Prove that $\int uv\,dx = u\int v\,dx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx$

Proof: We know that, if u_1, v_1 are functions of x then

$$\frac{d}{dx}(u_1 v_1) = u_1\frac{dv_1}{dx} + v_1\frac{du_1}{dx}$$

Integrating both sides with respect to x we get,

$$u_1 v_1 = \int\left(u_1\frac{dv_1}{dx}\right)dx + \int\left(v_1\frac{du_1}{dx}\right)dx$$

$$\therefore \int\left(u_1\frac{dv_1}{dx}\right)dx = u_1 v_1 - \int\left(v_1\frac{du_1}{dx}\right)dx \quad \text{—— (1)}$$

Suppose, $u_1 = u$ and $\frac{dv_1}{dx} = v$ $\therefore v_1 = \int v\,dx$.

From (1) we have,

$$\int uv\,dx = u\int v\,dx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx$$

Proved

(i) $\int x e^x \, dx$

(ii) $\int \log x \, dx$

(iii) $\int \tan^{-1} x \, dx$

(iv) $\int \log\left(x + \sqrt{x^2+a^2}\right) dx$

(v) $\int x^3 e^x \, dx$

(vi) $\int x \sin x \, dx$

(vii) $\int x^n \log x \, dx$

Sol^n (iv) $\quad I = \int 1 \cdot \log\left(x+\sqrt{x^2+a^2}\right) \cdot 1 \, dx$

$$= \log\left(x+\sqrt{x^2+a^2}\right) \int dx - \int \left\{ \frac{d}{dx} \log\left(x+\sqrt{x^2+a^2}\right) \int dx \right\} dx$$

$$= x \log\left(x+\sqrt{x^2+a^2}\right) - \cancel{\int \frac{1}{x+\sqrt{x^2+a^2}}} \int \frac{1}{\sqrt{x^2+a^2}} \cdot x \, dx$$

$$= x \log\left(x+\sqrt{x^2+a^2}\right) - \frac{1}{2} \int \frac{2x}{\sqrt{x^2+a^2}} \, dx$$

$$= x \log\left(x+\sqrt{x^2+a^2}\right) - \frac{1}{\cancel{2}} \cdot \cancel{2} \sqrt{x^2+a^2} + c$$

—x—

Formula: $$\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$$

$$= \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left(bx - \tan^{-1}\frac{b}{a}\right)$$

Proof: Let $I = \int e^{ax}\cos bx\,dx$

$$\therefore I = e^{ax}\frac{\sin bx}{b} - \int\left(a e^{ax}\frac{\sin bx}{b}\right)dx$$

$$= e^{ax}\frac{\sin bx}{b} - \frac{a}{b}\int e^{ax}\sin bx\,dx$$

$$= e^{ax}\frac{\sin bx}{b} - \frac{a}{b}\left[e^{ax}\frac{-\cos bx}{b} - \int\left(a e^{ax}\frac{-\cos bx}{b}\right)dx\right]$$

$$= e^{ax}\frac{\sin bx}{b} + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}\int e^{ax}\cos bx\,dx$$

$$= \frac{e^{ax}(a\cos bx + b\sin bx)}{b^2} - \frac{a^2}{b^2}I$$

$$\Rightarrow I\left(1+\frac{a^2}{b^2}\right) = \frac{e^{ax}(a\cos bx + b\sin bx)}{b^2}$$

$$\Rightarrow I = \frac{b^2}{a^2+b^2}\cdot\frac{e^{ax}(a\cos bx + b\sin bx)}{b^2}$$

$$= \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$$

Put, $a = r\cos\alpha$ $\qquad a^2+b^2 = r^2(\cos^2\alpha+\sin^2\alpha) = r^2$

$b = r\sin\alpha$

$\therefore\ r^2 = a^2+b^2$

$$\frac{b}{a} = \frac{r\sin\alpha}{r\cos\alpha}$$

$$\boxed{I = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}}$$

$\Rightarrow\ \tan\alpha = \frac{b}{a}$

$\therefore\ \alpha = \tan^{-1}\frac{b}{a}$

$$\therefore I = \frac{e^{ax}(r\cos\alpha\cos bx + r\sin\alpha\sin bx)}{r^2}$$

$$= \frac{e^{ax}}{r}\cos(bx-\alpha)$$

$$= \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left(bx-\tan^{-1}\frac{b}{a}\right)$$

W. Prove that $\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$

$$= \frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left(bx-\tan^{-1}\frac{b}{a}\right)$$

4 Show that $\int \sqrt{x^2+a^2}\,dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\log|x+\sqrt{x^2+a^2}|$

Proof:

Let, $I = \int \sqrt{x^2+a^2}\,dx$, then

$$I = \int \sqrt{x^2+a^2}\,.\,1\,dx$$

$$= \sqrt{x^2+a^2}\int dx - \int\left\{\frac{d}{dx}\sqrt{x^2+a^2}\int dx\right\}dx$$

$$= \sqrt{x^2+a^2}\,.\,x - \int\left\{\frac{1}{\cancel{2}\sqrt{x^2+a^2}}\,.\,\cancel{2}x\,.\,x\right\}dx$$

$$= x\sqrt{x^2+a^2} - \int\left\{\frac{x^2+a^2-a^2}{\sqrt{x^2+a^2}}\right\}dx$$

$$= x\sqrt{x^2+a^2} - \int\sqrt{x^2+a^2}\,dx + a^2\int\frac{dx}{\sqrt{x^2+a^2}}$$

$$= x\sqrt{x^2+a^2} - I + a^2\log|x+\sqrt{x^2+a^2}|$$

$$\Rightarrow \quad 2I = x\sqrt{x^2+a^2} + a^2\log|x+\sqrt{x^2+a^2}|$$

$$\therefore \quad I = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log|x+\sqrt{x^2+a^2}|$$

Proved

Show that $\int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\log|x+\sqrt{x^2-a^2}|$

Proof: Let $I = \int \sqrt{x^2-a^2}\,dx$, then

$$I = \int \sqrt{x^2-a^2}\,.\,1\,dx$$

$$= \sqrt{x^2-a^2}\int dx - \int \left\{\frac{d}{dx}\sqrt{x^2-a^2}\int dx\right\}dx$$

$$= x\sqrt{x^2-a^2} - \int\left\{\frac{1}{\cancel{2}\sqrt{x^2-a^2}}\,.\,\cancel{2}x\,.\,x\right\}dx$$

$$= x\sqrt{x^2-a^2} - \int\left\{\frac{x^2-a^2+a^2}{\sqrt{x^2-a^2}}\right\}dx$$

$$= x\sqrt{x^2-a^2} - \int\sqrt{x^2-a^2}\,dx - a^2\int\frac{dx}{\sqrt{x^2-a^2}}$$

$$= x\sqrt{x^2-a^2} - I - a^2\int\frac{dx}{\sqrt{x^2-a^2}}$$

$$\Rightarrow \quad 2I = x\sqrt{x^2-a^2} - a^2\log|x+\sqrt{x^2-a^2}|$$

$$\therefore \quad I = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\log|x+\sqrt{x^2-a^2}|$$

Proved

Prove that $\int \sqrt{a^2-x^2}\,dx = \dfrac{x\sqrt{a^2-x^2}}{2} + \dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a}$

<u>Proof:</u> Let $I = \int \sqrt{a^2-x^2}\,dx$, Then

$$I = \int \sqrt{a^2-x^2}\,.\,1\,dx$$

$$= \sqrt{a^2-x^2}\,.\int dx - \int \left\{\frac{d}{dx}\sqrt{a^2-x^2}\int dx\right\}dx$$

$$= \sqrt{a^2-x^2}\,.\,x - \int \left\{\frac{1}{\cancel{2}\sqrt{a^2-x^2}}\,.\,(-\cancel{2}x)\,.\,x\right\}dx$$

$$= x\sqrt{a^2-x^2} - \int \frac{a^2-x^2-a^2}{\sqrt{a^2-x^2}}\,dx$$

$$= x\sqrt{a^2-x^2} - \int \sqrt{a^2-x^2}\,dx + a^2\int \frac{dx}{\sqrt{a^2-x^2}}$$

$$= x\sqrt{a^2-x^2} - I + a^2\sin^{-1}\frac{x}{a}$$

$$\Rightarrow 2I = x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}$$

$$\therefore I = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

<u>Proved</u>

Another way

$$I = \int \sqrt{a^2 - x^2}\, dx$$

Put $x = a\sin\theta$ $\therefore dx = a\cos\theta\, d\theta$

$$\therefore I = \int \sqrt{a^2 - a^2\sin^2\theta}\; a\cos\theta\, d\theta$$

$$= \int a^2 \cos\theta . \cos\theta\, d\theta$$

$$= \frac{a^2}{2}\int 2\cos^2\theta\, d\theta$$

$$= \frac{a^2}{2}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)$$

$$= \frac{a^2}{2}\left(\theta + \frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}}\right]$$

$$= \frac{a^2}{2}\left[\sin^{-1}\frac{x}{a} + \sin\theta\sqrt{1-\sin^2\theta}\right]$$

$$= \frac{a^2}{2}\left[\sin^{-1}\frac{x}{a} + \frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\right]$$

$$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x\sqrt{a^2-x^2}}{\cancel{a}\,2} = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

Proved

$a\sin\theta = x$

$\sin\theta = \frac{x}{a}$

$\theta = \sin^{-1}\frac{x}{a}$

Type 9

$I = \int \sqrt{ax^2+bx+c}\, dx$

ax^2+bx+c has the form $a[(x+l)^2 \pm m^2]$

or $a[m^2-(x+l)^2]$

Then we can use the formula for

$\int \sqrt{x^2 \pm a^2}\, dx$ or $\int \sqrt{a^2-x^2}\, dx$

Type 10

$I = \int (px+q)\sqrt{ax^2+bx+c}\, dx$

Put, $px+q = \frac{p}{2a}(2ax+b) + \left(q - \frac{bp}{2a}\right)$

Then $I = \frac{p}{2a}\int (2ax+b)\sqrt{ax^2+bx+c}\, dx$

$+ \left(q-\frac{bp}{2a}\right)\int \sqrt{ax^2+bx+c}\, dx$ Type 9

Put $z = ax^2+bx+c$ in the first integral.

Type 11

formula:

$$\int e^x \{f(x) + f'(x)\}\,dx = e^x f(x)$$

Proof: $\frac{d}{dx} e^x f(x) = e^x f'(x) + f(x)\, e^x$

$$= e^x \{f(x) + f'(x)\}$$

$$\therefore \int e^x \{f(x) + f'(x)\}\,dx = e^x f(x)$$

Formula

$$2\tan^{-1} x = \sin^{-1}\frac{2x}{1+x^2} = \cos^{-1}\frac{1-x^2}{1+x^2} = \tan^{-1}\frac{2x}{1-x^2}$$

Integrate :

(i) $\int e^{2x} \sin 3x \cos x \, dx$

(ii) $\int e^{x} \sin x \sin 2x \, dx$

(iii) $\int e^{x} \sin^2 x \, dx$

(iv) $\int e^{x} \sin x \, dx$

(v) $\int e^{x} \cos x \, dx$

$$\int e^{ax} \cos bx \, dx$$
$$= \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$$
$$= \frac{e^{ax}}{\sqrt{a^2+b^2}} \cos\left(bx - \tan^{-1}\frac{b}{a}\right)$$

$$\int e^{ax} \sin bx \, dx$$
$$= \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$$
$$= \frac{e^{ax}}{\sqrt{a^2+b^2}} \sin\left(bx - \tan^{-1}\frac{b}{a}\right)$$

Solⁿ (i)

$$I = \int e^{2x} \sin 3x \cos x \, dx$$
$$= \frac{1}{2}\int e^{2x} \cdot 2\sin 3x \cos x \, dx$$
$$= \frac{1}{2}\int e^{2x}(\sin 4x + \sin 2x)\, dx$$
$$= \frac{1}{2}\int e^{2x} \sin 4x \, dx + \frac{1}{2}\int e^{2x} \sin 2x \, dx$$
$$= \frac{1}{2}\,\frac{e^{2x}(2\sin 4x - 4\cos 4x)}{2^2+4^2} + \frac{1}{2}\cdot\frac{e^{2x}(2\sin 2x - 2\cos 2x)}{2^2+2^2} + c$$
$$= \frac{e^{2x}}{40}(2\sin 4x - 4\cos 4x + 5\sin 2x - 5\cos 2x) + c$$

Sol^n (iii)

$$I = \int e^x \sin^2 x \, dx$$

$$= \frac{1}{2}\int e^x \, 2\sin^2 x \, dx$$

$$= \frac{1}{2}\int e^x (1-\cos 2x)\, dx$$

$$= \cancel{\frac{1}{2}e^x - \frac{1}{2}\int e^x \sin}$$

$$= \frac{1}{2}e^x - \frac{1}{2}\int e^x \cos 2x \, dx$$

$$= \frac{1}{2}e^x - \frac{1}{2}\frac{e^x}{\sqrt{1^2+2^2}}\cos(2x - \tan^{-1}2)$$

$$= \frac{e^x}{2}\left[\cancel{e^x} \; 1 - \frac{1}{\sqrt{5}}\cos(2x - \tan^{-1}2)\right] + c$$

Formula

$$\int e^{ax}\cos bx \, dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$$

$$= \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left(bx - \tan^{-1}\frac{b}{a}\right)$$

Integral by parts:

(i) $I = \int \dfrac{x\,dx}{1+\cos x}$

(ii) $I = \int \sin x \log(\sec x + \tan x)\,dx$

(iii) $I = \int \cos x \log(\operatorname{cosec} x + \cot x)\,dx$

(iv) $I = \int \cos^{-1} \dfrac{1-x^2}{1+x^2}\,dx$

(v) $I = \int \sin^{-1} \dfrac{2x}{1+x^2}\,dx$

(vi) $I = \int \tan^{-1} \dfrac{2x}{1-x^2}\,dx$

$$2\tan^{-1}x = \cos^{-1}\frac{1-x^2}{1+x^2}$$
$$= \sin^{-1}\frac{2x}{1+x^2}$$
$$= \tan^{-1}\frac{2x}{1-x^2}$$

Soln: (1)

$$I = \int \frac{x\,dx}{1+\cos x} = \int \frac{x\,dx}{2\cos^2\frac{x}{2}}$$

$$= \frac{1}{2}\int x\sec^2\frac{x}{2}\,dx$$

$$= \frac{1}{2}\left[x\,\frac{\tan\frac{x}{2}}{\frac{1}{2}} - \int\left\{1 . \frac{\tan\frac{x}{2}}{\frac{1}{2}}\right\}dx\right]$$

$$= x\tan\frac{x}{2} - \frac{\log\left|\sec\frac{x}{2}\right|}{\frac{1}{2}} + c$$

$$= x\tan\frac{x}{2} - 2\log\left|\sec\frac{x}{2}\right| + c$$

—x—

Soln (ii)

$$I = \int \sin x \log(\sec x + \tan x)\,dx$$

$$= -\cos x\log(\sec x+\tan x) - \int \sec x . (-\cos x)\,dx$$

$$= -\cos x\log(\sec x+\tan x) + \int dx$$

$$= x - \cos x\log(\sec x+\tan x) + c$$

—x—

Solⁿ: (iii), (v), (vi)

$$I = \int 2\tan^{-1}x\,dx$$

$$= 2\int \tan^{-1}x \cdot 1\,dx$$

$$= 2\left[\tan^{-1}x\int dx - \int\left\{\frac{d}{dx}\tan^{-1}x\int dx\right\}dx\right]$$

$$= 2\left[x\tan^{-1}x - \int \frac{x}{1+x^2}\,dx\right]$$

$$= 2x\tan^{-1}x - \int \frac{2x}{1+x^2}\,dx$$

$$= 2x\tan^{-1}x - \log|1+x^2| + c$$

—x—

Type 9

$$I = \int \sqrt{ax^2+bx+c}\, dx$$

(i) $I = \int \sqrt{4+8x-5x^2}\, dx$

(ii) $I = \int \sqrt{4-3x-2x^2}\, dx$

(iii) $I = \int \sqrt{(x-1)(2-x)}\, dx$

(iv) $I = \int \sqrt{25-9x^2}\, dx$

(v) $I = \int \sqrt{5-2x+x^2}\, dx$

(vi) $I = \int \sqrt{10-4x+4x^2}\, dx$

(vii) $I = \int \sqrt{18x-65-x^2}\, dx$

(viii) $I = \int \sqrt{4-3x-2x^2}\, dx$

(ix) $I = \int \sqrt{5x^2+8x+4}\, dx$

(x) $I = \int \sqrt{2ax-x^2}\, dx$

(xi) $I = \int \sqrt{(x-\alpha)(\beta-x)}\, dx$

Solⁿ (i)

$$I = \int \sqrt{4+8x-5x^2}\,dx$$

$$= \int \sqrt{-5\left(x^2 - 2.\tfrac{4}{5}x + \tfrac{16}{25}\right) + 4 + \tfrac{16}{5}}\,dx$$

$$= \int \sqrt{\tfrac{36}{5} - 5\left(x-\tfrac{4}{5}\right)^2}\,dx$$

$$= \sqrt{5} \int \sqrt{\left(\tfrac{6}{5}\right)^2 - \left(x-\tfrac{4}{5}\right)^2}\,dx$$

$$= \sqrt{5}\left[\frac{\left(x-\frac{4}{5}\right)\sqrt{\left(\frac{6}{5}\right)^2 - \left(x-\frac{4}{5}\right)^2}}{2} + \frac{\left(\frac{6}{5}\right)^2}{2}\sin^{-1}\frac{\left(x-\frac{4}{5}\right)}{\frac{6}{5}}\right] + c$$

$$= \cancel{\sqrt{5}}\left[\frac{5x-4}{10}\sqrt{4+8x-5x^2} + \sqrt{5}.\frac{\overset{6}{\cancel{36}} \times \cancel{5}}{\underset{5}{\cancel{25}} \times \cancel{6}}\sin^{-1}\left(\frac{5x-4}{5}\right)\right] + c$$

$$= \frac{1}{10}\left[(5x-4)\sqrt{4+8x-5x^2} + 12\sqrt{5}\sin^{-1}\left(\frac{5x-4}{5}\right)\right] + c$$

—x—

Soln: (X1) $I = \int \sqrt{(x-\alpha)(\beta-x)}\,dx$

Put $x = \alpha\cos^2\theta + \beta\sin^2\theta$

$dx = \{\alpha\, 2\cos\theta(-\sin\theta) + \beta\, 2\sin\theta\cos\theta\}\,d\theta$

$= (-\alpha\sin 2\theta + \beta\sin 2\theta)\,d\theta$

$= (\beta-\alpha)\sin 2\theta\,d\theta$

$x - \alpha = \beta\sin^2\theta + \alpha\cos^2\theta - \alpha$

$= \beta\sin^2\theta - \alpha(1-\cos^2\theta)$

$= (\beta-\alpha)\sin^2\theta$

$\beta - x = \beta - \beta\sin^2\theta - \alpha\cos^2\theta$

$= \beta(1-\sin^2\theta) - \alpha\cos^2\theta$

$= (\beta-\alpha)\cos^2\theta$

$\therefore I = \int (\beta-\alpha)\sin\theta\cos\theta.(\beta-\alpha)\sin 2\theta\,d\theta$

$= \int \frac{(\beta-\alpha)^2}{2}\sin 2\theta\sin 2\theta\,d\theta$

$$= \frac{(\beta-\alpha)^2}{4} \int 2 \sin^2 2\theta \, d\theta$$

$$= \frac{(\beta-\alpha)^2}{4} \int (1 - \cos 4\theta) \, d\theta$$

$$= \frac{(\beta-\alpha)^2}{4} \left[\theta - \frac{\sin 4\theta}{4} \right] + c$$

$$x - \alpha = (\beta - \alpha) \sin^2\theta$$

$$\Rightarrow \sin^2\theta = \frac{x-\alpha}{\beta-\alpha}$$

$$\therefore \sin\theta = \sqrt{\frac{x-\alpha}{\beta-\alpha}}$$

$$\therefore \theta = \sin^{-1}\left(\sqrt{\frac{x-\alpha}{\beta-\alpha}}\right)$$

$$\beta - x = (\beta-\alpha) \cos^2\theta$$

$$\therefore \cos^2\theta = \frac{\beta - x}{\beta-\alpha}$$

$$\therefore \cos\theta = \sqrt{\frac{\beta-x}{\beta-\alpha}}$$

$$\sin 4\theta = 2 \sin 2\theta \cos 2\theta$$

$$= 4 \sin\theta \cos\theta (\cos^2\theta - \sin^2\theta)$$

$$\frac{\sin 4\theta}{4} = \sqrt{\frac{x-\alpha}{\beta-\alpha}}\sqrt{\frac{\beta-x}{\beta-\alpha}}\left(\frac{\beta-x}{\beta-\alpha} - \frac{x-\alpha}{\beta-\alpha}\right)$$

$$= \frac{\sqrt{(x-\alpha)(\beta-x)}}{\beta-\alpha}\,\frac{\beta-x-x+\alpha}{\beta-\alpha}$$

$$= \frac{(\alpha+\beta-2x)\sqrt{(x-\alpha)(\beta-x)}}{(\beta-\alpha)^2}$$

$$\therefore I = \frac{(\beta-\alpha)^2}{4}\left[\sin^{-1}\sqrt{\frac{x-\alpha}{\beta-\alpha}} - \frac{(\alpha+\beta-2x)\sqrt{(x-\alpha)(\beta-x)}}{(\beta-\alpha)^2}\right] + c$$

$$= \frac{(\beta-\alpha)^2}{4}\sin^{-1}\sqrt{\frac{x-\alpha}{\beta-\alpha}} - \frac{(\alpha+\beta-2x)}{4}\sqrt{(x-\alpha)(\beta-x)} + c$$

—x—

Sol^n (ii) $I = \int \sqrt{4-3x-2x^2}\,dx$

$$= \int \sqrt{-2\left(x^2+2.x.\frac{3}{4}+\frac{9}{16}\right)+\frac{9}{8}+4}\,dx$$

$$= \int \sqrt{\frac{41}{8} - 2\left(x+\frac{3}{4}\right)^2}\,dx$$

$$= \sqrt{2}\int \sqrt{\left(\frac{\sqrt{41}}{4}\right)^2 - \left(x+\frac{3}{4}\right)^2}\,dx$$

$$= \sqrt{2}\left[\frac{\left(x+\frac{3}{4}\right)\sqrt{\left(\frac{\sqrt{41}}{4}\right)^2-\left(x+\frac{3}{4}\right)^2}}{2} + \frac{\left(\frac{\sqrt{41}}{4}\right)^2}{2}\sin^{-1}\frac{\left(x+\frac{3}{4}\right)}{\frac{\sqrt{41}}{4}}\right] + C$$

$$= \frac{4x+3}{8}\sqrt{4-3x-2x^2} + \frac{41\sqrt{2}}{32}\sin^{-1}\left(\frac{4x+3}{\sqrt{41}}\right) + C$$

— x —

$$I = \int_1^2 \sqrt{(x-1)(2-x)}\, dx$$

Put $x = \cos^2\theta + 2\sin^2\theta$ #

$\therefore dx = \{2\cos\theta(-\sin\theta) + 4\sin\theta\cos\theta\} d\theta$

$= \sin 2\theta \, d\theta$

$x - 1 = 2\sin^2\theta - (1-\cos^2\theta) = 2\sin^2\theta - \sin^2\theta = \sin^2\theta$

$\therefore \sin\theta = \sqrt{x-1}$ $\quad\therefore \theta = \sin^{-1}\sqrt{x-1}$

$x = 1 \Rightarrow \theta = \sin^{-1} 0 = 0$

$x = 2 \Rightarrow \theta = \sin^{-1}(1) = \frac{\pi}{2}$

$2 - x = 2 - 2\sin^2\theta - \cos^2\theta$

$= 2(1-\sin^2\theta) - \cos^2\theta$

$= 2\cos^2\theta - \cos^2\theta$

$= \cos^2\theta$

$\therefore \cos\theta = \sqrt{2-x}$

$$I = \int_0^{\pi/2} \sqrt{\sin^2\theta\cos^2\theta} \cdot \sin 2\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^2 2\theta \, d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} (1 - \cos 4\theta)\, d\theta$$

$$= \frac{1}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2}$$

$$= \frac{1}{4}\left[\frac{\pi}{2} - 0 - 0 + 0\right]$$

$$= \frac{\pi}{8}$$

— x —

Type 10

(i) $\int (3x-2)\sqrt{x^2-x+1}\,dx$

(ii) $\int (x-1)\sqrt{x^2-1}\,dx$

(iii) $\int (x+b)\sqrt{x^2+a^2}\,dx$

(iv) $\int (x-1)\sqrt{x^2-x+1}\,dx$

(v) $\int (x+2)\sqrt{2x^2+2x+1}\,dx$

(vi) $\int (3x-1)\sqrt{x^2-x+1}\,dx$

(vii) $\int x\sqrt{x^2+1}\,dx$

Sol^n: (vi) Let,

$$I = \int (3x-1)\sqrt{x^2-x+1}\,dx$$

$$= \int \left\{\frac{3}{2}(2x-1)+\frac{3}{2}-1\right\}\sqrt{x^2-x+1}\,dx$$

$$= \frac{3}{2}\int (2x-1)\sqrt{x^2-x+1}\,dx - \frac{1}{2}\int\sqrt{x^2-2\cdot x\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2+1-\frac{1}{4}}\,dx$$

$$\cancel{= \frac{3}{2}\cdot\left(\sqrt{x^2-x+1}\right.}$$

$$= \frac{3}{2}\cdot\frac{(x^2-x+1)^{3/2}}{3/2} - \frac{1}{2}\int\sqrt{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

$$\cancel{= (x^2-x+1)^{3/2} - \frac{1}{2}\cdot\frac{\left(x-\frac{1}{2}\right)\sqrt{x^2-x+1}}{2} + \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2}\log\left|\left(x-\frac{1}{2}\right)+\sqrt{}\right.}$$

$$= \frac{3}{2}\cdot\frac{(x^2-x+1)^{3/2}}{\frac{3}{2}} - \frac{1}{2}\,\frac{\left(x-\frac{1}{2}\right)\sqrt{x^2-x+1}}{2} + \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2}\log\left|\left(x-\frac{1}{2}\right)+\sqrt{x^2-x+1}\right| + c$$

$$= (x^2-x+1)^{3/2} - \frac{(2x-1)\sqrt{x^2-x+1}}{8} + \frac{3}{8}\log\left|\left(x-\frac{1}{2}\right)+\sqrt{x^2-x+1}\right| + c$$

— x —

Solⁿ (iii) Let, $I = \int x\sqrt{x^2+1}\,dx$

$$= \frac{1}{2}\int 2x\sqrt{x^2+1}\,dx$$

$$= \frac{1}{2}\cdot\frac{(x^2+1)^{3/2}}{\frac{3}{2}} + c$$

$$= \frac{1}{3}(x^2+1)^{3/2} + c$$

—x—

Solⁿ: (v) $I = \int (x+2)\sqrt{2x^2+2x+1}\,dx$

$$\cancel{= \frac{1}{4}\int\left\{(4x+2)-\frac{1}{2}+2\right\}\sqrt{2x^2+2x+1}\,dx}$$

$$= \int\left\{\frac{1}{4}(4x+2)-\frac{1}{2}+2\right\}\sqrt{2x^2+2x+1}\,dx$$

$$= \frac{1}{4}\int(4x+2)\sqrt{2x^2+2x+1}\,dx + \frac{3}{2}\int\sqrt{2}\sqrt{x^2+x+\frac{1}{2}}\,dx$$

$$= \frac{1}{4}\frac{(2x^2+2x+1)^{3/2}}{\frac{3}{2}} + \frac{3\sqrt{2}}{2}\int\sqrt{x^2+2.x.\frac{1}{2}+\frac{1}{4}+\frac{1}{4}}\,dx$$

$$= \frac{1}{6}(2x^2+2x+1)^{3/2} + \frac{3\sqrt{2}}{2}\int\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}\,dx$$

$$= \frac{1}{6}(2x^2+2x+1)^{3/2} + \frac{3\sqrt{2}}{2}\left[\frac{\left(x+\frac{1}{2}\right)\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}}{2} + \frac{\left(\frac{1}{2}\right)^2}{2}\log\left|x+\frac{1}{2}+\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}\right|\right. + c$$

$$= \frac{1}{6}(2x^2+2x+1)^{3/2} + \frac{3}{8}(2x+1)\sqrt{2x^2+2x+1}$$

$$+ \frac{3\sqrt{2}}{16}\log\left|x+\frac{1}{2}+\sqrt{x^2+x+\frac{1}{2}}\right| + c$$

<u>Soln: (ii)</u> Let, $I = \int (x-1)\sqrt{x^2-1}\,dx$

$$= \int \left(\frac{1}{2}\cdot 2x - 1\right)\sqrt{x^2-1}\,dx$$

$$= \frac{1}{2}\int 2x\sqrt{x^2-1}\,dx - \int \sqrt{x^2-1}\,dx$$

$$= \frac{1}{2}\frac{(x^2-1)^{3/2}}{3/2} - \left(\frac{x\sqrt{x^2-1}}{2} - \frac{1^2}{2}\log\left|x+\sqrt{x^2-1}\right|\right) + c$$

$$= \frac{1}{3}(x^2-1)^{3/2} - \frac{1}{2}x\sqrt{x^2-1} + \frac{1}{2}\log\left|x+\sqrt{x^2-1}\right| + c$$

—x—

Type 11 $\int e^x(f(x)+f'(x))\,dx = e^x f(x)$

(i) $\int e^x(\cos x+\sin x)\,dx$

(ii) $\int \frac{e^x}{x}(1+x\log x)\,dx$

(iii) $\int e^x(\tan x-\log\cos x)\,dx$

(iv) $\int e^x\{\log(\sec x+\tan x)+\sec x\}\,dx$

(v) $\int e^x \sec x(1+\tan x)\,dx$

(vi) $\int \left\{\frac{1}{\log x}-\frac{1}{(\log x)^2}\right\}dx$

(vii) $\int \frac{xe^x}{(x+1)^2}\,dx$

(viii) $\int e^x\,\frac{x^2+1}{(x+1)^2}\,dx$

Soln (vi) Let, $I = \int \left\{ \frac{1}{\log x} - \frac{1}{(\log x)^2} \right\} dx$

Put $\log x = z$

$\therefore \quad x = e^z$

$\therefore \quad dx = e^z dz$

$\therefore \quad I = \int e^z \left(\frac{1}{z} - \frac{1}{z^2} \right) dz$

let $f(z) = \frac{1}{z} \qquad \therefore f'(z) = -\frac{1}{z^2}$

$\therefore \quad I = \int e^z \{ f(z) + f'(z) \} dz$

$= e^z f(z) + c$

$= e^z \cdot \frac{1}{z} + c$

$= \frac{x}{\log x} + c$

—x—

Sol^n (vii) Let, $I = \int \frac{x e^x}{(x+1)^2} dx$

$$= \int e^x \frac{(x+1-1)}{(x+1)^2} dx$$

$$= \int e^x \left\{ \frac{1}{x+1} - \frac{1}{(x+1)^2} \right\} dx$$

Put $f(x) = \frac{1}{x+1}$ $\therefore f'(x) = -\frac{1}{(x+1)^2}$

$$\therefore \quad I = \int e^x \{ f(x) + f'(x) \} dx$$

$$= e^x f(x) + c$$

$$= \frac{e^x}{x+1} + c$$

—x—

Sol^n: (viii) Let, $I = \int e^x \frac{x^2+1}{(x+1)^2} dx$

$$= \int e^x \frac{x^2+2x+1-2x}{(x+1)^2} dx$$

$$= \int e^x \left\{ 1 - \frac{2x}{(x+1)^2} \right\} dx$$

(vii)

$$= \int e^x dx - 2 \int \frac{x e^x}{(x+1)^2} dx$$

$$\int \frac{dx}{a+b\cos x}$$

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

$$\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

Sol^n: $I = \int \frac{dx}{a+b\cos x}$

$$= \int \frac{dx}{a+b\cdot\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$$

$$= \int \frac{\left(1+\tan^2\frac{x}{2}\right)}{a+a\tan^2\frac{x}{2}+b-b\tan^2\frac{x}{2}}\,dx$$

$$= \int \frac{\sec^2\frac{x}{2}}{(a+b)+(a-b)\tan^2\frac{x}{2}}\,dx$$

Case I. $a>b$

Put $\sqrt{a-b}\tan\frac{x}{2} = z$

$$\therefore \frac{\sqrt{a-b}}{2}\sec^2\frac{x}{2}\,dx = dz$$

$$\sec^2\frac{x}{2}\,dx = \frac{2}{\sqrt{a-b}}\,dz$$

~~$\therefore$ $I =$~~

$$\therefore I = \frac{2}{\sqrt{a-b}}\int \frac{dz}{(\sqrt{a+b})^2+z^2}$$

$$= \frac{2}{\sqrt{a-b}} \cdot \frac{1}{\sqrt{a+b}} \tan^{-1} \frac{z}{\sqrt{a+b}}$$

$$= \frac{2}{\sqrt{a^2-b^2}} \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}} \tan \frac{x}{2}\right)$$

We know that $2 \tan^{-1} x = \cos^{-1} \frac{1-x^2}{1+x^2}$

$$\therefore \quad I = \frac{1}{\sqrt{a^2-b^2}} \cos^{-1} \left\{ \frac{1 - \frac{a-b}{a+b} \tan^2 \frac{x}{2}}{1 + \frac{a-b}{a+b} \cdot \tan^2 \frac{x}{2}} \right\}$$

$$\tan^2 x = \frac{1-\cos 2x}{1+\cos 2x}$$

$$= \frac{1}{\sqrt{a^2-b^2}} \cos^{-1} \left\{ \frac{1 - \frac{a-b}{a+b} \cdot \frac{1-\cos x}{1+\cos x}}{1 + \frac{a-b}{a+b} \cdot \frac{1-\cos x}{1+\cos x}} \right\}$$

$$= \frac{1}{\sqrt{a^2-b^2}} \cos^{-1} \left\{ \frac{a+b+(a+b)\cos x - (a-b) + (a-b)\cos x}{a+b+(a+b)\cos x + (a-b) - (a-b)\cos x} \right\}$$

$$= \frac{1}{\sqrt{a^2-b^2}} \cos^{-1} \left\{ \frac{\cancel{a}+b-\cancel{a}+b + (a+\cancel{b}+a-\cancel{b})\cos x}{a+\cancel{b}+a-\cancel{b} + (\cancel{a}+b-\cancel{a}+b)\cos x} \right\}$$

$$= \frac{1}{\sqrt{a^2-b^2}} \cos^{-1} \left(\frac{b + a\cos x}{a + b\cos x} \right)$$

Case II: $a < b$

$$I = \int \frac{\sec^2 \frac{x}{2}\, dx}{(a+b) - (b-a)\tan^2 \frac{x}{2}}$$

Put $\sqrt{b-a} \tan \frac{x}{2} = z$

$$\therefore \frac{\sqrt{b-a}}{2} \sec^2 \frac{x}{2}\, dx = dz$$

$$\therefore \sec^2 \frac{x}{2}\, dx = \frac{2}{\sqrt{b-a}}\, dz$$

$$\therefore I = \frac{2}{\sqrt{b-a}} \int \frac{dz}{(\sqrt{a+b})^2 - z^2}$$

$$= \frac{2}{\sqrt{b-a}} \cdot \frac{1}{2\sqrt{a+b}} \log \left| \frac{\sqrt{a+b} + z}{\sqrt{a+b} - z} \right|$$

$$= \frac{1}{\sqrt{b^2-a^2}} \log \left| \frac{\sqrt{a+b} + \sqrt{b-a}\tan\frac{x}{2}}{\sqrt{a+b} - \sqrt{b-a}\tan\frac{x}{2}} \right|$$

—x—

H.W.

(I) $\int \frac{dx}{a+b\sin x}$

(II) $\int \frac{dx}{\sin x+\cos x}$

$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$\tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}$

(III) $\int \frac{dx}{a\sin x+b\cos x}$

(IV) $\int \frac{dx}{5-13\sin x}$

(V) $\int \frac{dx}{13+3\cos x+4\sin x}$

(VI) $\int \frac{dx}{5+4\sin x}$

(VII) $\int \frac{dx}{4+5\sin x}$

(VIII) $\int \frac{dx}{5+4\cos x}$

(IX) $\int \frac{dx}{3+5\cos x}$

(X) $\int \frac{dx}{\sin x+\tan x}$

$$\int \frac{a\sin x + b\cos x}{e\sin x + d\cos x}\,dx$$

Sol^n: $a\sin x + b\cos x = l(\text{denominator}) + m(\text{diff. of denominator})$

$= l(e\sin x + d\cos x) + m(e\cos x - d\sin x)$

$= (le - md)\sin x + (ld + me)\cos x$

$\therefore\ le - md = a$

$ld + me = b$

Solve to find l and m

let, $l = l'$, $m = m'$

$$\therefore I = \int l'\,dx + m'\int \frac{e\cos x - d\sin x}{e\sin x + d\cos x}\,dx$$

$$= l'x + m'\log|e\sin x + d\cos x|$$

(I) $\int \frac{2\sin x + 3\cos x}{3\sin x + 4\cos x}\,dx$

(II) $\int \frac{\sin x + 2\cos x}{2\sin x + 3\cos x}\,dx$

Sol^n (II) Let $I = \int \frac{\sin x + 2\cos x}{2\sin x + 3\cos x}\,dx$

Let, $\sin x + 2\cos x = l(2\sin x + 3\cos x) + m(2\cos x - 3\sin x)$

$= (2l - 3m)\sin x + (3l + 2m)\cos x$

$\therefore$ $2l - 3m = 1$

and $3l + 2m = 2$

$$l = \frac{\begin{vmatrix} 1 & -3 \\ 2 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ 3 & 2 \end{vmatrix}} = \frac{2+6}{4+9} = \frac{8}{13}$$

$$m = \frac{\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ 3 & 2 \end{vmatrix}} = \frac{4-3}{4+9} = \frac{1}{13}$$

$$\therefore I = \frac{8}{13}\int dx + \frac{1}{13}\int \frac{2\cos x - 3\sin x}{2\sin x + 3\cos x}\,dx$$

$$= \frac{8}{13}x + \frac{1}{3}\log|2\sin x + 3\cos x| + c$$

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$$\int \frac{dx}{\sin x + \tan x}, \quad I = \int \frac{dx}{\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} + \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}}$$

$$= \int \frac{\left(1+\tan^2\frac{x}{2}\right)\left(1-\tan^2\frac{x}{2}\right)dx}{2\tan\frac{x}{2} - \cancel{2\tan^3\frac{x}{2}} + 2\tan\frac{x}{2} + \cancel{2\tan^3\frac{x}{2}}}$$

$$= \int \frac{\left(1-\tan^2\frac{x}{2}\right)\sec^2\frac{x}{2}\,dx}{4\tan\frac{x}{2}}$$

Put, $\tan\frac{x}{2} = z$

$$\therefore \frac{1}{2}\sec^2\frac{x}{2} = dz$$

$$\therefore I = \frac{1}{2}\int \frac{(1-z^2)\,dz}{z}$$

$$= \frac{1}{2}\left[\int \frac{dz}{z} - \int z\,dz\right] = \frac{1}{2}\log|z| - \frac{1}{2}z^2$$

$$= \frac{1}{2}\log\left|\tan\frac{x}{2}\right| - \frac{1}{4}\tan^2\frac{x}{2} + c$$

Partial fraction

7. $$\int_0^{\pi/2} \frac{\cos x}{(1+\sin x)(2+\sin x)}\,dx = I$$

Put, $\sin x = z$ $\therefore$ $\cos x\,dx = dz$

$x=0,\quad z=0,\quad x=\frac{\pi}{2},\quad z=1$

$$\therefore I = \int_0^1 \frac{z}{(1+z)(2+z)}\,dz$$

Let, $$\frac{z}{(1+z)(2+z)} = \frac{A}{1+z} + \frac{B}{2+z} \quad \text{——} (1)$$

$$\Rightarrow z = A(2+z) + B(1+z) \quad \text{——} (2)$$

Put, $z=-1$ in (2)

$-1 = A + 0,\quad A=-1$

Put, $z=-2$ in (2)

$-2 = 0 - B \Rightarrow B = 2$

$\therefore$

$$\therefore\ I = \int_0^1 \left(\frac{-1}{1+z} + \frac{2}{2+z} \right) dz$$

$$= \Big[-\log|1+z| + 2\log|2+z| \Big]_0^1$$

$$= -\log 2 + 2\log 3 + \log 1 - 2\log 2$$

$$= 2\log 3 - 3\log 2$$

$\boxed{4}$ $$I = \int \frac{x^2}{x^4+x^2-2}\, dx$$

Let, $x^2 = z$, Then

$$\frac{x^2}{x^4+x^2-2} = \frac{z}{z^2+z-2} = \frac{z}{z^2+2z-z-2} = \frac{z}{(z+2)(z-1)}$$

$$= \frac{z}{z(z+2)-1(z+2)}$$

Let, $\dfrac{z}{(z+2)(z-1)} = \dfrac{A}{z+2} + \dfrac{B}{z-1}$ —— (1)

$\Rightarrow \quad z = A(z-1) + B(z+2)$ —— (2)

Put, $z = -2$ in (2),

$-2 = -3A + 0$

$\Rightarrow \quad A = \dfrac{2}{3}$

Put, $z = 1$ in (2),

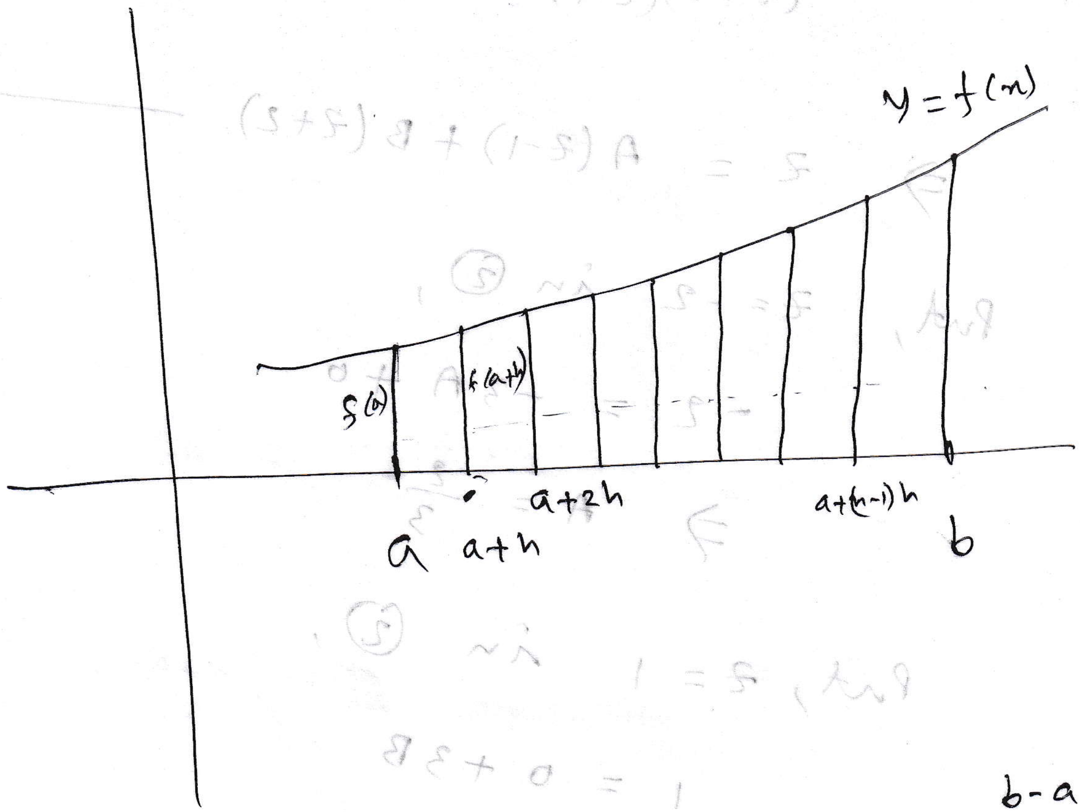
$1 = 0 + 3B$

$\Rightarrow \quad B = \dfrac{1}{3}$

$$I = \frac{2}{3}\int \frac{dx}{x^2+2} + \frac{1}{3}\int \frac{dx}{x^2-1}$$

$$= \frac{2}{3} \cdot \frac{1}{\sqrt{2}} \cdot \tan^{-1}\frac{x}{\sqrt{2}} + \frac{1}{3}\cdot\frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + c$$

$$= \frac{3}{3\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \frac{1}{6}\log\left|\frac{x-1}{x+1}\right| + c$$



$$\frac{b-a}{n} = h$$

$b - a = nh$

$b = a + nh$

$a + (n-1)h$

$a + nh - h$

$a + b - a - h$

$b - h$