Geometrical Interpretation of $\int_a^b f(x)\,dx$:

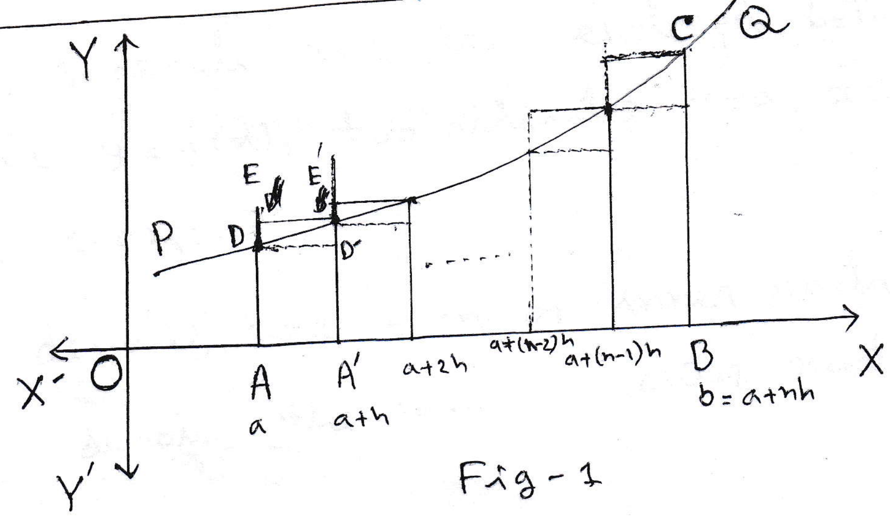

Fig-1

Let $f(x)$ be a bounded single valued function defined on the interval (a,b). Let $y=f(x)$ be the equation of the continuous curve PQ presented in the Fig-1. Let AD and BC be two ordinates corresponding to the points $x=a$ and $x=b$.

From the figure we have, $OA=a$, $OB=b$.

$\therefore AB = b-a$. Let AB be divided into n equal parts each of length h.

$\therefore h = \frac{b-a}{n}$ or, $nh = b-a$ or, $a+nh=b$.

Draw the ordinates through the points $x=a+h$, $x=a+2h, \ldots, x=a+(n-1)h$.

Complete ~~to~~ the inner ractangles $ADD'A'$, ...

And outer rectangles $AEE'A'$, ...

Let, S denote the area enclosed between the curve $y=f(x)$, two ordinates $x=a$, $x=b$, and the x-axis.

S_1 denotes the sum of inner ractangles.

S_2 denotes the sum of outer ractangles.

~~Then $S_1 < S < S_2$~~.

Here, $S_1 = hf(a) + hf(a+h) + \cdots + hf[a+(n-1)h]$

$$= h\sum_{r=0}^{n-1} f(a+rh)$$

$$S_2 = hf(a+h) + hf(a+2h) + \cdots + hf(a+nh)$$

$$= hf(a) + hf(a+h) + hf(a+2h) + \cdots + hf(a+(n-1)h) + hf(b) - hf(a)$$

$$= h\sum_{r=0}^{n-1} f(a+rh) + hf(b) - hf(a)$$

Here $S_1 < S < S_2$.

Now, if $n \to \infty$ then $h \to 0$, $hf(b) \to 0$

and $hf(a) \to 0$

$$\therefore \quad S_1 \rightarrow \lim_{h \to 0} h \sum_{r=0}^{n-1} f(a+rh) = \int_a^b f(x)\,dx$$

and $$S_2 \rightarrow \lim_{h \to 0} h \sum_{r=0}^{n-1} f(a+rh) = \int_a^b f(x)\,dx$$

Since we have always $S_1 < S < S_2$.

$$\therefore \quad S = \int_a^b f(x)\,dx$$

i.e. $\int_a^b f(x)\,dx$ geometrically represents the area of the space enclosed by the curve $y = f(x)$, the ordinates $x = a$, $x = b$, and the x-axis.

Fundamental theorem of Integral Calculus:

Statement: If $f(x)$ is integrable in (a, b), and if there exists a function $\phi(x)$ such that $\phi'(x) = f(x)$ in (a, b), then

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a).$$

Proof: Divide the interval (a, b) into n equal parts of length h by taking intermediate points

$$a = x_0 < x_1 < x_2 < \cdots\cdots < x_n = b.$$

where, $x_r = a + rh$, $r = 0, 1, 2, \ldots, n$

~~To~~ on each ~~interval~~ (x_{r-1}, x_r), using the mean value theorem we get

$$\phi'(\xi_r) = \frac{\phi(x_r) - \phi(x_{r-1})}{x_r - x_{r-1}} \quad \text{where } \xi_r \in (x_{r-1}, x_r)$$
$$[x_{r-1} < \xi_r < x_r]$$

$$\Rightarrow (x_r - x_{r-1})\,\phi'(\xi_r) = \phi(x_r) - \phi(x_{r-1})$$

$$\Rightarrow \cancel{h\phi'} \Rightarrow h f(\xi_r) = \phi(x_r) - \phi(x_{r-1}) \quad [\phi'(x) = f(x)]$$

$$\Rightarrow h f(\xi_1) = \phi(\cancel{x_1}) - \phi(x_0)$$

$$h f(\xi_2) = \phi(\cancel{x_2}) - \phi(\cancel{x_1})$$

$$\text{---} \quad \text{---} \quad \text{---}$$

$$h f(\xi_n) = \phi(x_n) - \phi(\cancel{x_{n-1}})$$

$$\sum_{r=1}^{n} h f(\xi_r) = \phi(x_n) - \phi(x_0)$$
$$= \phi(b) - \phi(a)$$

$$\lim_{n \to 0} \sum_{r=1}^{n} f(\xi_r) = \phi(b) - \phi(a)$$

$$\Rightarrow \int_a^b f(x)\,dx = \phi(b) - \phi(a)$$

General properties of definite integrals :

(i) $\int_a^b f(x)\,dx = \int_a^b f(z)\,dz$

Proof: By the fundamental theorem $\exists$ a function $\phi(x)$ such that $\phi'(x) = f(x)$ and

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a) \quad \text{--- (1)}$$

Now, $\phi'(x) = f(x) \Rightarrow \phi'(z) = f(z)$ then

$$\int_a^b f(z)\,dz = \phi(b) - \phi(a) \quad \text{--- (2)}$$

(1) and (2) gives

$$\int_a^b f(x)\,dx = \int_a^b f(z)\,dz$$

—x—

(ii) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Proof: By the fundamental theorem $\exists$ a function $\phi(x)$ such that $\phi'(x) = f(x)$ and

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a)$$

$$= -\{\phi(a) - \phi(b)\}$$

$$= -\int_b^a f(x)\,dx$$

—x—

(iii) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad (a<c<b).$

<u>Proof:</u> By the fundamental theorem $\exists$ a function $\phi(x)$ such that $\phi'(x) = f(x)$ and

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a)$$

$$\int_a^c f(x)\,dx = \phi(c) - \phi(a)$$

$$\int_c^b f(x)\,dx = \phi(b) - \phi(c)$$

Now, $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \cancel{\phi(c)} - \phi(a) + \phi(b) - \cancel{\phi(c)}$

$$= \phi(b) - \phi(a)$$

$$= \int_a^b f(x)\,dx$$

—x—

<u>Generalization:</u>

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \cdots$$

$$\cdots + \int_{c_{n-1}}^{c_n} f(x)\,dx + \int_{c_n}^{b} f(x)\,dx$$

$$a < c_1 < c_2 < \cdots < c_n < b$$

(iv) $\int_0^a f(x) = \int_0^a f(a-x)\,dx$

Proof: Put $a-x = z$ $\therefore -dx = dz$, $dx = -dz$

x	0	a
z	a	0

$$\therefore \int_0^a f(a-x)\,dx = \int_a^0 f(z)\,(-dz)$$

$$= -\int_a^0 f(z)\,dz$$

$$= \int_0^a f(z)\,dz$$

$$= \int_0^a f(x)\,dx$$

—x—

$$\int_0^{\pi/2} \sin x\,dx = \int_0^{\pi/2} \sin\left(\frac{\pi}{2}-x\right)dx$$

$$= \int_0^{\pi/2} \cos x\,dx$$

(v) $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$ if $f(x) = f(a+x)$

Proof: $\int_0^{na} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx + \cdots$

$\cdots + \int_{(n-1)a}^{na} f(x)\,dx$

Consider $\int_a^{2a} f(x)\,dx$

Put $x = z + a$ $\therefore dx = dz$

$x = a \Rightarrow z = 0$

$x = 2a \Rightarrow z = a$

$\therefore \int_a^{2a} f(x)\,dx = \int_0^a f(a+z)\,dz$

$= \int_0^a f(a+x)\,dx$

$= \int_0^a f(x)\,dx$ $[\because f(a+x) = f(x)$

Similarly, it can be shown that

$\int_{2a}^{3a} f(x)\,dx = \int_0^a f(x)\,dx$

$\int_{3a}^{4a} f(x)\,dx = \int_0^a f(x)\,dx$

$\cdots$

$\int_{(n-1)a}^{na} f(x)\,dx = \int_0^a f(x)\,dx$

$\therefore \int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$

(vi) $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx.$

Proof: $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$

Put $x = 2a - z$ in the 2nd integral; then $dx = -dz$, also when $x = a$, $z = a$ and when $x = 2a$, $z = 0$.

$$\therefore \int_a^{2a} f(x)\,dx = -\int_a^0 f(2a-z)\,dz = \int_0^a f(2a-z)\,dz$$
$$= \int_0^a f(2a-x)\,dx$$

$$\therefore \int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

—x—

(vii) $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$, if $f(2a-x) = f(x)$,

and $\int_0^{2a} f(x)\,dx = 0$, if $f(2a-x) = -f(x)$

Proof: $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$
$$= I_1 + I_2$$

where $I_1 = \int_0^a f(x)\,dx$ and $I_2 = \int_a^{2a} f(x)\,dx$

Put $x = 2a - z$ $\therefore dx = -dz$

x	a	$2a$
z	a	0

$$I_2 = -\int_a^0 f(2a-z)\,dz = \int_0^a f(2a-x)\,dx$$

if $f(2a-x) = f(x)$ then $I_2 = \int_0^a f(x)\,dx$

$$\therefore \int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$$

Again, if $f(2a-x) = -f(x)$ then $I_2 = -\int_0^a f(x)\,dx$

$$\therefore \int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx - \int_0^a f(x)\,dx = 0.$$

—x—

(viii) $\int_{-a}^{+a} f(x)\,dx = \int_0^a \{f(x) + f(-x)\}\,dx$

<u>Proof:</u> $\int_{-a}^{+a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^{+a} f(x)\,dx$

$$= I_1 + I_2$$

where, $I_1 = \int_{-a}^{0} f(x)\,dx$ and $I_2 = \int_0^a f(x)\,dx$

Put $x = -z$ $\therefore dx = -dz$

x	$-a$	0
z	a	0

$$\therefore I_1 = -\int_a^0 f(-z)\,dz = \int_0^a f(-z)\,dz = \int_0^a f(-x)\,dx$$

$$\therefore \int_{-a}^{+a} f(x)\,dx = \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx$$

$$= \int_0^a \{f(x) + f(-x)\}\,dx$$

—x—

Odd function:

A function $f(x)$ is called an odd function if
$$f(-x) = -f(x)$$

Even function:

A function $f(x)$ is called an even function if
$$f(-x) = f(x)$$

Show that
$$\int_{-a}^{+a} f(x)\,dx = \begin{cases} 0, & \text{if } f(x) \text{ is an odd function} \\ 2\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is an even function} \end{cases}$$

Proof: Let, $I = \int_{-a}^{+a} f(x)\,dx$

$$\therefore I = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$$
$$= I_1 + I_2$$

Where $I_1 = \int_{-a}^{0} f(x)\,dx$ and $I_2 = \int_0^a f(x)\,dx$

Put, $x = -z$ then $dx = -dz$,

x	$-a$	0
z	a	0

$$\therefore I_1 = -\int_a^0 f(-z)\,dz = \int_0^a f(-z)\,dz = \int_0^a f(-x)\,dx$$

If $f(x)$ is an odd function, then $f(-x) = -f(x)$

$\therefore \; I_1 = -\int_0^a f(x)\,dx$

$\therefore \; I = -\int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 0$

If $f(x)$ is an even function, then $f(-x) = f(x)$

$\therefore \; I_1 = \int_0^a f(x)\,dx$

$\therefore \; I = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$

$$\therefore \int_{-a}^{a} f(x)\,dx = \begin{cases} 0, & \text{if } f(x) \text{ is an odd function} \\ 2\int_0^a f(x)\,dx & \text{if } f(x) \text{ is an even function} \end{cases}$$

—x—

Show that $\int_0^{\pi/2} \log \tan x \, dx = 0$

Proof: Let, $I = \int_0^{\pi/2} \log \tan x \, dx$

$$= \int_0^{\pi/2} \log \tan\left(\frac{\pi}{2} - x\right) dx$$

$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$= \int_0^{\pi/2} \log \cot x \, dx$$

$$= \int_0^{\pi/2} \log (\tan x)^{-1} dx$$

$$= -\int_0^{\pi/2} \log \tan x \, dx$$

$$= -I$$

$\therefore \quad I + I = 0$

$\Rightarrow \quad 2I = 0$

$\Rightarrow \quad I = 0$

$\therefore \quad \int_0^{\pi/2} \log \tan x \, dx = 0$

—x—

1. Show that $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx = \frac{\pi}{4}$

Proof: Let, $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$ ——— (1)

Then, $I = \int_0^{\pi/2} \frac{\sqrt{\sin(\frac{\pi}{2} - x)}}{\sqrt{\sin(\frac{\pi}{2} - x)} + \sqrt{\cos(\frac{\pi}{2} - x)}}\, dx$ $\quad \because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$= \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\, dx$ ——— (2)

Adding (1) and (2), we get,

$$2I = \int_0^{\pi/2} \left(\frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} \right) dx$$

$$= \int_0^{\pi/2} dx$$

$$= \left[x \right]_0^{\pi/2}$$

$$= \frac{\pi}{2}$$

$\therefore I = \frac{\pi}{4}$ proved

Show that $\int_0^{\pi/2} \log \sin x \, dx = \int_0^{\pi/2} \log \cos x = \frac{\pi}{2} \log \frac{1}{2}$

Proof: Let $I = \int_0^{\pi/2} \log \sin x \, dx$ ——— (1)

Then $I = \int_0^{\pi/2} \log \sin\left(\frac{\pi}{2} - x\right) dx$ $\quad \because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$= \int_0^{\pi/2} \log \cos x \, dx$ ——— (2)

Adding (1) and (2) we get,

$$2I = \int_0^{\pi/2} (\log \sin x + \log \cos x)\, dx$$

$$= \int_0^{\pi/2} \log(\sin x \cos x)\, dx$$

$$= \int_0^{\pi/2} \log\left(\frac{1}{2} \cdot 2 \sin x \cos x\right) dx$$

$$= \int_0^{\pi/2} \left(\log \frac{1}{2} + \log \sin 2x\right) dx$$

$$= \log \frac{1}{2} \int_0^{\pi/2} dx + \int_0^{\pi/2} \log \sin 2x \, dx$$

$$= \log \frac{1}{2} \left[x\right]_0^{\pi/2} + \int_0^{\pi/2} \log \sin 2x \, dx$$

$$= \frac{\pi}{2} \log \frac{1}{2} + I_1$$

where $I_1 = \int_0^{\pi/2} \log \sin 2x \, dx$

Put $2x = z$ $\therefore 2dx = dz$, $dx = \frac{1}{2}dz$

x	0	$\pi/2$
z	0	π

$$\therefore \quad I_1 = \frac{1}{2}\int_0^{\pi} \log \sin z \, dz$$

$$= \frac{1}{2}\int_0^{\pi} \log \sin x \, dx$$

$$= \frac{1}{2}\cdot 2\int_0^{\pi/2} \log \sin x \, dx$$

$$= \int_0^{\pi/2} \log \sin x \, dx$$

$$= I$$

$\int_0^{2a} f(x)\,dx$

$= 2\int_0^{a} f(x)\,dx$

if $f(2a-x) = f(x)$

Here $f(x) = \log \sin x$

$a = \frac{\pi}{2}$

$\therefore f(2a-x) = f(\pi - x)$

$= \log \sin(\pi - x)$

$= \log \sin x$

$$\therefore \quad 2I = \frac{\pi}{2}\log\frac{1}{2} + I$$

$$\Rightarrow \quad I = \frac{\pi}{2}\log\frac{1}{2} \quad \text{——— (3)}$$

From, (1), (2) and (3), we have

$$\int_0^{\pi/2} \log \sin x \, dx = \int_0^{\pi/2} \log \cos x \, dx = \frac{\pi}{2}\log\frac{1}{2}$$

Proved ——X——

Show that $\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\log 2.$

Proof: Let, $I = \int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$

Put, $x = \tan\theta$ $\therefore dx = \sec^2\theta\,d\theta$

$\theta = \tan^{-1}x$

x	0	1
θ	0	$\pi/4$

$$\therefore I = \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{1+\tan^2\theta}\cdot\sec^2\theta\,d\theta$$

$$\text{i.e. } I = \int_0^{\pi/4} \log(1+\tan\theta)\,d\theta \quad \text{---} (1)$$

$$= \int_0^{\pi/4} \log\left\{1+\tan\left(\frac{\pi}{4}-\theta\right)\right\}d\theta$$

$$= \int_0^{\pi/4} \log\left\{1+\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right\}d\theta$$

$$= \int_0^{\pi/4} \log\left\{1+\frac{1-\tan\theta}{1+\tan\theta}\right\}d\theta$$

$$= \int_0^{\pi/4} \log\left(\frac{1+\cancel{\tan\theta}+1-\cancel{\tan\theta}}{1+\tan\theta}\right)d\theta$$

$$= \int_0^{\pi/4} \left\{\log 2 - \log(1+\tan\theta)\right\}d\theta$$

$$= \log 2\,[\theta]_0^{\pi/4} - \int_0^{\pi/4} \log(1+\tan\theta)\,d\theta$$

$\therefore\ I = \frac{\pi}{4}\log 2 - I$

$\Rightarrow\ 2I = \frac{\pi}{4}\log 2$

$\therefore\ I = \frac{\pi}{8}\log 2$

—x—

Show that $\int_{-a}^{+a} \frac{x e^{x^2}}{1+x^2}\,dx = 0.$

Proof: Let, $I = \int_{-a}^{+a} \frac{x e^{x^2}}{1+x^2}\,dx$

$= \int_{-a}^{0} \frac{x e^{x^2}}{1+x^2}\,dx + \int_{0}^{a} \frac{x e^{x^2}}{1+x^2}\,dx$

$= I_1 + I_2$ (say)

where, $I_1 = \int_{-a}^{0} \frac{x e^{x^2}}{1+x^2}\,dx$ and $I_2 = \int_{0}^{a} \frac{x e^{x^2}}{1+x^2}\,dx$

Put $x = -z,\ dx = -dz$

x	$-a$	0
z	a	0

$\therefore\ I_1 = \int_{a}^{0} \frac{-z e^{z^2}}{1+z^2}(-dz)$

$= -\int_{0}^{a} \frac{z e^{z^2}}{1+z^2}\,dz = -\int_{0}^{a} \frac{x e^{x^2}}{1+x^2}\,dx = -I_2$

$\therefore\ I = -I_2 + I_2 = 0$ Proved

If n is a positive integer,

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} & \text{when } n \text{ is even} \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{4}{5}\cdot\frac{2}{3}\cdot 1 & \text{when } n \text{ is odd} \end{cases}$$

Proof: Let $I_n = \int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \{\sin(\frac{\pi}{2}-x)\}^n dx$

$= \int_0^{\pi/2} \cos^n x\,dx$

Now,

$$I_n = \int_0^{\pi/2} \sin^{n-1} x \sin x\,dx$$

$$= \left[\sin^{n-1} x\,(-\cos x)\right]_0^{\pi/2} - \int_0^{\pi/2} \left\{\frac{d}{dx}\sin^{n-1} x\,(-\cos x)\right\} dx$$

$$= +\int_0^{\pi/2} (n-1)\sin^{n-2} x\,.\cos x.\cos x\,dx$$

$$= (n-1)\int_0^{\pi/2} \sin^{n-2} x\,(1-\sin^2 x)\,dx$$

$$= (n-1)\int_0^{\pi/2} \sin^{n-2} x\,dx - (n-1)\int_0^{\pi/2} \sin^n x\,dx$$

$$= (n-1)\,I_{n-2} - (n-1)\,I_n$$

$I_n = \int_0^{\pi/2} \sin^n x\,dx$

$I_{n-2} = \int_0^{\pi/2} \sin^{n-2} x\,dx$

$$\Rightarrow \cancel{I_n} + nI_n - \cancel{I_n} = (n-1)I_{n-2}$$

$$\therefore I_n = \frac{n-1}{n} I_{n-2}$$

$$\therefore I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot I_4$$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot I_{n-6}$$

$$I_n = \frac{n-1}{n} I_{n-2}$$

$$I_{n-2} = \frac{n-3}{n-2} \cdot I_{n-4}$$

$$I_{n-4} = \frac{n-5}{n-4} \cdot I_{n-6}$$

If n is odd, then

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot I_1$$

Now,

$$I_n = \int_0^{\pi/2} \sin^n x \, dx \quad \therefore I_1 = \int_0^{\pi/2} \sin x \, dx$$

$$= \left[-\cos x\right]_0^{\pi/2}$$

$$= -0 + 1$$

$$= 1$$

$$\therefore I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots \frac{4}{5} \cdot \frac{2}{3} \cdot 1$$

If n is even, then

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot I_0$$

Now, $I_0 = \int_0^{\pi/2} \sin^0 x \, dx = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2}$

$$\therefore I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

Hence
$$I_n = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots\cdots \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1 \end{cases}$$

Proved

Evaluate $\int_0^{\pi} \cos^n x \, dx$

Soln: Let $I_n = \int_0^{\pi} \cos^n x \, dx$

$$= \int_0^{\pi} \{\cos(\pi - x)\}^n dx = \begin{cases} \int_0^{\pi} \cos^n x \, dx & \text{if } n \text{ is even} \\ -\int_0^{\pi} \cos^n x \, dx & \text{if } n \text{ is odd} \end{cases}$$

Now, if n is odd then

$$I_n = -\int_0^{\pi} \cos^n x \, dx$$

**We know that $\int_0^{2a} f(x) \, dx = \int_0^a f(x) \, dx + \int_0^a f(2a - x) \, dx$

$$\therefore I_n = -\int_0^{2(\pi/2)} \cos^n x \, dx = -\int_0^{\pi/2} \cos^n x \, dx - \int_0^{\pi/2} \{\cos(2 \cdot \tfrac{\pi}{2} - x)\}^n dx$$

$$= -\int_0^{\pi/2} \cos^n x \, dx + \int_0^{\pi/2} \cos^n x \, dx$$

$$= 0$$

If n is even then

$$I_n = \int_0^{\pi} \cos^n x \, dx = \int_0^{\pi/2} \cos^n x \, dx + \int_0^{\pi/2} \{\cos(2 \cdot \tfrac{\pi}{2} - x)\}^n dx$$

$$= \int_0^{\pi/2} \cos^n x \, dx + \int_0^{\pi/2} \cos^n x \, dx = 2\int_0^{\pi/2} \cos^n x \, dx$$

$$= 2 \cdot \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

Ans

Show that

(i) $\displaystyle\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx = \frac{\pi}{4}$

(ii) $\displaystyle\int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \sin x \cos x}\, dx = 0$

(iii) $\displaystyle\int_0^{1} \frac{\log x}{\sqrt{1-x^2}}\, dx = \frac{\pi}{2} \log \frac{1}{2}$

(iv) $\displaystyle\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}\, dx = \frac{\pi}{2}(\pi - 2)$

(v) $\displaystyle\int_0^{1} \cot^{-1}(1 - x + x^2)\, dx = \frac{\pi}{2} - \log 2$

If $I_n = \displaystyle\int_0^{\pi/4} \tan^n\theta \, d\theta$, show that $I_n = \dfrac{1}{n-1} - I_{n-2}$.

Hence find the value of $\displaystyle\int_0^{\pi/4} \tan^6 x \, dx$.

~~Show that if m and n are~~

(i) Show that $\int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x}\,dx = \frac{\pi}{4}$ (135)

(i) Proof: Let, $I = \int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x}\,dx$ ——— (1)

$\therefore I = \int_0^{\pi/2} \frac{\sin(\frac{\pi}{2}-x)}{\sin(\frac{\pi}{2}-x)+\cos(\frac{\pi}{2}-x)}\,dx$ $\quad \because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$= \int_0^{\pi/2} \frac{\cos x}{\cos x+\sin x}\,dx$ ——— (2)

Adding, (1) and (2) we get,

$$2I = \int_0^{\pi/2} \frac{\sin x+\cos x}{\sin x+\cos x}\,dx$$

$$= \int_0^{\pi/2} dx$$

$$= [x]_0^{\pi/2}$$

$$= \frac{\pi}{2}$$

$\therefore I = \frac{\pi}{4}$. proved

(ii) Show that $\int_0^{\pi/2} \frac{\cos x-\sin x}{1+\sin x\cos x}\,dx = 0$

Proof: Let, $I = \int_0^{\pi/2} \frac{\cos x-\sin x}{1+\sin x\cos x}\,dx$ ——— (1)

$\therefore I = \int_0^{\pi/2} \frac{\cos(\pi/2-x)-\sin(\pi/2-x)}{1+\sin(\pi/2-x)\cos(\pi/2-x)}\,dx$

$$\therefore \quad I = \int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \cos x \sin x} dx \quad \text{——— (2)}$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{\cancel{\cos x} - \cancel{\sin x} + \cancel{\sin x} - \cancel{\cos x}}{1 + \sin x \cos x} dx$$

$$= \int_0^{\pi/2} 0 \, dx$$

$$= 0$$

$$\therefore I = 0$$

Proved

~~Sol~~

(iii) Show that $\int_0^1 \frac{\log x}{\sqrt{1-x^2}} dx = \frac{\pi}{2} \log \frac{1}{2}$

Proof: Let, $I = \int_0^1 \frac{\log x}{\sqrt{1-x^2}} dx$

Put, $x = \sin\theta \quad \therefore dx = \cos\theta \, d\theta$

x	0	1
θ	0	$\frac{\pi}{2}$

$$\therefore \quad I = \int_0^{\pi/2} \frac{\log \sin\theta}{\sqrt{1 - \sin^2\theta}} \cdot \cos\theta \, d\theta$$

$$= \int_0^{\pi/2} \log \sin\theta \, d\theta \quad \text{——— (1)}$$

$$\therefore \quad I = \int_0^{\pi/2} \log \sin\left(\frac{\pi}{2}-\theta\right) d\theta$$

$$= \int_0^{\pi/2} \log \cos\theta \, d\theta \quad \text{———} \quad (2)$$

Adding (1) and (2) we get,

$$2I = \int_0^{\pi/2} (\log \sin\theta + \log \cos\theta) \, d\theta$$

$$= \int_0^{\pi/2} \log(\sin\theta \cos\theta) \, d\theta$$

$$= \int_0^{\pi/2} \log\left(\frac{1}{2} \cdot \sin 2\theta\right) d\theta$$

$$= \int_0^{\pi/2} \left(\log\frac{1}{2} + \log \sin 2\theta\right) d\theta$$

$$= \log\frac{1}{2} \int_0^{\pi/2} d\theta + \int_0^{\pi/2} \log \sin 2\theta \, d\theta$$

$$= \log\frac{1}{2} \left[\theta\right]_0^{\pi/2} + I_1$$

$$= \frac{\pi}{2} \log\frac{1}{2} + I_1$$

where $I_1 = \int_0^{\pi/2} \log \sin 2\theta \, d\theta$

Put $2\theta = z \quad \therefore \quad d\theta = \frac{1}{2} dz$

θ	0	$\pi/2$
z	0	π

$$\therefore \quad I_1 = \frac{1}{2}\int_0^{\pi} \log \sin z \, dz$$

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

$$\int_a^b f(x)\,dx = \int_a^b f(z)\,dz$$

$$I_1 = \frac{1}{2}\int_0^{2\cdot\frac{\pi}{2}} \log\sin\theta\,d\theta$$

$$= \frac{1}{2}\left[\int_0^{\pi/2}\log\sin\theta\,d\theta + \int_0^{\pi/2}\log\sin(\pi-\theta)\,d\theta\right]$$

$$= \frac{1}{2}\left[\int_0^{\pi/2}\log\sin\theta\,d\theta + \int_0^{\pi/2}\log\sin\theta\,d\theta\right]$$

$$= \frac{1}{2}\cdot 2\int_0^{\pi/2}\log\sin\theta\,d\theta$$

$$= I$$

$$\therefore\ 2I = \frac{\pi}{2}\log\frac{1}{2} + I$$

$$\Rightarrow\ 2I - I = \frac{\pi}{2}\log\frac{1}{2}$$

$$\therefore\ I = \frac{\pi}{2}\log\frac{1}{2}$$ proved

(iv) Show that $\displaystyle\int_0^{\pi}\frac{x\tan x}{\sec x+\tan x}\,dx = \frac{\pi}{2}(\pi-2)$.

Proof: Let, $\displaystyle I = \int_0^{\pi}\frac{x\tan x}{\sec x+\tan x}\,dx$

$$= \int_0^{\pi}\frac{(\pi-x)\tan(\pi-x)}{\sec(\pi-x)+\tan(\pi-x)}\,dx$$

$$= \int_0^{\pi}\frac{-(\pi-x)\tan x}{-[\sec x+\tan x]}\,dx$$

$$\therefore I = \pi \int_0^{\pi} \frac{\tan x}{\sec x + \tan x}\, dx - I$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x} + \frac{\sin x}{\cos x}}\, dx$$

$$= \pi \int_0^{\pi} \frac{\sin x}{1 + \sin x}\, dx$$

$$= \pi \int_0^{\pi} \frac{1 + \sin x - 1}{1 + \sin x}\, dx$$

$$= \pi \int_0^{\pi} \left(1 - \frac{1}{1+\sin x}\right) dx$$

$$= \pi \int_0^{\pi} dx - \pi \int_0^{\pi} \frac{dx}{1 + \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$$

$$\left[\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} \right.$$

$$= \pi \left[x\right]_0^{\pi} - \pi \int_0^{\pi} \frac{\sec^2\frac{x}{2}\, dx}{1 + \tan^2\frac{x}{2} + 2\tan\frac{x}{2}}$$

$$= \pi^2 - \pi \int_0^{\pi} \frac{\sec^2\frac{x}{2}\, dx}{\left(1+\tan\frac{x}{2}\right)^2}$$

Put $1 + \tan\frac{x}{2} = z$ $\quad \therefore \frac{1}{2}\sec^2\frac{x}{2}\, dx = dz \Rightarrow \sec^2\frac{x}{2}\, dx = 2dz$

x	0	π
z	1	∞

$$\therefore \quad 2I = \pi^2 - 2\pi \int_1^\infty \frac{dz}{z^2}$$

$$= \pi^2 - 2\pi \left[\frac{z^{-2+1}}{-2+1} \right]_1^\infty$$

$$= \pi^2 + 2\pi \left[\frac{1}{z} \right]_1^\infty$$

$$= \pi^2 + 2\pi [0 - 1]$$

$$= \pi^2 - 2\pi$$

$$= \pi(\pi - 2)$$

$$\therefore \quad I = \frac{\pi}{2}(\pi - 2)$$

Proved

(v) Show that $\int_0^1 \cot^{-1}(1-x+x^2)\,dx = \frac{\pi}{2} - \log 2$

Proof: Let, $I = \int_0^1 \cot^{-1}(1-x+x^2)\,dx$

$$= \left[x \cdot \cot^{-1}(1-x+x^2) \right]_0^1 - \int_0^1 \left[x \cdot \frac{-1 \cdot (-1+2x)}{1+(1-x+x^2)^2} \right] dx$$

$$= 1 \cdot \frac{\pi}{4} + \int_0^1 \frac{(2x^2 - x)\,dx}{1+(1+x^2+x^4-2x-2x^3+2x^2)}$$

$$= \frac{\pi}{4} + I_1$$

Where $I_1 = \int_0^1 \frac{2x^2-x}{x^4-2x^3+3x^2-2x+2}\,dx$

Now, we have, $x^4-2x^3+3x^2-2x+2$

$= x^4+x^2-2x^3-2x+2x^2+2$

$= x^2(x^2+1)-2x(x^2+1)+2(x^2+1)$

$= (x^2+1)(x^2-2x+2)$

Let, $\frac{2x^2-x}{(x^2+1)(x^2-2x+2)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2-2x+2}$ —— (1)

Multiplying both sides of (1) by $(x^2+1)(x^2-2x+2)$ we get

$\therefore 2x^2-x = (Ax+B)(x^2-2x+2) + (Cx+D)(x^2+1)$

$\therefore 2x^2-x = Ax^3-2Ax^2+2Ax+Bx^2-2Bx+2B$
$+ Cx^3+Cx+Dx^2+D$

$\therefore 0.x^3+2x^2-x+0 = (A+C)x^3+(-2A+B+D)x^2+(2A-2B+C)x$
$+(2B+D)$ —— (2)

Equating the coefficient of like power in x, from both sides of (2) we get,

$A+C=0$ —— (3)
$-2A+B+D=2$ —— (4)
$2A-2B+C=-1$ —— (5)
$2B+D=0$ —— (6)

From (3) we have, $C = -A$ ——— (7)

From (6) we have, ~~$d =$~~ $D = -2B$ ——— (8)

From (4) we have, $-2A + B - 2B = 2$

$\Rightarrow -2A - B = 2$

$\Rightarrow B = -2A - 2$ ——— (9)

From (5) we have,

$2A - 2B + C = -1$

$\Rightarrow 2A - 2(-2A-2) + (-A) = -1$

$\Rightarrow 2A + 4A + 4 - A = -1$

$\Rightarrow 5A = -5$

$\therefore A = -1$

$\therefore B = -2A - 2 = 2 - 2 = 0$

$C = -A = 1$

$D = -2B = 0$

From (1) we have, $\dfrac{2x^2 - x}{(x^2+1)(x^2-2x+2)} = \dfrac{-x}{x^2+1} + \dfrac{x}{x^2-2x+2}$

$$\therefore I_1 = \int_0^1 \frac{-x\,dx}{x^2+1} + \int_0^1 \frac{x\,dx}{x^2-2x+2}$$

$$= -\frac{1}{2}\int_0^1 \frac{2x}{x^2+1}\,dx + \frac{1}{2}\int_0^1 \frac{2x-2}{x^2-2x+2}\,dx + \int_0^1 \frac{dx}{x^2-2x+2}$$

$$= -\frac{1}{2}\Big[\log(x^2+1)\Big]_0^1 + \frac{1}{2}\Big[\log(x^2-2x+2)\Big]_0^1 + \int_0^1 \frac{dx}{(x-1)^2+1^2}$$

$$= -\frac{1}{2}\Big[\log(2) - \log 1\Big] + \frac{1}{2}\Big[\log(1) - \log(2)\Big] + \Big[\tan^{-1}(x-1)\Big]_0^1$$

$$= -\frac{1}{2}\log 2 - \frac{1}{2}\log 2 + 0 + \frac{\pi}{4}$$

$$= -\log 2 + \frac{\pi}{4}$$

$$\therefore I = \frac{\pi}{4} + I_1$$

$$= \frac{\pi}{4} - \log 2 + \frac{\pi}{4}$$

$$= \frac{\pi}{2} - \log 2$$

Proved

If $I_n = \int_0^{\pi/4} \tan^n\theta\, d\theta$, show that $I_n = \frac{1}{n-1} - I_{n-2}$.
Hence find the value of $\int_0^{\pi/4} \tan^6 x\, dx$.

Proof: We have, $I_n = \int_0^{\pi/4} \tan^n\theta\, d\theta$

$$= \int_0^{\pi/4} \tan^{n-2}\theta\,(\sec^2\theta - 1)\, d\theta$$

$$= \int_0^{\pi/4} \tan^{n-2}\theta \sec^2\theta\, d\theta - \int_0^{\pi/4} \tan^{n-2}\theta\, d\theta$$

$$= \left[\frac{\tan^{n-1}\theta}{n-1}\right]_0^{\pi/4} - I_{n-2}$$

$$\therefore \quad I_n = \frac{1}{n-1} - I_{n-2}$$

Proved

2nd part: $I_6 = \int_0^{\pi/4} \tan^6 x\, dx$

We have, $I_6 = \frac{1}{6-1} - I_4$

$$= \frac{1}{5} - \left[\frac{1}{4-1} - I_2\right]$$

$$= \frac{1}{5} - \frac{1}{3} + \frac{1}{2-1} - I_0 \qquad \left[\tan^0 x = 1\right.$$

$$= \frac{3-5+15}{15} - \int_0^{\pi/4} dx$$

$$= \frac{13}{15} - \frac{\pi}{4} \quad \text{Ans}$$

"Improper Integral"

Def^n: If in an integral either the range is infinite or the integrand has an infinite discontinuity in the range, then the integral is called improper integral.

Infinite range:

(i) $\int_0^\infty f(x)\,dx$ is defined as $\lim\limits_{\varepsilon \to \infty} \int_a^\varepsilon f(x)\,dx$,

provided $f(x)$ is integrable in (a, ε), and this limit exists.

(ii) $\int_{-\infty}^{b} f(x)\,dx$ is defined as $\lim\limits_{\varepsilon \to -\infty} \int_\varepsilon^b f(x)\,dx$,

provided $f(x)$ is integrable in (ε, b), and this limit exists.

(iii) If the infinite integrals $\int_{-\infty}^{a} f(x)\,dx$ and $\int_a^\infty f(x)\,dx$ both exist, we ~~sh~~ say that $\int_{-\infty}^{\infty} f(x)\,dx$ exists, and

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^\infty f(x)\,dx.$$