[1] Find the length of the loop of the curve $x = t^2, y = t - \frac{1}{3}$

Sol$^n$: Given, $x = t^2, y = t\left(1 - \frac{t^2}{3}\right)$

Changing to cartesian coordinates, we get

$$y^2 = t^2\left(1 - \frac{t^2}{3}\right)^2 = x\left(1 - \frac{x}{3}\right)^2$$

$$\Rightarrow y^2 = x\left(1 - \frac{x}{3}\right)^2 \qquad (1)$$

$$\Rightarrow 9y^2 = x(3-x)^2$$

Putting $y = 0$ in (1), we get,

$$x\left(1 - \frac{x}{3}\right)^2 = 0$$

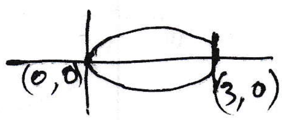
$$x = 0, \quad x = 3$$

If we put $x = 0$ in (1) then we get

$$y = 0$$

Thus the curve (1) has a symmetrical loop about $x$-axis ~~for~~ from $x = 0$ to $x = 3$.

i.e. from $t = 0$ to ~~to~~ $= \sqrt{3}$.




$\therefore$ The length of the loop $= 2\int_{t=0}^{t=\sqrt{3}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

Here, $\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 1 - t^2$

$\therefore$ The required length $= 2\int_0^{\sqrt{3}} \sqrt{4t^2+(1-t^2)^2}\, dt$

$$= 2\int_0^{\sqrt{3}} \sqrt{4t^2+1-2t^2+t^4}\, dt$$

$$= 2\int_0^{\sqrt{3}} \sqrt{t^4+2t^2+1}\, dt$$

$$= 2\int_0^{\sqrt{3}} (t^2+1)\, dt$$

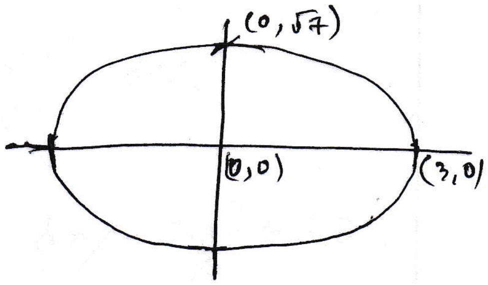
$$= 2\left[\frac{t^3}{3}+t\right]_0^{\sqrt{3}}$$

$$= 2\left[\frac{3\sqrt{3}}{3}+\sqrt{3}\right]$$

$= 4\sqrt{3}$ <u>Answer</u>

Compute the perimeter of the conic $\frac{x^2}{9}+\frac{y^2}{7}=1$.

<u>Soln:</u>




The required perimeter, p

$$= 4\int_0^{3} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

Given, $\frac{x^2}{9}+\frac{y^2}{7}=1$

$\Rightarrow 7x^2+9y^2=63$

$\therefore 14x+18y\frac{dy}{dx}=0$

Show that the perimeter of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ($x=a\cos t$, $y=b\sin t$) is given by $2a\pi\left\{1-\left(\frac{1}{2}\right)^2\frac{e^2}{1}-\left(\frac{1.3}{2.4}\right)^2\frac{e^4}{3}-\left(\frac{1.3.5}{2.4.6}\right)^2\frac{e^6}{5}-\cdots\right\}$ where $e$ is the eccentricity of the ellipse. $\left[e^2=1-\frac{b^2}{a^2}\right]$

<u>Proof:</u> Given,

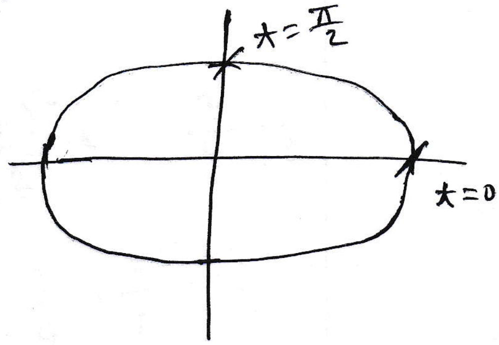
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{———} \quad (1)$$

The parametric equation of (1) is given by

$$x=a\cos t, \quad y=b\sin t.$$

$$\therefore \quad \frac{dx}{dt}=-a\sin t \quad \text{and} \quad \frac{dy}{dt}=b\cos t$$

$$\therefore \text{Perimeter}=4\int_{t=0}^{t=\frac{\pi}{2}}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$




$$=4\int_0^{\pi/2}\sqrt{a^2\sin^2 t+b^2\cos^2 t}\,dt$$

$$=4\int_0^{\pi/2}\sqrt{a^2(1-\cos^2 t)+b^2\cos^2 t}\,dt$$

$$=4\int_0^{\pi/2}\sqrt{a^2\left\{1-\cos^2 t+\frac{b^2}{a^2}\cos^2 t\right\}}\,dt$$

$$=4a\int_0^{\pi/2}\sqrt{1-\left(1-\frac{b^2}{a^2}\right)\cos^2 t}\,dt$$

$$\int_0^{\pi/2}\sin^n x\,dx = \int_0^{\pi/2}\cos^n x\,dx = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\,\frac{n-5}{n-4}\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} & [n \text{ is even}] \\ \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4}\cdots\frac{4}{5}\cdot\frac{2}{3}\cdot 1 & [n \text{ is odd}] \end{cases}$$

(195)

$$= 4a\int_0^{\pi/2}\left(1 - e^2\cos^2 t\right)^{1/2} dt \qquad \Big|\ e^2 = 1 - \frac{b^2}{a^2}$$

$$= 4a\int_0^{\pi/2}\Big[1 + \frac{1}{2}(-e^2\cos^2 t) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{\lfloor 2}(-e^2\cos^2 t)^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{\lfloor 3}(-e^2\cos^2 t)^3 + \cdots\cdots\Big]dt$$

$$= 4a\Big[\int_0^{\pi/2}dt - \frac{e^2}{2}\int_0^{\pi/2}\cos^2 t\,dt - \frac{(\frac{1}{2})^2}{\lfloor 2}e^4\int_0^{\pi/2}\cos^4 t\,dt - \frac{(\frac{1}{2})^3\cdot 3}{\lfloor 3}e^6\int_0^{\pi/2}\cos^6 t\,dt + \cdots\cdots\Big]$$

$$= 4a\Big[\frac{\pi}{2} - \frac{e^2}{2}\cdot\frac{1}{2}\cdot\frac{\pi}{2} - \frac{(\frac{1}{2})^2}{\lfloor 2}e^4\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} - \frac{(\frac{1}{2})^3\cdot 3}{\lfloor 3}\cdot e^6\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} - \cdots\cdots\Big]$$

$$= 2a\pi\Big[1 - \left(\frac{1}{2}\right)^2\frac{e^2}{1} - \left(\frac{1.3}{2.4}\right)^2\frac{e^4}{3} - \left(\frac{1.3.5}{2.4.6}\right)^2\frac{e^6}{5} - \cdots\cdots\Big]$$

**Proved**

H.W.

Compute the perimeters of the two conics

$$\frac{x^2}{9} + \frac{y^2}{7} = 1 \quad \text{and} \quad \frac{x^2}{36} + \frac{y^2}{28} = 1.$$

## Area

1. The area enclosed by

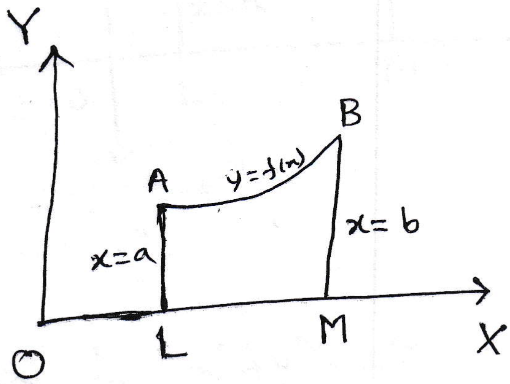
(i) the curve $y = f(x)$

(ii) the $x$-axis

(iii) the ordinate $x = a$

(iv) the ordinate $x = b$

is given by $\int_a^b y\,dx$




2. The area enclosed by

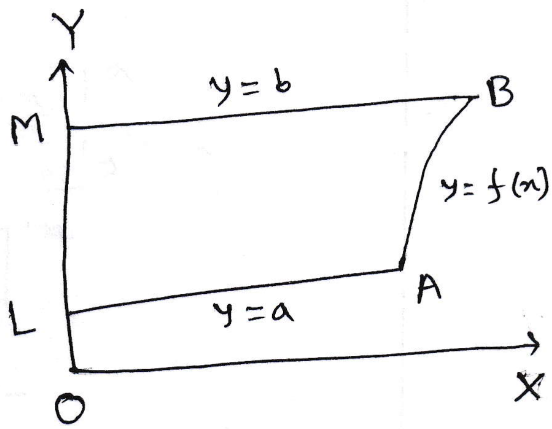
(i) the curve $y = f(x)$

(ii) the $y$-axis

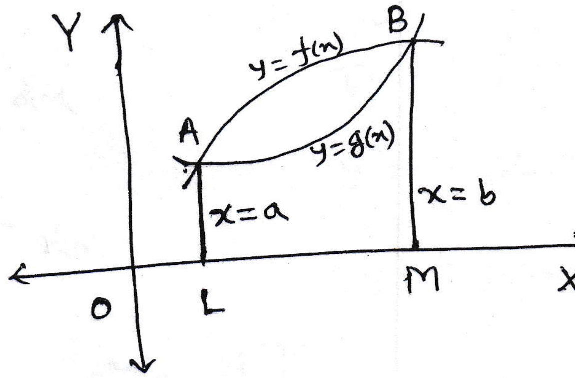
(iii) the absissa $y = a$

(iv) the absissa $y = b$

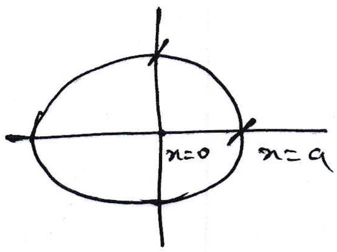
is given by $\int_a^b x\,dy$



3. The area between the curve $y = f(x)$ and $y = g(x)$ is given by $\int_a^b [f(x) - g(x)]\,dx$.




Find the area ~~fot~~ of the circle $x^2+y^2=a^2$.




Soln: Area of the circle

$$= 4\int_0^a y\,dx$$

$$= 4\int_0^a \sqrt{a^2-x^2}\,dx$$

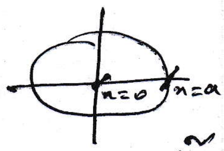
$$= 4\left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= 4\left[\left(0 + \frac{a^2}{2}\cdot\frac{\pi}{2}\right) - (0+0)\right]$$

$$= 4\cdot\frac{\pi a^2}{4}$$

$$= \pi a^2 \quad \text{Ans}$$

Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.




$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$

$\therefore y^2 = \frac{b^2}{a^2}(a^2 - x^2)$

$\therefore y = \frac{b}{a}\sqrt{a^2 - x^2}$

<u>Solⁿ:</u>

$$\text{Area} = 4\int_0^a y \, dx$$

$$= 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx$$

$$= 4\frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\, dx$$

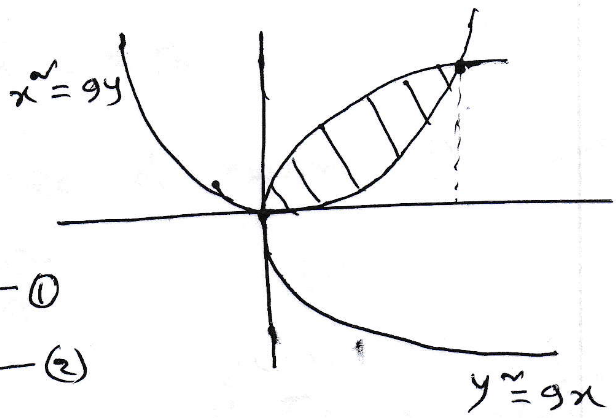
$$= 4\frac{b}{a}\left[\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= 4\frac{b}{a}\left[\left(0 + \frac{a^2}{2}\cdot\frac{\pi}{2}\right) - (0+0)\right]$$

$$= \frac{4b}{a}\cdot\frac{\pi a^2}{4}$$

$$= \pi ab$$

Find the area bounded by the curves $y^2 = 9x$ and $x^2 = 9y$.




Soln:

Given, $y^2 = 9x$ ——— ①

$x^2 = 9y$ ——— ②

From ① we get, $x = \frac{y^2}{9}$ ——— ③

Put ③ in ②

$$\frac{y^4}{81} = 9y$$

$$\Rightarrow y^4 = 9^3 y$$

$$\Rightarrow y(y^3 - 9^3) = 0$$

$$\Rightarrow y = 0, 9$$

when $y = 0$, ③ gives $x = 0$

when $y = 9$ from ③ we have $x = \frac{81}{9} = 9$

$\therefore$ The points of intersection of ① and ② are $(0,0)$ and $(9,9)$.

Find the area included between the curve $xy^2 = a^2(a-x)$ and its asymptotes.

Solⁿ: Given,

$$xy^2 = a^2(a-x)$$

$$\Rightarrow y^2 = \frac{a^2(a-x)}{x} \quad \text{——— (1)}$$

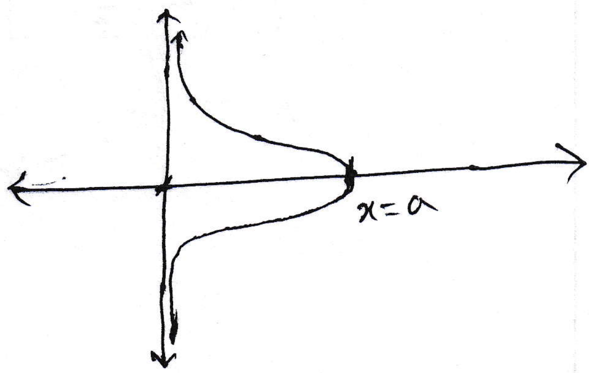
If $y=0$ then we have

$$\frac{a^2(a-x)}{x} = 0$$

$$\Rightarrow x = a$$

If $x=0$ then $y = \infty$

$\therefore$ $x=0$ is the asymptote of the curve (1). From (1) we see that the curve is symmetrical about $x$-axis as the left side of (1) is $y^2$.



The limits of the upper half are

$y = 0, \quad y = \infty$

$\therefore$ The required area $= 2\int_0^\infty x\,dy$

From ① we get,

$$xy^2 = a^3 - a^2x$$

$$\Rightarrow x(y^2 + a^2) = a^3$$

$$\Rightarrow x = \frac{a^3}{y^2+a^2}$$

$$\therefore \text{Area} = 2\int_0^\infty \frac{a^3}{y^2+a^2}\,dy$$

$$= 2a^3\left[\frac{1}{a}\tan^{-1}\frac{y}{a}\right]_0^\infty$$

$$= 2a^2\left[\tan^{-1}\infty - \tan^{-1}0\right]$$

$$= 2a^2\left[\frac{\pi}{2} - 0\right]$$

$$= \pi a^2$$

—x—

Find the area bounded by the curves $y^2 = x^3$ and the line $y = 2x$.

<u>Sol$^n$:</u> Given,

$$y^2 = x^3 \text{ ——— } (1)$$
$$y = 2x \text{ ——— } (2)$$

Putting (2) in (1) we get,

$$4x^2 = x^3 \Rightarrow x^3 - 4x^2 = 0$$
$$\Rightarrow x^2(x-4) = 0$$
$$\Rightarrow x = 0, 4$$

~~$\therefore y = 0, 8$~~

If $x = 0$ then $y = 0$

If $x = 4$ then $y = 2.4 = 8$

$\therefore$ Point of intersection of (1) and (2) are $(0,0)$ and $(4,8)$

From (1) we get, $y = f(x) = \sqrt{x^3}$

From (2) we get, $y = g(x) = 2x$

$\therefore$ The required area $= \int_{x=0}^{x=4} [f(x) - g(x)]\, dx$

$$= \int_0^4 \left(\sqrt{x^3} - 2x\right) dx$$

$$= \cancel{\int_0^4} \left[ \frac{x^{3/2+1}}{\frac{3}{2}+1} - \frac{\cancel{2}\, x^2}{\cancel{2}} \right]_0^4$$

$$= \left[ \left( \frac{2}{5} (4)^{\frac{5}{2}} - (4)^2 \right) - (0 - 0) \right]$$

$$= \left( \frac{64}{5} - 16 \right)$$

$$= \frac{64 - 80}{5}$$

$$= -\frac{16}{5}$$

Area cannot be negative.

$\therefore$ The required area $= \frac{16}{5}$

Find the area of the portion of the circle $x^2+y^2=1$ which lies inside the parabola $y^2=1-x$.

Sol$^n$: Given,

$$x^2+y^2=1 \quad \text{———} \quad (1)$$

$$y^2=1-x \quad \text{———} \quad (2)$$

Putting (2) in (1) we get,

$$x^2+1-x=1$$

$$\Rightarrow x^2-x=0$$

$$\Rightarrow x(x-1)=0$$

$$\therefore x=0,\ 1$$

when $x=0$, from (2) we get

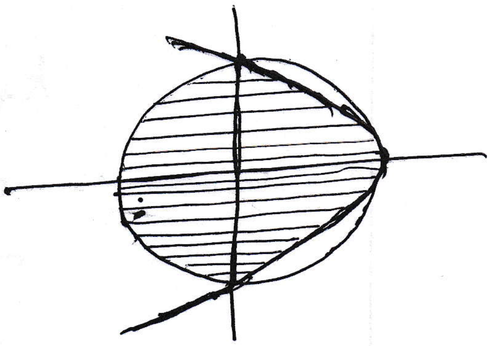
$$y^2=1-0$$

$$\Rightarrow y=\pm 1$$

When $x=1$, from (2) we get,

$$y^2=1-1=0$$

$$\Rightarrow y=0$$



Required area

~~$= \pi a^2$~~

$$= \pi.1^2-2\int_0^1 [f(x)-g(x)]\,dx$$

$$= \pi - 2\int_0^1 \left(\sqrt{1-x^2} - \sqrt{1-x}\right) dx$$

$$= \pi - 2\left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x - \frac{(1-x)^{3/2}}{\frac{3}{2}\cdot(-1)}\right]_0^1$$

$$= \pi - 2\left[\left(0 + \frac{1}{2}\frac{\pi}{2} - 0\right) - \left(0 + 0 + \frac{2}{3}\right)\right]$$

$$= \pi - \frac{\pi}{2} + \frac{4}{3}$$

$$= \frac{\pi}{2} + \frac{4}{3}$$

Ans

** $\boxed{y^2 = 4x}$

Find the area of the segment cut off from $y^2 = 4ax$ by the straight line $y = x$.

Soln: Given,

$$y^2 = 4ax \quad \text{--- (1)}$$

$$y = x \quad \text{--- (2)}$$

Putting (2) in (1) we get,

$$x^2 - 4ax = 0$$

$$\Rightarrow x(x-4a) = 0$$

$$\therefore x = 0, 4a$$

When $x = 0$, then $y = 0$

when $x = 4a$ then $y = 4a$

$\therefore$ The points of intersection of ① and ②
are $(0,0)$ and $(4a, 4a)$.

$\therefore$ The required area $= \int_0^{4a} [f(x) - g(x)]\, dx$

Here from ① we get,

$$y = f(x) = \sqrt{4ax}$$

From ② we get,

$$y = g(x) = x$$

$$\therefore \quad \text{Area} = \int_0^{4a} (\sqrt{4ax} - x)\, dx$$

$$= \int_0^{4a} \left[\sqrt{4a}\, x^{\frac{1}{2}} - x\right] dx$$

$$= \left[\sqrt{4a}\, \frac{x^{3/2}}{3/2} - \frac{x^2}{2}\right]_0^{4a}$$

$$= \left(2\sqrt{a} \cdot \frac{2}{3} \cdot (4a)^{3/2} - \frac{(4a)^2}{2}\right) - (0-0)$$

$$= \frac{32a^2}{3} - 8a^2 = \frac{8}{3}a^2$$ Ans

## Volume of Solid of Revolution:

$$V = \int_{x_1}^{x_2} \pi y^2 \, dx \quad \left(\text{about } x\text{-axis}\right)$$

Parametric

$$V = \int_{t_1}^{t_2} \pi y^2 \frac{dx}{dt} \, dt$$

$$V = \int_{x_1}^{x_2} \pi x^2 \, dy \quad (\text{about } y\text{-axis})$$

$$V = \int_{t_1}^{t_2} \pi x^2 \frac{dy}{dt} \, dt$$

## Surface Area of solids of Revolution:

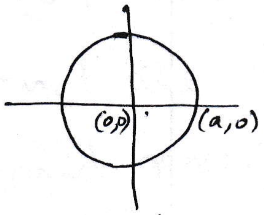
$$S = \int_{x_1}^{x_2} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \quad (\text{about } x\text{-axis})$$

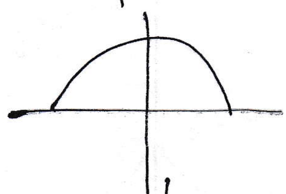
$$S = \int_{t_1}^{t_2} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

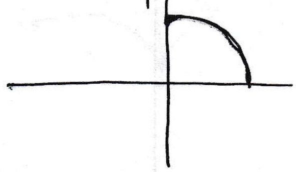
$$S = \int_{y_1}^{y_2} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy \quad (\text{about } y\text{-axis})$$

$$S = \int_{t_1}^{t_2} 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

Show that the volume and the surface-area of the whole sphere generated by revolving $x^2+y^2=a^2$ about the $x$-axis are respectively $\frac{4}{3}\pi a^3$ and $4\pi a^2$.





**Proof:** $V = 2\int_0^a \pi y^2 \, dx$

$$= 2\pi \int_0^a (a^2 - x^2) \, dx$$

$$= 2\pi \left[a^2 x - \frac{x^3}{3}\right]_0^a$$

$$= 2\pi \left(a^3 - \frac{a^3}{3}\right) = \frac{4}{3}\pi a^3$$

Here $x^2+y^2=a^2 \Rightarrow 2x+2y\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{x}{y}$, $y=\sqrt{a^2-x^2}$

$$S = 2\int_0^a 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 4\pi\int_0^a \sqrt{a^2-x^2}\sqrt{1+\frac{x^2}{y^2}}\,dx$$

$$= 4\pi\int_0^a \sqrt{a^2-x^2}\,\frac{\sqrt{y^2+x^2}}{y}\,dx$$

$$= 4\pi\int_0^a \sqrt{a^2-x^2}\,\frac{a}{\sqrt{a^2-x^2}}\,dx$$

$$= 4\pi a\,[x]_0^a$$

$$= 4\pi a(a-0)$$

$$= 4\pi a^2$$

—x—

$x^2+y^2=a^2$

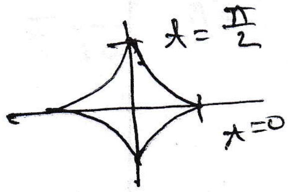
$y=\sqrt{a^2-x^2}$

$2x+2y\frac{dy}{dx}=0$

$\Rightarrow \frac{dy}{dx}=-\frac{x}{y}$

The astroid $x=a\cos^3 t$, $y=a\sin^3 t$ $\left(\text{or } x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}\right)$ revolves round the $x$-axis, show that the volume and surface-area of the whole solid generated are respectively $\frac{32}{105}\pi a^3$ and $\frac{12}{5}\pi a^2$.

Sol$^n$: $V = 2\int_0^{\pi/2} \pi y^2 \frac{dx}{dt}\,dt$



$$\int_0^{\pi/2} \sin^p t \cos^q t \, dt = \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}$$

$x = a\cos^3 t$

$\therefore \frac{dx}{dt} = 3a\cos^2 t.(-\sin t)$

$$\therefore V = 2\int_0^{\pi/2} \pi (a\sin^3 t)^2\, 3a\cos^2 t\,(-\sin t)\,dt$$

$$= -6\pi a^3 \int_0^{\pi/2} \sin^7 t \cos^2 t\,dt$$

$$= -6\pi a^3 \frac{\Gamma\frac{7+1}{2}\,\Gamma\frac{2+1}{2}}{2\,\Gamma\frac{7+2+2}{2}}$$

$$= -3\pi a^3 \frac{\Gamma 4\,.\,\Gamma 3/2}{\Gamma\frac{11}{2}}$$

$\Gamma n+1 = n\Gamma n$

$$= -\cancel{3}\pi a^3 \frac{\cancel{3}.2.1\,\cancel{\tfrac{1}{2}}.\cancel{\Gamma\tfrac{1}{2}}}{\cancel{3}\,\frac{\cancel{9}}{2}.\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\cancel{\tfrac{1}{2}}\cancel{\Gamma\tfrac{1}{2}}}$$

$= \cancel{64}$

$$= -\frac{32\pi a^3}{105}$$

Volume cannot be negative $\therefore V = \frac{32}{105}\pi a^3.$

$x = a\cos^3 t, \qquad y = a\sin^3 t$

$\therefore \dfrac{dx}{dt} = 3a\cos^2 t.(-\sin t) \qquad \therefore \dfrac{dy}{dt} = 3a\sin^2 t\cos t$

$\therefore$ Surface area $S = 2\displaystyle\int_0^{\pi/2} 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

$= 4\pi\displaystyle\int_0^{\pi/2} a\cos^3 t\sqrt{9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t}\;dt$

$= 4\pi\displaystyle\int_0^{\pi/2} a\cos^3 t \,.\, 3a\sin t\cos t\sqrt{\cos^2 t+\sin^2 t}\;dt$

$= 12\pi a^2\displaystyle\int_0^{\pi/3}\cos^4 t\sin t\,dt$

$\displaystyle\int_0^{\pi/2}\sin^p t\cos^q t\,dt = \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}$

$= 12\pi a^2\,\dfrac{\Gamma\frac{5}{2}\,\Gamma\frac{2}{2}}{2\,\Gamma\frac{7}{2}}$

$= \cancel{12\pi a^2\,\Gamma}$

$\Gamma 1 = 1$

$= 12\pi a^2\,\dfrac{\cancel{\frac{3}{2}}\cdot\cancel{\frac{1}{2}}\,\cancel{\Gamma\frac{1}{2}}}{\cancel{2}\cdot\frac{5}{\cancel{2}}\cdot\cancel{\frac{3}{2}}\cdot\cancel{\frac{1}{2}}\,\cancel{\Gamma\frac{1}{2}}}$

$= \dfrac{12}{5}\pi a^2$

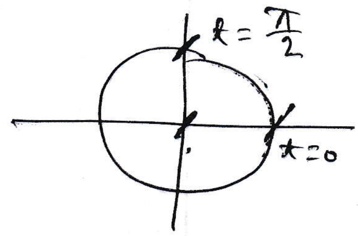
~~Ans~~ Proved

The circle $x = a\cos t$, $y = a\sin t$ revolves round the x-axis, show that the volume and the surface-area of the whole ~~area~~ sphere generated are resp. $\frac{4}{3}\pi a^3$ and $4\pi a^2$.

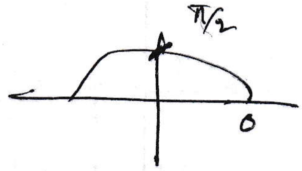
Proof: $x = a\cos t$, $y = a\sin t$

$\therefore \frac{dx}{dt} = -a\sin t$ $\therefore \frac{dy}{dt} = +a\cos t$

$$V = 2\int_0^{\pi/2} \pi y^2 \frac{dx}{dt} \cdot dt$$




$$= 2\pi \int_0^{\pi/2} a^2 \sin^2 t\,(-a\sin t)\,dt$$




$$= -2\pi a^3 \int_0^{\pi/2} \sin^3 t \cos^0 t\, dt$$

$$= -\cancel{2}\pi a^3 \frac{\Gamma\frac{4}{2}\,\Gamma\frac{1}{2}}{\cancel{2}\,\Gamma\frac{5}{2}}$$

$\Gamma 2 = 1! = 1$

$$= -\pi a^3 \frac{\Gamma 2\,\cancel{\Gamma\frac{1}{2}}}{\frac{3}{2}\cdot\frac{1}{2}\,\cancel{\Gamma\frac{1}{2}}}$$

$$= -\frac{4}{3}\pi a^3,$$ Volume cannot be negative

$\therefore V = \frac{4}{3}\pi a^3$ Proved

Surface area $S = 2\int_0^{\pi/2} 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$= 4\pi \int_0^{\pi/2} a\cos t \sqrt{a^2\sin^2 t + a^2\cos^2 t}\; dt$

$= 4\pi a^2 \int_0^{\pi/2} \cos t\, dt$

$= 4\pi a^2 \left[\sin t\right]_0^{\pi/2}$

$= 4\pi a^2$

Proved

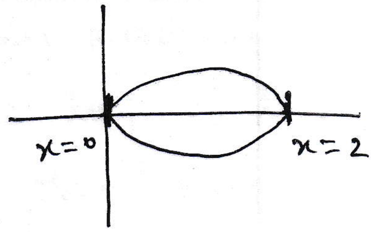
Show that the volume of the solid produced by the revolution of the upper half of the loop of the curve $y^2 = x^2(2-x)$ about the $x$-axis OX is $\frac{4}{3}\pi$.

Solⁿ: Given $y^2 = x^2(2-x)$ ——— (1)

If $y = 0$ then $x^2(2-x) = 0$ i.e. $x = 0, 2$

$\therefore$ The loop is formed between $x = 0$ to $x = 2$

$\therefore$ Required volume,




$$V = \int_0^2 \pi y^2 \, dx$$

$$= \int_0^2 \pi \; x^2(2-x)\, dx$$

$$= \pi \int_0^2 (2x^2 - x^3)\, dx$$

$$= \pi \left[ \frac{2}{3}x^3 - \frac{x^4}{4} \right]_0^2$$

$$= \pi \left[ \left( \frac{2}{3} \cdot 8 - \frac{\overset{4}{\cancel{16}}}{\cancel{4}} \right) - (0-0) \right]$$

$$= \pi \cdot \left( \frac{16-12}{3} \right)$$

$$= \frac{4}{3}\pi$$

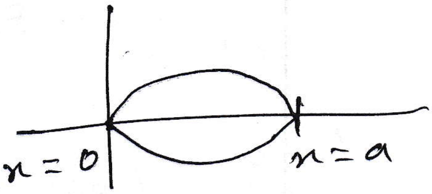
Proved

Show that the volume of the solid produced by revolution of the upper half of the loop of the curve $y^2(a+x) = x^2(a-x)$ about the x-axis is $2\pi a^3\left(\log 2 - \frac{2}{3}\right)$.

<u>Proof:</u> Given $y^2(a+x) = x^2(a-x)$ —— ①

If $y=0$ then, $0 = x^2(a-x)$ i.e. $x = 0, a$

$\therefore$ The loop is ~~sealed~~ formed from $x=0$ to $x=a$.




$\therefore$ Required volume

$$V = \int_0^a \pi y^2 \, dx$$

$$= \int_0^a \pi \frac{x^2(a-x)}{a+x} \, dx$$

Put $a + x = z$ $\quad \therefore dx = dz$

When $x = 0$, $z = a$

When $x = a$, $z = 2a$

$$\therefore V = \pi \int_a^{2a} \frac{(z-a)^2(a-z+a)}{z} \, dz$$

$$= \pi \int_a^{2a} \frac{(z^2 - 2za + a^2)(2a - z)}{z}\, dz$$

$$= \pi \int_a^{2a} \frac{2az^2 - 4a^2 z + 2a^3 - z^3 + 2az^2 - a^2 z}{z}\, dz$$

$$= \pi \int_a^{2a} \left( \frac{2a^3}{z} - 5a^2 + 4az - z^2 \right) dz$$

$$\cancel{= \pi \int_a^{2a}}$$

$$= \pi \left[ 2a^3 \log z - 5a^2 z + 2az^2 - \frac{z^3}{3} \right]_a^{2a}$$

$$= \pi \left[ \left( 2a^3 \log 2a - 5a^2 . 2a + 2a(4a^2) - \frac{8a^3}{3} \right) - \left( 2a^3 \log a - 5a^2 . a + 2a . a^2 - \frac{a^3}{3} \right) \right]$$

$$= \pi \left[ 2a^3 \log \frac{2\cancel{a}}{\cancel{a}} - 5a^2 (2a - a) + 2a(4a^2 - a^2) - \frac{a^3}{3}(8 - 1) \right]$$

$$= \pi \left[ 2a^3 \log 2 - 5a^3 + 6a^3 - \frac{7}{3} a^3 \right]$$

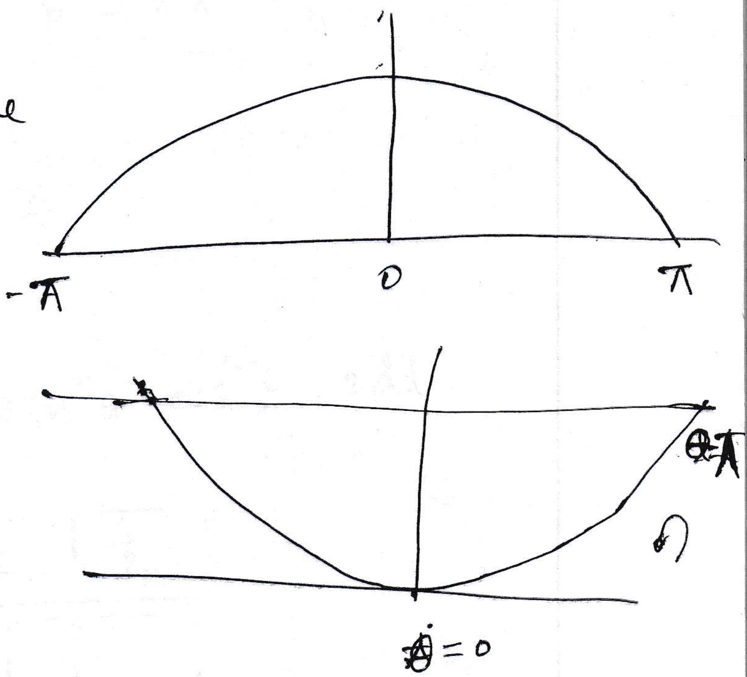
$$= \pi a^3 \left[ 2 \log 2 + \frac{-15 + 18 - 7}{3} \right]$$

$$= 2\pi a^3 \left[ \log 2 - \frac{2}{3} \right]$$

Proved

A cycloid $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$ revolves round the tangent at the vertex, show that the volume and the surface-area of the solid generated are $\pi^2 a^3$ and $\frac{32}{3}\pi a^2$ respectively. [a being the radius of the generating circle]

Sol$^n$: The vertex is the origin and the tangent at the vertex is the x-axis. Thus we revolute the cycloid about x-axis.




$\therefore$ Required volume

$$V = 2\int_{\theta=0}^{\pi} \pi y^2 \frac{dx}{d\theta}\, d\theta$$

Here

$x = a(\theta + \sin\theta)$ $\therefore$ $\frac{dx}{d\theta} = a(1+\cos\theta)$

$$V = 2\int_0^{\pi} \pi\, a^2(1-\cos\theta)^2\, a(1+\cos\theta)\, d\theta$$

$$= 2\pi a^3 \int_0^{\pi} \left(2\sin^2\frac{\theta}{2}\right)^2 \cdot 2\cos^2\frac{\theta}{2}\, d\theta$$

$$= 2\pi a^3 \int_0^{\pi} 4 \sin^4\frac{\theta}{2} \cdot 2\cos^2\frac{\theta}{2}\, d\theta$$

$$= 16\pi a^3 \int_0^{\pi} \sin^4\frac{\theta}{2} \cos^2\frac{\theta}{2}\, d\theta$$

Put $\frac{\theta}{2} = t \quad \therefore \quad \theta = 2t \quad \therefore\ d\theta = 2dt$

$\theta = 0 \quad \Rightarrow \quad t = 0$

$\theta = \pi \quad \Rightarrow \quad t = \frac{\pi}{2}$

$$\therefore\ V = 16\pi a^3 \int_0^{\pi/2} \sin^4 t \cos^2 t \cdot 2dt$$

$$= 32\pi a^3 \cdot \frac{\Gamma\frac{5}{2}\,\Gamma\frac{3}{2}}{2\,\Gamma\frac{8}{2}}$$

$$= 32\pi a^3 \frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{2\cdot 3\cdot 2\cdot 1}$$

$\Gamma\frac{1}{2} = \sqrt{\pi}$

$$= \cancel{4}\,\pi^2 a^3$$

Required surface area

$$S = 2\int_0^{\pi} 2\pi y \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$x = a(\theta + \sin\theta)$ $\qquad$ $y = a(1-\cos\theta)$

$\therefore \dfrac{dx}{d\theta} = a(1+\cos\theta)$ $\qquad$ $\therefore \dfrac{dy}{d\theta} = a\sin\theta$

$$\therefore S = 2\int_0^{\pi} 2\pi \, a(1-\cos\theta)\sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}\; d\theta$$

$$\cancel{= 4\pi a^2\int_0^{\pi} 2\sin^2\frac{\theta}{2}\sqrt{4\cos^2\frac{\theta}{2} + 4\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}\; d\theta}$$

$$\cancel{= 4\pi a^2\int_0^{\pi} 2\sin^2\frac{\theta}{2} \cdot 2\cos\frac{\theta}{2}\sqrt{1+\sin^2\frac{\theta}{2}}}$$

$$= 2\int_0^{\pi} 2\pi a(1-\cos\theta) \cdot a\sqrt{1+2\cos\theta + \cos^2\theta + \sin^2\theta}\; d\theta$$

$$= 4\pi a^2 \int_0^{\pi} (1-\cos\theta)\sqrt{2(1+\cos\theta)}\; d\theta$$

$$= 4\pi a^2 \int_0^{\pi} 2\sin^2\theta/2 \sqrt{2.2\cos^2\frac{\theta}{2}}\; d\theta$$

$$= 16\pi a^2 \int_0^{\pi} \sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\, d\theta$$

Put $\sin\frac{\theta}{2} = z$ $\quad \therefore \frac{1}{2}\cos\frac{\theta}{2}\, d\theta = dz$

$\therefore \cos\frac{\theta}{2}\, d\theta = 2dz$

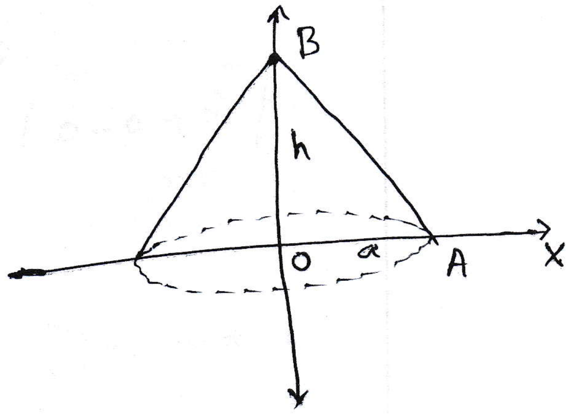
$\theta = 0 \Rightarrow z = 0$

$\theta = \pi \Rightarrow z = 1$

$$\therefore S = 16\pi a^2 \int_0^1 z^2\, 2dz = 32\pi a^2\left[\frac{z^3}{3}\right]_0^1$$

$$= \frac{32}{3}\pi a^2$$ **Proved**

Show that the volume of the right ~~angle~~ circular cone of height $h$ and base of radius $a$ is $\frac{1}{3}\pi a^2 h$.




Proof: Let O be the origin and $OA = a$ and $OB = h$.

Then the equation of AB is given by

$$\frac{x}{a} + \frac{y}{h} = 1 \quad \text{--- (1)}$$

The volume of the ~~right cir~~ required cone can be ~~found~~ obtained by revolving ~~AB~~ the area OAB about y-axis (OB).

$\therefore$ Required volume,

$$V = \int_{y=0}^{y=h} \pi x^2 \, dy$$

$$\frac{x}{a} + \frac{y}{h} = 1$$

$$hx + ay = ah$$

$$x = \frac{a(h-y)}{h}$$

$$= \int_0^h \pi \frac{a^2}{h^2} (h-y)^2 \, dy$$

$$= \frac{\pi a^2}{h^2} \int_0^h (h^2 - 2hy + y^2) \, dy$$

$$\cancel{= \frac{\pi a^2}{h^2} \left[ \frac{h^3}{3} - hy^2 + \frac{y^3}{3} \right]_0^h}$$

$$\cancel{\frac{\pi a^2}{h^2}\left[\frac{h^3}{3}\right]}$$

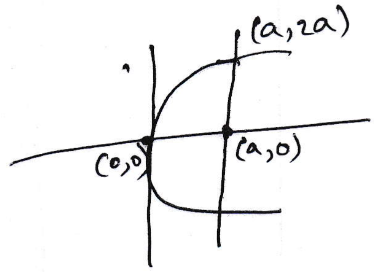
$$= \frac{\pi a^2}{h^2}\left[h^2 y - h y^2 + \frac{y^3}{3}\right]_0^h$$

$$= \frac{\pi a^2}{h^2}\left[\left(\cancel{h^3} - \cancel{h^3} + \frac{h^3}{3}\right) - (0 - 0 + 0)\right]$$

$$= \frac{1}{3}\pi a^2 h$$

Proved

The part of the parabola $y^2 = 4ax$ bounded by the latus rectum revolves about the $x$-axis. Find the volume and area of the curved surface of the generated paraboloid.




Solⁿ:

The required volume

$$V = \int_0^a \pi y^2\,dx = \int_0^a \pi \cdot 4ax\,dx = 4\pi a\left[\frac{x^2}{2}\right]_0^a$$

$$= 2\pi a^3$$

Required surface area

$$S = \int_0^a 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

Here $y^2 = 4ax$

$$\therefore 2y\frac{dy}{dx} = 4a$$

$$\therefore \frac{dy}{dx} = \frac{2a}{y}$$

$$\therefore S = \int_0^a 2\pi y\sqrt{1+\frac{4a^2}{y^2}}\,dx$$

$$= \int_0^a 2\pi \cancel{y}\cdot\frac{\sqrt{y^2+4a^2}}{\cancel{y}}\,dx$$

$$= 2\pi\int_0^a \sqrt{4ax+4a^2}\,dx$$

$$= 4\pi\sqrt{a}\int_0^a \sqrt{x+a}\,dx$$

Put $x+a = z \quad \therefore dx = dz$

$x=0 \Rightarrow z = a$

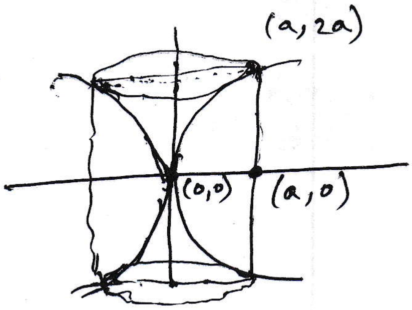
$x=a \Rightarrow z = 2a$

$$\therefore S = 4\pi\sqrt{a}\int_a^{2a} z^{\frac{1}{2}}\,dz$$

$$= 4\pi\sqrt{a}\left[\frac{z^{3/2}}{3/2}\right]_a^{2a}$$

$$= \frac{8\pi\sqrt{a}}{3}\left(2\sqrt{2}\,a\sqrt{a} - a\sqrt{a}\right) = \frac{8\pi a^2}{3}\left(2\sqrt{2}-1\right)$$

The part of the parabola $y^2=4ax$ bounded by the latus rectum revolves about the tangent at the vertex. Find the volume and area of the curved surface of the generated solid.




Proof: The tangent at the vertex is $x$ axis.

The required volume

$$V = 2\int_0^{2a} \pi x^2\, dy$$

$$= 2\pi \int_0^{2a} \left(\frac{y^2}{4a}\right)^2 dy$$

$$= \frac{\cancel{2}\pi}{\cancel{16}_{8}a^2}\left[\frac{y^5}{5}\right]_0^{2a}$$

$$= \cancel{\frac{1}{40}}$$

$$= \frac{\pi}{8a^2}\left[\frac{32a^5}{5} - 0\right]$$

$$= \frac{4}{5}\pi a^3$$

Also the required surface area is

$$S = 2\int_0^{2a} 2\pi x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

Here $y^2 = 4ax$

$$\therefore\ 2y = 4a\frac{dx}{dy}$$

$$\therefore\ \frac{dx}{dy} = \frac{y}{2a}$$

$$\therefore\ S = 4\pi\int_0^{2a} \frac{y^2}{4a}\sqrt{1+\frac{y^2}{4a^2}}\, dy$$

$$= \frac{4\pi}{8a^2}\int_0^{2a} y^2\sqrt{4a^2+y^2}\, dy$$

$$= \frac{\pi}{2a^2}\int_0^{2a} y^2\sqrt{4a^2+y^2}\, dy$$

Put $y = 2a\tan\theta$ $\quad\therefore\ dy = 2a\sec^2\theta\, d\theta$

$y = 0 \Rightarrow \theta = 0$

$y = 2a \Rightarrow \theta = \frac{\pi}{4}$

$$S = \frac{\pi}{\cancel{2}\cancel{a^2}}\int_0^{\pi/4} \overset{2}{\cancel{4}}\cancel{a^2}\tan^2\theta\sqrt{4a^2(1+\tan^2\theta)}\; 2a\sec^2\theta\, d\theta$$

$$= 8\pi a^2\int_0^{\pi/4}\tan^2\theta\sec^3\theta\, d\theta$$

We know that if $I_n = \int \sec^n x\,dx$ then

$$I_n = \frac{\sec^{n-1}x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

$$\therefore I_5 = \int \sec^5 x\,dx$$

$$= \frac{\sec^4 x \tan x}{4} + \frac{3}{4} \cdot I_3$$

$$\therefore I_5 - I_3 = \frac{\sec^4 x \tan x}{4} + \frac{3}{4} I_3 - I_3$$

$$= \frac{\sec^4 x \tan x}{4} + \left(\frac{3}{4} - 1\right) I_3$$

$$= \frac{\sec^4 x \tan x}{4} - \frac{1}{4}\left[\frac{\sec^2 x \tan x}{2} + \frac{1}{2} I_1\right]$$

$$= \frac{\sec^4 x \tan x}{4} - \frac{1}{8}\sec^2 x \tan x - \frac{1}{8}\int \sec x\,dx$$

$$\cancel{= \frac{\sec^4 x \tan x}{4} - \frac{1}{8}\sec^2 x \tan x - \frac{1}{8}\log\left|\tan\left(\frac{\pi}{4} + \frac{1}{2}x\right)\right|}$$

$$= \frac{1}{4}\sec^4 x \tan x - \frac{1}{8}\sec^2 x \tan x - \frac{1}{8}\log|\sec x + \tan x|$$

$$\therefore S = 8\pi a^2 \left[\frac{1}{4}\sec^4 x \tan x - \frac{1}{8}\sec^2 x \tan x - \frac{1}{8}\log|\sec x+\tan x|\right]_0^{\pi/4}$$

$$= 8\pi a^2 \left[\left(\frac{1}{4}(\sqrt{2})^4 \cdot 1 - \frac{1}{8}(\sqrt{2})^2 \cdot 1 - \frac{1}{8}\log|\sqrt{2}+1|\right) - (0-0-0)\right]$$

$$= 8\pi a^2 \left[1 - \frac{1}{4} - \frac{1}{8}\log|\sqrt{2}+1|\right]$$

$$\boxed{\begin{array}{l} \sec 0 = 1 \\ \log 1 = 0 \end{array}}$$

$$= 8\pi a^2 \left[\frac{3}{4} - \frac{1}{8}\log|\sqrt{2}+1|\right]$$

$$= \pi a^2 \left[6 - \log|\sqrt{2}+1|\right]$$

—x—