Fourier Analysis in the Binary World
The Discrete Heisenberg-Weyl Group

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Abstract: We introduce a parallel universe where there are binary counterparts of time and frequency shifts that are interchanged by a transform called the Walsh-Hadamard transform that plays the role of the Fourier transform in classical analysis

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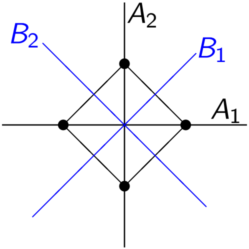


$$x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}: \text{ reflection in } B_1$$

eigenvectors $(1, 1)$ and $(1, -1)$
corresponding to B_1 and B_2

$$z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}: \text{ reflection in } A_1$$

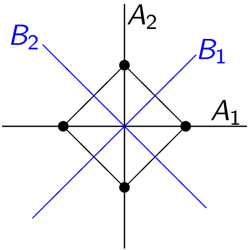
eigenvectors $(1, 0)$ and $(0, 1)$
corresponding to A_1 and A_2

Generators and Relations

$$x^2 = z^2 = I_2$$

$$\left.\begin{aligned} zx &= \begin{pmatrix} + & \\ & - \end{pmatrix}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \begin{pmatrix} & + \\ - & \end{pmatrix} \\ xz &= \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} + & \\ & - \end{pmatrix} = \begin{pmatrix} & - \\ + & \end{pmatrix} \end{aligned}\right] xz = -zx$$

The Hadamard Transform H_2



$H_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} + & + \\ + & - \end{pmatrix}$: represents rotation through $\pi/4$

Interchanges Coordinate Frames:

$$A_1, A_2 \longleftrightarrow B_1, B_2$$

$$\begin{tikzcd}
\text{Linear Transformation} & x \arrow[r, leftrightarrow, "g \rightarrow H_2 g H_2"] \arrow[d, dash] & z \arrow[d, dash] \\
\text{Eigenvector, Eigenvalue} & \begin{matrix} B_1, 1 \\ B_2, -1 \end{matrix} \arrow[r, leftrightarrow, "v \rightarrow vH_2"] & \begin{matrix} A_1, 1 \\ A_2, -1 \end{matrix}
\end{tikzcd}$$

Kronecker Products of Matrices

Given a $p \times p$ matrix $X = [x_{ij}]$ and a $q \times q$ matrix $Y = [Y_{ij}]$, the Kronecker product $X \otimes Y$ is defined by

$$X \otimes Y = \begin{bmatrix} x_{11}Y & \cdots & x_{1p}Y \\ \vdots & & \vdots \\ x_{p1}Y & \cdots & x_{pp}Y \end{bmatrix}$$

Proposition: $(X \otimes Y)(X' \otimes Y') = (XX') \otimes (YY')$ and in general $(X_1 \otimes \cdots \otimes X_m)(Y_1 \otimes \cdots \otimes Y_m) = X_1Y_1 \otimes \cdots \otimes X_mY_m$

$$\begin{pmatrix} x_{11}Y & x_{12}Y \\ x_{21}Y & x_{22}Y \end{pmatrix} \begin{pmatrix} x'_{11}Y' & x'_{12}Y' \\ x'_{21}Y' & x'_{22}Y' \end{pmatrix} = \begin{pmatrix} \bullet & \\ & \end{pmatrix}$$

$x_{11}x'_{11}YY' + x_{12}x'_{21}YY' = (XX')_{11}YY'$

Walsh-Hadamard Matrix: $H_{2^m} = H_2 \otimes \cdots \otimes H_2$ (m copies)

Exercise

Identify

$$\left[\begin{array}{cc|cc} + & & & \\ & + & & \\ \hline & & + & \\ & & & + \end{array}\right], \left[\begin{array}{cc|cc} & + & & \\ - & & & \\ \hline & & & - \\ & & + & \end{array}\right], \left[\begin{array}{cc|cc} & & - & \\ & & & - \\ \hline + & & & \\ & + & & \end{array}\right] \text{ and } \left[\begin{array}{cc|cc} & & & - \\ & & + & \\ \hline & - & & \\ + & & & \end{array}\right]$$

as Kronecker products of elements of the dihedral group D_4

Exercise

Identify

$$\left[\begin{array}{cc|cc} + & & & \\ & + & & \\ \hline & & + & \\ & & & + \end{array}\right], \left[\begin{array}{cc|cc} & + & & \\ - & & & \\ \hline & & & - \\ & & + & \end{array}\right], \left[\begin{array}{cc|cc} & & - & \\ & & & - \\ \hline + & & & \\ & + & & \end{array}\right] \text{ and } \left[\begin{array}{cc|cc} & & & - \\ & & + & \\ \hline & - & & \\ + & & & \end{array}\right]$$

as Kronecker products of elements of the dihedral group D_4

$$\left[\begin{array}{cc|cc} & & & - \\ & & + & \\ \hline & - & & \\ + & & & \end{array}\right]$$

Exercise

Identify

$$\left[\begin{array}{cc|cc} + & & & \\ & + & & \\ \hline & & + & \\ & & & + \end{array}\right], \left[\begin{array}{cc|cc} & + & & \\ - & & & \\ \hline & & & - \\ & & + & \end{array}\right], \left[\begin{array}{cc|cc} & & - & \\ & & & - \\ \hline + & & & \\ & + & & \end{array}\right] \text{ and } \left[\begin{array}{cc|cc} & & & - \\ & & + & \\ \hline & - & & \\ + & & & \end{array}\right]$$

as Kronecker products of elements of the dihedral group D_4

$$\left[\begin{array}{cc|cc} & & & - \\ & & + & \\ \hline & - & & \\ + & & & \end{array}\right] = \begin{pmatrix} & + \\ + & \end{pmatrix} \otimes \begin{pmatrix} & - \\ + & \end{pmatrix}$$

Identify

$$\left[\begin{array}{cc|cc} + & & & \\ & + & & \\ \hline & & + & \\ & & & + \end{array}\right], \left[\begin{array}{cc|cc} & + & & \\ - & & & \\ \hline & & & - \\ & & + & \end{array}\right], \left[\begin{array}{cc|cc} & & - & \\ & & & - \\ \hline + & & & \\ & + & & \end{array}\right] \text{ and } \left[\begin{array}{cc|cc} & & & - \\ & & + & \\ \hline & - & & \\ + & & & \end{array}\right]$$

as Kronecker products of elements of the dihedral group D_4

$$\left[\begin{array}{cc|cc} & & & - \\ & & + & \\ \hline & - & & \\ + & & & \end{array}\right] = \begin{pmatrix} & + \\ + & \end{pmatrix} \otimes \begin{pmatrix} & - \\ + & \end{pmatrix} = x \otimes xz$$

Assigning Coordinate Indices

Coordinates are ordered from left to right and the i^{th} coordinate is labeled by the binary expansion of i

$$\begin{aligned} e_0 &= (1,0) \quad \text{and} \quad e_1 = (0,1) \\ e_{10} &= (0100) = (10) \otimes (01) = e_0 \otimes e_1 \\ e_{011} &= (00000010) = (01) \otimes (01) \otimes (10) = e_1 \otimes e_1 \otimes e_0 \end{aligned}$$

Exercise: Given a binary m-tuple $v = (v_0, \ldots, v_{m-1})$ prove that

$$e_v = e_{v_{m-1}} \otimes e_{v_{m-2}} \otimes \cdots \otimes e_{v_0}$$

The Time Shift Group X_{2^m}

Given a binary m-tuple $a = (a_0, \ldots, a_{m-1})$ set

$$D(a, \mathbf{0}) = x^{a_{m-1}} \otimes x^{a_{m-2}} \otimes \cdots \otimes x^{a_0}$$

$$D(11, 00) = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \otimes \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \begin{array}{cc} \left[\begin{array}{cc|cc} & & & 1 \\ & & 1 & \\ \hline & 1 & & \\ 1 & & & \end{array}\right] & \begin{array}{c} 00 \\ 10 \\ 01 \\ 11 \end{array} \end{array}$$

$$D(10, 00) = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \otimes \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \begin{array}{cc} \left[\begin{array}{cc|cc} & 1 & & \\ 1 & & & \\ \hline & & & 1 \\ & & 1 & \end{array}\right] & \begin{array}{c} 00 \\ 10 \\ 01 \\ 11 \end{array} \end{array}$$

Exercise: Prove that $e_v D(a, \mathbf{0}) = e_{v \otimes a}$

The Frequency Shift Group Z_{2^m}

Given a binary m-tuple $b = (b_0, \ldots, b_{m-1})$ set

$$D(\mathbf{0}, b) = z^{b_{m-1}} \otimes z^{b_{m-2}} \otimes \cdots \otimes z^{b_0}$$

$$D(00,11) = \begin{pmatrix} + & \\ & - \end{pmatrix} \otimes \begin{pmatrix} + & \\ & - \end{pmatrix} = \begin{array}{c} \left[\begin{array}{cc|cc} + & & & \\ & - & & \\ \hline & & - & \\ & & & + \end{array}\right] \end{array} \begin{array}{c} 00 \\ 10 \\ 01 \\ 11 \end{array}$$

$$D(00,10) = \begin{pmatrix} + & \\ & + \end{pmatrix} \otimes \begin{pmatrix} + & \\ & - \end{pmatrix} = \begin{array}{c} \left[\begin{array}{cc|cc} + & & & \\ & - & & \\ \hline & & + & \\ & & & - \end{array}\right] \end{array} \begin{array}{c} 00 \\ 10 \\ 01 \\ 11 \end{array}$$

Exercise: Prove that $e_v D(\mathbf{0}, b) = (-1)^{bv^T} e_v$

The Heisenberg-Weyl Group HW_{2^m}

Given binary m-tuples a and b define

$$D(a, b) = D(a, \mathbf{0})D(\mathbf{0}, b)$$

Heisenberg-Weyl Group HW_{2^m}: all $i^\lambda D(a, b)$ where $a, b \in \mathbb{F}_2^m$ and $\lambda = 0, 1, 2$ or 3

$$D(\overset{a}{11010}, \overset{b}{10110}) = I_2 \otimes XZ \otimes Z \otimes X \otimes XZ$$

Exercise: Prove that $D(a, b)D(a', b') = (-1)^{a' b^T} D(a \oplus a', b \oplus b')$

$$D(a, b)D(a', b') = \left[\cdots \otimes x^{a_j} z^{b_j} \otimes \cdots \right] \left[\cdots \otimes x^{a'_j} z^{b'_j} \otimes \cdots \right]$$

Exercise: $D(a, b)D(a', b') = (-1)^{ab'^T + a' b^T} D(a', b')D(a, b)$

Note that every non-identity element of HW_{2^m} has order 2 or 4

Walsh Hadamard Matrix H_{2^m}: $H_2 \otimes \cdots \otimes H_2$
H_{2^m} interchanges time and frequency shifts

$$H_{2^m} D(a, \mathbf{0}) H_{2^m} = D(\mathbf{0}, a)$$
$$H_{2^m} D(a, b) H_{2^m} = (-1)^{ab^T} D(b, a)$$

Walsh Functions: Rows (Columns) of H_{2^m} and their negatives
Proposition: The v^{th} entry of the b^{th} row of H_{2^m} is $(-1)^{bv^T}$
Proof: The first ($\mathbf{0}^{\text{th}}$) row is $\underline{j} = (1, \ldots, 1)$
The b^{th} row is the first row of $D(b, \mathbf{0}) H_{2^m}$

$$D(b, \mathbf{0}) H_{2^m} = H_{2^m} D(\mathbf{0}, b)$$

Hence the b^{th} row is $\underline{j} D(\mathbf{0}, b)$ and the v^{th} diagonal entry of $D(\mathbf{0}, b)$ is $(-1)^{bv^T}$

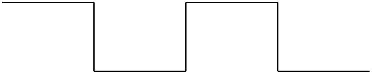
Walsh Functions as Sinusoids

Walsh Functions: $\hat{e}_b = e_b H_{2^m}$

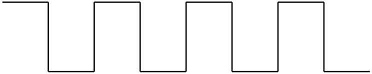
$$\begin{aligned}\hat{e}_b &= (e_{b_{m-1}} \otimes \cdots \otimes e_{b_0})(H_2 \otimes \cdots \otimes H_2) \\ &= \hat{e}_{b_{m-1}} \otimes \cdots \otimes \hat{e}_{b_0}\end{aligned}$$

Proposition: Walsh functions are eigenvectors of binary time shifts

$$\hat{e}_{010} = \frac{1}{\sqrt{8}}(1,1) \otimes (1,-1) \otimes (1,1)$$



$$\hat{e}_{100} = \frac{1}{\sqrt{8}}(1,1) \otimes (1,1) \otimes (1,-1)$$



Proof:

$$\begin{aligned}\hat{e}_b D(a,\mathbf{0}) &= \left(\bigotimes_j \hat{e}_{b_j}\right)\left(\bigotimes_j x^{a_j}\right) \\ &= (-1)^{ab^T}\left(\bigotimes_j \hat{e}_{b_j}\right) \\ &= (-1)^{ab^T}\hat{e}_b\end{aligned}$$

Maximal commutative subgroups and orthonormal bases of eigenvectors

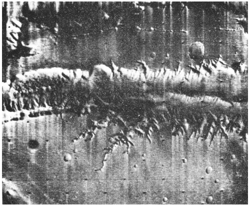
Maximal Commutative Subgroup	X_{2^m}	Z_{2^m}
Orthonormal Basis of Eigenvectors	$\hat{e}_b,\, b \in \mathbb{F}_2^m$	$e_b,\, b \in \mathbb{F}_2^m$
	TIME	FREQUENCY

$$
\begin{tikzcd}
X_{2^m} \arrow[rr, leftrightarrow, "g \longrightarrow H_{2^m} g H_{2^m}"] \arrow[d, dash] & & Z_{2^m} \arrow[d, dash] \\
\hat{e}_b,\, b \in \mathbb{F}_2^m \arrow[rr, leftrightarrow, "v \longrightarrow v H_{2^m}"] & & e_b,\, b \in \mathbb{F}_2^m
\end{tikzcd}
$$

Planetary Exploration

$$H_{2^m}^T = H_2^T \otimes \cdots \otimes H_2^T = H_{2^m}$$

Walsh functions of length 2^m are the rows (columns) of H_{2^m} and their negatives



Part of the Grand Canyon and Mars. This photograph was transmitted by the Mariner 9 spacecraft on January 19th, 1972 — gray levels are mapped to Walsh functions of length 32.

The closest Walsh function c to the received vector r is the one that maximizes the inner product (r, c):

$$\|r - c\|_2^2 = \|r\|_2^2 + \|c\|_2^2 - 2(r, c).$$

Exhaustive search requires about $2^m \times 2^m = 2^{2m}$ additions and subtractions to find the closest Walsh function to the received vector r.

Fast Hadamard Transform only requires about $m2^m$ operations

Example (m = 3):

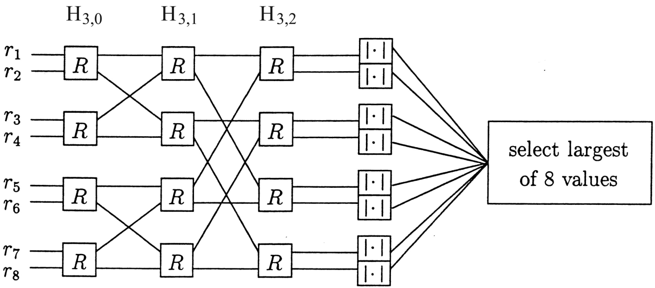
$$H_8 = \underset{H_{3,0}}{(I_2 \otimes I_2 \otimes H_2)}\underset{H_{3,1}}{(I_2 \otimes H_2 \otimes I_2)}\underset{H_{3,2}}{(H_2 \otimes I_2 \otimes I_2)}$$

$$H_8 = \underbrace{\left[\begin{array}{cc|cc|cc|cc} + & + & & & & & & \\ + & - & & & & & & \\ \hline & & + & + & & & & \\ & & + & - & & & & \\ \hline & & & & + & + & & \\ & & & & + & - & & \\ \hline & & & & & & + & + \\ & & & & & & + & - \end{array}\right]}_{2^m \text{ operations}} \underbrace{\left[\begin{array}{cc|cc|cc|cc} + & & + & & & & & \\ & + & & + & & & & \\ \hline + & & - & & & & & \\ & + & & - & & & & \\ \hline & & & & + & & + & \\ & & & & & + & & + \\ \hline & & & & + & & - & \\ & & & & & + & & - \end{array}\right]}_{2^m \text{ operations}} \underbrace{\left[\begin{array}{cc|cc|cc|cc} + & & & & + & & & \\ & + & & & & + & & \\ \hline & & + & & & & + & \\ & & & + & & & & + \\ \hline + & & & & - & & & \\ & + & & & & - & & \\ \hline & & + & & & & - & \\ & & & + & & & & - \end{array}\right]}_{2^m \text{ operations}}$$

Fast Hadamard Transform: Circuit Level Description

The component $\boxed{R}$ produces outputs $(a+b, a-b)$ from inputs (a, b). The component $\boxed{|\cdot|}$ produces $|a|$ from input a and stores the sign.

Received
Vector
r



The eight outputs of the third stage are the eight inner products of the vector r with the rows of H_8.