Sol^n(iii) $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{3n}\right]$

$$= \lim_{n\to\infty}\left[\frac{1}{n+0}+\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}+\frac{1}{n+n+1}+\frac{1}{n+n+2}+\cdots+\frac{1}{n+n+n}\right]$$

$$= \lim_{n\to\infty}\sum_{r=0}^{n}\frac{1}{n+r}+\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{2n+r}$$

$$= \lim_{n\to\infty}\sum_{r=0}^{n}\frac{1}{n(1+\frac{r}{n})}+\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n(2+\frac{r}{n})}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n}\frac{1}{1+\frac{r}{n}}+\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{2+\frac{r}{n}}$$

$$= \int_0^1\frac{1}{1+x}\,dx+\int_0^1\frac{1}{2+x}\,dx$$

$$= \left[\log(1+x)\right]_0^1+\left[\log(2+x)\right]_0^1$$

$$= \log 2-\log 1+\log 3-\log 2$$

$$= \log 3$$

Ans

Sol^n (iv) $\lim_{n\to\infty}\left\{\left(1+\frac{1}{n^2}\right)^{\frac{2}{n^2}}\left(1+\frac{2^2}{n^2}\right)^{\frac{4}{n^2}}\left(1+\frac{3^2}{n^2}\right)^{\frac{6}{n^2}}\cdots\left(1+\frac{n^2}{n^2}\right)^{\frac{2n}{n^2}}\right\}$

Let $A=\left\{\left(1+\frac{1}{n^2}\right)^{\frac{2.1}{n^2}}\left(1+\frac{2^2}{n^2}\right)^{\frac{2.2}{n^2}}\left(1+\frac{3^2}{n^2}\right)^{\frac{2.3}{n^2}}\cdots\left(1+\frac{n^2}{n^2}\right)^{\frac{2.n}{n^2}}\right\}$

$$\therefore \log A = \frac{2.1}{n^2}\log\left(1+\frac{1}{n^2}\right)+\frac{2.2}{n^2}\log\left(1+\frac{2^2}{n^2}\right)+\frac{2.3}{n^2}\log\left(1+\frac{3^2}{n^2}\right)$$
$$+\cdots+\frac{2n}{n^2}\log\left(1+\frac{n^2}{n^2}\right)$$

$$=\sum_{r=1}^{n}\frac{2r}{n^2}\log\left(1+\frac{r^2}{n^2}\right)$$

$$\lim_{n\to\infty}\log A=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}2.\frac{r}{n}\log\left\{1+\left(\frac{r}{n}\right)^2\right\}$$

$$=\int_0^1 2x\log(1+x^2)\,dx$$

Put $1+x^2=z$

$\therefore\ 2x\,dx=dz$

x	0	1
z	1	2

$$\therefore \lim_{n\to\infty}\log A=\int_1^2\log z\,dz$$

$\left[\int \log x\,dx = x\log x - x\right]$

$$=\left[z\log z-z\right]_1^2$$
$$=(2\log 2-2)-(1\log 1-1)$$
$$=2\log 2-1$$

$$\therefore \lim_{n\to\infty} \log A = \log 4 - \log e = \log \frac{4}{e}$$

$$\Rightarrow \log\left(\lim_{n\to\infty} A\right) = \log\left(\frac{4}{e}\right)$$

$$\therefore \lim_{n\to\infty} A = \frac{4}{e}$$

—x—

Sol^n (xix)

$$L.H.S. = \lim_{n\to\infty} \frac{1^m + 2^m + 3^m + \cdots + n^m}{n^{m+1}}$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^m + \left(\frac{2}{n}\right)^m + \cdots + \left(\frac{n}{n}\right)^m\right]$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \left(\frac{r}{n}\right)^m$$

$$= \int_0^1 x^m \, dx$$

$$= \left[\frac{x^{m+1}}{m+1}\right]_0^1$$

$$= \frac{1}{m+1} - \frac{0}{m+1}$$

$$= \frac{1}{m+1} = R.H.S.$$

Proved

(xi) $\lim_{n\to\infty} \sum_{r=1}^{n-1} \frac{1}{n} \sqrt{\frac{n+r}{n-r}}$

$$\left[\lim_{n\to\infty} \frac{1}{n}\sqrt{\frac{n+0}{n-0}} = \lim_{n\to\infty} \frac{1}{n} = 0 \right]$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n-1} \frac{1}{n} \sqrt{\frac{n(1+\frac{r}{n})}{n(1-\frac{r}{n})}}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n-1} \sqrt{\frac{1+\frac{r}{n}}{1-\frac{r}{n}}}$$

$$= \int_0^1 \sqrt{\frac{1+x}{1-x}}\, dx$$

$$= \int_0^1 \frac{1+x}{\sqrt{1-x^2}}\, dx$$

$$= \int_0^1 \frac{dx}{\sqrt{1-x^2}} + \int_0^1 \frac{x}{\sqrt{1-x^2}}\, dx$$

$$= \left[\sin^{-1}x\right]_0^1 - \frac{1}{2}\int_0^1 \frac{-2x}{\sqrt{1-x^2}}\, dx$$

$$= \left[\sin^{-1}x\right]_0^1 - \left[\frac{1}{\cancel{2}} \cdot \cancel{2}\sqrt{1-x^2}\right]_0^1$$

$$= \sin^{-1}(1) - \sin^{-1}(0) - \left(\sqrt{1-1^2} - \sqrt{1-0^2}\right)$$

$$= \frac{\pi}{2} - 0 - 0 + 1$$

$$= 1 + \frac{\pi}{2}$$

—x—

(xviii) $\lim_{n\to\infty} \left[\frac{n!}{n^n}\right]^{1/n}$

Let, $A = \left[\frac{n!}{n^n}\right]^{1/n} = \left[\frac{1}{n}\cdot\frac{2}{n}\cdot\frac{3}{n}\cdot\ldots\ldots\frac{n}{n}\right]^{1/n}$

$$\therefore \log A = \frac{1}{n}\left[\log\left(\frac{1}{n}\right)+\log\left(\frac{2}{n}\right)+\cdots+\log\left(\frac{n}{n}\right)\right]$$

$$= \frac{1}{n}\sum_{r=1}^{n}\log\left(\frac{r}{n}\right)$$

$$\therefore \lim_{n\to\infty}\log A = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\log\left(\frac{r}{n}\right)$$

$$= \int_0^1 \log x \, dx$$

$$= \left[x\log x - x\right]_0^1$$

$$= 1\log 1 - 1$$

$$= -\log e$$

$$= \log e^{-1}$$

$$\therefore \log\left(\lim_{n\to\infty} A\right) = \log\left(\frac{1}{e}\right)$$

$$\therefore \cancel{\log} \lim_{n\to\infty} A = \frac{1}{e}$$ Ans

Sol^n (v) $\lim_{n\to\infty}\left[\frac{1}{n+m}+\frac{1}{n+2m}+\cdots\cdots+\frac{1}{n+nm}\right]$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n+rm}$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n(1+m\frac{r}{n})}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+m\frac{r}{n}}$$

$$= \int_0^1 \frac{dx}{1+mx}$$

$$= \Big[\log(1+mx)\Big]_0^1$$

$$= \log(1+m) - \log 1$$

$$= \log(1+m)$$

—x—

Sol^n (vi) $\lim\limits_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\cdots+\frac{n}{n^2+n^2}\right]$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{n}{n^2+r^2}$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{n}{n^2\left(1+\frac{r^2}{n^2}\right)}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+\left(\frac{r}{n}\right)^2}$$

$$= \int_0^1 \frac{dx}{1+x^2}$$

$$= \left[\tan^{-1}x\right]_0^1$$

$$= \tan^{-1}(1)-\tan^{-1}(0)$$

$$= \frac{\pi}{4}-0$$

$$= \frac{\pi}{4}$$ Ans

Sol^n (vii) $\lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2-1^2}}+\frac{1}{\sqrt{n^2-2^2}}+\cdots+\frac{1}{\sqrt{n^2-(n-1)^2}}\right]$

$$= \lim_{n\to\infty}\sum_{r=1}^{n-1}\frac{1}{\sqrt{n^2-r^2}}$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n-1}\frac{1}{\sqrt{n^2\left(1-\frac{r^2}{n^2}\right)}}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n-1}\frac{1}{\sqrt{1-\left(\frac{r}{n}\right)^2}}$$

$$= \int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

$$= \left[\sin^{-1}x\right]_0^1$$

$$= \sin^{-1}(1)-\sin^{-1}(0)$$

$$= \frac{\pi}{2}-0$$

$$= \frac{\pi}{2}$$ Ans

Sol^n ~~(viii)~~ (ix)

$$\lim_{n\to\infty}\left[\frac{1^2}{n^3+1^3}+\frac{2^2}{n^3+2^3}+\cdots+\frac{n^2}{2n^3}\right]$$

$$=\lim_{n\to\infty}\left[\frac{1^2}{n^3+1^3}+\frac{2^2}{n^3+2^3}+\cdots\cdots+\frac{n^2}{n^3+n^3}\right]$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{r^2}{n^3+r^3}$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{r^2}{n^3\left(1+\frac{r^3}{n^3}\right)}$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{\frac{r^2}{n^2}}{n\left(1+\frac{r^3}{n^3}\right)}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{\left(\frac{r}{n}\right)^2}{1+\left(\frac{r}{n}\right)^3}$$

$$=\int_0^1\frac{x^2}{1+x^3}\,dx$$

$$=\frac{1}{3}\int_0^1\frac{3x^2}{1+x^3}\,dx$$

$$=\frac{1}{3}\left[\log(1+x^3)\right]_0^1$$

$$=\frac{1}{3}\left[\log 2-\log 1\right]$$

$$=\frac{1}{3}\log 2$$ Ans

Sol^n (viii)

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{2n-1^2}}+\frac{1}{\sqrt{4n-2^2}}+\cdots+\frac{1}{n}\right]$$

$$=\lim_{n\to\infty}\left[\frac{1}{\sqrt{2n.1-1^2}}+\frac{1}{\sqrt{2n.2-2^2}}+\cdots+\frac{1}{\sqrt{2n.n-n^2}}\right]$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\sqrt{2nr-r^2}}$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\sqrt{n^2\left(2\frac{r}{n}-\frac{r^2}{n^2}\right)}}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{\sqrt{2\frac{r}{n}-\left(\frac{r}{n}\right)^2}}$$

$$=\int_0^1\frac{dx}{\sqrt{2x-x^2}}$$

$$=\int_0^1\frac{dx}{\sqrt{1-x^2+2x-1}}$$

$$=\int_0^1\frac{dx}{\sqrt{1-(x-1)^2}}$$

$$=\left[\sin^{-1}(x-1)\right]_0^1$$

$$=\sin^{-1}(0)-\sin^{-1}(-1)=0+\frac{\pi}{2}=\frac{\pi}{2}$$ Ans

Sol^n(X) $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{\sqrt{n^2-1^2}}{n^2}+\dots+\frac{\sqrt{n^2-(n-1)^2}}{n^2}\right]$

$$=\lim_{n\to\infty}\left[\frac{\sqrt{n^2-0^2}}{n^2}+\frac{\sqrt{n^2-1^2}}{n^2}+\dots\dots+\frac{\sqrt{n^2-(n-1)^2}}{n^2}\right]$$

$$=\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{\sqrt{n^2-r^2}}{n^2}$$

$$=\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{\sqrt{n^2\left(1-\frac{r^2}{n^2}\right)}}{n^2}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\sqrt{1-\left(\frac{r}{n}\right)^2}$$

$$=\int_0^1\sqrt{1-x^2}\,dx$$

$$\int\sqrt{a^2-x^2}\,dx=\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

$$=\left[\frac{x\sqrt{1-x^2}}{2}+\frac{1}{2}\sin^{-1}x\right]_0^1$$

$$=\left(0+\frac{1}{2}\sin^{-1}(1)\right)-\left\{0+\frac{1}{2}\sin^{-1}(0)\right\}$$

$$=\frac{1}{2}\left(\pi/2\right)$$

$$=\frac{\pi}{4}$$ Ans

Sol^n (XII)

$$\lim_{n\to\infty}\left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\cdots\left(1+\frac{n}{n}\right)\right\}^{1/n}$$

Let, $A = \left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\cdots\left(1+\frac{n}{n}\right)\right\}^{1/n}$

$$\log A = \frac{1}{n}\left[\log\left(1+\frac{1}{n}\right)+\log\left(1+\frac{2}{n}\right)+\cdots+\log\left(1+\frac{n}{n}\right)\right]$$

$$\therefore \lim_{n\to\infty}\log A = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\log\left(1+\frac{r}{n}\right)$$

$$= \int_0^1 \log(1+x)\,dx$$

Put $1+x = z$ $\quad\therefore dx = dz$

x	0	1
z	1	2

$$\therefore \lim_{n\to\infty}\log A = \int_1^2 \log z\,dz$$

$$\Rightarrow \log\left(\lim_{n\to\infty} A\right) = \left[z\log z - z\right]_1^2$$

$$= (2\log 2 - 2) - (1\log 1 - 1)$$

$$= 2\log 2 - 2 - 0 + 1$$

$$= \log 4 - \log e$$

$$= \log\frac{4}{e}$$

$$\therefore \lim_{n\to\infty} A = \frac{4}{e}$$ Ans

Solⁿ (xiii) $\lim_{n\to\infty}\left\{\left(1+\frac{1^2}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\cdots\left(1+\frac{n^2}{n^2}\right)\right\}^{1/n}$

Let, $A=\left\{\left(1+\frac{1^2}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\cdots\left(1+\frac{n^2}{n^2}\right)\right\}^{1/n}$

$\therefore \log A = \frac{1}{n}\left\{\log\left(1+\frac{1^2}{n^2}\right)+\log\left(1+\frac{2^2}{n^2}\right)+\cdots+\log\left(1+\frac{n^2}{n^2}\right)\right\}$

$\therefore \lim_{n\to\infty}\log A = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\log\left(1+\frac{r^2}{n^2}\right)$

$\Rightarrow \log\left(\lim_{n\to\infty}A\right) = \int_0^1 \log(1+x^2)\,dx$

$\cancel{= \log(1+x^2)\int_0^1 dx - \int}$

Now, $\int \log(1+x^2)\,dx$

$= \log(1+x^2)\int dx - \int\left\{\frac{d}{dx}\log(1+x^2)\cdot\int dx\right\}dx$

$= x\log(1+x^2) - \int\left\{\frac{2x}{1+x^2}\cdot x\right\}dx$

$= x\log(1+x^2) - \int \frac{2(x^2+1)-2}{1+x^2}\,dx$

$= x\log(1+x^2) - 2\int dx + 2\int\frac{dx}{1+x^2}$

$= x\log(1+x^2) - 2x + 2\tan^{-1}x$

$\therefore \log\left(\lim_{n\to\infty}A\right) = \left[x\log(1+x^2)-2x+2\tan^{-1}x\right]_0^1$

$= \left(\log 2 - 2 + 2\cdot\frac{\pi}{4}\right) - (0-0+0)$

$= \log 2 - \log e^2 + \log e^{\pi/2} = \log\left(\frac{2e^{\pi/2}}{e^2}\right)$

$$\therefore \lim_{n\to\infty} A = 2e^{\frac{\pi}{2}-2}$$

Ans

Sol^n (XV)

$$\lim_{n\to\infty} \sum_{r=1}^{n} \frac{n}{(n+r)\sqrt{r(2n+r)}}$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{n}{n\left(1+\frac{r}{n}\right)\sqrt{n^2\left(2\frac{r}{n}+\frac{r^2}{n^2}\right)}}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{\left(1+\frac{r}{n}\right)\sqrt{2\frac{r}{n}+\left(\frac{r}{n}\right)^2}}$$

$$= \int_0^1 \frac{dx}{(1+x)\sqrt{2x+x^2}}$$

Put, $1+x = \frac{1}{z}$ $\qquad \therefore dx = -\frac{1}{z^2}dz$

x	0	1
z	1	$\frac{1}{2}$

$$\therefore \int_0^1 \frac{dx}{(1+x)\sqrt{2x+x^2}} = \int_1^{1/2} \frac{-\frac{1}{z^2}dz}{\frac{1}{z}\sqrt{2\left(\frac{1}{z}-1\right)+\left(\frac{1}{z}-1\right)^2}}$$

$$= \int_1^{1/2} \frac{-\frac{1}{z^2}dz}{\frac{1}{z}\sqrt{\frac{2-2z}{z}+\frac{1-2z+z^2}{z^2}}}$$

$$= -\int_1^{1/2} \frac{dz}{\sqrt{2z-2z^2+1-2z+z^2}}$$

$$= -\int_1^{1/2} \frac{dz}{\sqrt{1-z^2}}$$

$$= -\left[\sin^{-1} z\right]_1^{1/2}$$

$$= -\left[\sin^{-1}\left(\tfrac{1}{2}\right) - \sin^{-1}(1)\right]$$

$$= -\left[\frac{\pi}{6} - \frac{\pi}{2}\right]$$

$$= -\left[\frac{\pi - 3\pi}{6}\right]$$

$$= \frac{\pi}{3} \quad \underline{\underline{\text{Ans}}}$$

Sol^n: (xvi) $\lim_{n\to\infty} \frac{1}{\sqrt{n}}\left[1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}\right]$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n}}\left[\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}\right]$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{r=1}^{n} \frac{1}{\sqrt{r}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{r=1}^{n} \frac{1}{\sqrt{n \cdot \frac{r}{n}}}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{\sqrt{\frac{r}{n}}}$$

$$= \int_0^1 \frac{dx}{\sqrt{x}} = \left[2\sqrt{x}\right]_0^1 = 2 \quad \underline{\underline{\text{Ans}}}$$

Soln: (xvii) $\lim_{n\to\infty}\left[\frac{\sqrt{n+1}+\sqrt{n+2}+\cdots+\sqrt{2n}}{n\sqrt{n}}\right]$

$$= \lim_{n\to\infty}\left[\frac{\sqrt{n+1}}{n\sqrt{n}}+\frac{\sqrt{n+2}}{n\sqrt{n}}+\cdots+\frac{\sqrt{n+n}}{n\sqrt{n}}\right]$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{\sqrt{n+r}}{n\sqrt{n}}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\sqrt{\frac{n+r}{n}}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\sqrt{1+\frac{r}{n}}$$

$$= \int_0^1\sqrt{1+x}\,dx$$

$$= \left[\frac{(1+x)^{3/2}}{\frac{3}{2}}\right]_0^1$$

$$= \frac{2}{3}\left[2^{3/2}-1^{3/2}\right]$$

$$= \frac{2}{3}\left[2\sqrt{2}-1\right]$$

$$= \frac{4}{3}\sqrt{2}-\frac{2}{3}$$ Ans

Geometrical Interpretation of $\int_a^b f(x)\,dx$:

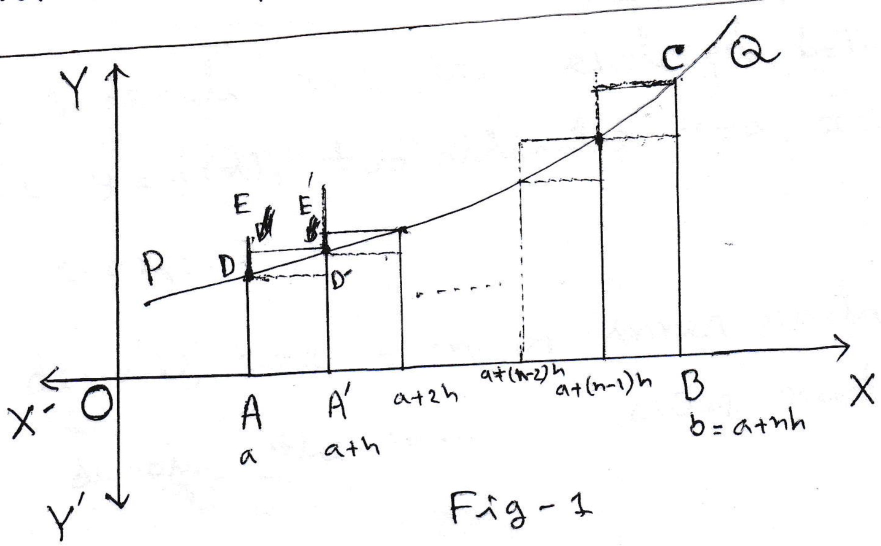

Fig - 1

Let $f(x)$ be a bounded single valued function defined on the interval (a, b). Let $y=f(x)$ be the equation of the continuous curve PQ presented in the Fig-1. Let AD and BC be two ordinates corresponding to the points $x=a$ and $x=b$.

From the figure we have, $OA=a$, $OB=b$.

$\therefore AB = b-a$. Let AB be divided into n equal parts each of length h.

$\therefore h = \frac{b-a}{n}$ or, $nh = b-a$ or, $a+nh=b$.

Draw the ordinates through the points $x=a+h$, $x=a+2h, \ldots, x=a+(n-1)h$.

Complete ~~te~~ the inner ractangles $ADD'A'$, ...