

Suppose that $r : [a, b] \rightarrow C$ is smooth and f is continuous on r^* . Then

$$\int_r f(z)dz = \int_a^b f(r(t))r'(t)dt$$

ML Inequality: Suppose that $r : [a, b] \rightarrow C$ is a smooth curve and f is continuous on r^* . If $|f| \leq M$ on r^* and L is the length of r then

$$\left| \int_r f(z) dz \right| \leq ML$$

Cauchy's Theorem for derivatives: Suppose V is an open subset of the plane, $f : V \rightarrow \mathbb{C}$ is continuous, and there exists an $F : V \rightarrow \mathbb{C}$ such that $f = F'$ in V . Then

$$\int_{\gamma} f(z) dz = 0$$

for any smooth closed curve γ in V

Cauchy's Integral Formula for disks:

$$f(z) = \frac{1}{2\pi i} \int_r \frac{f(w)dw}{w - z}$$

Cauchy Integral Formula for derivatives: Suppose $f \in H(V)$ and $\overline{D(z_0, r)} \subset V$. Define $r : [0, 2\pi] \rightarrow V$ by $r(t) = z_0 + re^{it}$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_r \frac{f(w)dw}{(w-z)^{n+1}}$$

for all $z \in D(z_0, r)$

Cauchy's Estimates: suppose that f is holomorphic in a neighborhood of the closed disk $\overline{D}(z_0, r)$ and $|f| \leq M$ in $\overline{D}(z_0, r)$. Then

$$\left| f^{(n)}(z_0) \right| \leq \frac{M n!}{r^n}$$

Liouville's Theorem: A bounded entire function must be constant.

Fundamental Theorem of Algebra:

Suppose that $f \in H(V)$ and V is connected. If all the derivatives of f vanish at some point of V then f is constant.

suppose that $f \in H(V)$ and f has a zero of order N at $z \in V$. Then there exists $g \in H(V)$ with $g(z) \neq 0$ such that

$$f(w) = (w - z)^N g(w)$$

for all $w \in V$

Maximum Modulus Theorem: Suppose that $f \in H(V)$ and V is connected. Then $|f|$ cannot achieve a (local) maximum in V unless f is constant: If f is nonconstant then for every $a \in V$ and $\delta > 0$ there exists $z \in V$ with $|f(z)| > |f(a)|$ and $|z - a| < \delta$.

Parseval Formula: suppose that the power series

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n$$

converges for $|z - a| < r$. Then

$$\frac{1}{2\pi} \int_0^{2\pi} |f(a + pe^{it})|^2 dt = \sum_{n=0}^{\infty} |c_n|^2 p^{2n}$$

for every $p \in [0, r)$.

洛朗级数 $f(w) = \sum_{n=-\infty}^{\infty} c_n(w-z)^n$ 可以分为两部分:

解析部分 (Analytic Part): $\sum_{n=0}^{\infty} c_n(w-z)^n$ 。这部分在 z 点是全纯的 (如果收敛半径足够大)。

主要部分 (Principal Part): $\sum_{n=-\infty}^{-1} c_n(w-z)^n$ 。这部分决定了奇点的性质。对于极点, 主要部分是有限项的。

可去奇点 (Removable Singularity): 如果洛朗级数中没有负幂次项 (即主要部分为零), 那么函数在该点可以被定义 (或重新定义) 为一个全纯函数。这相当于说, 在 z 点的奇点 “可以被移除”。

极点 (Pole): 如果洛朗级数中只有有限多项负幂次项, 并且最高负幂次项的系数不为零, 那么函数在该点有一个极点。这个极点的阶数就是最高负幂次的指数 (取绝对值)。

本质奇点 (Essential Singularity): 如果洛朗级数中有无限多项负幂次项, 那么函数在该点有一个本质奇点。

