Use the precise definition of limit to show that

(a) $\lim_{x \to 2} (5x-4) = 6$, (b) $\lim_{x \to 2} (4x-5) = 3$

(c) $\lim_{x \to 3} (2x-1) = 5$.

Solⁿ (a): Given, $f(x) = 5x-4$,

$a = 2$

and $L = 6$

Choose $\varepsilon > 0$, then $|f(x) - L| < \varepsilon$

$\Rightarrow |5x-4-6| < \varepsilon$

$\Rightarrow |5x-10| < \varepsilon$

$\Rightarrow 5|x-2| < \varepsilon$

$\Rightarrow |x-2| < \frac{\varepsilon}{5}$.

If we choose $\delta = \frac{\varepsilon}{5}$ then, we have

$|(5x-4)-6| < \varepsilon$ whenever $|x-2| < \delta$.

Hence, $\lim_{x \to a} (5x-4) = 6$.

—x—

If $\lim_{x \to a} f(x)$ exists, prove that it must be unique.

Solⁿ: Suppose that L_1 and L_2 are two limits of $f(x)$ as $x \to a$. Then, for any $\varepsilon > 0$ there exist $\delta_1 > 0$, $\delta_2 > 0$ such that

$$|f(x) - L_1| < \frac{\varepsilon}{2}, \quad \text{if } 0 < |x-a| < \delta_1,$$

$$|f(x) - L_2| < \frac{\varepsilon}{2}, \quad \text{if } 0 < |x-a| < \delta_2.$$

Hence, if $\delta = \min\{\delta_1, \delta_2\}$, then

$$\begin{aligned} |L_1 - L_2| &= |L_1 - f(x) + f(x) - L_2| \\ &\leq |f(x) - L_1| + |f(x) - L_2| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

whenever $0 < |x-a| < \delta$.

Now, since $|L_1 - L_2| < \varepsilon$ for every $\varepsilon > 0$.

$\therefore \quad |L_1 - L_2| = 0 \quad \Rightarrow L_1 = L_2.$

Hence limit must be unique.

Right hand limit:

A function $f(x)$ is said to possesses a right hand limit L at $x = a$, if for every $\varepsilon > 0$ (however small) $\exists\ \delta > 0$ (depending on ε and a) such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < x - a < \delta$$

$$\text{or} \quad a < x < a + \delta.$$

Mathematically, $\lim\limits_{x \to a^+} f(x) = L.$

Left hand limit:

A function $f(x)$ is said to possesses a left hand limit L at $x = a$, if for every $\varepsilon > 0$ (however small) $\exists\ \delta > 0$ (depending on ε and a) such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad -\delta < x - a < 0$$

$$\text{or,} \quad a - \delta < x < a.$$

Mathematically,

$$\lim_{x \to a^-} f(x) = L$$

Note: $\lim\limits_{x \to a} f(x)$ exists if and only if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

Does $\lim_{x \to a} f(x)$ exist for the following functions and values of a.

(a) $a = 0$ and $f(x) = \begin{cases} 2x+1 & \text{if } x \le 0 \\ x^2 - x & \text{if } x > 0, \end{cases}$

(b) $a = -1$ and $f(x) = \begin{cases} 1 & \text{if } x \le -1 \\ x+2 & \text{if } x > -1, \end{cases}$

(c) $a = 1$ and $f(x) = \begin{cases} x^2+1 & \text{if } x < 1 \\ 3 & \text{if } x = 1, \\ 4-2x & \text{if } x > 1 \end{cases}$

(d) $a = 3$ and $f(x) = \begin{cases} x^2 & \text{if } x < 3 \\ 7 & \text{if } x = 3. \\ 2x+3 & \text{if } x > 3 \end{cases}$

Solⁿ: (a) $\text{L.H.L.} = \lim_{x \to 0^-} f(x)$

$= \lim_{x \to 0} (2x+1)$

$= 1$

$\text{R.H.L.} = \lim_{x \to 0^+} f(x)$

$= \lim_{x \to 0} (x^2 - x)$

$= 0$

Here, L.H.L. = R.H.L.

Therefore, $\lim_{x \to 0} f(x)$ does not exist.

Define the continuity of a function $f(x)$ at a point $x = a$.

Sol^n: Continuity of a function:

A function $f(x)$ is said to be continuous at $x=a$ if limit at that point exists and equal to the value of the function.

i.e. L.H.L. = R.H.L. = F.V.

$$\lim_{x \to a^-} = \lim_{x \to a^+} = f(a)$$

A function $f(x)$ is said to be continuous at $x = a$ if for every $\varepsilon > 0$ (However small) $\exists$ $\delta > 0$ (depending on ε and a) such that $|f(x) - f(a)| < \varepsilon$ whenever $|x-a| < \delta$.

i.e., $\lim_{x \to a} f(x) = f(a)$.

A function $f(x)$ is said to be continuous at $x=a$, if (i) $f(x)$ is well defined at $x=a$, and

(ii) $\lim_{x \to a} f(x) = f(a)$.

$a = 0, \frac{3}{2}$ and $f(x) = \begin{cases} 3+2x & \text{if } -\frac{3}{2} \leq x < 0 \\ 3-2x & \text{if } 0 \leq x < \frac{3}{2}. \\ -3-2x & \text{if } x \geq \frac{3}{2} \end{cases}$

Solⁿ: For $a = 0$,

$$\text{L.H.L.} = \lim_{x \to 0^-} f(x) = \lim_{x \to 0} (3+2x) = 3$$

$$\text{R.H.L.} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0} (3-2x) = 3$$

$$\text{F.V.} = f(0) = 3 - 2 \times 0 = 3$$

Since, L.H.L. = R.H.L. = F.V.

$\therefore$ $f(x)$ is continuous at $x = 0$.

For $a = \frac{3}{2}$,

$$\text{L.H.L.} = \lim_{x \to \frac{3}{2}^-} f(x) = \lim_{x \to \frac{3}{2}} (3-2x) = 3 - 2\left(\frac{3}{2}\right)$$

$$= 3 - 3 = 0$$

$$\text{R.H.L.} = \lim_{x \to \frac{3}{2}^+} f(x) = \lim_{x \to \frac{3}{2}} (-3-2x) = -3 - 2\left(\frac{3}{2}\right)$$

$$= -3 - 3 = -6$$

Since L.H.L. $\neq$ R.H.L, then $f(x)$ is not continuous at $x = \frac{3}{2}$.

$f(x) = c \quad \forall\, x \in \mathbb{R}$

At $x = x_0$,

$$\text{L.H.L.} = \lim_{x \to x_0^-} f(x) = \lim_{x \to x_0} c = c$$

$$\text{R.H.L.} = \lim_{x \to x_0^+} f(x) = \lim_{x \to x_0} c = c$$

$\text{F.V.} = f(x_0) = c$

Since L.H.L = R.H.L = F.V and x_0 is arbitrary, then $f(x) = c$ is continuous at every $x = x_0$.

$f(x) = x \quad \forall\, x \in \mathbb{R}$

At $x = x_0$,

$$\text{L.H.L.} = \lim_{x \to x_0^-} f(x) = \lim_{x \to x_0} x = x_0$$

$$\text{R.H.L.} = \lim_{x \to x_0^+} f(x) = \lim_{x \to x_0} x = x_0$$

$\text{F.V.} = f(x_0) = x_0$

$\boxed{f(x) = x}$

Continuity of $f(x)$ in the closed interval $[a,b]$:

A ~~$f(x)$~~ function $f(x)$ is said to be continuous ~~at~~ x in $[a,b]$ if $f(x)$ is right continuous at $x=a$ [i.e. $\lim_{x\to a^+} f(x) = f(a)$], $f(x)$ is left continuous at $x=b$ [i.e. $\lim_{x\to b^-} f(x) = f(b)$] and $f(x)$ is continuous at every point in (a,b).

Intermediate value theorem:

If $f(x)$ is continuous in $[a,b]$ and if $f(a) = A$ and $f(b) = B$, then ~~$\exists$ a~~ for every number C between A and B, there exists at least one number $c \in [a,b]$ such that $f(c) = C$.

Extreme value theorem:

If a real-valued function f is continuous on the closed interval [a,b], then f must attain a maximum and a minimum, each at least once. That is, there exist numbers c and d in [a,b] such that:
f(c) <= f(x)<= f(d) for all x $\in$ [a,b].

Types of discontinuity

There are two types of discontinuity.

1. Discontinuity of the first kind.
2. Discontinuity of the second kind.

1. Discontinuity of the first kind:

A function $f(x)$ has discontinuity of the first kind at $x=a$ if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist, but at least one of them is different from $f(a)$.

There are two types of of the discontinuity of the first kind.

(i) Solvable discontinuity:

$$\text{if } \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$$

Ex: $f(x) = \dfrac{x^2+4x-12}{x(x-2)}$ has solvable discontinuity at $x=2$, as

$$\lim_{x\to 2^-} f(x) = 4 = \lim_{x\to 2^+} f(x).$$

(ii) Jump discontinuity:

$$\text{if } \lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x).$$

Ex: $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$

has a Jump discontinuity at $x=0$,

as $\lim_{x\to 0^-} f(x) = 0$ and $\lim_{x\to 0^+} f(x) = 1$.

(2) $f(x)$ has discontinuity of the second Kind if at least one of $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ does not exist.

Ex: $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

has discontinuity of the second Kind at every real number $x=a$.