Previously, we solved $y' = f(x,y)$; $y(x_0) = y_0$ where y', x, y are scalars.

For a system of ODE's we could write

$$\bar{y}' = \bar{f}(x,\bar{y}) \; ; \; \bar{y}(x_0) = \bar{y}_0$$

which is a vector equation

$$\bar{y}' = \begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_m' \end{bmatrix} \qquad \bar{f}(x,\bar{y}) = \begin{bmatrix} f_1(x,\bar{y}) \\ f_2(x,\bar{y}) \\ \vdots \\ f_m(x,\bar{y}) \end{bmatrix} \qquad \bar{y}(x_0) = \begin{bmatrix} y_1(x_0) \\ y_2(x_0) \\ \vdots \\ y_m(x_0) \end{bmatrix} = \begin{bmatrix} y_{10} \\ y_{20} \\ \vdots \\ y_{m0} \end{bmatrix}$$

Note: if we have a higher order ODE : $y^{(m)} = f(x, y, y', y'', \dots, y^{m-1})$
we can write : $y_1 = y$, $y_1' = y_2$, $y_2' = y_3$, ..., $y_m = y^{(m-1)}$

$$y_1' = y_2$$
$$y_2' = y_3$$
$$y_3' = y_4$$
$$\vdots$$
$$y_{m-1}' = y_m$$
$$y_m' = f(x, y, y_1, y_2, \dots, y_{m-1})$$

and $y_1(x_0) = k_1$, $y_2(x_0) = k_2$, ..., $y_m(x_0) = k_m$

Euler method for systems

$$y'' + 2y' + 0.75y = 0 \quad ; \; y(0) = 3, \; y'(0) = -2.5$$

we can write:

$$y_1 = y, \quad y_1' = y_2, \quad y_2'' = y_3$$

for a system of first order ODE's:

$$\bar{y}_{n+1} = \bar{y}_n + h\bar{f}(x_n, \bar{y}_n)$$

or

$$y_{1,n+1} = y_{1,n} + hf_1(x_n, y_{1,n}, y_{2,n})$$
$$y_{2,n+1} = y_{2,n} + hf_2(x_n, y_{1,n}, y_{2,n})$$

for 2^{nd} order ODE above:

$$y_3 + 2y_2 + 0.75y_1 = 0 \rightarrow y_3 = f_2(x, y_1, y_2) = -2y_2 - 0.75y_1$$

$$y_2 = y_1' = f_1(x, y_1, y_2)$$

so $y_{1,n+1} = y_{1,n} + hy_{2,n}$

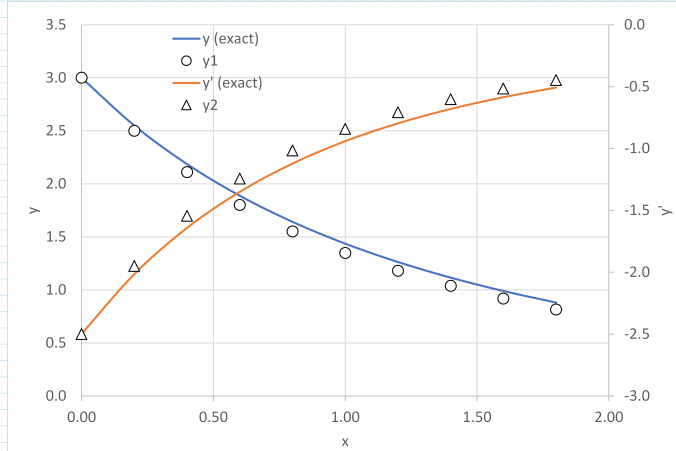
$$y_{2,n+1} = y_{2,n} + h(-2y_{2,n} - 0.75y_{1,n})$$

so $y_{1,n+1} = y_{1,n} + h y_{2,n}$

$$y_{2,n+1} = y_{2,n} + h(-2y_{2,n} - 0.75 y_{1,n})$$

Exact solution: $y = y_1 = 2e^{-0.5x} + e^{-1.5x}$; $y' = y_2 = -e^{-0.5x} - 1.5e^{-1.5x}$

				Exact	Err	Exact	Err
n	**x_n**	**$y_{1,n}$**	**$y_{2,n}$**	**y_1**		**y_2**	
0	0.00	3.00000	-2.500000	3.000000	0.000000	-2.500000	0.000000
1	0.20	2.50000	-1.950000	2.550493	0.050493	-2.016065	-0.066065
2	0.40	2.11000	-1.545000	2.186273	0.076273	-1.641948	-0.096948
3	0.60	1.80100	-1.243500	1.888206	0.087206	-1.350673	-0.107173
4	0.80	1.55230	-1.016250	1.641834	0.089534	-1.122111	-0.105861
5	1.00	1.34905	-0.842595	1.436191	0.087141	-0.941226	-0.098631
6	1.20	1.18053	-0.707915	1.262922	0.082391	-0.796760	-0.088845
7	1.40	1.03895	-0.601828	1.115627	0.076679	-0.680270	-0.078442
8	1.60	0.91858	-0.516939	0.989376	0.070793	-0.585406	-0.068467
9	1.80	0.81519	-0.447951	0.880345	0.065150	-0.507378	-0.059427



Runge-Kutta Methods for Systems

$\bar{y}(x_0) = \bar{y}_0$ (initial values)

$$\bar{k}_1 = h\bar{f}(x_n, \bar{y}_n)$$

$$\bar{k}_2 = h\bar{f}(x_n + \tfrac{1}{2}h, \bar{y}_n + \tfrac{1}{2}\bar{k}_1)$$

$$\bar{k}_3 = h\bar{f}(x_n + \tfrac{1}{2}h, \bar{y}_n + \tfrac{1}{2}\bar{k}_2)$$

$$\bar{k}_4 = h\bar{f}(x_n + h, \bar{y}_n + \bar{k}_3)$$

$$\bar{y}_{n+1} = \bar{y}_n + \tfrac{1}{2}(\bar{k}_1 + 2\bar{k}_2 + 2\bar{k}_3 + \bar{k}_4)$$

Example: Airy's equation and Airy function $Ai(x)$

$y'' = xy$, $y(0) = 1/(3^{2/3} \cdot \Gamma(\tfrac{2}{3})) = 0.35502805$, $y'(0) = -1/(3^{1/3} \cdot \Gamma(\tfrac{1}{3})) = -0.25881940$

$h = 0.2$ Note: $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$; $\alpha > 0$ (Excel: $\Gamma(\alpha)$ = Gamma(α))

$y_1 = y$
$y_1' = y_2 = y' = f(x, y_2)$
$y_2' = x y_1 = f(x, y_1)$

$y_{1,n} = y_{1,n} + \frac{1}{6}(a_{1,n} + 2b_{1,n} + 2c_{1,n} + d_{1,n})$
$y_{2,n} = y_{2,n} + \frac{1}{6}(a_{2,n} + 2b_{2,n} + 2c_{2,n} + d_{2,n})$

				k_1: [a1 a2]		k_2: [b1 b2]		k_3: [c1 c2]		K_4: [d1 d2]		Exact		Error
n	**x_n**	**$y_{1,n}$**	**$y_{2,n}$**	**a1**	**a2**	**b1**	**b2**	**c1**	**c2**	**d1**	**d2**	y1	y2	y1*10^8
0	0.00	0.35503	-0.258819	-0.051764	0.000000	-0.051764	0.006583	-0.051106	0.006583	-0.050447	0.012157	0.355028054	-0.259	0
1	0.20	0.30370	-0.25240	-0.050481	0.012148	-0.049266	0.016708	-0.048810	0.016744	-0.047132	0.020391	0.30370313	-0.252	10
2	0.40	0.25474	-0.23583	-0.047166	0.020379	-0.045128	0.023116	-0.044855	0.023218	-0.042523	0.025187	0.25474235	-0.236	24
3	0.60	0.20980	-0.21279	-0.042558	0.025176	-0.040041	0.026393	-0.039919	0.026569	-0.037245	0.027181	0.20980006	-0.213	33
4	0.80	0.16985	-0.18641	-0.03728	0.027175	-0.034565	0.027217	-0.03456	0.027461	-0.031790	0.027057	0.16984632	-0.186	36

				6				5						
3	0.60	0.20980	-0.21279	-0.042558	0.025176	-0.040041	0.026393	-0.039919	0.026569	-0.037245	0.027181	0.20980006	-0.213	33
4	0.80	0.16985	-0.18641	-0.03728	0.027175	-0.034565	0.027217	-0.03456	0.027461	-0.031790	0.027057	0.16984632	-0.186	36
5	1.00	0.13529	-0.15915	-0.0[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]5461	0.13529242	-0.159	35