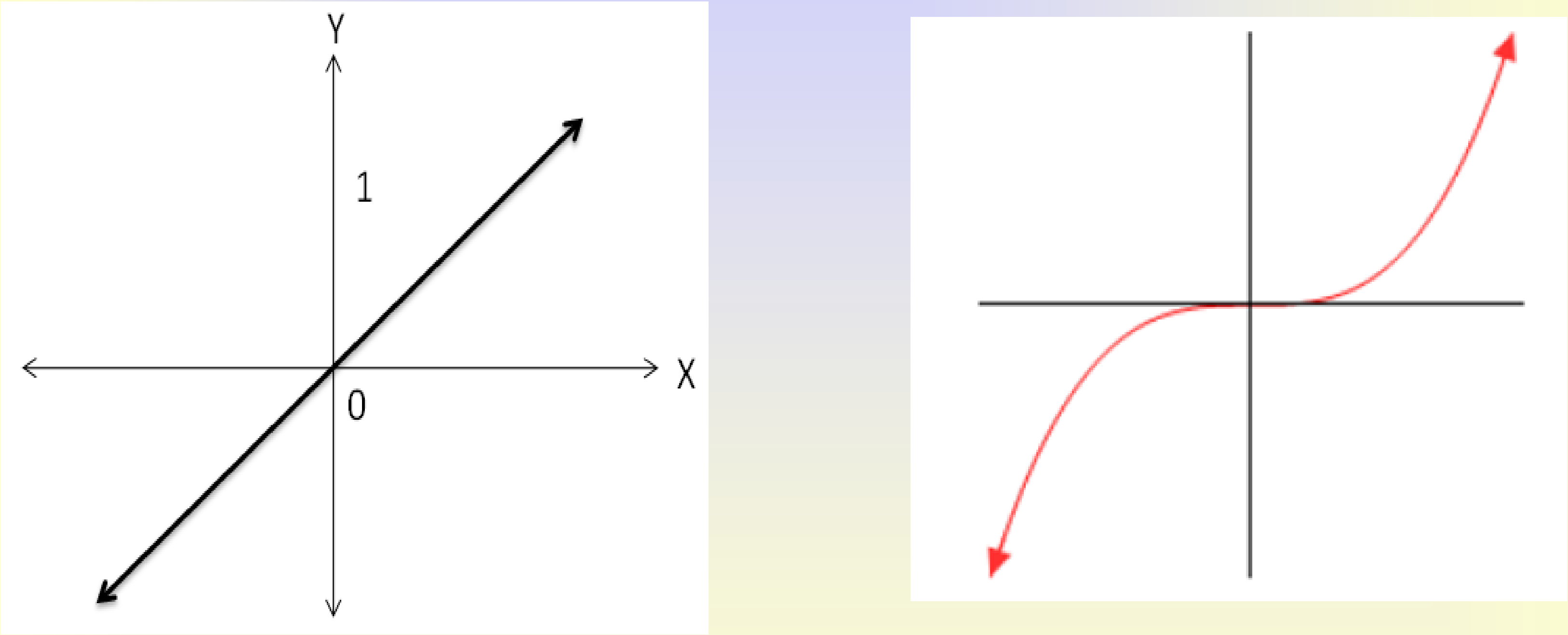

Figure- 17 Graphs of un-bounded functions.

Monotonic Function

Monotonic functions

A function f is said to be **increasing** on a set S if $f(x) \le f(y)$ for every pair of points x and y in S with $x \le y$. If the strict inequality $f(x) < f(y)$ holds for all $x < y$ in S, the function is said to be **strictly increasing** on S.

Similarly, f is called **decreasing** on S if $f(x) \ge f(y)$ for all $x \le y$ in S. If $f(x) > f(y)$ for all $x < y$ in S, then f is called **strictly decreasing** on S.

A function is called **monotonic** on S if it is **increasing** on S or if it is **decreasing** on S.

The term **strictly monotonic** means that is **strictly increasing** on S or **strictly decreasing** on S. Ordinarily, the set S under consideration is either an open interval or a closed interval.

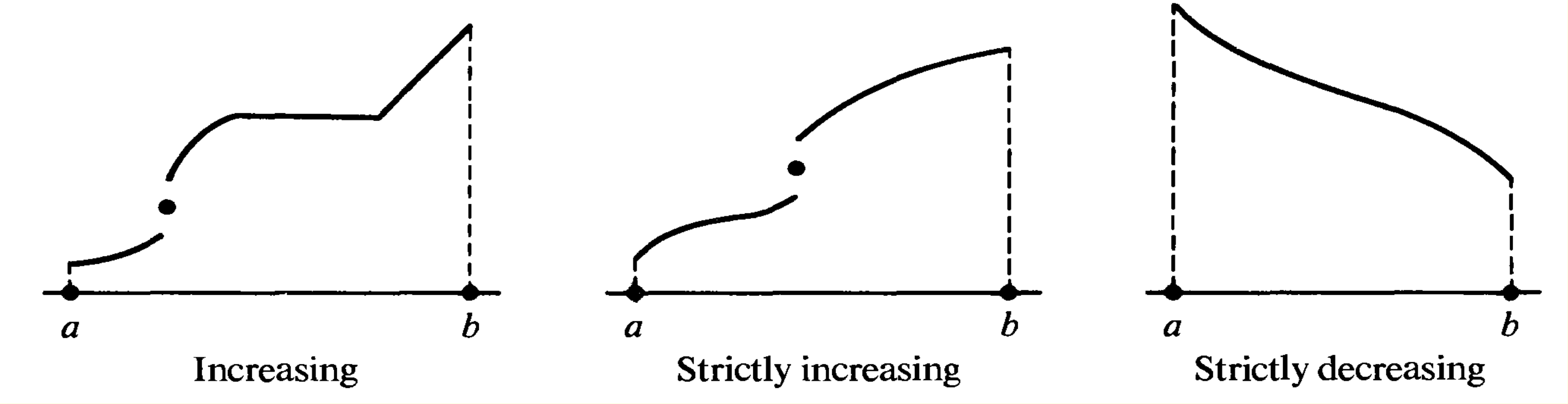

Figure- 18 Monotonic functions.

Monotonic Function

Piecewise Monotonic functions

A function *f* is said to be **piecewise monotonic** on an interval if its **graph** consists of a **finite** number of **monotonic pieces**.

That is to say, *f* is **piecewise monotonic** on [a, b] if there is a partition P of [a, b] such that *f* is **monotonic** on each of the open subintervals of P.

In particular, **step functions** are **piecewise monotonic**, as are all the examples shown in Figures- 23 and 24.

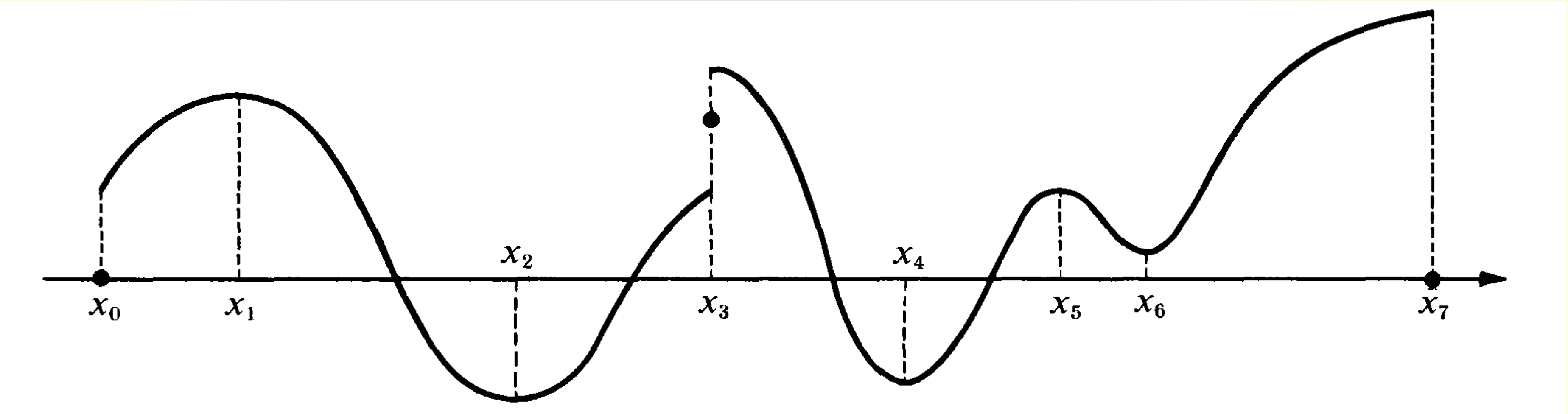

Figure- 19 Piecewise Monotonic function.

Monotonic Function

Example 1.

The power functions.

If p is a positive integer, we have the inequality $x^p < y^p$ if $0 \le x < y$, which is easily proved by mathematical induction. This shows that the power function f, defined for all real x by the equation $f(x) = x^p$, is strictly increasing on the nonnegative real axis. It is also strictly monotonic on the negative real axis (it is decreasing if p is even and increasing if p is odd). Therefore, f is **piecewise monotonic** on every finite interval.

Example 2.

The square-root function.

Let $f(x) = \sqrt{x}$ for $x > 0$. This function is strictly increasing on the nonnegative real axis. In fact, if $0 \le x < y$, we have

$$\sqrt{y} - \sqrt{x} = \frac{y - x}{\sqrt{y} + \sqrt{x}} ;$$ hence, $\sqrt{y} - \sqrt{x} > 0.$

Example 3.

The graph of the function g defined by the equation $g(x) = \sqrt{(r^2 - x^2)}$ if $-r \le x \le r$ is a semicircle of radius r. This function is strictly increasing on the interval $-r \le x \le 0$ and strictly decreasing on the interval $0 \le x \le r$. Hence, g is **piecewise monotonic** on $[-r, r]$.

Outline

- **Functions and its graphs.**
- **One-one, Onto and inverse functions.**
- **Principal values.**
- **Transcendental functions.**
- **Bounded and monotonic functions.**
- **Limits of functions.**
- **Right and left hand limits.**
- **Special limits.**
- **Continuity.**
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- **Uniform continuity, Lipschitz continuity.**

Limit of a Function

Example

$$\frac{1-\cos(\theta)}{\theta}$$

θ	$f(\theta)$	θ	$f(\theta)$
1	0.45969769	-1	-0.45969769
0.1	0.04995835	-0.1	-0.04995835
0.01	0.00499996	-0.01	-0.00499996
0.001	0.00049999	-0.001	-0.00049999

$$\lim_{\theta \to 0} \frac{1-\cos(\theta)}{\theta} = 0$$

Limit of a Function

Example

$$\frac{x^2+4x-12}{x^2-2x}$$

x	$f(x)$	x	$f(x)$
2.5	3.4	1.5	5.0
2.1	3.857142857	1.9	4.157894737
2.01	3.985074627	1.99	4.015075377
2.001	3.998500750	1.999	4.001500750
2.0001	3.999850007	1.9999	4.000150008
2.00001	3.999985000	1.99999	4.000015000

$$\lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x}=4\,.$$

Limit of a Function

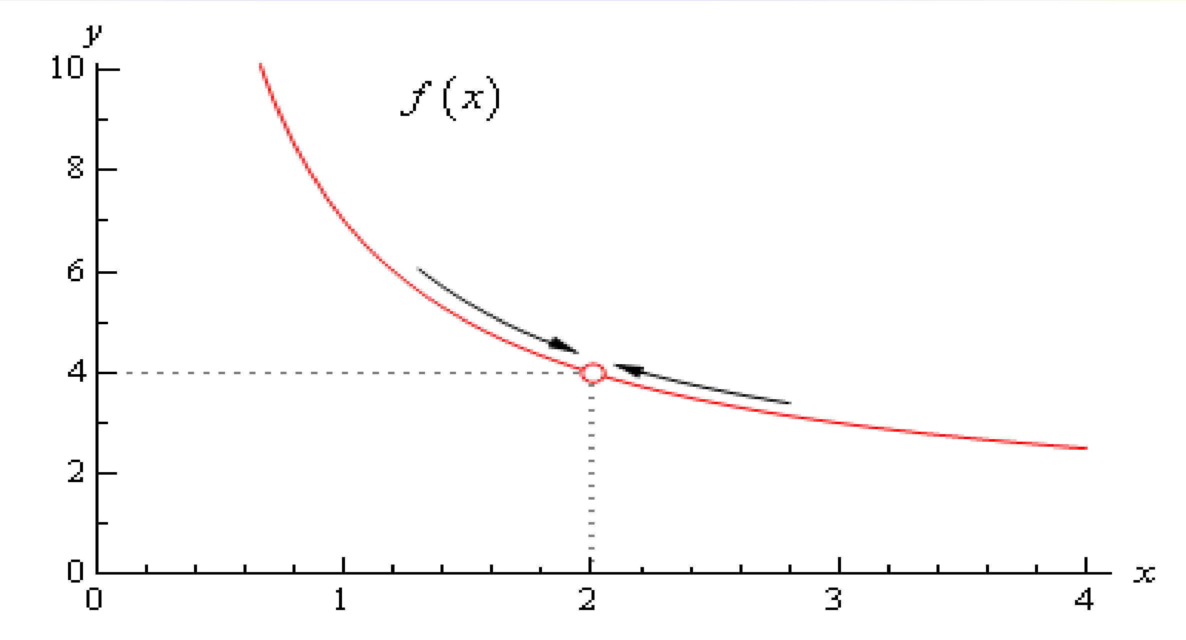


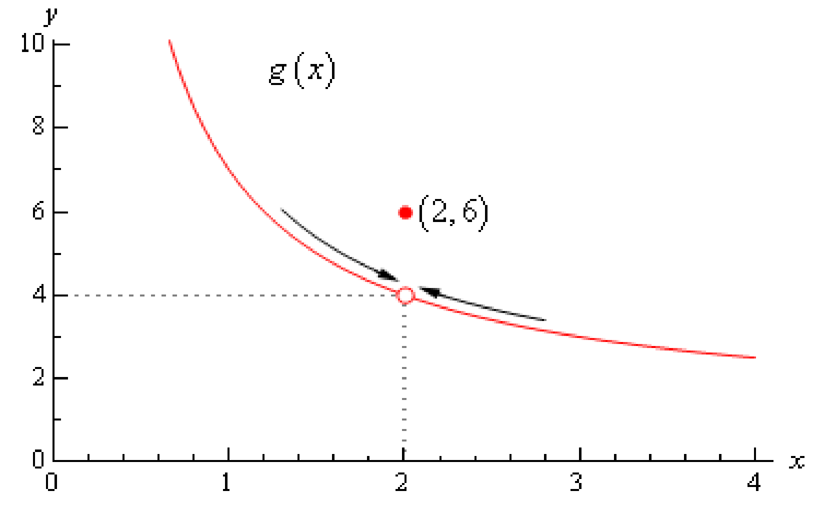
Figure- 20

Limit of a Function

Example

$$g(x)=\begin{cases}\dfrac{x^2+4x-12}{x^2-2x} & \text{if } x\neq 2\\ 6 & \text{if } x=2\end{cases}$$

x	$f(x)$	x	$f(x)$
2.5	3.4	1.5	5.0
2.1	3.857142857	1.9	4.157894737
2.01	3.985074627	1.99	4.015075377
2.001	3.998500750	1.999	4.001500750
2.0001	3.999850007	1.9999	4.000150008
2.00001	3.999985000	1.99999	4.000015000



$$\lim_{x\to 2} g(x)=4$$

Figure- 21

Limit of a Function

Example

$$\cos\left(\frac{\pi}{t}\right)$$

t	$f(t)$	t	$f(t)$
1	-1	-1	-1
0.1	1	-0.1	1
0.01	1	-0.01	1
0.001	1	-0.001	1

$$f\left(\frac{1}{2001}\right)=-1 \qquad f\left(\frac{2}{2001}\right)=0 \qquad f\left(\frac{4}{4001}\right)=\frac{\sqrt{2}}{2}$$

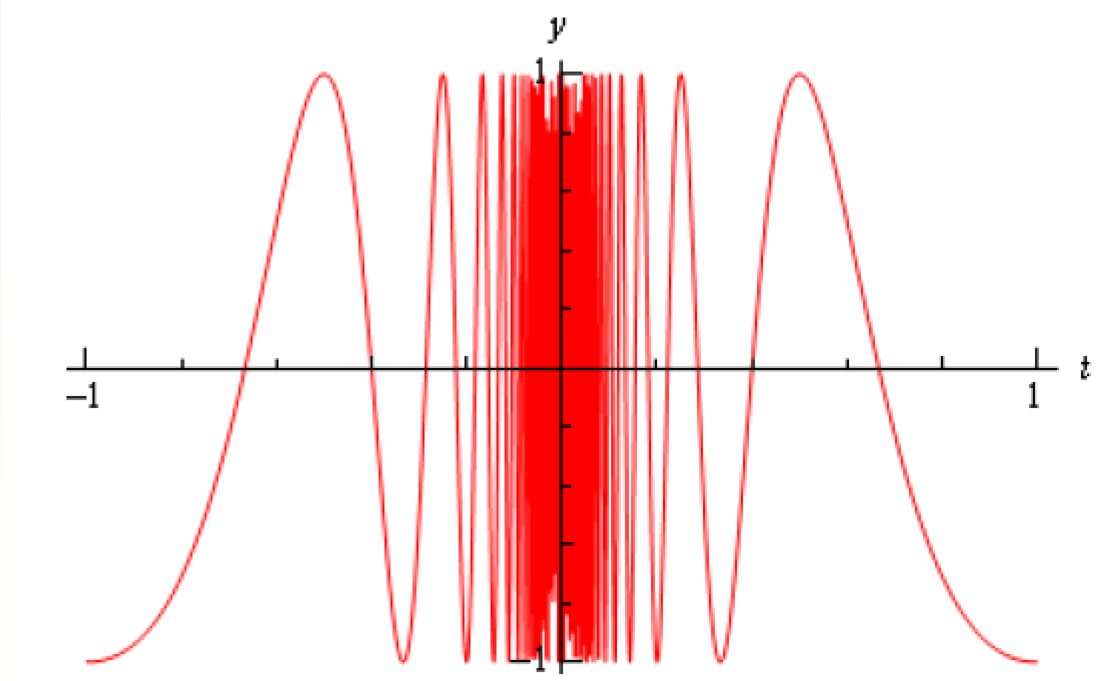


Figure- 22

Limit of a Function

Example

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

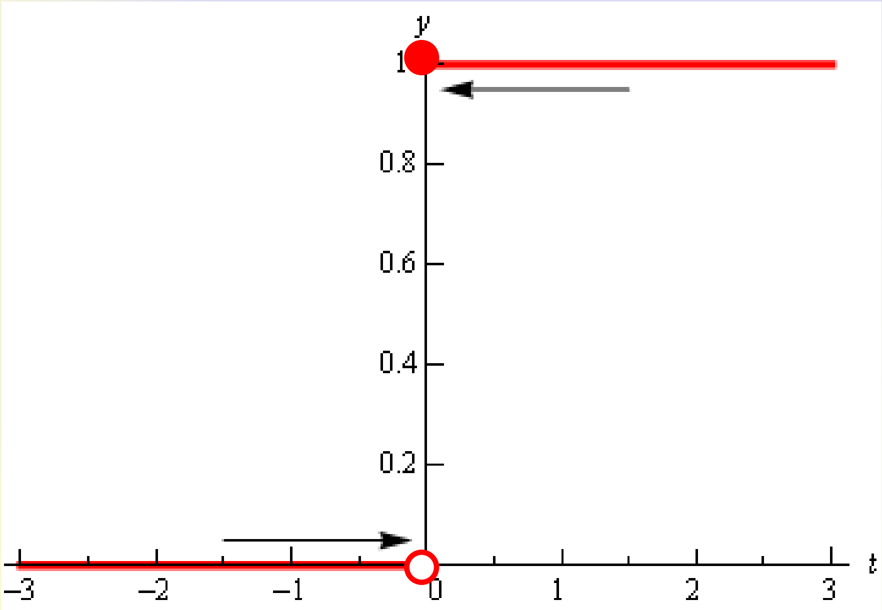


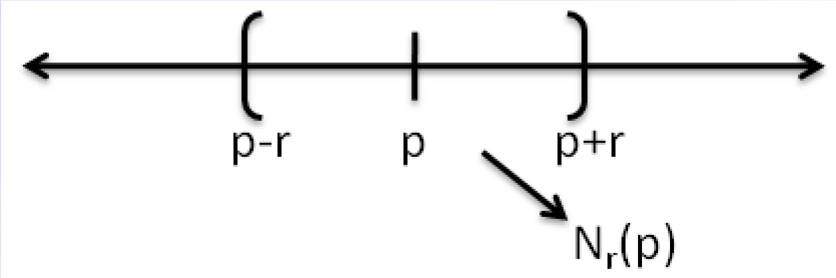
Figure- 23

Limit of a Function

Neighborhood of a point

Any **open interval** containing a point p as its **midpoint** is called a **neighborhood** of p. The **neighborhood** of p with radius r is denoted by $N_r(p)$ and defined by

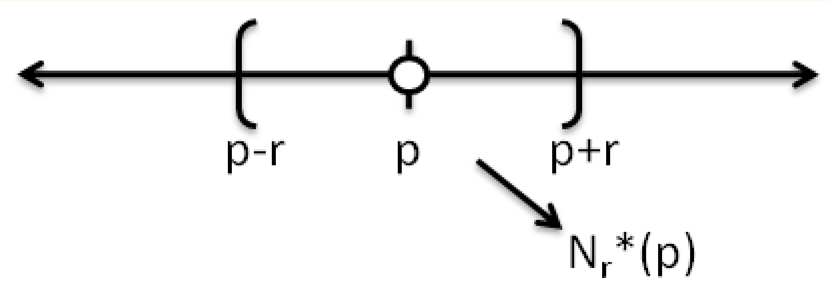
$N_r(p) = \{x \in \mathbf{R} : |x - p| < r\} = (p - r, p + r)$.



Deleted neighborhood of a point

$N_r{}^*(p) = N_r(p) \setminus \{p\}$ is called the **deleted neighborhood** of p, i.e.

$N_r{}^*(p) = \{x \in \mathbf{R} : x \neq p \text{ and } |x - p| < r\}$.



Limit of a Function

Let $f(x)$ be a function defined on an interval that contains $x = a$, except possibly at $x = a$. Then we say that, $\lim_{x \to a} f(x) = L$

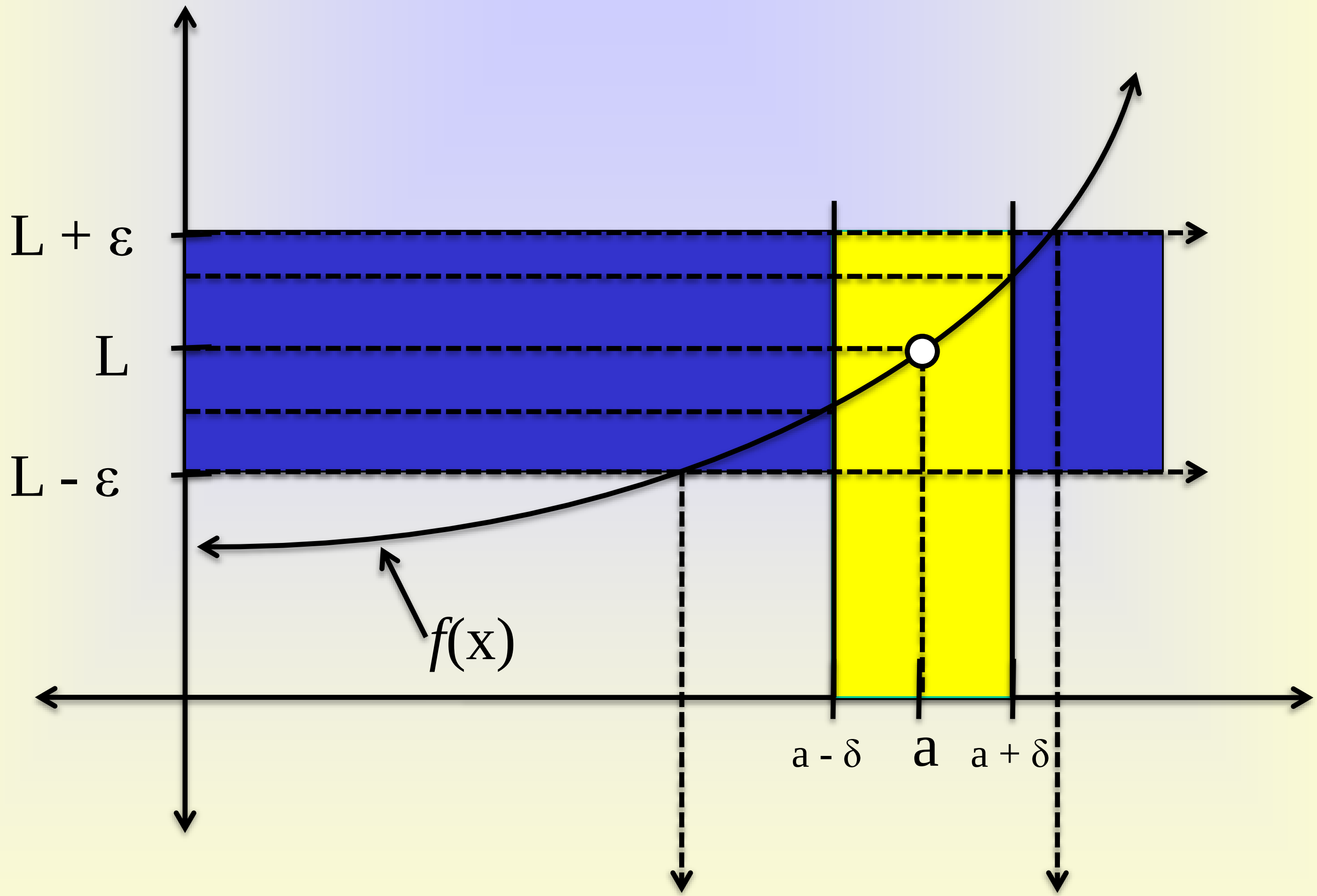
if for every number $\varepsilon > 0$ (however small) there is some number $\delta > 0$ (usually depending on ε and a) such that

$$|f(x) - L| < \varepsilon \qquad \text{whenever} \qquad 0 < |x-a| < \delta$$

Or equivalently $\lim_{x \to a} f(x) = L$

if for every neighborhood $N_\varepsilon(L)$ there is some deleted neighborhood $N_\delta(a)$ such that if $x \in N_\delta(a)$ then $f(x) \in N_\varepsilon(L)$, i.e. for every neighborhood $N_\varepsilon(L)$ there is some neighborhood $N_\delta(a)$ such that if $x \in N_\delta(a) \setminus \{a\}$ then $f(x) \in N_\varepsilon(L)$.

Limit of a Function



$$\lim_{x \to a} f(x) = L$$

Figure- 24

Limit of a Function

Find the value of delta (δ) and Epsilon (ε)

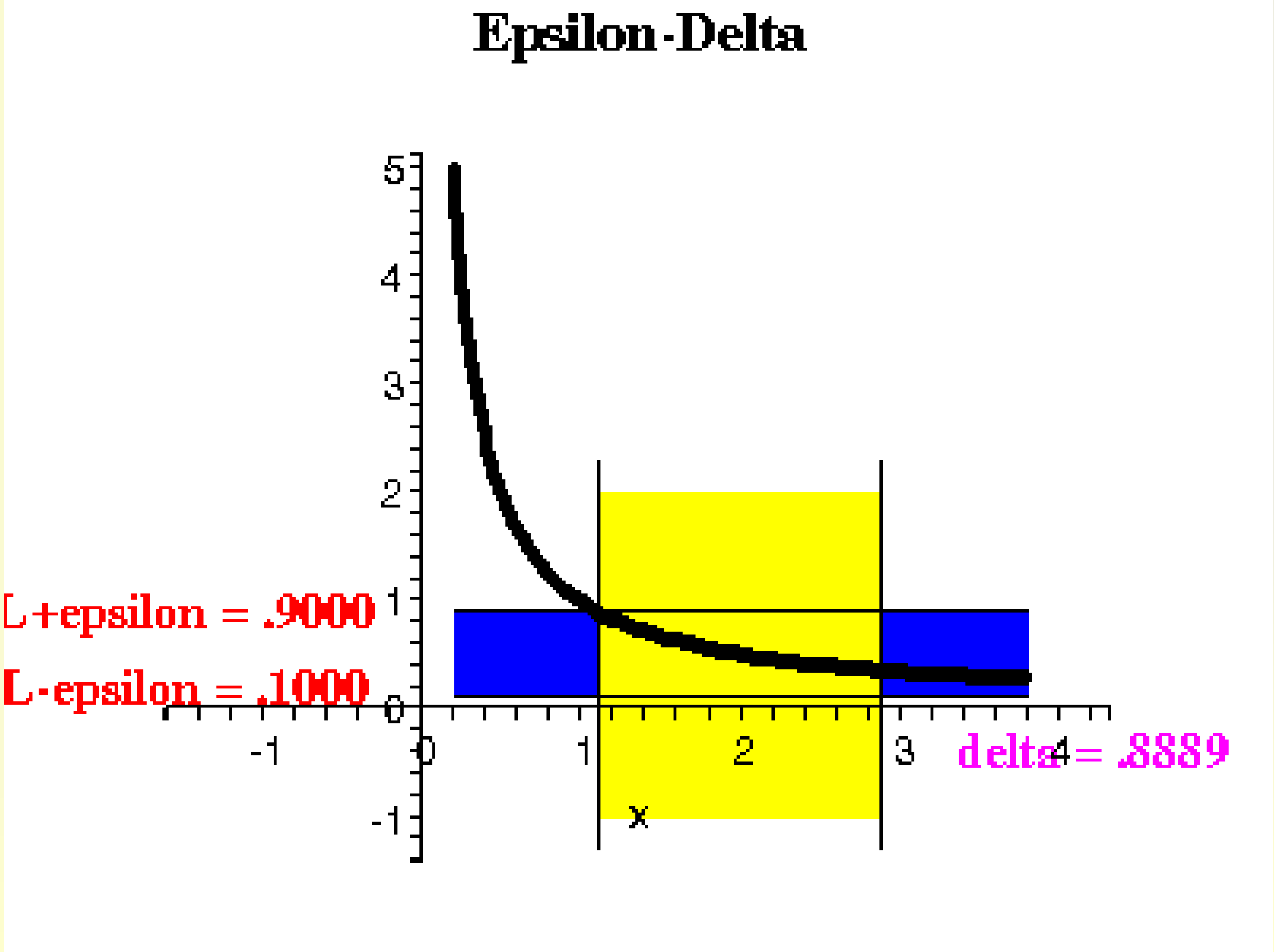


Figure- 25

Limit of a Function

Procedure for finding limit

$$\lim_{x \to a} f(x) = L$$

Step-1 Guess L

Step-2 Choose an arbitrary (small) number $\varepsilon > 0$.

Step-3 Find the value of $\delta > 0$ which satisfies the following condition

$|f(x) - L| < \varepsilon$ whenever $0 < |x-a| < \delta$

Limit of a Function

Example

Use the definition of the limit to prove the following limit.

$$\lim_{x \to 2} 5x - 4 = 6$$

Solution:

Let ε > 0 be any arbitrary (small) number then we need to find a number $\delta > 0$ so that the following will be true.

$$|(5x - 4) - 6| < \varepsilon \qquad \text{whenever} \qquad 0 < |x-2| < \delta$$

Simplifying the left inequality in an attempt to get a guess for δ, we get,

$|(5x - 4) - 6| = |5x - 10| = 5|x - 2| < \varepsilon \quad \Rightarrow \quad |x - 2| < \varepsilon/5$.

So, it looks like if we choose $\delta = \varepsilon/5$ we should get what we want.

Let's now verify this guess. So, again let ε > 0 be any number and then choose $\delta = \varepsilon/5$.

Now, assume $0 < |x-2| < \delta = \varepsilon/5$ then we get,

$|(5x - 4) - 6| = |5x - 10| = 5|x - 2| < 5(\varepsilon/5) = \varepsilon$.

So, we've shown that

$$|(5x - 4) - 6| < \varepsilon \qquad \text{whenever} \qquad 0 < |x-2| < \delta = \varepsilon/5$$

Hence by definition,

$$\lim_{x \to 2} 5x - 4 = 6$$

Limit of a Function

Example

If c and x are arbitrary real numbers and $f(x) = cx$ then

$$\lim_{x \to x_0} f(x) = cx_0.$$

Solution:

Let ε > 0 be any arbitrary (small) number then we need to find a number $\delta > 0$ so that the following will be true.

$$|f(x) - cx_0| < \epsilon \quad \text{whenever} \quad |x - x_0| < \delta$$

Simplifying the left inequality in an attempt to get a guess for δ, we get

$$|f(x) - cx_0| = |cx - cx_0| = |c||x - x_0| < \varepsilon$$

If $c \neq 0$, this yields

$$|x - x_0| < \epsilon/|c|$$

So, we have

$$|f(x) - cx_0| < \epsilon \quad \text{whenever} \quad |x - x_0| < \delta \quad \text{...(1)}$$

where δ is any number such that $0 < \delta \leq \epsilon/|c|$.

If $c = 0$, then $f(x) - cx_0 = 0$ for all x, so (1) holds for all x.

Limit of a Function

Example

Use the definition of the limit to prove the following limit.

$$\lim_{x \to 2} \frac{x^2 + 4x - 12}{x^2 - 2x} = 4 .$$

Solution:

Let ε > 0 be any arbitrary (small) number then we need to find a number $\delta > 0$ so that the following will be true.

$$\left| \frac{x^2 + 4x - 12}{x^2 - 2x} - 4 \right| < \varepsilon \qquad \text{whenever} \qquad 0 < |x-2| < \delta$$

Since x ≠ 2 then for all x , $f(x) = \dfrac{x+6}{x}$

$$\left| \frac{x+6}{x} - 4 \right| = \left| \frac{-3x+6}{x} \right| = \left| \frac{-3}{x} \right| |x-2| = \left| \frac{3}{x} \right| |x-2|$$

Assume $\delta \leq 1$, then $|x-2| < \delta$ implies $1 < x < 3$. Thus

$$\left| \frac{3}{x} \right| |x-2| < 1\delta = \delta$$ So, we can choose $\delta = \varepsilon$.

Then we have, $|f(x) - 4| < \varepsilon$ whenever $0 < |x-2| < \delta$

Hence by definition, $\displaystyle \lim_{x \to 2} \frac{x^2 + 4x - 12}{x^2 - 2x} = 4 .$

Limit of a Function

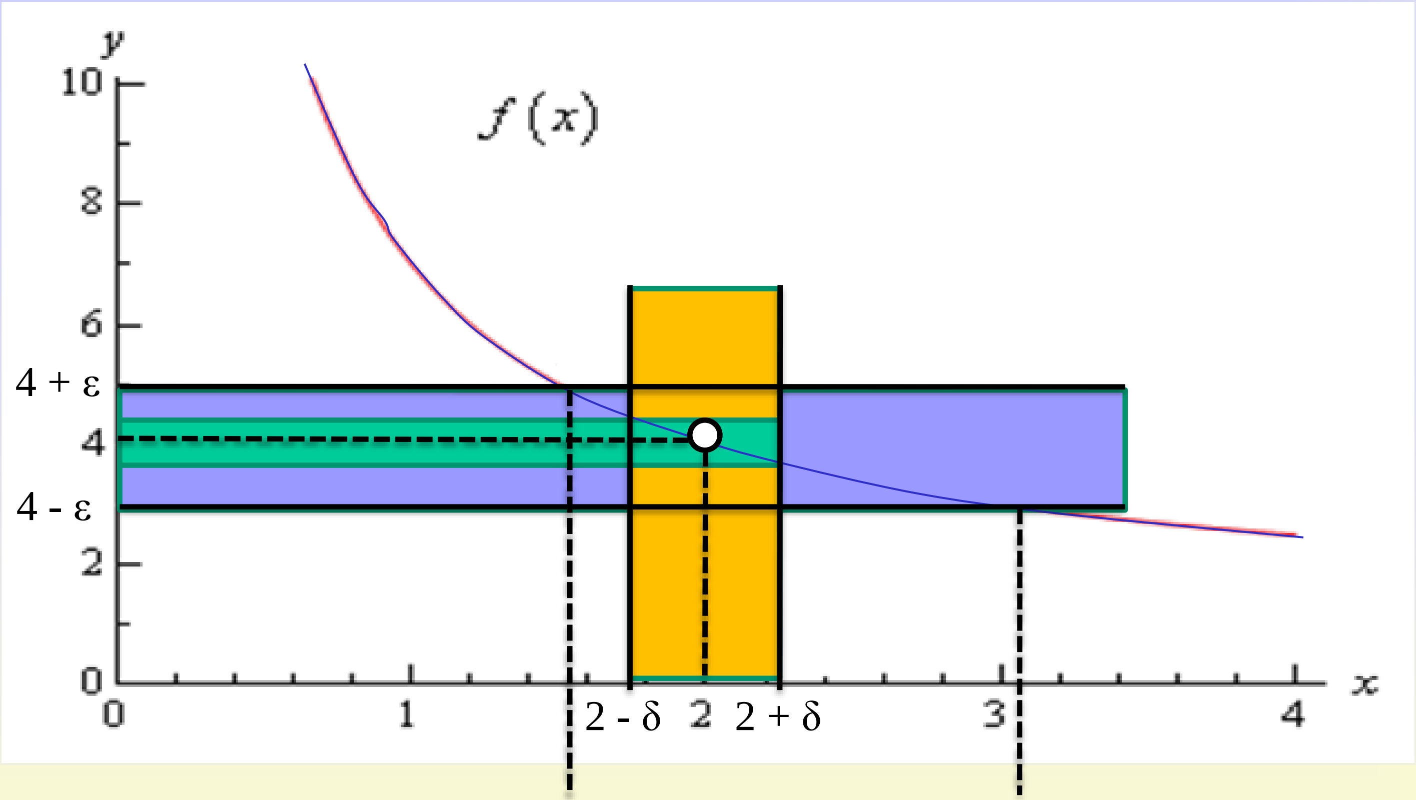


Figure- 26

Limit of a Function

Example If

$$f(x) = x \sin \frac{1}{x}, \quad x \neq 0,$$

then

$$\lim_{x \to 0} f(x) = 0$$

Solution:
Let ε > 0 be any arbitrary (small) number then we need to find a number $\delta > 0$ so that the following will be true.

$|f(x) - 0| < \varepsilon$ whenever $0 < |x-0| < \delta$

Simplifying the left inequality in an attempt to get a guess for δ, we get,

$$|f(x) - 0| = \left| x \sin \frac{1}{x} \right| = |x| \left| \sin \frac{1}{x} \right| \leq |x| < \epsilon$$ since $\left| \sin \frac{1}{x} \right| \leq 1$

So, we can choose $\delta = \varepsilon$.

Then we have, $|f(x) - 0| < \varepsilon$ whenever $0 < |x-0| < \delta$

Hence by definition, $\lim_{x \to 0} f(x) = 0$

Limit of a Function

Theorem

If $f(x)$ has a limit L as $x \to a$, then L must be unique.

Proof

suppose that L_1 and L_2 are two limits of $f(x)$ as $x \to a$. Then, for any $\epsilon > 0$ there exist $\delta_1 > 0, \delta_2 > 0$ such that

$$|f(x) - L_1| < \frac{\epsilon}{2}, \qquad \text{if } 0 < |x-a| < \delta_1,$$

$$|f(x) - L_2| < \frac{\epsilon}{2}, \qquad \text{if } 0 < |x-a| < \delta_2.$$

Hence, if $\delta = \min(\delta_1, \delta_2)$, then

$$\begin{aligned} |L_1 - L_2| &= |L_1 - f(x) + f(x) - L_2| \\ &\le |f(x) - L_1| + |f(x) - L_2| \\ &< \epsilon \end{aligned}$$

for all x for which $0 < |x-a| < \delta$. Since $|L_1 - L_2|$ is smaller than ϵ, which is an arbitrary positive number, we must have $L_1 = L_2$

Outline

- **Functions and its graphs.**
- **One-one, Onto and inverse functions.**
- **Principal values.**
- **Transcendental functions.**
- **Bounded and monotonic functions.**
- **Limits of functions.**
- **Right and left hand limits.**
- **Special limits.**
- **Continuity.**
- **Right and left hand continuity.**
- **Sectional continuity.**
- **Uniform continuity, Lipschitz continuity.**

Right hand limit

Let $f(x)$ be a function defined on an interval that contains x = a, except possibly at x = a . Then for **right hand limit** we say,

$$\lim_{x \to a^+} f(x) = L$$

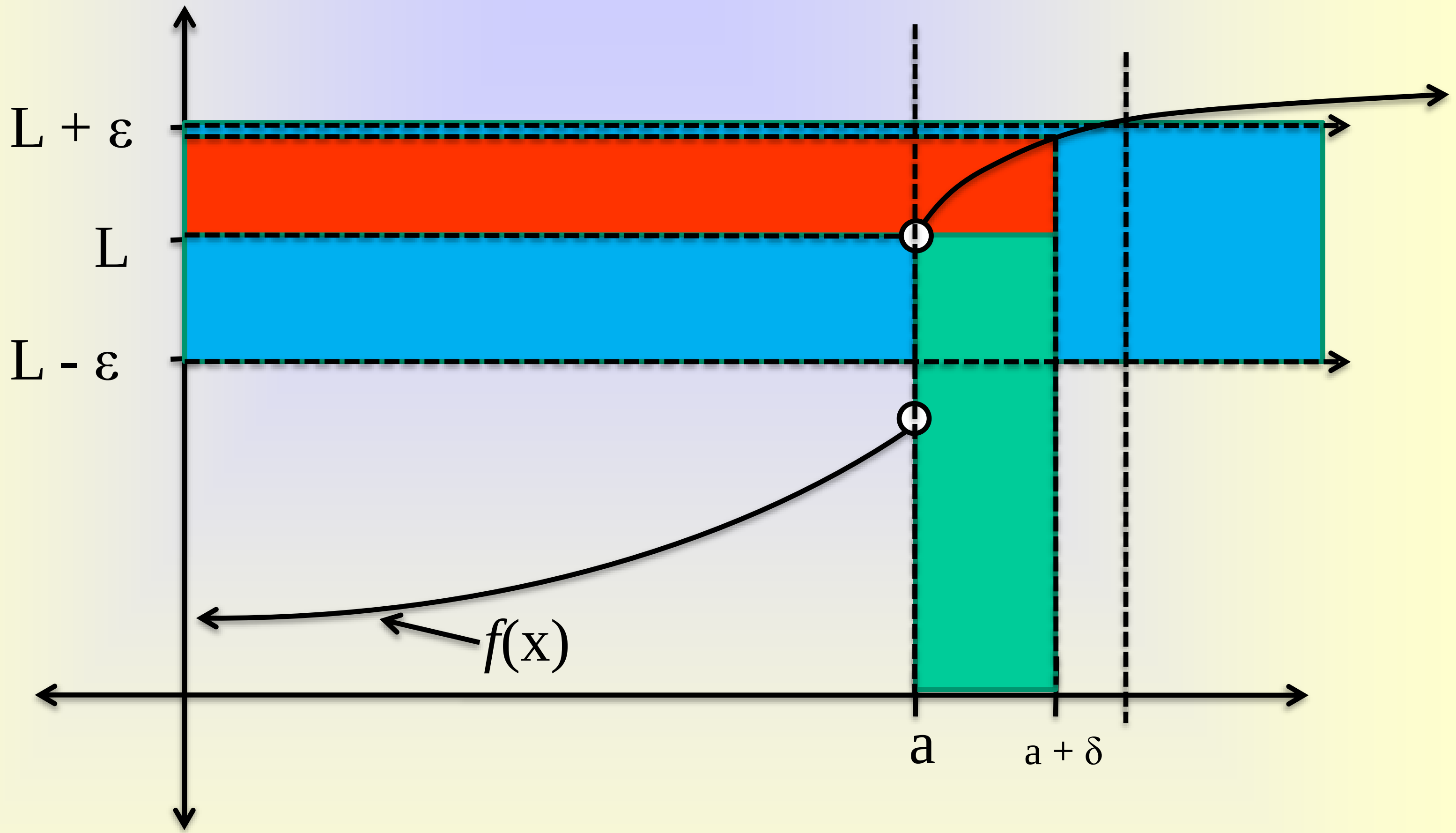
if for every number $\varepsilon > 0$ (however small) there is some number $\delta > 0$ (usually depending on ε and a) such that

$|f(x) - L| < \varepsilon$ whenever $0 < x-a < \delta$ (Or $a < x < a + \delta$)

Or equivalently $$\lim_{x \to a^+} f(x) = L$$

if for every neighborhood $N_\varepsilon(L)$ there is some number $\delta > 0$ such that if $x \in (a, a + \delta)$ then $f(x) \in N_\varepsilon(L)$.

Right hand limit



$$\lim_{x \to a^+} f(x) = L$$

Figure- 27

Right hand limit

Example

Use the definition of the limit to prove the following limit.

$$\lim_{x \to 0^+} \sqrt{x} = 0$$

Solution

Let $\varepsilon > 0$ be any arbitrary (small) number then we need to find a number $\delta > 0$ so that the following will be true.

$$|\sqrt{x} - 0| < \varepsilon \qquad \text{whenever} \qquad 0 < x - 0 < \delta$$

i.e. we need to show that

$|\sqrt{x}| < \varepsilon$ whenever $0 < x < \delta$

Simplifying the left inequality in an attempt to get a guess for δ, we get,

$|\sqrt{x}| < \varepsilon \Rightarrow x < \varepsilon^2$.

So, it looks like we can choose $\delta = \varepsilon^2$.

Let's now verify this guess. So, again let $\varepsilon > 0$ be any number and then choose $\delta = \varepsilon^2$.

Now, assume $0 < x - 0 < \delta = \varepsilon^2$ then we get,

$|\sqrt{x} - 0| = |\sqrt{x}| < |\sqrt{\varepsilon^2}| = \varepsilon$.

So, we've shown that

$|\sqrt{x} - 0| < \varepsilon$ whenever $0 < x - 0 < \delta = \varepsilon^2$

Hence by definition, $\lim_{x \to 0^+} \sqrt{x} = 0$

Left hand limit

Let $f(x)$ be a function defined on an interval that contains x = a, except possibly at x = a . Then for **left hand limit** we say,

$$\lim_{x \to a^-} f(x) = L$$

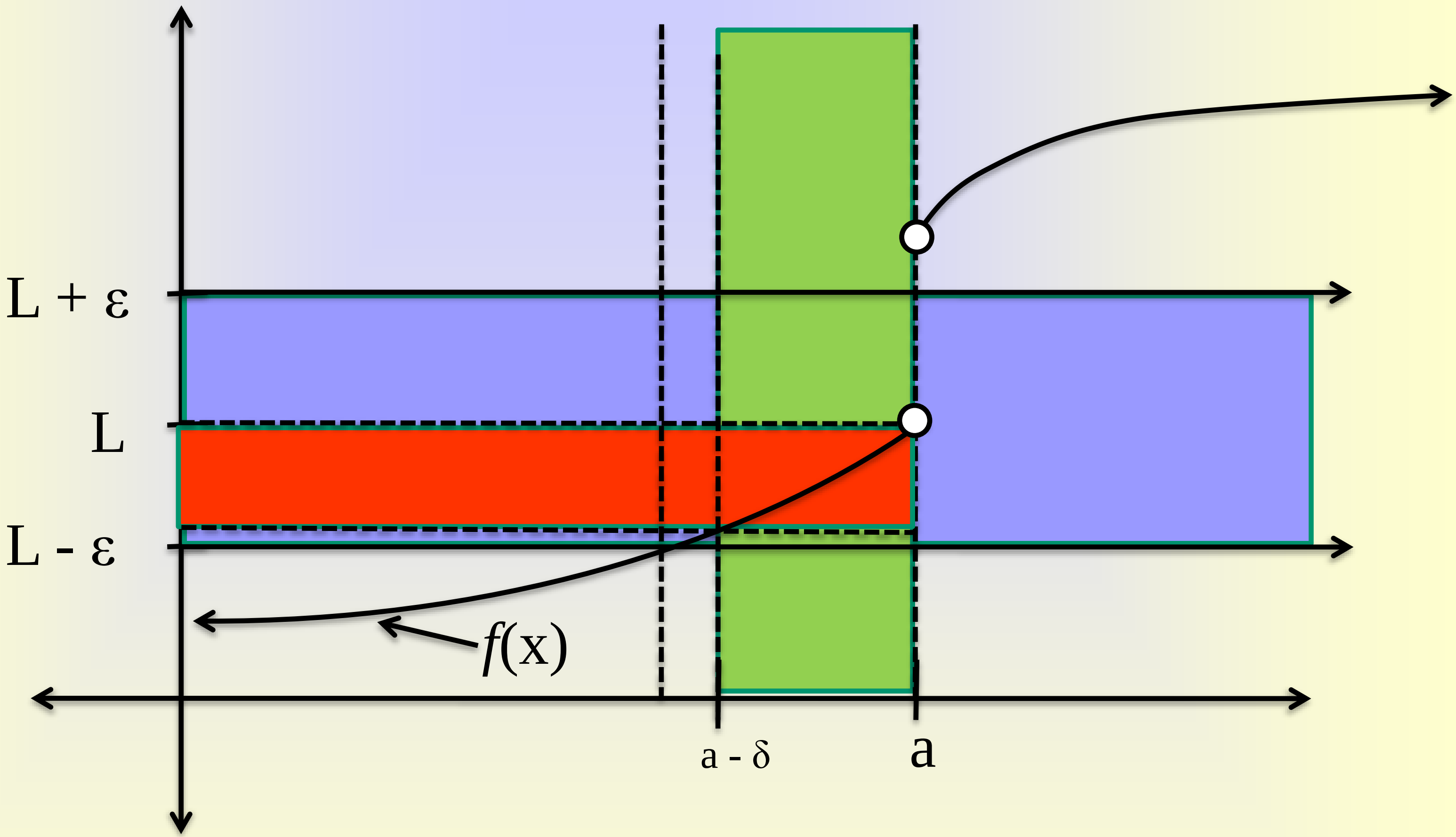
if for every number ε > 0 (however small) there is some number δ > 0 (usually depending on ε and a) such that

$|f(x) - L| < \varepsilon$ whenever $-\delta < x-a < 0$ (Or $a - \delta < x < a$)

Or equivalently $\lim_{x \to a^-} f(x) = L$

if for every neighborhood $N_\varepsilon(L)$ there is some number δ > 0 such that if $x \in (a - \delta, a)$ then $f(x) \in N_\varepsilon(L)$.

Left hand limit



$$\lim_{x \to a^-} f(x) = L$$

Figure- 28

Right and left hand limits

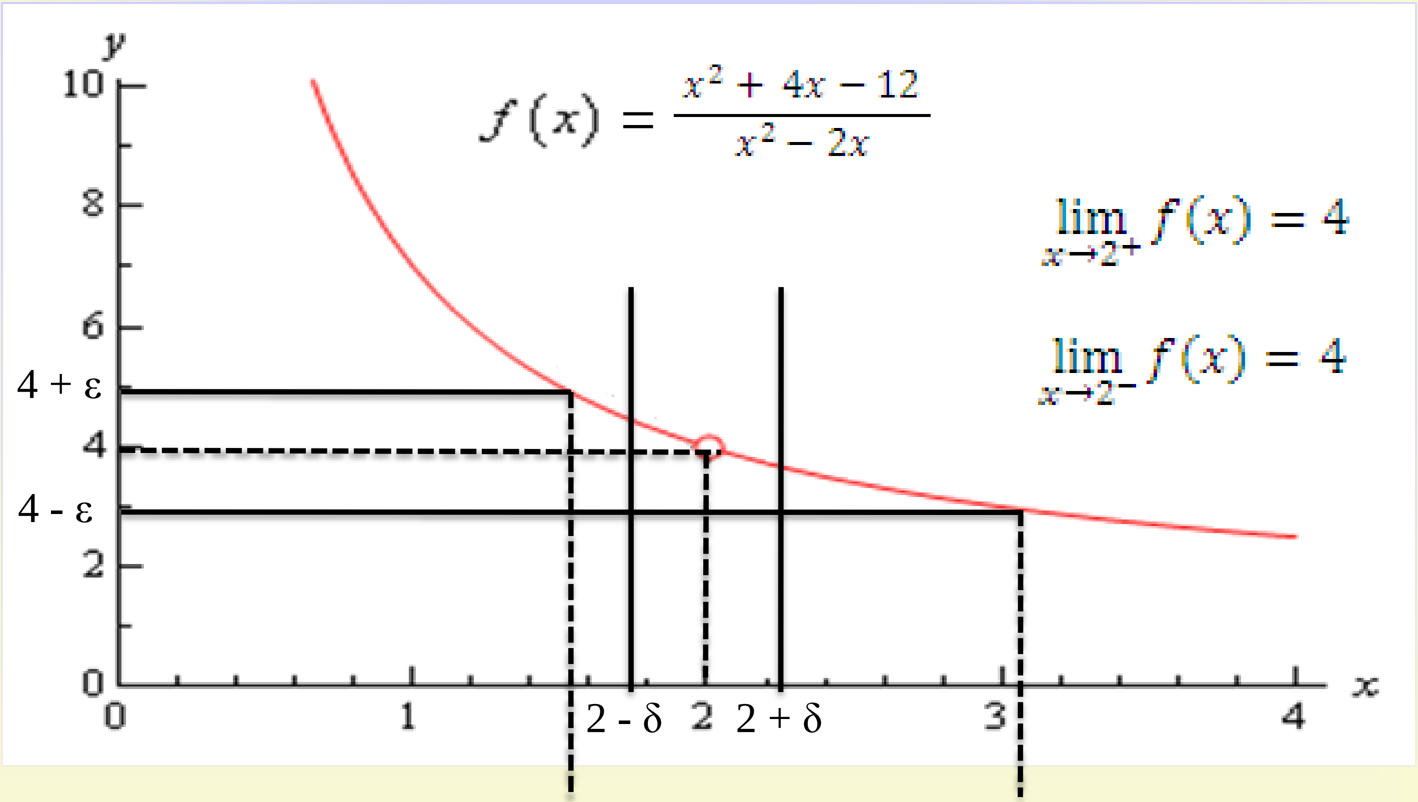


Figure- 29

Right and left hand limits

Example

$$f(x) = \frac{x}{|x|}, \quad x \neq 0.$$

If $x < 0$, then $f(x) = -x/x = -1$, so

$$\lim_{x \to 0-} f(x) = -1.$$

If $x > 0$, then $f(x) = x/x = 1$, so

$$\lim_{x \to 0+} f(x) = 1.$$

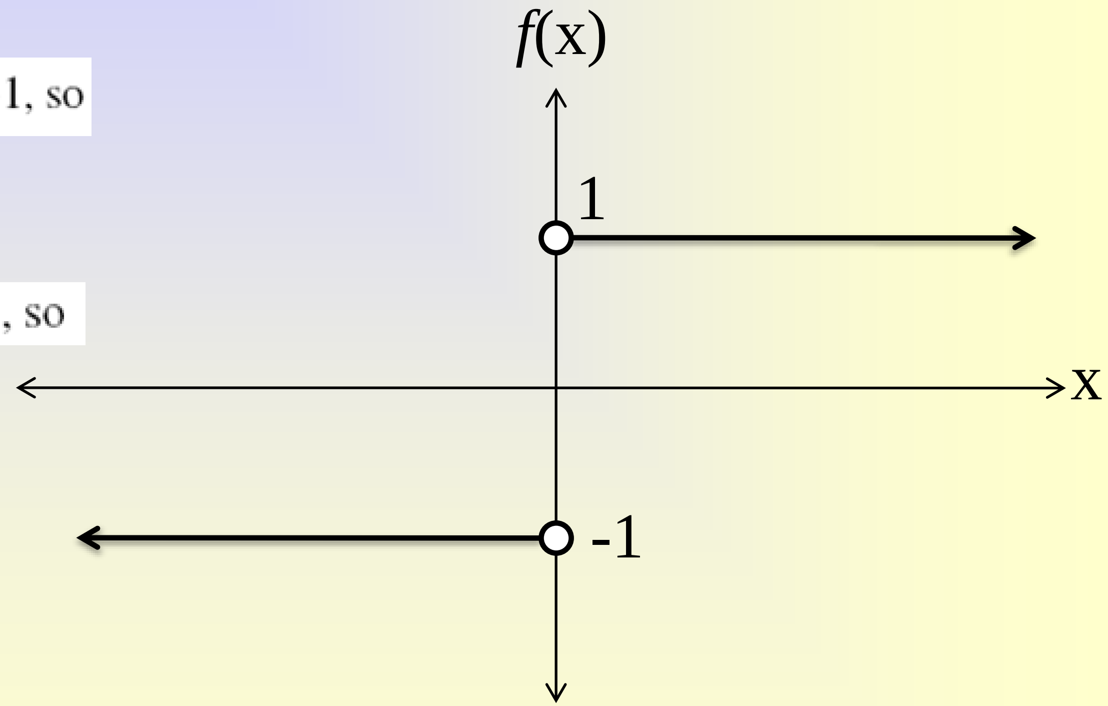


Figure- 30

Right and left hand limits.

Example Let

$$g(x) = \frac{x + |x|(1+x)}{x} \sin\frac{1}{x}, \quad x \neq 0.$$

If $x < 0$, then

$$g(x) = -x \sin\frac{1}{x},$$

so

$$\lim_{x \to 0-} g(x) = 0,$$

since

$$|g(x) - 0| = \left| x \sin\frac{1}{x} \right| \leq |x| < \epsilon \quad \text{if } -\epsilon < x < 0$$

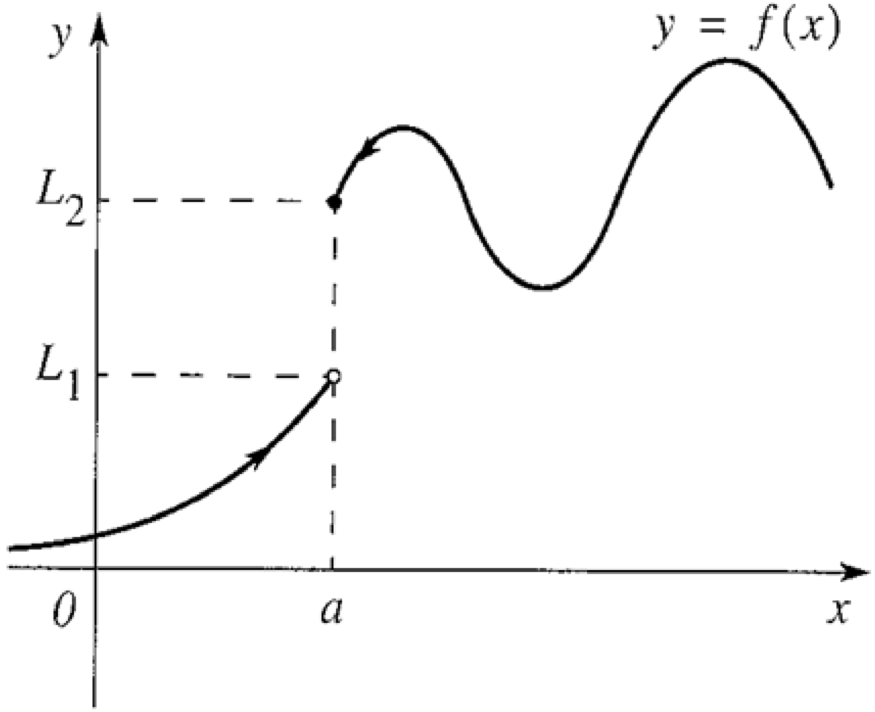
That is the definition of left hand limit is satisfied for $\delta = \varepsilon$.

If $x > 0$, then

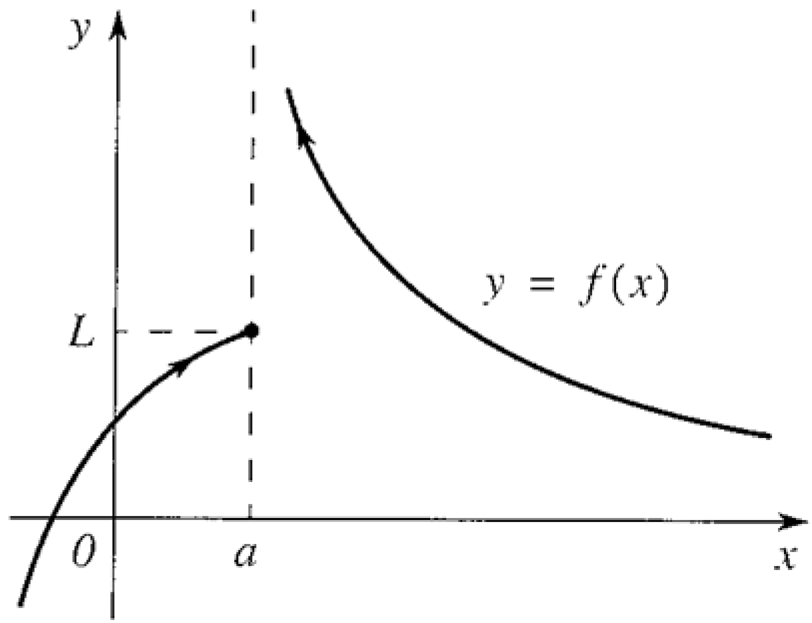
$$g(x) = (2 + x) \sin\frac{1}{x},$$

which takes on every value between -2 and 2 in every interval $(0, \delta)$. Hence, $g(x)$ does not approach a right-hand limit at x approaches 0 from the right. This shows that a function may have a limit from one side at a point but fail to have a limit from the other side.

Right and left hand limits



$\lim_{x \to a^-} f(x) = L_1$ and $\lim_{x \to a^+} f(x) = L_2$



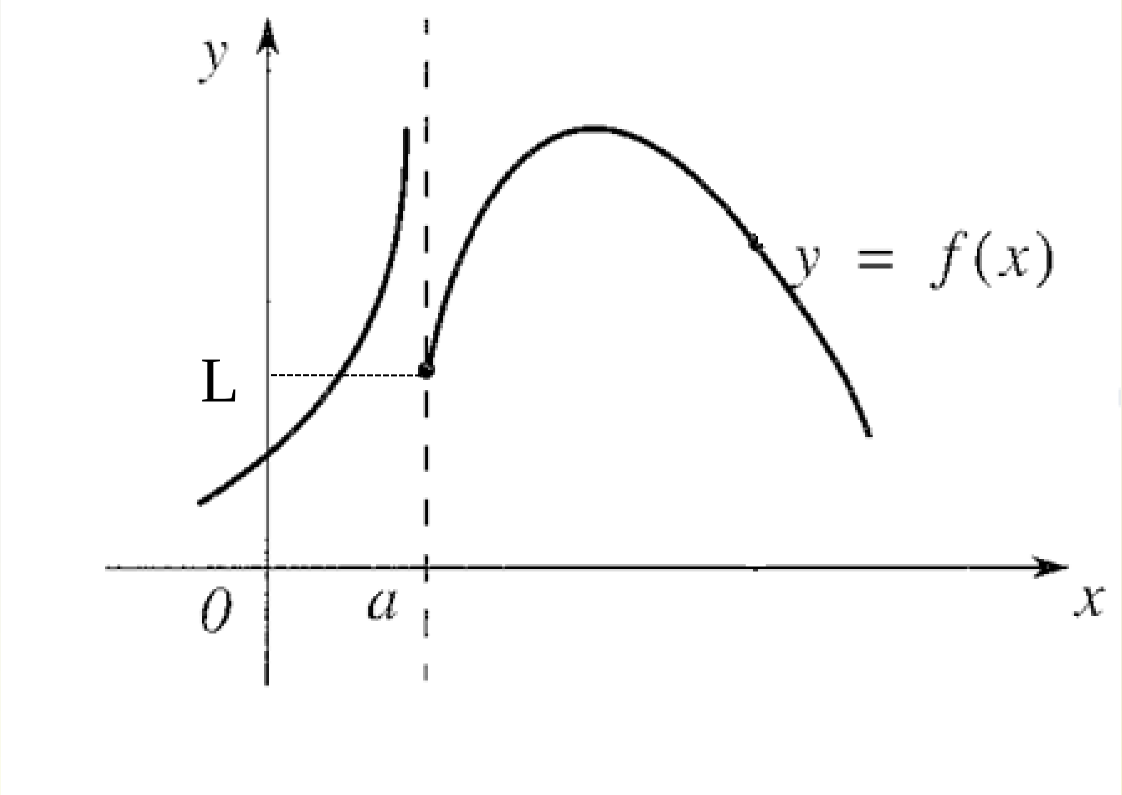
$\lim_{x \to a^-} f(x) = L$

but $\lim_{x \to a^+} f(x)$ does not exist.

Figure- 31

Figure- 32

Right and left hand limits

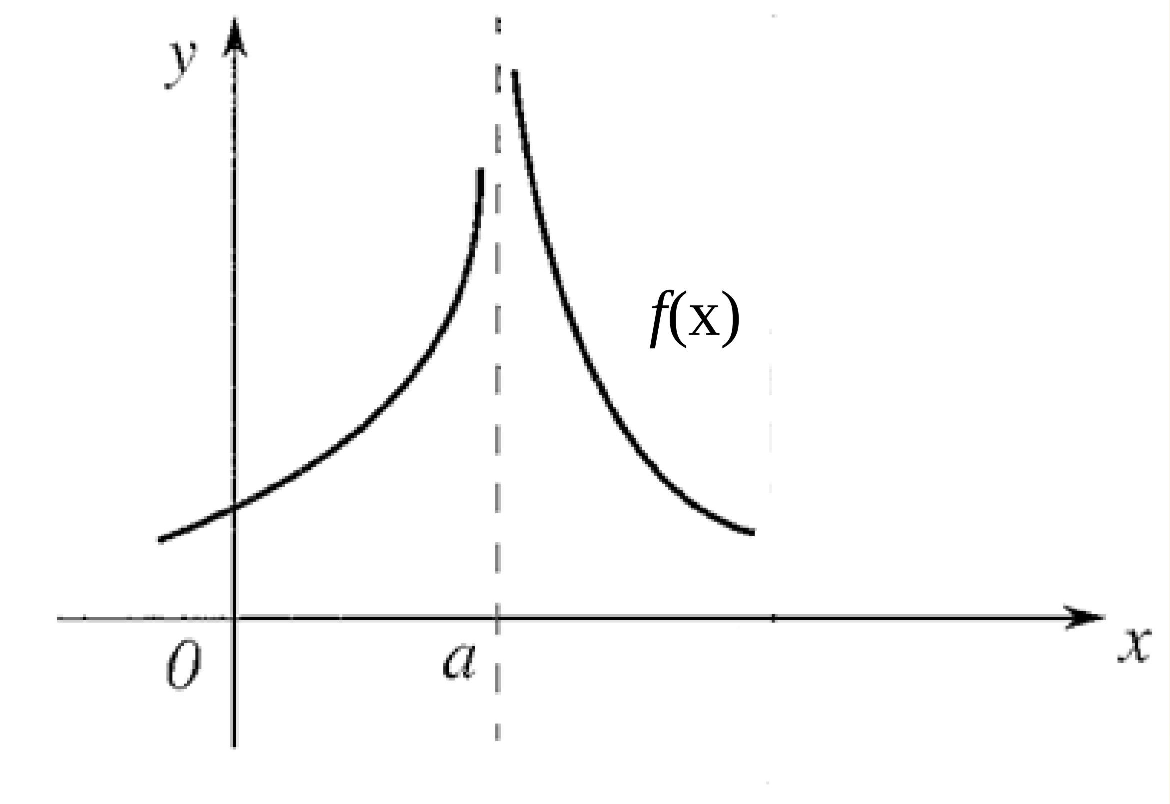


$$\lim_{x \to a^+} f(x) = L$$

but $\lim_{x \to a^-} f(x)$ does not exist.

Figure- 33

Right and left hand limits



$\lim_{x \to a^-} f(x)$ does not exist

Figure- 34

$\lim_{x \to a^+} f(x)$ does not exist.

Right and left hand limits

$$\lim_{x \to a} f(x) = L \text{ exists if and only if}$$

$$\lim_{x \to a^-} f(x) \text{ and } \lim_{x \to a^+} f(x) \text{ both exist}$$

and equal to L.

Limit Properties

Assume that $\lim_{x \to a} f(x) = K$ and $\lim_{x \to a} g(x) = L$ exist and that c is any constant. Then,

1. $\lim_{x \to a} \left[cf(x) \right] = c \lim_{x \to a} f(x) = cK$

2. $\lim_{x \to a} \left[f(x) \pm g(x) \right] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = K \pm L$

3. $\lim_{x \to a} \left[f(x) g(x) \right] = \lim_{x \to a} f(x) \; \lim_{x \to a} g(x) = KL$

4. $\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{K}{L}$, provided $L = \lim_{x \to a} g(x) \neq 0$

5. $\lim_{x \to a} \left[f(x) \right]^n = \left[\lim_{x \to a} f(x) \right]^n = K^n$, where n is any real number

6. $\lim_{x \to a} \left[\sqrt[n]{f(x)} \right] = \sqrt[n]{\lim_{x \to a} f(x)}$

Limit Properties

7. $\lim\limits_{x\to a} c = c$

8. $\lim\limits_{x\to a} x = a$

9. $\lim\limits_{x\to a} x^n = a^n$

Using the limit properties We can easily find the following limits

$$\lim_{x\to 2}(x^2-6x+4) = \lim_{x\to 2} x^2 + \lim_{x\to 2}(-6x) + \lim_{x\to 2} 4$$

$$= (\lim_{x\to 2} x)(\lim_{x\to 2} x) + (\lim_{x\to 2} -6)(\lim_{x\to 2} x) + \lim_{x\to 2} 4$$

$$= (2)(2) + (-6)(2) + 4 = -4$$

$$\lim_{x\to -1}\frac{(x+3)(2x-1)}{x^2+3x-2} = \frac{\lim\limits_{x\to -1}(x+3)\lim\limits_{x\to -1}(2x-1)}{\lim\limits_{x\to -1}(x^2+3x-2)} = \frac{2\cdot(-3)}{-4} = \frac{3}{2}$$

$$\lim_{h\to 0}\frac{\sqrt{4+h}-2}{h} = \lim_{h\to 0}\frac{\sqrt{4+h}-2}{h}\cdot\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}$$

$$= \lim_{h\to 0}\frac{4+h-4}{h(\sqrt{4+h}+2} = \lim_{h\to 0}\frac{1}{\sqrt{4+h}+2} = \frac{1}{2+2} = \frac{1}{4}$$

Limit Properties

Fact

If $f(x) \le g(x)$ for all x on $[a, b]$ (except possibly at $x = c$) and $a \le c \le b$ then,

$$\lim_{x \to c} f(x) \le \lim_{x \to c} g(x)$$

We can take this fact one step farther to get the following theorem.

Squeeze Theorem

Suppose that for all x on $[a, b]$ (except possibly at $x = c$) we have,

$$f(x) \le h(x) \le g(x)$$

Also suppose that,

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = L$$

for some $a \le c \le b$. Then,

$$\lim_{x \to c} h(x) = L$$

Limit Properties

The following figure illustrates what is happening in **Squeeze Theorem**.

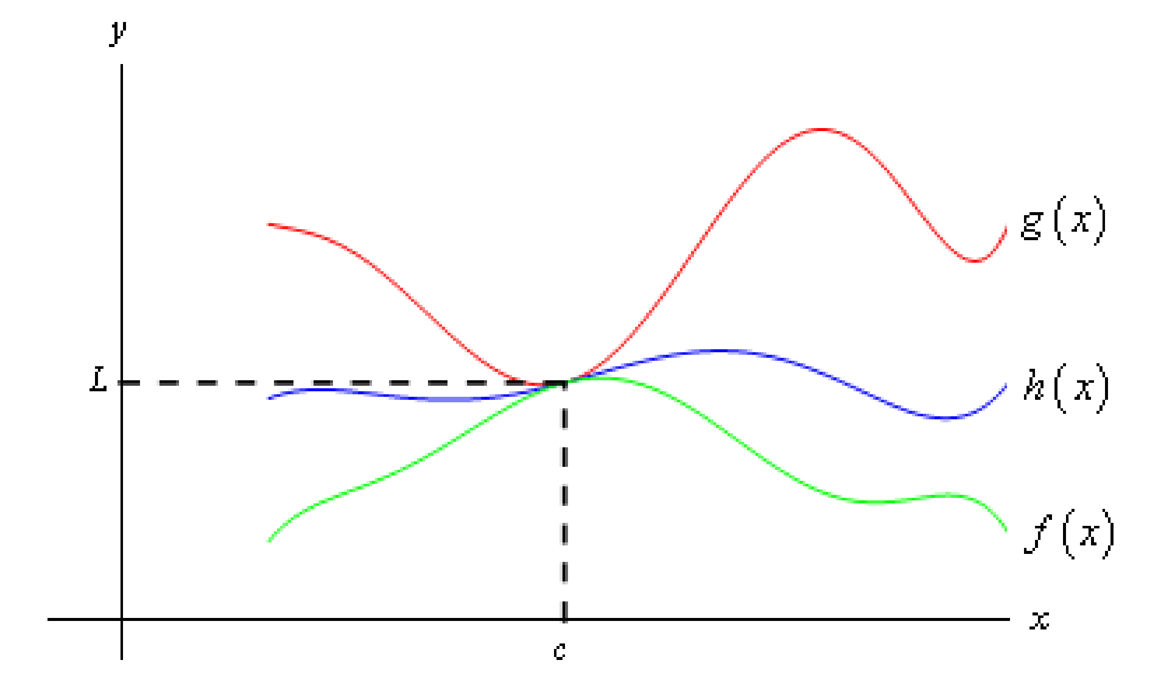


Figure- 35

The **Squeeze theorem** is also known as the **Sandwich Theorem** and the **Pinching Theorem.**

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Special Limits

Let $f(x)$ be a function defined on an interval that contains x = a, except possibly at x = a . Then we say that, $\lim_{x \to a} f(x) = \infty$

if for every number M > 0 there is some number δ > 0 (usually depending on M and a) such that $f(x) > M$ whenever $0 < |x-a| < \delta$

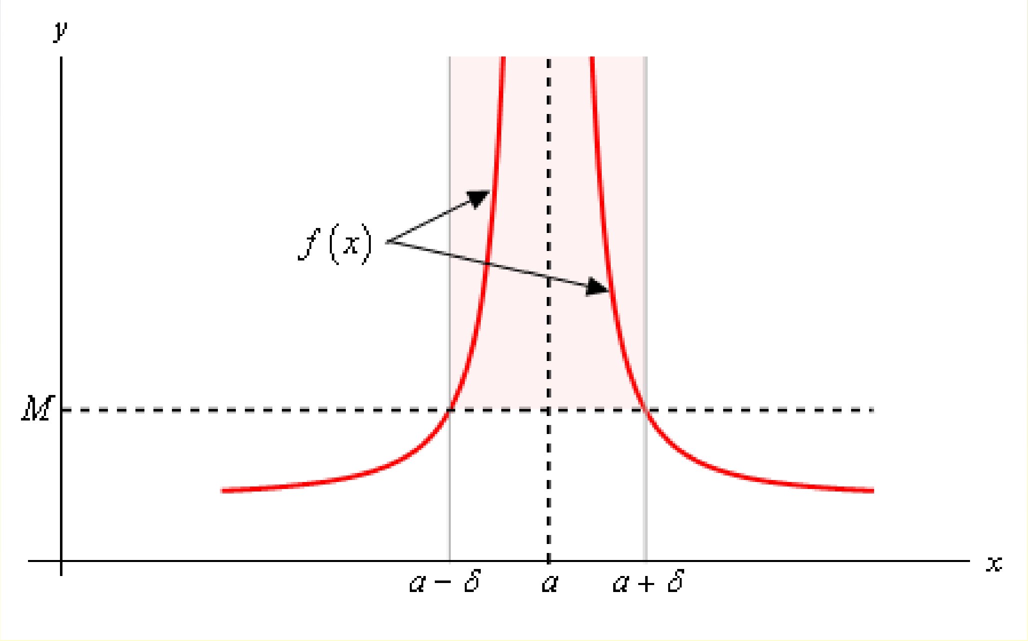


Figure- 36

Special Limits

Example

Use the definition of the limit to prove the following limit.

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

Solution

Let M > 0 be any number and we'll need to choose a δ > 0 so that,

$$\frac{1}{x^2} > M \qquad \text{whenever} \qquad 0 < |x - 0| = |x| < \delta$$

i.e. $\frac{1}{x^2} > M \quad \Rightarrow \quad x^2 < \frac{1}{M} \quad \Rightarrow \quad |x| < \frac{1}{\sqrt{M}}$

So, it looks like we can chose $\delta = \frac{1}{\sqrt{M}}$

Let $M > 0$ be any number, choose $\delta = \frac{1}{\sqrt{M}}$ and assume that $0 < |x| < \frac{1}{\sqrt{M}}$.

Special Limits

Thus we have

$$|x| < \frac{1}{\sqrt{M}}$$

$$|x|^2 < \frac{1}{M} \qquad \text{square both sides}$$

$$x^2 < \frac{1}{M} \qquad \text{acknowledge that } |x|^2 = x^2$$

$$\frac{1}{x^2} > M \qquad \text{solve for } x^2$$

So, we've managed to show that,

$$\frac{1}{x^2} > M \qquad \text{whenever} \qquad 0 < |x - 0| < \frac{1}{\sqrt{M}}$$

and so by the definition of the limit we have,

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

Special Limits

Let $f(x)$ be a function defined on an interval that contains x = a, except possibly at x = a . Then we say that,

$$\lim_{x \to a} f(x) = -\infty$$

if for every number $N < 0$ there is some number $\delta > 0$ (usually depending on N and a) such that $f(x) < N$ whenever $0 < |x-a| < \delta$

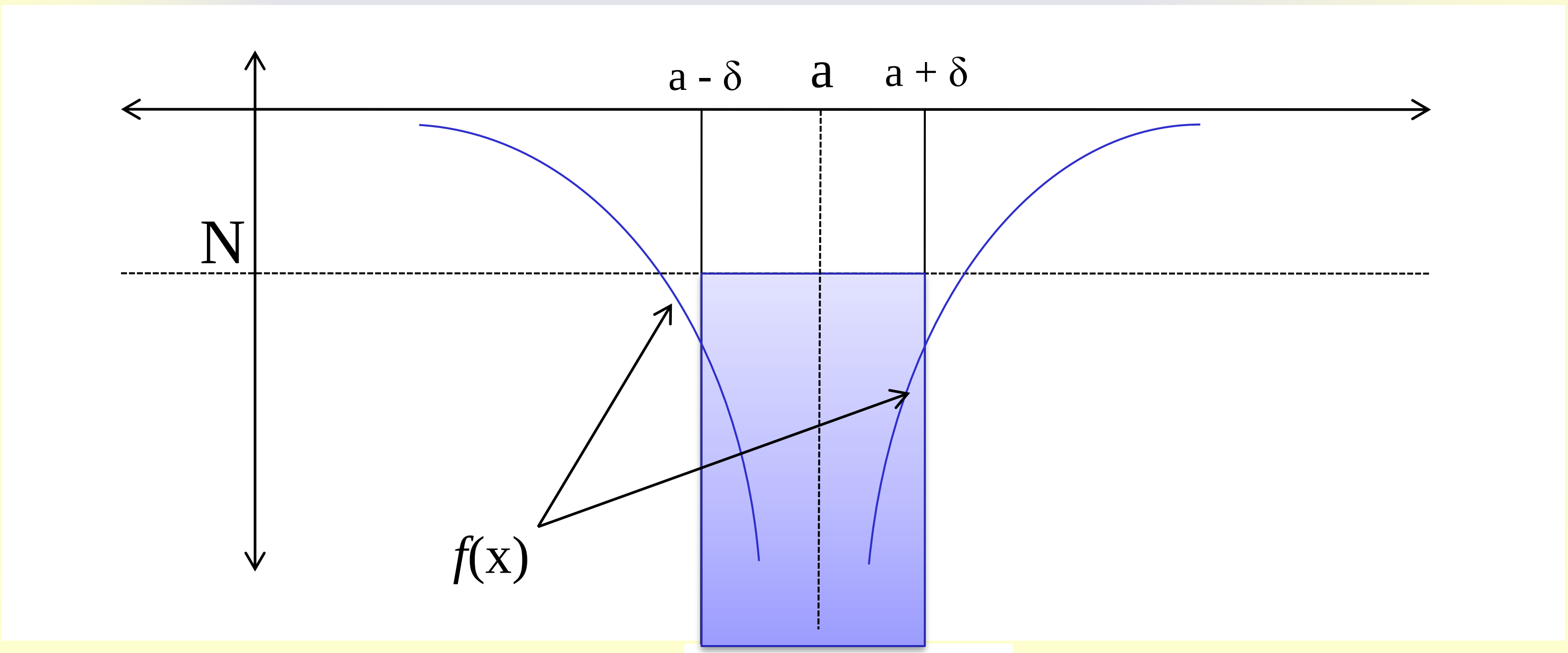


Figure- 37

Special Limits

Let $f(x)$ be a function defined on an interval that contains x = a, except possibly at x = a . Then we say that, $\lim_{x \to \infty} f(x) = L$

if for every number ε > 0 (however small) there is some number M > 0 (usually depending on ε and a) such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad x > M.$$

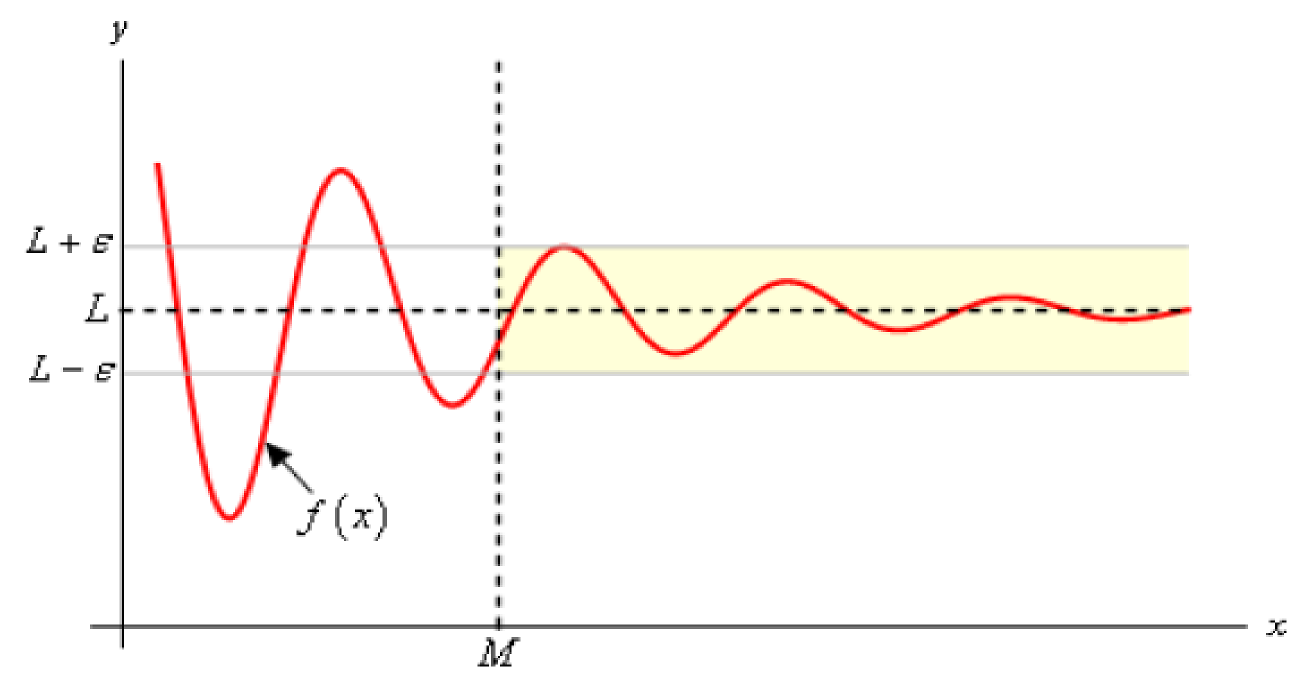


Figure- 38

Special Limits

e.g. $\lim_{x \to +\infty} e^{-x} = 0,$

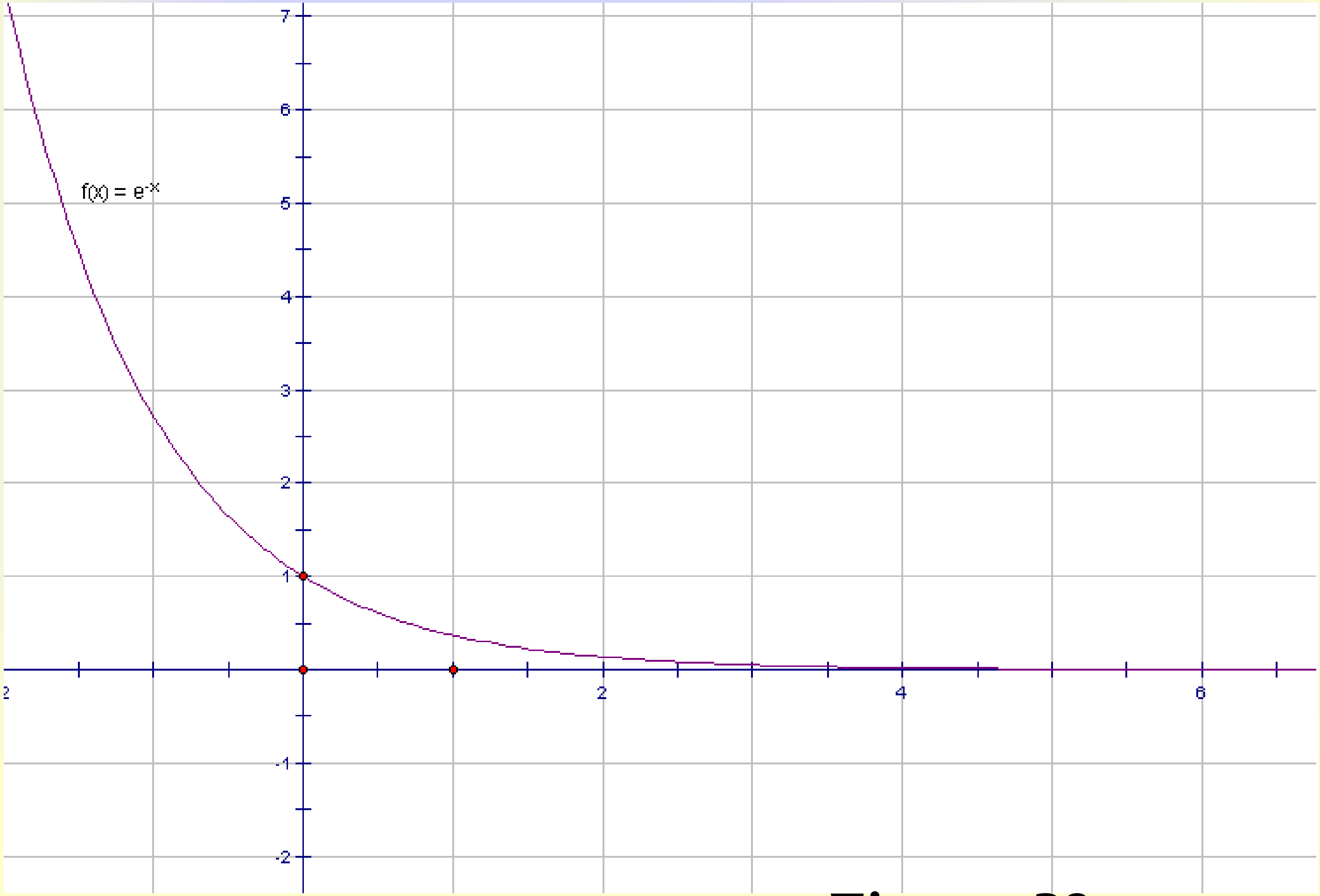


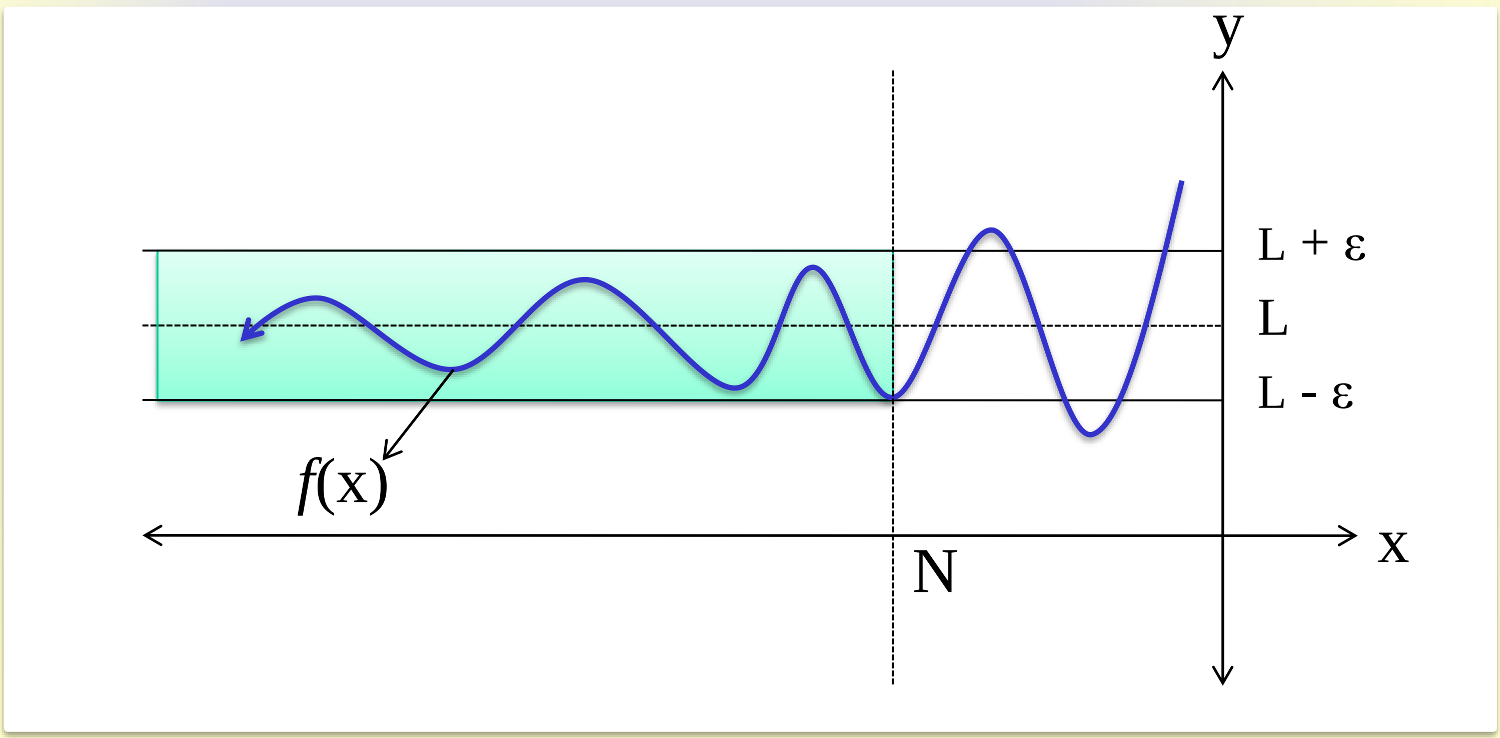
Figure- 39

Special Limits

Let f(x) be a function defined on an interval that contains x = a, except possibly at x = a . Then we say that, $\lim_{x \to -\infty} f(x) = L$

if for every number ε > 0 (however small) there is some number N < 0 (usually depending on ε and a) such that

$$| f(x) - L| < \varepsilon \quad \text{whenever} \quad x < N.$$



Md. Masum Murshed **Figure- 40**

Special Limits

Let $f(x)$ be a function defined on an interval that contains x = a, except possibly at x = a . Then we say that,

$$\lim_{x \to \infty} f(x) = \infty$$

if for every number N > 0 there is some number M > 0 (usually depending on N) such that

$f(x) > N$ whenever $x > M$.

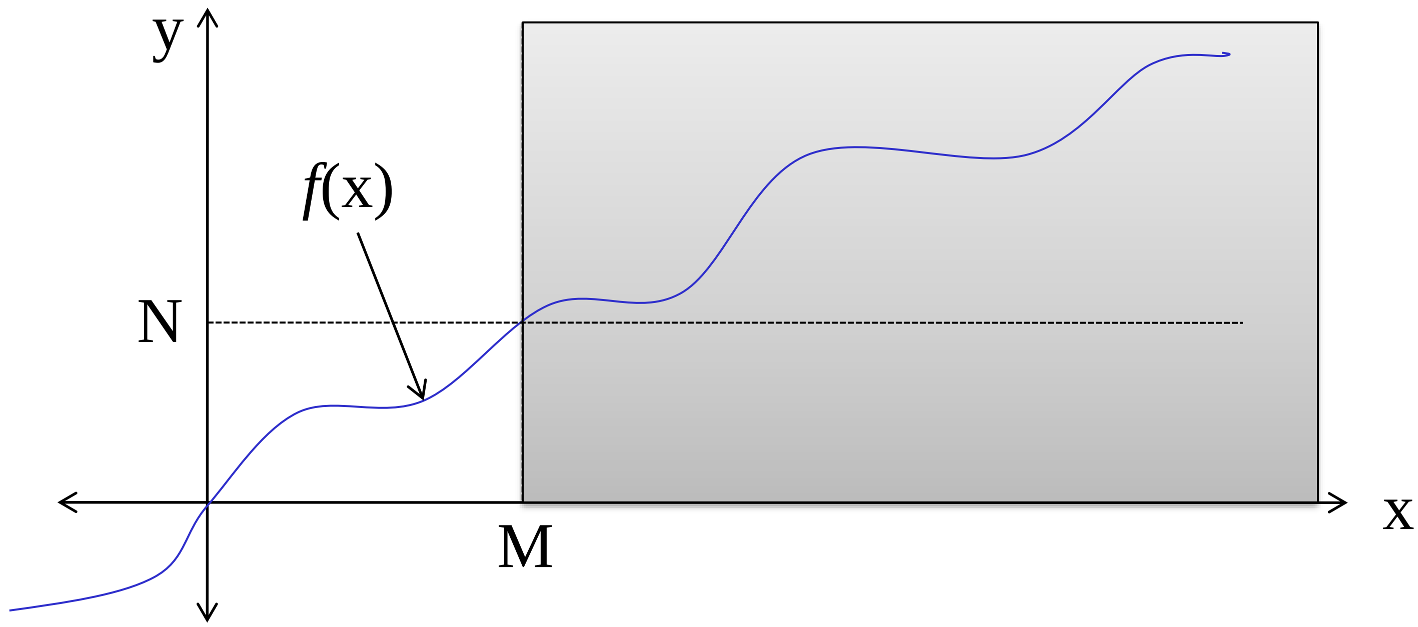


Figure- 41

Special Limits

If

$$\lim_{x \to a} f(x) = \infty \quad \text{or} \quad \lim_{x \to a} f(x) = -\infty$$

we say that limit of *f*(x) does not exist as x tends to a. These limits exist in the **extended real line**.

Special Limits

Properties of limit of functions :

(1) If $\lim_{x \to x_0} f(x) = 0$ and $f(x) \neq 0$, then $\lim_{x \to x_0} \frac{1}{f(x)} = \infty$.

(2) If $\lim_{x \to x_0} f(x) = \infty$ (or $\pm\infty$), then $\lim_{x \to x_0} \frac{1}{f(x)} = 0$.

(3) If $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = +\infty$ (resp. $-\infty$), then

$$\lim_{x \to x_0} [f(x) + g(x)] = +\infty \text{ (resp. } -\infty).$$

Special Limits

Properties of limit of functions :

(4) If $\lim\limits_{x\to x_0} f(x)=\infty$, then $\lim\limits_{x\to x_0}[f(x)\pm k]=\infty$

for any real constant k.

(5) If $\lim\limits_{x\to x_0} f(x)=\infty$, then $\lim\limits_{x\to x_0}[kf(x)]=\infty$

for any non - zero constant k.

(6) If $\lim\limits_{x\to x_0} f(x)=\lim\limits_{x\to x_0} g(x)=\infty$, then

$\lim\limits_{x\to x_0}[f(x)g(x)]=\infty$.

Special Limits

1. $$\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad \lim_{x\to 0}\frac{1-\cos x}{x}=0$$

2. $$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e \qquad \lim_{x\to 0+}(1+x)^{1/x}=e$$

3. $$\lim_{x\to 0}\frac{e^{x}-1}{x}=1 \qquad \lim_{x\to 1}\frac{x-1}{\ln x}=1$$

Special Limits

Present a **geometric proof** of the following limit.

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.$$

Proof

Construct a circle with center at O and radius OA = OD = 1, as in Figure 33. Choose point B on OA extended and point C on OD so that lines BD and AC ar perpendicular to OD.

It is geometrically evident that

Area of triangle OAC < Area of sector OAD < Area of triangle OBD that is,

$$\frac{1}{2}\sin\theta\cos\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$$

or

$$\cos\theta < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

$$\cos\theta < \frac{\sin\theta}{\theta} < \frac{1}{\cos\theta}$$

As $\theta \to 0$, $\cos\theta \to 1$, and it follows that $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.$

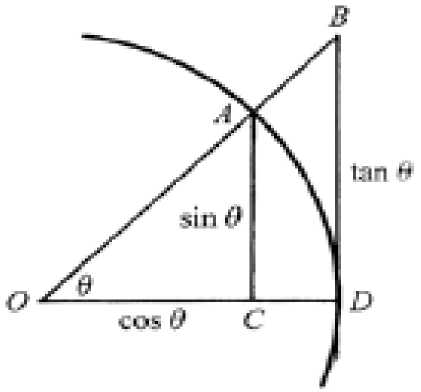


Figure- 42

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Continuous Function

A function $f(x)$ is said to be continuous at $x = a$, if

(i) $f(x)$ is well defined at $x = a$, and

(ii) $\lim_{x \to a} f(x) = f(a)$

A function $f(x)$ is said to be continuous $x = a$, if for every number $\varepsilon > 0$ (however small) there is some number $\delta > 0$ (usually depending on ε and a) such that

$$|f(x) - f(a)| < \varepsilon \quad \text{whenever} \quad |x-a| < \delta$$

A function $f(x)$ is said to be continuous $x = a$, if for every neighborhood $N_\varepsilon(f(a))$ there is some neighborhood $N_\delta(a)$ such that if $x \in N_\delta(a)$ then $f(x) \in N_\varepsilon(f(a))$.

Continuous Function

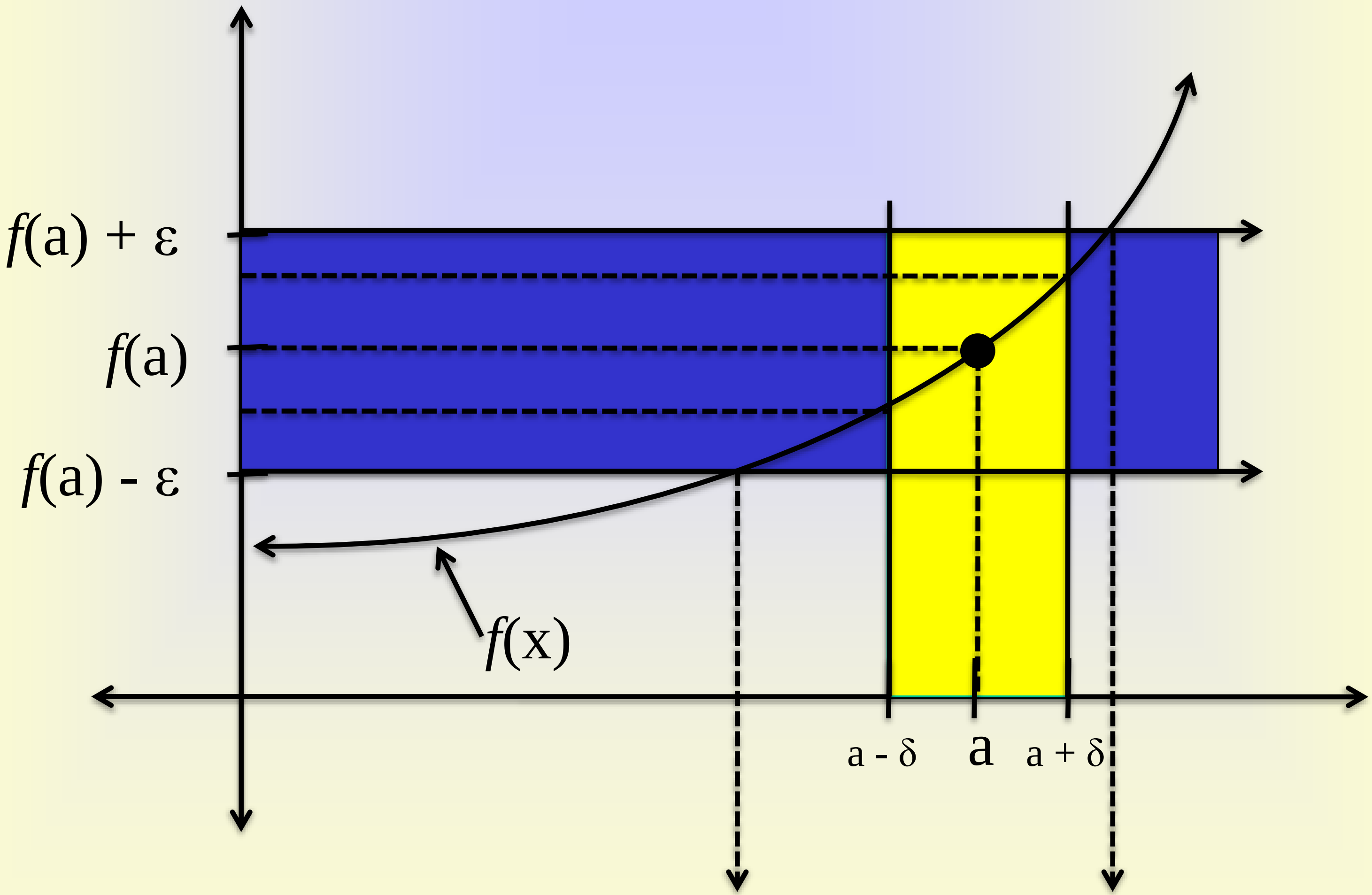


Figure- 43

Continuous Function

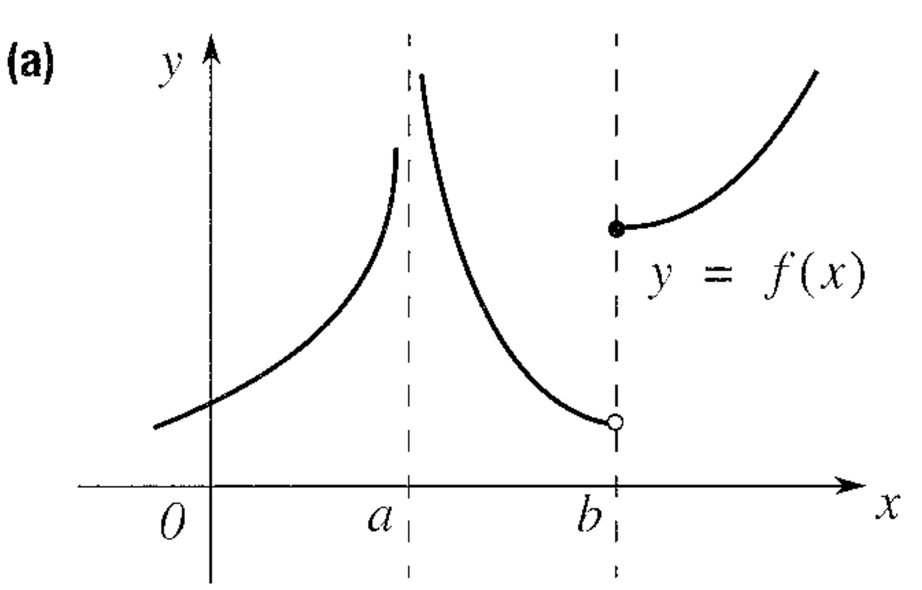
$f(x)$ is said to be continuous in the open interval (a,b) if and only if $f(x)$ is continuous at evey point in (a,b),

i.e. $\lim\limits_{x \to c} f(x) = f(c)$, for all $c \in (a,b)$.

Continuous Function

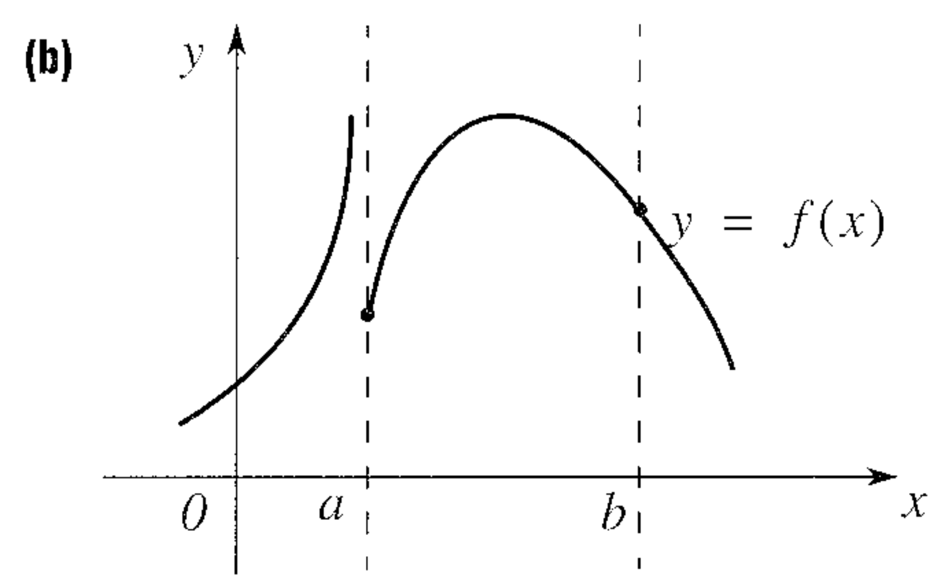
$f(x)$ is said to be continuous on the closed interval $[a,b]$ if and only if $f(x)$ is continuous in (a,b), and $\lim_{x\to a^+} f(x) = f(a)$, and $\lim_{x\to b^-} f(x) = f(b)$.

Continuous Function



$f(x)$ is continuous on (a, b).

Figure- 44



$f(x)$ is continuous on $[a, b]$.

Figure- 45

Outline

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Right hand continuity

If $\lim\limits_{x\to a^+} f(x) = f(a)$, $f(x)$ is said to be right continuous at $x = a$.

Let $f(x)$ be a function defined on an interval that contains $x = a$. Then $f(x)$ is said to be **right hand continuous** or **right continuous at $x = a$**, if for every number $\varepsilon > 0$ (however small) there is some number $\delta > 0$ (usually depending on ε and a) such that

$$|f(x) - f(a)| < \varepsilon \quad \text{whenever} \quad 0 \le x-a < \delta \text{ (Or } a \le x < a + \delta)$$

Or equivalently

if for every neighborhood $N_\varepsilon(f(a))$ there is some number $\delta > 0$ such that if $x \in [a, a + \delta)$ then $f(x) \in N_\varepsilon(f(a))$.

Right hand continuity

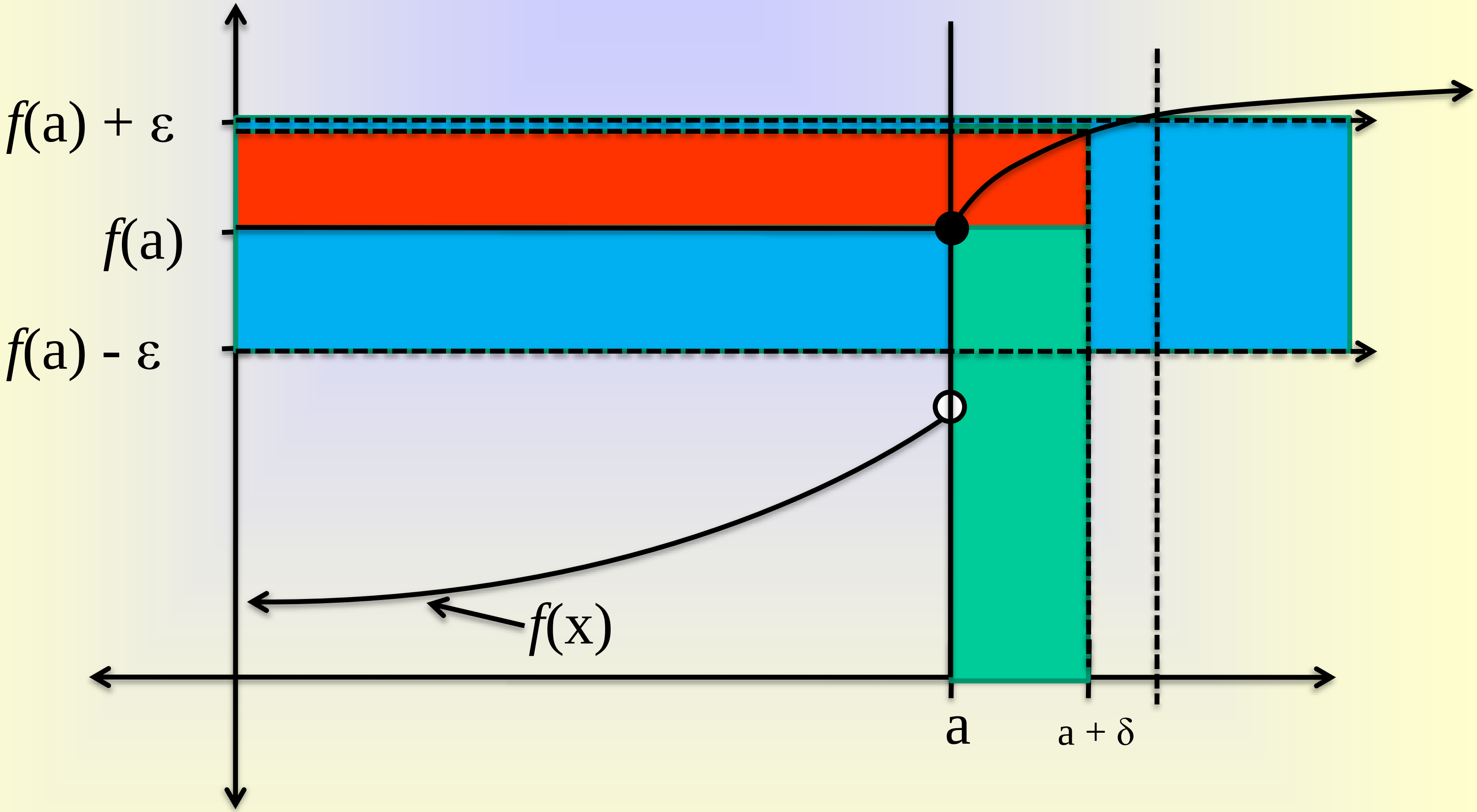
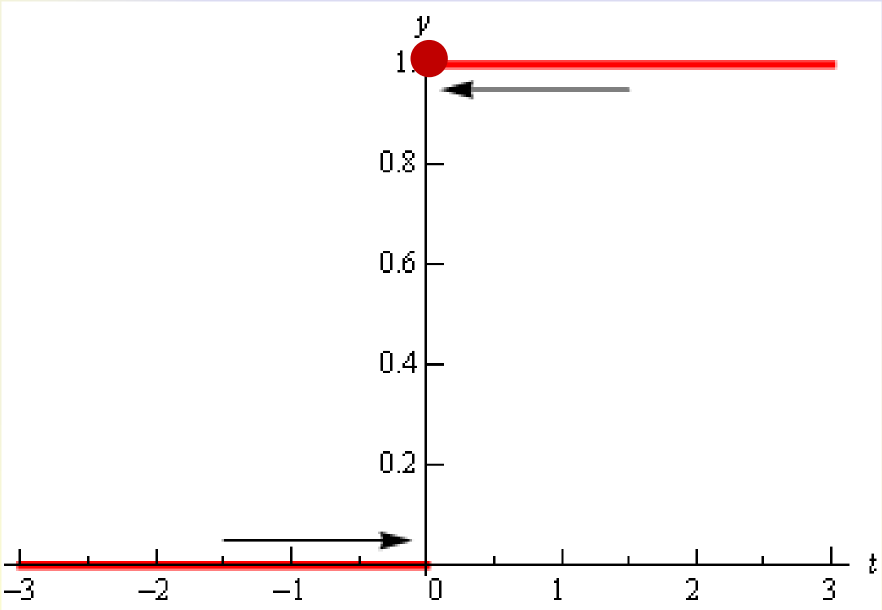


Figure- 46

Right hand continuity

Example

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$



H(t) is right continuous at t = 0.

Figure- 47

Left hand continuity

If $\lim_{x \to a^-} f(x) = f(a)$, $f(x)$ is said to be left continuous at $x = a$.

Let f(x) be a function defined on an interval that contains x = a. Then f(x) is said to be **left hand continuous** or **left continuous at x = a**, if for every number ε > 0 (however small) there is some number δ > 0 (usually depending on ε and a) such that

$$| f(x) - f(a) | < \varepsilon \quad \text{whenever } -\delta < x - a \leq 0 \ (\text{Or } a - \delta < x \leq a)$$

Or equivalently
if for every neighborhood $N_\varepsilon(f(a))$ there is some number δ > 0 such that if $x \in (a - \delta, a\,]$ then $f(x) \in N_\varepsilon(f(a))$.

Left hand continuity

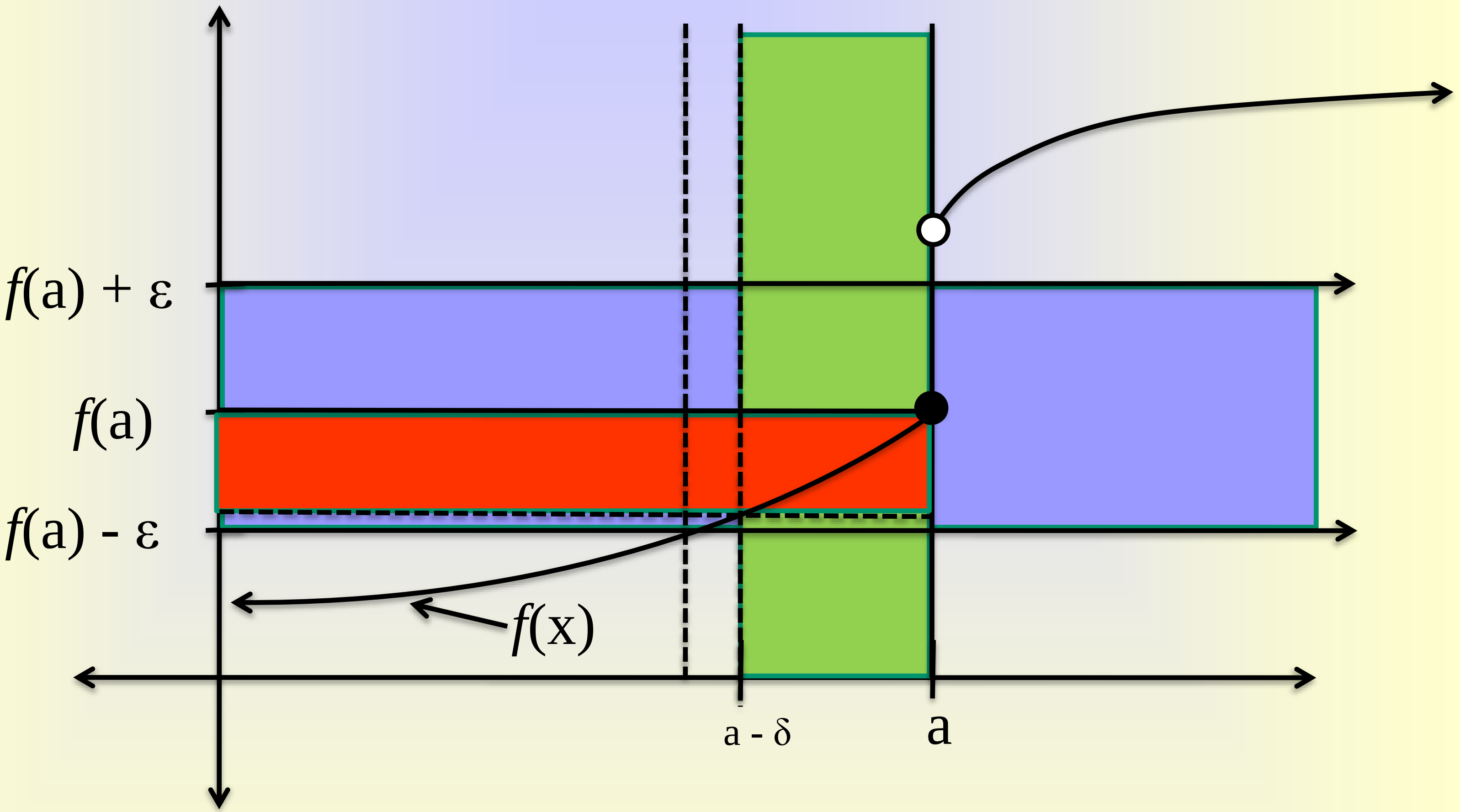


Figure- 48

Left hand continuity

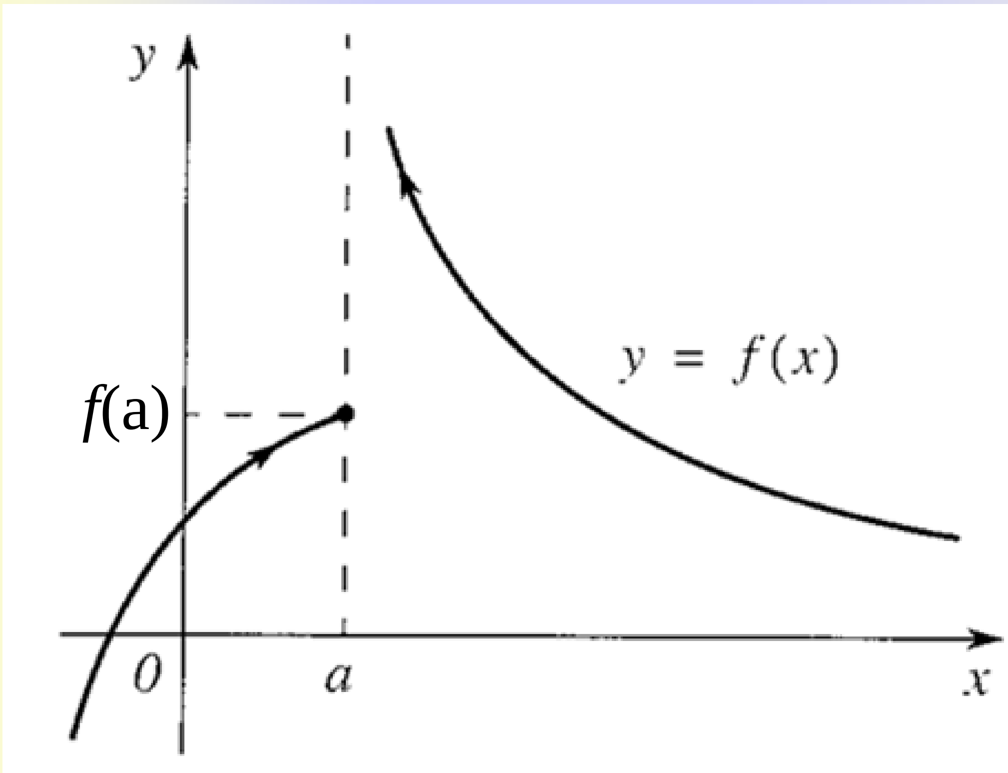


Figure- 49

f(x) is **left continuous** at x = a.

Continuous Functions

To show **continuity** of $f(x)$ at x = a, the following conditions must be verified:

(1) $f(x)$ is defined at all points inside a neighborhood of the point a.

(2) **Left hand limit** and **right hand limit** of $f(x)$ exist at x = a and both have equal values,
i.e. $f(a^+) = f(a^-) = L$, Where $f(a^+)$ and $f(a^-)$ denote the **left hand limit** and the **right hand limit** of $f(x)$ at x = a respectively.

(3) The value of $f(x)$ at x = a is equal to L, i.e. $f(a) = L$.

If any of the above conditions is violated, then $f(x)$ is said to be **discontinuous** at x = a.

Continuous Functions

There are two kinds of **discontinuity**.

(1) $f(x)$ has **discontinuity of the first kind** at $x = a$ if $f(a^+)$ and $f(a^-)$ exist, but at least one of them is different from $f(a)$.

There are two types of the **Discontinuity of the first kind**.

(i) **Solvable discontinuity:** if $f(a^+) = f(a^-)$.

(ii) **Jump discontinuity:** if $f(a^+) \neq f(a^-)$.

(2) $f(x)$ has **discontinuity of the second kind** at $x = a$ if at least one of $f(a^+)$ and $f(a^-)$ does not exist.

Continuous Functions

Solvable discontinuity

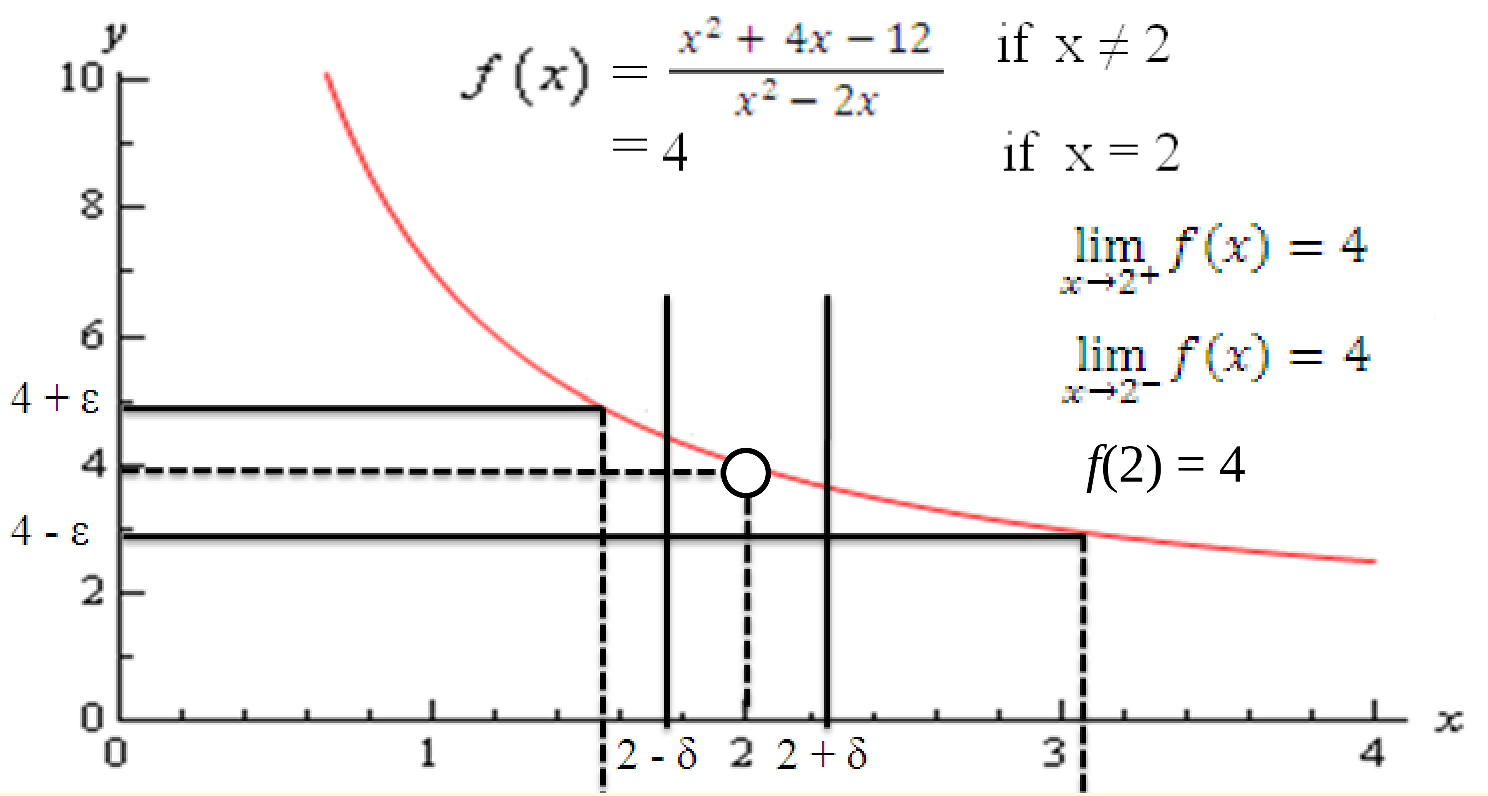


Figure- 50

Continuous Functions

Example

Jump discontinuity

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

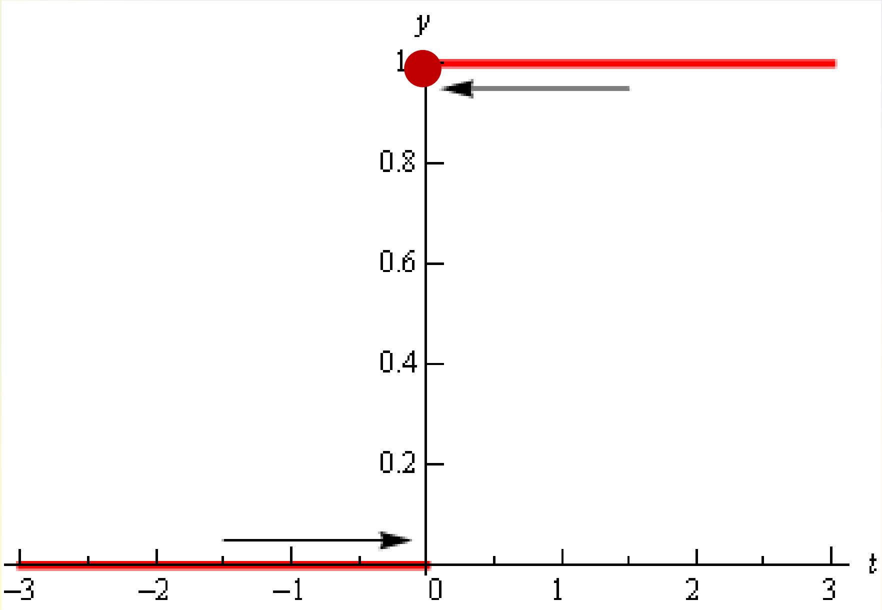


Figure- 51

Continuous Functions

Example

Jump discontinuity

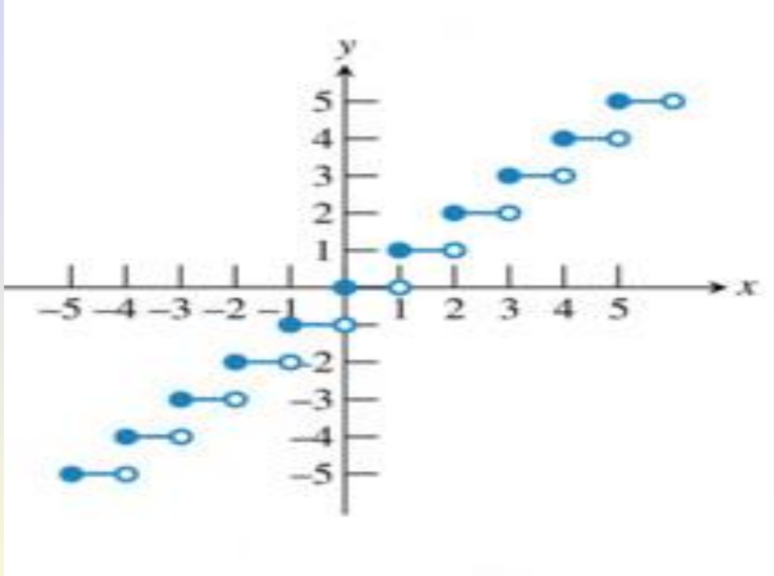


Figure- 52

Continuous Functions

Let us now consider the function

$$f(x) = \begin{cases} x+1, & x>0, \\ 0, & x=0, \\ x-1, & x<0. \end{cases}$$

This function is continuous everywhere except at $x = 0$, since it has no limit as $x \to 0$ by the fact that $f(0^-) = -1$ and $f(0^+) = 1$. The discontinuity at this point is therefore of the first kind.

An example of a discontinuity of the second kind is given by the function

$$f(x) = \begin{cases} \cos\dfrac{1}{x}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

which has a discontinuity of the second kind at $x = 0$, since neither $f(0^-)$ nor $f(0^+)$ exists.

Continuous Functions

Example

Discontinuity of the second kind

Dirichlet Function

The **Dirichlet Function** $d : \mathbf{R} \to \mathbf{R}$ is defined as follows:

$$d(x) = \begin{cases} 1 & \text{if x is rational} \\ 0 & \text{if x is irrational} \end{cases}$$

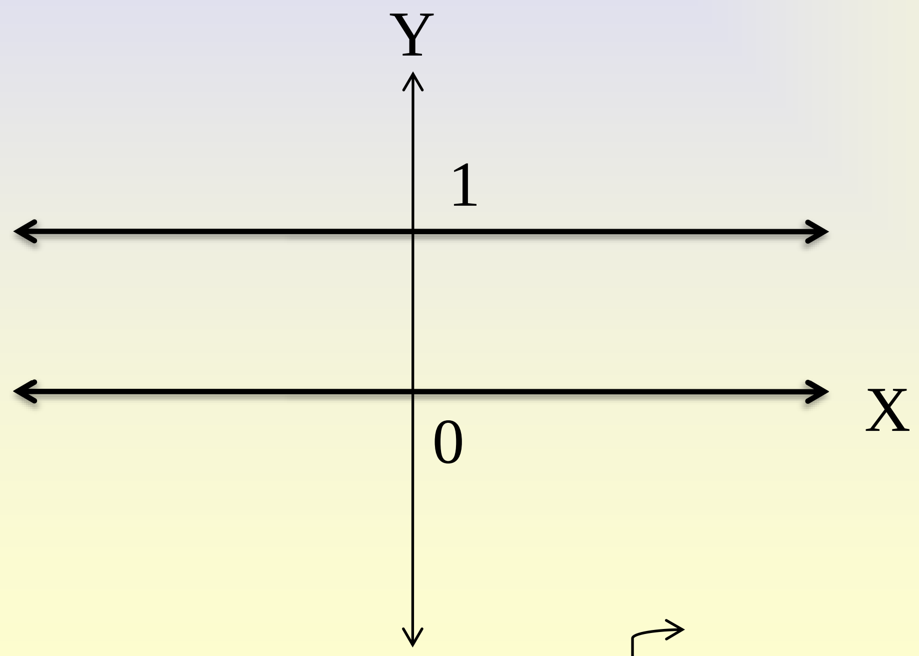


Figure-53 Graph of the function $\boldsymbol{d(x) = \begin{cases} 1 & \text{if x is rational} \\ 0 & \text{if x is irrational} \end{cases}}$

Continuous Functions

Example

Modification-1 of Dirichlet Function

The **Modification-1 of Dirichlet Function** $f : \mathbf{R} \to \mathbf{R}$ is defined as follows:

$$f(\mathrm{x}) = \mathrm{x}.\, d(\mathrm{x}) \quad \begin{cases} \mathrm{x} & \text{if x is rational} \\ 0 & \text{if x is irrational} \end{cases}$$

This function is continuous at only x = 0.

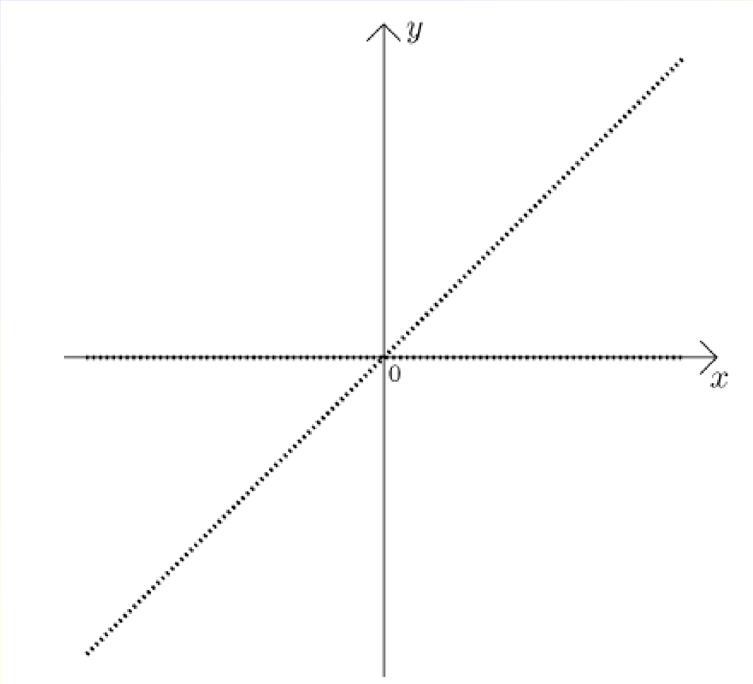


Figure-54 Graph of the function $\boldsymbol{f(\mathrm{x}) = \begin{cases} \mathrm{x} & \text{if x is rational} \\ 0 & \text{if x is irrational} \end{cases}}$

Continuous Functions

Example

Modification-2 of Dirichlet Function

The **Modification-2 of Dirichlet Function** $f : \mathbf{R} \to \mathbf{R}$ is defined as follows:

$$f(x) = x^2 .\, d(x) \quad \begin{cases} x^2 & \text{if x is rational} \\ 0 & \text{if x is irrational} \end{cases}$$

This function is continuous at only x = 0.

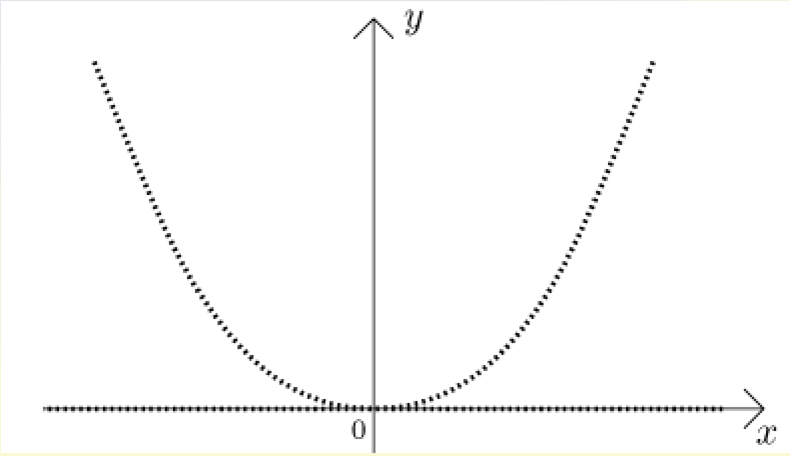


Figure-55 Graph of the function $f(x) = \begin{cases} x^2 & \text{if x is rational} \\ 0 & \text{if x is irrational} \end{cases}$

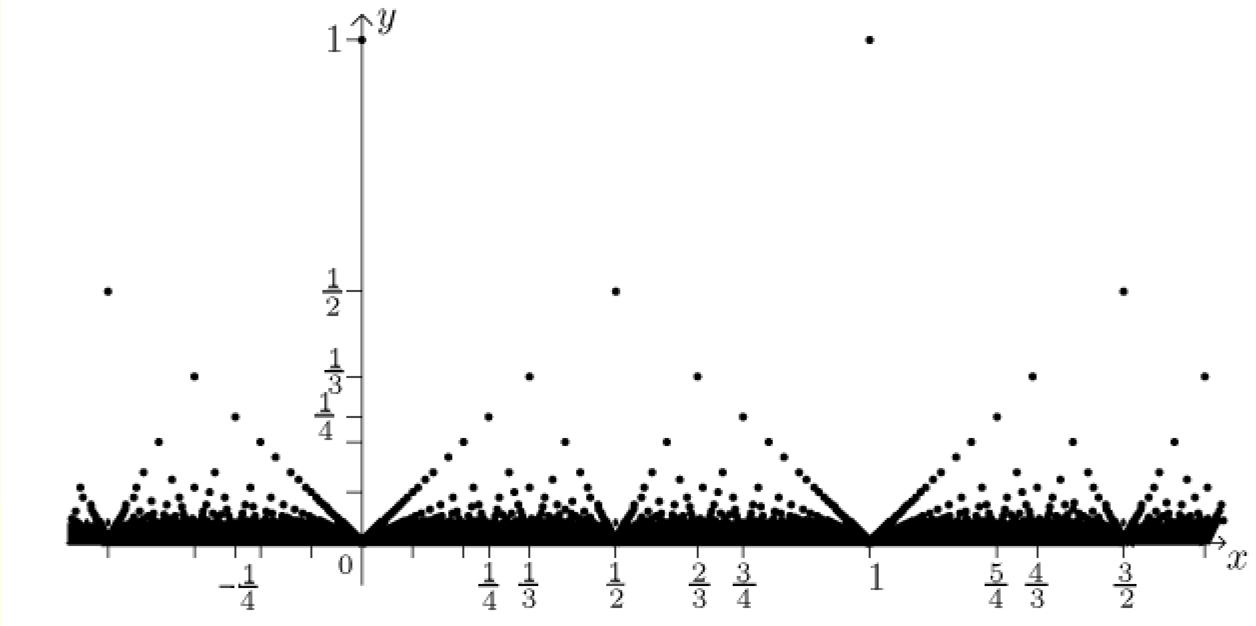
Continuous Functions

Example

Modification-3 of Dirichlet Function. It is also known as **ruler function**.

The **Modification-3 of Dirichlet Function** f : **R** $\rightarrow$ **R** is defined as follows:

$$f(x) = \begin{cases} 1, & x = 0; \\ \frac{1}{q}, & x = \frac{p}{q},\ p, q \neq 0 \text{ coprime integers},\ q > 0; \\ 0, & x \in \mathbb{R} - \mathbb{Q}. \end{cases}$$



This function is continuous at every irrational numbers.

Figure-56 Graph of the ruler function.

Continuous Functions

Theorems on Continuity

Theorem 1 If f and g are continuous at $x = x_0$, so also are the functions whose image values satisfy the relations $f(x) + g(x), f(x) - g(x), f(x)g(x)$, and $\dfrac{f(x)}{g(x)}$, the last only if $g(x_0) \neq 0$. Similar results hold for continuity in an interval.

Theorem 2 Functions described as follows are continuous in every finite interval: (*a*) all polynomials; (*b*) $\sin x$ and $\cos x$; and (*c*) a^x, $a > 0$.

Theorem 3 Let the function f be continuous at the domain value $x = x_0$. Also suppose that a function g, represented by $z = g(y)$, is continuous at y_0, where $y = f(x)$ (i.e., the range value of f corresponding to x_0 is a domain value of g). Then a new function, called a *composite function*, $f(g)$, represented by $z = g[f(x)]$, may be created which is continuous at its domain point $x = x_0$. (One says that *a continuous function of a continuous function is continuous*.)

Theorem 4 If $f(x)$ is continuous in a closed interval, it is bounded in the interval.

Theorem 5 If $f(x)$ is continuous at $x = x_0$ and $f(x_0) > 0$ [or $f(x_0) < 0$], there exists an interval about $x = x_0$ in which $f(x) > 0$ [or $f(x) < 0$].

Continuous Functions

Theorem 6 If a function $f(x)$ is continuous in an interval and either strictly increasing or strictly decreasing, the inverse function $f^{-1}(x)$ is single-valued, continuous, and either strictly increasing or strictly decreasing.

Theorem 7 If $f(x)$ is continuous in $[a, b]$ and if $f(a) = A$ and $f(b) = B$, then corresponding to any number C between A and B there exists at least one number c in $[a, b]$ such that $f(c) = C$. This is sometimes called the *intermediate value theorem*.

Theorem 8 If $f(x)$ is continuous in $[a, b]$ and if $f(a)$ and $f(b)$ have opposite signs, there is at least one number c for which $f(c) = 0$ where $a < c < b$. This is related to Theorem 7.

Theorem 9 If $f(x)$ is continuous in a closed interval, then $f(x)$ has a maximum value M for at least one value of x in the interval and a minimum value m for at least one value of x in the interval. Furthermore, $f(x)$ assumes all values between m and M for one or more values of x in the interval.

Theorem 10 If $f(x)$ is continuous in a closed interval and if M and m are, respectively, the least upper bound (l.u.b.) and greatest lower bound (g.l.b.) of $f(x)$, there exists at least one value of x in the interval for which $f(x) = M$ or $f(x) = m$. This is related to Theorem 9.

Outline

- **Functions and its graphs.**
- **One-one, Onto and inverse functions.**
- **Principal values.**
- **Transcendental functions.**
- **Bounded and monotonic functions.**
- **Limits of functions.**
- **Right and left hand limits.**
- **Special limits.**
- **Continuity.**
- **Right and left hand continuity.**
- **Sectional continuity.**
- **Uniform continuity, Lipschitz continuity.**

Sectional continuity or piecewise continuity

Piecewise Continuity

A function is called *piecewise continuous* in an interval $a \leq x \leq b$ if the interval can be subdivided into a finite number of intervals in each of which the function is continuous and has finite right- and left-hand limits. Such a function has only a finite number of discontinuities. An example of a function which is piecewise continuous in $a \leq x \leq b$ is shown graphically in Figure. This function has discontinuities at x_1, x_2, x_3, and x_4.

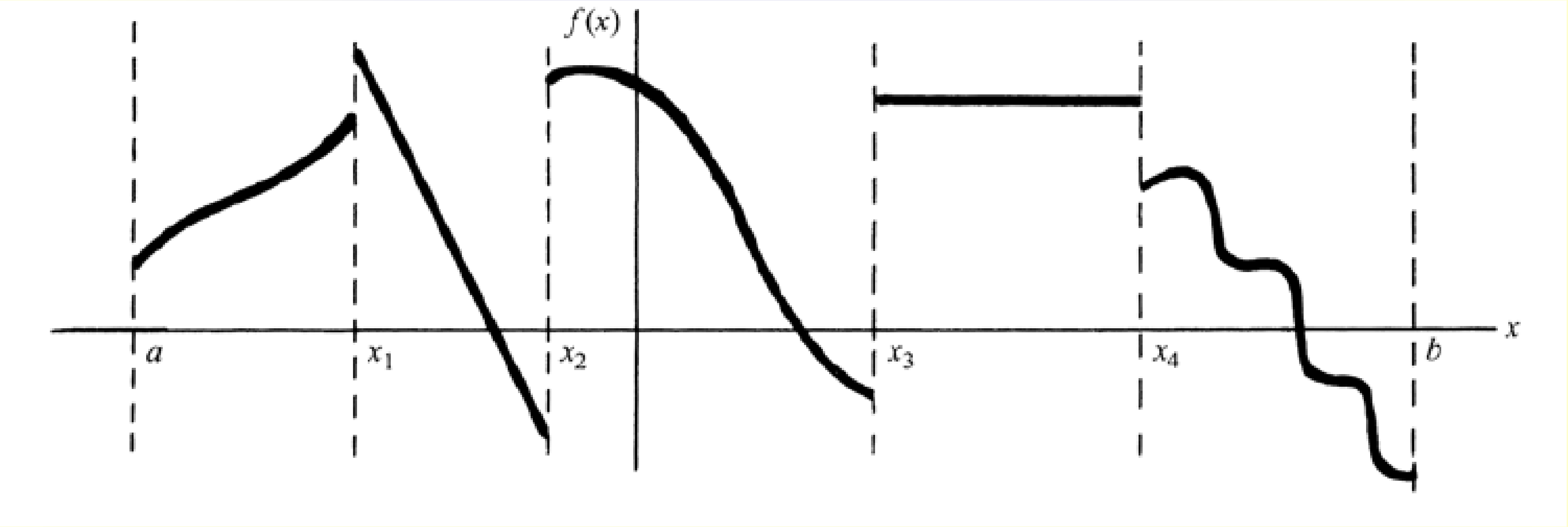


Figure- 57

Outline

- **Functions and its graphs.**
- **One-one, Onto and inverse functions.**
- **Principal values.**
- **Transcendental functions.**
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- **Special limits.**
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- **Right and left hand continuity.**
- **Sectional continuity.**
- **Uniform continuity, Lipschitz continuity.**

Uniform Continuity

Let $f(x)$ be continuous in an interval. Then, by definition, at each point a of the interval and for every number $\varepsilon > 0$ (however small) there is some number $\delta > 0$ (usually depending on ε and a) such that

$|f(x) - f(a)| < \varepsilon$ whenever $|x-a| < \delta$. If we can find δ for each ε which holds for all points of the interval (i.e. δ depends only one ε and not on a) we say that is uniformly continuous in the interval.

A function $f(x)$ is said to be uniformly continuous in an interval if for every number $\varepsilon > 0$ (however small) there is some number $\delta > 0$ (depending on ε) such that

$|f(x_1) - f(x_2)| < \varepsilon$ whenever $|x_1-x_2| < \delta$, where $x_1$and x_2 are any two points in the interval.

Uniform Continuity

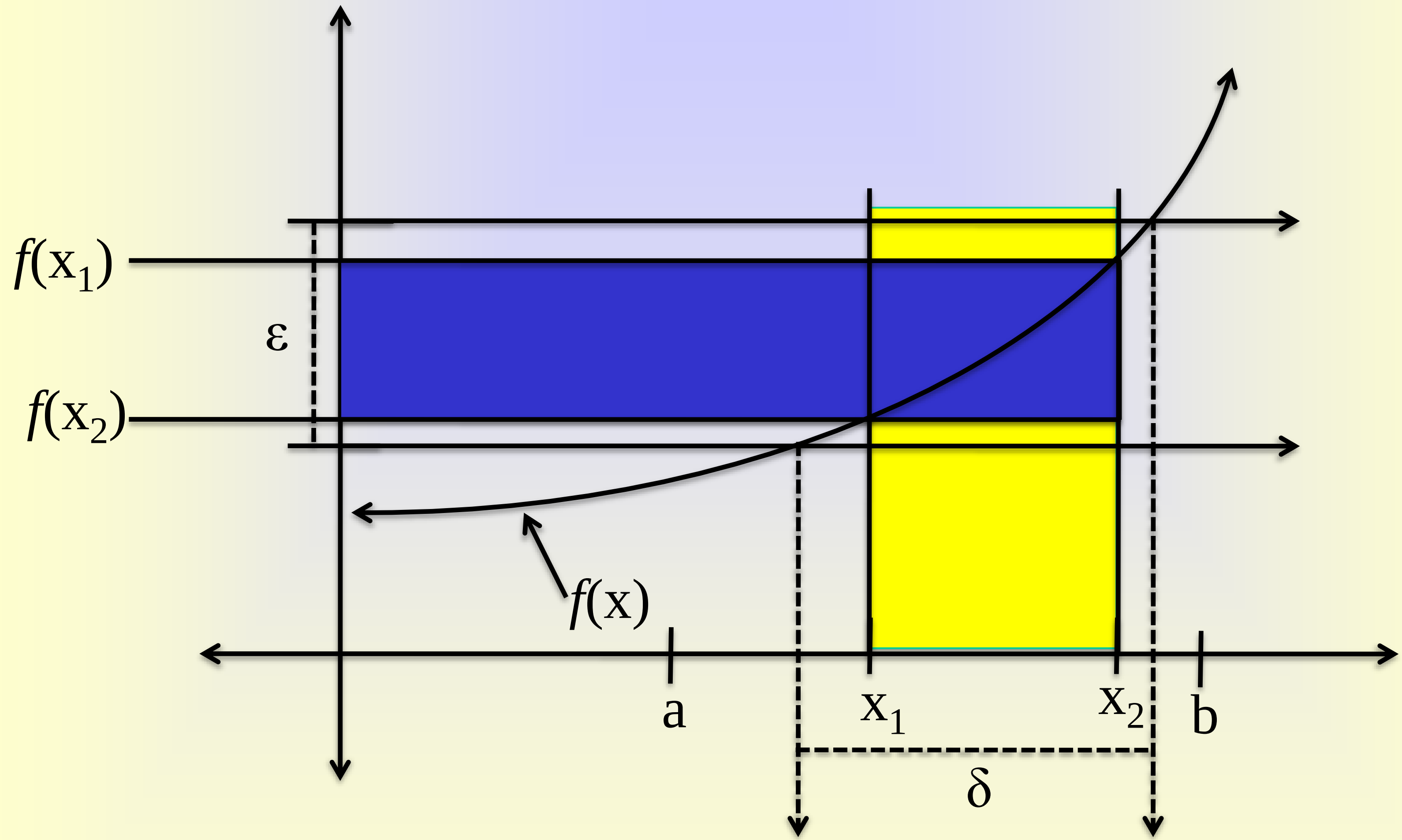

Figure- 58

Uniform Continuity

Theorem

If f is continuous in a closed interval, it is **uniformly continuous** in the interval.

Problem

Prove that $f(x) = x^2$ is uniformly continuous in $0 < x < 1$.

Solution

We must show that, given any $\varepsilon > 0$, we can find $\delta > 0$ such that

$|x_1^2 - x_2^2| < \varepsilon$ whenever $|x_1 - x_2| < \delta$, where $x_1, x_2 \in (0, 1)$.

Now

$|x_1^2 - x_2^2)| = |x_1 + x_2|\,|x_1 - x_2| \leq (|x_1| + |x_2|) < (1 + 1)\,|x_1 - x_2| = 2\,|x_1 - x_2|$

Therefore given $\varepsilon > 0$ let $\delta = \varepsilon/2$. Then if $|x_1 - x_2| < \delta$, then $|x_1^2 - x_2^2| < \varepsilon$.

Uniform Continuity

Problem

Prove that $f(x) = x^2$ is not **uniformly continuous** in $\mathbf{R} = (-\infty, \infty)$.

Solution

Suppose that $f(x) = x^2$ is **uniformly continuous** in **R**, then for all $\varepsilon > 0$, there would exist a $\delta > 0$ such that $|x_1^2 - x_2^2| < \varepsilon$ whenever $|x_1 - x_2| < \delta$ where $x_1, x_2 \in (0, 1)$.

Take $x_1 > 0$ and let $x_2 = x_1 + \delta/2$. Write

$\varepsilon \geq | x_1^2 - x_2^2)| = |x_1 + x_2| \, |x_1 - x_2| = (2x_1 + \delta/2)\delta/2 > x_1\delta$

Therefore $x_1 \leq \varepsilon/\delta$ for all $x_1 > 0$, which is a contradiction. It follows that $f(x) = x^2$ cannot be uniformly continuous in $\mathbf{R} = (-\infty, \infty)$.

Uniform Continuity

Problem

Prove that $f(x) = 1/x$ is not **uniformly continuous** in $0 < x < 1$.

Solution

Suppose that $f(x) = 1/x$ is **uniformly continuous** in **R**, then for all $\varepsilon > 0$, there would exist a $\delta > 0$ such that $|1/x_1 - 1/x_2| < \varepsilon$ whenever $|x_1 - x_2| < \delta$ where $x_1, x_2 \in (0, 1)$.

$\varepsilon > |1/x_1 - 1/x_2| = (|x_2 - x_1|)/(x_1 x_2)$

Or $|x_1 - x_2| < x_1 x_2 \varepsilon$

Therefore, to satisfy the definition of uniform continuity we would have to have $\delta \leq x_1 x_2 \varepsilon$ for all $x_1, x_2 \in (0, 1)$, but that would mean that $\delta \leq 0$. Therefore there is no single $\delta > 0$. Therefore $f(x) = 1/x$ is not **uniformly continuous** in $0 < x < 1$.

Lipschitz continuity

Lipschitz Continuous Functions

Lipschitz continuity is a specialized form of uniform continuity.

Definition The function $f\colon D \to R$ is said to satisfy the Lipschitz condition on a set $E \subset D$ if there exist constants, K and α, where $K > 0$ and $0 < \alpha \leq 1$ such that

$$|f(x_1) - f(x_2)| \leq K|x_1 - x_2|^{\alpha}$$

for all $x_1, x_2 \in E$.

Notationally, whenever $f(x)$ satisfies the Lipschitz condition with constants K and α on a set E, we say that it is $\mathrm{Lip}(K, \alpha)$ on E. In this case, $f(x)$ is called a Lipschitz continuous function. It is easy to see that a Lipschitz continuous function on E is also uniformly continuous there.

Lipschitz continuity

As an example of a Lipschitz continuous function, consider $f(x) = \sqrt{x}$, where $x \geq 0$. We claim that $\sqrt{x}$ is Lip$(1, \frac{1}{2})$ on its domain. To show this, we first write

$$\left| \sqrt{x_1} - \sqrt{x_2} \right| \leq \sqrt{x_1} + \sqrt{x_2} \, .$$

Hence,

$$\left| \sqrt{x_1} - \sqrt{x_2} \right|^2 \leq |x_1 - x_2| \, .$$

Thus,

$$\left| \sqrt{x_1} - \sqrt{x_2} \right| \leq |x_1 - x_2|^{1/2} \, ,$$

which proves our claim.

Lipschitz continuity

Lipschitz continuous functions

Definition Let $f: S \to \mathbb{R}$ be a function such that there exists a number K such that for all x and y in S we have

$$|f(x) - f(y)| \leq K\,|x - y|.$$

Then f is said to be *Lipschitz continuous*.

Proposition *A Lipschitz continuous function is uniformly continuous.*

Proof. Let $f: S \to \mathbb{R}$ be a function and let K be a constant such that for all x, y in S we have $|f(x) - f(y)| \leq K\,|x - y|$.

Let $\varepsilon > 0$ be given. Take $\delta = \varepsilon/K$. For any x and y in S such that $|x - y| < \delta$ we have that

$$|f(x) - f(y)| \leq K\,|x - y| < K\delta = K\frac{\varepsilon}{K} = \varepsilon.$$

Therefore f is uniformly continuous.

Lipschitz continuity

Example The functions $\sin(x)$ and $\cos(x)$ are Lipschitz continuous.

since

$$|\sin(x)-\sin(y)| \leq |x-y| \qquad \text{and} \qquad |\cos(x)-\cos(y)| \leq |x-y|.$$

Hence sin and cos are Lipschitz continuous with $K = 1$.

Example The function $f\colon [1,\infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is Lipschitz continuous.

$$|\sqrt{x}-\sqrt{y}| = \left|\frac{x-y}{\sqrt{x}+\sqrt{y}}\right| = \frac{|x-y|}{\sqrt{x}+\sqrt{y}}.$$

As $x \geq 1$ and $y \geq 1$, we can see that $\frac{1}{\sqrt{x}+\sqrt{y}} \leq \frac{1}{2}$. Therefore

$$|\sqrt{x}-\sqrt{y}| = \left|\frac{x-y}{\sqrt{x}+\sqrt{y}}\right| = \frac{1}{2}|x-y|.$$

Lipschitz continuity

On the other hand $f\colon [0,\infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is not Lipschitz continuous.

Suppose that we have

$$\left|\sqrt{x} - \sqrt{y}\right| \leq K\,|x - y|,$$

for some K. Let $y = 0$ to obtain $\sqrt{x} \leq Kx$. If $K > 0$, then for $x > 0$ we then get $1/K \leq \sqrt{x}$. This cannot possibly be true for all $x > 0$. Thus no such $K > 0$ can exist and f is not Lipschitz continuous.

Note that the last example shows an example of a function that is uniformly continuous but not Lipschitz continuous.

since $\sqrt{x}$ is uniformly continuous on $[0,\infty)$.

Thank you all