Some Math 347 sample questions for review

Know the definitions of the terms used and the proofs of the main theorems- see various review materials on Blackboard for some guidance there.

Q 1 State Lagrange's theorem and use it to prove that every group of prime order is cyclic.

Q 2 Find a group isomorphism between the real numbers $\mathbb{R}$ under addition and the positive real numbers $\mathbb{R}^+$ under multiplication or show that no such isomorphism exists.

Q 3 Use the definition of normality to show that if H is a normal subgroup of G, then $ghg^{-1} \in H$ for all $g \in G, h \in H$.

Q 4 Draw an example of a Frieze pattern (plane figure with a symmetry group containing parallel translations) which contains vertical reflections but no horizontal reflection.

Q 5 Prove the following or give a counterexample to show that it is false:
there is no injective automorphism from the cyclic group of order 4 to the non-zero complex numbers $\mathbb{C}^*$ under multiplication.

Q 6 For a, b and c non-zero integers with a and b relatively prime, prove that if $a|bc$ then $a|c$.

Q 7 Each statement is FALSE. Give an example to show that it is false.

- Every group of order 24 is abelian.
- The product of a two-cycle and a three-cycle in S_6 is always a five-cycle.
- If ϕ is a homomorphism from G to H, then for every element g in G, the order of g is the same as the order of $\phi(g)$.
- Every abelian group is a cyclic group.
- In any group G, the product of two elements of order 2 is either order 2 or the identity.
- In the alternating group A_6, there is no element of order 4.
- In the alternating group A_4, there is no element of order 2.
- In the symmetric group S_{10}, there is no element of order 21.
- In the symmetric group S_5, the product of two odd permutations is an even permutation.
- In a finite abelian group, the product of two elements has order which is the product of the orders of the two elements.

Q 8 Prove the following or give a counterexample to show that it is false:
every monomorphism (injection homomorphism) from $\mathbb{Z}$ to $\mathbb{Z}$ is an automorphism.

Q 9 Find an embedding of the cyclic group of order 6 into a group of permutations.

Q 10 Draw a figure with a symmetry group which is isomorphic to the dihedral group D_4 which contains a letter from your first name.

Q 11 List the left cosets of the alternating group A_4 as a subgroup of S_4.

Q 12 Let G be the cyclic group of order 12 generated by a. List all subgroups of G and indicate the order and index of each subgroup. For all nontrivial subgroups of order 5 or fewer, list all generators of each subgroup.

Q 13 Prove the following or give a counterexample to show that it is false:
If ϕ is a homomorphism from G to H, then the order of $\phi(g)$ must divide the order of g for all $g \in G$.

Q 14 Prove the following or give a counterexample to show that it is false:
If $\phi : G \to H$ is a homomorphism of groups, then the kernel of ϕ is a normal subgroup of G.

Q 15 Recall that the center of a group $Z(G)$ is the set of $a \in G$ such that $ax = xa$ for all $x \in G$.
Prove the following or give a counterexample to show that it is false:
the center of a group is a subgroup of G.

Q 16 Give an example of non-identity automorphisms of the integers $\mathbb{Z}$, the rationals $\mathbb{Q}$, and the real numbers $\mathbb{R}$.

Q 17 State the definition of a group.

Q 18 Prove that every finite integral domain is a field.

Q 19 Prove that a group where every non-identity element has order 2 is abelian.

Q 20 Complete the chart about rings below.

Ring	Set of units	Integral domain?	Field?
$\mathbb{Z}, +, *$			
$5\mathbb{Z}, +, *$			
$\mathbb{Z}_{18}, +, *$			
$\mathbb{C}, +, *$			
$M_2(\mathbb{R}), +, *$			

For the ring of two-by-two real matrices $M_2(\mathbb{R})$, $+$ and $*$ indicate matrix addition and multiplication. For rings that are not integral domains, give examples showing how they fail to be integral domains.

Q 21 a) Prove the following: If $gcd(a, n) = 1$, then the equation $ax \equiv b$ has a solution in the integers modulo n.
b) Use the method of proof from part a) to find all integer solutions to the equation $9x \equiv 7$ modulo 25.

Q 22 Let G be the cyclic group of order 18 generated by a. List all subgroups of G and indicate the order and index of each subgroup. For all nontrivial subgroups of order 6 or fewer, list all generators of each subgroup.
Find two subgroups H and K such that the group $\mathbb{Z}_{18}$ is the direct sum of subgroups $H \oplus K$.

Q 23 State the definition of an integral domain by continuing the definition below.

A commutative ring D with unity 1 is an *integral domain* if
Use the definition above to show that the cancellation law holds in an integral domain D as follows: for nonzero a in D, $ab = ac$ implies $b = c$.

Q 24 Examples: for each part, give an example that satisfies the conditions:

- An example of a finite ring that is not an integral domain is:
- An example of an abelian group which is not cyclic is:
- An example of a commutative ring with no unity (multiplicative identity) is:
- An example of a binary operation on the integers which is not associative is:
- An example of a field which is not an ordered integral domain is:
- An example of a subring of a ring which is not an ideal of that ring is:
- An example of a subgroup of a group which is not normal is:
- An example of a non-abelian group of order 12 is:
- An example of a figure with a nontrivial abelian symmetry group is:

Q 25
a) Express the following permutation in disjoint cycle form:

$$(1, 5, 3, 6, 2, 4)(6, 7)(1, 7, 6)(2, 5, 3)(2, 4)(6, 7)$$

b) What is the largest possible order of an element in S_9? Given an example of an element realizing this maximal order.

Q 26 Two elements x and y in a group G are said to be *conjugate* if there is a $g \in G$ such that $x = gyg^{-1}$. Prove that conjugacy in G is an equivalence relation.

Q 27 The *centralizer* of an element x in a group G is the set of all elements of g which commute with x. Show that the centralizer of an element is a subgroup. Is that subgroup necessarily a normal subgroup? If not, give a counterexample. If so, prove that.

Q 28 Find all subgroups of the dihedral group D_3 of symmetries of the equilateral triangle. For each subgroup, list its order and whether or not it is a normal subgroup of D_3.

Q 29 State the definition of an ideal in a ring R and show that the kernel of a ring homomorphism $\phi : R \to R'$ is an ideal in R.

Q 30 Let $R = \mathbb{Z}_2[x]$ be the ring of polynomials in an indeterminate x with coefficients in $\mathbb{Z}_2$. Let $I = (x^2 + x + 1)$ be the principal ideal of R generated by the polynomial $x^2 + x + 1$.
a) Give a list of the cosets in the quotient ring and construct the table for multiplication in the quotient ring.
b) Is the quotient ring R/I an integral domain? Is it a field?

Q 31 Let a be the permutation with cycle notation $(1, 2, 4, 7, 8)(3, 5, 6)$ in the symmetric group S_8 on $\{1, \ldots, 8\}$, and let b be the element $(1, 7, 3)(2, 5, 8)$.
1) Evaluate the product ab^2 and put it into disjoint cycle form.
2) Give an example of an element of order 44 in S_{16} or show that no such element can exist.

Q 32 Draw an example of a Frieze pattern (plane figure with a symmetry group containing parallel translations) containing at least one triangle, with a nonabelian symmetry group. Give two specific elements in your example's symmetry group which do not commute.

Q 33 Let G be the cyclic group of order 24 generated by a. List all subgroups of G and indicate the order and index of each subgroup. For all nontrivial subgroups of order 6 or fewer, list all generators of each subgroup.

Q 34 Prove the following or give a counterexample to show that it is false:
If $\phi : G \to H$ is an isomorphism of groups, and a is an element of G, then the order of a in G and the order of $\phi(a)$ in H are the same.

Q 35 Recall that the center of a group $Z(G)$ is the set of $a \in G$ such that $ax = xa$ for all $x \in G$.
Prove the following or give a counterexample to show that it is false:
the center of a group is a subgroup of G.

Q 36 Prove the following or give a counterexample to show that it is false:
For each n satisfying $23 \leq n \leq 29$, the set $\mathbb{Z}_n \setminus \{[0]\}$ of non-zero congruence classes modulo n is a group under multiplication.

Q 37 Consider the group $GL_2(\mathbb{Z}_5)$, the set of two-by-two matrices with entries in Z_5 which are invertible under matrix multiplication.

a) Find the inverse of the element a in G, with $a = \begin{pmatrix} [2] & [2] \\ [1] & [0] \end{pmatrix}$

b) What is the order of a?

Q 38 Find all integer solutions to the system of equations
$10x \equiv 3$ modulo 21, $2x \equiv 3$ modulo 5.