Data 345

Applied Linear Algebra for Statistical Learning

Class 7 (Sep. 18, 2025)

Vector (Sub)Spaces

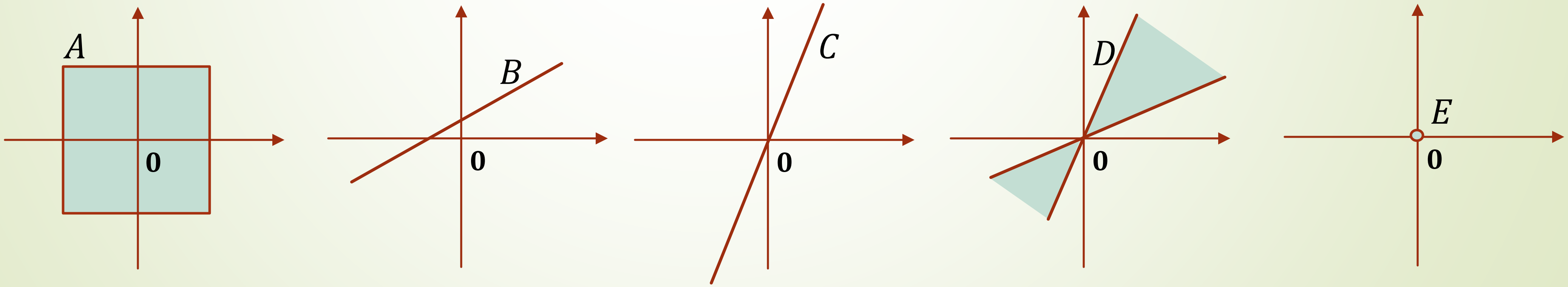
- Recall: we denoted by $\mathbb{R}^n$ the set of all n-dimensional (real) vectors. We defined two operations for vectors: $+$ (vector addition) and $\cdot$ (scalar multiplication).
- This is a specific instance of a more general setting of "vector spaces."
- We will not worry extensively about the abstract definition. A real finite-dimensional vector space "is" some version of $\mathbb{R}^n$ under appropriate re-labeling.
- We will be primarily concerned with vector *sub*spaces, which are smaller vector spaces embedded in a larger one.

Vector (Sub)Spaces

- Let $V = \mathbb{R}^n$ be a vector space and suppose that U is a subset of V. Then we say that U is a **subspace** of V if:
  - U is not empty, i.e., U has at least 1 vector in it;
  - U is "closed" under vector addition, meaning that if $\mathbf{x}, \mathbf{y} \in U$, then $\mathbf{x} + \mathbf{y} \in U$ as well;
  - U is "closed" under scalar multiplication, meaning that if $\boldsymbol{x} \in U$ and $\lambda \in \mathbb{R}$, then $\lambda\mathbf{x} \in U$ as well.
- Note that these conditions always imply that if U is a subspace then $\mathbf{0} \in U$: take any $\mathbf{x} \in U$. Then $(-1) \cdot \mathbf{x} \in U$ and $\mathbf{x} + (-1)\mathbf{x} \in U$, but $\mathbf{x} + (-1)\mathbf{x} = 0$.

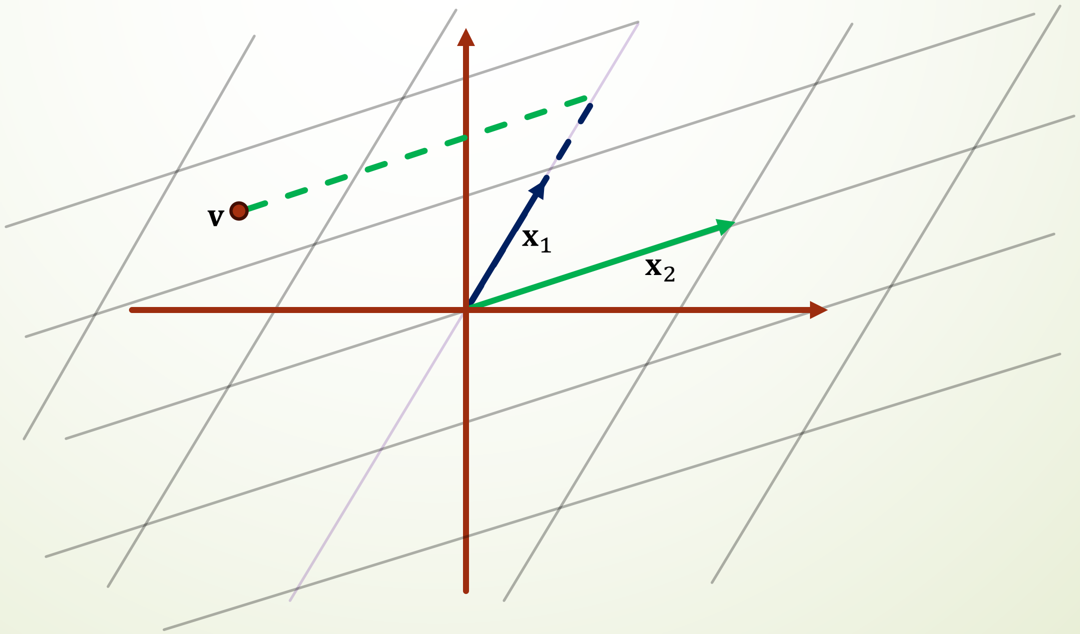
Vector (Sub)Spaces - Examples

- For any vector space V, the sets V and $\{\mathbf{0}\}$ are both trivial subspaces of V.
- The solution set of a system of linear equations $A\mathbf{x} = \mathbf{0}$ with n unknowns is a subspace of $\mathbb{R}^n$.
- Any two subspaces of the same vector space overlap in a subspace. In other words, if U and W are subspaces of V, then the intersection $U \cap W$ is also a subspace of V.



Linear Combinations

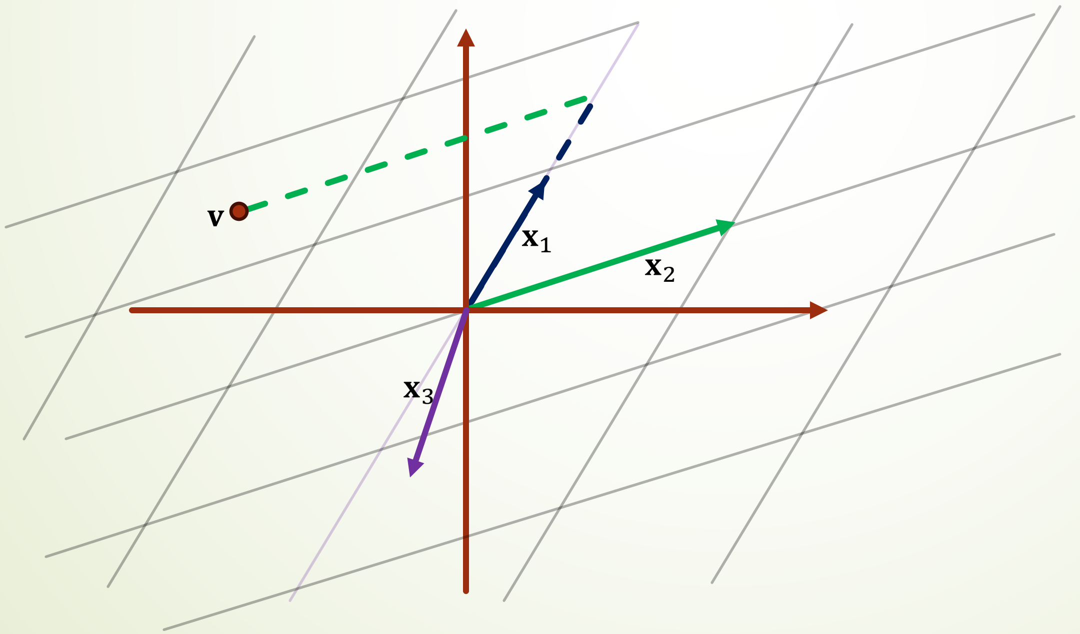
- Let V be a vector space, and suppose that $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ are vectors in V. Then every vector $\mathbf{v} \in V$ of the form $\mathbf{v} = \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \cdots + \lambda_k\mathbf{x}_k$ (where λ_1 through λ_k are real numbers) is called a **linear combination** of $\mathbf{x}_1$ through $\mathbf{x}_k$.
- If you imagine the set of vectors $\mathbf{x}_i$ as directions, then a linear combination of the vectors $\mathbf{x}_i$ is simply any location on a "grid" in which the gridlines are lines parallel to the $\mathbf{x}_i$.



(Note: since λ_i may be any real number you can travel between gridlines)

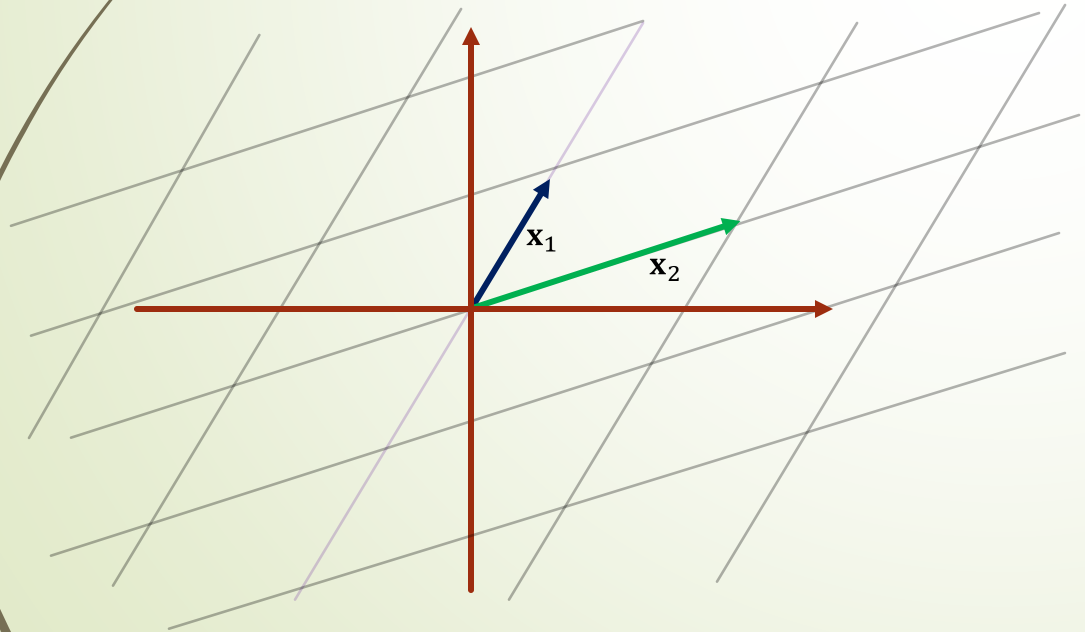
Linear Combinations

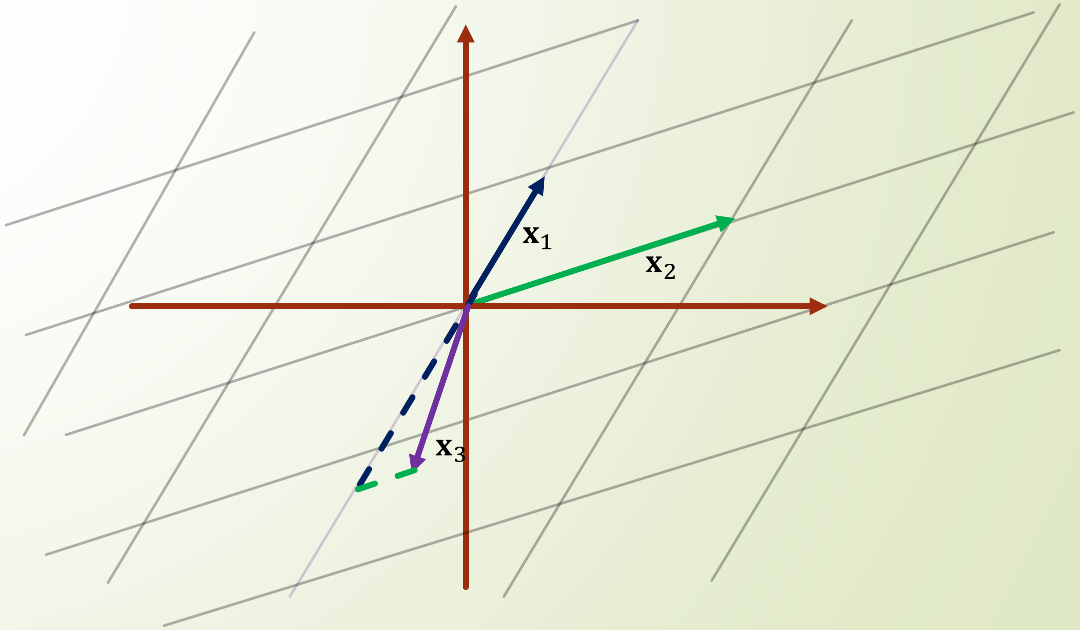
- We are particularly interested in linear combinations without redundancy of information.
- For instance, in our earlier picture, adding a new vector $\mathbf{x}_3$ introduces some redundancy, because we could already travel to any point in the Euclidean plane with just $\mathbf{x}_1$ and $\mathbf{x}_2$.



Linear Independence

- Let V be a vector space and $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ a set of vectors in V. These vectors are said to be **linearly independent** if $\mathbf{0} = \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \cdots + \lambda_k\mathbf{x}_k$ implies that $\lambda_1 = \lambda_2 = \cdots = \lambda_k = 0$. In other words, the only way to reach the vector $\mathbf{0}$ by the directions $\mathbf{x}_i$ is by traveling a net distance of 0 in each direction.
- If the set $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ is not linearly independent then it is said to be **linearly dependent**. This means the equation $\mathbf{0} = \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \cdots \lambda_k\mathbf{x}_k$ has some non-trivial solution (at least one λ_i is not 0).





Linear Independence

- A set of k vectors can only be linearly independent or linearly dependent. There is no third option.
- If any of the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ is the zero vector then the set is automatically linearly dependent. The same is true if any two vectors are equal.
- A set of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ is linearly dependent if and only if at least one of them is a linear combination of the others.
- **We already have a way** to check if a set of vectors is independent!

Checking Linear Independence

- Let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ be a set of vectors in $\mathbb{R}^n$. The set is linearly dependent if and only if there is a nontrivial solution to the equation $0 = \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \cdots + \lambda_k\mathbf{x}_k$. If we let A be the $n \times k$ matrix $[\mathbf{x}_1|\mathbf{x}_2|\cdots|\mathbf{x}_k]$, then this is literally equivalent to getting a non-trivial solution to the system of equations:

$$A \cdot \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- Since $[0, 0, \dots, 0]^T$ is <u>always</u> a solution to this equation, this implies the solution set is infinite. In particular, not every column of A contains a pivot when A is row-reduced via Gaussian elimination.
- Therefore, a set $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ of vectors is linearly independent if and only if we arrange the set into a matrix A (where the columns of A are the vectors $\mathbf{x}_i$) and every column contains a pivot after performing Gaussian elimination.

Checking Linear Independence

- Example: Determine whether $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\} \subset \mathbb{R}^4$ is linearly independent.

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix} \qquad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} \qquad \mathbf{x}_3 = \begin{bmatrix} -1 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

$$\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\} \longrightarrow \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & -2 \\ -3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix} \xrightarrow[\text{(Gaussian elimination)}]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

```python
A = matrix([[1, 1, -1], [2, 1, -2], [-3, 0, 1], [4, 2, 1]])
A.row_add(1,0,-2)
A.row_add(2,0,3)
A.row_add(3,0,-4)
A.row_scale(1,-1)
A.row_add(2,1,-3)
A.row_add(3,1,2)
A.row_scale(2,-1/2)
A.row_add(3,2,-5)
A.row_add(0,2)
A.row_add(0,1,-1)
A
```

✓ 0.0s Python

```
matrix([  1  0.0  0.0]
       [0.0    1  0.0]
       [0.0  0.0    1]
       [0.0  0.0  0.0])
```

3 columns, 3 pivots. Therefore, the set is linearly independent.

Checking Linear Independence

- Determine whether the set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\} \subset \mathbb{R}^4$ is linearly independent:

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ -1 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} -4 \\ -2 \\ 0 \\ 4 \end{bmatrix} \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 3 \\ -1 \\ -3 \end{bmatrix} \quad \mathbf{x}_4 = \begin{bmatrix} 17 \\ -10 \\ 11 \\ 1 \end{bmatrix}$$

$$\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\} \longrightarrow \begin{bmatrix} 1 & -4 & 2 & 17 \\ -2 & -2 & 3 & -10 \\ 1 & 0 & -1 & 11 \\ -1 & 4 & -3 & 1 \end{bmatrix} \xrightarrow[\text{(Gaussian elimination)}]{} \begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -15 \\ 0 & 0 & 1 & -18 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

```python
A = matrix([[1, -4, 2, 17], [-2, -2, 3, -10], [1, 0, -1, 11], [-1, 4, -3, 1]])
A.rref()
```

✓ 0.0s Python

$$\begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -15 \\ 0 & 0 & 1 & -18 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$