Maxima and Minima

What do you mean by the maximum or minimum value of a function $f(x)$ at $x = c$? If $f'(x)$ exists, what will be the value of $f'(c)$? Is it necessary as well as sufficient condition?

Soln: A function $f(x)$ is said to have a (local) maximum value at $x = c$ if there exists some $h > 0$ such that $f(x) < f(c)$ whenever $|x-c| < h$.

A function $f(x)$ is said to have a (local) minimum value at $x = c$ if there exists some $h > 0$ such that $f(x) > f(c)$ whenever $|x-c| < h$.

2nd part: If $f(x)$ has a maximum or a minimum value at $x = c$, and $f'(x)$ exists, then $f'(c) = 0$.

3rd part: Yes, ~~th~~ it is the necessary and sufficient condition for $f(x)$ having maximum or minimum value at $x = c$.

Theorem: If $f(x)$ be a maximum or minimum at $x=c$, and if $f'(c)$ exists, then $f'(c)=0$.

Theorem: The necessary and sufficient condition for $f(x)$ being maximum or minimum at $x=c$ is $f'(c)=0$.

Theorem: If $f(x)$ be defined in any interval containing the point $x=c$, and $f'(c)=0$, but $f''(c)\neq 0$, then $f(x)$ has its one maximum value of $f(c)$ if $f''(c)<0$ and minimum value of $f(c)$ if $f''(c)>0$.

Note: Let $f'(c) = f''(c) = \cdots = f^{n-1}(c) = 0$

Then (i) if n is even, $f(c)$ is ~~maxi~~ a maximum or ~~a~~ minimum according as $f^{n}(c)$ is negative or positive.

(ii) if n is odd, $f(c)$ is neither a maximum nor a minimum.

Procedure for finding maxima or minima

1. find $f'(x)$ or $\frac{dy}{dx}$

2. Put $\frac{dy}{dx} = 0$ or $f'(x) = 0$ and find the values of x (say $x_1, x_2, \ldots$)

3. find $f''(x)$ or $\frac{d^2y}{dx^2}$

4. find $f''(x_1), f''(x_2) \ldots$

   or $\left.\frac{d^2y}{dx^2}\right|_{x=x_1}, \left.\frac{d^2y}{dx^2}\right|_{x=x_2} \ldots$

5. Then $f(x)$ is maximum or minimum according as the above values ~~are~~ being negative or positive.

~~The~~ maximum or minimum values ~~value~~ are $f(x_1), f(x_2) \ldots$

Q.
Show that the maximum value of xy subject to the condition $3x+4y=5$ is $\frac{25}{48}$.

Soln: Let $u = xy$ ——— (1)

and $3x+4y=5$ ——— (2)

From (2) we have,

$$y = \frac{5-3x}{4} \quad \text{——— (3)}$$

Using (3) in (1) we get,

$$u = x\left(\frac{5-3x}{4}\right)$$

i.e. $u = \frac{1}{4}(5x - 3x^2)$ ——— (4)

$\therefore \frac{du}{dx} = \frac{1}{4}(5-6x)$ ——— (5)

$\therefore \frac{d^2u}{dx^2} = -\frac{3}{2}$ ——— (6)

For maxima or minima

$$\frac{du}{dx} = 0$$

$\Rightarrow \frac{1}{4}(5-6x) = 0$

$\Rightarrow 5 = 6x \Rightarrow x = \frac{5}{6}$

For $x = \frac{5}{6}$, $\frac{d^2u}{dx^2} = -\frac{3}{2} < 0$

$\therefore$ u has a maximum value at $x = \frac{5}{6}$.

Now, if $x = \frac{5}{6}$ then from (4) we have,

$$u = \frac{1}{4}\left\{5 \cdot \frac{5}{6} - 3 \cdot \left(\frac{5}{6}\right)^2\right\}$$

$$= \frac{1}{4}\left\{\frac{25}{6} - \frac{75}{36}\right\}$$

$$= \frac{1}{4} \cdot \frac{150 - 75}{36}$$

$$= \frac{\cancel{75}^{\,25}}{\cancel{144}_{\,48}}$$

$$= \frac{25}{48}$$

$\therefore$ The maximum value is $\frac{25}{48}$. Proved

Q. Show that the maximum value of $\left(\frac{1}{x}\right)^x$ is $e^{1/e}$.

Soln: Let, $y = \left(\frac{1}{x}\right)^x$ ——— (1)

$\therefore \log y = x \log\left(\frac{1}{x}\right)$

$= x(\log 1 - \log x)$

$= -x \log x$ ——— (2)

$\therefore \frac{1}{y}\frac{dy}{dx} = -\left[\cancel{x}\cdot\frac{1}{\cancel{x}} + \log x\right]$ ——— (3)

$\therefore \frac{dy}{dx} = \left(\frac{1}{x}\right)^x(-1 - \log x)$ ——— (4)

If $\frac{dy}{dx} = 0$ then $\left(\frac{1}{x}\right)^x(-1 - \log x) = 0$

$\Rightarrow -1 - \log x = 0$

$\Rightarrow \log x = -1$

$\Rightarrow \log x = \log e^{-1}$

$\therefore x = e^{-1} = \frac{1}{e}$

Differentiating (3) w.r.to x we get,

$\frac{1}{y}\frac{d^2y}{dx^2} - \frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 = -\frac{1}{x}$

$$\Rightarrow \frac{1}{y}\frac{d^2y}{dx^2} = -\frac{1}{x} + (1+\log x)^2$$

$$\Rightarrow \frac{d^2y}{dx^2} = y\left[-\frac{1}{x} + (1+\log x)^2\right]$$

$$= \left(\frac{1}{x}\right)^x \left[-\frac{1}{x} + (1+\log x)^2\right] \quad \text{——— (5)}$$

Put $x = \frac{1}{e}$ in (5), then

$$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{e}} = e^{\frac{1}{e}}\left[-e + \left(1+\log\frac{1}{e}\right)^2\right]$$

$$= e^{\frac{1}{e}}\left[-e + \left(1+\log e^{-1}\right)\right]$$

$$= e^{\frac{1}{e}}\left[-e + (1-1)\right]$$

$$= -e^{\frac{1}{e}}.e$$

$$= -e^{\frac{1}{e}+1} < 0$$

$\therefore$ y ~~is max~~ has a maximum value at $x = \frac{1}{e}$. The maximum value is $\left(\frac{1}{\frac{1}{e}}\right)^{\frac{1}{e}} = e^{1/e}$. Proved

Q. Find the maximum or minimum values of u where, $u = \frac{4}{x} + \frac{36}{y}$ and $x + y = 2$.

Sol^n: Given,

$$u = \frac{4}{x} + \frac{36}{y} \quad \text{------------} \quad (1)$$

and

$$x + y = 2 \quad \text{------------} \quad (2)$$

From (2) we have,

$$y = 2 - x \quad \text{------------} \quad (3)$$

Using (3) in (1) we get,

$$u = \frac{4}{x} + \frac{36}{2-x} \quad \text{------------} \quad (4)$$

$$\therefore \frac{du}{dx} = -\frac{4}{x^2} + \frac{36}{(2-x)^2} \quad \text{------------} \quad (5)$$

$$\therefore \frac{d^2u}{dx^2} = \frac{8}{x^3} + \frac{72}{(2-x)^3} \quad \text{------------} \quad (6)$$

For maxima or minima, we have,

$$\frac{du}{dx} = 0$$

...

i.e. $-\frac{4}{x^2}+\frac{36}{(2-x)^2}=0$

$$\Rightarrow \frac{36}{4-4x+x^2}=\frac{4}{x^2}$$

$$\Rightarrow 36x^2=16-16x+4x^2$$

$$\Rightarrow 32x^2+16x-16=0$$

$$\Rightarrow 2x^2+x-1=0$$

$$\Rightarrow 2x^2+2x-x-1=0$$

$$\Rightarrow 2x(x+1)-1(x+1)=0$$

$$\Rightarrow (x+1)(2x-1)=0$$

$$\therefore x=-1, \frac{1}{2}$$

Now, if $x=-1$ then from (6), we get,

$$\frac{d^2u}{dx^2}=\frac{8}{-1}+\frac{72}{3^3}=-8+\frac{72}{27}$$

$$=\frac{-216+72}{27}=-\frac{144}{27}<0$$

$\therefore$ u is maximum at $x=-1$

from (4)

and the ~~?~~ maximum value is

$$\frac{4}{-1} + \frac{36}{2+1}$$

$$= -4 + 12$$

$$= 8$$

If $x = \frac{1}{2}$ then from (6), we get,

$$\frac{d^2u}{dx^2} = \frac{8}{\left(\frac{1}{2}\right)^3} + \frac{72}{\left(2-\frac{1}{2}\right)^3}$$

$$= \frac{8}{\left(\frac{1}{2}\right)^3} + \frac{72}{\left(\frac{3}{2}\right)^3} > 0$$

$\therefore$ u has a minimum value at $x = \frac{1}{2}$

and from (4) the

minimum value is,

$$\frac{4}{\frac{1}{2}} + \frac{36}{2-\frac{1}{2}} = 8 + \frac{\overset{12}{\cancel{36}} \times 2}{\cancel{3}}$$

$$= 32.$$

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