"Improper Integral"

<u>Def^n</u>: If in an integral either the range is infinite or the integrand has an infinite discontinuity in the range, then the integral is called improper integral.

Infinite range:

(i) $\int_0^\infty f(x)\,dx$ is defined as $\lim_{\varepsilon \to \infty} \int_a^\varepsilon f(x)\,dx$, provided $f(x)$ is integrable in (a, ε), and this limit exists.

(ii) $\int_{-\infty}^{b} f(x)\,dx$ is defined as $\lim_{\varepsilon \to -\infty} \int_\varepsilon^b f(x)\,dx$, provided $f(x)$ is integrable in (ε, b), and this limit exists.

(iii) If the infinite integrals $\int_{-\infty}^{a} f(x)\,dx$ and $\int_a^\infty f(x)\,dx$ both exist, we ~~sh~~ say that $\int_{-\infty}^{\infty} f(x)\,dx$ exists, and

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^\infty f(x)\,dx.$$

Integrand infinitely discontinuous at a point:

(i) If $f(x)$ is infinitely discontinuous only at the end point a, i.e., if $f(x) \to \infty$ as $x \to a$, then

$\int_a^b f(x)\,dx$ is defined as $\lim\limits_{\varepsilon \to 0} \int_{a+\varepsilon}^{b} f(x)\,dx$, $\varepsilon > 0$,

provided $f(x)$ is integrable in $(a+\varepsilon, b)$ and this limit exists.

(ii) If $f(x)$ is infinitely discontinuous only at the end point b, i.e., if $f(x) \to \infty$ as $x \to b$, then

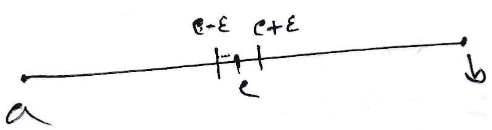
$\int_a^b f(x)\,dx$ is defined as $\lim\limits_{\varepsilon \to 0} \int_a^{b-\varepsilon} f(x)\,dx$, $\varepsilon > 0$,

provided $f(x)$ is integrable in $(a, b-\varepsilon)$ and this limit exists.

(iii) If $f(x)$ is infinitely discontinuous only at ~~the~~ an internal point c, $(a<c<b)$, i.e., if $f(x) \to \infty$ as $x \to c$,

then $\int_a^b f(x)\,dx = \lim\limits_{\varepsilon \to 0} \int_a^{c-\varepsilon} f(x)\,dx + \lim\limits_{\varepsilon' \to 0} \int_{c+\varepsilon'}^{b} f(x)\,dx$

when $\varepsilon \to 0$, $\varepsilon' \to 0$ ~~intependent~~ independently.



Evaluate:

(i) $\int_0^\infty e^{-x}\,dx$

(ii) $\int_0^\infty \cos tx\,dx$

(iii) $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

(iv) $\int_0^1 \frac{dx}{x^{2/3}}$

(v) $\int_{-1}^{1} \frac{dx}{x^2}$

(vi) $\int_{-1}^{1} \frac{dx}{x}$

Solⁿ: (i) $I = \lim_{\varepsilon \to \infty} \int_0^\varepsilon e^{-x}\,dx$

$$= \lim_{\varepsilon \to \infty} \left[-e^{-x}\right]_0^\varepsilon$$

$$= \lim_{\varepsilon \to \infty} \left[-\frac{1}{e^\varepsilon} + \frac{1}{e^0}\right]$$

$$= -0 + 1 = 1$$

Sol^n (ii) Let $I = \int_0^\infty \cos tx \, dx$

$$= \lim_{\varepsilon \to \infty} \int_0^\varepsilon \cos tx \, dx$$

$$= \lim_{\varepsilon \to \infty} \left[\frac{\sin tx}{t} \right]_0^\varepsilon$$

$$= \lim_{\varepsilon \to \infty} \left[\frac{\sin t\varepsilon}{t} - 0 \right]$$

$$= \lim_{\varepsilon \to \infty} \frac{\sin t\varepsilon}{t}$$

Since $\lim_{\varepsilon \to \infty} \frac{\sin t\varepsilon}{t}$ does not exist,

$\therefore \int_0^\infty \cos tx \, dx$ does not exist.

Sol^n (iii) $I = \int_{-\infty}^\infty \frac{dx}{1+x^2}$

$$= \int_{-\infty}^a \frac{dx}{1+x^2} + \int_a^\infty \frac{dx}{1+x^2}$$

$$= \lim_{\varepsilon \to -\infty} \int_\varepsilon^a \frac{dx}{1+x^2} + \lim_{\varepsilon \to \infty} \int_a^{\varepsilon'} \frac{dx}{1+x^2}$$

$$= \lim_{\varepsilon \to -\infty} \left[\tan^{-1} x \right]_{\varepsilon}^{a} + \lim_{\varepsilon' \to \infty} \left[\tan^{-1} x \right]_{a}^{\varepsilon'}$$

$$\cancel{= \tan^{-1} a - \tan}$$

$$= \lim_{\varepsilon \to -\infty} \left(\tan^{-1} a - \tan^{-1} \varepsilon \right) + \lim_{\varepsilon' \to \infty} \left(\tan^{-1} \varepsilon' - \tan^{-1} a \right)$$

$$= \tan^{-1} a + \frac{\pi}{2} + \frac{\pi}{2} - \tan^{-1} a$$

$$= \pi$$

<u>Sol^n: (iv)</u> $I = \int_0^1 \frac{dx}{x^{2/3}}$

Here $\frac{1}{x^{2/3}} \to \infty$ as $x \to 0$

$$\therefore \quad I = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{1} \frac{dx}{x^{2/3}}$$

$$= \lim_{\varepsilon \to 0} \left[\frac{x^{-2/3+1}}{-\frac{2}{3}+1} \right]_{\varepsilon}^{1}$$

$$= \lim_{\varepsilon \to 0} \left[3x^{1/3} \right]_{\varepsilon}^{1} = \lim_{\varepsilon \to 0} \left[3 - 3\varepsilon^{1/3} \right] = 3$$

<u>Ans</u>

Sol^n (v) $\quad I = \int_{-1}^{1} \frac{dx}{x^2}$

Here $\frac{1}{x^2} \to \infty$ as $x \to 0$

$$\therefore \quad I = \int_{-1}^{0} \frac{dx}{x^2} + \int_{0}^{1} \frac{dx}{x^2}$$

$$= \lim_{\varepsilon \to 0} \int_{-1}^{\varepsilon} \frac{dx}{x^2} + \lim_{\varepsilon \to 0} \int_{\varepsilon}^{1} \frac{dx}{x^2}$$

$$= \lim_{\varepsilon \to 0} \left[-\frac{1}{x} \right]_{-1}^{\varepsilon} + \lim_{\varepsilon \to 0} \left[-\frac{1}{x} \right]_{\varepsilon}^{1}$$

$$= \lim_{\varepsilon \to 0} \left[-\frac{1}{\varepsilon} - 1 \right] + \lim_{\varepsilon \to 0} \left[-1 + \frac{1}{\varepsilon} \right]$$

since $\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}$ ~~is undefined~~.

does not exist.

Hence $\int_{-1}^{1} \frac{dx}{x^2}$ does not exist.

Usual mistake: $I = \int_{-1}^{1} \frac{dx}{x^2} = \left[-\frac{1}{x} \right]_{-1}^{1} = -1 - 1 = -$

which is not correct.

(vi) $I = \int_{-1}^{+1} \frac{dx}{x}$

$= \int_{-1}^{0} \frac{dx}{x} + \int_{0}^{+1} \frac{dx}{x}$

$= \lim_{\varepsilon \to 0} \int_{-1}^{\varepsilon} \frac{dx}{x} + \lim_{\varepsilon' \to 0} \int_{\varepsilon'}^{+1} \frac{dx}{x}$

$= \lim_{\varepsilon \to 0} \Big[\log|x|\Big]_{-1}^{\varepsilon} + \lim_{\varepsilon' \to 0} \Big[\log|x|\Big]_{\varepsilon'}^{1}$

$= \lim_{\varepsilon \to 0} \left(\log|\varepsilon| - \log|-1|\right)$

$\quad + \lim_{\varepsilon' \to 0} \left(\log|1| - \log|\varepsilon'|\right)$

Since $\lim_{\varepsilon \to 0} \log|\varepsilon|$ does not exist.

$\therefore \int_{-1}^{+1} \frac{dx}{x}$ does not exist.

—x—

☑ Show that $\int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}$, $a>0$.

<u>Proof:</u> $I = \int_0^\infty e^{-ax}\cos bx\,dx$

$$= \lim_{\varepsilon\to\infty}\int_0^\varepsilon e^{-ax}\cos bx\,dx$$

We know that $\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$

$$\therefore I = \lim_{\varepsilon\to\infty}\left[\frac{e^{-ax}(-a\cos bx + b\sin bx)}{a^2+b^2}\right]_0^\varepsilon$$

$$= \lim_{\varepsilon\to\infty}\left[\frac{1}{e^{a\varepsilon}}\left(\frac{-a\cos b\varepsilon + b\sin b\varepsilon}{a^2+b^2}\right) - \cdot 1\left(\frac{-a+0}{a^2+b^2}\right)\right]$$

$$= 0 + \frac{a}{a^2+b^2}$$

[since $\lim_{\varepsilon\to\infty}\frac{1}{e^{a\varepsilon}} = 0$

and $\lim_{\varepsilon\to\infty}$ $\cos b\varepsilon$ and $\sin b\varepsilon$

are bounded]

$$= \frac{a}{a^2+b^2}$$

<u>Proved</u>

Show that $\int_0^\infty e^{-x} x^n dx = n!$, n being a +ve integer.

__Proof:__ Let, $I_n = \int_0^\infty e^{-x} x^n dx$

$$= \lim_{\varepsilon \to \infty} \int_0^\varepsilon e^{-x} x^n dx$$

$$= \lim_{\varepsilon \to \infty} \left\{ \left[e^{-x} x^n \right]_0^\varepsilon - \int_0^\varepsilon n x^{n-1} . (-e^{-x}) dx \right\}$$

$$= \lim_{\varepsilon \to \infty} \left\{ -e^{-\varepsilon} \varepsilon^n + n \int_0^\varepsilon e^{-x} x^{n-1} dx \right\}$$

since $\lim_{\varepsilon \to \infty} e^{-\varepsilon} \varepsilon^n = 0$, we have

$$I_n = n I_{n-1}$$

$$= n.(n-1) I_{n-2}$$

$$= n(n-1) \ldots 3.2.1. I_0$$

$$= n! \int_0^\infty e^{-x} dx$$

$\because I_0 = \int_0^\infty e^{-x} x^0 dx$

$= \int_0^\infty e^{-x} dx$

$$= n! \lim_{\varepsilon \to \infty} \int_0^\varepsilon e^{-x} dx$$

$$= n! \lim_{\varepsilon \to \infty} \left[-e^{-x} \right]_0^\varepsilon \cancel{= n!}$$

$= n! [$[illegible]$] = n!$ proved

"Reduction formula"

Def^n: The formula in which a certain integral involving some parameters is connected with some integrals of lower order is called a reduction formula.

☐ ~~Find~~ Obtain the reduction formula for $\int x^n e^{ax} dx$

Sol^n: Let, $I_n = \int x^n e^{ax} dx$

$$= x^n \frac{e^{ax}}{a} - \int n x^{n-1} \frac{e^{ax}}{a} dx$$

$$= \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx$$

$$= \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$$

This is the reduction formula.

☐ Find the reduction formula for $\int \sin^n x \, dx$

Sol^n: Let $I_n = \int \sin^n x \, dx$

$$\cancel{= \sin^n x (x) - \int n \sin^{n-1} x ..}$$

$$= \int \sin^{n-1} x . \sin x \, dx$$

$$= \sin^{n-1}x\,(-\cos x) - \int (n-1)\sin^{n-2}x \cdot \cos x \cdot (-\cos x)\,dx$$

$$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,(1-\sin^2 x)\,dx$$

$$= -\sin^{n-1}x\cos x + (n-1)\,I_{n-2} - (n-1)\,I_n$$

$$\Rightarrow (1+n-1)\,I_n = -\sin^{n-1}x\cos x + (n-1)\,I_{n-2}$$

$$\Rightarrow \boxed{I_n = -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}\,I_{n-2}}$$

This is the reduction formula.

H.W. Find the reduction formula for (i) $\int_0^{\pi/2} \sin^n x\,dx$

(ii) $\int \cos^n x\,dx$ (iii) $\int_0^{\pi/2} \cos^n x\,dx$.

Find the reduction formula for (i) $\int \tan^n x\,dx$

(ii) $\int_0^{\pi/4} \tan^n x\,dx$

Solⁿ: (i) Let, $I_n = \int \tan^n x\,dx = \int \tan^{n-2}x\,(\sec^2 x - 1)\,dx$

$$= \frac{\tan^{n-1}x}{n-1} - I_{n-2}$$ This is the reduction formu

$\therefore I_6 = \int \tan^6 x \, dx$

$$I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

$$\therefore I_6 = \frac{\tan^{6-1} x}{6-1} - I_{6-2}$$

$$= \frac{\tan^5 x}{5} - I_4$$

$$= \frac{\tan^5 x}{5} - \left[\frac{\tan^3 x}{3} - I_2\right]$$

$$= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \frac{\tan x}{1} - I_0$$

$$= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - \int dx$$

$$= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x$$

Solⁿ (ii) Let $I_n = \int_0^{\pi/4} \tan^n x \, dx$

$$= \int_0^{\pi/4} \tan^{n-2} x \,(\sec^2 x - 1)\, dx$$

$$= \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\pi/4} - \int_0^{\pi/4} \tan^{n-2} dx$$

$$= \left[\frac{1}{n-1} - 0\right] - I_{n-2}$$

$$= \frac{1}{n-1} - I_{n-2}$$

This is the reduction formula.

H.W. Obtain the reduction formula for $\int \sec^n x \, dx$

Hint: $I_n = \int \sec^{n-2} x \,.\, \sec^2 x \, dx$

$$= \sec^{n-2} x . \tan x - \int (n-2) . \sec^{n-3} x . \sec x \tan x . \tan x dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \,(\sec^2 x - 1)\, dx$$

$$= \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$$

$$\therefore (1+n-2) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$$

$$\therefore I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

—x—

Obtain the reduction formula for $\int e^{ax}\cos^n x\,dx$

Sol^n: Let, $I_n = \int e^{ax}\cos^n x\,dx$

$$= \frac{e^{ax}}{a}\cos^n x - \int n\cos^{n-1}x(-\sin x).\frac{e^{ax}}{a}\,dx$$

$$= \frac{1}{a}e^{ax}\cos^n x + \frac{n}{a}\int e^{ax}\cos^{n-1}x\sin x\,dx$$

$$= \frac{1}{a}e^{ax}\cos^n x + \frac{n}{a}\Big[\frac{e^{ax}}{a}\cos^{n-1}x\sin x - \int\{(n-1)\cos^{n-2}x(-\sin x)\sin x$$
$$+ \cos^{n-1}x.\cos x\}\frac{e^{ax}}{a}\,dx\Big]$$

$$= \frac{e^{ax}\cos^n x}{a} + \frac{n}{a}\Big[\frac{e^{ax}\cos^{n-1}x\sin x}{a} - \int\{+(n-1)\cos^n x \cancel{-(n-1)}$$
$$-(n-1)\cos^{n-2}x + \cos^n x\}\frac{e^{ax}}{a}\,dx\Big]$$

$$= \frac{e^{ax}\cos^n x}{a} + \frac{n e^{ax}\cos^{n-1}x\sin x}{a^2} - \frac{n(n-1)}{a^2}I_n + \frac{n(n-1)}{a^2}I_{n-2}$$
$$- \frac{n}{a^2}I_n$$

$$\Rightarrow I_n + \frac{n(n-1)}{a^2}I_n + \frac{n}{a^2}I_n = \frac{e^{ax}\cos^{n-1}x(a\cos x + n\sin x)}{a^2}$$
$$+ \frac{n(n-1)}{a^2}I_{n-2}$$

$$\Rightarrow I_n\left(1+\frac{n^2-n}{a^2}+\frac{n}{a^2}\right) = \frac{e^{ax}\cos^{n-1}x\,(a\cos x+n\sin x)}{a^2}$$

$$+\frac{n(n-1)}{a^2}I_{n-2}$$

$$\Rightarrow I_n(a^2+n^2-\cancel{n}+\cancel{n}) = e^{ax}\cos^{n-1}x\,(a\cos x+n\sin x)$$
$$+n(n-1)I_{n-2}$$

$$\therefore\ I_n = \frac{e^{ax}\cos^{n-1}x\,(a\cos x+n\sin x)}{a^2+n^2}+\frac{n(n-1)}{a^2+n^2}I_{n-2}$$

This is the reduction formula. —x—

Find the reduction formula for $\int_0^\infty e^{-ax}\cos^n x\,dx$.

Soln: Let $I_n=\int_0^\infty e^{-ax}\cos^n x\,dx$

$$= \left[\frac{e^{-ax}}{-a}\cos^n x\right]_0^\infty - \int_0^\infty\left\{n\cos^{n-1}x\sin x\,\frac{e^{-ax}}{a}\right\}dx$$

$$=\left[0+\frac{1}{a}\right]-\frac{n}{a}\left\{\left[\frac{e^{-ax}}{-a}\cos^{n-1}x\sin x\right]_0^\infty\right.$$

$$\left.-\int_0^\infty\left[(n-1)\cos^{n-2}x(-\sin x)\sin x+\cos^{n-1}x\cdot\cos x\right]\frac{e^{-ax}}{-a}\right\}$$

$$=\frac{1}{a}-0+\frac{n}{a}\int_0^\infty (n-1)\cos^{n-2}x\,(1-\cos^2 x)\cdot\frac{e^{-ax}}{a}\,dx$$
$$-\frac{n}{a}\int_0^\infty \cos^n x\,\frac{e^{-ax}}{a}\,dx$$

$$\therefore\ I_n = \frac{1}{a} + \frac{n(n-1)}{a^2} I_{n-2} - \frac{n(n-1)}{a^2} I_n - \frac{n}{a^2} I_n$$

$$\Rightarrow \left(1 + \frac{n(n-1)}{a^2} + \frac{n}{a^2}\right) I_n = \frac{1}{a} + \frac{n(n-1)}{a^2} I_{n-2}$$

$$\Rightarrow \frac{a^2 + n^2 \cancel{-n} + \cancel{n}}{a^2} I_n = \frac{1}{a} + \frac{n(n-1)}{a^2} I_{n-2}$$

$$\Rightarrow (a^2+n^2) I_n = a + n(n-1) I_{n-2}$$

$$\therefore\ I_n = \frac{a}{a^2+n^2} + \frac{n(n-1)}{a^2+n^2} I_{n-2}$$

This is the reduction formula.

4. Obtain the reduction formula for $\int \sin^m x \cos^n x\, dx$.

Sol^n: Let, $I_{m,n} = \int \sin^m x \cos^n x\, dx$

$$\therefore\ I_{m,n} = \int \cos^{n-1} x\, (\sin^m x \cos x)\, dx$$

$$= \cos^{n-1} x \frac{\sin^{m+1} x}{m+1} - \int \left\{ (n-1) \cos^{n-2} x . (-\sin x) . \frac{\sin^{m+1} x}{m+1} \right\} dx$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \cos^{n-2} x\, (1-\cos^2 x) \sin^n x\, dx$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} . I_{m,n-2} - \frac{n-1}{m+1} I_{m,n}$$

$$\Rightarrow \left(1+\frac{n-1}{m+1}\right) I_{m,n} = \frac{\cos^{n-1}x \sin^{m+1}x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$$

$$\therefore (m+n) I_{m,n} = \cos^{n-1}x \sin^{m+1}x + (n-1) I_{m,n-2}$$

$$\therefore I_{m,n} = \frac{\cos^{n-1}x \sin^{m+1}x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}$$

This is the reduction formula.

2nd way

Sol^n: let $I_{m,n} = \int \sin^m x \cos^n x \, dx$

$$= \int \sin^{m-1}x \, (\cos^n x \sin x) \, dx$$

$$= -\sin^{m-1}x \frac{\cos^{n+1}x}{n+1} - \int \left\{ (m-1)\sin^{m-2}x \cdot \cos x \cdot \frac{-\cos^{n+1}x}{n+1} \right\} dx$$

$$= -\frac{\sin^{m-1}x \cos^{n+1}x}{n+1} + \frac{m-1}{n+1} \int \sin^{m-2}x \, (1-\sin^2 x) \cos^n x \, dx$$

$$= -\frac{\sin^{m-1}x \cos^{n+1}x}{n+1} + \frac{m-1}{n+1} I_{m-2,n} - \frac{m-1}{n+1} I_{m,n}$$

$$\Rightarrow \left(1+\frac{m-1}{n+1}\right) I_{m,n} = -\frac{\sin^{m-1}x \cos^{n+1}x}{n+1} + \frac{m-1}{n+1} \cdot I_{m-2,n}$$

$$\Rightarrow (m+n) I_{m,n} = -\sin^{m-1}x \cos^{n+1}x + (m-1) I_{m-2,n}$$

$$\therefore I_{m,n} = -\frac{\sin^{m-1}x \cos^{n+1}x}{m+n} + \frac{m-1}{m+n} I_{m-2,n}$$

This is the reduction formula

Q. Obtain the reduction formula for $\int_0^{\pi/2} \sin^m x \cos^n x\, dx$

Solⁿ: Let $I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x\, dx$

$$= \int_0^{\pi/2} \cos^{n-1} x\, (\sin^m x \cos x)\, dx$$

$$= \left[\cos^{n-1} x \; \frac{\sin^{m+1} x}{m+1} \right]_0^{\pi/2} - \int_0^{\pi/2} \left\{ (n-1) \cos^{n-2} x \cdot (-\sin x) \right\} \frac{\sin^{m+1} x}{m+1}$$

$$= \frac{n-1}{m+1} \int_0^{\pi/2} \cos^{n-2} x\, (1-\cos^2 x) \sin^m x\, dx$$

$$= \frac{n-1}{m+1} \cdot I_{m,n-2} - \frac{n-1}{m+1} I_{m,n}$$

$$\therefore \left(1 + \frac{n-1}{m+1}\right) I_{m,n} = \frac{n-1}{m+1} I_{m,n-2}$$

$$\Rightarrow (m+n) I_{m,n} = (n-1) I_{m,n-2}$$

$$\therefore I_{m,n} = \frac{n-1}{m+n} I_{m,n-2}$$

This is the reduction formula.

—x—

2nd way

Let, $I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x \, dx$

$$= \int_0^{\pi/2} \sin^{m-1} x \, (\cos^n x \sin x) \, dx$$

$$= \left[\sin^{m-1} x \left(-\frac{\cos^{n+1} x}{n+1} \right) \right]_0^{\pi/2} + \int_0^{\pi/2} \left\{ (m-1) \sin^{m-2} x \cdot (\cos x) \frac{\cos^{n+1} x}{n+1} \right\} dx$$

$$= 0 + \frac{m-1}{n+1} \int_0^{\pi/2} \sin^{m-2} x \, (1 - \sin^2 x) \cos^n x \, dx$$

$$= \frac{m-1}{n+1} I_{m-2,n} - \frac{m-1}{n+1} I_{m,n}$$

$$\therefore \left(1 + \frac{m-1}{n+1}\right) I_{m,n} = \frac{m-1}{n+1} I_{m-2,n}$$

$$\Rightarrow (m+n) I_{m,n} = (m-1) I_{m-2,n}$$

$$\therefore I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}$$

This is the reduction formula.

4. Obtain the reduction formula for $\int \frac{\sin^m x}{\cos^n x}\, dx$, $(n \neq 1)$.

<u>Sol^n:</u> Let, $I_{m,n} = \int \sin^m x \cos^{-n} x \, dx$

$$= \int (\sin^m x \cos x) \cos^{-n-1} x \, dx$$

$$= \frac{\sin^{m+1} x}{m+1} \cdot \cos^{-n-1} x - \int (-n-1) \cos^{-n-1-1} x \cdot (-\sin x) \cdot \frac{\sin^{m+1} x}{m+1} \, dx$$

$$= \frac{\sin^{m+1} x}{(m+1)\cos^{n+1} x} - \frac{n+1}{m+1} \int \frac{1}{\cos^{n+2} x} \cdot (1-\cos^2 x) \sin^m x \, dx$$

$$= \frac{\sin^{m+1} x}{(m+1)\cos^{n+1} x} - \frac{n+1}{m+1} I_{m,n+2} + \frac{n+1}{m+1} I_{m,n}$$

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$$\Rightarrow \frac{n+1}{m+1} I_{m,n+2} = \frac{\sin^{m+1} x}{(m+1)\cos^{n+1} x} + \left(\frac{n+1}{m+1} - 1\right) I_{m,n}$$

$$\Rightarrow (n+1) I_{m,n+2} = \frac{\sin^{m+1} x}{\cos^{n+1} x} + (n-m) I_{m,n}$$

$$\Rightarrow I_{m,n+2} = \frac{\sin^{m+1} x}{(n+1)\cos^{n+1} x} + \frac{n-m}{n+1} I_{m,n}$$

Replace n by $n-2$, then we have

$$I_{m,n} = \frac{1}{n-1} \frac{\sin^{n+1} x}{\cos^{n-1} x} - \frac{m-n+2}{n-1} I_{m,n-2}$$

Obtain the reduction formula for $\int \frac{\sin^m x}{\cos^n x} dx$, $(n \neq 1)$.

Soln: Let, $I_{m,n} = \int \frac{\sin^m x}{\cos^n x} dx = \int \sin^m x \cos^{-n} x \, dx$

Consider $I'_{p,q} = \int \sin^p x \cos^q x \, dx$

$$= \frac{\sin^{p+1} x \cos^{q-1} x}{p+q} + \frac{q-1}{p+q} I'_{p,q-2}$$

Replace q by $q+2$

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Using the reduction formula for $\int \sin^m x \cos^n x \, dx$ evaluate $\int_0^{\pi/2} \sin^4 x \cos^5 x \, dx$.

Soln: Let, $I_{m,n} = \int \sin^m x \cos^n x \, dx$, then we have

$$I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}$$

And also if $I'_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x \, dx$ then we have

$$I'_{m,n} = \left[\frac{\cos^{n-1} x \sin^{m+1} x}{m+n} \right]_0^{\pi/2} + \frac{n-1}{m+n} \cdot I'_{m,n-2}$$

$$= \frac{n-1}{m+1} \cdot I'_{m,n-2}$$

If $m=4$ and $n=5$ then we have

$$I'_{m,n} = I'_{4,5} = \frac{5-1}{4+1} I'_{4,5-2} = \frac{4}{5} \cdot I'_{4,3} = \frac{4}{5} \cdot \frac{3}{5} \cdot I'_{4,1}$$

$$= \frac{4}{5} \cdot \frac{3}{5} \cdot \int_0^{\pi/2} \sin^4 x \cos x \, dx$$

$$= \frac{12}{25} \left[\frac{\sin^5 x}{5} \right]_0^{\pi/2}$$

$$= \frac{12}{25} \left[\frac{1}{5} - 0 \right]$$

$$= \frac{12}{75}$$

Obtain a reduction formula for $\int \frac{\sin^m x}{\cos^n x}\,dx, (n \neq 1)$.

Sol^n: Let, $I_{m,n} = \int \frac{\sin^m x}{\cos^n x}\,dx = \int \sin^m x \cos^{-n} x\,dx$

$$= \int (\sin^m x \cos x) \cos^{-(n+1)} x\,dx$$

$$= \frac{\sin^{m+1} x}{m+1} \cdot \cos^{-(n+1)} x - \int \{-(n+1)\} \cos^{-(n+1)-1} x \cdot (-\sin x) \frac{\sin^{m+1} x}{m+1}\,dx$$

$$= \frac{1}{m+1} \cdot \frac{\sin^{m+1} x}{\cos^{n+1} x} - \frac{n+1}{m+1} \int \cos^{-n-2} x \,(1-\cos^2 x) \sin^m x\,dx$$

$$= \frac{1}{m+1} \cdot \frac{\sin^{m+1} x}{\cos^{n+1} x} - \frac{n+1}{m+1} \left[\int \frac{\sin^m x}{\cos^{n+2} x}\,dx - \int \frac{\sin^m x}{\cos^n x}\,dx \right]$$

$$= \frac{1}{m+1} \cdot \frac{\sin^{m+1} x}{\cos^{n+1} x} - \frac{n+1}{m+1} I_{m,n+2} + \frac{n+1}{m+1} I_{m,n}$$

$$\Rightarrow \frac{n+1}{m+1} I_{m,n+2} = \frac{1}{m+1} \cdot \frac{\sin^{m+1} x}{\cos^{n+1} x} + \left(\frac{n+1}{m+1} - 1 \right) I_{m,n}$$

$$\Rightarrow I_{m,n+2} = \frac{1}{n+1} \cdot \frac{\sin^{m+1} x}{\cos^{n+1} x} + \frac{n-m}{n+1} I_{m,n}$$

Replacing n by $n-2$, we get,

$$I_{m,n} = \frac{1}{n-2+1} \frac{\sin^{m+1} x}{\cos^{n-2+1} x} + \frac{n-2-m}{n-2+1} I_{m,n-2}$$

$$\therefore I_{m,n} = \frac{1}{n-1} \cdot \frac{\sin^{m+1} x}{\cos^{n-1} x} - \frac{m-n+2}{n-1} I_{m,n-2}$$

This is the reduction formula.

H.W. Obtain the reduction formula for $\int \frac{dx}{\sin^m x \cos^n x}$, $(n \neq 1)$.

$\boxed{\text{II}}$ Obtain a reduction formula for $\int \cos^m x \cos nx\, dx$ and hence evaluate $\int \cos^3 x \cos 7x\, dx$.

<u>Solⁿ:</u> Let $I_{m,n} = \int \cos^m x \cos nx\, dx$

$$= \frac{\cos^m x \sin nx}{n} - \int (m \cancel{-1}) \cos^{m-1} x(-\sin x) \frac{\sin nx}{n}\, dx$$

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n} \int \cos^{m-1} x \{\sin nx \sin x\}\, dx$$

Since $\cos(nx - x) = \cos nx \cos x + \sin nx \sin x$

$$\Rightarrow \sin nx \sin x = \cos(n-1)x - \cos nx \cos x$$

$$\therefore I_{m,n} = \frac{\cos^m x \sin nx}{n} + \frac{m}{n} \int \cos^{m-1} x \{\cos(n-1)x - \cos nx \cos x\} dx$$

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n} I_{m-1,n-1} - \frac{m}{n} I_{m,n}$$

$$\Rightarrow \left(1 + \frac{m}{n}\right) I_{m,n} = \frac{\cos^m x \sin nx}{n} + \frac{m}{n} I_{m-1,n-1}$$

$$\Rightarrow (m+n) I_{m,n} = \cos^m x \sin nx + m I_{m-1,n-1}$$

$$\therefore I_{m,n} = \frac{\cos^m x \sin nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$$

This is the reduction formula.

$$I_{m,n} = \int \cos^m x \cos nx\,dx = \frac{\cos^m x \sin nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$$

Here $I_{3,7} = \int \cos^3 x \cos 7x\,dx$

$$= \frac{\cos^3 x \sin 7x}{3+7} + \frac{3}{3+7} I_{2,6}$$

$$= \frac{1}{10}\cos^3 x \sin 7x + \frac{3}{10}\left[\frac{\cos^2 x \sin 6x}{2+6} + \frac{2}{2+6} I_{1,5}\right]$$

$$= \frac{1}{10}\cos^3 x \sin 7x + \frac{3}{80}\cos^2 x \sin 6x + \frac{6}{80}\left[\frac{\cos x \sin 5x}{1+5} + \frac{1}{1+5} I_{0,4}\right]$$

$$= \frac{1}{10}\cos^3 x \sin 7x + \frac{3}{80}\cos^2 x \sin 6x + \frac{1}{80}\cos x \sin 5x + \frac{1}{80}\int \sin 4x\,dx$$

$$= \frac{1}{10}\cos^3 x \sin 7x + \frac{3}{80}\cos^2 x \sin 6x + \frac{1}{80}\cos x \sin 5x - \frac{1}{320}\cos 4x + c$$

where c is the integrating constant.

—x—

Use reduction formula to find the value of $\int \sin^6 x \cos^8 x\,dx$.

Sol^n: If $I_{m,n} = \int \sin^m x \cos^n x\,dx$ then

$$I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}$$

Now, $I_{6,8} = \int \sin^6 x \cos^8 x\,dx$,

Using the reduction formula, we have,

$$I_{6,8} = \frac{\cos^7 x \sin^7 x}{6+8} + \frac{7}{6+8} \cdot I_{6,6}$$

$$= \frac{1}{14} \cos^7 x \sin^7 x + \frac{7}{14}\left[\frac{\cos^{6-1} x \sin^{6+1} x}{6+6} + \frac{6-1}{6+6} I_{6,4}\right]$$

$$= \frac{1}{14} \cos^7 x \sin^7 x + \frac{7}{168} \cos^5 x \sin^7 x + \frac{7.5}{168}\left[\frac{\cos^{4-1} x \sin^{6+1} x}{6+4} + \frac{4-1}{6+4} I_{6,2}\right]$$

$$= \frac{1}{14} \cos^7 x \sin^7 x + \frac{\cancel{7}^{\,1}}{\cancel{168}_{\,24}} \cos^5 x \sin^7 x + \frac{\cancel{7}^{\,1}}{\cancel{336}_{\,48}} \cos^3 x \sin^7 x + \cancel{\;7\;} + \frac{\cancel{7}^{\,1}}{\cancel{112}_{\,16}}\left[\frac{\cos x \sin^7 x}{6+2} + \frac{2-1}{6+2} I_{6,0}\right]$$

$$= \frac{1}{14} \cos^7 x \sin^7 x + \frac{1}{24} \cos^5 x \sin^7 x + \frac{1}{48} \cos^3 \sin^7 x + \frac{1}{128} \cos x \sin^7 x + \frac{1}{128} I_{6,0}$$

Now, $I_{6,0} = \int \sin^6 x \, dx$

Again
We know that if $I_n = \int \sin^n x \, dx$, then

$$I_n = -\frac{\sin^{n-1}x \cos x}{n} + \frac{n-1}{n} I_{n-2}$$

Here $I_6 = \int \sin^6 x \, dx$, then using the reduction formula we have,

$$I_6 = -\frac{\sin^5 x \cos x}{6} + \frac{6-1}{6} I_4$$

$$= -\frac{1}{6}\sin^5 x \cos x + \frac{5}{6}\left[-\frac{\sin^3 x \cos x}{4} + \frac{4-1}{4} I_2\right]$$

$$= -\frac{1}{6}\sin^5 x \cos x - \frac{5}{24}\sin^3 x \cos x + \frac{5\cdot\cancel{3}}{\cancel{6}\cdot 4}\left[-\frac{\sin x \cos x}{2} + \frac{2-1}{2} I_0\right]$$

$$= -\frac{1}{6}\sin^5 x \cos x - \frac{5}{24}\sin^3 x \cos x - \frac{5}{16}\sin x \cos x + \frac{5}{16}\int dx$$

$$= -\frac{1}{6}\sin^5 x \cos x - \frac{5}{24}\sin^3 x \cos x - \frac{5}{16}\sin x \cos x + \frac{5}{16}x + c_1$$

$$\therefore I_{6,8} = \frac{1}{14}\cos^7 x\sin^7 x + \frac{1}{24}\cos^5 x\sin^7 x + \frac{1}{48}\cos^3 x\sin^7 x$$

$$+\frac{1}{128}\cos x\sin^7 x - \frac{1}{768}\sin^5 x\cos x - \frac{5}{3072}\sin^3 x\cos x$$

$$-\frac{5}{2048}\sin x\cos x + \frac{5}{2048}x + c \qquad \left[c = \frac{1}{128}c_1\right]$$

Gamma and Beta function

Beta function:

The first Eulerian integral or Beta function is denoted by $\beta(m,n)$ and is defined by

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx,$$

where $m>0$ and $n>0$.

Gamma function:

The second Eulerian integral or Gamma function is denoted by Γn and is defined by

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx, \quad \text{where } n>0.$$

Show that $\beta(m,n) = \beta(n,m)$.

<u>Proof:</u> We have $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

Put $1-x = z$ then $x = 1-z$ and $dx = -dz$

x	0	1
z	1	0

$$\therefore \beta(m,n) = \int_1^0 (1-z)^{m-1} z^{n-1}(-dz)$$

$$= \int_0^1 (1-z)^{m-1} z^{n-1} dz \qquad \left| \because \int_a^b f(x)dx = -\int_b^a f(x)dx \right.$$

$$= \int_0^1 x^{n-1}(1-x)^{m-1}dx \qquad \left| \because \int_a^b f(z)dz = \int_a^b f(x)dx \right.$$

$$= \beta(n,m)$$

<u>Proved</u>

Show that $\Gamma 1 = 1$.

Proof: We know that $\Gamma n = \int_0^\infty e^{-x}x^{n-1}dx$

$$\therefore \Gamma 1 = \int_0^\infty e^{-x}x^{1-1}dx = \left[-e^{-x}\right]_0^\infty = -\left[\frac{1}{e^x}\right]_0^\infty$$

$$= -[0-1] = 1$$ <u>proved</u>

(iii) Show that $\Gamma(n+1) = n\Gamma(n)$. If n is a positive integer then show that $\Gamma(n+1) = n!$.

Proof: We know that $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$

$$\therefore \Gamma(n+1) = \int_0^\infty e^{-x} x^{n+1-1} dx = \int_0^\infty e^{-x} x^n dx$$

$$= \left[-e^{-x} . x^n\right]_0^\infty - \int_0^\infty n x^{n-1}(-e^{-x}) dx$$

$$= 0 + n\int_0^\infty x^{n-1} e^{-x} dx \qquad \Big| \because \lim_{x\to\infty} \frac{x^n}{e^x} = 0$$

$$= n\int_0^\infty e^{-x} x^{n-1} dx$$

$$= n\Gamma(n)$$

Proved

2nd part: If n is a positive integer, then we have,

$$\Gamma(n+1) = n\Gamma(n)$$
$$= n(n-1)\Gamma(n-1)$$
$$= n(n-1)(n-2)\Gamma(n-2)$$
$$= n(n-1)(n-2) \cdots\cdots 2.1\,\Gamma(1)$$
$$= n(n-1)(n-2) \cdots\cdots 2.1 \qquad [\because \Gamma(1) = 1]$$
$$= n!$$

(iv) Show that $\int_0^\infty e^{-kx} x^{n-1} dx = \frac{\Gamma n}{k^n}$; $k>0, n>0$.

Proof: We know that $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$

Let, $I = \int_0^\infty e^{-kx} x^{n-1} dx$

Put $kx = z$ $\therefore$ $dx = \frac{1}{k} dz$

x	0	∞
z	0	∞

$$\therefore I = \int_0^\infty e^{-z} \left(\frac{z}{k}\right)^{n-1} \cdot \frac{1}{k} dz$$

$$= \frac{1}{k^n} \int_0^\infty e^{-z} z^{n-1} dz$$

$$= \frac{1}{k^n} \int_0^\infty e^{-x} x^{n-1} dx$$

$$= \frac{\Gamma n}{k^n}$$

Proved

$$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

Proof: We know that

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Put, $x = \frac{1}{1+y}$ $\therefore dx = \frac{-1}{(1+y)^2}dy$

$\therefore 1+y = \frac{1}{x} \Rightarrow y = \frac{1-x}{x}$

$x = 0 \Rightarrow y = \infty$

$x = 1 \Rightarrow y = 0$

$$\therefore \beta(m,n) = \int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1}\left(1-\frac{1}{1+y}\right)^{n-1}\frac{-1}{(1+y)^2}\,dy$$

$$= \int_0^\infty \frac{1}{(1+y)^{m-1}} \cdot \frac{y^{n-1}}{(1+y)^{n-1}\cdot(1+y)^2}\,dy$$

$$= \int_0^\infty \frac{y^{n-1}\,dy}{(1+y)^{m+n}}$$

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

We know that $\beta(m,n) = \beta(n,m) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

$$\int_0^\infty e^{-kx} x^{n-1}\, dx = \frac{\Gamma n}{k^n} \qquad (176)$$

Show that $\beta(m,n) = \dfrac{\Gamma m\,\Gamma n}{\Gamma m+n}$

<u>Proof:</u> We know that

$$\frac{\Gamma n}{z^n} = \int_0^\infty e^{-zx} x^{n-1}\, dx$$

$$\Rightarrow \Gamma n = z^n \int_0^\infty e^{-zx} x^{n-1}\, dx$$

Multiply both sides by $e^{-z} z^{m-1}$ and integrating w.r.t z from 0 to ∞, we get,

$$\Gamma n \int_0^\infty e^{-z} z^{m-1}\, dz = \int_0^\infty x^{n-1}\, dx \int_0^\infty e^{-zx} . e^{-z} . z^{m+n-1}\, dz$$

$$\Rightarrow \Gamma n\,\Gamma m = \int_0^\infty x^{n-1}\, dx \int_0^\infty e^{-z(x+1)} z^{m+n-1}\, dz$$

$$= \int_0^\infty x^{n-1} \frac{\Gamma m+n}{(x+1)^{m+n}}\, dx$$

$$\Rightarrow \frac{\Gamma m\,\Gamma n}{\Gamma m+n} = \int_0^\infty \frac{x^{n-1}}{(x+1)^{m\ldots}}$$

$= \beta(m \ldots$

$\ldots x$

Hence $\beta(m, \ldots$

$$\beta(m,n) = \beta(n,m) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$$

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

<u>Proof:</u> We know that, $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

Put $x = \sin^2\theta \quad \therefore \theta = \sin^{-1}(\sqrt{x})$

$dx = 2\sin\theta\cos\theta \, d\theta$

when, $x = 0$, $\theta = 0$

when $x = 1$, $\theta = \frac{\pi}{2}$

$$\beta(m,n) = \int_0^{\pi/2} \sin^{2m-2}\theta \,(1-\sin^2\theta)^{n-1} \cdot 2\sin\theta\cos\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

—x—

$$\Gamma\tfrac{1}{2} = \sqrt{\pi}$$

<u>Proof:</u> We know that $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

Put, $m = n = \frac{1}{2}$, then we have

$$\beta\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = 2\int_0^{\pi/2} \sin^0\theta \cos^0\theta \, d\theta$$

$$\Rightarrow \frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = 2\,[\theta]_0^{\pi/2} \Rightarrow \left(\Gamma\tfrac{1}{2}\right)^2 = \pi \quad [\Gamma 1 = 1$$

$\therefore \Gamma\frac{1}{2} = \sqrt{\pi}$.

—x—

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}$$

Proof: We know that, $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

Put, $2m-1 = p$ and $2n-1 = q$ then we have,

$m = \frac{p+1}{2}$ and $n = \frac{q+1}{2}$

$$\therefore \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$$

$$\Rightarrow \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2} \cdot \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{\Gamma\frac{p+q+2}{2}} \qquad \left[\because \beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}\right]$$

Put ~~a~~ $q = 0$

$$\int_0^{\pi/2} \sin^p\theta \, d\theta = \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{1}{2}}{2\,\Gamma\frac{p+2}{2}} = \frac{\sqrt{\pi}}{2} \frac{\Gamma\frac{p+1}{2}}{\Gamma\frac{p+2}{2}}$$

Show that $\int_0^\infty e^{-x^2} dx = \frac{1}{2}\sqrt{\pi}$

Proof: ~~px^2~~ Put $x^2 = z$ $\therefore$ ~~$2x\,dx = dz$~~ $x = \sqrt{z}$ $\therefore dx = \frac{1}{2\sqrt{z}} dz$

x	0	∞
z	0	∞

$$\therefore I = \int_0^\infty e^{-z} \cdot \cancel{2x} \frac{1}{2\sqrt{z}} dz$$

$$= \frac{1}{2}\int_0^\infty e^{-z} z^{-\frac{1}{2}} dz = \frac{1}{2}\Gamma\left(-\tfrac{1}{2}+1\right)$$

$$= \frac{1}{2}\Gamma\tfrac{1}{2} = \frac{1}{2}\sqrt{\pi}$$

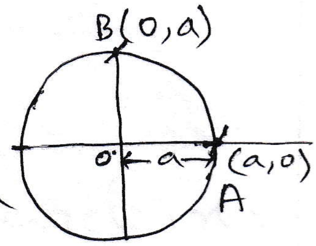
Arc length:

Formula $$s = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$s = \int_{y_1}^{y_2} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\, dt$$

☐ Find the perimeter of the circle $x^2+y^2=a^2$.



Sol^n:

We know that $s = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

Given, $x^2+y^2=a^2$

$\therefore\ 2x + 2y\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$

Now, perimeter of the circle $= 4\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$= 4\int_0^a \sqrt{1+\frac{x^2}{y^2}}\, dx$

$$= 4\int_0^a \frac{\sqrt{x^2+y^2}}{y}\,dx$$

$$= 4\int_0^a \frac{a}{\sqrt{a^2-x^2}}\,dx \qquad \left|\; \begin{aligned} &x^2+y^2=a^2 \\ &\therefore y=\sqrt{a^2-x^2} \end{aligned}\right.$$

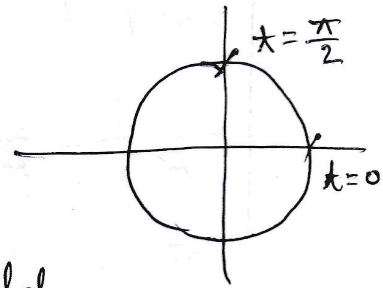
$$= 4a\int_0^a \frac{dx}{\sqrt{a^2-x^2}}$$

$$= 4a\left[\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= 4a\left[\frac{\pi}{2}-0\right]$$

$$= 2\pi a$$

☐ Find the perimeter of the circle $x = a\cos t$, $y = a\sin t$.



Sol^n: The perimeter of the circle is given by, $4\int_{t=0}^{t=\pi/2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

~~= 4~~ Here, $x = a\cos t$ and $y = a\sin t$

$\therefore \frac{dx}{dt} = -a\sin t$ and $\frac{dy}{dt} = a\cos t$

$\therefore$ Perimeter $= 4\int_0^{\pi/2}\sqrt{a^2\sin^2 t + a^2\cos^2 t}\,dt$

$= 4\int_0^{\pi/2} a\, dt$

$= 4a\,[t]_0^{\pi/2}$

$= 4a\,[\frac{\pi}{2} - 0]$

$= 2\pi a.$

☐ Prove that the arc length of the curve $y = \log \sec x$ from $x=0$ to $x = \frac{\pi}{3}$ is $\log_e(2+\sqrt{3})$.

Proof: We know that, ~~the~~

arc length $= \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

Here $x_1 = 0$, $x_2 = \frac{\pi}{3}$, $y = \log \sec x$

$\therefore \frac{dy}{dx} = \frac{1}{\sec x} \cdot \sec x \tan x$

$= \tan x$

$\therefore$ required Arc length $= \int_0^{\pi/3} \sqrt{1 + \tan^2 x}\, dx$

$= \int_0^{\pi/2} \sec x\, dx$

$= \left[\log(\sec x + \tan x)\right]_0^{\pi/3}$

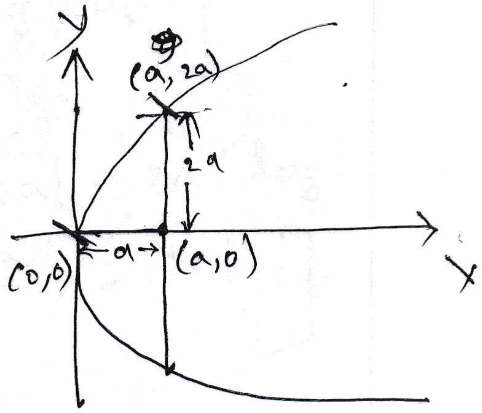
$$= \log\left(\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right) - \log(\sec 0 + \tan 0)$$

$$= \log(2+\sqrt{3}) - \log 1$$

$$= \log(2+\sqrt{3}) \qquad [\because \log 1 = 0]$$

Proved

▣ Find the length of the parabola $y^2 = 4ax$ from the vertex to an extremity of the latus rectum.



Sol^n: We know that

$$s = \int_{y_1}^{y_2} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

Here, $y_1 = 0$, $y_2 = 2a$, $y^2 = 4ax$

differentiating w.r.t. y we get

$$2y = 4a\frac{dx}{dy} \Rightarrow \frac{dx}{dy} = \frac{y}{2a}$$

$$\therefore s = \int_0^{2a} \sqrt{1+\left(\frac{y}{2a}\right)^2}\, dy$$

(183) $\int \sqrt{x^2+a^2}\,dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|$

$$= \frac{1}{2a}\int_0^{2a}\sqrt{(2a)^2+y^2}\,dy$$

$$= \frac{1}{2a}\left[\frac{y\sqrt{y^2+4a^2}}{2} + \frac{4a^2}{2}\log\left|y+\sqrt{y^2+4a^2}\right|\right]_0^{2a}$$

$$= \frac{1}{2a}\left[\frac{\cancel{2}a\sqrt{8a^2}}{\cancel{2}} + \frac{4a^2}{2}\log\left|2a+\sqrt{8a^2}\right| - \left(0 + 2a^2\log\left|0+\sqrt{0+4a^2}\right|\right)\right]$$

$$= \frac{1}{2a}\left[2\sqrt{2}a^2 + 2a^2\log\left|2a(1+\sqrt{2})\right| - 2a^2\log|2a|\right]$$

$$= \frac{1}{2a}\left[2\sqrt{2}a^2 + \cancel{2a^2\log|2a|} + 2a^2\log|1+\sqrt{2}| - \cancel{2a^2\log|2a|}\right]$$

$$= \sqrt{2}a + a\log|1+\sqrt{2}|$$

$$= a\left(1+\log|1+\sqrt{2}|\right)$$

4 Show that the length of the arc of the parabola $y^2=4ax$ which is intercepted between the points of intersection of the parabola and the straight line $3y=8x$ is $a(\log 2+15/16)$.

Proof: Given,

$$y^2=4ax \quad \text{——— (1)}$$

$$\text{and } 3y=8x \quad \text{——— (2)}$$

From (2) we have,

$$y=\frac{8}{3}x \quad \text{——— (3)}$$

Using (3) in (1) we get,

$$\left(\frac{8}{3}x\right)^2=4ax$$

$$\Rightarrow \frac{64}{9}x^2-4ax=0$$

$$\Rightarrow 4x\left(\frac{16}{9}x-a\right)=0$$

$$\Rightarrow x=0 \quad \text{or} \quad x=\frac{9a}{16}$$

$$\therefore y=\frac{8}{3}x=0 \quad \text{or} \quad y=\frac{8}{3}x=\frac{\cancel{8}^{\,1}}{\cancel{3}}\cdot\frac{\cancel{9}^{\,3}a}{\cancel{16}_{\,2}}=\frac{3a}{2}$$

$\therefore$ (1) and (2) meets at $(0,0)$ and $\left(\frac{9a}{16},\frac{3a}{2}\right)$

~~∴ The required~~

Differentiating ① w.r.t y we get,

$$2y = 4a \frac{dx}{dy}$$

$$\Rightarrow \frac{dx}{dy} = \frac{y}{2a}$$

∴ The required arc length

$$= \int_{y=0}^{y=\frac{3a}{2}} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_0^{\frac{3a}{2}} \sqrt{1+\frac{y^2}{4a^2}}\, dy$$

$$= \frac{1}{2a}\int_0^{\frac{3a}{2}} \sqrt{(2a)^2+y^2}\, dy$$

$$= \frac{1}{2a}\left[\frac{y\sqrt{y^2+4a^2}}{2} + \frac{\cancel{4}a^2}{\cancel{2}}\log\left|y+\sqrt{y^2+4a^2}\right|\right]_0^{\frac{3a}{2}}$$

$$= \frac{1}{2a}\left[\frac{\frac{3a}{2}\sqrt{\frac{9a^2}{4}+4a^2}}{2} + 2a^2\log\left|\frac{3a}{2}+\sqrt{\frac{9a^2}{4}+4a^2}\right| - \left(0+2a^2\log\left|0+\sqrt{4a^2}\right|\right)\right]$$

$$= \frac{1}{2a}\left[\frac{\frac{3a}{2}\cdot\frac{5a}{2}}{2} + 2a^2\log\left|\frac{3a}{2}+\frac{5a}{2}\right| - 2a^2\log|2a|\right]$$

$$= \frac{1}{2a}\left[\frac{15a^2}{8} + 2a^2\log|2| + \cancel{2a^2\log|2a|} - \cancel{2a^2\log|2a|}\right]$$

$$= \frac{15a}{16} + a\log 2$$

$$= a\left(\frac{15}{16} + \log 2\right)$$

$$= a\left(\log 2 + \frac{15}{16}\right)$$

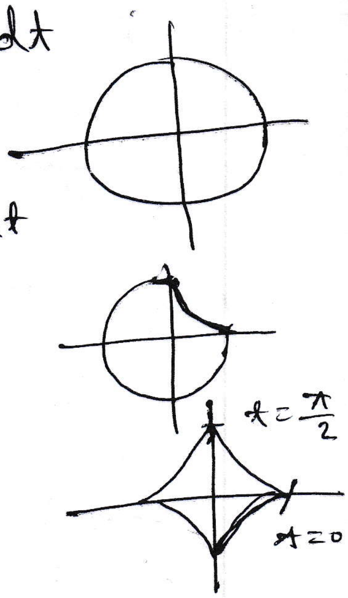
Proved

Prove that the whole length of the curve $x = a\cos^3 t$, $y = a\sin^3 t$ $\left(x^{2/3} + y^{2/3} = (a)^{2/3}\right)$ is $6a$.

or

Find the perimeter of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$.

Solⁿ: Perimeter $= 4\int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$



$= 4\int_0^{\pi/2} \sqrt{\left(-3a\cos^2 t \sin t\right)^2 + \left(3a \sin^2 t \cos t\right)^2}\, dt$

$= 12a \int_0^{\pi/2} \sqrt{\sin^2 t \cos^2 t \left(\cos^2 t + \sin^2 t\right)}\, dt$

$= 6a \int_0^{\pi/2} 2\sin t \cos t \, dt$

$= 6a \int_0^{\pi/2} \sin 2t \, dt$

$= 6a \left[-\frac{\cos 2t}{2}\right]_0^{\pi/2}$

$= 3a \left[-(-1) + (1)\right]$

$= 3a(2)$

$= 6a$

\# For the second question

Solⁿ We know that the parametric equation of

4. Show that the whole length of the curve $8a^2y^2 = x^2(a^2-x^2)$ is $\sqrt{2}\pi a$.

Sol^n: Given $8a^2y^2 = x^2(a^2-x^2)$ ——— (1)

Here the power of both x and y is even, therefore the curve is symmetric ~~wrt~~ w.r.t both x and y-

Put $y=0$ in (1), Then we have,

$$x^2(a^2-x^2) = 0$$

$$\Rightarrow x = 0, a, -a.$$

$\therefore$ The curve (1) cuts x-axis at $x=0$, $x=a$ and $x=-a$.

Now, put $x=0$ in (1), then we have,

$$8a^2y^2 = 0$$

$$\Rightarrow y = 0$$

~~Thus the~~

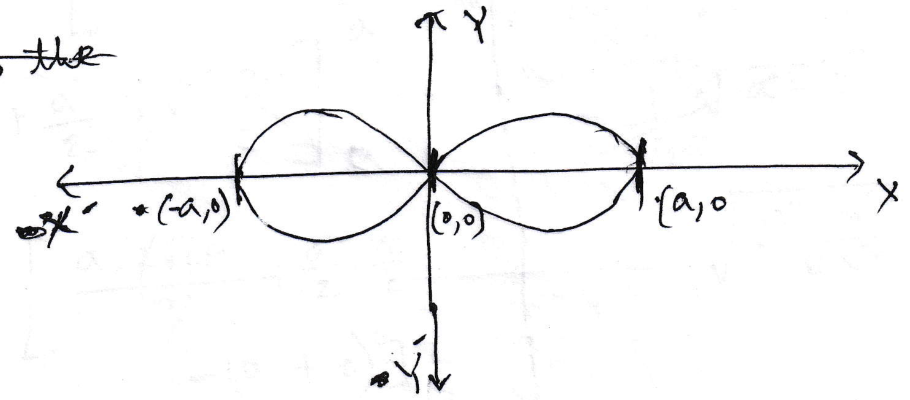


Figure-1

$$8a^2y^2 = x^2(a^2-x^2)$$

$$\therefore y = \sqrt{\frac{x^2(a^2-x^2)}{8a^2}}$$

The whole length of the curve (1) is,

$$s = 4\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Differentiating (1), w.r.t. x we get,

$$8a^2\cdot 2y\frac{dy}{dx} = 2a^2x - 4x^3 = \frac{\cancel{2}x(a^2-2x^2)}{\cancel{16}a^2y}$$

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$$= \frac{x(a^2-2x^2)}{8a^2\sqrt{\frac{x^2(a^2-x^2)}{8a^2}}}$$

$$= \frac{2\sqrt{2}\,ax(a^2-2x^2)}{8a^2x\sqrt{a^2-x^2}}$$

$$= \frac{a^2-\cancel{x^2}-x^2}{\sqrt{2}a\sqrt{\cancel{a^2}-x^2}}$$

$$= \frac{a^2-2x^2}{2\sqrt{2}\,a\sqrt{a^2-x^2}}$$

$$\therefore s = 4\int_0^a \sqrt{1+\frac{(a^2-2x^2)^2}{8a^2(a^2-x^2)}}\,dx$$

$$= 4\int_0^a \sqrt{\frac{8a^4-8a^2x^2+a^4-4a^2x^2+4x^4}{8a^2(a^2-x^2)}}\,dx$$

$$= 4\int_0^a \sqrt{\frac{9a^4 - 12a^2x^2 + 4x^4}{8a^2(a^2-x^2)}}\,dx$$

$$= \frac{4}{2\sqrt{2}\,a}\int_0^a \frac{3a^2 - 2x^2}{\sqrt{a^2-x^2}}\,dx$$

$$= \frac{12a^2}{2\sqrt{2}\,a}\int_0^a \frac{dx}{\sqrt{a^2-x^2}} + \frac{8}{2\sqrt{2}\,a}\int_0^a \frac{x^2}{\sqrt{a^2-x^2}}\,dx$$

$$= 3\sqrt{2}\,a\left[\sin^{-1}\frac{x}{a}\right]_0^a + \frac{2\sqrt{2}}{a}\int_0^a a^2$$

$$= \frac{\sqrt{2}}{a}\int_0^a \frac{a^2 + 2(a^2-x^2)}{\sqrt{a^2-x^2}}\,dx$$

$$= \sqrt{2}\,a\int_0^a \frac{dx}{\sqrt{a^2-x^2}} + \frac{2\sqrt{2}}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

$$= \sqrt{2}a\left[\sin^{-1}\frac{x}{a}\right]_0^a + \frac{2\sqrt{2}}{a}\left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \sqrt{2}a\left[\frac{\pi}{2} - 0\right] + \frac{2\sqrt{2}}{a}\left[\left(0 + \frac{a^2}{2}\cdot\frac{\pi}{2}\right) - (0+0)\right]$$

$$= \frac{\sqrt{2}\pi a}{2} + \frac{\sqrt{2}\pi a}{2}$$

$$= \sqrt{2}\pi a$$ **Proved**