Exploring for typical phenomena in Thompson's groups

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grateful to NSF #1417820, PSC-CUNY research support

Plan

- Stratifications of groups
- Random subgroups of F
- Random elements and subgroups of T and V
- Word lengths of random tree pair diagrams in F
- Sprawl and expected length combinations in F

Size of elements

G a finitely generated group, with an associated *size* $|x|$ of representatives of elements x.

Desired properties of size:

- Only finitely many elements of a fixed size.
- Increasing size exhausts the group.

Examples: word length, diagram size, ...

Density of elements

Given a set of elements of a group G, a measure of their density with respect to a specified notion of size with balls B_n is

Def The density δ of a set X in G with respect to a size:

$$\delta = \lim_{n \to \infty} \frac{|X \cap B_n|}{|B_n|}$$

if the limit exists.

Size of k-tuples

Fix k, we want to understand a subgroup generated by a tuple of k elements chosen at "random."

Def The sphere of radius n in X^k in the "sum" stratification $= \mathrm{Sph}_k^{\mathrm{sum}}(n) = \{(x_1, \ldots, x_k) \in X^k$ such that $\sum |x_i| = n\}$

Def The sphere of radius n in X^k in the "max" stratification $= \mathrm{Sph}_k^{\mathrm{max}}(n)$ is the set of unordered k-tuples of X such that $\max |x_i| = n$

These spheres give k-tuples of elements of G, generating subgroups.

Asymptotic density

Def The *asymptotic density* of a subset T of k-tuples of $X =$

$$\lim_{n\to\infty} \frac{|T \cap \mathsf{Sph}_k(n)|}{|\mathsf{Sph}_k(n)|}$$

if the limit exists.

Typical questions: what is the asymptotic density of tuples which generate finite subgroups, abelian subgroups, free subgroups, or particular isomorphism classes of subgroups?

Can depend upon notions of size- these can create biases.

Subgroup spectra

Fix a number of generators k.

Def A subgroup H of G is visible if its isomorphism class has positive asymptotic density with respect to a particular stratification.

Def A subgroup H of G is negligible if its isomorphism class has 0 asymptotic density with respect to a particular stratification.

Def The k-generator spectrum $= \mathrm{Spec}_k(G)$ of a group is the set of subgroup isomorphism classes which are visible.

Random free subgroups

Arzhantseva, Arzhantseva and Olshanskii: random subgroup of free group is free on those generators.

Jitsukawa, Miasnikov and Ushakov, Fine, Roseberger, and Miasnokov: In finite rank free groups, non-elementary torsion-free hyperbolic groups, pure braid groups, various free products: a random k-tuple of elements generate a free group of rank k with probability 1.

(Size = word length, spheres from max stratification)

Weil and Ventura (2013): size is size of Whitehead graph, different distributions.

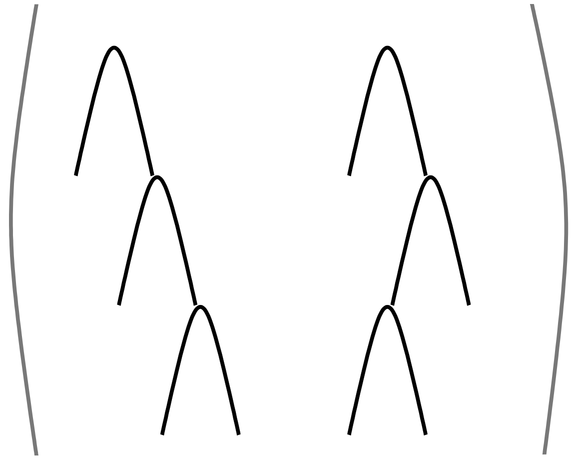
Thompson's group F

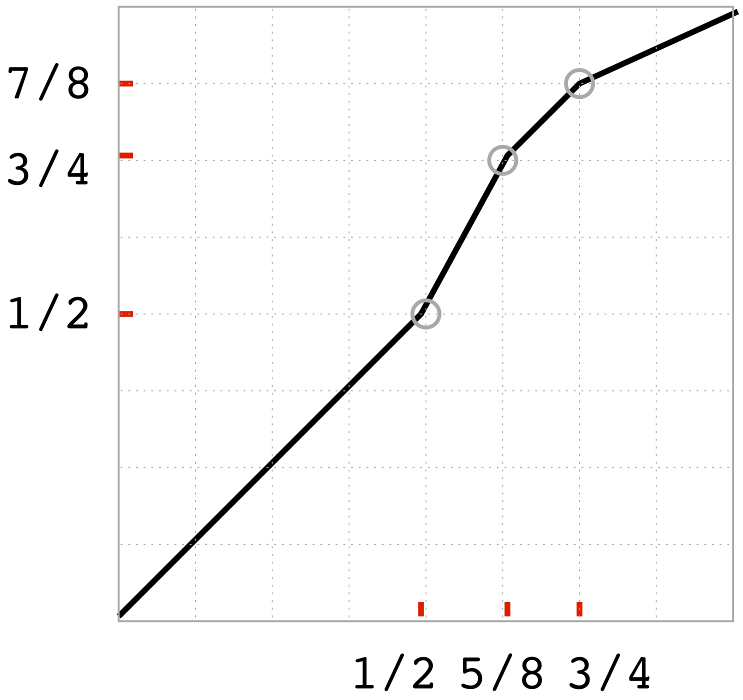
F = group of PL homeos of $[0,1]$ with slopes 2^i and breakpoints in $Z[\frac{1}{2}]$.

$F = \langle x_0, x_1, x_2, \ldots | x_n x_i = x_i x_{n+1}$ for all $i < n \rangle$

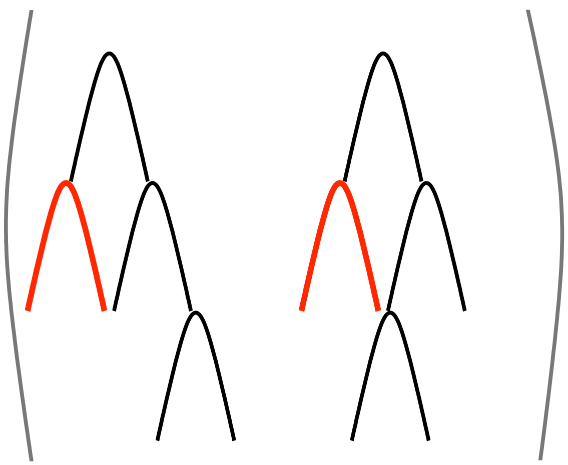
F generated by $\{x_0, x_1\}$, finitely presented with 2 relators

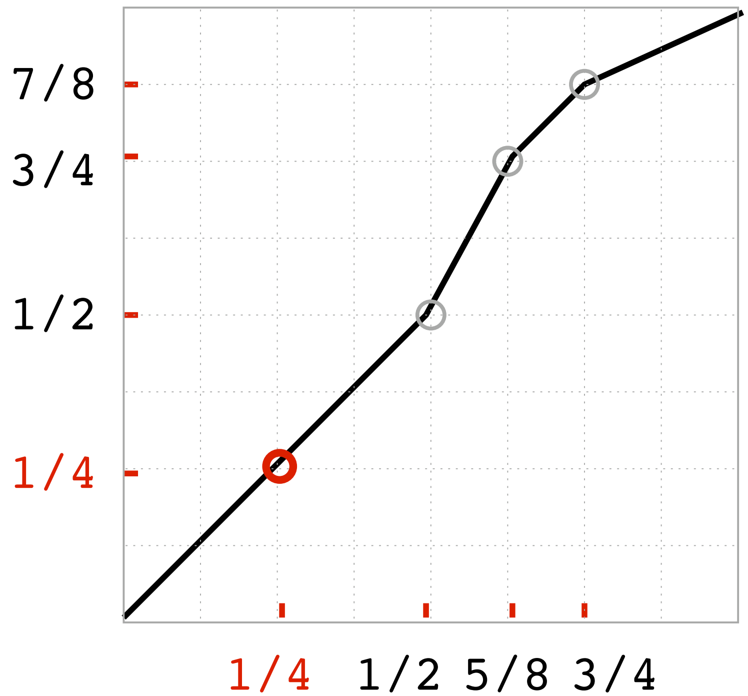
Thompson's group F





Thompson's group F





Red caret: unreduced, same resulting homeo

Some relevant properties of F

Contains free abelian subgroups of large rank

Contains $F \times F$

No free subgroups (Brin-Squier)

Contains a free sub-semigroup

Ker ϕ slope at endpoints/exponent sum map is simple.

Word length in F

Normal forms w.r.t. infinite generating set

$|w|$ = word length in F w.r.t. $\{x_0, x_1\}$

Algorithm for word length exactly: Fordham, from tree pairs

Word length is q.i. to # carets in reduced tree pair diagram

Word length in F

Exact word length w.r.t $\{x_0, x_1\}$: Fordham's method involves classifying carets into 7 types, and summing weights according to the table:

weight	R_0	R_{NI}	R_I	L_L	I_0	I_R
R_0	0	2	2	1	1	3
R_{NI}	2	2	2	1	1	3
R_I	2	2	2	1	3	3
L_L	1	1	1	2	2	2
I_0	1	1	3	2	2	4
I_R	3	3	3	2	4	4

This gives word length is q.i. to the number of carets in a minimal tree pair diagram.

Spheres in F

We want to compute

$$\lim_{n\to\infty} \frac{|T \cap \mathsf{Sph}_k(n)|}{|\mathsf{Sph}_k(n)|}$$

Word length for F- exact growth is unknown, rationality unknown, growth rate unknown

Burillo: exact growth for positive words.

Guba: bounds for exponential rate of growth

Meaningful positive limits- need the growth precisely.

Exponential growth rate alone is insufficient- we want dominant polynomial coefficients as well, if present.

Size of tree pairs

Catalan numbers: c_n= number of rooted binary trees with n carets

Tree pair diagrams: c_n^2 = number of tree pair diagrams with n carets in each tree

Reduced tree pair diagrams: r_n = number of reduced tree pair diagrams with n carets

We considered stratifications with respect to r_n.

Random subgroup classes in F

(joint with Murray Elder, Andrew Rechnitzer, and Jennifer Taback)

We considered Thompson's group F, which has no free non-abelian subgroups.

We found a diversity of subgroups in the spectra of F w.r.t. diagram size, for example:

$\mathrm{Spec}_3^{\mathrm{sum}}(F)$ contains all three-generator subgroups of F.

$\mathrm{Spec}_3^{\mathrm{max}}(F)$ contains F, $F \times \mathbb{Z}, (\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z}, (\mathbb{Z} \wr \mathbb{Z}) \times \mathbb{Z}, F \wr \mathbb{Z}, (\mathbb{Z} \times \mathbb{Z}) \wr \mathbb{Z}, \ldots$

Some subgroups are *persistent* and present in all large rank spectra.

Generating function methods

Analysis uses `WZ`, `GFUN` package in Maple, singularity analysis, methods of Flajolet.

Lemma: The generating function $R(z)$ satisfies the following linear ordinary differential equation

$$\begin{aligned} z^2(1-z)(16z^2-16z+1)(2z-1)^2\frac{\mathrm{d}^3R}{\mathrm{d}z^3} \\ -z(2z-1)(16z^2-16z+1)(8z^2-11z+5)\frac{\mathrm{d}^2R}{\mathrm{d}z^2} \\ -(128z^5-320z^4+365z^3-232z^2+76z-4)\frac{\mathrm{d}R}{\mathrm{d}z} \\ +36z(z-1)R(z)=0. \end{aligned}$$

It follows that $R(z)$ is D-finite.

Asymptotics of ratios of reduced tree pair counts

Recurrence:

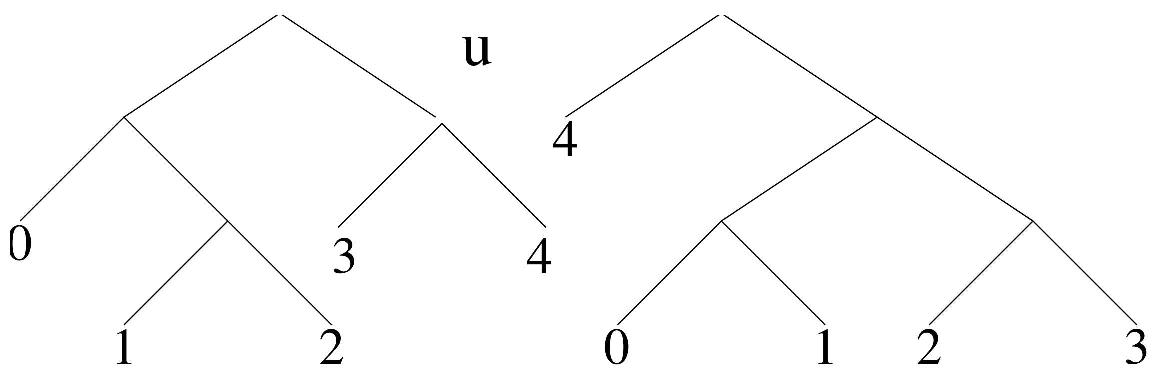
$$\begin{aligned} 0 = & -64n^2(n+1)r_n + 32(6n+5)(n+2)(n+1)r_{n+1} \\ & - 4(n+3)(53n^2+208n+195)r_{n+2} \\ & + 2(4n+15)(n+4)(13n+33)r_{n+3} \\ & - (n+5)(n+4)(21n+101)r_{n+4} + (n+5)(n+6)^2 r_{n+5}. \end{aligned}$$

Convergence: $\dfrac{n^3}{A\mu^n} r_n \sim 1 + \dfrac{33/2 - 11\sqrt{3}}{n}$ with $\mu = 8 + 4\sqrt{3} \sim$ 14.9282...

Lemma: For any $k \in \mathbb{Z}$, $\displaystyle\lim_{n\to\infty} \frac{r_{n-k}}{r_n} = \mu^{-k}$

Thompson's group T

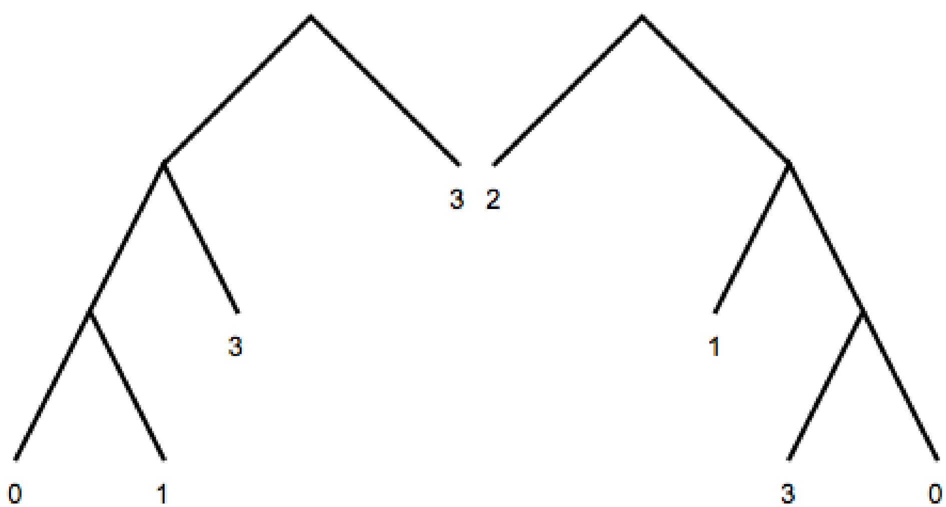
Thompson's group T: have a cyclic order on leaves.



Regard as PL maps from S^1 to itself, include rotations.

Thompson's group V

Thompson's group V: have no order, allow permutation on leaves.



Regard as PL right-continuous bijections from S^1 to itself, includes permutations.

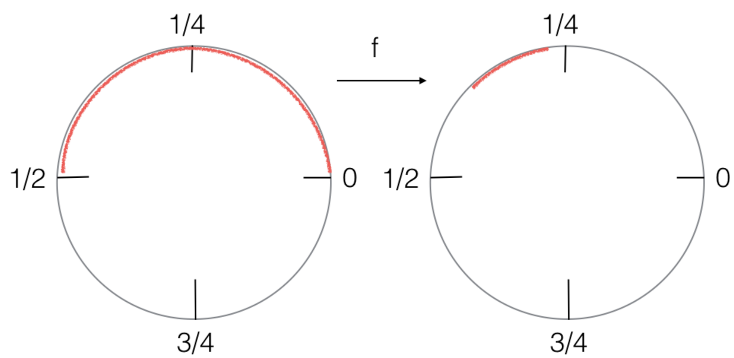
Free subgroups

Both T and V: finitely presented, simple, have free subgroups.

Known cases that contain free subgroups (free, torsion-free hyperbolic) the free groups are generic (asympt. density 1)

Dynamics of groups acting on the circle

Def An element acting on the unit circle has *north-south dynamics* it if has exactly two fixed points, one of which is attracting for positive iterates of the action and one of which is attracting for iterates of the inverse of the action.



Random elements and subgroups of T and V

Results (joint with Ariadna Fossas Tenas):

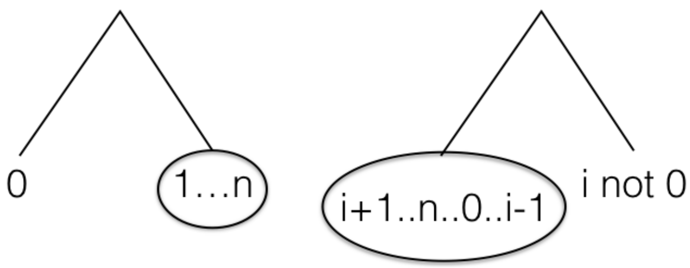
- T has a positive density of elements with north-south dynamics.

- T has positive densities of free subgroups

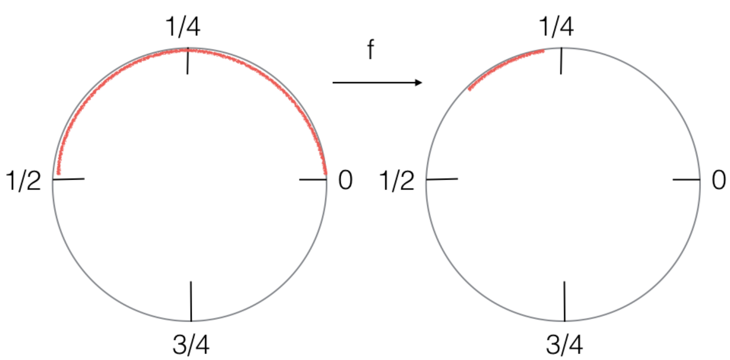
- V has a positive density of elements with north-south dynamics.

- V has positive densities of free subgroups

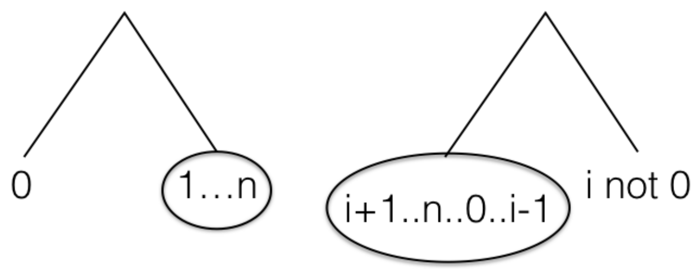
Elements with north-south dynamics



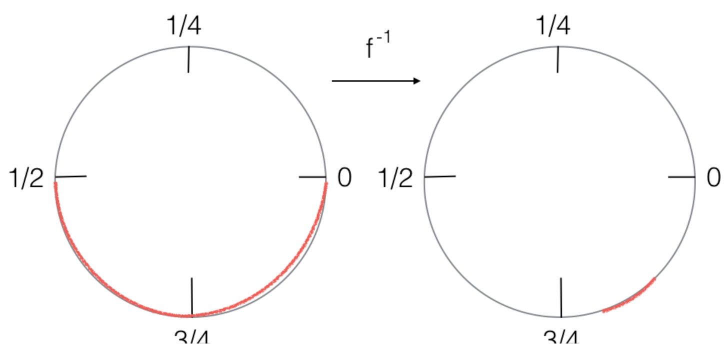
where the circled parts are any trees of size $n-1$



Elements with north-south dynamics



inverse element contracts into part of the second half



Fractions of north-south elements in T

Total number of elements of size n is $C_n^2(n+1)$ labelled tree pairs.

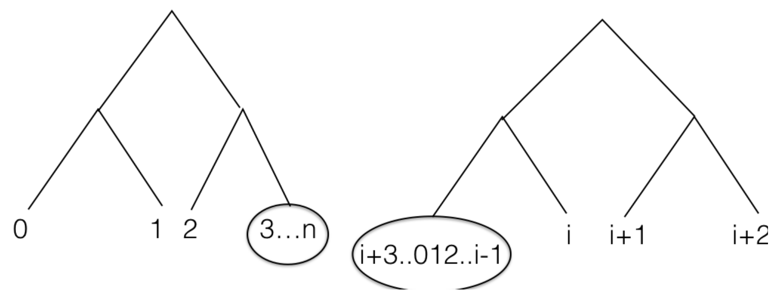
Number of elements of this particular type is $C_{n-1}^2(n)$ with subtree of size $n-1$ and labelled appropriately.

$$\lim_{n\to\infty} \frac{nC_{n-1}^2}{(n+1)C_n^2} = \frac{1}{16} > 0$$

giving positive density

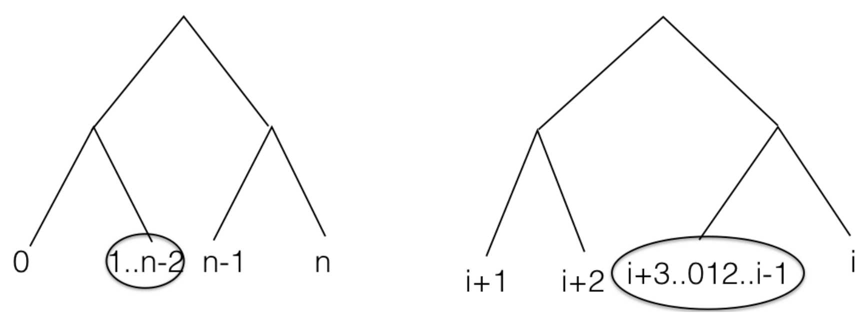
Free subgroups in T via ping-pong elements

Element u:



attracting fp in $[0, \frac{1}{4}]$ repelling fp in $[\frac{3}{4}, 1]$

Element v:



attracting fp in $[\frac{1}{2}, \frac{3}{4}]$ repelling fp in $[\frac{1}{4}, \frac{1}{2}]$

Counting 2-generator ping-pong subgroups in T

Total number of 2-tuples each with n total carets is $c_n^4(n+1)^2$.

Total number of ping-pong pairs of the specific type shown above is $c_{n-3}^4(n-5)^2$

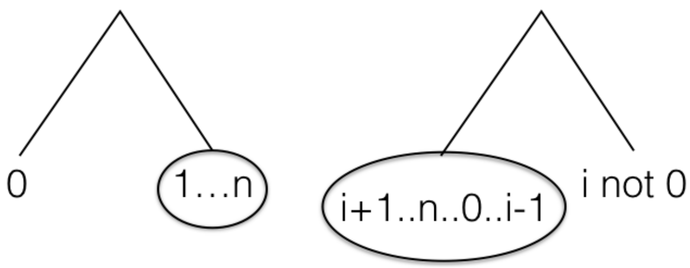
Gives fraction:

$$\lim_{n\to\infty} \frac{c_{n-3}^4(n-5)^2}{c_n^4(n+1)^2} = \frac{1}{2^{24}} > 0$$

and so rank 2 free groups are visible.

Higher rank: similar constructions give visibility of F_k in k-generator spectra.

Fractions of north-south elements in V



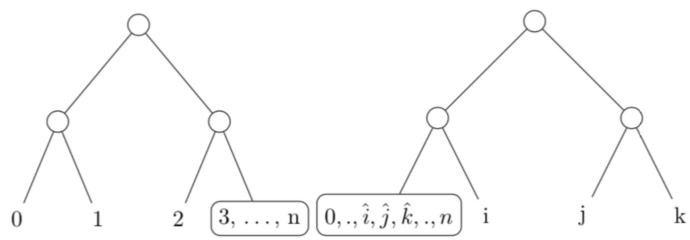
Total number of elements of size n is $C_n^2(n+1)!$ labelled tree pairs.

Number of elements of this particular type is $C_{n-1}^2(n!)$ with subtree of size $n-1$ and labelled appropriately. Gives positive density:

$$\lim_{n\to\infty} \frac{n!(n-2)C_{n-1}^2}{(n+1)!C_n^2} = \frac{1}{16} > 0$$

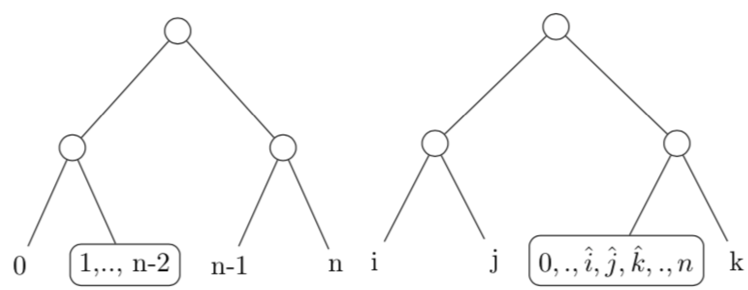
Free subgroups in V via ping-pong elements

Element u:



attracting fp in $[0, \frac{1}{4}]$, repelling fp in $[\frac{3}{4}, 1]$

Element v:



attracting fp in $[\frac{1}{2}, \frac{3}{4}]$, repelling fp in $[\frac{1}{4}, \frac{1}{2}]$

Ping-pong element pairs in V

Total fraction of 2-tuples with pairs generating ping-pong free subgroups of the types above:

$$\lim_{n\to\infty} \frac{C^4_{n-3}(n-4)!^2(n-1)^2(n-2)(n-3)^5(n-4)^2}{C^4_n(n+1)!^2} = \frac{1}{2^{24}} > 0$$

so 2-generator free subgroups are visible in the 2-spectrum of V with respect to the max stratification.

Higher rank: similar constructions give visibility of F_k in k-generator spectra.

Reduced tree pair counts

Could ask: what about corresponding fractions for analogues of r_n: fractions of reduced pairs.

In T, not so easy to count these exactly, about 1/3 are fully reduced on average, more than F.

In V, unlike F and T, most tree pairs are already reduced, expect limits to be the same.

Connections with word length: T is linear in the number of nodes, V is more complicated and bounded above by $n \ln(n)$ rather than linear.

Word length of random tree pairs in F

Select tree pair of size n at random : c_n^2 ways. Asymptotically not reduced, fraction falls exponentially.

Select reduced tree pair of size n: r_n ways.

Word length $|w|$ of resulting word w.r.t. std. $\{x_0, x_1\}$ for pair with N carets

Satisfies $N - 2 \leq |w| \leq 4N - 8$

Word length of random tree pairs in F

Drawing c_n^2 tree pairs: reduces on average to $\frac{5\pi-16}{\pi} \sim 0.91$ size (analyzed by C, Rechnitzer, Wong)

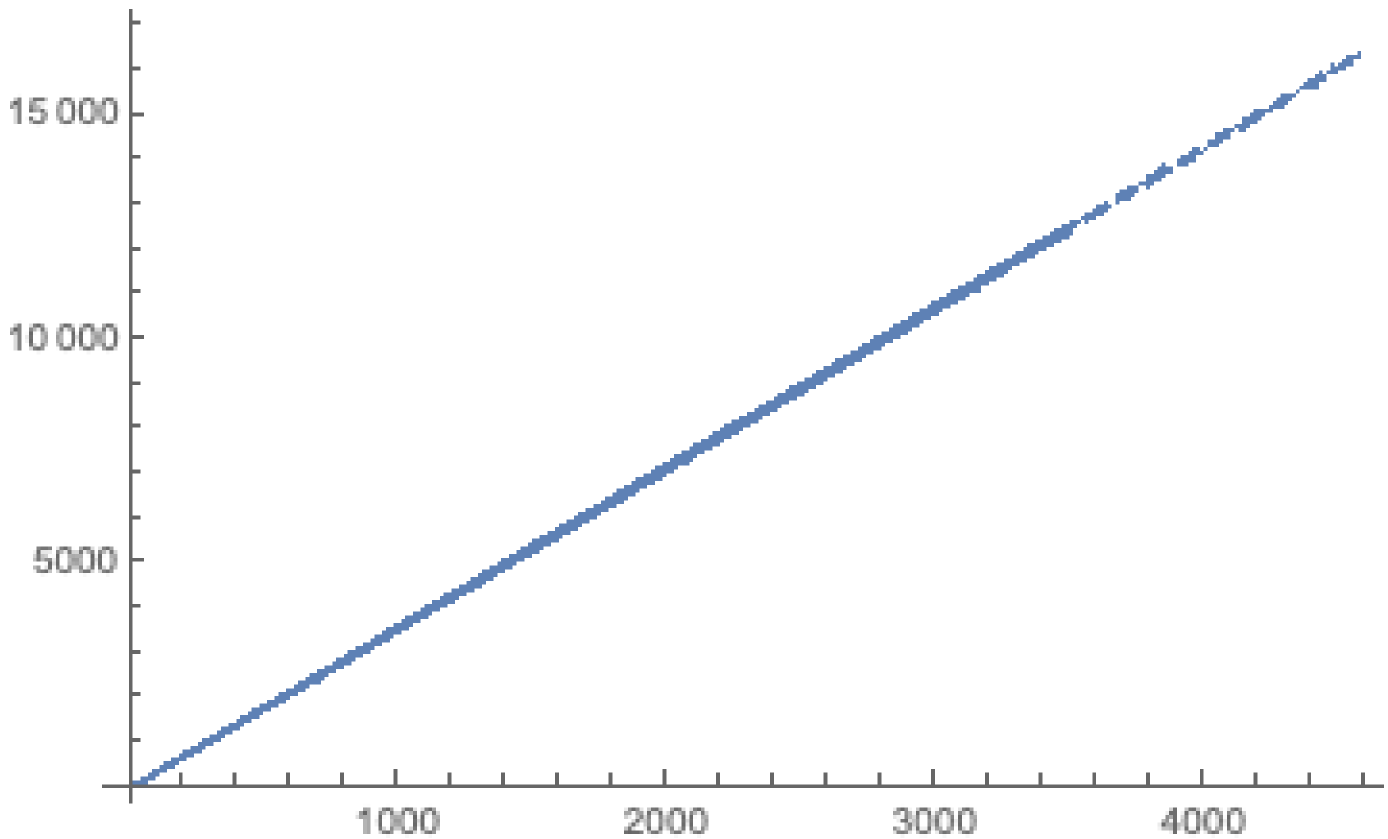
Investigate word lengths of resulting tree pairs: (joint with Haris Nadeem, building on earlier work joint with Timothy Chu, Roland Maio)

Experiments show that word length distributions for reduced tree pair diagrams of size n are tightly concentrated near $\sim 3.57n$

Summary statistics for word length vs. tree size

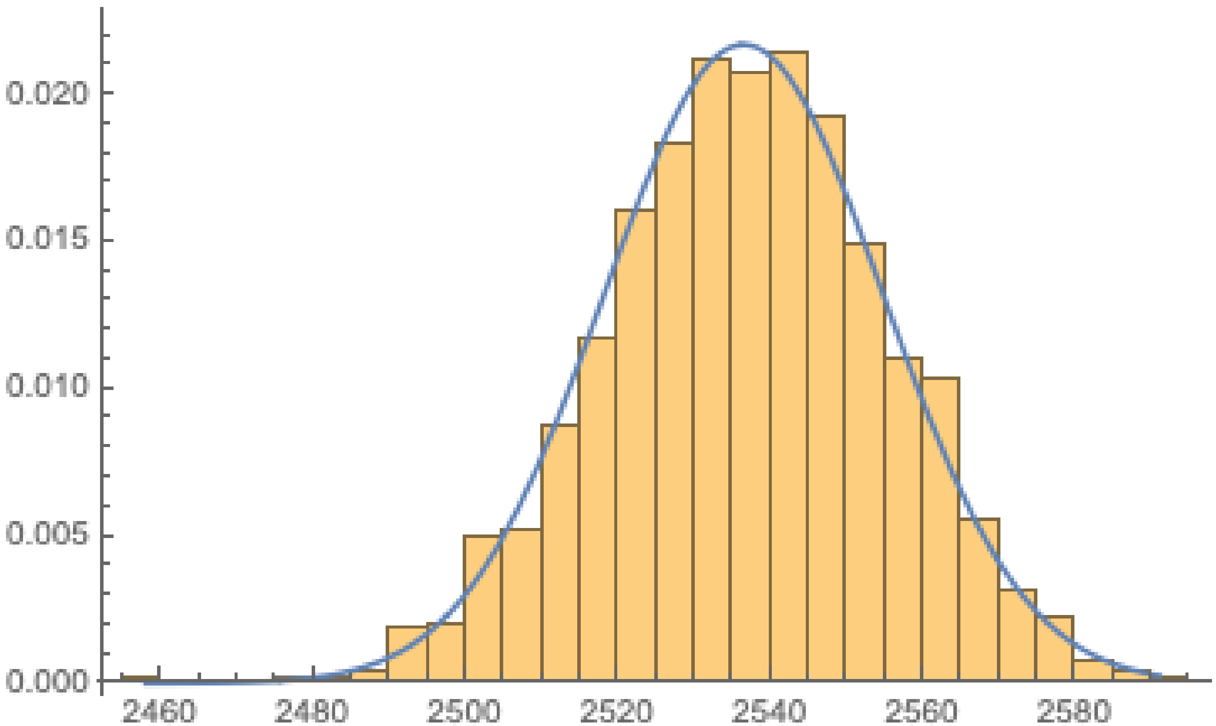
Tree size range	Number of tree pairs sampled	Avg. size ratio
800–899	48099	3.55662
900–999	40737	3.55859
1000–1249	94865	3.56172
1250–1499	100074	3.56451
1500–1749	77950	3.56616
1750–1999	22109	3.56769
2000–2249	21630	3.56872
2250–2499	20622	3.5695
2500–2749	15851	3.57045
2750–2999	13158	3.57073
3000–3249	8342	3.57162
3250–3499	2607	3.57268
3500–3999	821	3.57276
4000–4500	717	3.57285

Word length vs. tree size



Lengths concentrated near the line of slope ~ 3.57

Distributions of lengths



1200 randomly-produced reduced tree pairs of size 714. Mean about 2536.4, std. dev about 18.4

Sprawl and expected cancellation

Statistical hyperbolicity introduced by Duchin, Lelièvre, and Mooney:

<u>Def</u> The sprawl of a group G with respect to a generating set X in the "sum" stratification

$$= \lim_{n\to\infty} \frac{1}{|S_n|^2} \sum_{x,y\in S_n} \frac{d(x,y)}{n}$$

if the limit exists.

Average value of quantities ranging from 0: complete cancelation to 2: complete extension.

Can depend on generating sets but for non-elem hyp. groups: all 2 with respect to any generating set. Lamplighter groups: not hyperbolic but have sprawl 2.

Expected cancellation in F via experiments

Ideally: select two elements x, y of size n in F, calculate word length of xy, tabulate

Not currently possible: growth not precisely known, selecting uniformly at random not known, ...

To generate elements of a specified size is not feasible, but we can generate elements in a range of sizes:

take two tree pairs of size 1000, reduce and get maybe one of size 917 and the other of 931. Calculate word lengths and get two words of lengths say 3250 and 3302. Multiply and see how long the resulting word is, say 6548: sprawl = ?. Repeat.

Sprawl for words of different lengths

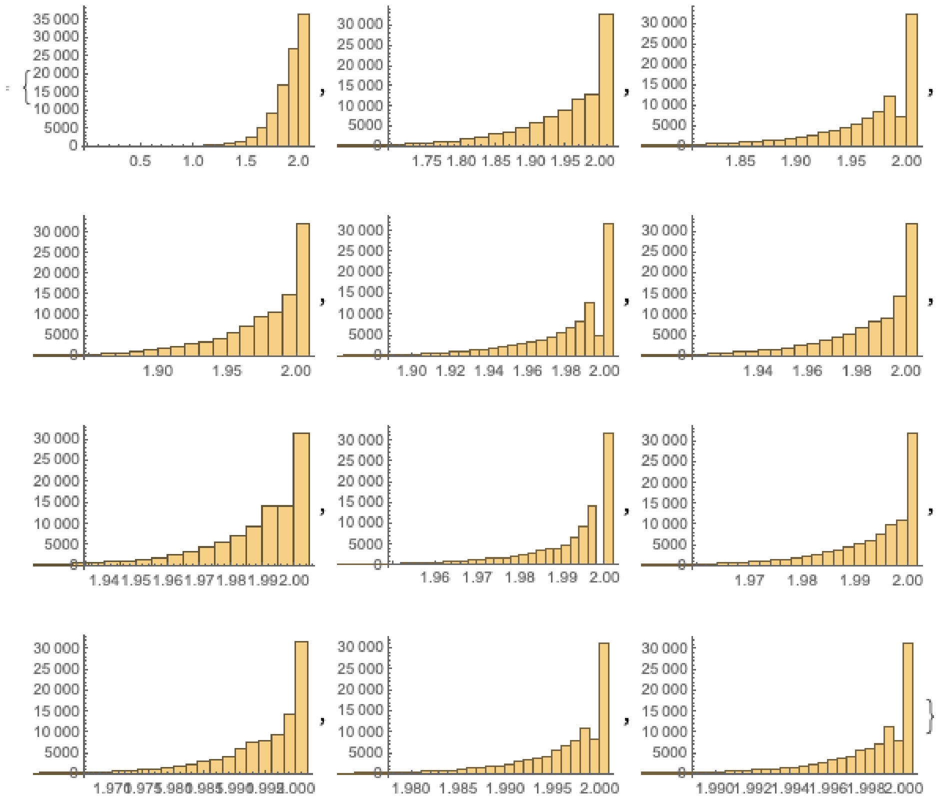
If $w = xy$, with $|x| \geq |y|$, then $|x| - |y| \leq |w| \leq |x| + |y|$

Scale cancellation to sprawl range $[0, 2]$ by tabulating $\frac{|w| - |(|x| - |y|)|}{\min(|x|, |y|)}$

Gives some measure of statistical hyperbolicity of F (which does contain free sub-semigroups):

Take two tree pairs of size 1000, reduce and get maybe one of size 917 and the other of 931. Calculate word lengths and get two words of lengths say 3250 and 3302. Multiply and see how long the resulting word is, say 6548: sprawl $=$ 1.9988... Repeat.

Observed sprawl increasingly closer to 2



Lengths from 20 total to 25k total, in 100k blocks of incr. size

Acknowledgements, References

grateful to NSF #1417820, PSC-CUNY research support, DASNY GRTI funding for computational nodes.

"Elements with north-south dynamics and free subgroups have positive density in Thompson's groups T and V" (with Adriana Fossas Tenas), *Münster Journal of Mathematics*, to appear.

"Distributions of restricted rotation distances" (with Haris Nadeem), *The Art of Discrete and Applied Mathematics*, to appear.

"Counting difficult tree pairs with respect to the rotation distance problem" (with Roland Maio), *Journal of Combinatorial Mathematics and Combinatorial Computing*, to appear.

Growth of F

Word length still open. Elder, Fusy, Rechnitzer:

Starts: 1,4,12,36,108,324,952,2800,8132,23608,67884, 195132,556932,1588836, 4507524,12782560,36088224,101845032, 286372148,804930196,2255624360, 6318588308,17654567968, . . . , 6015840076078706884412, . . .

calculated up to $n = 1500$ (600+ digits), within 10^{-48} of Guba's lower bound growth rate of $\frac{3+\sqrt{5}}{2}$