Differentiability:

A function $f(x)$ is said to be differentiable at $x = a$ if $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ exist and finite for very small h.

If $f(x)$ is differentiable at $x = a$ then

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$ is called the differential coefficient or derivative of $f(x)$ at $x = a$.

Note: A function $f(x)$ is differentiable at $x = a$ if L.H.D. = R.H.D, where

$$\text{L.H.D} = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{f[a+(-h)] - f(a)}{(-h)}$$

$$= \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}$$

and $$\text{R.H.D} = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Q. If a function $f(x)$ is differentiable at $x=a$ then show that it must be continuous at $x=a$.

Proof: We have, $f(a+h)-f(a) = \frac{f(a+h)-f(a)}{h} \times h$

$$\therefore \lim_{h\to 0} \{f(a+h)-f(a)\} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \times h$$

$$\Rightarrow \lim_{h\to 0} f(a+h) - \lim_{h\to 0} f(a)$$

$$= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \times \lim_{h\to 0} h$$

$$\Rightarrow \lim_{h\to 0} f(a+h) - f(a) = f'(a) \times 0$$

$$\left[\because f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}\right.$$

$$\Rightarrow \lim_{h\to 0} f(a+h) - f(a) = 0$$

$$\Rightarrow \lim_{h\to 0} f(a+h) = f(a)$$

$\Rightarrow$ $f(x)$ is continuous at $x=a$.

(Proved)

$$\left[\left.\begin{matrix} a+h=x & \Rightarrow h = x-a \\ h\to 0 & \Rightarrow x\to a \end{matrix}\right\} \therefore \lim_{x\to a} f(x) = f(a)\right]$$

Limiting value = functional value

Q. Show that the function $f(x) = |x|$ is continuous but not differentiable at $x = 0$.

Solⁿ: We have, $f(x) = |x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$

Now, $\text{L.H.L.} = \lim_{x \to 0^-} f(x) = \lim_{x \to 0} (-x) = 0$

$\text{R.H.L.} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0} (x) = 0$

$\text{F.V.} = f(0) = 0$

Since L.H.L = R.H.L. = F.V. therefore $f(x)$ is continuous at $x = 0$.

2nd part:

$$\text{L.H.D.} = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - 0}{-h}$$

$$= \lim_{h \to 0} \frac{h}{-h} = \lim_{h \to 0} (-1) = -1$$

$$\text{R.H.D.} = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - 0}{h}$$

$$= \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} (1) = 1.$$

Since L.H.D $\neq$ R.H.D. therefore $f(x)$ is not differentiable at $x = 0$. (Proved)

Q. Does $f'(0)$ exist for $f(x) = \begin{cases} 2x+1 & \text{if } x \le 0 \\ x^2+1 & \text{if } x > 0 \end{cases}$?

Sol^n:

$$L.H.D. = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{\{2(-h)+1\} - \{2(0)+1\}}{-h}$$

$$= \lim_{h \to 0} \frac{-2h+1-0-1}{-h}$$

$$= \lim_{h \to 0} \frac{-2h}{-h}$$

$$= \lim_{h \to 0} 2$$

$$= 2$$

$$R.H.D. = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{(h^2+1) - \{2 \times 0 + 1\}}{h}$$

$$= \lim_{h \to 0} \frac{h^2}{h}$$

$$= \lim_{h \to 0} h = 0$$

Since L.H.D $\neq$ R.H.D. therefore $f(x)$ is not differentiable at $x = 0$. i.e. $f'(0)$ does not exist.

Q. Does $f'(2)$ exist for $f(x) = \begin{cases} x & \text{if } 0<x<1 \\ 2-x & \text{if } 1 \le x < 2 \\ x - \frac{x^2}{2} & \text{if } x \geq 2 \end{cases}$?

Solⁿ: L.H.D = $\lim_{h \to 0} \frac{f(2-h) - f(2)}{-h}$

$$= \lim_{h \to 0} \frac{\{2-(2-h)\} - \left(2 - \frac{2^2}{2}\right)}{-h}$$

$$= \lim_{h \to 0} \frac{h}{-h}$$

$$= \lim_{h \to 0} (-1)$$

$$= -1$$

R.H.D $= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$

$$= \lim_{h \to 0} \frac{\left\{(2+h) - \frac{(2+h)^2}{2}\right\} - \left(2 - \frac{2^2}{2}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{4+2h-4-4h-h^2}{2} - 0}{h}$$

$$= \lim_{h \to 0} \frac{-h(2-h)}{2h}$$

$$= \lim_{h \to 0} \frac{-2+h}{2} = \frac{-2}{2} = -1$$

Since L.H.D. = R.H.D therefore $f'(2)$ exists.

Q. Does $f'(-1)$ exist for $f(x) = \begin{cases} 1 & \text{if } x \le -1 \\ x+2 & \text{if } x > -1 \end{cases}$?

Solⁿ: $\text{L.H.D} = \lim_{h \to 0} \dfrac{f(-1-h) - f(-1)}{-h}$

$= \lim_{h \to 0} \dfrac{1-1}{-h}$

$= \lim_{h \to 0} 0 = 0$

$\text{R.H.D.} = \lim_{h \to 0} \dfrac{f(-1+h) - f(-1)}{h}$

$= \lim_{h \to 0} \dfrac{\{(-1+h)+2\} - 1}{h}$

$= \lim_{h \to 0} \dfrac{-1+h+2-1}{h}$

$= \lim_{h \to 0} \dfrac{h}{h}$

$= \lim_{h \to 0} 1$

$= 1$

Since L.H.D $\neq$ R.H.D therefore $f'(-1)$ does not exist.

Q. Does $f'(1)$ exist for $f(x) = \begin{cases} x^2+1 & \text{if } x<1 \\ 2 & \text{if } x=1 \\ 4-2x & \text{if } x>1 \end{cases}$?

Soln: L.H.D $= \lim_{h\to 0} \dfrac{f(1-h)-f(1)}{-h}$

$= \lim_{h\to 0} \dfrac{\{(1-h)^2+1\}-2}{-h}$

$= \lim_{h\to 0} \dfrac{1-2h+h^2+1-2}{-h}$

$= \lim_{h\to 0} \dfrac{-h(2-h)}{-h}$

$= \lim_{h\to 0} (2-h)$

$= 2$

R.H.D $= \lim_{h\to 0} \dfrac{f(1+h)-f(1)}{h}$

$= \lim_{h\to 0} \dfrac{\{4-2(1+h)\}-2}{h}$

$= \lim_{h\to 0} \dfrac{4-2-2h-2}{h}$

$= \lim_{h\to 0} (-2) = -2$

Since L.H.D $\neq$ R.H.D, therefore $f'(1)$ doesnot exist.

Q. Does $f'(3)$ exist for $f(x) = \begin{cases} x^2 & \text{if } x < 3 \\ 9 & \text{if } x = 3 \\ 2x+3 & \text{if } x > 3 \end{cases}$?

Sol^n: L.H.D. $= \lim_{h \to 0} \frac{f(3-h) - f(3)}{-h}$

$= \lim_{h \to 0} \frac{(3-h)^2 - 9}{-h}$

$= \lim_{h \to 0} \frac{9 - 6h + h^2 - 9}{-h}$

$= \lim_{h \to 0} \frac{-h(6-h)}{-h}$

$= \lim_{h \to 0} (6-h)$

$= 6$

R.H.D $= \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$

$= \lim_{h \to 0} \frac{\{2(3+h) + 3\} - 9}{h}$

$= \lim_{h \to 0} \frac{6 + h + 3 - 9}{h}$

$= \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} (1) = 1$

Since L.H.D $\neq$ R.H.D therefore $f'(3)$ does not exist.

Q. Does $f'(1)$ exist for $f(x) = \begin{cases} x & \text{if } 0 < x < 1 \\ 2-x & \text{if } 1 \leq x < 2 \\ x - \dfrac{x^2}{2} & \text{if } x \geq 2 \end{cases}$?

$$\text{L.H.D.} = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \to 0} \frac{(1-h) - (2-1)}{-h}$$

$$= \lim_{h \to 0} \frac{1-h-1}{-h}$$

$$= \lim_{h \to 0} (1)$$

$$= 1$$

$$\text{R.H.D.} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{\{2-(1+h)\} - (2-1)}{h}$$

$$= \lim_{h \to 0} \frac{2-1-h-1}{h}$$

$$= \lim_{h \to 0} (-1)$$

$$= -1$$

Since L.H.D $\neq$ R.H.D therefore $f'(1)$ does not exist.

Q. Does $f'(0)$ exist for the following function $f(x) = \begin{cases} 3+2x & \text{if } -\frac{3}{2} \le x < 0 \\ 3-2x & \text{if } 0 \le x < \frac{3}{2} \\ -3-2x & \text{if } x \geq \frac{3}{2} \end{cases}$?

Sol^n:

For $f'(0)$

$$\text{L.H.D.} = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{f(-h) - (3-0)}{-h}$$

$$= \lim_{h \to 0} \frac{3+2(-h) - 3}{-h}$$

$$= \lim_{h \to 0} (2)$$

$$= 2$$

$$\text{R.H.D.} = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{(3-2h) - (3-0)}{h}$$

$$= \lim_{h \to 0} (-2)$$

$$= -2$$

Since L.H.D. $\neq$ R.H.D therefore $f'(0)$ does not exist.