Expansion of Function

MM

Expansion of function

Rolle's Theorem :

Statement: If

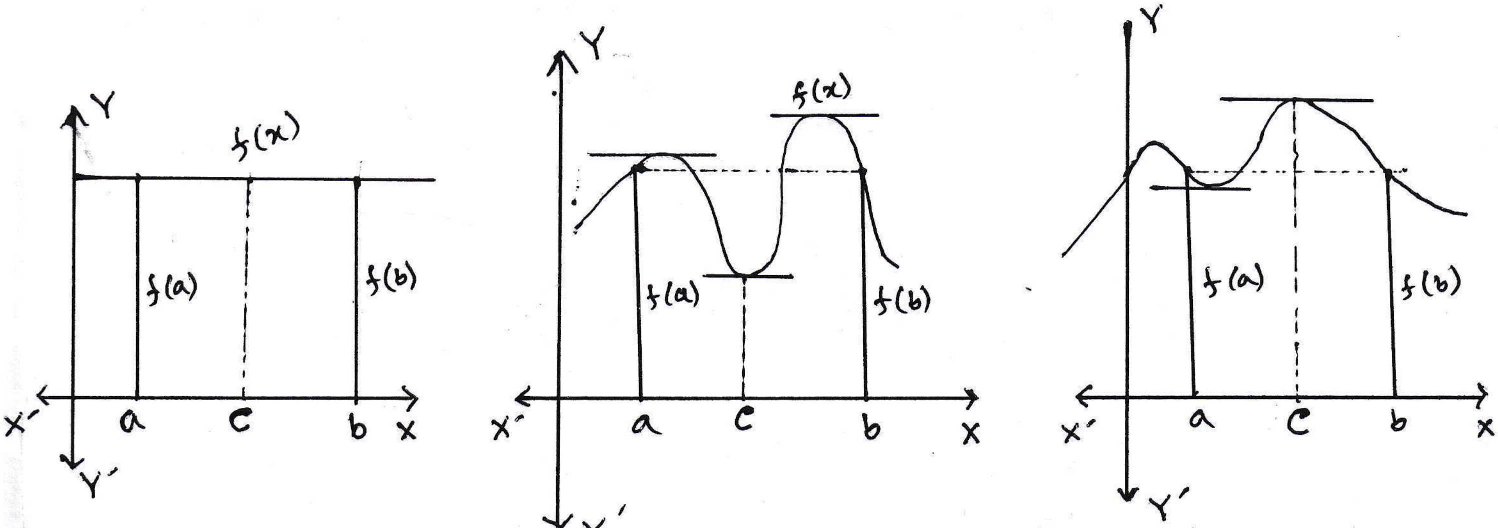
(i) $f(x)$ is continuous in $[a,b]$,

(ii) $f'(x)$ exists in (a,b), and

(iii) $f(a) = f(b)$,

then $\exists$ at least one value of x (say c) in (a,b) such that $f'(c) = 0$.

Geometrical interpretation of Rolle's theorem :



If $f(x)$ is a continuous curve on $[a,b]$, ~~and~~ $f'(x)$ exists in (a,b), and $f(a) = f(b)$, then by Rolle's theorem $\exists$ at least one point $c \in (a,b)$ such that tangent ~~at $(c, f(c))$~~ to the curve at c is parallel to x-axis.

Q. State and prove Rolle's theorem.

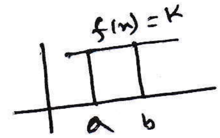
Sol^n: **Statement:** If

(i) $f(x)$ is continuous in $[a,b]$,

(ii) $f'(x)$ exists in (a,b), and

(iii) $f(a) = f(b)$,

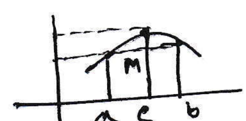
then there exists at least one value of x (say c) in (a,b), such that $f'(c) = 0$.



Proof:

Since $f(a) = f(b)$, let $f(x) = K$ in $[a,b]$, where K is a constant. Then $f(a) = K = f(b)$ and $f'(x) = 0$ at every point in (a,b).

Consider that ~~Let~~ $f(x)$ is not constant in $[a,b]$. Then $f(x)$ must have values either greater than or less than $f(a)$ or both in $[a,b]$.



First, suppose that $f(x)$ has values greater than $f(a)$ in (a,b). Since $f(x)$ is continuous in $[a,b]$, it must be bounded. Then $\exists$ $M \in \mathbb{R}$ such that $f(x) \leq M$ for all $x \in [a,b]$. By the intermediate value theorem $\exists$ $c \in (a,b)$ such that $f(c) = M$. [Here $M > f(a)$].

$\therefore \quad f(c+h) - f(c) \leq 0 \quad$ for $\quad h > 0 \quad$ or $\quad h < 0.$

$$\therefore \quad \frac{f(c+h) - f(c)}{h} \geq 0 \quad \text{if} \quad h < 0 \quad \text{and}$$

$$\frac{f(c+h) - f(c)}{h} \leq 0 \quad \text{if} \quad h > 0$$

~~lim~~

$$\therefore \quad Lf'(c) = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \geq 0 \quad \text{——— (1)}$$

$$\text{and} \quad Rf'(c) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq 0 \quad \text{——— (2)}$$

Since $f'(c)$ exists, we must have

$$Lf'(c) = Rf'(c)$$

Then from (1) and (2) we have

$$Lf'(c) = Rf'(c) = 0$$

$$\text{i.e. } f'(c) = 0$$

Similarly, We can show that $f'(c) = 0$ when $f(x)$ has values less than $f(a)$ in the interval (a, b).

—x—

Q. Verify Rolle's theorem for $f(x) = x^2 - 6x + 8$ in $[2,4]$.

Soln: Given,

$$f(x) = x^2 - 6x + 8 \quad \text{—— (1)}$$

$$\therefore f'(x) = 2x - 6 \quad \text{—— (2)}$$

From (1) it is clear that $f(x)$ is continuous in $[2,4]$.

From (2) it is clear that $f'(x)$ exists in $(2,4)$.

Now, $f(2) = 2^2 - 6(2) + 8 = 4 - 12 + 8 = 0$

and $f(4) = 4^2 - 6(4) + 8 = 16 - 24 + 8 = 0$

i.e. $f(2) = f(4)$ —— (3)

To verify Rolle's theorem, we need to find a $c \in (2,4)$ such that

$$f'(c) = 0$$

i.e. $2c - 6 = 0$

i.e. $c = 3$

Now, we see that $3 \in (2,4)$ and

$$f'(3) = 2(3) - 6 = 6 - 6 = 0.$$

Thus the Rolle's theorem is verified.

—x—

Q. Verify Rolle's theorem for $f(x) = x^3 - 7x^2 + 36$ in $[3,6]$.

Sol^n: Given, $f(x) = x^3 - 7x^2 + 36$ ——— (1)

$\therefore f'(x) = 3x^2 - 14x$ ——— (2)

From (1) it is clear that $f(x)$ is continuous in $[3,6]$.

From (2) it is clear that $f'(x)$ exists in $(3,6)$.

Now, $f(3) = 3^3 - 7(3)^2 + 36 = 3^2(3 - 7 + 4) = 9 \times 0 = 0$

and $f(6) = 6^3 - 7(6)^2 + 36 = 6^2(6 - 7 + 1) = 36 \times 0 = 0$

i.e. $f(3) = f(6)$ ——— (3)

To verify Rolle's theorem, we need to find a $c \in (3,6)$ such that

$$f'(c) = 0$$

i.e. $3c^2 - 14c = 0$

i.e. $c = 0, \frac{14}{3}$

Here we see that $c = \frac{14}{3} \in (3,6)$ and

$$f'\left(\frac{14}{3}\right) = 3\left(\frac{14}{3}\right)^2 - 14\left(\frac{14}{3}\right) = \frac{(14)^2}{3} - \frac{(14)^2}{3} = 0$$

Thus the Rolle's theorem is verified.

Q. If $f(x) = \tan x$, then $f(0) = 0$ and $f(\pi) = 0$. Is Rolle's theorem applicable to $f(x)$ in $(0, \pi)$?

Sol^n: Since $f(x) = \tan x$ is not continuous at $x = \frac{\pi}{2}$ and $\frac{\pi}{2} \in (0, \pi)$, therefore Rolle's theorem is not applicable for $f(x)$ in $(0, \pi)$.

—x—

Q. If $f(x)$ and $g(x)$ are differentiable in the interval (a, b), then prove that there is a number p, $a < p < b$ such that

$$\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (b-a) \begin{vmatrix} f(a) & f'(p) \\ g(a) & g'(p) \end{vmatrix}.$$

Sol^n: suppose,

$$\phi(x) = \begin{vmatrix} f(a) & f(x) \\ g(a) & g(x) \end{vmatrix} - \frac{x-a}{b-a} \begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} \quad \text{——— (1)}$$

$$\therefore \phi'(x) = \begin{vmatrix} f(a) & f'(x) \\ g(a) & g'(x) \end{vmatrix} - \frac{1}{b-a} \begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} \quad \text{——— (2)}$$

From (1) it is clear that $\phi(x)$ is continuous in $[a, b]$.
From (2) it is " " $\phi'(x)$ exists in $[a, b]$ as $f(x)$ and $g(x)$ are differentiable in (a, b).

Now,
$$\phi(a) = \begin{vmatrix} f(a) & f(a) \\ g(a) & g(a) \end{vmatrix} - \frac{a-a}{b-a}\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix}$$

$$= f(a)g(a) - f(a)g(a) - 0$$

$$= 0$$

$$\phi(b) = \begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} - \frac{b-a}{b-a}\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix}$$

$$= \begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} - \begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix}$$

$$= 0$$

$$\therefore \quad \phi(a) = \phi(b) \quad \text{——————— (3)}$$

Then by Rolle's theorem there ~~is~~ exists $p \in (a,b)$ such that $\phi'(p) = 0$

i.e. $$\begin{vmatrix} f(a) & f'(p) \\ g(a) & g'(p) \end{vmatrix} - \frac{1}{b-a}\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = 0$$

$$\therefore \begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (b-a)\begin{vmatrix} f(a) & f'(p) \\ g(a) & g'(p) \end{vmatrix}$$

Proved

Q. State and prove the mean value theorem.

or

State and prove the Lagrange's Mean Value Theorem

Sol^n: Statement:

If (i) $f(x)$ is continuous in the closed interval $[a,b]$,

(ii) $f'(x)$ exists in the open interval (a,b),

then there is at least one value of x (say c) in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Proof: Consider, $\phi(x) = f(b) - f(x) - (b-x)\frac{f(b)-f(a)}{b-a}$ ——— (1)

$\therefore \phi'(x) = -f'(x) + \frac{f(b)-f(a)}{b-a}$ ——— (2)

From, (1), it is clear that $\phi(x)$ is ~~closed~~ continuous in $[a,b]$ as $f(x)$ and $(b-x)$ are continuous in $[a,b]$.

From, (2), it is clear that $\phi'(x)$ exists in (a,b) as $f'(x)$ exists in (a,b).

Now, $\phi(a) = f(b) - f(a) - (b-a)\frac{f(b)-f(a)}{(b-a)}$

$= f(b) - f(a) - f(b) + f(a)$

$= 0$

and $\phi(b) = f(b) - f(b) - (b-b)\frac{f(b)-f(a)}{b-a}$

$= 0$

$\therefore \quad \phi(a) = \phi(b)$ ——— (3)

Then by Rolle's theorem there is at least one value of x (say c) in (a, b) such that

$$\phi'(c) = 0$$

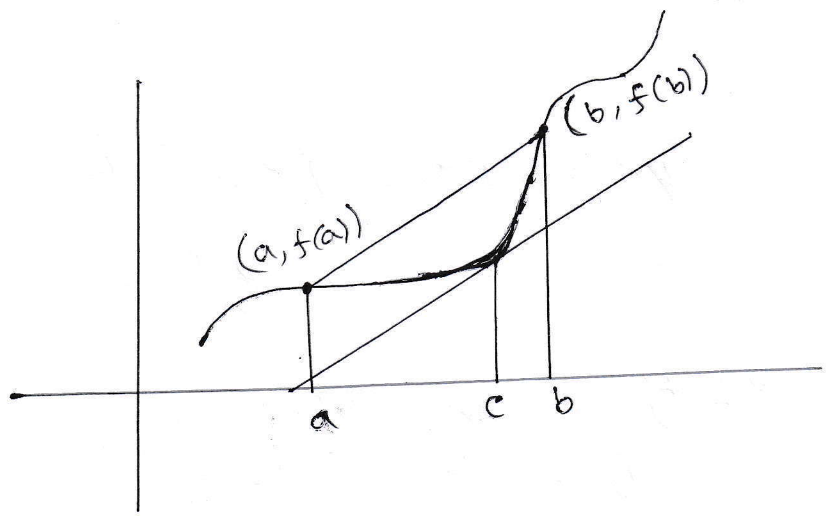
$$\Rightarrow \quad -f'(c) + \frac{f(b)-f(a)}{b-a} = 0$$

Hence $\quad f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

Proved

Q. Explain Lagrange's mean value theorem geometrically.

Sol^n:



Slope of the tangent passing through $(a, f(a))$ and $(b, f(b))$ is $\dfrac{f(b)-f(a)}{b-a}$

If $f(x)$ is continuous in $[a, b]$ and $f'(x)$ exists in (a, b), then by the mean value theorem there is at least one value of x (say c) in (a, b) such that tangent at $(c, f(c))$ is parallel to the line passing through $(a, f(a))$ and $(b, f(b))$.

Corollary:

If $c \in (a,b)$ then $c = a + \theta(b-a)$ for $\theta \in (0,1)$.

Put $b = a+h$, then $b-a = a+h-a = h$. Then

from the mean value theorem, we get,

$$f'(a+\theta h) = \frac{f(a+h)-f(a)}{h} \quad \text{for some } \theta \in (0,1)$$

$$\Rightarrow f(a+h)-f(a) = hf'(a+\theta h) \quad \text{for some } \theta \in (0,1)$$

$$\Rightarrow f(a+h) = f(a) + hf'(a+\theta h) \quad \text{for some } \theta \in (0,1).$$

$$\therefore f(x+h) = f(x) + hf'(x+\theta h) \quad \text{for some } \theta \in (0,1).$$

This is another form of the mean value theorem.

Cauchy's Mean Value Theorem

or

Generalized Mean Value Theorem

Statement:

If (i) $f(x)$ and $g(x)$ are both continuous in the closed interval $a \le x \le b$,

(ii) $f'(x)$ and $g'(x)$ ~~are~~ exist in the open interval $a < x < b$,

and (iii) $g'(x) \neq 0$ anywhere in the open interval $a < x < b$,

then there is a value of x say c between a and b, i.e., $a < c < b$, such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof: Consider the function,

$$\phi(x) = \{f(b) - f(a)\}\, g(x) - \{g(b) - g(a)\}\, f(x) \quad \text{--- (1)}$$

$$\therefore \phi'(x) = \{f(b) - f(a)\}\, g'(x) - \{g(b) - g(a)\}\, f'(x) \quad \text{--- (2)}$$

Put, $x = a$ in (1), then

$$\phi(a) = \{f(b) - f(a)\} g(a) - \{g(b) - g(a)\} f(a)$$

$$= f(b)\,g(a) - \cancel{f(a)\,g(a)} - g(b)\,f(a) + \cancel{g(a)\,f(a)}$$

$$= f(b)\,g(a) - f(a)\,g(b)$$

Put $x = b$ in (1), then

$$\phi(b) = \{f(b) - f(a)\} g(b) - \{g(b) - g(a)\} f(b)$$

$$= \cancel{f(b)\,g(b)} - f(a)\,g(b) - \cancel{f(b)\,g(b)} + f(b)\,g(a)$$

$$= f(b)\,g(a) - f(a)\,g(b)$$

$$\therefore \quad \phi(a) = \phi(b) \quad \text{———— (3)}$$

Since $f(x)$ and $g(x)$ are continuous in $a \le x \le b$, then by (1) $\phi(x)$ is continuous in $a \le x \le b$.

Again, since $f'(x)$ and $g'(x)$ exist in $a < x < b$, then by (2) $\phi'(x)$ exists in $a < x < b$.

By (3) $\phi(a) = \phi(b)$.

Then by Rolle's Theorem there ~~is a~~ exists a value $x = c$, such that

$$\phi'(c) = 0$$

$$\Rightarrow \{f(b) - f(a)\} g'(c) - \{g(b) - g(a)\} f'(c) = 0$$

$$\therefore \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)} \quad \text{------} \quad (4)$$

— x —

State and prove Cauchy's Mean Value Theorem and hence deduce the first (Lagrange's) Mean Value Theorem.

Solⁿ: Cauchy's Mean Value Theorem

+

Let $g(x) = x$ ——— (5)

$\therefore g'(x) = 1$ ——— (6)

$\therefore g(a) = a$

$g(b) = b$

From 4 we get,

$$\frac{f(b)-f(a)}{b-a} = \frac{f'(c)}{1}$$

$$\therefore f'(c) = \frac{f(b)-f(a)}{b-a}$$

which is the first Mean Value Theorem.

Q. Find the value of θ in the Mean value theorem $f(x+h) = f(x) + hf'(x+\theta h)$, where $f(x) = \frac{1}{x}$.

Sol^n: Given, $f(x) = \frac{1}{x}$, $f(x+h) = \frac{1}{x+h}$

$$\therefore f'(x) = -\frac{1}{x^2}$$

$$\therefore f'(x+\theta h) = -\frac{1}{(x+\theta h)^2}$$

We have,

$$f(x+h) = f(x) + hf'(x+\theta h)$$

$$\Rightarrow \frac{1}{x+h} = \frac{1}{x} + h \cdot \frac{-1}{(x+\theta h)^2}$$

$$\Rightarrow \frac{h}{(x+\theta h)^2} = \frac{1}{x} - \frac{1}{x+h}$$

$$\Rightarrow \frac{h}{\cancel{x}\cdot(x+\theta h)^2} = \frac{\cancel{x}+h-\cancel{x}}{x(x+h)}$$

$$\Rightarrow (x+\theta h)^2 = x(x+h)$$

$$\Rightarrow x+\theta h = \sqrt{x(x+h)}$$

$$\Rightarrow \theta h = \sqrt{x(x+h)} - x$$

$$\therefore \theta = \frac{\sqrt{x(x+h)} - x}{h}$$

Answer

Q. Find the value of c in Lagrange's mean value theorem for $f(x) = \sqrt{x}$; $a=9$ and $b=16$.

Sol^n: Given, $f(x) = \sqrt{x}$

$$\therefore f(a) = f(9) = \sqrt{9} = 3$$

$$\therefore f(b) = f(16) = \sqrt{16} = 4$$

$$\therefore f'(x) = \frac{1}{2\sqrt{x}}$$

$$\therefore f'(c) = \frac{1}{2\sqrt{c}}$$

By Lagrange's mean value theorem, we have,

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$\Rightarrow \frac{1}{2\sqrt{c}} = \frac{4-3}{16-9}$$

$$\Rightarrow \frac{1}{2\sqrt{c}} = \frac{1}{7}$$

$$\Rightarrow 2\sqrt{c} = 7$$

$$\Rightarrow \sqrt{c} = \frac{7}{2}$$

$$\therefore c = \frac{49}{4}$$ Answer

Q. Verify mean value theorem for $f(x) = 2x^2 - 7x + 10$ for $a = 2$ and $b = 5$.

Solⁿ: Given, $f(x) = 2x^2 - 7x + 10$ —— (1)

$\therefore f'(x) = 4x - 7$ —— (2)

From (1) and (2) it is clear that $f(x)$ is continuous in $[2, 5]$ and $f'(x)$ exists in $(2, 5)$.

To verify mean value theorem, we need to find a $c \in (2, 5)$ such that,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow f'(c) = \frac{f(5) - f(2)}{5 - 2}$$

$$\Rightarrow 4c - 7 = \frac{\{2(5)^2 - 7(5) + 10\} - \{2(2)^2 - 7(2) + 10\}}{3}$$

$$\Rightarrow 4c - 7 = \frac{50 - 35 + 10 - 8 + 14 - 10}{3}$$

$$\cancel{\Rightarrow 4c = \frac{21}{3} + 7} \Rightarrow 4c - 7 = 7$$

$$\Rightarrow c = \frac{14}{4} = \frac{7}{2}$$

Here $2 < \frac{7}{2} < 5$, i.e. $\frac{7}{2} \in (2, 5)$ and

$$f'\left(\frac{7}{2}\right) = 4\left(\frac{7}{2}\right) - 7 = 14 - 7 = 7 = \frac{f(5) - f(2)}{5 - 2}$$

Hence the mean value theorem is verified.

Q. If $f'(x) = 0$ for all values of x in $[a, b]$, then show that $f(x)$ is constant in $[a, b]$.

Solⁿ: Suppose $f'(x) = 0$ ~~in~~ at every point in $[a, b]$.

Let, $x_1, x_2 \in [a, b]$ and $x_1 < x_2$.

By mean value theorem, we get,

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad \text{for some } c \in (x_1, x_2).$$

$$\Rightarrow \quad 0 = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad \left[\text{Since } f'(x) = 0 \ \forall x \in [a, b]\right]$$

$$\Rightarrow \quad f(x_2) - f(x_1) = 0$$

$$\Rightarrow \quad f(x_2) = f(x_1)$$

Since x_1, x_2 are arbitrary points in $[a, b]$, therefore $f(x)$ must be constant in $[a, b]$.

• If f is differentiable on (a,b) and $f'(x) \geq 0$ for every $x \in (a,b)$ then show that f is monotonically increasing on (a,b).

<u>Proof:</u> Suppose, $x_1, x_2 \in (a,b)$ where $x_2 > x_1$.

By mean value theorem, we get,

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad \text{for some } c \in (x_1, x_2).$$

Since $f'(x) \geq 0 \quad \forall \; x \in (a,b)$

$$\therefore \quad f'(c) \geq 0$$

$$\Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} \geq 0 \qquad [x_2 > x_1 \Rightarrow x_2 - x_1 > 0]$$

$$\Rightarrow f(x_2) - f(x_1) \geq 0$$

$$\Rightarrow f(x_2) \geq f(x_1)$$

Since x_1, x_2 are arbitrary points in (a,b) and $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$, hence ~~f is~~ ~~f(x) is~~ f is monotonically increasing on (a,b).

Q. Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x. How large can $f(2)$ possible be?

Solⁿ: Since $f'(x)$ exists for all values of x then $f(x)$ is continuous for all values of x.

Thus we have, (i) $f(x)$ is continuous in $[0,2]$

(ii) $f'(x)$ exists in ~~(a,b)~~ $(0,2)$

Then by the mean value theorem $\exists$ $c \in (0,2)$ such that

$$f'(c) = \frac{f(2) - f(0)}{2 - 0}$$

$$\Rightarrow f(2) - f(0) = 2f'(c)$$

$$\Rightarrow f(2) = f(0) + 2f'(c)$$

$$\Rightarrow f(2) = -3 + 2f'(c)$$

$$\Rightarrow f(2) \leq -3 + 2(5) \quad [\because f'(x) \leq 5 \ \forall x]$$

$$\therefore f(2) \leq 7$$

Hence $f(2)$ can be at most 7.

Taylor's Series in finite form :

If $f^{n-1}(x)$ is continuous in $[a, b]$ and $f^n(x)$ exists in (a, b), then

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!} f''(a) + \cdots\cdots + \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(b-a)^n}{n!} f^n(c) \quad \text{——— (1)}$$

where $c \in (a, b)$.

Now, if $b = a+h$, then $b-a = h$,

We have from (1),

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \cdots\cdots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) \cancel{+ \frac{h^n}{n!} f^n(a)} + \frac{h^n}{n!} f^n(a+\theta h) \quad \text{——— (2)}$$

where $\theta \in (0, 1)$.

Putting $a = x$ in (2) we get,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \cdots\cdots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(x) + \frac{h^n}{n!} f^n(x+\theta h) \quad \text{——— (3)}$$

The series (1) or (2) or (3) is called Taylor's series with remainder in Lagrange's form, the remainder (after n terms) being $\frac{(b-a)^n}{n!} f^n(c)$ or $\frac{h^n}{n!} f^n(a+\theta h)$ or $\frac{h^n}{n!} f^n(x+\theta h)$, $\theta \in (0,1)$ is denoted by R_n.

Putting $n = 1$ in ①, we get,

$$f(b) = f(a) + \frac{(b-a)^1}{1!} f'(c)$$

$$\Rightarrow (b-a) f'(c) = f(b) - f(a)$$

$$\therefore f'(c) = \frac{f(b) - f(a)}{b-a}$$

This is the mean value theorem.

Putting $n = 1$ in ②, we get,

$$f(a+h) = f(a) + \frac{h^1}{1!} f'(a+\theta h), \quad \theta \in (0,1).$$

$$\Rightarrow f(a+h) = f(a) + h f'(a+\theta h), \quad \theta \in (0,1).$$

This is the mean value theorem.

Putting $n = 2$ in ②, we get,

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a+\theta h), \quad \theta \in (0,1).$$

This is called Mean Value theorem of the second order.

Maclaurin's series in finite form :

Putting $x=0$, $h=x$ in Taylor's series in finite form ③,

we get,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0)$$

$$+ \frac{x^n}{n!} f^n(\theta x), \quad 0<\theta<1. \quad \text{—— (4)}$$

This is Maclaurin's series in finite form with the remainder in Lagrange's form.

Putting $n=1$, in ④ we get,

$$f(x) = f(0) + \frac{x^1}{1!} f'(\theta x), \quad \theta \in (0,1).$$

$$\therefore f(x) = f(0) + x f'(\theta x), \quad \theta \in (0,1).$$

Putting $n=2$ in ④, we get,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(\theta x), \quad \theta \in (0,1).$$

$$* \quad \therefore f(h) = f(0) + h f'(0) + \frac{h^2}{2!} f''(\theta h), \quad \theta \in (0,1).$$

Putting x for b in ① we get,

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots\cdots$$

$$\cdots + \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(x-a)^n}{n!} f^n\{a+\theta(x-a)\}, \quad 0<\theta<1. \quad \text{—— (5)}$$

Q. Expand $\sin x$ in a finite series in powers of x, with the remainder in Lagrange's form.

<u>Sol^n</u>: The Maclaurin's series in finite form with the remainder in Lagrange's form is given by

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \frac{x^5}{5!}f^{v}(0) + \cdots\cdots + \frac{x^n}{n!}f^n(\theta x),\ 0<\theta<1, \quad \text{——— (1)}$$

Here

$f(x) = \sin x,$ $\qquad f(0) = 0$

$f'(x) = \cos x = \sin(\frac{\pi}{2}+x),$ $\qquad f'(0) = 1$

$f''(x) = \cos(\frac{\pi}{2}+x) = \sin(\frac{2\pi}{2}+x),$ $\qquad f''(0) = 0$

$f'''(x) = \cos(\frac{2\pi}{2}+x) = \sin(\frac{3\pi}{2}+x),$ $\qquad f'''(0) = -1$

$f^{iv}(x) = \cos(\frac{3\pi}{2}+x) = \sin(\frac{4\pi}{2}+x),$ $\qquad f^{iv}(0) = 0$

$f^{v}(x) = \cos(\frac{4\pi}{2}+x) = \sin(\frac{5\pi}{2}+x),$ $\qquad f^{v}(0) = 1$

$\cdots\cdots\cdots\cdots\cdots\cdots$

$f^n(x) = \sin(\frac{n\pi}{2}+x),$ $\qquad f^n(\theta x) = \sin(\frac{n\pi}{2}+\theta x)$

$\therefore$ From (1), we get,

$$f(x) = 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + \cdots\cdots + \frac{x^n}{n!}\sin(\frac{n\pi}{2}+\theta x)$$

$$\therefore \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\cdots + \frac{x^n}{n}\sin(\frac{n\pi}{2}+\theta x)$$

<u>Answer</u>

Q. Expand $e^x \cos x$ in power of x with Lagrange's remainder using Maclaurin's series.

Sol^n: The Maclaurin's series in finite form with the remainder in Lagrange's form is given by

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots\cdots + \frac{x^n}{n!} f^n(\theta x),$$

$$0 < \theta < 1, \quad \text{——— (1)}$$

Here,

$f(x) = e^x \cos x,$ $\qquad f(0) = 1$

$f'(x) = e^x \cos x - e^x \sin x$

$= \sqrt{2} e^x \left(\frac{1}{\sqrt{2}} \cos x - \frac{1}{\sqrt{2}} \sin x \right)$

$= \sqrt{2} e^x \left(\cos x \cos \frac{\pi}{4} - \sin x \sin \frac{\pi}{4} \right)$ $\qquad f'(0) = \sqrt{2} \cos\left(\frac{\pi}{4}\right)$

$= \sqrt{2} e^x \cos\left(x + \frac{\pi}{4}\right),$

$\therefore f''(x) = (\sqrt{2})^2 e^x \cos\left(x + \frac{2\pi}{4}\right),$ $\qquad f''(0) = (\sqrt{2})^2 \cos\left(\frac{2\pi}{4}\right)$

$\cdots\cdots\cdots$

$\therefore f^n(x) = (\sqrt{2})^n e^x \cos\left(x + \frac{n\pi}{4}\right),$ $\qquad f^n(\theta x) = (\sqrt{2})^n e^{\theta x} \cos\left(\theta x + \frac{n\pi}{4}\right)$

From (1), we get,

$$e^x \cos x = 1 + \frac{x}{1!} \sqrt{2} \cos\left(\frac{\pi}{4}\right) + \frac{x^2}{2!} (\sqrt{2})^2 \cos\left(2 \frac{\pi}{4}\right) + \cdots\cdots + \frac{x^n}{n!} (\sqrt{2})^n e^{\theta x} \cos\left(\theta x + \frac{n\pi}{4}\right)$$

Answer

Q. Expand e^x in powers of $(x-1)$.

Soln: From the Taylor's series, we get,

$$f(b) = f(a) + \frac{(b-a)}{1!} f'(a) + \frac{(b-a)^2}{2!} f''(a) + \cdots$$

$$\cdots + \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(b-a)^n}{n!} f^n\{a + \theta(b-a)\}, \quad 0<\theta<1 \qquad (1)$$

Putting x for b we get,

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots$$

$$\cdots + \frac{(x-a)^n}{n!} f^n\{a + \theta(x-a)\}, \quad 0<\theta<1 \qquad (2)$$

Given, $f(x) = e^x$

Let $a = 1$, then $f(1) = e^1 = e$

$\therefore f'(x) = e^x$, $\quad f'(1) = e^1 = e$

$\therefore f''(x) = e^x$, $\quad f''(1) = e^1 = e$

$\cdots$

$\therefore f^n(x) = e^x$, $\quad f^n\{1 + \theta(x-1)\} = e^{1+\theta(x-1)} = e \cdot e^{\theta(x-1)}$

From (2), we get,

$$e^x = e\left[1 + \frac{(x-1)}{1!} + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \cdots + \frac{(x-1)^n}{n!} e^{\theta(x-1)}\right]$$

Answer

Q. If $f(h) = f(0) + hf'(0) + \frac{h^2}{2!} f''(\theta h)$, $0 < \theta < 1$, find θ, when $h = 7$ and $f(x) = \frac{1}{1+x}$.

Solⁿ: Given, $f(x) = \frac{1}{1+x}$

$$\therefore f'(x) = -\frac{1}{(1+x)^2}$$

$$\therefore f''(x) = \frac{2}{(1+x)^3}$$

$$\therefore f(h) = \frac{1}{1+h}$$

$$\therefore f(0) = \frac{1}{1+0} = 1$$

$$\therefore f'(0) = -\frac{1}{(1+0)^2} = -1$$

$$\therefore f''(\theta h) = \frac{2}{(1+\theta h)^3}$$

We have,

$$f(h) = f(0) + hf'(0) + \frac{h^2}{2!} f''(\theta h)$$

$$\Rightarrow \frac{1}{1+h} = 1 - h + \frac{h^2}{2} \cdot \frac{2}{(1+\theta h)^3}$$

$$\Rightarrow \frac{1}{1+7} = 1 - 7 + \frac{49}{(1+7\theta)^3} \quad [\because h = 7]$$

$$\Rightarrow \frac{1}{8} + 6 = \frac{49}{(1+7\theta)^3}$$

$$\Rightarrow \frac{49}{8} = \frac{49}{(1+7\theta)^3}$$

$$\Rightarrow (1+7\theta)^3 = 8 \quad \Rightarrow 1+7\theta = 2 \quad \therefore \theta = \frac{1}{7}$$

Answer

Q. If $f(h) = f(0) + hf'(0) + \frac{h^2}{2!} f''(\theta h)$, $0 < \theta < 1$, find θ, when $h = 1$ and $f(x) = (1-x)^{\frac{5}{2}}$

<u>Sol</u>

Sol^n: Given, $f(x) = (1-x)^{5/2}$

$$\therefore f'(x) = -\frac{5}{2}(1-x)^{3/2}$$

$$\therefore f''(x) = \frac{15}{4}(1-x)^{\frac{1}{2}}$$

$$\therefore f(h) = (1-h)^{\frac{5}{2}}$$

$$\therefore f(0) = (1-0)^{\frac{5}{2}} = 1$$

$$\therefore f'(0) = -\frac{5}{2}(1-0)^{3/2} = -\frac{5}{2}$$

$$\therefore f''(\theta h) = \frac{15}{4}(1-\theta h)^{\frac{1}{2}}$$

We have, $f(h) = f(0) + hf'(0) + \frac{h^2}{2!} f''(\theta h)$

$$\Rightarrow (1-h)^{\frac{5}{2}} = 1 + h\left(-\frac{5}{2}\right) + \frac{h^2}{2} \cdot \frac{15}{4}(1-\theta h)^{\frac{1}{2}}$$

$$\Rightarrow 0 = 1 - \frac{5}{2} + \frac{15}{8}(1-\theta)^{\frac{1}{2}} \qquad [\because h = 1]$$

$$\Rightarrow \frac{15}{8}\sqrt{1-\theta} = \frac{3}{2}$$

$$\Rightarrow \sqrt{1-\theta} = \frac{3}{2} \times \frac{8}{15}$$

$$\Rightarrow \sqrt{(1-\theta)} = \frac{4}{5}$$

$$\Rightarrow 1 - \theta = \frac{16}{25} \qquad \therefore \theta = 1 - \frac{16}{25} = \frac{9}{25}$$

<u>Answer</u>