Partial Differentiation

Partial derivative:

Let $u = f(x,y)$ be a function of x and y.

Then the derivative of u with respect to x, treating y as constant, is called partial derivative of u with respect to x.

It is denoted as

$$\frac{\partial u}{\partial x}, \frac{\partial f}{\partial x}, f_x(x,y), f_x \text{ or } u_x.$$

Mathematically, $\dfrac{\partial u}{\partial x} = \dfrac{\partial f}{\partial x} = \lim\limits_{h \to 0} \dfrac{f(x+h,y) - f(x,y)}{h}$

Similarly, the partial derivative of u with respect to y is defined.

$$\frac{\partial^2 u}{\partial x^2} = u_{xx} = \frac{\partial^2 f}{\partial x^2} = f_{xx}$$

$$\frac{\partial^3 u}{\partial x^3} = u_{xxx} = \frac{\partial^3 f}{\partial x^3} = f_{xxx}$$

$$\frac{\partial^2 u}{\partial x \partial y} = u_{xy} = \frac{\partial^2 f}{\partial x \partial y} = f_{xy}$$

$\vdots$

If $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$, then show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

Proof: Given,

$$u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right) \quad \text{——— (1)}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\cdot\frac{1}{y} + \frac{1}{1+\frac{y^2}{x^2}}\cdot\left(-\frac{y}{x^2}\right)$$

$$\therefore x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} \quad \text{——— (2)}$$

$$\therefore \frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\cdot\left(-\frac{x}{y^2}\right) + \frac{1}{1+\frac{y^2}{x^2}}\cdot\left(\frac{1}{x}\right)$$

$$\therefore y\frac{\partial u}{\partial y} = \frac{-x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2} \quad \text{——— (3)}$$

Adding (2) and (3) we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

Proved

If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, then show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0.$$

Proof: Given,

$$u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right) \qquad (1)$$

$$\therefore \frac{\partial u}{\partial x} = -\frac{1}{\sqrt{1-\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)^2}} \cdot \frac{(\sqrt{x}+\sqrt{y}).1-(x+y).\frac{1}{2\sqrt{x}}}{(\sqrt{x}+\sqrt{y})^2}$$

$$= \frac{-(\sqrt{x}+\sqrt{y})}{\sqrt{(\sqrt{x}+\sqrt{y})^2-(x+y)^2}} \cdot \frac{2\sqrt{x}(\sqrt{x}+\sqrt{y})-(x+y)}{2\sqrt{x}(\sqrt{x}+\sqrt{y})^2}$$

$$= \frac{(x+y)-2\sqrt{x}(\sqrt{x}+\sqrt{y})}{2\sqrt{x}(\sqrt{x}+\sqrt{y})\sqrt{(\sqrt{x}+\sqrt{y})^2-(x+y)^2}}$$

$$\therefore x\frac{\partial u}{\partial x} = \frac{\sqrt{x}(x+y)-2x(\sqrt{x}+\sqrt{y})}{2(\sqrt{x}+\sqrt{y})\sqrt{(\sqrt{x}+\sqrt{y})^2-(x+y)^2}} \qquad (2)$$

$$\therefore \frac{\partial u}{\partial y} = -\frac{1}{\sqrt{1-\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)^2}} \cdot \frac{(\sqrt{x}+\sqrt{y}).1-(x+y).\frac{1}{2\sqrt{y}}}{(\sqrt{x}+\sqrt{y})^2}$$

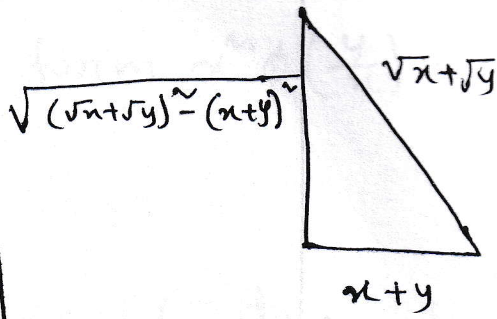
$$= \frac{-(\sqrt{x}+\sqrt{y})}{\sqrt{(\sqrt{x}+\sqrt{y})^2-(x+y)^2}} \cdot \frac{2\sqrt{y}(\sqrt{x}+\sqrt{y})-(x+y)}{2\sqrt{y}(\sqrt{x}+\sqrt{y})^2}$$

$$= \frac{(x+y)-2\sqrt{y}(\sqrt{x}+\sqrt{y})}{2\sqrt{y}(\sqrt{x}+\sqrt{y})\sqrt{(\sqrt{x}+\sqrt{y})^2-(x+y)^2}}$$

$$\therefore y\frac{\partial u}{\partial y} = \frac{\sqrt{y}(x+y) - 2y(\sqrt{x}+\sqrt{y})}{2(\sqrt{x}+\sqrt{y})\sqrt{(\sqrt{x}+\sqrt{y})^2 - (x+y)^2}} \quad \text{--- (3)}$$

$$\frac{1}{2}\cot u = \frac{1}{2}\cot\cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$$

$$= \frac{1}{2}\cot\cot^{-1}\left(\frac{x+y}{\sqrt{(\sqrt{x}+\sqrt{y})^2-(x+y)^2}}\right)$$



$$\therefore \frac{1}{2}\cot u = \frac{x+y}{2\sqrt{(\sqrt{x}+\sqrt{y})^2-(x+y)^2}} \quad \text{--- (4)}$$

Adding (2), (3) and (4) we get,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u$$

$$= \frac{\sqrt{x}(x+y) - 2x(\sqrt{x}+\sqrt{y}) + \sqrt{y}(x+y) - 2y(\sqrt{x}+\sqrt{y}) + (\sqrt{x}+\sqrt{y})(x+y)}{2(\sqrt{x}+\sqrt{y})\sqrt{(\sqrt{x}+\sqrt{y})^2 - (x+y)^2}}$$

$$= \frac{(x+y)(\sqrt{x}+\sqrt{y}) - 2(\sqrt{x}+\sqrt{y})(x+y) + (x+y)(\sqrt{x}+\sqrt{y})}{2(\sqrt{x}+\sqrt{y})\sqrt{(\sqrt{x}+\sqrt{y})^2-(x+y)^2}}$$

$$= 0$$

Proved

Homogeneous function:

A function $f(x,y)$ is said to be homogeneous of degree n ~~if~~ in the variables x and y if it can be expressed in the form $x^n\phi\left(\frac{y}{x}\right)$ or in the form $y^n\phi\left(\frac{x}{y}\right)$.

$\boxed{\;}$ If $f(x,y)$ is a homogeneous function of degree n then $f(tx,ty) = t^n f(x,y)$.

Ex: $f(x,y) = x^2y + xy^2 + y^3$

$$\therefore f(tx,ty) = (tx)^2(ty) + tx(ty)^2 + (ty)^3$$
$$= t^3x^2y + t^3xy^2 + t^3y^3$$
$$= t^3(x^2y + xy^2 + y^3)$$
$$= t^3 f(x,y).$$

$\therefore f$ is a homogeneous function of degree 3.

$\boxed{\;}$ $$f(x,y) = x^3\left(\frac{y}{x} + \frac{y^2}{x^2} + \frac{y^3}{x^3}\right)$$
$$= x^3\left\{\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right)^3\right\}$$
$$= x^3\,\phi\left(\frac{y}{x}\right)$$

$\therefore f$ is a homogeneous function of degree 3.

Euler's Theorem :

Statement : If $f(x,y)$ is a homogeneous function of x and y of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x,y).$$

Proof : Since $f(x,y)$ is a homogeneous function of degree n, then

$$f(x,y) = x^n \phi\left(\frac{y}{x}\right) \quad \text{——— (1)}$$

Let, $$v = \frac{y}{x} \quad \text{——— (2)}$$

$$\therefore \frac{\partial v}{\partial x} = -\frac{y}{x^2} \quad \text{——— (3)}$$

$$\therefore \frac{\partial v}{\partial y} = \frac{1}{x} \quad \text{——— (4)}$$

From (1) and (2) we get,

$$f(x,y) = x^n \phi(v) \quad \text{——— (5)}$$

$$\therefore \frac{\partial f}{\partial x} = nx^{n-1}\phi(v) + x^n\phi'(v)\cdot\frac{\partial v}{\partial x}$$

$$= nx^{n-1}\phi(v) - x^{n-2}y\,\phi'(v)$$

$$\therefore x\frac{\partial f}{\partial x} = n x^n \phi(v) - x^{n-1} y \phi'(v) \quad \text{------} \quad (6)$$

Again, differentiating (5) w.r. to y, we get,

$$\frac{\partial f}{\partial y} = x^n \phi'(v) \frac{\partial v}{\partial y}$$

$$= x^n \phi'(v) \cdot \frac{1}{x} \quad [\text{From } (4)]$$

$$= x^{n-1} \phi'(v)$$

$$\therefore y\frac{\partial f}{\partial y} = x^{n-1} y \phi'(v) \quad \text{------} \quad (7)$$

Adding, (6) and (7) we get,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n x^n \phi(v)$$

$$= n x^n \phi\left(\frac{y}{x}\right) \quad [\text{from } (2)]$$

$$= n f(x, y) \quad [\text{from } (1)]$$

Proved

If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, then show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0.$$

Proof : [2nd way]

Given, $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

$$\Rightarrow \cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \frac{x\left(1+\frac{y}{x}\right)}{\sqrt{x}\left(1+\sqrt{\frac{y}{x}}\right)}$$

$$= x^{1/2}\,\phi\left(\frac{y}{x}\right)$$

$\therefore$ $\cos u$ is a homogeneous function of degree $\frac{1}{2}$. Then by Euler's theorem we get,

$$x\frac{\partial(\cos u)}{\partial x} + y\frac{\partial(\cos u)}{\partial y} = n\cos u$$

$$\Rightarrow -x\sin u\frac{\partial u}{\partial x} - y\sin u\frac{\partial u}{\partial y} = \frac{1}{2}\cos u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$$

proved

If $u = x\,\phi(y/x) + \psi(y/x)$, by using Euler's Theorem find the value of $x^2 u_{xx} + 2xy\,u_{xy} + y^2 u_{yy}$.

Solⁿ: Let, $u = v + w$ ——— (1)

where $v = x\,\phi\left(\frac{y}{x}\right)$ and $w = \psi(y/x)$

$\therefore$ v is a homogeneous function of degree ~~1~~ 1.

and w is a homogeneous function of degre 0.

$$\therefore \quad x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 1.v \qquad \text{——— (2)}$$

$$\text{and} \quad x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 0 \qquad \text{——— (3)}$$

~~Differentiating (2)~~

Now,
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\frac{\partial}{\partial x}(v+w) + y\frac{\partial}{\partial y}(v+w)$$
$$= x\frac{\partial v}{\partial x} + x\frac{\partial w}{\partial x} + y\frac{\partial v}{\partial y} + y\frac{\partial w}{\partial y}$$
$$= v + 0 \qquad \text{[From (2) and (3)}$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = v \qquad \text{——— (4)}$$

Differentiating partially both sides of (4) w.r. to x we get,

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial v}{\partial x} \quad \text{——— (5)}$$

Again differentiating (4) w.r.to y we get,

$$x\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial v}{\partial y} \quad \text{——— (6)}$$

Apply, $(5)\times x + (6)\times y$

$$x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + xy\frac{\partial^2 u}{\partial x \partial y} + xy\frac{\partial^2 u}{\partial x \partial y}$$
$$+ y\frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} = x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}$$

$$\Rightarrow x^2 u_{xx} + \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) + 2xy\, u_{xy}$$
$$+ y\, u_{yy} = v \qquad [\text{ from } (2)]$$

$$\Rightarrow x^2 u_{xx} + \cancel{v} + 2xy\, u_{xy} + y\, u_{yy} - \cancel{v} = 0$$
[from (4)]

$$\therefore x u_{xx} + 2xy\, u_{xy} + y\, u_{yy} = 0.$$

Ans

If $u = F(y-z,\ z-x,\ x-y)$, Prove that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

Proof: Given, $u = F(y-z,\ z-x,\ x-y)$ ——— ①

Let, $y-z = x_1$, $z-x = x_2$ and $x-y = x_3$

$$\therefore \frac{\partial x_1}{\partial x} = 0, \quad \frac{\partial x_2}{\partial x} = -1, \quad \frac{\partial x_3}{\partial x} = 1$$

$$\frac{\partial x_1}{\partial y} = 1, \quad \frac{\partial x_2}{\partial y} = 0, \quad \frac{\partial x_3}{\partial y} = -1$$

$$\frac{\partial x_1}{\partial z} = -1, \quad \frac{\partial x_2}{\partial z} = 1, \quad \frac{\partial x_3}{\partial z} = 0$$

From ①, we get,

$$u = F(x_1, x_2, x_3)$$

$$\therefore \frac{\partial u}{\partial x} = \cancel{\frac{\partial F}{\partial x}}\ \frac{\partial u}{\partial x_1}\cdot\frac{\partial x_1}{\partial x} + \frac{\partial u}{\partial x_2}\cdot\frac{\partial x_2}{\partial x} + \frac{\partial u}{\partial x_3}\cdot\frac{\partial x_3}{\partial x}$$

$$= 0 - \frac{\partial u}{\partial x_2} + \frac{\partial u}{\partial x_3}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial x_3} - \frac{\partial u}{\partial x_2} \quad \text{——— (3)}$$

similarly,

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x_1} - \frac{\partial u}{\partial x_3} \quad \text{——— (4)}$$

and $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x_2} - \frac{\partial u}{\partial x_1} \quad \text{——— (5)}$$

Adding (3), (4) and (5) we get,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \cancel{\frac{\partial u}{\partial x_3}} - \cancel{\frac{\partial u}{\partial x_2}} + \cancel{\frac{\partial u}{\partial x_1}} - \cancel{\frac{\partial u}{\partial x_3}} + \cancel{\frac{\partial u}{\partial x_2}} - \cancel{\frac{\partial u}{\partial x_1}}$$

$$= 0$$

Proved

If $u = F(x^2+y^2+z^2)\, f(xy+yz+zx)$; then prove that $(y-z)\frac{\partial u}{\partial x} + (z-x)\frac{\partial u}{\partial y} + (x-y)\frac{\partial u}{\partial z} = 0$.

Proof: Let, $x^2+y^2+z^2 = p$ and $xy+yz+zx = q$

$$\therefore u = F(p)\, f(q) \quad \text{——— (1)}$$

$$\therefore \frac{\partial u}{\partial x} = F'(p).2x.f(q) + F(p) f'(q)(y+z)$$

$$\therefore \frac{\partial u}{\partial y} = F'(p).2y.f(q) + F(p) f'(q)(x+z)$$

$$\therefore \frac{\partial u}{\partial z} = F'(p).2z.f(q) + F(p).f'(q)(x+y)$$

Now, $(y-z)\frac{\partial u}{\partial x} + (z-x)\frac{\partial u}{\partial y} + (x-y)\frac{\partial u}{\partial z}$

$$= F'(p) f(q)\, 2\left[x(y-z)+y(z-x)+z(x-y)\right]$$

$$+ F(p) f'(q)\,(y^2-z^2+z^2-x^2+x^2-y^2)$$

$$= F'(p) f(q)\, 2(xy-zx+yz-xy+zx-yz)$$

$$+ 0$$

$$= 0$$

Proved

$f(x, y) = 0$

Note

Formula: $$\frac{dy}{dx} = -\frac{f_x}{f_y} \quad \text{—— (1)}$$

$$u = f(x, y) \Rightarrow du = f_x\,dx + f_y\,dy$$

$$u = f(x, y, z) \Rightarrow du = f_x\,dx + f_y\,dy + f_z\,dz$$