Data 345

Applied Linear Algebra for Statistical Learning

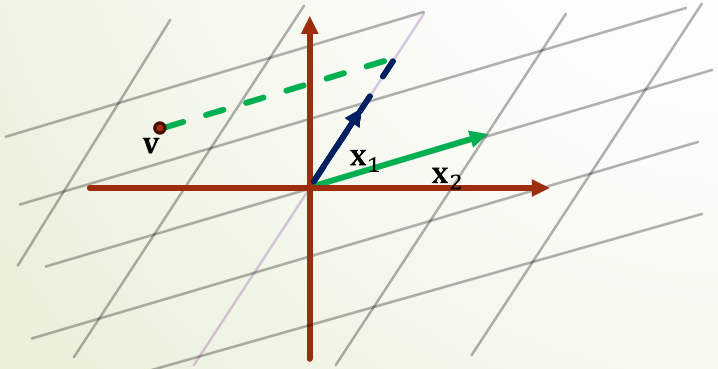
Class 8 (Sep. 22, 2025)

Generating Sets

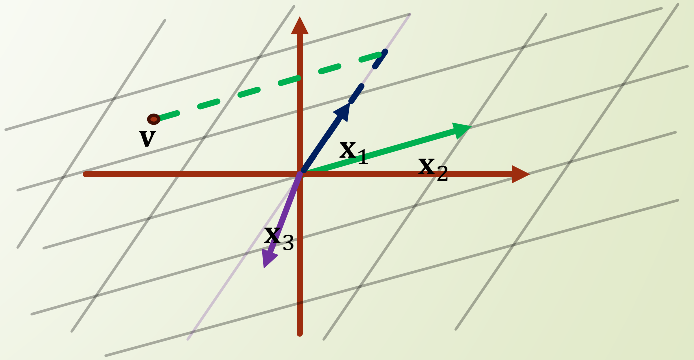
- Remember that a linear combination of vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k$ is any vector $\mathbf{v}$ of the form $\mathbf{v} = \lambda_1\mathbf{x}_1 + \cdots + \lambda_k\mathbf{x}_k$. In other words, it's a recipe for reaching the point $\mathbf{v}$ in $\mathbb{R}^n$ by only traveling along certain directions $\mathbf{x}_1$ through $\mathbf{x}_k$.
- Consider the vectors $\mathbf{x} = [1,0,0], \mathbf{y} = [0,1,0] \in \mathbb{R}^3$. Then, a linear combination of these two vectors will allow you to travel to any point in the (flat) Euclidean plane, but since x and y don't allow for vertical travel, not every vector in $\mathbb{R}^3$ is a linear combination of x and y.
- We are primarily interested in finding, for a particular vector space V, a set of vectors $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ so that every vector in V is a linear combination of $\mathbf{x}_1$ through $\mathbf{x}_k$.

Generating Sets

- Let V be a real vector space and $\mathcal{A} = \{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ a set of vectors in V. The **linear span** of $\mathcal{A}$, written $\text{span}[\mathcal{A}]$, is the set of all linear combinations of vectors $\mathbf{x}_1$ through $\mathbf{x}_k$. (This may also be written as $\text{span}[\mathbf{x}_1, \ldots, \mathbf{x}_k]$)
- If $\text{span}[\mathcal{A}] = V$, then we call $\mathcal{A}$ a **generating set** for the vector space V.
- Suppose that V is a vector space and $\mathcal{A}$ is a generating set for V. If the set $\mathcal{A}$ is linearly independent, then the set of vectors $\mathcal{A}$ is said to be a **basis** for the vector space V.
- Keeping with the "travel" analogy, a basis is a set of directions for traveling to any point in a vector space V where no directions are redundant.



$\{\mathbf{x}_1, \mathbf{x}_2\}$ is a basis for $\mathbb{R}^2$



$\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is not a basis for $\mathbb{R}^2$

Basis Vectors - Examples

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad \mathcal{B}_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \qquad \mathcal{B}_3 = \left\{ \begin{bmatrix} 0.5 \\ 0.8 \\ 0.4 \end{bmatrix}, \begin{bmatrix} 1.8 \\ 0.3 \\ 0.3 \end{bmatrix}, \begin{bmatrix} -2.2 \\ -1.3 \\ 3.5 \end{bmatrix} \right\}$$

- Each of the sets $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$ is a linearly independent generating set for $\mathbb{R}^3$ (not obvious by looking – it would need to be verified). Therefore, each is a basis for the vector space $\mathbb{R}^3$. The set $\mathcal{B}_1$ is called the **standard** or **canonical** basis for $\mathbb{R}^3$.

$$\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -4 \end{bmatrix} \right\}$$

- The set $\mathcal{A}$ is linearly independent, but is not a generating set for $\mathbb{R}^4$ (and therefore not a basis for $\mathbb{R}^4$).

Basis Vectors and Dimension

- Every vector space V has at least one basis*, but as we saw with the previous example, it is possible that a vector space has multiple sets which form a basis (i.e., you should never assume a basis to be a unique set of vectors). However, every basis will contain the same number of vectors.
- Remember that previously we noted that $\mathbb{R}^n$ is an "n-dimensional vector space." This is certainly believable because there are n components to each vector, but this definition isn't equipped to handle subspaces.
- Let V be a vector space and $\mathcal{B}$ a basis for V. We say that the **dimension** of V, written $\dim(V)$, is the number of vectors in $\mathcal{B}$. If $\mathcal{B}$ is a finite set then V is said to be **finite-dimensional**, and if $\mathcal{B}$ is infinite then V is said to be **infinite-dimensional**.
- For finite-dimensional vector spaces, if $U \subseteq V$ is a subspace of V, then $\dim(U) \leq \dim(V)$ and $\dim(U) = \dim(V)$ if and only if $U = V$. The statement isn't true when $\dim(V) = \infty$.

*For finite-dimensional vector spaces this is easily observable. It's still true for infinite-dimensional vector spaces but is a deeper fact.

Basis Vectors and Dimension

- We said that $\mathbb{R}^n$ is an n-dimensional vector space because each vector has n components. In reality, the number of components doesn't fully reflect the dimension.
- Using the number of components as a proxy for $\dim(\mathbb{R}^n)$ is allowable because of the "standard basis" which consists of n vectors e_i, where e_i has a "1" in the i^{th} component and 0 elsewhere.

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\{e_1, e_2, e_3, e_4\}$ is a basis for $\mathbb{R}^4$

- However, a vector like $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ has two components, yet $\operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$ is only a one-dimensional vector space, because it is generated by 1 vector.

Determining a Basis

- Suppose that $U = \text{span}[\mathbf{x}_1, \dots, \mathbf{x}_m] \subseteq \mathbb{R}^n$. To determine what subset of $\{\mathbf{x}_1, \dots, \mathbf{x}_m\}$ is a basis for U, we follow this procedure:
- Write the spanning vectors as columns of a matrix A.
- Determine the row-echelon form of A.
- The spanning vectors associated with the pivot columns of A are a basis of U.

Determining a Basis - Example

- Suppose that $U \subseteq \mathbb{R}^5$ is spanned by the vectors below. Determine a basis for U.

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -1 \\ -1 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \\ -2 \end{bmatrix} \quad \mathbf{x}_3 = \begin{bmatrix} 3 \\ -4 \\ 3 \\ 5 \\ -3 \end{bmatrix} \quad \mathbf{x}_4 = \begin{bmatrix} -1 \\ 8 \\ -5 \\ -6 \\ 1 \end{bmatrix}$$

$$\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & -1 & -4 & 8 \\ -1 & 1 & 3 & -5 \\ -1 & 2 & 5 & -6 \\ -1 & -2 & -3 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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A = matrix([[1,2,3,-1],[2,-1,-4,8], [-1,1,3,-5], [-1,2,5,-6], [-1,-2,-3,1]])
A.rref()
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$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivots for $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_4$. Thus the set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4\}$ is a basis for U.

Matrix Rank

- Let $A \in \mathbb{R}^{m \times n}$ be a matrix. The number of linearly independent columns of A is called the **rank** of A, written $\mathrm{rank}(A)$. This means that $0 \leq \mathrm{rank}(A) \leq n$ for any matrix A.
- This number has some extremely important and useful properties and interpretations.
- $\mathrm{rank}(A) = \mathrm{rank}(A^T)$. This means that the number of linearly independent columns is always equal to the number of linearly independent rows. Thus, $0 \leq \mathrm{rank}(A) \leq \min\{m, n\}$ for any matrix A.
- The columns of A span a subspace U of $\mathbb{R}^m$, and $\dim(U) = \mathrm{rank}(A)$.
- The rows of A span a subspace W of $\mathbb{R}^n$, and $\dim(W) = \mathrm{rank}(A)$.
- For any $A \in \mathbb{R}^{n \times n}$, the matrix A is invertible (has an inverse A^{-1}) if and only if $\mathrm{rank}(A) = n$.
- For all $A \in \mathbb{R}^{m \times n}$ and any $\mathbf{b} \in \mathbb{R}^m$, the linear equation $A\mathbf{x} = \mathbf{b}$ is solvable if and only if $\mathrm{rank}(A) = \mathrm{rank}(A|\mathbf{b})$.