Announcements

- PP2 is released, due March 5th
- Midterm 1 is in two weeks

Topic 3: Graphs

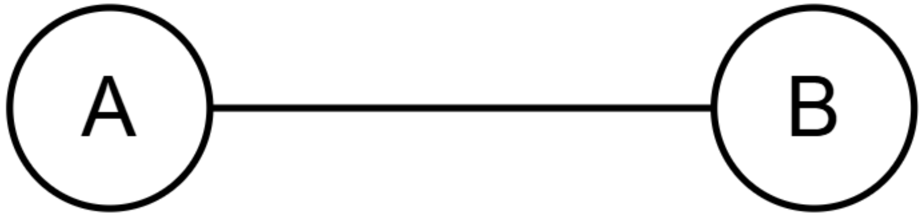
By Professor Hudson Lynam

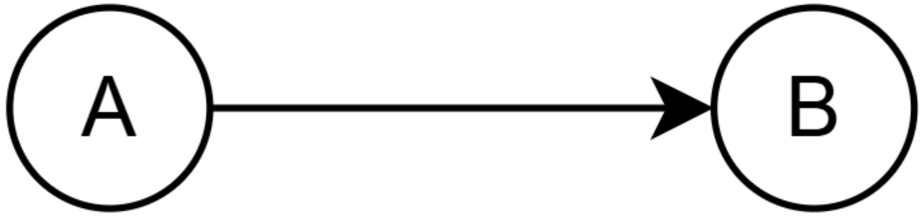
Graph Definition

- Definition: *Graph*
- *A graph G=(V,E) where V is a finite non-empty set of vertices (a.k.a. nodes) and E is a binary relation on V.*
- Note that definitions of “graph” vary: this is a common definition we’ll be using for this class

Undirected versus Directed Graphs

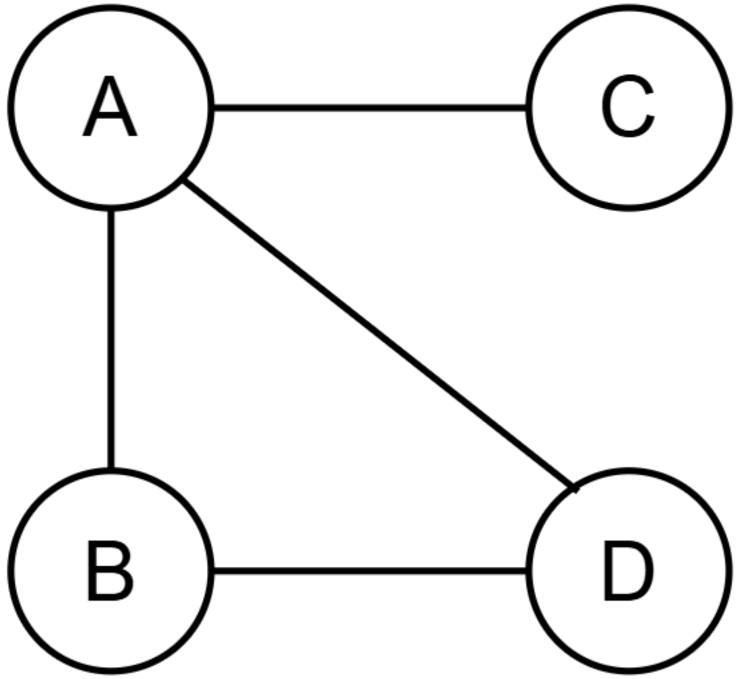
- Undirected:
  - The pairs of vertices in E are **<u>unordered</u>**. In other words, each edge is a set, like {A,B}.
  - When we say "G is a graph," the default assumption is that G is undirected
- Directed
  - Each edge has an order, like a tuple (A,B).
  - Because directed graphs are common, we usually abbreviate to "digraph."

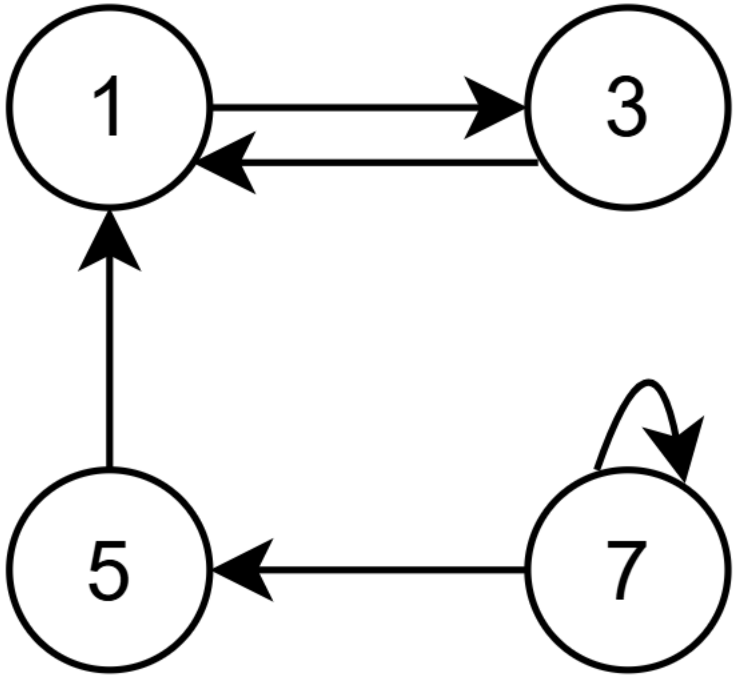




Graphs Example

- What would V be for this graph?
- V={A, B, C, D}
- E={{A,C}, {A,B}, {A,D}, {B, D}}
- What about V and E for this digraph?
- V={1, 3, 5, 7}
- E={(1,3), (3,1), (5,1), (7, 7), (7, 5)}
- Note: Don’t use bidirectional arrows for undirected graphs, and always use arrows to show two edges between vertices in a digraph





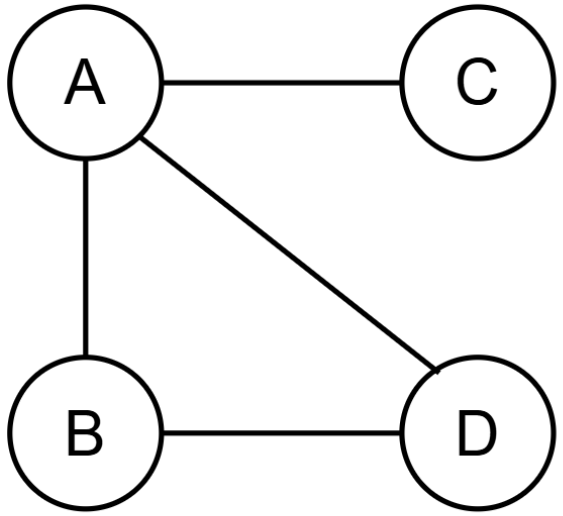
Graph Terminology

- Definition: *Adjacent Vertices*
- *2 vertices are adjacent when an edge exists between them*

- In our example graph, C and D are not adjacent, B and C are not adjacent, every other pair of vertices are adjacent

- Definition: *Path (a.k.a. Walk)*
- *A path is a sequence of adjacent vertices*

- Example: C-A-B-A-D-B.
- A **simple** path's vertices appear exactly once. The above example is *not* simple. What would be a simple path here?

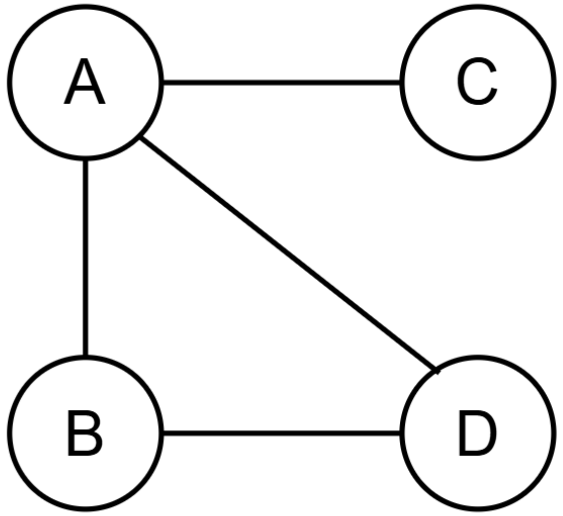


Graph Terminology

- Definition: *Cycle*
- *A cycle is a path of length ≥ 1 that begins and ends with the same vertex*

- In our example graph, A-C-A is a cycle. D-B-A-C-A-D is a cycle.
- A cycle is **simple** if all the vertices in the path only occur *once* (except for the first and last vertices)

- Notes: “the cycle ABD” means A-B-D-**A**
- Common assumption: simple cycles in undirected graphs must have length ≥ 3

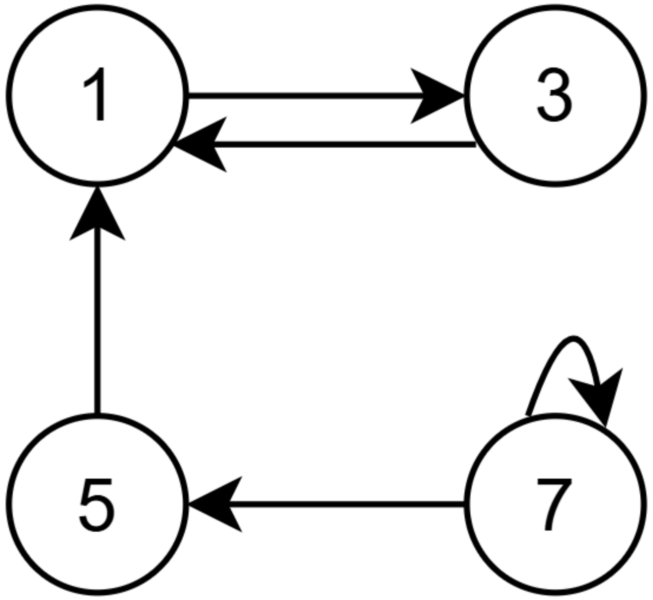


Graph Terminology

- Definition: *Incident Edges*
- *In any graph, an edge is incident on both participating vertices*
- *In digraphs, an edge is* ***incident from*** *the source and* ***incident to*** *the destination vertex*

- Definition: *Degree*
- *The degree of a vertex is its # of incident edges*

- Note: a self-loop adds to a vertex's degree
- What is the degree of 1? In versus out?
- In-degree of 1: 2, out-degree of 1: 1



Graph Representations

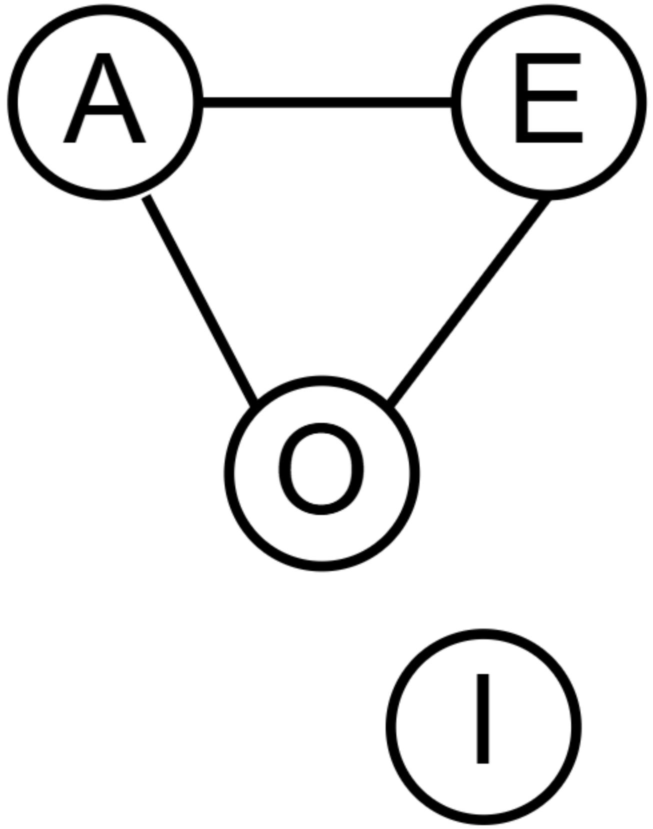
- How can we store all of a graph’s vertices and edges? Three common representations:

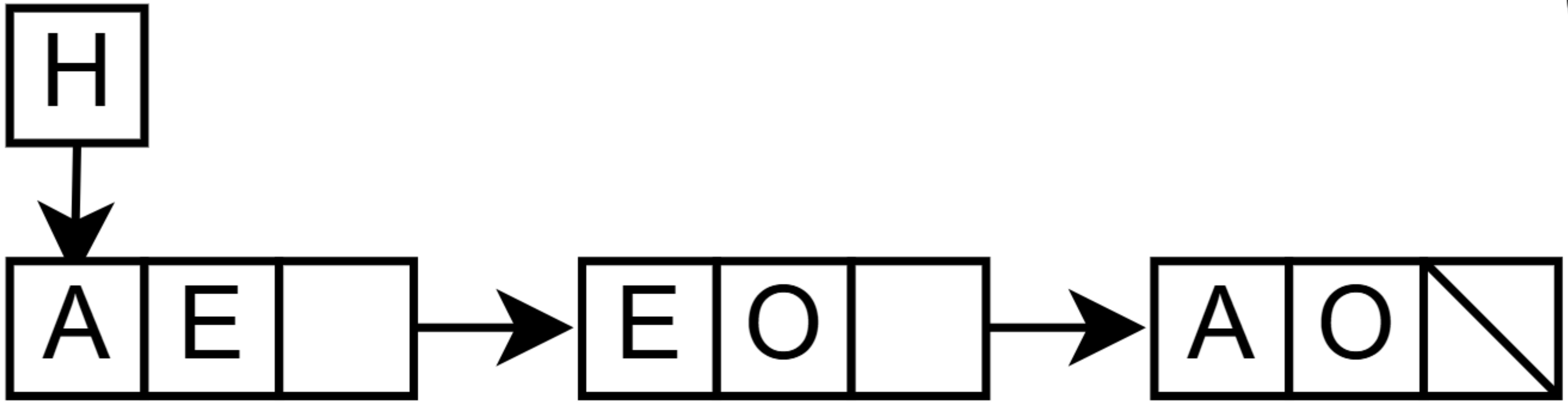
1. Edge Lists
2. Adjacency Matrices
3. Adjacency Lists

- Which should we choose? Depends on the operations we want to be efficient.

Edge Lists

- An edge list is a collection of edges within a graph. Example:
- I is an isolate in this graph. Obviously, not an ideal representation for graphs with isolates
- Note: In a digraph, just allow 1st vertex in each edge to the source or “from” vertex



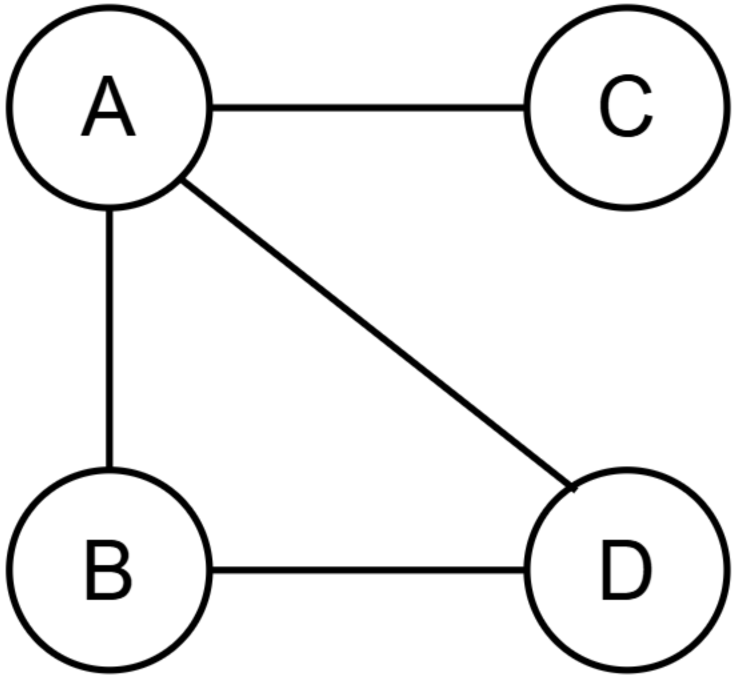


Adjacency Matrices

- Adjacency Matrices are a good choice when the quantity of vertices is bounded and the graph has a relatively large quantity of edges.
- The **adjacency matrix** of a graph G=(V, E) is a |V| x |V| matrix M in which element M[i][j] = 1 iff $\{v_i, v_j\}$ or (v_i, v_j) ∈ E. Otherwise, M[i][j] = 0.

Adjacency Matrices

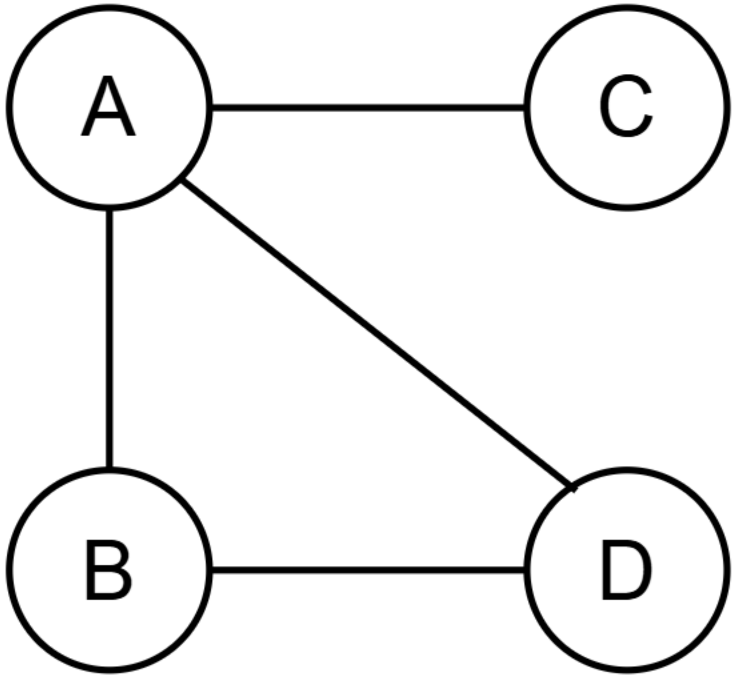
- Let’s take a look at our example graph:



	A	B	C	D
A	0	1	1	1
B	1	0	0	1
C	1	0	0	0
D	1	1	0	0

Adjacency Matrices

- Did we even need the full matrix? Could we represent the edges more efficiently? How?

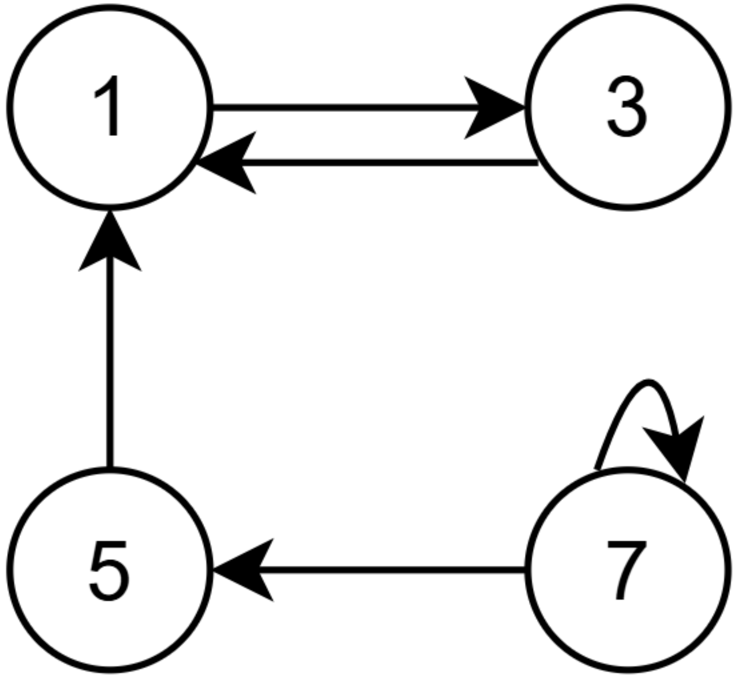


	A	B	C	D
A	0	1	1	1
B	1	0	0	1
C	1	0	0	0
D	1	1	0	0

- Note: this is a lower left triangular matrix

Adjacency Matrices

- Try to make an adjacency matrix for this digraph. Cols are To, Rows are From



	1	3	5	7
1	0	1	0	0
3	1	0	0	0
5	1	0	0	0
7	0	0	1	1

Adjacency Matrices

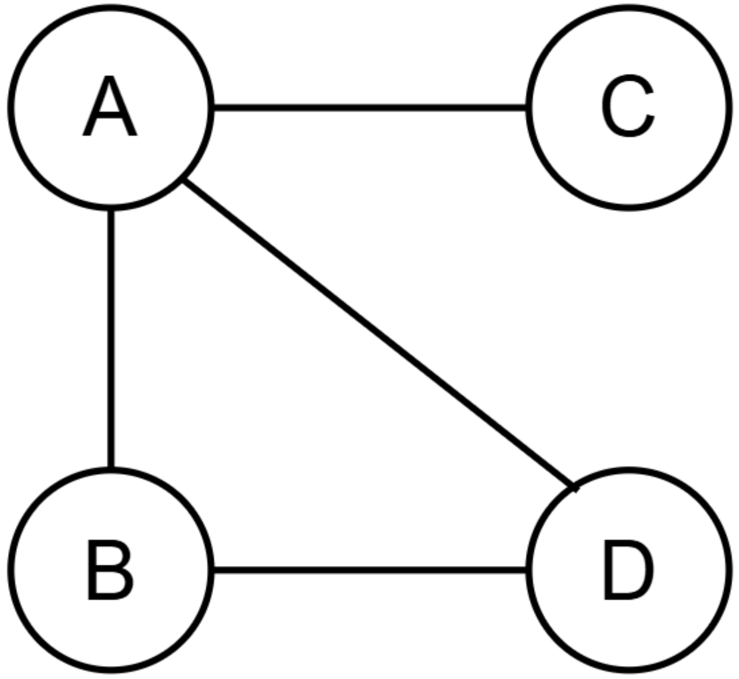
- Adjacency matrices make many operations easy. For example…
- How would we find the degree of a vertex in an AM?
- Count the row or col for the vertex if it’s a graph, count the row for out-degree in a digraph, or col for the in-degree
- Which edges are incident to a given vertex?
- Same solution as finding degree!
- But there’s a downside:
- We need $O(|V|^2)$ space to store an AM. Which is why AM is a good choice when the number of vertices is low

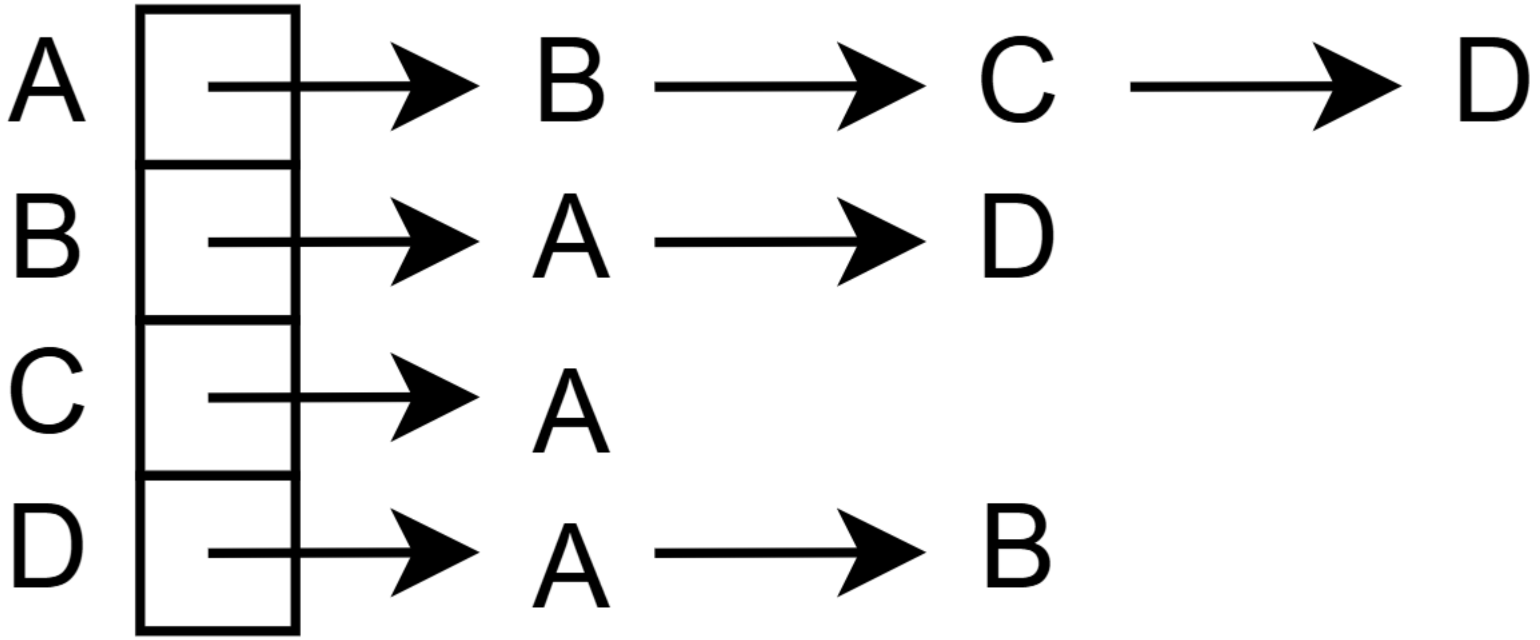
Adjacency Lists

- The **adjacency list** of a graph G=(V, E) is an array of |V| lists of vertices, with each list holding the vertices to which the vertex is adjacent (for a graph) or the vertices incident from a vertex (for a digraph).

Adjacency Lists

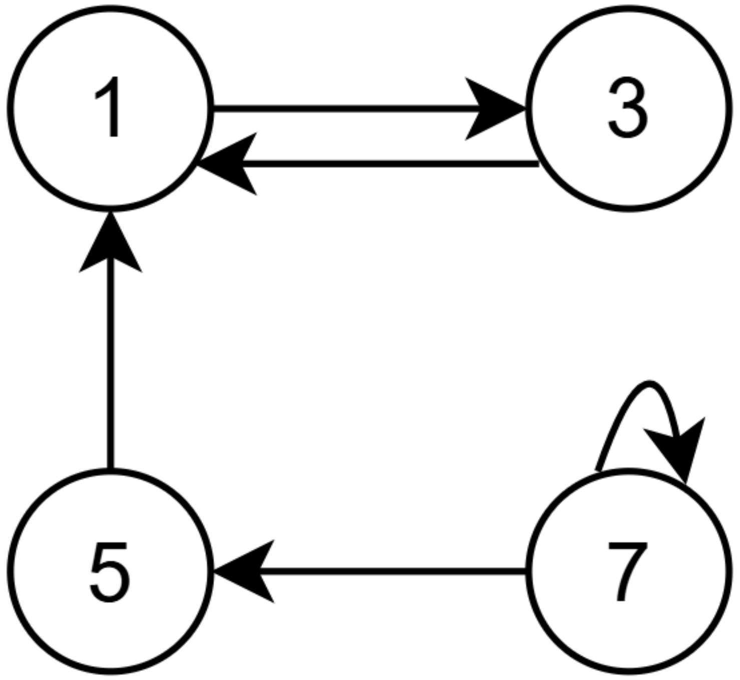
- Let’s take a look at our example graph:

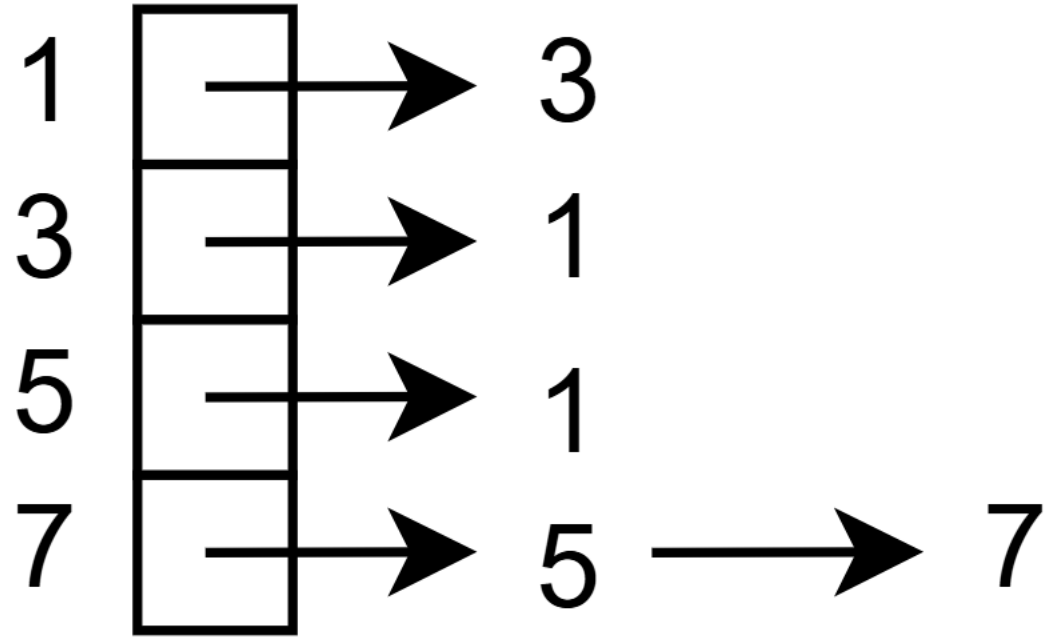




Adjacency Lists

- Now go ahead and try to construct the AL for our digraph example.





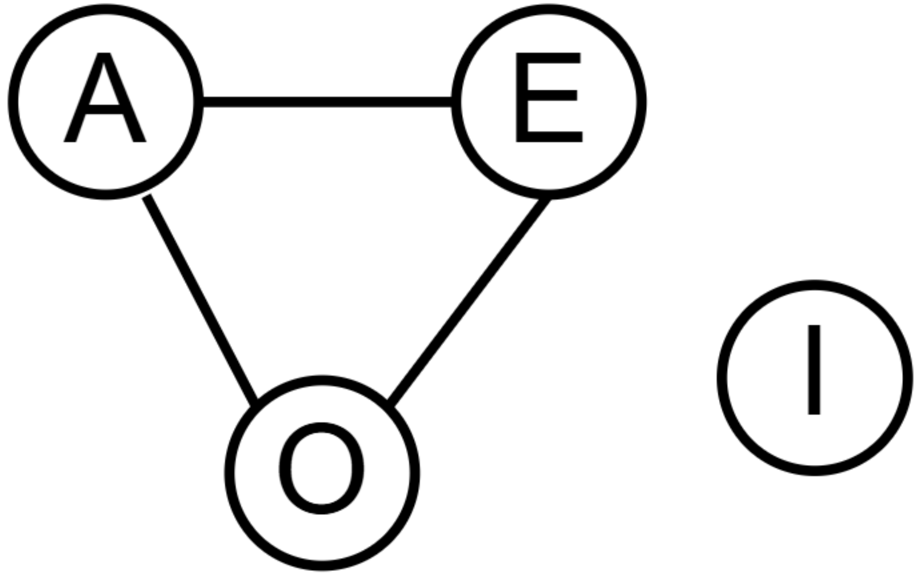
Adjacency Lists

- ALs are efficient for some operations, but not others
- How about finding the degree for a given vertex?
- Easy for graphs, but digraphs require searching the entire AL!
- What's the storage requirement for AL?
- $O(|V|+|E|)$; this is an example of having *2* instance characteristics!
- In summary, when choosing a graph representation, consider the graph (# of vertices vs edges) and the operations you expect to do on the graph (and how expensive they're likely to be)

Connectedness of Graphs

- Definition: *Connected Graph*
- *A graph G is connected when every pair of vertices in G is connected.*

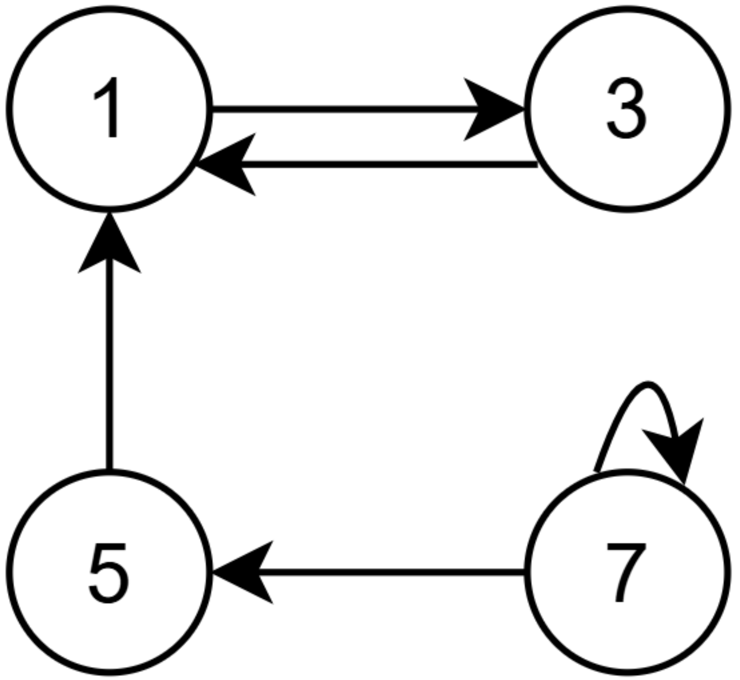
- Is this graph connected?
- How about this graph?
- You can't get to I from any other vertex, so no, not a connected graph



Connectedness of Digraphs

- Definition: Weakly *Connected Digraph*
- *A digraph G is a weakly connected digraph if it is connected when edge direction is ignored.*

- Definition: Strongly *Connected Digraph*
- *A digraph G is a strongly connected digraph if, for every pair of vertices there is a path between them in both directions.*



Determining that a Graph is Connected

- What are some of the tree traversals you've learned?
- Preorder, Postorder, Inorder, and Level Order
- Would any of these work for graphs?
- No; because graphs can have <u>cycles</u>
- We need a different way to traverse (or search through) graphs!

Breadth-First Search (BFS)

Breadth-First Search (BFS)

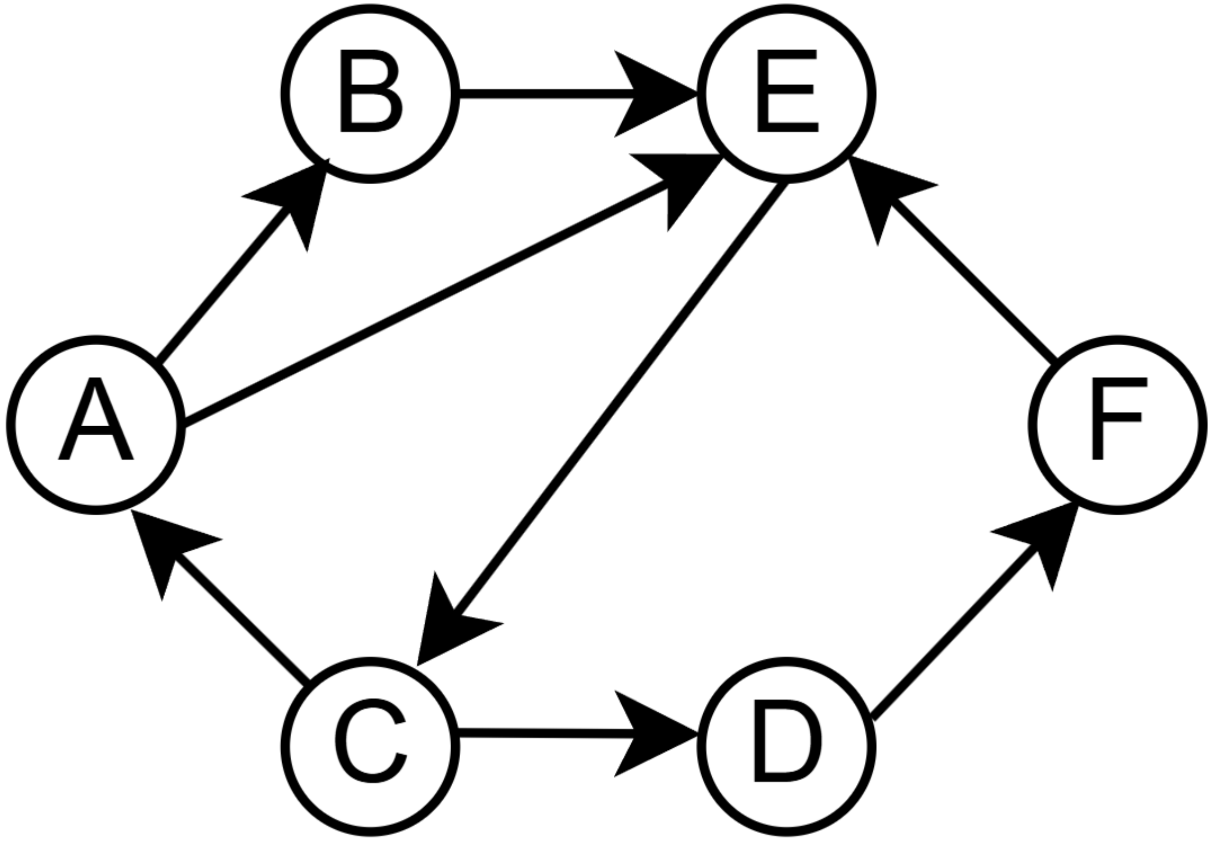
- Like the spread of liquid after popping a water balloon, BFS starts at some "center" and fans in all directions
- BFS relies on a **queue**

BFS:

```
Enqueue the source vertex
While the queue is not empty:
        Dequeue the first vertex
        Act on the vertex
        Enqueue the adjacent but previously undiscovered vertices
```

Breadth-First Search (BFS)

- Let's do a BFS on this graph, with A as the source vertex
- Queue: A, prev seen: A. Dequeue A.
- Queue: B, E prev seen: A, B, E. Dequeue B.
- Queue: E, prev seen: A, B, E. Dequeue E.
- Queue: C, prev seen A, B, E, C. Dequeue C.
- Queue D, prev seen all but F. Dequeue D.
- Queue: F, prev seen all. Dequeue F, queue is empty, finished.



Breadth-First Search (BFS)

- Some notes on BFS:
- If vertices and edges are not reachable from the source vertex, BFS cannot find them.
    - To deal with this, we can re-apply BFS to the remaining portions of the graph to find additional connected components of the graph
- How efficient is BFS? Depends on the graph representation. Discuss: Adjacency Matrix versus Adjacency List?
- AM: $O(|V|^2)$
- AL: $O(|V| + |E|)$

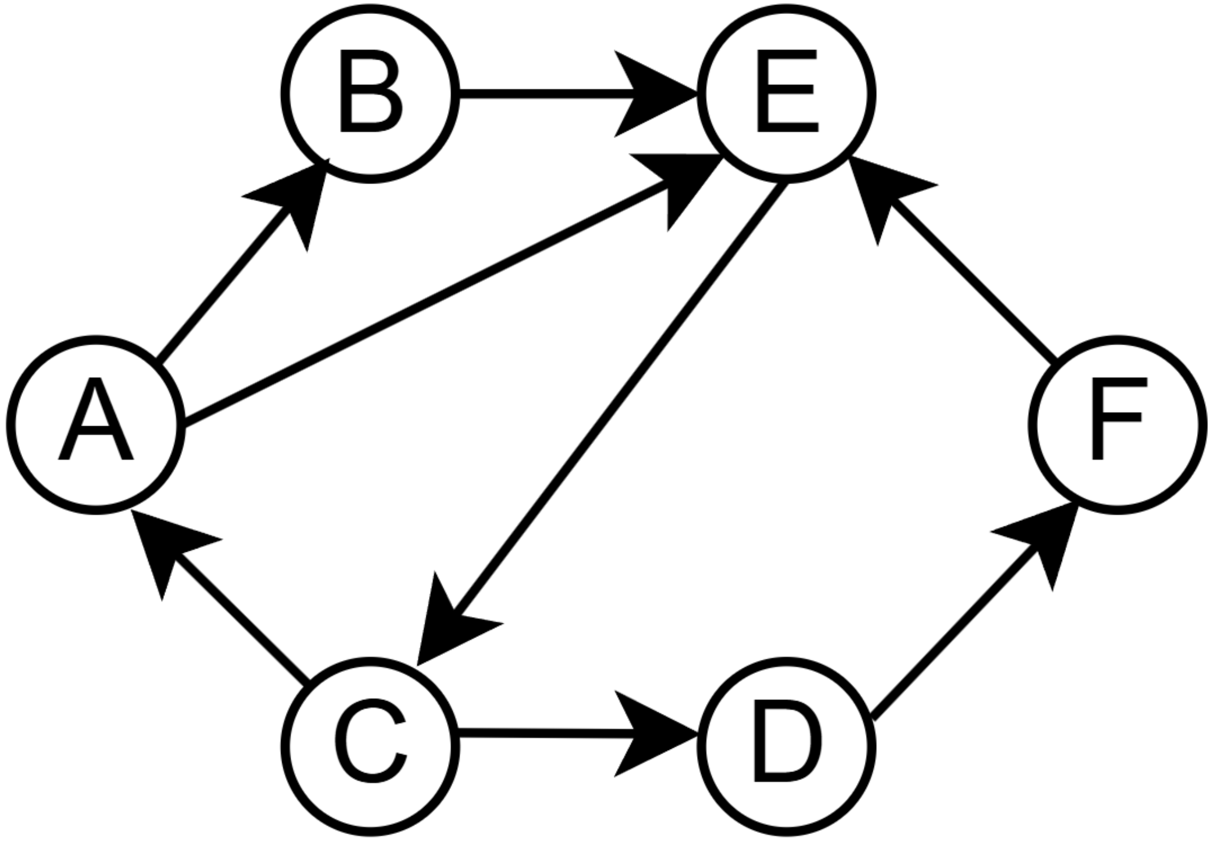
Depth-First Search (DFS)

- The idea is: when we see a new vertex, we visit that vertex!
- DFS relies on a **stack**

```
DFS:
Push the source vertex
While the stack is not empty:
        Pop the first vertex
        Act on the popped vertex
        Push its adjacent but previously undiscovered vertices
```

Depth-First Search (DFS)

- Let's do a DFS on this graph, with A as the source vertex
- Push: A, prev seen: A. Pop A.
- Push: B, E prev seen: A, B, E. Pop E.
- Push: C, prev seen: A, B, E, C. Pop C.
- Push: D, prev seen A, B, E, C, D. Pop D.
- Push: F, prev seen all. Pop F.
- Pop: B. Stack is empty, finished.



Depth-First Search (DFS)

- Some notes on DFS, first like BFS:
- If vertices and edges are not reachable from the source vertex, DFS cannot find them.
  - To deal with this, we can re-apply DFS to the remaining portions of the graph to find additional connected components of the graph
- Unlike BFS: DFS is easy to code recursively.
- How efficient is DFS? Depends on the graph representation. Discuss: Adjacency Matrix versus Adjacency List?
- AM: $O(|V|^2)$
- AL: $O(|V| + |E|)$

More Graph Terminology

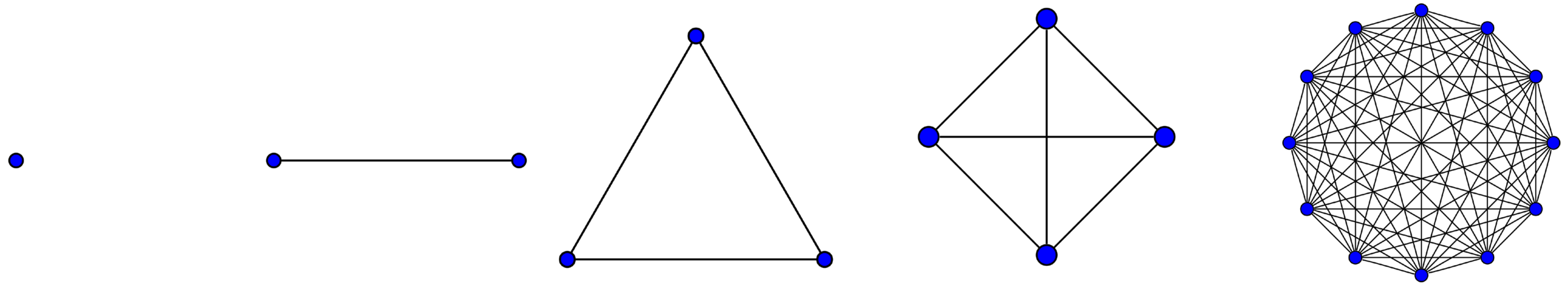
- Definition: Simple *Graph*
- *A simple graph is a graph w/o self-loops and w/o multiple edges between a pair of vertices*

- Definition: Acyclic *Graph*
- *An acyclic graph is a graph with no cycles.*
- *In undirected acyclic graphs, we'll ignore non-simple cycles*
- *A directed acyclic graph is often called a "DAG"*

More Graph Terminology

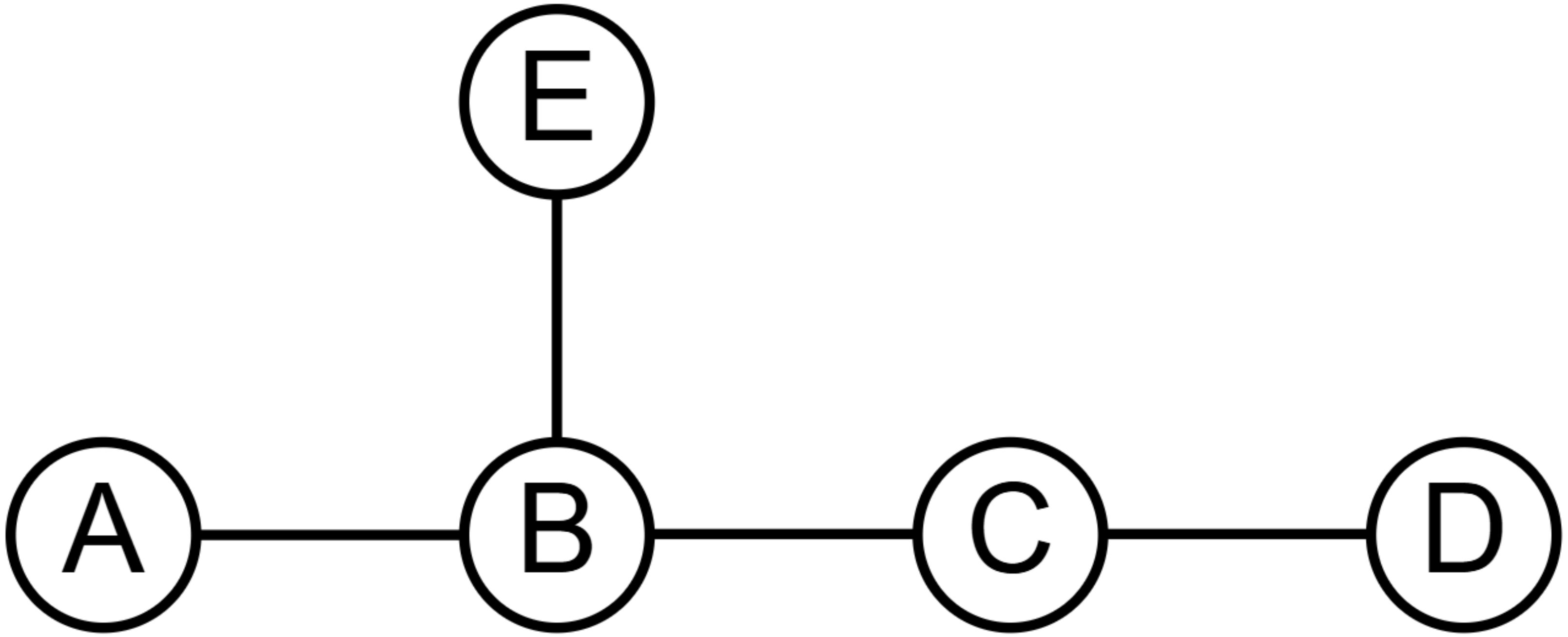
- Definition: Subgraph
- *G' is a subgraph of G if $V' \leq V$, $E' \leq E$, and E' uses only the vertices in V'*

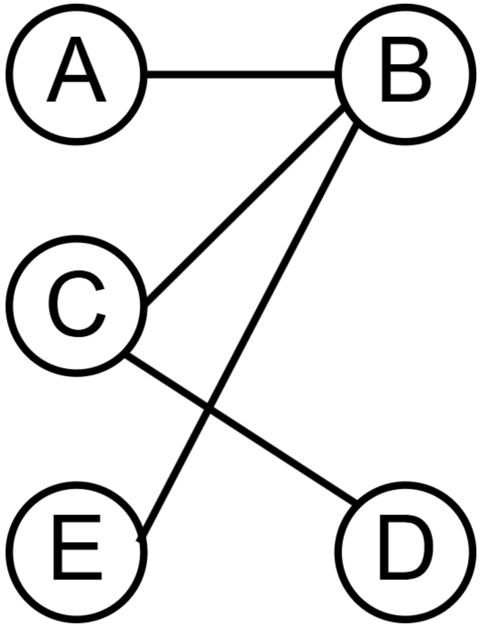
- Definition: Complete *Graph*
- *A complete graph is an undirected graph in which all pairs of n vertices are adjacent is a complete graph and is denoted K_n*



More Graph Terminology

- Definition: Bipartite *Graphs*
- *A graph G whose vertices are partitionable into two subsets V_1 and V_2 such that all of G's edges connect a vertex from V_1 to one from V_2 is a bipartite graph.*





More Graph Terminology

- Definition: Weighted *Graph*
- *Any graph whose edges have assigned costs weighted graph. (On an unweighted graph, edges have the same weight (usually 1).)*

- Definition: Forest
- *A forest is a simple undirected acyclic graph*

- Definition: Tree
- *A tree is a <u>connected</u> simple undirected acyclic graph*

The Single-Source Shortest Path Problem

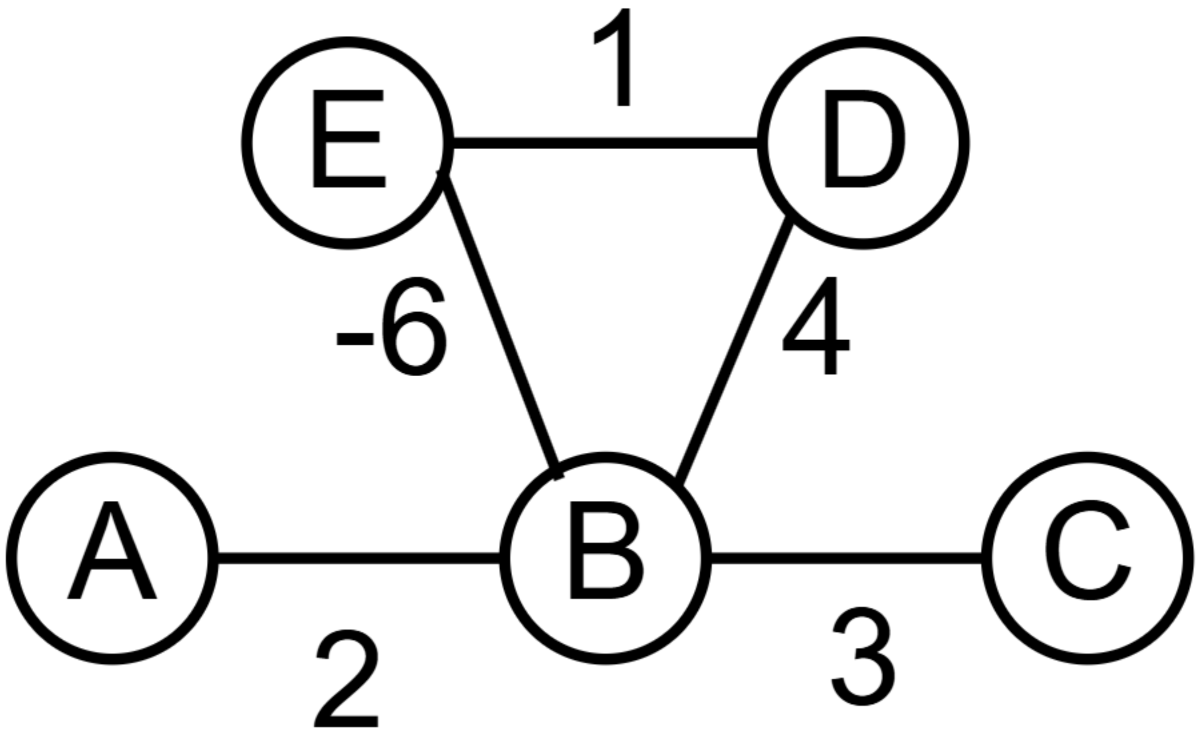
- After class, say you want to walk to Gould-Simpson taking the fewest # of steps. What route do you take?
- This is an example of the Single-Pair Shortest Path (SPSP) problem.
- A more general problem is the Single-Source Shortest Path problem: what is the shortest path from the start to <u>all</u> destinations?
- We would like to solve this problem for…
  - Weighted and unweighted graphs
  - Directed and undirected graphs

Negative Edge Weights

- In some cases, negative edge weights make sense:
- Example 1: Elevation changes. A graph of cities relative elevation levels would have negative edge weights for edges from higher elevation cities (Denver) to lower elevation cities (Tucson)
- Example 2: Costs versus Profit. In a graph of items you want to sell versus items you want to buy. Positive edge weights might represent money gained, negative edge weights might represent money lost

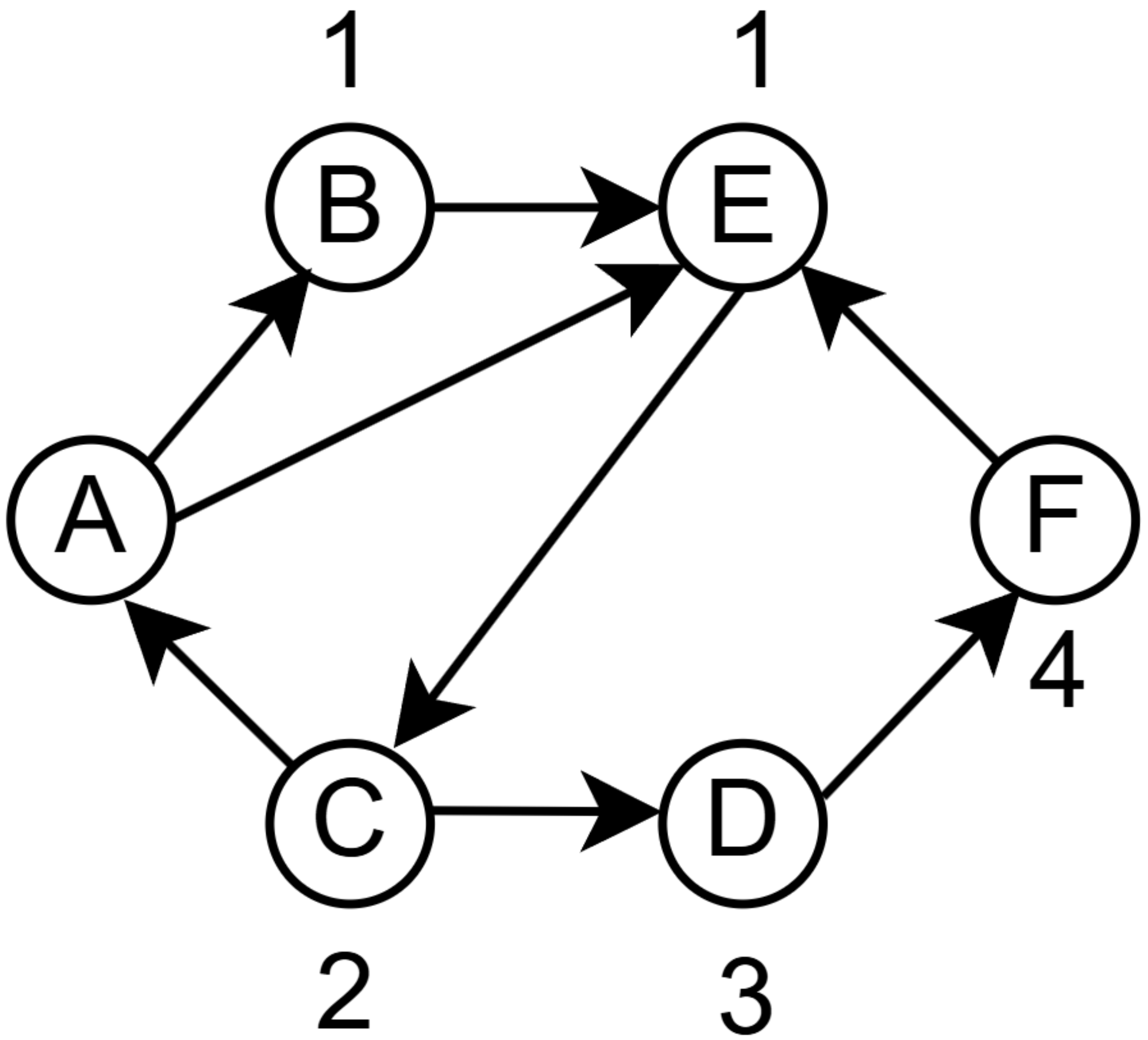
Negative Edge Weights

- Can a shortest path include a cycle?
- If the cycle weight is >0, the cycle is skipped
- If the cycle weight is =0, cycle *may* be included
- If the cycle weight is <0, we'll *never* leave it!



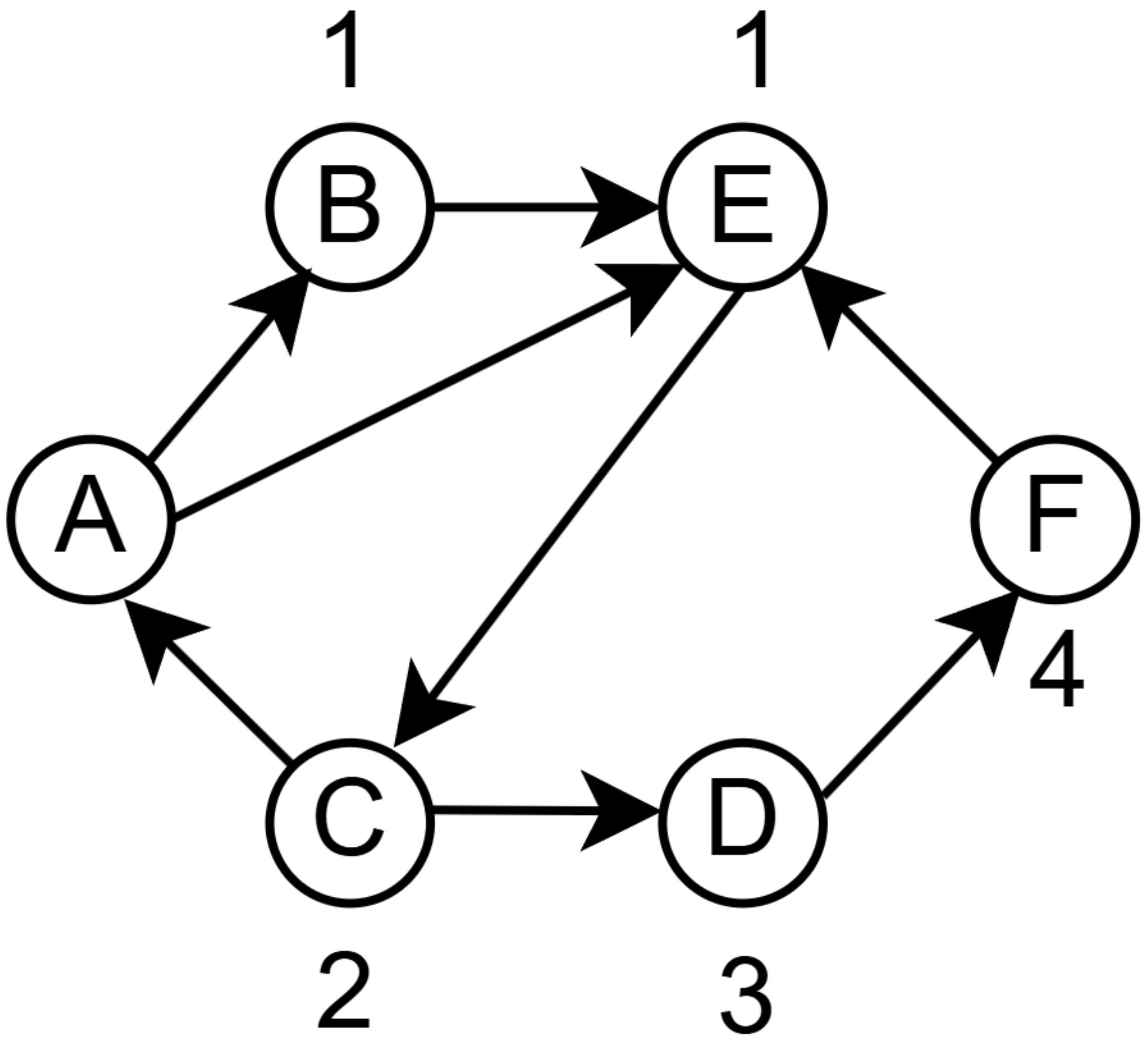
SSSP

- In a simple unweighted graph, finding the shortest paths from the source is easy! What could we use to do it, and how?
- BFS! Update when we reach each vertex



SSSP

- A path with <u>more</u> edges can be <u>cheaper</u>, in a **weighted** graph
- What if the edge weight from A to B was 1, and B to E also 1, but A to E was weight 5?
- BFS won't work, because we need to update the costs of paths as we search
- We need a different algorithm…



Dijkstra’s (SSSP) Algorithm

- Dijkstra’s Algorithm relies on four data structures:
- w(x, y): The weight of the edge connecting vertices x and y.
- d(a, b): The path cost (distance) from vertex a to vertex b.
- Fringe: The set of vertices that we know we can reach from the source
- Known: The set of vertices whose shortest paths from the source are know.

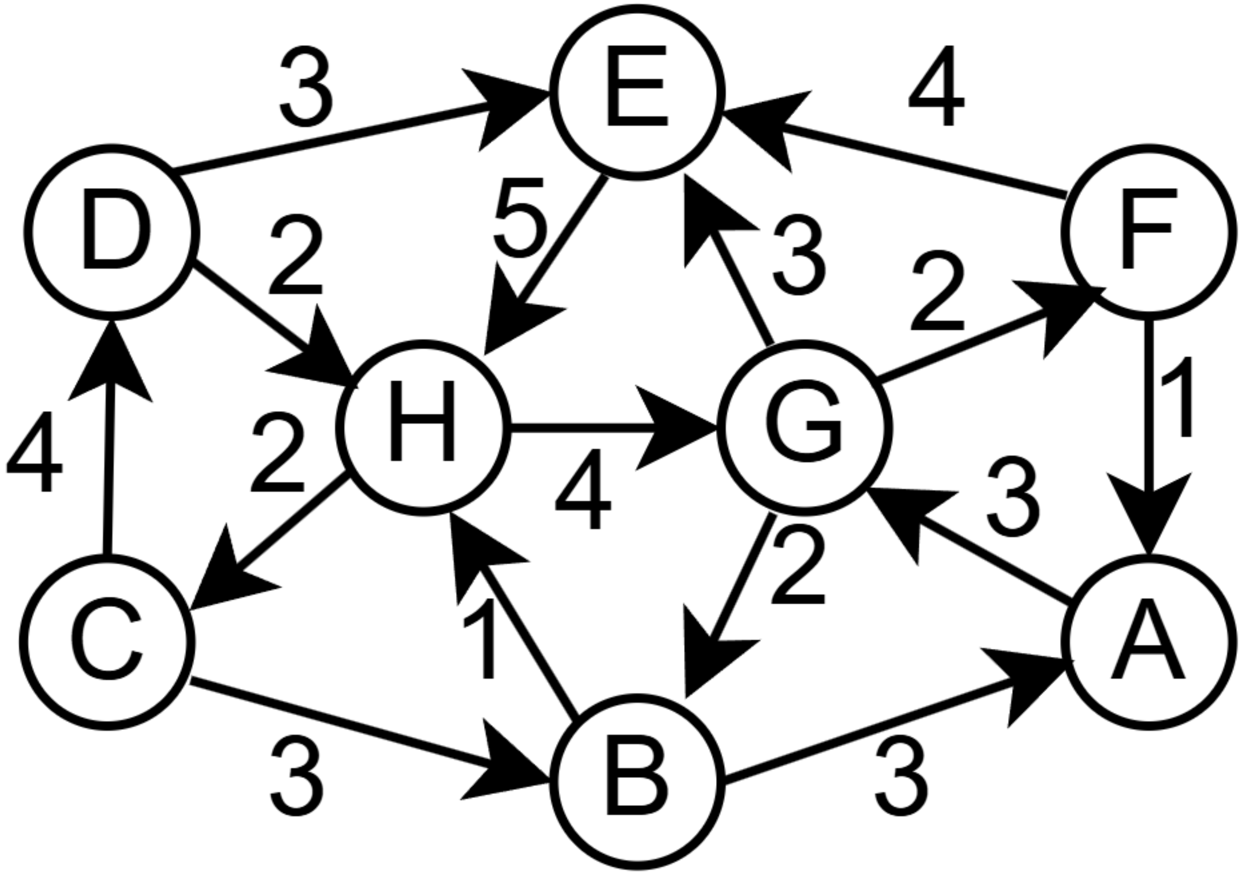
Dijkstra’s (SSSP) Algorithm

```
Dijkstra’s:
d(source, source) = 0
d(source, x) = +infinity, for all non-source vertices
Known ← source
Fringe ← vertices adjacent to source
While |Fringe| != 0:
        f ← fringe vertex with smallest d(source, f)
        Move f from Fringe to Known
        Add unKnown, unFringe vertices that are
adjacent to f to the Fringe
        Update Fringe content as necessary
```

- Notes:
- d(source, f) = d(source, t) + w(t, f) where t ∈ Known
- Must update Fringe when another path to a vertex is found.

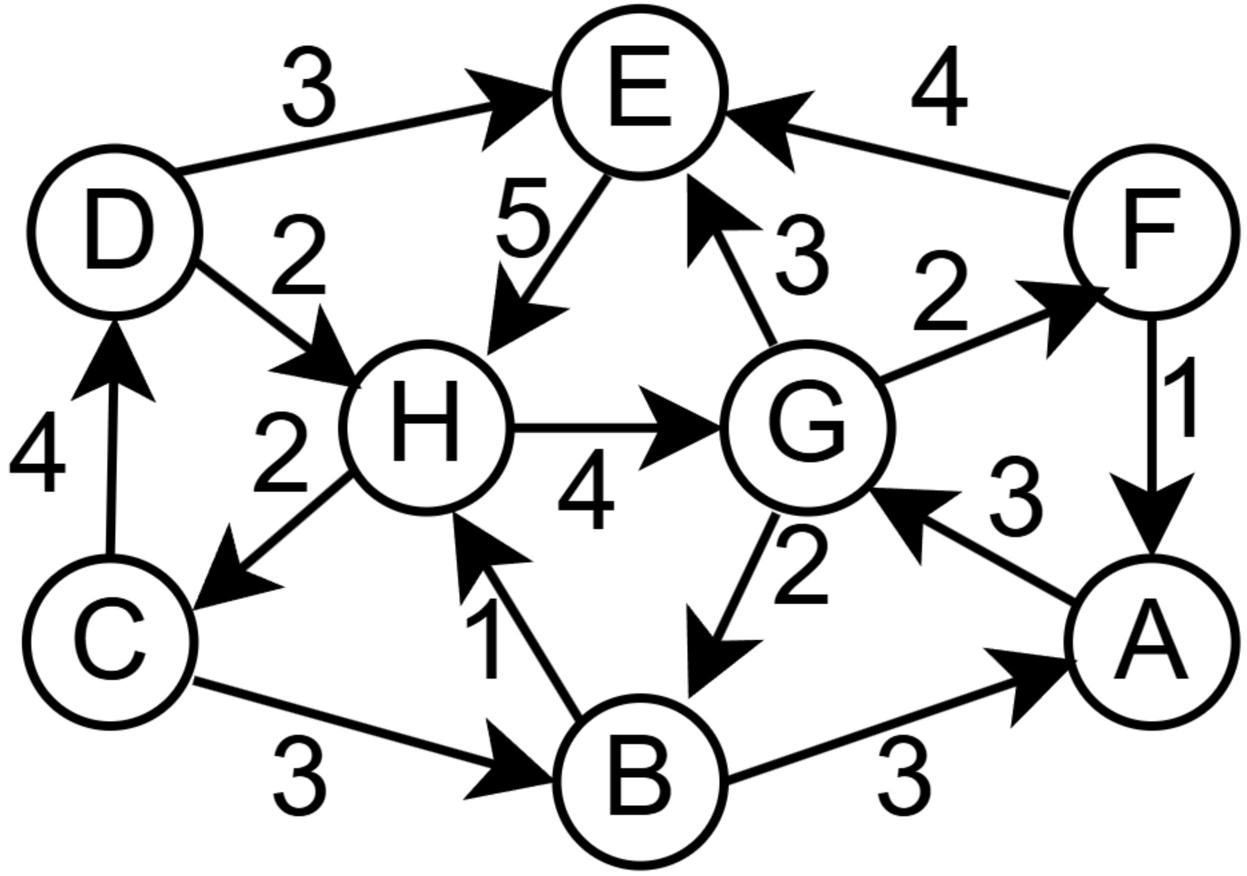
Dijkstra’s (SSSP) Algorithm

- Let’s try it with Source=D. Notation $V_{\#,P}$ where V is a vertex, # = d(source, v), and p = immediate previous vertex



Dijkstra's (SSSP) Algorithm

Known	**Fringe**
$D_{0,-}$	$H_{2,D}$ $E_{3,D}$
$H_{2,D}$	$E_{3,D}$ $C_{4,H}$ $G_{6,H}$
$E_{3,D}$	$C_{4,H}$ $G_{6,H}$
$C_{4,H}$	$G_{6,H}$ $B_{7,C}$
$G_{6,H}$	$B_{7,C}$ $F_{8,G}$
$B_{7,C}$	$F_{8,G}$ $A_{10,B}$
$F_{8,G}$	$A_{9,F}$
$A_{9,F}$	---



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