DYNAMICAL SYSTEMS WITH APPLICATIONS USING MATLAB

THIRD EDITION

ANSWERS and HINTS to EXERCISES

Chapter 26

Answers and Hints to Exercises

26.1 Chapter 1

1. (a) 3; (b) 531441; (c) 0.3090; (d) 151; (e) $-\frac{1}{10}$.
2. (a) `u=4:3:199;`
   (b) `v=[6.4:0.8:60]';`
   (c) 18;
   (d) `u=[1,0,1,2];v=repmat(u,1,100);`
   (e) `a=1:100;b=a.^2./a.^a.`
3. (a)
$$A + 4BC = \begin{pmatrix} 57 & 38 & 19 \\ 40 & 25 & 16 \\ 35 & 19 & 14 \end{pmatrix}.$$
   (b)
$$A^{-1} = \begin{pmatrix} 0.4 & -0.6 & 0.2 \\ 0 & 1 & 0 \\ -0.6 & 1.4 & 0.2 \end{pmatrix}, \qquad B^{-1} = \begin{pmatrix} 0 & 1 & -1 \\ 2 & -2 & -1 \\ -1 & 1 & 1 \end{pmatrix}.$$
   The matrix C is singular.
   (c)
$$A^3 = \begin{pmatrix} -11 & 4 & -4 \\ 0 & 1 & 0 \\ 12 & 20 & -7 \end{pmatrix}.$$
   (d) Determinant of $C = 0$.
   (e) Eigenvalues and corresponding eigenvectors are

   $\lambda_1 = -0.3772,\ (0.4429, -0.8264, 0.3477)^T$;

   $\lambda_2 = 0.7261,\ (0.7139, 0.5508, -0.4324)^T$;

   $\lambda_3 = 3.6511,\ (0.7763, 0.5392, 0.3266)^T$.

4. (a) $-1+3i$; (b) $1-3i$; (c) $1.4687+2.2874i$; (d) $0.3466+0.7854i$; (e) $-1.1752i$.

5. (a) 1; (b) $\frac{1}{2}$; (c) 0; (d) ∞; (e) 0.

6. (a) $9x^2+4x$; (b) $\frac{2x^3}{\sqrt{1+x^4}}$; (c) $e^x(\sin(x)\cos(x)+\cos^2(x)-\sin^2(x))$; (d) $1-\tanh^2 x$; (e) $\frac{2\ln x x^{\ln x}}{x}$.

7. (a) $-\frac{43}{12}$; (b) 1; (c) $\sqrt{\pi}$; (d) 2; (e) divergent.

8. (a) cubic curve; (b) exponential decay; (c) implicit plot; (d) surface plot; (e) parametric plot.

9. (a) $y(x) = \frac{1}{2}\sqrt{2x^2+2}$; (b) $y(x) = \frac{6}{x}$; (c) $y(x) = \frac{(108x^3+81)^{1/4}}{3}$; (d) $x(t) = -2e^{-3t}+3e^{-2t}$; (e) $\frac{16}{5}e^{-2t} - \frac{21}{10}e^{-3t} - \frac{1}{10}\cos t + \frac{1}{10}\sin t$.

10. (a) `sigmoid=@(x) 1/(1+e^(-x)); sigmoid(0.5)`

(b)
```
function hsgn(x)
if x>0
    1
elseif x==0
    0
else -1
end

>> hsgn(-6)

ans = -1
```

(c) This iterative map is sensitive to initial conditions. See Chapter 3.

```
(i) When x(0):=0.2,          (ii) when x(0):=0.2001,

x(91):=0.8779563852          x(91):=0.6932414820
x(92):=0.4285958836          x(92):=0.8506309185
x(93):=0.9796058084          x(93):=0.5082318360
x(94):=0.7991307420e-1       x(94):=0.9997289475
x(95):=0.2941078991          x(95):=0.1083916122e-2
x(96):=0.8304337709          x(96):=0.4330964991e-2
x(97):=0.5632540923          x(97):=0.1724883093e-1
x(98):=0.9839956791          x(98):=0.6780523505e-1
x(99):=0.6299273044e-1       x(99):=0.2528307406
x(100):=0.2360985855         x(100):=0.7556294285
```

(d)
```
> # Euclid's algorithm
  a:=12348:b:=14238:
  while b<>0 do
  d:=irem(a,b):
  a:=b:b:=d:
  end do:
  lprint("The greatest common divisor is",a);
```

(e) Pythagorean triples.

```
3 4 5
5 12 13
7 24 25
9 40 41
11 60 61
13 84 85
9 12 15
15 36 39
21 72 75
27 120 123
33 180 183
```

26.2 Chapter 2

1. The general solution is $x_n = \pi(4n + cn(n-1))$.
2. (a) $2 \times 3^n - 2^n$; (b) $2^{-n}(3n+1)$; (c) $2^{\frac{n}{2}}\left(\cos(n\pi/4) + \sin(n\pi/4)\right)$;

   (d) $F_n = \frac{1}{2^n\sqrt{5}}\left[(1+\sqrt{5})^n - (1-\sqrt{5})^n\right]$;

   (e) (i) $x_n = 2^n + 1$;
   (ii) $x_n = \frac{1}{2}(-1)^n + 2^n + n + \frac{1}{2}$;
   (iii) $x_n = \frac{1}{3}(-1)^n + \frac{5}{3}2^n - \frac{1}{6}e^n(-1)^n - \frac{1}{3}e^n 2^n + \frac{1}{2}e^n$.
3. The dominant eigenvalue is $\lambda_1 = 1.107$ and

   (a)
   $$X^{(15)} = \begin{pmatrix} 64932 \\ 52799 \\ 38156 \end{pmatrix};$$
   (b)
   $$X^{(50)} = \begin{pmatrix} 2.271 \times 10^6 \\ 1.847 \times 10^6 \\ 1.335 \times 10^6 \end{pmatrix};$$
   (c)
   $$X^{(100)} = \begin{pmatrix} 3.645 \times 10^8 \\ 2.964 \times 10^8 \\ 2.142 \times 10^8 \end{pmatrix}.$$
4. The eigenvalues are $\lambda_1 = 1$ and $\lambda_{2,3} = \frac{-1\pm\sqrt{3}}{2}$. There is no dominant eigenvalue since $|\lambda_1| = |\lambda_2| = |\lambda_3|$. The population stabilizes.
5. The eigenvalues are 0,0,$-0.656 \pm 0.626i$, and $\lambda_1 = 1.313$. Therefore the population increases by 31.3% every 15 years. The normalized eigenvector is given by
   $$\hat{X} = \begin{pmatrix} 0.415 \\ 0.283 \\ 0.173 \\ 0.092 \\ 0.035 \end{pmatrix}.$$

7. Before insecticide is applied, $\lambda_1 = 1.465$, which means that the population increases by 46.5% every 6 months. The normalized eigenvector is

$$\hat{X} = \begin{pmatrix} 0.764 \\ 0.208 \\ 0.028 \end{pmatrix}.$$

After the insecticide is applied, $\lambda_1 = 1.082$, which means that the population increases by 8.2% every 6 months. The normalized eigenvector is given by

$$\hat{X} = \begin{pmatrix} 0.695 \\ 0.257 \\ 0.048 \end{pmatrix}.$$

8. For this policy, $d_1 = 0.1, d_2 = 0.4$, and $d_3 = 0.6$. The dominant eigenvalue is $\lambda_1 = 1.017$ and the normalized eigenvector is

$$\hat{X} = \begin{pmatrix} 0.797 \\ 0.188 \\ 0.015 \end{pmatrix}.$$

9. Without any harvesting the population would double each year since $\lambda_1 = 2$.

   (a) $\lambda_1 = 1; \quad \hat{X} = \begin{pmatrix} 24/29 \\ 4/29 \\ 1/29 \end{pmatrix}.$

   (b) $h_1 = 6/7; \quad \hat{X} = \begin{pmatrix} 2/3 \\ 2/9 \\ 1/9 \end{pmatrix}.$

   (c) $\lambda_1 = 1.558; \quad \hat{X} = \begin{pmatrix} 0.780 \\ 0.167 \\ 0.053 \end{pmatrix}.$

   (d) $h_1 = 0.604, \lambda_1 = 1.433; \quad \hat{X} = \begin{pmatrix} 0.761 \\ 0.177 \\ 0.062 \end{pmatrix}.$

   (e) $\lambda_1 = 1.672; \quad \hat{X} = \begin{pmatrix} 0.668 \\ 0.132 \\ 0.199 \end{pmatrix}.$

10. Take $h_2 = h_3 = 1$, then $\lambda_1 = 1, \lambda_2 = -1$, and $\lambda_3 = 0$. The population stabilizes.

26.3 Chapter 3

1. The iterates give orbits with periods (i) one, (ii) one, (iii) three, and (iv) nine. There are two points of period one, two points of period two, six points of period three, and twelve points of period four. In general, there are 2^N-(sum of points of periods that divide N) points of period N.

2. (a) The functions are given by

$$T^2(x) = \begin{cases} \frac{9}{4}x & 0 \leq x < \frac{1}{3} \\ \frac{3}{2} - \frac{9}{4}x & \frac{1}{3} \leq x < \frac{1}{2} \\ \frac{9}{4}x - \frac{3}{4} & \frac{1}{2} \leq x < \frac{2}{3} \\ \frac{9}{4}(1-x) & \frac{2}{3} \leq x \leq 1 \end{cases}$$

and

$$T^3(x) = \begin{cases} \frac{27}{8}x & 0 \leq x < \frac{2}{9} \\ \frac{3}{2} - \frac{27}{8}x & \frac{2}{9} \leq x < \frac{1}{3} \\ \frac{27}{8}x - \frac{3}{4} & \frac{1}{3} \leq x < \frac{4}{9} \\ \frac{9}{4} - \frac{27}{8}x & \frac{4}{9} \leq x < \frac{1}{2} \\ \frac{27}{8}x - \frac{9}{8} & \frac{1}{2} \leq x < \frac{5}{9} \\ \frac{21}{8} - \frac{27}{8}x & \frac{5}{9} \leq x < \frac{2}{3} \\ \frac{27}{8}x - \frac{15}{8} & \frac{2}{3} \leq x < \frac{7}{9} \\ \frac{27}{8}(1-x) & \frac{7}{9} \leq x < 1. \end{cases}$$

There are two points of period one, two points of period two, and no points of period three.

(b) $x_{1,1} = 0$, $x_{1,2} = \frac{9}{14}$; $x_{2,1} = \frac{45}{106}$, $x_{2,2} = \frac{81}{106}$; $x_{3,1} = \frac{45}{151}$, $x_{3,2} = \frac{81}{151}$, $x_{3,3} = \frac{126}{151}$, $x_{3,4} = \frac{225}{854}$, $x_{3,5} = \frac{405}{854}$, $x_{3,6} = \frac{729}{854}$.

3. Edit Program_03d.m.

4. Use functions of functions to determine f_μ^N. There are two, two, six, and twelve points of periods one, two, three, and four, respectively.

5. Edit Program_03d.m.

6. (a) Fixed points at $O = (0, 0)$ and $A = \left(\frac{100}{17} \ln \frac{17}{10}, \frac{70}{17} \ln \frac{17}{10}\right)$. O is unstable and A is stable. (b) Edit Program_05c.m. Ramp the parameter b up and down using the second iterative method with feedback. There is period one behavior for, $0 < b < 20$, on ramp up and chaos with chaotic windows for, $20 < b < 50$. On ramp down, there is chaos, then period undoubling for, $15 < b < 23$, and then a return to period one behavior.

7. Points of period one are $(-3/10, -3/10)$ and $(1/5, 1/5)$. Two points of period two are given by $(x_1/2, (0.1 - x_1)/2)$, where x_1 is a root of $5x^2 - x - 1 = 0$. The inverse map is given by

$$x_{n+1} = y_n, \quad y_{n+1} = \frac{10}{9}\left(x_n - \frac{3}{50} + y_n^2\right).$$

8. (a) The eigenvalues are given by $\lambda_{1,2} = -\alpha x \pm \sqrt{\alpha^2 x^2 + \beta}$. A bifurcation occurs when one of the $|\lambda| = 1$. Take the case where $\lambda = -1$.

   (c) The program is listed in Section 3.6.

9. (a) (i) When $a = 0.2$, $c_{1,1} = 0$ is stable, $c_{1,2} = 0.155$ is unstable, and $c_{1,3} = 0.946$ is stable. (ii) When $a = 0.3$, $c_{1,1} = 0$ is stable, $c_{1,2} = 0.170$ is unstable, and $c_{1,3} = 0.897$ is unstable.

10. See the Ahmed paper in the Bibliography.

26.4 Chapter 4

1. (a) The orbit remains bounded forever, $z_{500} \approx -0.3829 + 0.1700i$;

   (b) the orbit is unbounded, $z_{10} \approx -0.6674 \times 10^{197} + 0.2396 \times 10^{197}$.

2. Fixed points of period one are given by

$$z_{1,1} = \frac{1}{2} + \frac{1}{4}\sqrt{10 + 2\sqrt{41}} - \frac{i}{4}\sqrt{2\sqrt{41} - 10},$$

$$z_{1,2} = \frac{1}{2} - \frac{1}{4}\sqrt{10 + 2\sqrt{41}} + \frac{i}{4}\sqrt{2\sqrt{41} - 10}.$$

   Fixed points of period two are given by

$$z_{2,1} = -\frac{1}{2} + \frac{1}{4}\sqrt{2 + 2\sqrt{17}} - \frac{i}{4}\sqrt{2\sqrt{17} - 2},$$

$$z_{2,2} = -\frac{1}{2} - \frac{1}{4}\sqrt{2 + 2\sqrt{17}} + \frac{i}{4}\sqrt{2\sqrt{17} - 2}.$$

3. Use the MATLAB program given in Section 4.4; $J(0,0)$ is a circle and $J(-2,0)$ is a line segment.

4. There is one fixed point located approximately at $z_{1,1} = 1.8202 - 0.0284i$.

5. See the example in the text. The curves are again a cardioid and a circle but the locations are different in this case.

7. Fixed points of period one are given by

$$z_{1,1} = \frac{3 + \sqrt{9 - 4c}}{2}, \quad z_{1,2} = \frac{3 - \sqrt{9 - 4c}}{2}.$$

   Fixed points of period two are given by

$$z_{2,1} = \frac{1 + \sqrt{5 - 4c}}{2}, \quad z_{2,2} = \frac{1 - \sqrt{5 - 4c}}{2}.$$

9. (i) Period four and (ii) period three.

26.5 Chapter 5

1. There are 11 points of period one.

3. Find an expression for E_n in terms of E_{n+1}.

5. See the paper of Li and Ogusu in the Bibliography.

6. (a) Bistable: $4.765 - 4.766$ Wm^{-2}. Unstable: $6.377 - 10.612$ Wm^{-2}. (b) Bistable: $3.936 - 5.208$ Wm^{-2}. Unstable: $4.74 - 13.262$ Wm^{-2}. (c) Bistable: $3.482 - 5.561$ Wm^{-2}. Unstable: $1.903 - 3.995$ Wm^{-2}.

8. Use the function $G(x) = ae^{-bx^2}$ to generate the Gaussian pulse. The parameter b controls the width of the pulse.

26.6 Chapter 6

1. (a) The length remaining at stage k is given by

$$L = 1 - \frac{2}{5} - \frac{2 \times 3}{5^2} - \ldots - \frac{2 \times 3^{k-1}}{5^k}.$$

   The dimension is $D_f = \frac{\ln 3}{\ln 5} \approx 0.6826$.

   (b) $D_f = \frac{\ln 2}{\ln \sqrt{2}} = 2$. If the fractal were constructed to infinity there would be no holes and the object would have the same dimension as a plane. Thus this mathematical object is not a fractal.

2. The figure is similar to the stage 3 construction of the Sierpiński triangle. In fact, this gives yet another method for constructing this fractal as Pascal's triangle is extended to infinity.

3. The dimension is $D_f = \frac{\ln 8}{\ln 3} \approx 1.8928$.

4. See Figure 6.11.

5. You get the Sierpiński triangle.

6. $S_1 = \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right]$, $S_2 = \left[0, \frac{1}{16}\right] \cup \left[\frac{3}{16}, \frac{1}{4}\right] \cup \left[\frac{3}{4}, \frac{13}{16}\right] \cup \left[\frac{15}{16}, 1\right]$. $D_f = 0.5$.

7. (i) The fractal is homogeneous; (ii) $\alpha_{\max} \approx 1.26$ and $\alpha_{\min} \approx 0.26$; (iii) $\alpha_{\max} \approx 0.83$ and $\alpha_{\min} \approx 0.46$. Take $k = 500$ in the plot commands.

8. Using the same methods as in Example 4:

$$D_0 = \frac{\ln 4}{\ln 3}, \quad \alpha_s = \frac{s \ln p_1 + (k-s) \ln p_2}{-k \ln 3}, \text{ and } -f_s = \frac{\ln\left(2^k \begin{pmatrix} k \\ s \end{pmatrix}\right)}{-k \ln 3}.$$

9. At the kth stage, there are 5^k segments of length 3^{-k}. A number

$$N_s = 3^{k-s} 2^s \begin{pmatrix} k \\ s \end{pmatrix}$$

   of these have weight $p_1^{k-s} p_2^s$. Use the same methods as in Example 4.

10. Using multinomials,

$$\alpha_s = \frac{n_1 \ln p_1 + n_2 \ln p_2 + n_3 \ln p_3 + n_4 \ln p_4}{\ln 3^{-k}} \text{ and } -f_s = \frac{\ln \frac{4!}{n_1! n_2! n_3! n_4!}}{\ln 3^{-k}},$$

   where $n_1 + n_2 + n_3 + n_4 = k$.

26.7 Chapter 7

2. The algorithm converges to: (i) $\mathbf{x_2}$; (ii) $\mathbf{x_1}$; (iii) $\mathbf{x_3}$; (iv) $-\mathbf{x_1}$.
5. Take the dot products.
8. The results improve slightly for one, two and three hidden neurons.
9. Bifurcation diagrams match with the stability diagram.
10. (iv) Edit Program_05c.m. Ramp ω up and down.

26.8 Chapter 8

1. The image is 512×512 pixels.
2. The image is 729×729 pixels.
5. Chaotic solutions.
9. $G(x,y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)$.

26.9 Chapter 9

1. (a) $y = \frac{C}{x}$; (b) $y = Cx^2$; (c) $y = C\sqrt{x}$; (d) $\frac{1}{y} = \ln\left(\frac{C}{x}\right)$; (e) $\frac{y^4}{4} + \frac{x^2y^2}{2} = C$; (f) $y = Ce^{-\frac{1}{x}}$.
2. The fossil is 8.03×10^6 years old.
3. (a) $\dot{d} = k_f(a_0 - d)(b_0 - d)(c_0 - d) - k_r(d_0 + d)$;

   (b) $\dot{x} = k_f(a_0 - 3x)^3 - k_r x$, where $a = [A], x = [A_3], b = [B], c = [C]$, and $d = [D]$.
4. (a) The current is $I = 0.733$ amps;

   (b) the charge is $Q(t) = 50(1 - \exp(-10t - t^2))$ coulombs.
5. (a) Time 1.18 hours. (b) The concentration of glucose is

$$g(t) = \frac{G}{100kV} - Ce^{-kt}.$$

6. Set $x(t) = \sum_{n=0}^{\infty} a_n t^n$.
7. The differential equations are

$$\dot{A} = -\alpha A,\ \dot{B} = \alpha A - \beta B,\ \dot{C} = \beta B.$$

9. The differential equations are

$$\dot{H} = -aH + bI,\ \dot{I} = aH - (b+c)I,\ \dot{D} = cI.$$

   The number of dead is given by

$$D(t) = acN\left(\frac{\alpha - \beta + \beta e^{\alpha t} - \alpha e^{\beta t}}{\alpha\beta(\alpha - \beta)}\right),$$

   where α and β are the roots of $\lambda^2 + (a+b+c)\lambda + ac = 0$. This is not realistic as the whole population eventually dies. In reality people recover and some are immune.

10. (a) (i) Solution is $x^3 = 1/(1-3t)$, with maximal interval (MI) $-\infty < t < \frac{1}{3}$; (ii) $x(t) = (e^t + 3)/(3 - e^t)$, with MI $-\infty < t < \ln 3$; (iii) $x(t) = 6/(3 - e^{2t})$, with MI $-\infty < t < \ln\sqrt{3}$.

    (b) Solution is $x(t) = (t + x_0^{1/2} - t_0)^2$, with MI $t_0 - x_0^{1/2} < t < \infty$.

26.10 Chapter 10

1. Analytical solution is: $y(x) = e^{-3x} + e^{4x}$.
2. Analytical solution is: $\theta(t) = \frac{125}{26} e^{-t/5} + \frac{5}{\sqrt{26}} \cos(t - \tan^{-1}(5)) - 3$.
3. Trajectories appear to be chaotic.
5. Periodic solutions. A Lotka-Volterra predator-prey system.
10. Use vectorized code.

26.11 Chapter 11

1. (a) Eigenvalues and eigenvectors are $\lambda_1 = -10$, $(-2, 1)^T$; $\lambda_2 = -3$, $(\frac{3}{2}, 1)^T$. The origin is a stable node.

   (b) Eigenvalues and eigenvectors are $\lambda_1 = -4$, $(1, 0)^T$; $\lambda_2 = 2$, $(-\frac{4}{3}, 1)^T$. The origin is a saddle point.

2. (a) All trajectories are vertical and there are an infinite number of critical points on the line $y = -\frac{x}{2}$.

   (b) All trajectories are horizontal and there are an infinite number of critical points on the line $y = -\frac{x}{2}$.

   (c) Eigenvalues and eigenvectors are $\lambda_1 = 5$, $(2, 1)^T$; $\lambda_2 = -5$, $(1, -2)^T$. The origin is a saddle point.

   (d) Eigenvalues are $\lambda_1 = 3 + i$, $\lambda_2 = 3 - i$, and the origin is an unstable focus.

   (e) There are two repeated eigenvalues and one linearly independent eigenvector: $\lambda_1 = -1$, $(-1, 1)^T$. The origin is a stable degenerate node.

   (f) This is a nonsimple fixed point. There are an infinite number of critical points on the line $y = x$.

3. (a) $\dot{x} = y,\ \dot{y} = -25x - \mu y$;

   (b) (i) unstable focus, (ii) center, (iii) stable focus, (iv) stable node;

   (c) (i) oscillations grow, (ii) periodic oscillations, (iii) damping, (iv) *critical damping*.

   The constant μ is called the *damping coefficient*.

4. (a) There is one critical point at the origin which is a col. Plot the isoclines. The eigenvalues are $\lambda = \frac{-1 \pm \sqrt{5}}{2}$ with eigenvectors $\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$.

   (b) There are two critical points at $A = (0, 2)$ and $B = (1, 0)$. A is a stable focus and B is a col with eigenvalues and corresponding eigenvectors given by $\lambda_1 = 1$, $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ and $\lambda_2 = -2$, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

   (c) There are two critical points at $A = (1, 1)$ and $B = (1, -1)$. A is an unstable focus and B is a stable focus. Plot the isoclines where $\dot{x} = 0$ and $\dot{y} = 0$.

   (d) There are three critical points at $A = (2, 0), B = (1, 1)$, and $C = (1, -1)$; A is a col and B and C are both stable foci.

(e) There is one nonhyperbolic critical point at the origin. The solution curves are given by $y^3 = x^3 + C$. The line $y = x$ is invariant, the flow is horizontal on $\dot{y} = x^2 = 0$, and the flow is vertical on the line $\dot{x} = y^2 = 0$. The slope of the trajectories is given by $\frac{dy}{dx} = \frac{x^2}{y^2}$.

(f) There is one nonhyperbolic critical point at the origin. The solution curves are given by $y = \frac{x}{1+Cx}$. The line $y = x$ is invariant.

(g) There is one nonhyperbolic critical point at the origin. The solution curves are given by $2y^2 = x^4 + C$. The slope of the orbits is given by $\frac{dy}{dx} = \frac{x^3}{y}$.

(h) When $\mu < 0$ there are no critical points. When $\mu = 0$, the solution curves are given by $|x| = Ce^{\frac{1}{y}}$. When $\mu > 0$, there are two critical points at $A = (0, \sqrt{\mu})$ and $B = (0, -\sqrt{\mu})$; A is a col and B is an unstable node.

5. One possible system is

$$\dot{x} = y^2 - x^2, \; \dot{y} = x^2 + y^2 - 2,$$

for example.

6. There are three critical points at $O = (0,0), A = (1,0)$, and $B = (-1,0)$. If $a_0 > 0$, since det $J_O > 0$ and trace $J_O < 0$, the origin is stable and A and B are cols because det $J < 0$ for these points. If $a_0 < 0$, the origin is unstable and A and B are still cols. Therefore, if $a_0 > 0$, the current in the circuit eventually dies away to zero with increasing time. If $a_0 < 0$, the current increases indefinitely, which is physically impossible.

7. There are three critical points at $O = (0,0)$, $A = (\frac{a}{b}, 0)$, and $B = \left(\frac{c+a}{b}, \frac{c(c+a)}{b}\right)$. The origin is an unstable node and A is a col. The critical point at B is stable since det $J_B > 0$ and trace $J_B < 0$. Therefore, the population and birth rate stabilize to the values given by B in the long term.

8. When $\alpha\beta > 1$, there is one stable critical point at $A = (0, \frac{1}{\beta})$. When $\alpha\beta < 1$, A becomes a col and $B = (\sqrt{1-\alpha\beta}, \alpha)$ and $C = (-\sqrt{1-\alpha\beta}, \alpha)$ are both stable. When $\alpha\beta > 1$, the power goes to zero and the velocity of the wheel tends to $\frac{1}{\beta}$ and when $\alpha\beta < 1$, the power and velocity stabilize to the point B.

9. (a) There is one critical point at $\left(\frac{KG_0}{K-C}, \frac{G_0}{K-C}\right)$, which is in the first quadrant if $K > C$. When $C = 1$, the critical point is nonhyberbolic. The system can be solved and there are closed trajectories around the critical point. The economy oscillates (as long as $I(t), S(t) > 0$). If $C \neq 1$, then the critical point is unstable if $0 < C < 1$ and stable if $C > 1$.

(b) The critical point is stable and the trajectory tends to this point. The choice of initial condition is important to avoid $I(t)$ or $S(t)$ from going negative, where the model is no longer valid.

10. Note that $\frac{d\eta}{d\tau} = e^t$ and $\frac{d^2\eta}{d\tau^2} = \frac{d\eta}{d\tau}\frac{dt}{d\tau}$. There are four critical points: $O = (0,0)$, an unstable node; $A = (-1,0)$, a col; $B = (0,2)$, a col; and $C = \left(-\frac{3}{2}, \frac{1}{2}\right)$, a stable focus.

26.12 Chapter 12

1. This is a competing species model. There are four critical points in the first quadrant at $O = (0,0)$, $P = (0,3)$, $Q = (2,0)$, and $R = (1,1)$. The point O

is an unstable node, P and Q are both stable nodes, and R is a saddle point. There is mutual exclusion and one of the species will become extinct depending on the initial populations.

2. This is a Lotka-Volterra model with critical points at $O = (0, 0)$ and $A = (3, 2)$. The system is structurally unstable. The populations oscillate but the cycles are dependent on the initial values of x and y.

3. This is a predator-prey model. There are three critical points in the first quadrant at $O = (0, 0)$, $F = (2, 0)$, and $G = (\frac{3}{2}, \frac{1}{2})$. The points O and F are saddle points and G is a stable focus. In terms of species behavior, the two species coexist and approach constant population values.

4. Consider the three cases separately.

   (i) If $0 < \mu < \frac{1}{2}$, then there are four critical points at $O = (0, 0)$, $L = (2, 0)$, $M = (0, \mu)$, and $N = \left(\frac{\mu-2}{\mu^2-1}, \frac{\mu(2\mu-1)}{\mu^2-1}\right)$. The point O is an unstable node, L and M are saddle points, and N is a stable point. To classify the critical points, consider det J and trace J. The two species coexist.

   (ii) If $\frac{1}{2} < \mu < 2$, then there are three critical points in the first quadrant, all of which lie on the axes. The point O is an unstable node, L is a stable node, and M is a saddle point. Species y becomes extinct.

   (iii) If $\mu > 2$, then there are four critical points in the first quadrant. The point O is an unstable node, L and M are stable nodes, and N is a saddle point. One species becomes extinct.

5. (a) A predator-prey model. There is coexistence; the populations stabilize to the point $(\frac{5}{4}, \frac{11}{4})$.

   (b) A competing species model. There is mutual exclusion; one species becomes extinct.

6. There are three critical points in the first quadrant if $0 \leq \epsilon < 1$: at $O = (0, 0)$, $A = (\frac{1}{\epsilon}, 0)$ and $B = (\frac{1+\epsilon}{1+\epsilon^2}, \frac{1-\epsilon}{1+\epsilon^2})$. There are two when $\epsilon \geq 1$. The origin is always a col. When $\epsilon = 0$, the system is Lotka-Volterra, and trajectories lie on closed curves away from the axes. If $0 < \epsilon < 1$, A is a col, and B is stable since the trace of the Jacobian is negative and the determinant is positive. When $\epsilon \geq 1$, A is stable.

7. There are three critical points at $O = (0, 0)$, $P = (1, 0)$, and $Q = (0.6, 0.24)$. Points O and P are cols and Q is stable. There is coexistence.

8. There is a limit cycle enclosing the critical point at $(0.48, 0.2496)$. The populations vary periodically and coexist.

9. One example would be the following. X and Y prey on each other; Y has cannibalistic tendencies and also preys on Z. See Figure 26.1.

10. Let species X, Y, and Z have populations $x(t)$, $y(t)$, and $z(t)$, respectively. The interactions are as follows: X preys on Y; Z preys on X; Y and Z are in competition.

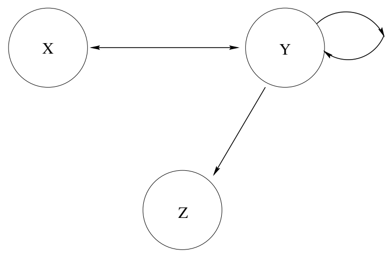


Figure 26.1: One possible interaction between three interacting insect species.

26.13 Chapter 13

1. Convert to polar coordinates to get

$$\dot{r} = r\left(1 - r^2 - \frac{1}{2}\cos^2\theta\right), \quad \dot{\theta} = -1 + \frac{1}{2}\cos\theta\sin\theta.$$

   Since $\dot{\theta} < 0$, the origin is the only critical point. On $r = \frac{1}{2}$, $\dot{r} > 0$, and on $r = 2$, $\dot{r} < 0$. Therefore, there exists a limit cycle by the corollary to the Poincaré-Bendixson Theorem.

2. Plot the graph of $y = x - x^3\cos^3(\pi x)$ to prove that the origin is the only critical point inside the square. Linearize to show that the origin is an unstable focus. Consider the flow on the sides of the rectangle, for example, on $x = 1$, with $-1 \leq y \leq 1$, $\dot{x} = -y + \cos\pi \leq 0$. Hence the flow is from right to left on this line. Show that the rectangle is invariant and use the corollary to the Poincaré-Bendixson Theorem.

3. Plot the graph of $y = x^8 - 3x^6 + 3x^4 - 2x^2 + 2$ to prove that the origin is a unique critical point. Convert to polar coordinates to get

$$\dot{r} = r\left(1 - r^2(\cos^4\theta + \sin^4\theta)\right), \quad \dot{\theta} = 1 - r^2\cos\theta\sin\theta(\sin^2\theta - \cos^2\theta).$$

   Now $\text{div}(\mathbf{X}) = 2 - 3r^2$ and so $\text{div}(\mathbf{X})$ is nonzero in the annulus $A = \{1 < r < 2\}$. On the circle $r = 1 - \epsilon$, $\dot{r} > 0$, and on the circle $r = 2 + \epsilon$, $\dot{r} < 0$. Therefore there is a unique limit cycle contained in the annulus by Dulac's criteria.

4. Use the Poincaré-Bendixson theorem.

5. Consider the isocline curves. If the straight line intersects the parabola to the right of the maximum, then there is no limit cycle. If the straight line intersects the parabola to the left of the maximum, then there exists a limit cycle.

6. (a) The limit cycle is circular. (b) The limit cycle has fast and slow branches.

7. It will help if you draw rough diagrams.

   (a) Now $\text{div}(\mathbf{X}) = -(1 + x^2 + x^4) < 0$. Hence there are no limit cycles by Bendixson's criteria.

   (b) Now $\text{div}(\mathbf{X}) = 2 - x$. There are four critical points at $(0,0), (1,0), (-1,1)$, and $(-1,-1)$. The x axis is invariant. On $x = 0$, $\dot{x} = 2y^2 \geq 0$. Hence there are no limit cycles in the plane.

(c) Now $\mathrm{div}(\mathbf{X}) = -6 - 2x^2 < 0$. Hence there are no limit cycles by Bendixson's criteria.

(d) Now $\mathrm{div}(\mathbf{X}) = -3 - x^2 < 0$. Hence there are no limit cycles by Bendixson's criteria.

(e) Now $\mathrm{div}(\mathbf{X}) = 3x - 2$, and $\mathrm{div}(\mathbf{X}) = 0$ on the line $x = \frac{2}{3}$. There are three critical points at $(1, 0), (-1, 0)$, and $(2, 3)$. The x-axis is invariant, and $\dot{x} < 0$ for $y > 0$ on the line $x = \frac{2}{3}$. Hence there are no limit cycles by Bendixson's criteria.

(f) Now $\mathrm{div}(\mathbf{X}) = -3x^2y^2$. Therefore there are no limit cycles lying entirely in one of the quadrants. However, $\dot{x} = -y^2$ on the line $x = 0$ and $\dot{y} = x^5$ on the line $y = 0$. Hence there are no limit cycles by Bendixson's criteria.

(g) Now $\mathrm{div}(\mathbf{X}) = (x - 2)^2$. On the line $x = 2$, $\dot{x} = -y^2$, and so no limit cycle can cross this line. Hence there are no limit cycles by Bendixson's criteria.

8. (a) The axes are invariant. Now $\mathrm{div}(\psi\mathbf{X}) = \frac{1}{xy^2}(2 - 2x)$ and so $\mathrm{div}(\psi\mathbf{X}) = 0$ when $x = 1$. There are four critical points and only one, $(-16, 38)$, lying wholly in one of the quadrants. Since the divergence is nonzero in this quadrant, there are no limit cycles.

   (b) Now $\mathrm{div}(\psi\mathbf{X}) = -\frac{\delta}{y} - \frac{d}{x}$ and so $\mathrm{div}(\psi\mathbf{X}) = 0$ when $y = -\frac{\delta x}{d}$. Since $\delta > 0$ and $d > 0$, there are no limit cycles contained in the first quadrant.

9. The one-term uniform expansion is $x(t, \epsilon) = a\cos(t)\left(1 - \epsilon\left(\frac{1}{2} + \frac{a^2}{8}\right) + \cdots\right) + O(\epsilon)$, as $\epsilon \to 0$.

26.14 Chapter 14

1. The Hamiltonian is $H(x, y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4}$. There are three critical points: $(0, 0)$, which is a saddle point and $(1, 0)$ and $(-1, 0)$, which are both centers.
2. There are three critical points: $(0, 0)$, which is a center, and $(1, 0)$ and $(-1, 0)$, which are both saddle points.
3. The critical points occur at $(n\pi, 0)$, where n is an integer. When n is odd, the critical points are saddle points, and when n is even the critical points are stable foci. The system is now damped and the pendulum swings less and less, eventually coming to rest at $\theta = 2n\pi$ degrees. The saddle points represent the unstable equilibria when $\theta = (2n + 1)\pi$ degrees.
4. The Hamiltonian is $H(x, y) = \frac{y^4}{4} - \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4}$. There are nine critical points.
5. (a) The origin is asymptotically stable.

   (b) The origin is asymptotically stable if $x < \alpha$ and $y < \beta$.

   (c) The origin is unstable.
6. The origin is asymptotically stable. The positive limit sets are either the origin or the ellipse $4x^2 + y^2 = 1$, depending on the value of p.
7. The function $V(x, y)$ is a Lyapunov function if $a > \frac{1}{4}$.
8. The basin of attraction of the origin is the circle $x^2 + y^2 < 4$.
9. Include the vectorfield.
10. Now $\dot{V} = -8(x^4 + 3y^6)(x^4 + 2y^2 - 10)^2$. The origin is unstable, and the curve $x^4 + 2y^2 = 10$ is an attractor.

26.15 Chapter 15

1. (a) There is one critical point when $\mu \leq 0$, and there are two critical points when $\mu > 0$. This is a saddle-node bifurcation.

   (b) When $\mu < 0$, there are two critical points and the origin is stable. When $\mu > 0$, there is one critical point at the origin which is unstable. The origin undergoes a transcritical bifurcation.

   (c) There is one critical point at the origin when $\mu \leq 0$, and there are three critical points—two are unstable—when $\mu > 0$. This is called a *subcritical pitchfork bifurcation*.

2. Possible examples include

   (a) $\dot{x} = \mu x(\mu^2 - x^2)$;

   (b) $\dot{x} = x^4 - \mu^2$; and

   (c) $\dot{x} = x(\mu^2 + x^2 - 1)$.

3. The critical points are given by $O = (0,0)$, $A = \frac{12+\sqrt{169-125h}}{5}$, and $B = \frac{12-\sqrt{169-125h}}{5}$. There are two critical points if $h \leq 0$, the origin is unstable, and A is stable (but negative harvesting is discounted). There are three critical points if $0 < h < 1.352$, the origin and A are stable, and B is unstable. There is one stable critical point at the origin if $h \geq 1.352$.

   The term $x(1 - \frac{x}{5})$ represents the usual logistic growth when there is no harvesting. The term $\frac{hx}{0.2+x}$ represents harvesting from h is zero up to a maximum of h, no matter how large x becomes (plot the graph).

   When $h = 0$, the population stabilizes to 5×10^5; when $0 < h < 1.352$, the population stabilizes to $A \times 10^5$; and when $h > 1.352$, the population decreases to zero. Use animation in MATLAB to plot $\dot{x}$ as h varies from zero to eight. The harvesting is *sustainable* if $0 < h < 1.352$, where the fish persist, and it is *unsustainable* if $h > 1.352$, when the fish become extinct from the lake.

4. (a) No critical points if $\mu < 0$. There is one nonhyperbolic critical point at $O = (0,0)$ if $\mu = 0$, and there are two critical points at $A = (0, \sqrt[4]{\mu})$ and $B = (0, -\sqrt[4]{\mu})$. Both A and B are unstable.

   (b) There are two critical points at $O = (0,0)$ and $A = (\mu^2, 0)$ if $\mu \neq 0$ (symmetry). O is stable and A is unstable. There is one nonhyperbolic critical point at $O = (0,0)$ if $\mu = 0$.

   (c) There are no critical points if $\mu < 0$. There is one nonhyperbolic critical point at $O = (0,0)$ if $\mu = 0$, and there are four critical points at $A = (2\sqrt{\mu}, 0)$, $B = (-2\sqrt{\mu}, 0)$, $C = (\sqrt{\mu}, 0)$, and $D = (-\sqrt{\mu}, 0)$ if $\mu > 0$. The points A and D are stable, while B and C are unstable.

5. (a) If $\mu < 0$, there is a stable critical point at the origin and an unstable limit cycle of radius $r = -\mu$. If $\mu = 0$, the origin is a center, and if $\mu > 0$, the origin becomes unstable. The flow is counterclockwise.

   (b) If $\mu \leq 0$, the origin is an unstable focus. If $\mu > 0$, the origin is unstable, and there is a stable limit cycle of radius $r = \frac{\mu}{2}$ and an unstable limit cycle of radius $r = \mu$.

   (c) If $\mu \neq 0$, the origin is unstable and there is a stable limit cycle of radius $|r| = \mu$. If $\mu = 0$, the origin is stable.

6. Take $\mathbf{x} = \mathbf{u} + \mathbf{f}_3(\mathbf{u})$. Then, if the eigenvalues of J are not resonant of order 3,

$$f_{30} = \frac{a_{30}}{2\lambda_1},\ f_{21} = \frac{a_{21}}{\lambda_1 + \lambda_2},\ f_{12} = \frac{a_{12}}{2\lambda_2},\ f_{03} = \frac{a_03}{3\lambda_2 - \lambda_1},$$

$$g_{30} = \frac{b_{30}}{3\lambda_1 - \lambda_2},\ g_{21} = \frac{b_{21}}{2\lambda_1},\ g_{12} = \frac{b_{12}}{\lambda_1 + \lambda_2},\ g_{03} = \frac{b_{03}}{2\lambda_2}$$

and all of the cubic terms can be eliminated from the system resulting in a linear normal form $\dot{\mathbf{u}} = J\mathbf{u}$.

7. See the book of Guckenheimer and Holmes referenced in Chapter 17.

8. (a) There is one critical point at the origin and there are at most two stable limit cycles. As μ increases through zero there is a Hopf bifurcation at the origin. Next there is a saddle-node bifurcation to a large-amplitude limit cycle. If μ is then decreased back through zero, there is another saddle-node bifurcation back to the steady state at the origin.

   (b) If $\mu < 0$, the origin is unstable, and if $\mu = 0$, $\dot{r} > 0$ if $r \neq 0$ the origin is unstable and there is a semistable limit cycle at $r = 1$. If $\mu > 0$, the origin is unstable, there is a stable limit cycle of radius $r = \frac{2+\mu-\sqrt{\mu^2+4\mu}}{2}$ and an unstable limit cycle of radius $r = \frac{2+\mu+\sqrt{\mu^2+4\mu}}{2}$. It is known as a fold bifurcation because a fold in the graph of $y = (r-1)^2 - \mu r$ crosses the r-axis at $\mu = 0$.

9. If $\mu < 0$, the origin is a stable focus and as μ passes through zero, the origin changes from a stable to an unstable spiral. If $\mu > 0$, convert to polars. The origin is unstable and a stable limit cycle bifurcates.

10. The critical points occur at $A = (0, -\frac{\alpha}{\beta})$ and $B = (\alpha+\beta, 1)$. Thus there are two critical points everywhere in the (α, β) plane apart from along the line $\alpha = -\beta$ where there is only one. The eigenvalues for the matrix J_A are $\lambda_1 = \beta$ and $\lambda_2 = -\frac{(\alpha+\beta)}{\beta}$. The eigenvalues for the matrix J_B are $\lambda = \frac{-\alpha\pm\sqrt{\alpha^2-4(\alpha+\beta)}}{2}$. There is a codimension-2 bifurcation along the line $\alpha = -\beta$ and it is a transcritical bifurcation.

26.16 Chapter 16

1. Eigenvalues and eigenvectors given by $[3, (-2,-2,1)^T], [-3, (-2,1,-2)^T]$, and $[9, (1,-2,-2)^T]$. The origin is unstable; there is a col in two planes and an unstable node in the other.

2. Eigenvalues are $\lambda_{1,2} = 1 \pm i\sqrt{6}, \lambda_3 = 1$. The origin is unstable and the flow is rotating. Plot solution curves using MATLAB.

3. There are two critical points at $O = (0,0,0)$ and $P = (-1,-1,-1)$. The critical points are both hyperbolic and unstable. The eigenvalues for O are $[1,1,-1]$ and those for P are $[1,-1,-1]$.

4. Consider the flow on $x = 0$ with $y \geq 0$ and $z \geq 0$, etc. The first quadrant is positively invariant. The plane $x + y + 2z = k$ is invariant since $\dot{x} + \dot{y} + 2\dot{z} = 0$. Hence if a trajectory starts on this plane, then it remains there forever. The critical points are given by $\left(\frac{\lambda y}{1+y}, y, y/2\right)$. Now on the plane $x + y + 2z = k$,

the critical point satisfies the equation $\frac{\lambda y}{1+y} + y + y = k$, which has solutions $y = \frac{(2-\lambda)\pm\sqrt{(2-\lambda)^2+32}}{4}$. Since the first quadrant is invariant, $\lambda^+(p)$ must tend to this critical point.

5. (a) Take $V = x^2+y^2+z^2$. Then $\dot{V} = -\left(x^2 + y^4 + (y-z^2)^2 + (z-x^2)^2\right) \leq 0$. Now $\dot{V} = 0$ if and only if $x = y = z = 0$; hence the origin is globally asymptotically stable.

   (b) Consider $V = ax^2+by^2+cz^2$. Now $\dot{V} = -2(a^2x^2+b^2y^2+c^2z^2)+2xyz(ax+by+cz)$. Hence $\dot{V} < \frac{V^2}{c} - 2cV$ and $\dot{V} < 0$ in the set $V < 2c^2$. Therefore the origin is asymptotically stable in the ellipsoid $V < 2c^2$.

6. A stiff system, use ode23s.

7. There are eight critical points at $(0,0,0), (0,0,1/2), (0,1/2,0), (0,1,-1), (1/2,0,0), (-1/3,0,1/3), (1/3,-1/3,0)$, and $(1/14,3/14,3/14)$. The plane $x+y+z = 1/2$ is a solution plane since $\dot{x}+\dot{y}+\dot{z} = (x+y+z)-2(x+y+z)^2 = 0$ on this plane. There are closed curves on the plane representing periodic behavior. The three species coexist and the populations oscillate in phase. The system is structurally unstable.

8. (i) The populations settle on to a period-2 cycle. (ii) The populations settle on to a period-4 cycle.

9. Plot a time series.

10. A Jordan curve lying wholly in the first quadrant exists, similar to the limit cycle for the Liénard system when a parameter is large. The choice of q and C are important.

26.17 Chapter 17

1. Starting with $r_0 = 4$, the returns are $r_1 = 1.13854$, $r_2 = 0.66373$, $\ldots$, $r_{10} = 0.15307$, to five decimal places.

2. The Poincaré map is given by $r_{n+1} = \mathbf{P}(r_n) = \frac{\mu r_n}{r_n + e^{-2\mu\pi}(\mu - r_n)}$.

3. Now $\left.\frac{d\mathbf{P}}{dr}\right|_{\mu} = e^{-2\mu\pi}$. Therefore the limit cycle at $r = \mu$ is hyperbolic stable if $\mu > 0$ and hyperbolic unstable if $\mu < 0$. What happens when $\mu = 0$?

4. The Poincaré map is given by $r_{n+1} = \mathbf{P}(r_n) = \left(\frac{r_n^2}{r_n^2 + e^{-4\pi}(1-r_n^2)}\right)^{\frac{1}{2}}$.

5. The limit cycle at $r = 1$ is stable since $\left.\frac{d\mathbf{P}}{dr}\right|_{r=1} = e^{-4\pi}$.

6. (a) The Poincaré section in the p_1q_1 plane is crossed 14 times. (b) The trajectory is quasiperiodic.

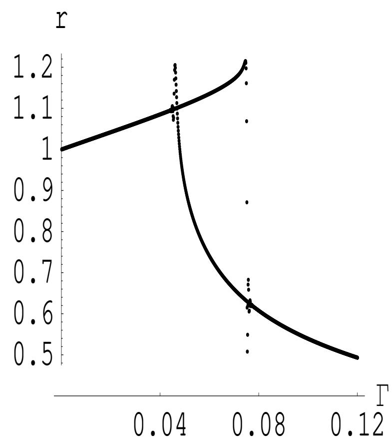
7. Edit program listed in Section 15.4.

8. Edit program listed in Section 15.4.

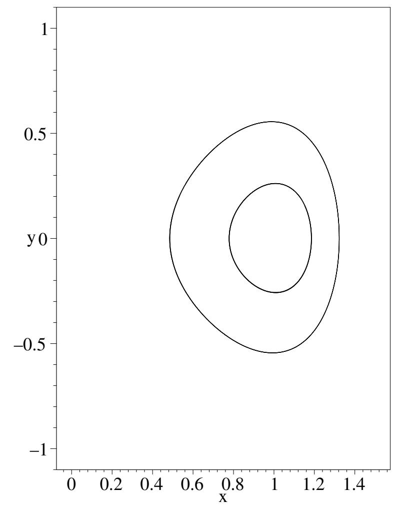
9. A chaotic attractor is formed.

10. (a) See Figure 26.2(a).

    (b) See Figure 26.2(b). Take $\Gamma = 0.07$. For example, choose initial conditions (i) $x_0 = 1.16, y_0 = 0.112$ and (ii) $x_0 = 0.585, y_0 = 0.29$.



(a)



(b)

Figure 26.2: (a) Bifurcation diagram. (b) Multistable behavior.

26.18 Chapter 18

1. Differentiate to obtain $2u\dot{u} = G'(x)\dot{x}$ and find $\frac{dy}{du}$.
2. Using MATLAB: $\left\{\left\{x^2 - 3xy + 9y^2, 0, -26y^2 - 36yz - 26z^2\right\}, -25z^3\right\}$ and $\left\{\left\{9, 9 + x^2 - 3xy, -27 + y^2 + yz + z^2\right\}, -27z + 2z^3\right\}$.
3. Lex $\left\{y^3 - y^4 - 2y^6 + y^9, x + y^2 + y^4 - y^7\right\}$; DegLex $\left\{-x^2 + y^3, -x + x^3 - y^2\right\}$; DegRevLex $\left\{-x^2 + y^3, -x + x^3 - y^2\right\}$. Solutions are $(0, 0)$, $(-0.471074, 0.605423)$, and $(1.46107, 1.28760)$.
4. The Lyapunov quantities are given by $L(i) = a_{2i+1}$, where $i = 0$ to 6.
5. See the Lloyd and Lynch paper in the Bibliography.
7. The Lyapunov quantities are given by $L(0) = -a_1, L(1) = -3b_{03} - b_{21}, L(2) = -3b_{30}b_{03} - b_{41}$, and $L(3) = b_{03}^3$.
8. The homoclinic loop lies on the curve $y^2 = x^2 + \frac{2}{3}x^3$.
10. There are three limit cycles when $\lambda = -0.9$.

26.19 Chapter 19

1. There is one critical point in the finite plane at the origin which is a stable node. The eigenvalues and eigenvectors are given by $\lambda_1 = -1$, $(1, -1)^T$ and $\lambda_2 = -4$, $(1, -4)^T$, respectively. The function $g_2(\theta)$ is defined as

$$g_2(\theta) = -4\cos^2\theta - 5\cos\theta\sin\theta - \sin^2\theta.$$

There are four critical points at infinity at $\theta_1 = \tan^{-1}(-1)$, $\theta_2 = \tan^{-1}(-1) + \pi$, $\theta_3 = \tan^{-1}(-4)$, and $\theta_4 = \tan^{-1}(-4) + \pi$. The flow in a neighborhood of a critical point at infinity is qualitatively equivalent to the flow on $X = 1$ given by

$$\dot{y} = -y^2 - 5y - 4, \quad \dot{z} = -yz.$$

There are two critical points at $(-1, 0)$, which is a col and $(-4, 0)$, which is an unstable node. Since n is odd, antinodal points are qualitatively equivalent.

2. There is one critical point in the finite plane at the origin which is a col. The eigenvalues and eigenvectors are given by $\lambda_1 = 1$, $(1,1)^T$ and $\lambda_2 = -1$, $(2,1)^T$, respectively. The function $g_2(\theta)$ is defined as

$$g_2(\theta) = -2\cos^2\theta + 6\cos\theta\sin\theta - 4\sin^2\theta.$$

There are four critical points at infinity at $\theta_1 = \tan^{-1}(1)$, $\theta_2 = \tan^{-1}(1) + \pi$, $\theta_3 = \tan^{-1}(1/2)$, and $\theta_4 = \tan^{-1}(1/2) + \pi$. The flow in a neighborhood of a critical point at infinity is qualitatively equivalent to the flow on $X = 1$ given by

$$\dot{y} = -4y^2 + 6y - 2, \quad \dot{z} = 3z - 4yz.$$

There are two critical points at $(1, 0)$, which is a stable node and $(1/2, 0)$, which is an unstable node. Since n is odd, antinodal points are qualitatively equivalent.

3. There are no critical points in the finite plane. The function $g_3(\theta)$ is given by

$$g_3(\theta) = 4\cos^2\theta\sin\theta - \sin^3\theta.$$

The function has six roots in the interval $[0, 2\pi)$, at $\theta_1 = 0$, $\theta_2 = 1.10715$, $\theta_3 = 2.03444$, $\theta_4 = 3.14159$, $\theta_5 = 4.24874$, and $\theta_6 = 5.1764$. All of the angles are measured in radians. The behavior on the plane $X = 1$ is determined from the system

$$\dot{y} = 4y - 5z^2 - y^3 + yz^2, \quad \dot{z} = -z - zy^2 + z^3.$$

There are three critical points at $O = (0, 0)$, $A = (2, 0)$, and $B = (-2, 0)$. Points A and B are stable nodes and O is a col. Since n is even, antinodal points are qualitatively equivalent, but the flow is reversed.

All of the positive and negative limit sets for this system are made up of the critical points at infinity.

4. There is one critical point at the origin in the finite plane which is a stable focus. The critical points at infinity occur at $\theta_1 = 0$ radians, $\theta_2 = \frac{\pi}{2}$ radians, $\theta_3 = -\frac{\pi}{2}$ radians, and $\theta_4 = \pi$ radians. Two of the points at infinity are cols and the other two are unstable nodes.

5. There is a unique critical point in the finite plane at the origin which is an unstable node. The critical points at infinity occur at $\theta_1 = 0$ radians, $\theta_2 = \frac{\pi}{2}$ radians, $\theta_3 = -\frac{\pi}{2}$ radians, and $\theta_4 = \pi$ radians. Two of the points at infinity are cols and the other two are unstable nodes. There is at least one limit cycle surrounding the origin by the corollary to the Poincaré-Bendixson Theorem.

7. If $a_1 a_3 > 0$, then the system has no limit cycles. If $a_1 a_3 < 0$, there is a unique hyperbolic limit cycle. If $a_1 = 0$ and $a_3 \neq 0$, then there are no limit cycles. If $a_3 = 0$ and $a_1 \neq 0$, then there are no limit cycles. If $a_1 = a_3 = 0$, then the origin is a center by the classical symmetry argument.

8. When ϵ is small one may apply the Melnikov theory of Chapter 11 to establish where the limit cycles occur. The limit cycles are asymptotic to circles centered at the origin. If the degree of F is $2m + 1$ or $2m + 2$, there can be no more

than m limit cycles. When ϵ is large, if a limit cycle exists, it shoots across in the horizontal direction to meet a branch of the curve $y = F(x)$, where the trajectory slows down and remains near the branch until it shoots back across to another branch of $F(x)$ where it slows down again. The trajectory follows this pattern forever. Once more there can be no more than m limit cycles.

9. Use a similar argument to that used in the proof to Theorem 4.
10. The function F has to satisfy the conditions $a_1 > 0$, $a_3 < 0$, and $a_3^2 > 4a_1$, for example. This guarantees that there are five roots for $F(x)$. If there is a local maximum of $F(x)$ at say $(\alpha_1, 0)$, a root at $(\alpha_2, 0)$, and a local minimum at $(\alpha_3, 0)$, then it is possible to prove that there is a unique hyperbolic limit cycle crossing $F(x)$ in the interval (α_1, α_2) and a second hyperbolic limit cycle crossing $F(x)$ in the interval (α_3, ∞). Use similar arguments to those used in the proof of Theorem 4.

26.20 Chapter 20

1. Use equation (20.3).
5. There is periodic, quasiperiodic and possibly chaotic behavior.
7. When the global warming trem is small we see no discernible difference in the steady-state solutions; however, when W gets too large, the oscillatory behavior disappears.
9. See the paper cited in the question.
10. See the paper cited in the question.

26.21 Chapter 21

1. Take the transformations $x_n = \frac{1}{a} u_n$ and $y_n = \frac{b}{a} v_n$.
2. There are one control range when $p = 1$, three control ranges when $p = 2$, seven control ranges when $p = 3$, and twelve control ranges when $p = 4$.
3. Points of period one are located at approximately $(-1.521, -1.521)$ and $(0.921, 0.921)$. Points of period two are located near $(-0.763, 1.363)$ and $(1.363, -0.763)$.
4. See the paper of Chau in the Bibliography.
5. See Section 21.3.
6. The two-dimensional mapping is given by
$$x_{n+1} = A + B(x_n \cos(x_n^2 + y_n^2) - y_n \sin(x_n^2 + y_n^2)),$$
$$y_{n+1} = B(x_n \sin(x_n^2 + y_n^2) + y_n \cos(x_n^2 + y_n^2)).$$
The one point of period one is located near $(2.731, 0.413)$.
7. (i) There are three points of period one; (ii) there are nine points of period one.
8. See our research paper on chaos control in the Bibliography.
9. The control region is very small and targeting is needed in this case. The chaotic transients are very long. Targeting is not required in Exercise 9, where the control region is much larger. Although there is greater flexibility (nine points of period one) with this system, the controllability is reduced.

26.22 Chapter 22

1. The analytic solution is $I(t) = -3.14\cos(t) + 9.29\sin(t) + 3.14e^{-2t}$.
2. The analytic solution is $I(t) = (-2\cos(4t) - \sin(4t) + 10e^{-2t} - 8e^{-3t})/50$.
3. The analytic solution is $I(t) = (2\sin(t) - \cos(t) + e^{-2t})/\sqrt{2}$.
4. As $\mu \to 0$, the limit cycle becomes a circle.
5. Period 4.
6. Chaotic behavior.
7. There are two distinct limit cycles. The system is multistable.

26.23 Chapter 23

1. The threshold voltage is approximately 6.3mV. (a) When $I = 8$mV, frequency is approximately 62.5Hz. (b) When $I = 20$mV, frequency is approximately 80Hz.
2. An example of a Fitzhugh-Nagumo system with a critical point at the origin is given by

$$\dot{x} = (x + 0.1) * ((x - 0.039)(0.9 - x)) - 0.0035 - y, \quad \dot{y} = 0.008(x - 2.54y).$$

3. The inequalities are given by:

$$\begin{array}{ll}
\text{for } I = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \left\{ \begin{array}{l} \sum Iw_1 - \hat{O}_2 x_1 < T \\ \sum Iw_2 < T \end{array} \right. \\
\text{for } I = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} & \left\{ \begin{array}{l} \sum Iw_1 - \hat{O}_2 x_1 > T \\ \sum Iw_2 < T \end{array} \right. \\
\text{for } I = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} & \left\{ \begin{array}{l} \sum Iw_1 - \hat{O}_2 x_1 < T \\ \sum Iw_2 > T \end{array} \right. \\
\text{for } I = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} & \left\{ \begin{array}{l} \sum Iw_1 - \hat{O}_2 x_1 > T \\ \sum Iw_2 > T. \end{array} \right.
\end{array} \tag{26.1}$$

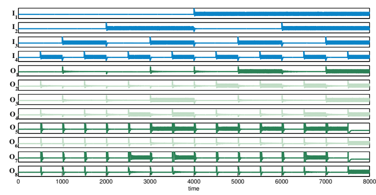
6. The truth table and time series are shown in Figure 26.3.
9. See Chapter 14 on Hamiltonian systems.

26.24 Chapter 24

1.

Input				Output			
I_1	I_2	I_3	I_4	O_1	O_5	O_7	O_8
0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0
0	0	1	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	0	0	0	0	0
0	1	0	1	0	0	1	0
0	1	1	0	0	1	0	0
0	1	1	1	0	1	1	0
1	0	0	0	0	0	0	0
1	0	0	1	0	1	0	0
1	0	1	0	1	0	0	0
1	0	1	1	1	1	0	0
1	1	0	0	0	0	0	0
1	1	0	1	0	1	1	0
1	1	1	0	1	1	0	0
1	1	1	1	1	0	0	1

(a)



(b)

Figure 26.3: (a) Truth table for a 2×2 bit binary multiplier. (b) Time series of a 2×2 bit binary multiplier based on Fitzhugh-Nagumo oscillations.

26.25 Chapter 25

Coursework-Type Questions

I still use these questions with my students. If any instructors have adopted my book, I can email solutions. Please contact me for further details.

Examination-Type Questions

Examination 1

1. (a) Eigenvalues and eigenvectors $\lambda_1 = 3.37$, $(1, 0.19)^T$; $\lambda_2 = -2.37$, $(1, -2.7)^T$. Saddle point, $\dot{x} = 0$ on $y = -\frac{3}{2}x$, $\dot{y} = 0$ on $y = \frac{1}{2}x$.

   (b) $\dot{r} > 0$ when $0 < \theta < \pi$, $\dot{r} < 0$ when $\pi < \theta < 2\pi$, $\dot{r} = 0$ when $\theta = 0, \pi$, $\dot{\theta} = 0$ when $\theta = \frac{(2n-1)}{4}\pi$, $n = 1, 2, 3, 4$.

2. (a) $\dot{V} = -(x - 2y)^2$, $\dot{V} = 0$ when $y = \frac{x}{2}$. On $y = \frac{x}{2}$, $\dot{x}, \dot{y} \neq 0$, therefore, the origin is asymptotically stable.

   (b) $r = \frac{1}{t+1}$, $\theta = t + 2n\pi$.

3. (a) $\lambda_1 = -1, \lambda_2 = -2+i, \lambda_3 = -2-i$. Origin is globally asymptotically stable.

   (b) $\dot{V} = -4y^4 - 2z^4 < 0$, if $y, z \neq 0$. Therefore, the origin is asymptotically stable, trajectories approach the origin forever.

4. (a) One limit cycle when $\mu < 0$, three limit cycles when $\mu > 0$, $\mu \neq 1$, and two limit cycles when $\mu = 1$.

   (b) Use Bendixson's criteria:

      (i) $\text{div}\mathbf{X} = -(1 + 3x^2 + x^4) < 0$;

      (ii) $\text{div}\mathbf{X} = 3x^3y^2$, on $x = 0$, $\dot{x} \geq 0$, on $y = 0$, $\dot{y} \geq 0$, no limit cycles in the quadrants and axes invariant;

      (iii) $\text{div}\mathbf{X} = (1 + y)^2$. On $y = -1$, $\dot{y} > 0$.

5. (a) $x_{1,1}=0, x_{1,2}=\frac{7}{11}$; $x_{2,1}=\frac{28}{65}, x_{2,2}=\frac{49}{65}$; $x_{3,1}=\frac{28}{93}, x_{3,2}=\frac{49}{93}, x_{3,3}=\frac{77}{93}, x_{3,4}=\frac{112}{407}, x_{3,5}=\frac{196}{407}, x_{3,6}=\frac{343}{407}$.

   (b) $z_{1,1}=\frac{1+\sqrt{13}}{2}, z_{1,2}=\frac{1-\sqrt{13}}{2}$; $z_{2,1}=1, z_{2,2}=-2$. Fixed points of period one are unstable.

6. (a) Area of inverted Koch snowflake is $\frac{\sqrt{3}}{10}$ units^2, $D_f=1.2619$.

   (b) Use L'Hopital.

7. (a) Period one $\left(\frac{5}{9},\frac{1}{9}\right)$, $\left(-1,-\frac{1}{5}\right)$, both fixed points are unstable.

   (b) See Chapter 6.

8. (a)

$$\mathbf{W}=\frac{1}{4}\begin{pmatrix} 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \end{pmatrix}.$$

   (b)

$$\mathbf{V}(\mathbf{a})=-\frac{1}{2}\left(a_1^2+2a_1a_2+4a_2^2+12a_1+20a_2\right) - \frac{4}{\gamma\pi^2}\left(\log\left(\cos(\pi a_1/2)\right)+\log\left(\cos(\pi a_2/2)\right)\right).$$

Examination 2

1. (a) $\dot{x}=-k_1x, \quad \dot{y}=k_1x-k_2y, \quad \dot{z}=k_2y$; $x(20)=4.54\times 10^{-5}, y(20)=0.3422, z(20)=0.6577$.

   (b) Period is approximately $T\approx -6.333$.

2. (a) See Section 10.5, Exercise 6.

   (b) $H(x,y)=\frac{y^2}{2}-\frac{x^2}{2}+\frac{x^3}{3}$, saddle point at origin, center at $(1,0)$.

3. (a) Three critical points when $\mu<0$, one when $\mu\geq 0$.

   (b) Chaos.

4. (a) $x_{1,1}=0, x_{1,2}=0.716$, $x_{2,1}=0.43, x_{2,2}=0.858$, no points of period three, $x_{4,1}=0.383, x_{4,2}=0.5, x_{4,3}=0.825, x_{4,4}=0.877$.

   (b) $z_{1,1}=-0.428+1.616i, z_{1,2}=1.428-1.616i$; $z_{2,1}=-1.312+1.847i, z_{2,2}=0.312-1.847i$; $z_{3,1}=-1.452+1.668i, z_{3,2}=-1.269+1.800i, z_{3,3}=-0.327+1.834i, z_{3,4}=0.352-1.891i, z_{3,5}=0.370-1.570i, z_{3,6}=1.326-1.845i$.

5. (a) Fixed points of period one $(0.888,0.888),(-1.688,-1.688)$; fixed points of period two $(1.410,-0.610),(-0.610,1.410)$.

   (b) Lyapunov exponent is approximately 0.4978.

6. (b) $J(0,1.3)$: Scattered dust, totally disconnected.

7. (a) Period-one points $(2.76,0.73),(3.21,-1.01),(3.53,1.05),(4.33,0.67)$.

8. (a)

$$\mathbf{W}=\frac{1}{6}\begin{pmatrix} 0 & -1 & 1 & 1 & -1 & 3 \\ -1 & 0 & 1 & 1 & 3 & -1 \\ 1 & 1 & 0 & -1 & 1 & 1 \\ 1 & 1 & -1 & 0 & 1 & 1 \\ -1 & 3 & 1 & 1 & 0 & -1 \\ 3 & -1 & 1 & 1 & -1 & 0 \end{pmatrix}.$$

(b) The width of the curve gives a measure of dispersion.

Examination 3

1. (a) Eigenvalues and eigenvectors $\lambda_1 = 2$, $(3,1)^T$; $\lambda_2 = -5$, $(1,-2)^T$. Saddle point.

   (b) $r_{n+1} = \frac{r_n}{r_n - (r_n - 1)e^{-2\pi}}$.

2. (a) Critical points at $(0,0)$ and $(0, \mu - 1)$. Critical point at origin is unstable for $\mu < 1$ and stable for $\mu > 1$. Other critical point has opposite stability.

   (b) One unstable limit cycle when $\mu < 0$. Critical point at origin is stable when $\mu < 0$ and unstable when $\mu > 0$.

3. (a) Critical points at $(0,0,0), (0,0,1), (0,1,0), (1,0,0), (1,0,1)$, and $(\frac{2}{3}, \frac{1}{3}, \frac{4}{3})$. Critical point away from axes is stable.

   (b) $z_{1,1} = 1.6939 - 0.4188i, z_{1,2} = -0.6939 + 0.4188i$, $z_{1,2} = -1.3002 + 0.6248i, z_{2,2} = 0.3002 - 0.6248i$.

4. (a) Critical point at $(0.5, 1.428)$, is an unstable focus.

   (b) There exists a limit cycle by the Poincaré-Bendixson theorem.

5. (a) Determine an expression for E_n in terms of E_{n+1}.

   (b) Fixed point at $k \approx 0.26$, is stable when $B = 1$, and fixed point $k \approx 0.92$ is unstable when $B = 4$. There is a bifurcation when $B \approx 0.36$.

6. (a) Period one fixed points at $(\frac{2}{5}, \frac{1}{5})$ and $(-\frac{2}{3}, -\frac{1}{3})$. Period two fixed points at $(\frac{10}{17}, -\frac{3}{17})$ and $(-\frac{6}{17}, \frac{5}{17})$.

   (b) $S_1 = \left[\frac{1}{6}, \frac{2}{6}\right] \cup \left[\frac{4}{6}, \frac{5}{6}\right]$. $D_f \approx 0.3869$.

7. (a) Eigenvalues are $\lambda_1 = 2.5150, \lambda_2 = -1.7635, \lambda_3 = -0.7516$. Long term population is $[0.8163; 0.1623; 0.0215]$.

   (b) Use fft.

8. (a) Lyapunov exponent=0.940166.

   (b) `mmax=2^k,h=sqrt(2)^(-k);angle(1)=pi/4;`

   `angle(2)=-pi/4;segment(b)=mod(m,2);m=floor(m/2).`