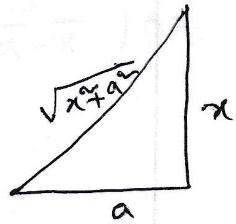
(A) $$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad (a \neq 0).$$

Proof: Put $x = a\tan\theta \quad \therefore dx = a\sec^2\theta\, d\theta$

Then $$I = \int \frac{dx}{x^2+a^2} = \int \frac{a\sec^2\theta\, d\theta}{a^2(\tan^2\theta+1)} = \frac{1}{a}\int \frac{\sec^2\theta}{\sec^2\theta}\, d\theta$$

$$= \frac{1}{a}\int d\theta = \frac{1}{a}\theta + c$$

$$= \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$



$x = a\tan\theta$

$\therefore \tan\theta = \frac{x}{a}$

$\theta = \tan^{-1}\left(\frac{x}{a}\right)$

Cor: $$\int \frac{dx}{1+x^2} = \tan^{-1}x$$

Cor: $$\int \frac{-dx}{x^2+a^2} = \frac{1}{a}\cot^{-1}\frac{x}{a}$$ | $x = a\cot\theta$, $dx = -a\,\mathrm{cosec}^2\theta\, d\theta$

(B) $$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| \quad (|x| > |a|)$$

Proof: $$I = \int \frac{dx}{x^2-a^2} = \int \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)dx$$

$$= \frac{1}{2a}\left(\int \frac{dx}{x-a} - \int \frac{dx}{x+a}\right)$$

$$= \frac{1}{2a}\{\log|x-a| - \log|x+a|\}$$

$$= \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|$$

(c) $$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| \qquad (|x| < |a|)$$

Proof: $$I = \int \frac{dx}{a^2-x^2} = \int \frac{1}{2a}\left(\frac{1}{a+x} + \frac{1}{a-x}\right)dx$$

$$= \frac{1}{2a}\left(\int \frac{dx}{a+x} + \int \frac{dx}{a-x}\right)$$

$$= \frac{1}{2a}\left\{\log|a+x| + \frac{\log|a-x|}{(-1)}\right\}$$

$$= \frac{1}{2a}\{\log|a+x| - \log|a-x|\}$$

$$= \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|$$

D) $$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log\left|\left(x + \sqrt{x^2 \pm a^2}\right)\right|$$

Proof: Put $\sqrt{x^2 \pm a^2} = z - x$

or, $z = x + \sqrt{x^2 \pm a^2}$

$$\therefore dz = dx + \frac{1}{\cancel{2}\sqrt{x^2 \pm a^2}} \cdot \cancel{2}x\,dx$$

$$\therefore dz = \left(\frac{\sqrt{x^2 \pm a^2} + x}{\sqrt{x^2 \pm a^2}} \right) dx$$

$$= \frac{z}{\sqrt{x^2 \pm a^2}} dx$$

$$\therefore dx = \frac{\sqrt{x^2 \pm a^2}}{z} dz$$

$$\therefore I = \int \frac{dx}{\sqrt{x^2 \pm a^2}}$$

$$= \int \frac{1}{\sqrt{x^2 \pm a^2}} \cdot \frac{\sqrt{x^2 \pm a^2}}{z} dz$$

$$= \int \frac{dz}{z}$$

$$= \log |z|$$

$$= \log \left| x + \sqrt{x^2 \pm a^2} \right|$$

$$I = \int \frac{dx}{\sqrt{x^2 - a^2}}$$

Put $x = a\sec\theta$ $\quad \therefore dx = a\sec\theta\tan\theta\, d\theta$

$$\therefore I = \int \frac{a\sec\theta\tan\theta}{a\tan\theta}\, d\theta$$

$$= \int \sec\theta\, d\theta$$

$$= \int \frac{\sec\theta(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta}\, d\theta$$

$z = \sec\theta + \tan\theta$

$\therefore dz = (\sec\theta\tan\theta + \sec^2\theta)\, d\theta = \sec\theta(\tan\theta + \sec\theta)\, d\theta$

$$\therefore I = \int \frac{dz}{z} = \log|z| = \log|\sec\theta + \tan\theta|$$

$$= \log\left|\sec\theta + \sqrt{\sec^2\theta - 1}\right|$$

$$= \log\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right|$$

$$= \log\left|\frac{x + \sqrt{x^2 - a^2}}{a}\right|$$

$$= \log\left|x + \sqrt{x^2 - a^2}\right| - \log|a|$$

constant

$$I = \int \frac{dx}{\sqrt{x^2+a^2}}$$

Put $x = a\tan\theta$

$\therefore\ dx = a\sec^2\theta\, d\theta$

$$\therefore\ I = \int \frac{a\sec^2\theta}{\sqrt{a^2(\tan^2\theta+1)}}\, d\theta$$

$$= \int \frac{a\sec^2\theta}{a\sec\theta}\, d\theta$$

$$= \int \sec\theta\, d\theta$$

$$= \int \frac{\sec\theta(\sec\theta+\tan\theta)}{\sec\theta+\tan\theta}\, d\theta$$

$$= \log|\sec\theta + \tan\theta|$$

$$= \log|\sqrt{1+\tan^2\theta} + \tan\theta|$$

$$= \log\left|\sqrt{1+\frac{x^2}{a^2}} + \frac{x}{a}\right|$$

$$= \log\left|\frac{\sqrt{x^2+a^2}+x}{a}\right|$$

$$= \log|x+\sqrt{x^2+a^2}| - \log|a|$$

Constant

(E) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$ $\quad (|x| < |a|)$

Put $x = a\sin\theta$ $\quad \therefore dx = a\cos\theta\, d\theta$

$$\therefore I = \int \frac{a\cos\theta}{\sqrt{a^2(1-\sin^2\theta)}}\, d\theta$$

$$= \int \frac{a\cos\theta\, d\theta}{a\cos\theta}$$

$$= \int d\theta$$

$$= \theta$$

$$= \sin^{-1}\frac{x}{a}$$

$\sin\theta = \frac{x}{a}$

$\therefore \theta = \sin^{-1}\frac{x}{a}$

** $\boxed{-\int \frac{dx}{\sqrt{a^2-x^2}} = \cos^{-1}\frac{x}{a} \quad \text{Put } x = a\cos\theta}$

Type-1

$$I = \int \frac{dx}{ax^2+bx+c} \quad (a \neq 0)$$

We can write, $I = \frac{1}{a}\int \frac{dx}{x^2+\frac{b}{a}x+\frac{c}{a}}$

$$= \frac{1}{a}\int \frac{dx}{x^2+2.x.\frac{b}{2a}+\left(\frac{b}{2a}\right)^2+\frac{c}{a}-\frac{b^2}{4a^2}}$$

$$= \frac{1}{a}\int \frac{dx}{\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}}$$

Putting $z = x+\frac{b}{2a}$ we have I in the form $\frac{1}{a}\int \frac{dz}{z^2 \pm k^2}$

Type-2

$$I = \int \frac{px+q}{ax^2+bx+c}\,dx \quad (a \neq 0,\ p \neq 0)$$

$$= \int \frac{p\left(x+\frac{q}{p}\right)}{ax^2+bx+c}\,dx$$

$$= \frac{p}{2a}\int \frac{2ax+\frac{2aq}{p}}{ax^2+bx+c}\,dx$$

$$= \frac{p}{2a}\int \frac{2ax+b+\frac{2aq}{p}-b}{ax^2+bx+c}\,dx$$

Type-1

$$= \frac{p}{2a}\left\{\int \frac{(2ax+b)}{ax^2+bx+c}\,dx + \left(\frac{2aq}{p}-b\right)\int \frac{dx}{ax^2+bx+c}\right\}$$

$$= \frac{p}{2a}\left\{\log|ax^2+bx+c| + \left(\frac{2aq}{p}-b\right)\int \frac{dx}{ax^2+bx+c}\right\}$$

Type-3

$$I = \int \frac{dx}{\sqrt{ax^2+bx+c}} \qquad (a \neq 0)$$

Similar to Type-1, I has of the form

$$\frac{1}{\sqrt{a}} \int \frac{dz}{\sqrt{z^2 \pm k^2}}$$

Type-4

$$I = \int \frac{px+q}{\sqrt{ax^2+bx+c}}\, dx$$

similar to Type 2 I can be written as

$$I = \frac{p}{2a} \left\{ \underbrace{\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}\, dx}_{\int \frac{dz}{\sqrt{z}} = 2\sqrt{z}} + \left(\frac{2aq}{p} - b\right) \underbrace{\int \frac{dx}{\sqrt{ax^2+bx+c}}}_{\text{Type - 3}} \right\}$$

Type - 5

$$I = \int \frac{dx}{(ax+b)\sqrt{ex+d}} \qquad (a \neq 0,\ c \neq 0)$$

Put $ex+d = z^2 \quad \therefore\ e\,dx = 2z\,dz$

$dx = \frac{2z}{e}\,dz$

$ex = z^2 - d$

$x = \frac{z^2-d}{e}$

$$I = \int \frac{\cancel{dx}\ \frac{2z}{e}\,dz}{\left(a\cdot\frac{z^2-d}{e} + b\right) z}$$

$$= 2\int \frac{dz}{az^2 - ad + be}$$

$$= \frac{2}{a}\int \frac{dz}{z^2 + \frac{be}{a} - d}$$

$\rightarrow \int \frac{dx}{x^2 \pm a^2}$ or $\int \frac{dx}{a^2 - x^2}$

Type - 6

$$I = \int \frac{dx}{(px+q)\sqrt{ax^2+bx+c}} \qquad (a \neq 0,\ p \neq 0)$$

Put $px+q = \frac{1}{z} \quad \therefore\ p\,dx = -\frac{1}{z^2}\,dz$

$\therefore\ dx = -\frac{1}{pz^2}\,dz$

$x = \frac{1}{p}\left(\frac{1}{z} - q\right)$

$$I = -\frac{1}{p}\int \frac{dz}{z^2 \cdot \frac{1}{z}\sqrt{\frac{a}{p^2}\left(\frac{1}{z}-q\right)^2 + \frac{b}{p}\left(\frac{1}{z}-q\right) + c}}$$

$$= -\frac{1}{p}\int \frac{dz}{\sqrt{\frac{a}{p^2}(1-qz)^2 + \frac{bz}{p}(1-qz) + cz^2}}$$

After simplification I has the form

$$-\int \frac{dz}{\sqrt{Az^2 + Bz + C}}$$

which is Type-3

Type-7

$$I = \int \sqrt{\frac{ax+b}{cx+d}}\,dx$$

$$= \int \sqrt{\frac{(ax+b)(ax+b)}{(cx+d)(ax+b)}}\,dx$$

$$= \int \frac{ax+b}{\sqrt{acx^2 + (bc+ad)x + bd}} \rightarrow \text{Type-4}$$

Type-8

$$I = \int \frac{dx}{\sqrt{(ax+b)(cx+d)}}$$

Put $ax+b = z^2$ $\therefore ax = z^2 - b$

$\therefore x = \frac{1}{a}(z^2-b)$

$\therefore a\,dx = 2z\,dz$

$$dx = \frac{2z}{a}\,dz$$

$$I = \int \frac{\frac{2z}{a}\,dz}{z\sqrt{\frac{c}{a}(z^2-b)+d}}$$

$$= \frac{2}{a}\int \frac{dz}{\sqrt{\frac{c}{a}\left(z^2-b+\frac{ad}{c}\right)}}$$

This is of the form

$$= \frac{2}{\sqrt{ac}}\int \frac{dz}{\sqrt{z^2 \pm k^2}}$$

$$\int \frac{dx}{ax^2+bx+c}, \quad a \neq 0$$

Type - 1

(i) $\int \frac{dx}{1+x+x^2}$

(ii) $\int \frac{dx}{4x^2+4x+5}$

(iii) $\int \frac{dx}{1+x-x^2}$

(iv) $\int \frac{dx}{6x^2+7x+2}$

(v) $\cancel{\int \frac{dx}{6x^2+7x+2}}$

$\int \frac{dx}{x\{10+7\log x+(\log x)^2\}}$

Put $\log x = z$

$\therefore \frac{dx}{x} = dz$

$I = \int \frac{dz}{10+7z+z^2}$

(vi) $\int \frac{x\,dx}{x^4+2x^2+2}$

(vii) $\int \frac{\cos x\,dx}{\sin^2 x+4\sin x+3}$

(viii) $\int \frac{e^x\,dx}{e^{2x}+2e^x+5}$

(ix) $\int \frac{x^2\,dx}{x^6-6x^3+5}$

Sol^n: (i)

$$I = \int \frac{dx}{1+x+x^2}$$

$$= \int \frac{dx}{x^2+2.x.\frac{1}{2}+\frac{1}{4}+1-\frac{1}{4}}$$

$$= \int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+c$$

$$= \frac{2\sqrt{3}}{3} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+c$$

$$= \frac{2\sqrt{3}}{3} \tan^{-1}\left\{\frac{\sqrt{3}(2x+1)}{3}\right\}+c$$

—x—

(iii) $I = \int \frac{dx}{1+x-x^2}$

$$= \int \frac{dx}{1-\left(x^2 - 2.x.\frac{1}{2}+\frac{1}{4}\right)+\frac{1}{4}}$$

$$= \int \frac{dx}{\left(\frac{\sqrt{5}}{2}\right)^2 - \left(x-\frac{1}{2}\right)^2}$$

$$= \frac{1}{2\left(\frac{\sqrt{5}}{2}\right)} \log \left| \frac{\frac{\sqrt{5}}{2} + x - \frac{1}{2}}{\frac{\sqrt{5}}{2} - \left(x-\frac{1}{2}\right)} \right| + c$$

$$= \frac{\sqrt{5}}{5} \log \left| \frac{\sqrt{5}-1+x}{\sqrt{5}+1-x} \right| + c$$

(vi) $I = \int \frac{x\,dx}{x^4+2x^2+2}$

Put $x^2 = z \quad \therefore 2x\,dx = dz$

$\therefore I = \frac{1}{2}\int \frac{dz}{z^2+2z+2}$

$= \frac{1}{2}\int \frac{dz}{(z+1)^2+1^2}$

$= \frac{1}{2}\cdot\frac{1}{1}\tan^{-1}\left(\frac{z+1}{1}\right) + c$

$= \frac{1}{2}\tan^{-1}(x^2+1) + c$

(vii) $I = \int \frac{\cos x\,dx}{\sin^2 x + 4\sin x + 3}$

Put $\sin x = z$

$\therefore \cos x\,dx = dz$

$I = \int \frac{dz}{z^2+4z+3} = \int \frac{dz}{(z+2)^2-1}$

$= \frac{1}{2.1}\log\left|\frac{z+2-1}{z+2+1}\right| + c = \frac{1}{2}\log\left|\frac{\sin x+1}{\sin x+3}\right| + c$

(viii) $$I = \int \frac{e^x\,dx}{e^{2x}+2e^x+5}$$

Put $e^x = z \quad \therefore e^x dx = dz$

$$\therefore I = \int \frac{dz}{z^2+2z+5}$$

$$= \int \frac{dz}{(z+1)^2+2^2}$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{z+1}{2}\right)+c$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{e^x+1}{2}\right)+c$$

—x—

(ix) $$I = \int \frac{x^2\,dx}{x^6-6x^3+5}$$

Put $x^3 = z \quad \therefore 3x^2 dx = dz$

$$I = \frac{1}{3}\int \frac{dz}{z^2-6z+5} = \frac{1}{3}\int \frac{dz}{(z-3)^2-2^2}$$

$$= \frac{1}{3}\cdot\frac{1}{2.2}\log\left|\frac{z-3-2}{z-3+2}\right|+c$$

$$= \frac{1}{12}\log\left|\frac{z-5}{z-1}\right|+c$$

(17) $\int \frac{px+q}{ax^2+bx+c}\,dx$ $a \neq 0,\ p \neq 0$

Type – 2

(i) $\int \frac{x\,dx}{x^2+2x+1}$

(ii) $\int \frac{x+1}{3+2x-x^2}\,dx$

(iii) $\int \frac{x+1}{x^2+4x+5}\,dx$

(iv) $\int \frac{2x+3}{4x^2+1}\,dx$

(v) $\int \frac{4x+3}{3x^2+3x+1}\,dx$

(vi) $\int \frac{x}{2-6x-x^2}\,dx$

(vii) $\int \frac{x^2-x+1}{x^2+x+1}\,dx$

Sol^n (i) $I = \int \frac{x\,dx}{x^2+2x+1}$

$$= \int \frac{\frac{1}{2}(2x+2) - 1}{x^2+2x+1}\,dx$$

$$= \frac{1}{2}\int \frac{2x+2}{x^2+2x+1}\,dx - \int \frac{dx}{(x+1)^2}$$

$$= \frac{1}{2}\log|x^2+2x+1| - \frac{(x+1)^{-2+1}}{-2+1} + c$$

$$= \log|x+1| + \frac{1}{x+1} + c$$

Sol^n (ii) $I = \int \frac{x+1}{3+2x-x^2}\,dx$

$$= \int \frac{-\frac{1}{2}(2-2x)+2}{3+2x-x^2}\,dx$$

$$= -\frac{1}{2}\int \frac{2-2x}{3+2x-x^2}\,dx + 2\int \frac{dx}{3-x^2+2x-1+1}$$

$$= -\frac{1}{2}\log|3+2x-x^2| + 2\int \frac{dx}{2^2-(x-1)^2}$$

$$= -\frac{1}{2}\log|3+3x-x-x^2| + \cancel{2}\cdot\frac{1}{\cancel{2}\cdot 2}\log\left|\frac{2+x-1}{2-x+1}\right| + c$$

$$= -\frac{1}{2}\log|(3-x)(1+x)| + \frac{1}{2}\log\left|\frac{1+x}{3-x}\right| + c$$

$$= -\frac{1}{2}\log|3-x| - \cancel{\frac{1}{2}\log|1+x|} + \cancel{\frac{1}{2}\log|1+x|} - \frac{1}{2}\log|3-x| + c$$

$$= -\log|3-x| + c$$

(vii) $$I = \int \frac{x^2 - x + 1}{x^2 + x + 1}\,dx$$

$$= \int \frac{x^2 + x + 1 - 2x}{x^2 + x + 1}\,dx$$

$$= \int \left(1 - \frac{2x}{x^2 + x + 1}\right)dx$$

$$= x - \int \frac{2x + 1 - 1}{x^2 + x + 1}\,dx$$

$$= x - \int \frac{2x+1}{x^2+x+1}\,dx + \int \frac{dx}{x^2+x+1}$$

$$= x - \log|x^2+x+1| + \int \frac{dx}{x^2 + 2.x.\frac{1}{2} + \frac{1}{4} + 1 - \frac{1}{4}}$$

$$= x - \log|x^2+x+1| + \int \frac{dx}{(x+1/2)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= x - \log|x^2+x+1| + \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + c$$

$$= x - \log|x^2+x+1| + \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c$$

—x—

28

Type-4

$$\boxed{\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx \quad p\neq 0,\ a\neq 0}$$

(i) $\int \frac{x+b}{\sqrt{x^2+a^2}}\,dx$

(ii) $\int \frac{2x+3}{\sqrt{x^2+x+1}}\,dx$

(iii) $\int \frac{x-2}{\sqrt{2x^2-8x+5}}\,dx$

(iv) $\int \frac{x+1}{\sqrt{4+8x-5x^2}}\,dx$

(v) $\int \frac{(2x-1)\,dx}{\sqrt{4x^2+4x+2}}$

Type – 3 & Type – 8

$\int \frac{dx}{\sqrt{ax^2+bx+c}}, a \neq 0, \quad \int \frac{dx}{\sqrt{(ax+b)(cx+d)}}$

Problem:

(i) $\int \frac{dx}{\sqrt{2+3x-2x^2}}$

(ii) $\int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$ $(\beta > \alpha)$

(iii) $\int \frac{x\,dx}{\sqrt{(x^2-a^2)(b^2-x^2)}}$ $(b^2 > a^2)$

(iv) $\int \frac{dx}{\sqrt{x^2-7x+12}}$

(v) $\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$

(vi) $\int \frac{dx}{\sqrt{2ax-x^2}}$

(vii) $\int \frac{dx}{\sqrt{2ax+x^2}}$

Solⁿ (i) Let, $I = \int \frac{dx}{\sqrt{2+3x-2x^2}}$

$= \int \frac{dx}{\sqrt{2+4x-x-2x^2}}$

$= \int \frac{dx}{(1+2x)(2-x)}$

Put $2-x = z^2$ $\therefore x = 2-z^2$ $\therefore dx = -2z\,dz$

$$\therefore\ I = \int \frac{-2\cancel{z}\,dz}{\sqrt{1+2(2-z^2)}\cdot\cancel{z}}$$

$$= -2\int \frac{dz}{\sqrt{1+4-2z^2}}$$

$$= -2\int \frac{dz}{\sqrt{2}\sqrt{\frac{5}{2}-z^2}}$$

$$= -\sqrt{2}\int \frac{dz}{\sqrt{\left(\sqrt{\frac{5}{2}}\right)^2-z^2}}$$

$$= -\sqrt{2}\,\sin^{-1}\left(\frac{z\sqrt{2}}{\sqrt{5}}\right)+c$$

$$= -\sqrt{2}\,\sin^{-1}\left(\frac{\sqrt{2}\sqrt{2-x}}{\sqrt{5}}\right)+c$$

$$= -\sqrt{2}\,\sin^{-1}\sqrt{\frac{2(2-x)}{5}}+c$$

$$= \sqrt{2}\,\cos^{-1}\left(\frac{z\sqrt{2}}{\sqrt{5}}\right)+c$$

$$= \sqrt{2}\,\cos^{-1}\sqrt{\frac{2(2-x)}{5}}+c$$

(ii) $\displaystyle\int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$ $\quad(\beta>\alpha)$

<u>Sol^n:</u> $\displaystyle I = \int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$

Put $x-\alpha = z^2$

$\therefore\ x = z^2+\alpha$

$\therefore\ dx = 2z\,dz$

$\beta - x = \beta - z^2 - \alpha = \beta-\alpha-z^2$

$$\therefore\ I = \int \frac{2\cancel{z}\,dz}{\cancel{z}\sqrt{\beta-\alpha-z^2}}$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$$

$$\cancel{= 2\sin^{-1}\left(\frac{\sqrt{\beta-\alpha}}{z}\right)} + c$$

$$\cancel{= 2\sin^{-1}\left(\sqrt{\frac{\beta-\alpha}{x-\alpha}}\right)} + c$$

$$= 2\int \frac{dz}{\sqrt{\left(\sqrt{\beta-\alpha}\right)^2 - z^2}}$$

$$= 2\sin^{-1}\left(\frac{z}{\sqrt{\beta-\alpha}}\right) + c = 2\sin^{-1}\sqrt{\frac{x-\alpha}{\beta-\alpha}} + c$$

(iii) $$I = \int \frac{x\,dx}{\sqrt{(x^2-a^2)(b^2-x^2)}}$$

Put $x^2 - a^2 = z^2$

$\Rightarrow x^2 = z^2 + a^2$

$\therefore\ 2x\,dx = 2z\,dz$

$\therefore\ x\,dx = z\,dz$

$b^2 - x^2 = b^2 - a^2 - z^2$

$$\therefore\ I = \int \frac{\cancel{z}\,dz}{\cancel{z}\sqrt{b^2-a^2-z^2}}$$

$$= \sin^{-1}\left(\frac{z}{\sqrt{b^2-a^2}}\right) + c$$

$$= \sin^{-1}\sqrt{\frac{x^2-a^2}{b^2-a^2}} + c$$

(iv) $$I = \int \frac{dx}{\sqrt{x^2-7x+12}} = \int \frac{dx}{\sqrt{x^2-4x-3x+12}}$$

$$= \int \frac{dx}{\sqrt{(x-4)(x-3)}}$$

Put $x-4 = z^2$

$\therefore\ x = z^2+4$

$\therefore\ dx = 2z\,dz$

$x-3 = z^2+4-3 = z^2+1$

$$\therefore\ I = \int \frac{2z\,dz}{z\sqrt{z^2+1}} = 2\int \frac{dz}{\sqrt{z^2+1}}$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right|$$

$$= 2\log\left|z+\sqrt{z^2+1}\right| + c$$

$$= 2\log\left|\sqrt{x-4}+\sqrt{x-3}\right| + c$$

Another way

$$I = \int \frac{dx}{\sqrt{x^2 - 7x + 12}}$$

$$= \int \frac{dx}{\sqrt{x^2 - 2.x.\frac{7}{2} + \left(\frac{7}{2}\right)^2 + 12 - \left(\frac{7}{2}\right)^2}}$$

$$= \int \frac{dx}{\sqrt{\left(x - \frac{7}{2}\right)^2 + \frac{48-49}{4}}}$$

$$= \int \frac{dx}{\sqrt{\left(x - \frac{7}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}$$

$$= \log\left| \left(x - \frac{7}{2}\right) + \sqrt{\left(x - \frac{7}{2}\right)^2 - \left(\frac{1}{2}\right)^2} \right| + c$$

$$= \log\left| \frac{2x - 7 + \sqrt{(2x-7)^2 - 1}}{2} \right| + c$$

$$= \log\left| \frac{2x - 7 + \sqrt{(2x-6)(2x-8)}}{2} \right| + c$$

$$= \log\left| \left(\sqrt{x-4}\right)^2 + 2\sqrt{x-4}\sqrt{x-3} + \left(\sqrt{x-3}\right)^2 \right| \cancel{+\log} - \log 2 + c$$

$$= \log\left| \sqrt{x-4} + \sqrt{x-3} \right|^2 + C_1 \qquad [c_1 = c - \log 2$$

$$= 2\log\left| \sqrt{x-4} + \sqrt{x-3} \right| + c_1$$

(v) $I = \int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$

Put $x - \alpha = z^2$

$\therefore \quad x = z^2 + \alpha$

$\therefore \quad dx = 2z\,dz$

$x - \beta = z^2 + \alpha - \beta$

$\therefore \quad I = \int \frac{2\cancel{z}\,d\cancel{z}}{\cancel{z}\sqrt{z^2+\alpha-\beta}}$

$= 2\int \frac{dz}{\sqrt{z^2+(\sqrt{\alpha-\beta})^2}}$

$= 2\log\left|z+\sqrt{z^2+\alpha-\beta}\right| + c$

$= 2\log\left|\sqrt{x-\alpha}+\sqrt{x-\beta}\right| + c$

—x—

Solⁿ: (i) $I = \int \frac{x+b}{\sqrt{x^2+a^2}}\,dx$

$$= \int \frac{\frac{1}{2}\cdot 2x + b}{\sqrt{x^2+a^2}}\,dx$$

$$= \frac{1}{2}\int \frac{2x}{\sqrt{x^2+a^2}}\,dx + b\int \frac{dx}{\sqrt{x^2+a^2}}$$

$\boxed{\int \frac{dz}{\sqrt{z}} = 2\sqrt{z}}$

$$\cancel{= \frac{1}{2}\log|}$$

$$= \frac{1}{2}\cdot 2\sqrt{x^2+a^2} + b\cdot\log\left|x+\sqrt{x^2+a^2}\right| + C$$

$$= \sqrt{x^2+a^2} + b\log\left|x+\sqrt{x^2+a^2}\right| + C$$

Solⁿ (ii) $I = \int \frac{2x+1+2}{\sqrt{x^2+x+1}}\,dx$

$$= \int \frac{2x+1}{\sqrt{x^2+x+1}}\,dx + 2\int \frac{dx}{\sqrt{x^2+x+1}}$$

$$= 2\sqrt{x^2+x+1} + 2\int \frac{dx}{\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}$$

$$= 2\sqrt{x^2+x+1} + 2\cdot\log\left|x+\sqrt{x^2+x+1}\right| + C$$

(iv) $I = \int \frac{x+1}{\sqrt{4+8x-5x^2}}\,dx$

$$= \int \frac{(8-10x)\left(\frac{-1}{10}\right) + 1 + \frac{8}{10}}{\sqrt{4+8x-5x^2}}\,dx$$

$$= -\frac{1}{10}\int \frac{8-10x}{\sqrt{4+8x-5x^2}}\,dx + \frac{9}{5}\int \frac{dx}{\sqrt{5\left(\frac{4}{5}+\frac{8}{5}x-x^2\right)}}$$

$$= -\frac{1}{10}\,2\sqrt{4+8x-5x^2} + \frac{9}{5\sqrt{5}}\int \frac{dx}{\sqrt{\frac{4}{5}-\left(x^2-2\cdot\frac{4}{5}x+\frac{16}{25}-\frac{16}{25}\right)}}$$

$$= -\frac{1}{5}\sqrt{4+8x-5x^2} + \frac{9}{5\sqrt{5}}\int \frac{dx}{\sqrt{\frac{20+16}{25}-\left(x-\frac{4}{5}\right)^2}} \cancel{+c}$$

$$= -\frac{1}{5}\sqrt{4+8x-5x^2} + \frac{9}{5\sqrt{5}}\cdot \sin^{-1}\left(\frac{x-\frac{4}{5}}{\frac{6}{5}}\right) + c$$

$$= -\frac{1}{5}\sqrt{4+8x-5x^2} + \frac{9}{5\sqrt{5}}\sin^{-1}\left(\frac{5x-4}{6}\right) + c$$

$$= \frac{9}{5\sqrt{5}}\sin^{-1}\left(\frac{5x-4}{6}\right) - \frac{1}{5}\sqrt{4+8x-5x^2} + c$$

(31) $\displaystyle\int \frac{dx}{(ax+b)\sqrt{cx+d}}$ Put $cx+d = z^2$

$\int$

Type – 5

(i) $\displaystyle\int \frac{dx}{(2+x)\sqrt{1+x}}$

(ii) $\displaystyle\int \frac{dx}{(2x+1)\sqrt{4x+3}}$

Sol^n (i) $\displaystyle I = \int \frac{dx}{(2+x)\sqrt{1+x}}$

Put $1+x = z^2$ $\therefore$ $x = z^2-1$

$\therefore$ $dx = 2z\,dz$

$$I = \int \frac{2\cancel{z}\,dz}{(2+z^2-1)\cancel{z}}$$

$$= 2\int \frac{dz}{z^2+1}$$

$$= 2\tan^{-1} z + c$$

$$= 2\tan^{-1}(\sqrt{1+x}) + c$$

$=$

Soln (ii) $I = \int \frac{dx}{(2x+1)\sqrt{4x+3}}$

Put $4x+3 = z^2 \Rightarrow x = \frac{z^2-3}{4}$

$\therefore dx = \frac{\cancel{2}z\,dz}{\cancel{4}_2} = \frac{1}{2}z\,dz$

$$\therefore I = \int \frac{\frac{1}{2}\cancel{z}\,dz}{\left(2\cdot\frac{z^2-3}{4}+1\right)\cancel{z}}$$

$$= \frac{1}{2}\int \frac{dz}{\frac{z^2-3+2}{2}}$$

$$= \int \frac{dz}{z^2-1}$$

$$= \frac{1}{2}\log\left|\frac{z-1}{z+1}\right| + c$$

$$= \frac{1}{2}\log\left|\frac{\sqrt{4x+3}-1}{\sqrt{4x+3}+1}\right| + c$$

$$\int \frac{dx}{(px+q)\sqrt{ax^2+bx+c}}$$ $p \neq 0,\ a \neq 0$

Put $px+q = \frac{1}{z}$

Type – 6

(i) $\int \frac{dx}{(1+x)\sqrt{1-x^2}}$

(ii) $\int \frac{dx}{x\sqrt{9x^2+4x+1}}$

(iii) $\int \frac{dx}{(1+x)\sqrt{1+2x-x^2}}$

(iv) $\int \frac{dx}{x\sqrt{x^2+2x-1}}$

(v) $\int \frac{dx}{(1+x)\sqrt{1+x-x^2}}$

(vi) $\int \frac{dx}{(x-3)\sqrt{x^2-6x+8}}$

(vii) $\int \frac{dx}{(2x+3)\sqrt{x^2+3x+2}}$

(viii) $\int \frac{2x\,dx}{(1-x^2)\sqrt{x^4-1}}$

(ix) Show that

$$\int \frac{dx}{(x-a)\sqrt{(x-a)(b-x)}} = \frac{2}{a-b}\sqrt{\frac{b-x}{x-a}}$$

Solⁿ (i) $I = \int \frac{dx}{(1+x)\sqrt{1-x^2}}$

Put $1+x = \frac{1}{z}$

$\therefore \quad x = \frac{1}{z} - 1$

$\therefore \quad dx = -\frac{1}{z^2}\,dz$

$z = \frac{1}{1+x}$

$2z = \frac{2}{1+x}$

$2z-1 = \frac{2}{1+x} - 1$

$= \frac{1-x}{1+x}$

$$I = \int \frac{-\frac{1}{z^2}\,dz}{\frac{1}{z}\sqrt{1-\left(\frac{1}{z}-1\right)^2}}$$

$$= \int \frac{-\frac{1}{z^2}\,dz}{\frac{1}{z}\sqrt{1-\frac{1}{z^2}+\frac{2}{z}-1}}$$

$$= \int \frac{-\frac{1}{z^2}\,dz}{\frac{1}{z^2}\sqrt{2z-1}}$$

$$= -\int \frac{dz}{\sqrt{2z-1}}$$

$$= -\frac{\cancel{2}\sqrt{2z-1}}{\cancel{2}} + c = -\sqrt{\frac{1-x}{1+x}} + c$$

Soln (iii) $I = \int \frac{dx}{(1+x)\sqrt{1+2x-x^2}}$

Put $1+x = \frac{1}{z}$, $z = \frac{1}{1+x}$

$\therefore x = \frac{1}{z} - 1$

$\therefore dx = -\frac{1}{z^2} dz$

$$I = \int \frac{-\frac{1}{z^2} dz}{\frac{1}{z}\sqrt{1+2\left(\frac{1}{z}-1\right) - \left(\frac{1}{z}-1\right)^2}}$$

$$= \int \frac{-\frac{1}{z^2} dz}{\frac{1}{z}\sqrt{\cancel{1}+\frac{2}{z} - 2 - \frac{1}{z^2} + \frac{2}{z} - \cancel{1}}}$$

$$= -\int \frac{dz}{\sqrt{2\left(-z^2+2z-\frac{1}{2}\right)}}$$

$$= -\frac{1}{\sqrt{2}} \int \frac{dz}{\sqrt{-(z^2-2z+1-1)-\frac{1}{2}}}$$

$$= -\frac{1}{\sqrt{2}} \int \frac{dz}{\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 - (z-1)^2}}$$

$$= -\frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{z-1}{\frac{1}{\sqrt{2}}}\right) + c$$

$$= -\frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{\frac{1}{1+x} - 1}{\frac{1}{\sqrt{2}}}\right) + c$$

$$= -\frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{\frac{1-1-x}{1+x}}{\frac{1}{\sqrt{2}}}\right) + c$$

$$= \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{\sqrt{2}x}{1+x}\right) + c$$

(viii) $I = \int \frac{2x\,dx}{(1-x^2)\sqrt{x^4-1}}$

Put $x^2 = u$ $\therefore 2x\,dx = du$

$$\therefore I = \int \frac{du}{(1-u)\sqrt{u^2-1}}$$

Put $1-u = \frac{1}{z}$ $\qquad z = \frac{1}{1-u}$

$\therefore \quad u = 1 - \frac{1}{z}$

$du = \frac{1}{z^2}dz$

$$\therefore I = \int \frac{\frac{1}{z^2}\,dz}{\frac{1}{z}\sqrt{\left(1-\frac{1}{z}\right)^2-1}}$$

$$= \int \frac{\frac{1}{z^2}\,dz}{\frac{1}{z^2}\sqrt{(z-1)^2-z^2}}$$

$$= \int \frac{dz}{\sqrt{1-2z}} = \frac{2\sqrt{1-2z}}{-2} + c$$

$$= -\sqrt{1-2\cdot\frac{1}{1-u}} + c = -\sqrt{\frac{1-x^2-2}{1-x^2}} + c$$

$$= -\sqrt{\frac{x^2+1}{x^2-1}} + c$$

—x—

Type – 7 $\int \sqrt{\frac{ax+b}{cx+d}}\, dx$

(i) $\int \sqrt{\frac{1+x}{1-x}}\, dx$

(ii) $\int \sqrt{\frac{x-3}{x-4}}\, dx$

(iii) $\int \sqrt{\frac{2x+1}{3x+2}}\, dx$

(iv) $\int \sqrt{\frac{a+x}{a-x}}\, dx$

(v) $\int \sqrt{\frac{x}{a-x}}\, dx$

(vi) $\int \sqrt{\frac{a+x}{x}}\, dx$

Soln (i)

$$I = \int \sqrt{\frac{1+x}{1-x}}\, dx$$

$$= \int \frac{1+x}{\sqrt{1-x^2}}\, dx$$

$$= \int \frac{dx}{\sqrt{1-x^2}} + \left(-\frac{1}{2}\right) \int \frac{-2x\, dx}{\sqrt{1-x^2}}$$

$$= \sin^{-1}x - \frac{1}{2} \cdot 2\sqrt{1-x^2} + c$$

$$= \sin^{-1}x - \sqrt{1-x^2} + c$$

Sol^n (ii) $\quad I = \int \sqrt{\dfrac{x-3}{x-4}}\,dx$

$$= \int \frac{x-3}{\sqrt{(x-3)(x-4)}}\,dx$$

$$= \int \frac{\frac{1}{2}(2x-7) - 3 + \frac{7}{2}}{\sqrt{x^2-7x+12}}\,dx$$

$$= \frac{1}{2}\int \frac{2x-7}{\sqrt{x^2-7x+12}}\,dx + \frac{1}{2}\int \frac{dx}{\sqrt{(x-3)(x-4)}}$$

$$= \frac{1}{\cancel{2}}\cdot\cancel{2}\sqrt{x^2-7x+12} + I_1$$

$$= \sqrt{(x-3)(x-4)} + I_1$$

Put $x-3 = z^2$ $\quad \therefore x = z^2+3$

$\therefore dx = 2z\,dz$

$$\therefore I_1 = \frac{1}{\cancel{2}}\int \frac{\cancel{2}\cancel{z}\,dz}{\cancel{z}\sqrt{z^2+3-4}}$$

$$I_1 = \int \frac{dz}{\sqrt{z^2-1}}$$

$$= \log\left|z + \sqrt{z^2-1}\right| + c$$

$$= \log\left|\sqrt{x-3} + \sqrt{x-4}\right| + c$$

$$\therefore\ I = \sqrt{(x-3)(x-4)} + \log\left|\sqrt{x-3} + \sqrt{x-4}\right| + c$$

—x—

(iv) $$I = \int \sqrt{\frac{a+x}{a-x}}\, dx$$

$$= \int \frac{a+x}{\sqrt{(a-x)(a+x)}}\, dx$$

$$= \int \frac{a+x}{\sqrt{a^2-x^2}}\, dx$$

$$= -\frac{1}{2}\int \frac{-2x}{\sqrt{a^2-x^2}}\, dx + a\int \frac{dx}{\sqrt{a^2-x^2}}$$

$$= -\frac{1}{\cancel{2}}\cdot \cancel{2}\sqrt{a^2-x^2} + a.\sin^{-1}\frac{x}{a} + c$$

$$= -\sqrt{a^2-x^2} + a\sin^{-1}\frac{x}{a} + c$$

(v) $I = \int \sqrt{\dfrac{x}{a-x}}\, dx$

$= \int \dfrac{x\, dx}{\sqrt{x(a-x)}}$

$= \int \dfrac{(a-2x)(-\frac{1}{2}) + \frac{a}{2}}{\sqrt{x(a-x)}}\, dx$

$= -\frac{1}{2} \cdot \int \dfrac{a-2x}{\sqrt{x(a-x)}}\, dx + \frac{a}{2} \int \dfrac{dx}{\sqrt{x(a-x)}}$

$= -\frac{1}{\cancel{2}} \cdot \cancel{2}\sqrt{x(a-x)} + \frac{a}{2} \int \dfrac{dx}{\sqrt{x(a-x)}}$

$= -\sqrt{x(a-x)} + I_1$

Put $x = z^2$ $\quad \therefore dx = 2z\, dz$

$\therefore I_1 = \frac{a}{\cancel{2}} \int \dfrac{\cancel{2}\cancel{z}\, dz}{\cancel{z}\sqrt{a - z^2}}$

$= a \sin^{-1} \dfrac{z}{\sqrt{a}} + c$

$= a \sin^{-1} \sqrt{\dfrac{x}{a}} + c$

$\therefore I = a \sin^{-1} \sqrt{\dfrac{x}{a}} - \sqrt{x(a-x)} + c$

(vi) $I = \int \sqrt{\frac{a+x}{x}}\, dx$

$= \int \frac{a+x}{\sqrt{x(a+x)}}\, dx$

$= \int \frac{a+x}{\sqrt{ax+x^2}}\, dx$

$= \int \frac{\frac{1}{2}(a+2x)+\frac{1}{2}a}{\sqrt{ax+x^2}}\, dx$

$= \frac{1}{2}\int \frac{a+2x}{\sqrt{ax+x^2}}\, dx + \frac{a}{2}\int \frac{dx}{\sqrt{x(a+x)}}$

$= \frac{1}{\cancel{2}}\cdot \cancel{2}\sqrt{x(a+x)} + I_1$

where $I_1 = \frac{a}{2}\int \frac{dx}{\sqrt{x(a+x)}}$

Put $x = z^2$ $\therefore$ $dx = 2z\, dz$

$\therefore I_1 = \frac{a}{\cancel{2}}\int \frac{\cancel{2}\cancel{z}\, dz}{\cancel{z}\sqrt{a+z^2}} = a\int \frac{dz}{\sqrt{z^2+(\sqrt{a})^2}}$

$= a.\log|z+\sqrt{z^2+a}|+c$

$$= a \log \left| \sqrt{x} + \sqrt{x+a} \right| + c$$

$$\therefore I = \sqrt{x(a+x)} + a \log \left| \sqrt{x} + \sqrt{x+a} \right| + c$$

—x—

Type - 4

3 $$I = \int \frac{x-2}{\sqrt{2x^2 - 8x + 5}} \, dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{x-2}{\sqrt{x^2 - 4x + \frac{5}{2}}} \, dx$$

$$= \frac{1}{2\sqrt{2}} \int \frac{2x-4}{\sqrt{x^2 - 4x + \frac{5}{2}}} \, dx$$

$$= \frac{1}{\cancel{2}\sqrt{2}} \cdot \cancel{2} \sqrt{x^2 - 4x + \frac{5}{2}} + c$$

$$= \frac{\sqrt{2x^2 - 8x + 5}}{2} + c$$

—x—

Type – 6

(ix) $$I = \int \frac{dx}{(x-a)\sqrt{(x-a)(b-x)}}$$

Put $x - a = \frac{1}{z}$ $\therefore dx = -\frac{1}{z^2}dz$ | $z = \frac{1}{x-a}$

$x = \frac{1}{z} + a$

$$\therefore I = \int \frac{-\frac{1}{z^2}dz}{\frac{1}{z}\sqrt{\frac{1}{z}\left(b - \frac{1}{z} - a\right)}}$$

$$= -\int \frac{dz}{\sqrt{bz - 1 - az}}$$

$$= -\int \frac{dz}{\sqrt{(b-a)z - 1}}$$

$$= \frac{-\cdot 2\sqrt{(b-a)z - 1}}{b-a}$$

$$= \frac{2}{a-b}\sqrt{(b-a)\frac{1}{x-a} - 1}$$

$$= \frac{2}{a-b}\sqrt{\frac{b - \cancel{a} - x + \cancel{a}}{x-a}}$$

$$= \frac{2}{a-b}\sqrt{\frac{b-x}{x-a}}$$ **Proved**