Data 345

Applied Linear Algebra for Statistical Learning

Class 5 (Sep. 11, 2025)

Systems of Linear Equations

- So: we have vectors and matrices; what's the connection to simultaneous linear equations?

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 39 \\ 2x_1 + 3x_2 + x_3 &= 34 \\ x_1 + 2x_2 + 3x_3 &= 26 \end{aligned} \longrightarrow \begin{aligned} \boldsymbol{v}_1 \cdot \boldsymbol{x} &= 39 \\ \boldsymbol{v}_2 \cdot \boldsymbol{x} &= 34 \\ \boldsymbol{v}_3 \cdot \boldsymbol{x} &= 26 \end{aligned}$$

$$\underbrace{\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{bmatrix} 39 \\ 34 \\ 26 \end{bmatrix}}_{\boldsymbol{b}}$$

- Taking the perspective of matrix multiplication and vector dot products, we see that a compact way to write the system of equations is $A\boldsymbol{x} = \boldsymbol{b}$, where A is a 3×3 matrix of coefficients, $\boldsymbol{x}$ is a 3×1 matrix (column vector) of unknowns, and $\boldsymbol{b}$ is a 3×1 matrix (column vector) representing the right-hand side of each equation.

Systems of Linear Equations

- Speaking generally, we may compactly write any system of m linear equations in n unknowns as:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n &= b_m \end{aligned} \quad \longrightarrow \quad \overset{m \times n}{A}\underset{n \times 1}{\boldsymbol{x}} = \overset{m \times 1}{\boldsymbol{b}}$$

- Remember that from the first day of class a single linear equation in n variables has an $n - 1$ dimensional solution set.
- Barring some special exceptions, adding more constraints (equations) reduces the maximal dimensionality of the solution set by ~1 per additional equation.
- Therefore, there are essentially 3 possibilities for solution sets: the system has no solution (over-constrained), the system has infinitely many solutions (under-constrained), or has a single unique solution (perfectly constrained).

Systems of Linear Equations

- Let's look at the following example:

$$\begin{bmatrix} 1 & 0 & 8 & -4 \\ 0 & 1 & 2 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 42 \\ 8 \end{bmatrix}$$

- This is a system which has 2 equations and 4 unknowns. Since there are fewer equations than variables, we'd generally expect an infinite solution set.
- Because of the way matrix multiplication works, this is the same as finding a vector $[x_1, x_2, x_3, x_4]$ where

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_2 + \begin{bmatrix} 8 \\ 2 \end{bmatrix} x_3 + \begin{bmatrix} -4 \\ 12 \end{bmatrix} x_4 = \begin{bmatrix} 42 \\ 8 \end{bmatrix}$$

- Since we're expecting infinitely many solutions it's hard to capture them right away – but if we let $x_3 = 0$ and $x_4 = 0$, the problem is suddenly much easier:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_2 = \begin{bmatrix} 42 \\ 8 \end{bmatrix} \longrightarrow \boldsymbol{x} = \begin{bmatrix} 42 \\ 8 \\ 0 \\ 0 \end{bmatrix}$$

Systems of Linear Equations

- So, $[42, 8, 0, 0]$ is *a* solution to our system of equations, but there are possibly many more. In this case we'd call $[42, 8, 0, 0]$ a **particular solution** or **special solution**.
- Our ability to find an easy particular solution hinged on the fact that we deleted two variables. We were able to do this because the first two columns had a special form, which would allow us to write the third column in terms of the first two:

$$\begin{bmatrix} 8 \\ 2 \end{bmatrix} = 8\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Thus, since $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 8\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 8 \\ 2 \end{bmatrix}$, adding any multiple of the vector $A \cdot \begin{bmatrix} 8 \\ 2 \\ -1 \\ 0 \end{bmatrix}$ would add $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to the right-hand side of the equation.

Systems of Linear Equations

$$A\boldsymbol{x} = \boldsymbol{b}$$

$$A\begin{bmatrix}42\\8\\0\\0\end{bmatrix} = \begin{bmatrix}42\\8\end{bmatrix}$$

$$A\begin{bmatrix}42\\8\\0\\0\end{bmatrix} + A \cdot \lambda_1\begin{bmatrix}8\\2\\-1\\0\end{bmatrix} = \begin{bmatrix}42\\8\end{bmatrix} + \begin{bmatrix}0\\0\end{bmatrix}$$

$$A\left(\begin{bmatrix}42\\8\\0\\0\end{bmatrix} + \lambda_1\begin{bmatrix}8\\2\\-1\\0\end{bmatrix}\right) = \begin{bmatrix}42\\8\end{bmatrix}$$

$$A\begin{bmatrix}42 + 8\lambda_1\\8 + 2\lambda_1\\-\lambda_1\\0\end{bmatrix} = \begin{bmatrix}42\\8\end{bmatrix}$$

A *more general* solution vector; can still get the initial particular solution by setting $\lambda_1 = 0$.

Systems of Linear Equations

- We could also apply this thinking to the fourth column, since:

$$\begin{bmatrix}-4\\12\end{bmatrix} = -4 \cdot \begin{bmatrix}1\\0\end{bmatrix} + 12 \cdot \begin{bmatrix}0\\1\end{bmatrix}$$

- Thus, any scalar multiple of the vector $\begin{bmatrix}-4\\12\\0\\-1\end{bmatrix}$ would also effectively "add 0" to both sides of the equation. This means we may write the equation as

$$A\left(\begin{bmatrix}42\\8\\0\\0\end{bmatrix} + \lambda_1\begin{bmatrix}8\\2\\-1\\0\end{bmatrix} + \lambda_2\begin{bmatrix}-4\\12\\0\\-1\end{bmatrix}\right) = \begin{bmatrix}42\\8\end{bmatrix} \qquad (\lambda_1, \lambda_2 \in \mathbb{R})$$

Systems of Linear Equations

- The set of all 4-dimensional vectors satisfying this condition is called the **general solution** of the system of equations.
- In this case, the general solution would be

$$\left\{ \boldsymbol{x} \in \mathbb{R}^4 : \ \boldsymbol{x} = \begin{bmatrix} 42 \\ 8 \\ 0 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} 8 \\ 2 \\ -1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} -4 \\ 12 \\ 0 \\ -1 \end{bmatrix} \quad (\lambda_1, \lambda_2 \in \mathbb{R}) \right\}$$

- Since λ_1 and λ_2 may be *any* real number, there are infinitely many possible solutions, and in particular, the defining expression constitutes a plane in $\mathbb{R}^4$.