Problem Books in Mathematics

Cornel Ioan Vălean

(Almost) Impossible Integrals, Sums, and Series

With foreword by
Paul J. Nahin

Springer

Problem Books in Mathematics

Series Editor:

Peter Winkler
Department of Mathematics
Dartmouth College
Hanover, NH
USA

More information about this series at http://www.springer.com/series/714

Cornel Ioan Vălean

(Almost) Impossible Integrals, Sums, and Series

With a Foreword by Paul J. Nahin

Springer

Cornel Ioan Vălean
Teremia Mare, Timiş County
Romania

ISSN 0941-3502 ISSN 2197-8506 (electronic)
Problem Books in Mathematics
ISBN 978-3-030-02461-1 ISBN 978-3-030-02462-8 (eBook)
https://doi.org/10.1007/978-3-030-02462-8

Library of Congress Control Number: 2018966810

This Springer imprint is published by the registered company Springer Nature Switzerland AG.
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

To my forever living parents,
Ileana Ursachi and Ionel Valean

Foreword

Question: "Can you tell me who can help me do some elliptic integrals?"
Answer: "We've tried to get rid of anyone like that."
—Exchange between a physics graduate student and a professor of mathematics[1]

Shortly after my book *Inside Interesting Integrals* was published by Springer in August 2014, I began to receive e-mails from all over the world. They were from readers who were writing to show me how to do one or the other of the problems in my book in a way "easier," or "more direct," than was the solution I gave. Almost all were fascinating reading, and those communications confirmed my belief that people who buy math books are quite different from those who don't.

One of those correspondents was the author of the book you now hold in your hands. Cornel, in fact, wrote to me (from his home in Romania) numerous times over the following months. One of his first e-mails was to take exception to my claim that a result attributed to the great Cauchy, himself,

$$\int_0^\infty \frac{e^{\cos(x)}\sin\{\sin(x)\}}{x}\mathrm{d}x = \frac{\pi}{2}(e-1),$$

would be "pretty darn tough [to do]" by means other than contour integration (which is how I do it in my book). Cornel's clever solution, however, using just "routine methods" available to any undergraduate in math, physics, or engineering by the end of their second year, *is* indeed much easier, and later in this book, you'll see just how he does it. I was impressed, yes, but soon put it aside and turned to other matters.

[1]From an e-mail I received, after the publication of *Inside Interesting Integrals*, from Lawrence Glasser, Professor Emeritus of Physics at Clarkson University, Potsdam, New York, USA. Larry was the grad student. I'll explain the significance of this quote by the end of this essay.

Later communications from Cornel, however, increased my interest. But what really convinced me that Cornel wasn't "just" a clever math aficionado but rather is a seriously talented mathematician was when he sent me the calculation of

$$\int_0^1 \int_0^1 \int_0^1 \left(\left\{\frac{x}{y}\right\}\left\{\frac{y}{z}\right\}\left\{\frac{z}{x}\right\}\right)^n \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

where $\{x\}$ denotes the fractional part of x and the integer $n \geq 1$. I had never seen anything like it before. It is, of course, immediately clear that the integral exists, as the integrand is always in the interval 0 to 1 over the entire finite region of integration (the volume of the unit cube). But how to *do* the triple integration completely baffled me. And then, turning to the final page of Cornel's five-page derivation that he had mercifully included in his e-mail (else I would have gone mad with frustration), his answer was just too wonderful to simply be made-up: using his general result that is a function of n, he gave the explicit solutions for the first five values of n, the first two of which I'll repeat here:

$$\int_0^1 \int_0^1 \int_0^1 \left\{\frac{x}{y}\right\}\left\{\frac{y}{z}\right\}\left\{\frac{z}{x}\right\} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 1 - \frac{3}{4}\zeta(2) + \frac{1}{6}\zeta(3)\zeta(2)$$

and

$$\begin{aligned}\int_0^1 \int_0^1 \int_0^1 \left(\left\{\frac{x}{y}\right\}\left\{\frac{y}{z}\right\}\left\{\frac{z}{x}\right\}\right)^2 \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 1 &- \frac{1}{2}\zeta(2) - \frac{1}{2}\zeta(3) + \frac{7}{48}\zeta(6)\\ &+ \frac{1}{18}\zeta(3)\zeta(2) + \frac{1}{18}\zeta^2(3)\\ &+ \frac{1}{12}\zeta(3)\zeta(4),\end{aligned}$$

where ζ denotes the famous Euler–Riemann zeta function, well-known to physicists and mathematicians alike.

You'll see later in this book how he arrived at these amazing results (wait until you see the $n = 5$ case!), but one thing I could immediately do was a computer check. The first two theoretical results on the right-hand side of the equality signs are:

$$(n = 1)\ 0.095850\ldots \quad \text{and} \quad (n = 2)\ 0.023409\ldots$$

while a direct numerical evaluation of the integrals gives the values of

$$(n = 1)\ 0.095844\ldots \quad \text{and} \quad (n = 2)\ 0.023335\ldots .$$

Here we have an agreement between Cornel's theory and numerical calculations out to "only" three or four decimal places, but, when you consider how wild[2] are the fluctuations of the integrand as we move from point to point in the unit cube, what impresses me is that we have even *that* much agreement. As I looked at these results, I knew I had a correspondent of real talent.

My curiosity now fully engaged, I plunged into Cornel's detailed derivation of the triple integral, and, for page after page, it was just one incredibly clever, occasionally devious, sly trick after another. I knew, as I staggered from one line to the next (often after scribbling away for half an hour or more before I caught on to what he was doing), that I was following the path of the most creative person. You'll find that this entire book is like that, with one spectacular computation after another. I predict you'll have a hard time in putting this book down once you start. You'll find results in here that you have never seen before or, if you have, with an ingenious derivation that you haven't seen before. I predict, if you love mathematics, that you are in for a great time.

As you must surely now be wondering, just as I did, who *is* Cornel Ioan Vălean? Not being a particularly subtle person, I simply asked him. Cornel revealed to me that he was 37 years old in 2015, holds a degree in financial accounting, and, while he has authored or co-authored several papers in various European math journals (including *The Gazette of the Royal Spanish Mathematical Society* and the *Journal of Classical Analysis*), he is not a professional mathematician in an academic post. Rather, in the words of the great French mathematician Henri Poincaré (1854–1912), he is a person blessed with a "special intuition [that allows someone] to perceive at a glance" the solution to a problem.[3] Today, alas, more than a century later, not much more than that is known about how the mathematical mind works. In Cornel's own words about this, words he wrote to me when I asked him specifically about how he works, "I have many ideas for research (I don't know where they come from but it's like a flood), maybe it's simply the natural expression of a crazy passion."

---

[2]If you make a three-dimensional sketch of $f = \left\{\frac{x}{y}\right\}$ as x and y each vary over the interval 0 to 1, you'll quickly appreciate why I use the word *wild*.

[3]From an address Poincaré gave in 1908, titled "Mathematical Creation," to the Société de Psychologie in Paris. You can find an English translation reprinted in a collection of Poincaré's works, *The Foundations of Science* (translation by G. B. Halstead), The Science Press 1929, pp. 383–394. Some years later the famous French mathematician Jacques Hadamard (1865–1963) tried his hand, too, at answering the puzzle of mathematical creation, with a little book called *The Psychology of Invention in the Mathematical Field*, Princeton 1945, but I don't believe he offered any new insights beyond Poincaré's.

Encountering someone like Cornel, with what Poincaré called a "special intuition," is an exciting experience. An illustration of such an experience was nicely described[4] a few years ago by Tulane University's mathematics professor Victor Moll, in the story of how his book *Irresistible Integrals*, written in collaboration with the late George Boros, came to be.[5] Moll had actually been aware of Boros long before Boros came to be Moll's doctoral student at Tulane. That is because Boros had a widespread reputation in the New Orleans math community as "the person who could do any integral." Still, when Boros approached Moll about becoming a PhD student, Moll felt obligated to warn him: "George, nobody is going to give you a doctorate in mathematics for doing integrals."[6] Boros said he understood that, and the two men eventually settled on a program that satisfied both themselves and the Tulane "math establishment." But neither one forgot their mutual love of "doing integrals," and so, along the way, they wrote their wonderful *Irresistible Integrals*. Sadly, Boros didn't live to see its 2004 publication, having died in February 2003 of cancer at age 55.

Clearly, Moll's friendship with Boros, combined with the special talents each man could bring to a sharp focus on the challenges of "doing integrals," had a profound impact on Moll.[7] As he relates at the end of his essay, when asked what he did in mathematics *before* Boros, he'd reply "classical analysis." *Now*, when asked the same question, his answer is "I compute integrals for a living."[8]

That change in Moll, due in part to Boros, goes a long way in explaining the attraction books on definite integrals (and sums and series, too) have. It's for the same reason for why you are reading this new one by Cornel—they are a lot of fun! Each new integral presents new difficulties, often ones never before encountered by anyone, and to succeed in finding the Excalibur that brings it to ground (perhaps

[4] Victor Moll, "The Evaluation of Integrals: A Personal Story," *Notices of the American Mathematical Society*, March 2002, pp. 311–317.

[5] Reading *Irresistible Integrals* (Cambridge 2004) was, in fact, the inspiration for my *Inside Interesting Integrals*, in which I told my readers where Moll and Boros almost certainly got their title: from a note written in 1926 by the great English mathematician G. H. Hardy (1877–1947) to a student at Trinity College, Cambridge, who had written Hardy to ask for help in doing some particularly tough integrals. A busy man, Hardy at first tried to ignore the request, but, in the end, he wrote to his correspondent, "I tried very hard not to spend time on your integrals, but to me the challenge of a definite integral is irresistible."

[6] This attitude toward "doing integrals" perhaps explains the dismissive quote that opens this Foreword. That attitude changes quickly, however, when someone needs the integral that has just appeared in *their* work evaluated. (I don't think Moll shared that dismissive attitude when he warned Boros but rather was simply alerting him to the prevailing atmosphere he'd find in any university math department.)

[7] The most famous such intellectual interaction in mathematics is, of course, that of Hardy and the Indian genius Srinivasa Ramanujan (1887–1920). In a talk Hardy gave in 1936 at Harvard University, long after Ramanujan's premature death, he described the collaboration with his friend as "the only romantic incident in my life."

[8] And, indeed, Moll's latest books are *Special Integrals in Gradshteyn and Ryzhik: The Proofs*, volume 1 (2014) and volume 2 (2015), both published by CRC Press. Every serious student in physics, engineering, and mathematics will learn a lot from them.

even to be the *first* person to do that) is every bit as thrilling as being the first to climb Mount Everest or to break the 4-minute mile. The only difference is that you don't have to risk falling off a cliff or having a heart attack while doing it!

Okay, enough of my thoughts—off you go now, for lots of thrilling adventures with Cornel.

Professor Emeritus of Electrical Engineering Paul J. Nahin
University of New Hampshire
Durham, NH, USA
paul.nahin@unh.edu
August 2018

Preface

Mathematical problems have the power to ignite passions, create dreams, and change destinies. The first encounter years ago with the splendid result,

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6},$$

about which I was not aware at all at that time, is still alive today. *Wow! This result looks amazing! Let's give it a try!*, was my first reaction, but I have to confess that my initial courage and desire to come up with a proof to the appealing result in front of my eyes were not enough. I was completely absorbed by the beauty of the result: *How could the sum of the reciprocals of the positive square integers give such a beautiful result involving π ?*

I found that value counterintuitive and very charming, appealing at the same time. *How is that connected to π ?* The questions were coming one after another, but I had to wait a while until I found some answers.

The series struck me so profoundly that things were never the same as before and the mathematical world seemed to me more amazing than at any time; I realized that behind the series there must be absolutely stunning mathematical connections, a magical world which I wanted to know so much.

Some months after the encounter with this earth-shaking result, I finally found the story about the problem, called the Basel problem, and it took humanity almost a century until the problem was solved. Initially the problem was posed by Pietro Mengoli in 1650 and solved for the first time by Leonhard Euler in 1734 after the unsuccessful attempts of more leading mathematicians. I was finally fulfilled because I knew more about the series that apparently moved my world!
Probably it's more than that, more fantastic results have been waiting for me to amaze me, the thought that steadily came to mind after the fascinating encounter with the Basel problem. And so it happened, I was right! Another two such examples to mention are the quadratic series of Au-Yeung,

$$H_1^2+\left(\frac{H_2}{2}\right)^2+\left(\frac{H_3}{3}\right)^2+\cdots=\frac{17}{4}\zeta(4)$$

and its more advanced version, the cubic version,

$$H_1^3+\left(\frac{H_2}{2}\right)^3+\left(\frac{H_3}{3}\right)^3+\cdots=\frac{1}{2}\left(\frac{93}{8}\zeta(6)-5\zeta^2(3)\right),$$

where the harmonic number is defined by $H_n=\sum_{k=1}^{n}\frac{1}{k}$ and ζ denotes the Riemann zeta function. *What gorgeous results!* I used to say to myself while I was admiring them. There are more challenges to take in front of me, which I took with such a great pleasure, and this life story has continued until today!

Coming now to the book, *why a book containing integrals, sums, and series and not only integrals, or sums, or series?* A direct answer would be that they are simply strongly interrelated and we need to combine them to solve the problems given in the book. For instance, let's use the power of the example to understand the importance of the connections between integrals and series. A good example from this category would be the calculation of the double integral,

$$\int_0^{\pi/2}\left(\int_0^{\pi/2}\frac{\log(\sin^{\cos(x)}(x))\log(\sin^{\cos(y)}(y))}{\sin(x)+\sin(y)-\sin(x)\sin(y)}\mathrm{d}x\right)\mathrm{d}y.$$

How does the integral seem to you at first sight? Friendly to some extent or not really? The integral might create difficulties without a proper approach, but observing that behind the scene there is the quadratic series of Au-Yeung, which in this book we calculate elegantly, our task becomes easier. How do we actually reduce the integral to the calculation of the quadratic series of Au-Yeung? First, we make the changes of variable $\sin(x)=1-u$ and $\sin(y)=1-v$ which, if we combine with the geometric series and a special logarithmic integral we calculate in the book, lead to

$$\int_0^1\left(\int_0^1\frac{\log(1-u)\log(1-v)}{1-uv}\mathrm{d}u\right)\mathrm{d}v$$

$$=\int_0^1\left(\int_0^1\sum_{n=1}^{\infty}(uv)^{n-1}\log(1-u)\log(1-v)\mathrm{d}u\right)\mathrm{d}v$$

$$=\sum_{n=1}^{\infty}\underbrace{\int_0^1 u^{n-1}\log(1-u)\mathrm{d}u}_{-H_n/n}\underbrace{\int_0^1 v^{n-1}\log(1-v)\mathrm{d}v}_{-H_n/n}=\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^2=\frac{17}{4}\zeta(4),$$

that finally led us to the quadratic series of Au-Yeung and its value. Often turning the integrals into series makes everything easier!

In this book the reader will meet plenty of examples where one needs to know how to craftily use and combine the integrals, sums, and series for obtaining solutions to the proposed problems.

To say a few words about the way I came to the title of the book, *(Almost) Impossible Integrals, Sums, and Series*, I tell you that some time ago, I found out about the release of a new book on integrals; it was Paul Nahin's book entitled *Inside Interesting Integrals*. One of the first things that impressed me profoundly before reading the mathematical stuff in that book was exactly this thing Paul wrote in its Preface: "Despite all the math in it, this book has been written in the spirit of 'let's have fun."' Doing mathematics, calculating all kind of integrals, some being pretty difficult, and at the same time having fun? I was amazed! The book was a journey that I took with much pleasure from cover to cover, and I remember that at that time in the *Contour Integration* chapter, I met the following integral:

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1-e^{x}} \mathrm{d}x, \ 0 < a < 1,$$

about which Paul said "what I'll do next is use indents to derive a result that would be extremely difficult to get by other means." That was a challenge to me, a lover of real methods! *How to do it by real methods?* Indeed, it looked like a wild merciless integral beast! After some work on it, I found a pretty simple way to do it, and then I was so glad to contact Paul and reveal to him my solution, which he enjoyed much, and since then we have kept talking about such problems. *How did I do it?* I'll give you a hint and then you can try it on your own, that is, arrange the integral such that you can use the integral representation of Digamma function,

$$\psi(x) = \int_0^{\infty} \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}}\right) \mathrm{d}t,$$

together with one of the Digamma function properties, $\psi(1-x) - \psi(x) = \dfrac{\pi}{\tan(\pi x)}$.

I recall that at some point, I told Paul I plan to write a book, and the very nice surprise to me was that he had shown much interest in my publishing idea, encouraging and giving me support in this respect for which I want to simply say:

> Paul Nahin, you're a wonderful person, writer, mathematician, physicist, thank you very much for all! It's nothing short of a blessing to have known you, and especially your work exposed through excellent books! You really cared about my work and trusted me!

If you did not write a book yet and plan to do it at some point, you might have the surprise to realize how hard finding the proper title for your book or writing a Preface is. I didn't have a final formula, a variant about to say *That's the title!*, but that lasted until the day I talked with Paul about the title and he asked me if I had a title. As you may anticipate, I owe Paul the *inspiration* for the present title which perfectly fits my attraction to the calculation of difficult integrals, sums, and series.

The present book represents the materialization of my burning passion and dedicated hard work, a book not written by a professional mathematician, I am licensed in economic sciences (programme Accounting and Business Informatics).

Now, the book is structured in accordance with the title; it has two major areas, *Integrals* and *Sums and Series*, and within each area were considered those problems that had something special, I usually call *wow elements*, those kinds of fascinating problems and/or solutions that you don't usually find in classical books and textbooks and that have the power to surprise you (profoundly) pleasantly and then possibly say something like *Wow! That was (really) nice!*; it's also one of the main goals of the book, and the readers will decide if this goal has been reached. In the book the reader will find both classical and new, original problems and solutions coming from my personal research. In the case of most classical problems, the goal was to come up with new, surprising ways of solving them, possibly not known, or less known, from those problems at the contests level to those proposed by famous mathematicians like Srinivasa Ramanujan. Besides the *Solution* section where the reader can find solutions with detailed steps for the problems, without entering into the technical details a real mathematician would expect, often finding helpful comments between the lines of the solutions such that everything can be easily understood (which was another important goal of the book), I also added a section called *Hints* where the reader might find guiding information, clues on how to approach the problems.

The book mainly addresses to all people with a good knowledge of calculus (from self-educated people, students, and undergraduate students to researchers, either used as an entertaining way, as part of the preparation for exams, or as research topics) and to all those that enjoy the world of integrals, sums, and series and that cannot resist them. As Hardy said, "I could never resist an integral."

Finally, I would like to thank all people I had beautiful discussions with about mathematics during the time, Paul Nahin, Rob Johnson, and Karl W. Heuer, who are excellent mathematicians, and the list continues.

Many special thanks to the mathematical journals that published my work during the time: *The American Mathematical Monthly* (contact editors, Kelly Minnis, Doug Hensley, Daniel Ullman), *La Gaceta de la RSME* (contact editor, Oscar Ciaurri), *MathProblems Journal* (contact editors, Valmir Krasniqi, Omran Kouba), *School Science and Mathematics Association Journal* (contact editor, Ted Eisenberg), *Journal of Classical Analysis* (contact editor-in-chief, Tibor K. Pogány), and *Mediterranean Journal of Mathematics* (contact editor-in-chief, Francesco Altomare).

I am very grateful to the publisher Springer, New York, for the acceptance of my book project, and to my editor Dr. Sam Harrison, his assistant editor Sanaa Ali-Virani, the project manager Lavanya Venkatesan and their colleagues for the outstanding work done during the publishing process of the book. Thank you everybody so much!

I would like to express my thanks to the anonymous referees for their many insightful comments and suggestions that were very useful in shaping the finished manuscript.

I want to thank my parents for fully trusting me and for all the given support, and I also particularly dedicate this book to my special mother, Ileana Ursachi.

I wrote and finalized the book from a beautiful place at the western border of Romania, commune Teremia Mare, where I presently live, and I sent the full manuscript to Springer in the first half of March, 2018.

For the readers I have the last words of the Preface: Let's start the journey in the fascinating world of integrals, sums, and series and have much fun!

Teremia Mare, Timiș County, Romania — Cornel Ioan Vălean
2018

Contents

Chapter 1
Integrals

"For God has not given us a spirit of fear, but of power and of love and of a sound mind."—2 Timothy 1:7

1.1 A Powerful Elementary Integral

Prove that

$$\int_0^1 \frac{1}{(1+yx)\sqrt{1-x^2}}\mathrm{d}x = \frac{\arccos(y)}{\sqrt{1-y^2}}, \quad y \in (-1,1). \tag{1.1}$$

A challenging question: How would we solve the Basel problem using the integral result?

1.2 A Pair of Elementary Logarithmic Integrals We Might Find Very Useful for Solving the Problems in the Book

Let m, n be natural numbers. Prove that

$$i) \int_0^1 x^m \log^n(x)\mathrm{d}x = (-1)^n \frac{n!}{(m+1)^{n+1}}; \tag{1.2}$$

(continued)

C. I. Vălean, *(Almost) Impossible Integrals, Sums, and Series*, Problem Books in Mathematics, https://doi.org/10.1007/978-3-030-02462-8_1

$$ii) \int_0^a x^m \log^n(x)\mathrm{d}x = a^{m+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{k!}{(m+1)^{k+1}} \times \log^{n-k}(a),\ a>0. \tag{1.3}$$

1.3 Four Logarithmic Integrals Strongly Connected with the League of Harmonic Series

Let n be a positive integer. Then prove the following equalities hold

$$I_n = \int_0^1 x^{n-1}\log(1-x)\mathrm{d}x = -\frac{H_n}{n}; \tag{1.4}$$

$$J_n = \int_0^1 x^{n-1}\log^2(1-x)\mathrm{d}x = \frac{2}{n}\sum_{k=1}^{n}\frac{H_k}{k} = \frac{H_n^2+H_n^{(2)}}{n}; \tag{1.5}$$

$$K_n = \int_0^1 x^{n-1}\log^3(1-x)\mathrm{d}x = -\frac{3}{n}\sum_{k=1}^{n}\frac{H_k^2+H_k^{(2)}}{k} = -\frac{H_n^3+3H_nH_n^{(2)}+2H_n^{(3)}}{n}; \tag{1.6}$$

$$L_n = \int_0^1 x^{n-1}\log^4(1-x)\mathrm{d}x = \frac{4}{n}\sum_{k=1}^{n}\frac{H_k^3+3H_kH_k^{(2)}+2H_k^{(3)}}{k} = \frac{H_n^4+6H_n^2H_n^{(2)}+8H_nH_n^{(3)}+3\left(H_n^{(2)}\right)^2+6H_n^{(4)}}{n}, \tag{1.7}$$

where $H_n^{(m)} = 1+\frac{1}{2^m}+\cdots+\frac{1}{n^m},\ m\geq 1$, is the nth generalized harmonic number of order m.

A challenging question: Is it possible to do the calculations with the high school knowledge only (supposing we know and use the notation of the generalized harmonic numbers)?

1.4 Two Very Useful Classical Logarithmic Integrals That May Arise in the Calculation of Some Tough Integrals and Series

Show that

$$i) \int_0^x \frac{\log^2(1-t)}{t} \mathrm{d}t$$

$$= \log(x)\log^2(1-x)+2\log(1-x)\mathrm{Li}_2(1-x)-2\mathrm{Li}_3(1-x)+2\zeta(3); \tag{1.8}$$

$$ii) \int_0^x \frac{\log^2(1+t)}{t} \mathrm{d}t$$

$$= \log(x)\log^2(1+x)-\frac{2}{3}\log^3(1+x)-2\log(1+x)\,\mathrm{Li}_2\left(\frac{1}{1+x}\right)-2\,\mathrm{Li}_3\left(\frac{1}{1+x}\right)$$
$$+2\zeta(3), \tag{1.9}$$

where ζ denotes the Riemann zeta function and Li_n represents the Polylogarithm function.

1.5 A Couple of Practical Definite Integrals Expressed in Terms of the Digamma Function

Let $s > 0$ be a real number. Prove that

$$i) \int_0^1 \frac{x^{s-1}}{1+x} \mathrm{d}x = \frac{1}{2}\left(\psi\left(\frac{1+s}{2}\right)-\psi\left(\frac{s}{2}\right)\right) = \psi(s)-\psi\left(\frac{s}{2}\right)-\log(2); \tag{1.10}$$

$$ii) \int_0^\infty \tanh(x)e^{-sx}\mathrm{d}x = \frac{1}{2}\left(\psi\left(\frac{s+2}{4}\right)-\psi\left(\frac{s}{4}\right)-\frac{2}{s}\right), \tag{1.11}$$

where $\psi(s)$ is the Digamma function.

1.6 A Useful Special Generalized Integral Expressed in Terms of the Polylogarithm Function

Let n be a positive integer and $y \in (-\infty, 1)$ be a real number. Then, prove that

$$y \int_0^1 \frac{\log^n(x)}{1-y+yx} \mathrm{d}x = (-1)^{n-1} n! \operatorname{Li}_{n+1}\left(\frac{y}{y-1}\right), \tag{1.12}$$

where Li_n denotes the Polylogarithm function.

A useful form during the calculations of various integrals:

$$\int_0^1 \frac{u \log^n(x)}{1-ux} \mathrm{d}x = (-1)^n n! \operatorname{Li}_{n+1}(u).$$

1.7 Two Little Tricky Classical Logarithmic Integrals

Prove that

$$i) \int_0^1 \frac{\log^2(1+x)}{x} \mathrm{d}x = \frac{1}{4}\zeta(3); \tag{1.13}$$

$$ii) \int_0^1 \frac{\log(1-x)\log(1+x)}{x} \mathrm{d}x = -\frac{5}{8}\zeta(3), \tag{1.14}$$

where ζ denotes the Riemann zeta function.

1.8 A Special Trio of Integrals with $\log^2(1-x)$ and $\log^2(1+x)$

Prove that

$$i) \int_0^1 \log^2(1-x)\log^2(1+x)\mathrm{d}x$$

$$= 24 - 8\zeta(2) - 8\zeta(3) - \zeta(4) + 8\log(2)\zeta(2) - 4\log^2(2)\zeta(2) + 8\log(2)\zeta(3)$$

(continued)

$$-24\log(2)+12\log^2(2)-4\log^3(2)+\log^4(2); \tag{1.15}$$

$$ii)\ \int_0^1 \frac{\log^2(1-x)\log^2(1+x)}{x}\mathrm{d}x$$

$$=\frac{2}{15}\log^5(2)-\frac{2}{3}\log^3(2)\zeta(2)+\frac{7}{4}\log^2(2)\zeta(3)-\frac{25}{8}\zeta(5)$$

$$+4\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right)+4\operatorname{Li}_5\left(\frac{1}{2}\right); \tag{1.16}$$

$$iii)\ \int_0^1 \frac{\log^2(1-x)\log^2(1+x)}{1+x}\mathrm{d}x$$

$$=\frac{63}{8}\zeta(5)-\frac{9}{2}\log(2)\zeta(4)+4\log^2(2)\zeta(3)-\frac{4}{3}\log^3(2)\zeta(2)-2\zeta(2)\zeta(3)$$

$$+\frac{7}{30}\log^5(2)-4\operatorname{Li}_5\left(\frac{1}{2}\right), \tag{1.17}$$

where ζ denotes the Riemann zeta function and Li_n represents the Polylogarithm function.

1.9 *A Darn Integral in Disguise* (Possibly Harder Than It Seems to Be?), an Integral with Two Squared Logarithms on the Half of the Unit Interval

Prove that

$$\int_0^{1/2} \frac{\log^2(x)\log^2(1-x)}{x}\mathrm{d}x$$

$$=\frac{1}{8}\zeta(5)-2\zeta(2)\zeta(3)-\frac{2}{3}\log^3(2)\zeta(2)+\frac{7}{4}\log^2(2)\zeta(3)-\frac{1}{15}\log^5(2)$$

$$+4\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right)+4\operatorname{Li}_5\left(\frac{1}{2}\right), \tag{1.18}$$

(continued)

where ζ denotes the Riemann zeta function and Li_n represents the Polylogarithm function.

A challenging question: Prove the result by using entirely real methods.

1.10 The Evaluation of a Class of Logarithmic Integrals Using a Slightly Modified Result from *Table of Integrals, Series and Products* by I.S. Gradshteyn and I.M. Ryzhik Together with a Series Result Elementarily Proved by Guy Bastien

Let $n \geq 1$ be a positive integer. Prove that

$$\begin{aligned}
&\int_0^1 \frac{\log(1-x)\log^{2n}(x)\log(1+x)}{x}\mathrm{d}x \\
&= \frac{1}{2}(2n)!\left(1-\frac{1}{2^{2n+1}}\right)\times\sum_{k=1}^{2n}\zeta(k+1)\zeta(2n-k+2) \\
&-(2n)!\sum_{k=1}^{n}\left(1-\frac{1}{2^{2k-1}}\right)\zeta(2k)\zeta(2n-2k+3) \\
&+\frac{1}{2^{2n+3}}(2n+3-2^{2n+3})(2n)!\zeta(2n+3),
\end{aligned} \tag{1.19}$$

where ζ represents the Riemann zeta function.

Examples:

For $n = 1$,

$$\int_0^1 \frac{\log(1-x)\log^{2}(x)\log(1+x)}{x}\mathrm{d}x = \frac{3}{4}\zeta(2)\zeta(3)-\frac{27}{16}\zeta(5);$$

(continued)

For $n = 2$,

$$\int_0^1 \frac{\log(1-x)\log^4(x)\log(1+x)}{x}\mathrm{d}x = \frac{9}{4}\zeta(3)\zeta(4)+\frac{45}{4}\zeta(2)\zeta(5)-\frac{363}{16}\zeta(7);$$

For $n = 3$,

$$\int_0^1 \frac{\log(1-x)\log^6(x)\log(1+x)}{x}\mathrm{d}x$$

$$= \frac{2835}{8}\zeta(2)\zeta(7) + \frac{135}{8}\zeta(3)\zeta(6) + \frac{675}{8}\zeta(4)\zeta(5) - \frac{22635}{32}\zeta(9).$$

1.11 Logarithmic Integrals Containing an Infinite Series in the Integrand, Giving Values in Terms of Riemann Zeta Function

Evaluate

$$i)\ \int_0^1 \left(4x^2 + 4^2x^{2^2} + 4^3x^{2^3} + 4^4x^{2^4} + \cdots\right)\frac{\log^2(x)}{x(1+x)}\mathrm{d}x; \tag{1.20}$$

$$ii)\ \int_0^1 \left(2^kx + 2^{2k}x^{2^2-1} + 2^{3k}x^{2^3-1} + 2^{4k}x^{2^4-1} + \cdots\right)\frac{\log^k(x)}{1+x}\mathrm{d}x,\ k \in \mathbb{N}. \tag{1.21}$$

1.12 Two Appealing Integral Representations of $\zeta(4)$ and $\zeta(2)G$

Prove that

$$I = 2\int_0^1 \frac{\operatorname{arctanh}(x)\operatorname{Li}_2(x)}{x}\mathrm{d}x - \int_0^1 \frac{\log(1-x)\log(x)\log(1+x)}{x}\mathrm{d}x$$

$$= \frac{23}{16}\zeta(4); \tag{1.22}$$

(continued)

$$J=\int_0^1 \frac{\arctan(x)\log(x)\log(1+x)}{x}\mathrm{d}x-\int_0^1 \frac{\arctan(x)\operatorname{Li}_2(-x)}{x}\mathrm{d}x$$
$$=\frac{1}{8}\zeta(2)G, \tag{1.23}$$

where ζ is the Riemann zeta function, G represents the Catalan's constant, and Li_2 denotes the Dilogarithm function.

A (super) challenging question: Prove both results without using harmonic series.

1.13 A Special Pair of Logarithmic Integrals with Connections in the Area of the Alternating Harmonic Series

Prove that

$$I=\int_0^1 \frac{x\log(1-x)}{1+x^2}\mathrm{d}x=\frac{1}{8}\left(\log^2(2)-\frac{5}{2}\zeta(2)\right); \tag{1.24}$$

$$J=\int_0^1 \frac{x\log(1+x)}{1+x^2}\mathrm{d}x=\frac{1}{8}\left(\log^2(2)+\frac{1}{2}\zeta(2)\right), \tag{1.25}$$

where ζ denotes the Riemann zeta function.

A (little) challenging question: How about calculating the integrals without using the differentiation under the integral sign, double integrals?

1.14 Another Special Pair of Logarithmic Integrals with Connections in the Area of the Alternating Harmonic Series

Show that

$$I=\int_0^1 \frac{x\log(1-x)\log(x)}{1+x^2}\mathrm{d}x=\frac{1}{16}\left(\frac{41}{4}\zeta(3)-9\log(2)\zeta(2)\right); \tag{1.26}$$

(continued)

$$J=\int_0^1 \frac{x\log(1+x)\log(x)}{1+x^2}\mathrm{d}x=\frac{1}{16}\left(3\log(2)\zeta(2)-\frac{15}{4}\zeta(3)\right), \tag{1.27}$$

where ζ denotes the Riemann zeta function.

1.15 A Class of Tricky and Useful Integrals with Consecutive Positive Integer Powers for the Logarithms

Let p be a positive integer. Show that

$$\int_0^{1/2} \frac{\log^p(1-x)\log^{p+1}(x)}{1-x}\mathrm{d}x=-\frac{1}{2(1+p)}\log^{2(p+1)}(2)$$
$$+\frac{1}{2}\lim_{\substack{x\to 0\\ y\to 1}}\frac{\partial^{2p+1}}{\partial x^p\partial y^{p+1}}\mathrm{B}(x,y), \tag{1.28}$$

where $\mathrm{B}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t$ is the Beta function.

Examples:

For $p=1$,

$$\int_0^{1/2}\frac{\log(1-x)\log^2(x)}{1-x}\mathrm{d}x=-\frac{1}{4}\left(\zeta(4)+\log^4(2)\right);$$

For $p=2$,

$$\int_0^{1/2}\frac{\log^2(1-x)\log^3(x)}{1-x}\mathrm{d}x=6\zeta^2(3)-\frac{69}{8}\zeta(6)-\frac{1}{6}\log^6(2);$$

For $p=3$,

$$\int_0^{1/2}\frac{\log^3(1-x)\log^4(x)}{1-x}\mathrm{d}x$$

$$=288\zeta(3)\zeta(5)-72\zeta(2)\zeta^2(3)-\frac{1497}{8}\zeta(8)-\frac{1}{8}\log^8(2),$$

where ζ denotes the Riemann zeta function.

1.16 A Double Integral and a Triple Integral, Beautifully Connected with the Advanced Harmonic Series

Calculate

$$i)\ \int_0^1\int_0^1 \frac{\log(1-x)\log(1-y)\log(1+xy)}{xy(1+xy)}\mathrm{d}x\mathrm{d}y; \tag{1.29}$$

$$ii)\ \int_0^1\int_0^1\int_0^1 \frac{\log(1-x)\log(1-y)\log(1-z)}{(1-xy)(1-xyz)}\mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{1.30}$$

1.17 Let's Take Two Double Logarithmic Integrals with Beautiful Values Expressed in Terms of the Riemann Zeta Function

Show that

$$i)\ \int_0^1\int_0^1 \frac{\log\left(\frac{1}{x}\right)-\log\left(\frac{1}{y}\right)}{\log\left(\log\left(\frac{1}{x}\right)\right)-\log\left(\log\left(\frac{1}{y}\right)\right)}\mathrm{d}x\mathrm{d}y=\frac{7\zeta(3)}{6\zeta(2)}; \tag{1.31}$$

$$ii)\ \int_0^1\int_0^1 \left(\frac{\log\left(\frac{1}{x}\right)-\log\left(\frac{1}{y}\right)}{\log\left(\log\left(\frac{1}{x}\right)\right)-\log\left(\log\left(\frac{1}{y}\right)\right)}\right)^2\mathrm{d}x\mathrm{d}y=\frac{31\zeta(5)}{15\zeta(4)}-\frac{7\zeta(3)}{6\zeta(2)}, \tag{1.32}$$

where ζ denotes the Riemann zeta function.

1.18 Interesting Integrals Containing the Inverse Tangent Function and the Logarithmic Function

Prove that

$$I=\int_0^1 \arctan(x)\log(1-x)\mathrm{d}x$$

(continued)

$$= \frac{1}{8}\left(4\log(2) - \log^2(2) + \frac{5}{2}\zeta(2) - 2\pi + \pi\log(2) - 8G\right); \qquad (1.33)$$

$$J = \int_0^1 \arctan(x)\log(1+x)\mathrm{d}x$$

$$= \frac{1}{8}(3\pi\log(2) + 4\log(2) - \frac{1}{2}\zeta(2) - \log^2(2) - 2\pi), \qquad (1.34)$$

where G denotes the Catalan's constant and ζ represents the Riemann zeta function.

1.19 Interesting Integrals Involving the Inverse Tangent Function and Dilogarithm Function

Prove that

$$I = \int_0^1 \arctan(x)\,\mathrm{Li}_2(x)\mathrm{d}x$$

$$= \frac{\pi}{4}G - G - \frac{\pi}{4} + \frac{5}{96}\pi^2 + \frac{\pi^3}{24} + \frac{\pi}{8}\log(2) + \frac{1}{2}\log(2) - \frac{1}{8}\log^2(2)$$

$$- \frac{\pi^2}{12}\log(2) - \frac{23}{64}\zeta(3); \qquad (1.35)$$

$$J = \int_0^1 \arctan(x)\,\mathrm{Li}_2(-x)\mathrm{d}x$$

$$= \frac{33}{64}\zeta(3) + \frac{\pi^2}{24}\log(2) - \frac{\pi^3}{48} - \frac{\pi^2}{96} - \frac{\pi}{4} - \frac{\pi}{4}G + \frac{3}{8}\log(2)\pi$$

$$+ \frac{1}{2}\log(2) - \frac{1}{8}\log^2(2), \qquad (1.36)$$

where G is the Catalan's constant, ζ denotes the Riemann zeta function, and Li_2 represents the Dilogarithm function.

A challenging question: Is it possible to calculate the integrals exclusively by real methods (without using complex numbers)?

1.20 More Interesting Integrals Involving the Inverse Tangent Function and the Logarithmic Function: The First Part

Prove that

$$I=\int_0^1 \frac{\arctan(x)\log(1-x)}{1+x^2}\mathrm{d}x=\frac{1}{8}\left(\frac{3}{4}\log(2)\zeta(2)-\frac{7}{8}\zeta(3)-\pi G\right); \tag{1.37}$$

$$J=\int_0^1 \frac{\arctan(x)\log(1+x)}{1+x^2}\mathrm{d}x=\frac{1}{8}\left(\frac{21}{8}\zeta(3)+\frac{3}{4}\log(2)\zeta(2)-\pi G\right), \tag{1.38}$$

where G denotes the Catalan's constant and ζ represents the Riemann zeta function.

1.21 More Interesting Integrals Involving the Inverse Tangent Function and the Logarithmic Function: The Second Part

Prove that

$$I=\int_0^1 \frac{\arctan^2(x)\log(1-x)}{1+x^2}\mathrm{d}x$$

$$=\frac{1}{1536}\psi^{(3)}\left(\frac{1}{4}\right)-\frac{\pi^4}{192}+\log(2)\frac{\pi^3}{384}-\frac{\pi^2}{32}G-\frac{35}{256}\pi\zeta(3); \tag{1.39}$$

$$J=\int_0^1 \frac{\arctan^2(x)\log(1+x)}{1+x^2}\mathrm{d}x=\log(2)\frac{\pi^3}{384}+\frac{21}{256}\pi\zeta(3)-\frac{3}{16}\zeta(2)G, \tag{1.40}$$

where G is the Catalan's constant, ζ denotes the Riemann zeta function, and $\psi^{(n)}$ represents the Polygamma function.

1.22 Challenging Integrals Involving arctan(x), log(x), log($1-x$), $\mathrm{Li}_2(x)$, and $\mathrm{Li}_2(x^2)$

Show that

$$i)\ \int_0^1 \arctan(x)\log(x)\log(1-x)\mathrm{d}x - \int_0^1 \frac{x\log(x)\arctan(x)}{1-x}\mathrm{d}x$$

$$= G - \frac{41}{64}\zeta(3) + \frac{3}{32}\log(2)\pi^2 - \frac{5}{96}\pi^2 - \frac{1}{8}\log(2)\pi + \frac{\pi}{4} - \frac{1}{2}\log(2) + \frac{1}{8}\log^2(2); \tag{1.41}$$

$$ii)\ \int_0^1 \arctan(x)\log(x)\,\mathrm{Li}_2(x)\mathrm{d}x - \int_0^1 \frac{x\log(x)\arctan(x)}{1-x}\mathrm{d}x$$

$$= 2G - \frac{1}{2}G^2 - \frac{\pi}{4}G + \frac{41}{7680}\pi^4 - \frac{\pi^3}{24} - \frac{9}{32}\zeta(3) - \frac{5\pi^2}{48} + \frac{\pi}{2} + \frac{\log^2(2)}{4}$$

$$+ \frac{17}{96}\log(2)\pi^2 - \frac{\pi}{4}\log(2) - \log(2); \tag{1.42}$$

$$iii)\ \int_0^1 \arctan(x)\log(x)\,\mathrm{Li}_2(x^2)\mathrm{d}x - 2\int_0^1 \frac{x\log(x)\arctan(x)}{1-x}\mathrm{d}x$$

$$= 4G + \log(2)G + \frac{\pi^4}{1920} - \frac{7}{96}\pi^3 - \frac{9}{8}\zeta(3) - \frac{5}{24}\log(2)\pi^2 - 2\log(2)\pi - \frac{5}{24}\pi^2$$

$$+ \frac{5}{2}\pi - 5\log(2) + \log^2(2), \tag{1.43}$$

where G is the Catalan's constant, ζ denotes the Riemann zeta function, and Li_2 represents the Dilogarithm function.

1.23 Two More Special Challenging Integrals Involving arctan(x), log(x), log($1+x$), and $\mathrm{Li}_2(-x)$

Show that

$$i)\ \int_0^1 \arctan(x)\log(x)\log(1+x)\mathrm{d}x$$

(continued)

$$= \frac{\log(2)}{2}G - \frac{\pi^3}{64} + \frac{15}{64}\zeta(3) - \log(2)\frac{\pi^2}{32} - \frac{\pi^2}{96} - \frac{3}{8}\log(2)\pi$$

$$+\frac{\pi}{2} + \frac{1}{8}\log^2(2) - \log(2); \tag{1.44}$$

$$ii) \int_0^1 \arctan(x)\log(x)\operatorname{Li}_2(-x)\mathrm{d}x$$

$$= \frac{1}{2}G^2 + \frac{\pi}{4}G + \frac{\log(2)}{2}G - \frac{13}{2560}\pi^4 + \frac{\pi^3}{192} - \frac{9}{32}\zeta(3) - \frac{7}{96}\log(2)\pi^2 - \frac{3}{4}\log(2)\pi$$

$$+\frac{3}{4}\pi - \frac{3}{2}\log(2) + \frac{1}{4}\log^2(2), \tag{1.45}$$

where G is the Catalan's constant, ζ denotes the Riemann zeta function, and Li_2 represents the Dilogarithm function.

1.24 A Challenging Integral with the Inverse Tangent Function and an Excellent Generalization According to the Even Positive Powers of the Logarithm

Show that

$$i) \int_0^1 \frac{\arctan(x)\log(x)}{1+x}\mathrm{d}x = \frac{\log(2)}{2}G - \frac{\pi^3}{64}, \tag{1.46}$$

where G represents the Catalan's constant.

Let $n \geq 1$ be a positive integer. Then, prove the generalization

$$ii) \int_0^1 \frac{\arctan(x)\log^{2n}(x)}{1+x}\mathrm{d}x$$

$$= \frac{\pi}{4}(1 - 2^{-2n})\zeta(2n+1)(2n)!$$

$$+\frac{1}{2}\beta(2n+2)(2n)! - \frac{\pi}{16}\lim_{s\to 0}\left(\frac{d^{2n}}{ds^{2n}}\left(\csc\left(\frac{\pi s}{2}\right)\left(\psi\left(\frac{3}{4} - \frac{s}{4}\right) - \psi\left(\frac{1}{4} - \frac{s}{4}\right)\right)\right)\right)$$

(continued)

$$+\sec\left(\frac{\pi s}{2}\right)\left(\psi\left(1-\frac{s}{4}\right)-\psi\left(\frac{1}{2}-\frac{s}{4}\right)\right)-2\pi\csc(\pi s)\Bigg)\Bigg), \tag{1.47}$$

where ζ represents the Riemann zeta function, ψ denotes the Digamma function, and β designates the Dirichlet beta function.

Examples:

For $n = 1$,

$$\int_0^1 \frac{\arctan(x)\log^2(x)}{1+x}\mathrm{d}x = \frac{21}{64}\pi\zeta(3) - \frac{\pi^3}{32}\log(2) - \frac{\pi^2}{24}G;$$

For $n = 2$,

$$\int_0^1 \frac{\arctan(x)\log^4(x)}{1+x}\mathrm{d}x$$

$$= \frac{1395}{256}\pi\zeta(5) - \frac{9}{128}\pi^3\zeta(3) - \frac{7}{480}\pi^4 G - \frac{5}{128}\pi^5\log(2) + \frac{\pi^6}{192}$$

$$-\frac{\pi^2}{1536}\psi^{(3)}\left(\frac{1}{4}\right);$$

For $n = 3$,

$$\int_0^1 \frac{\arctan(x)\log^6(x)}{1+x}\mathrm{d}x$$

$$= \frac{31}{1536}\pi^8 + \frac{360045}{2048}\pi\zeta(7) - \frac{675}{1024}\pi^3\zeta(5) - \frac{225}{1024}\pi^5\zeta(3) - \frac{61}{512}\log(2)\pi^7$$

$$-\frac{31}{2688}\pi^6 G - \frac{7}{12288}\pi^4\psi^{(3)}\left(\frac{1}{4}\right) - \frac{\pi^2}{16384}\psi^{(5)}\left(\frac{1}{4}\right).$$

1.25 Let's Tango with an Exciting Integral Involving the Inverse Tangent Integral, the Lerch Transcendent Function, and the Logarithm with Odd Positive Powers

Let $n \geq 1$ be a positive integer. Then, prove that

$$\int_0^1 \mathrm{Ti}_2(x)\frac{\log^{2n-1}(x)}{1+x}\mathrm{d}x = \frac{1}{4}\int_0^1 \Phi\left(-x^2, 2, \frac{1}{2}\right)\frac{x\log^{2n-1}(x)}{1+x}\mathrm{d}x$$

(continued)

$$= \frac{\pi}{4}(1-2^{-2n})\zeta(2n+1)(2n)! - \frac{1}{2}\beta(2n+2)(2n-1)!$$

$$-\frac{\pi}{64}\lim_{s\to 0}\left(\frac{d^{2n-1}}{ds^{2n-1}}\left(\csc\left(\frac{\pi s}{2}\right)\left(\psi^{(1)}\left(\frac{1}{4}-\frac{s}{4}\right)-\psi^{(1)}\left(\frac{3}{4}-\frac{s}{4}\right)\right)\right.\right.$$

$$\left.\left.+\sec\left(\frac{\pi s}{2}\right)\left(\psi^{(1)}\left(\frac{1}{2}-\frac{s}{4}\right)-\psi^{(1)}\left(1-\frac{s}{4}\right)\right)-32G\csc(\pi s)\right)\right), \tag{1.48}$$

where $\operatorname{Ti}_2(x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)^2}$ is the Inverse tangent integral, $\Phi(z,s,a) = \sum_{k=0}^{\infty}\frac{z^k}{(a+k)^s}$ is the Lerch transcendent function, G represents the Catalan's constant, ζ denotes the Riemann zeta function, $\psi^{(n)}$ is the Polygamma function, and β designates the Dirichlet beta function.

Examples:

For $n = 1$,

$$\int_0^1 \operatorname{Ti}_2(x)\frac{\log(x)}{1+x}\mathrm{d}x = \frac{21}{64}\pi\zeta(3) + \frac{\pi^2}{16}G + \frac{\pi^4}{48} - \frac{1}{384}\psi^{(3)}\left(\frac{1}{4}\right);$$

For $n = 3$,

$$\int_0^1 \operatorname{Ti}_2(x)\frac{\log^3(x)}{1+x}\mathrm{d}x$$

$$= \frac{29}{1280}\pi^6 + \frac{7}{128}\pi^4 G - \frac{9}{256}\pi^3\zeta(3) + \frac{1395}{256}\pi\zeta(5) - \frac{\pi^2}{2048}\psi^{(3)}\left(\frac{1}{4}\right)$$

$$-\frac{3}{40960}\psi^{(5)}\left(\frac{1}{4}\right);$$

For $n = 5$,

$$\int_0^1 \operatorname{Ti}_2(x)\frac{\log^5(x)}{1+x}\mathrm{d}x$$

$$= \frac{1417}{21504}\pi^8 + \frac{31}{256}\pi^6 G - \frac{75}{1024}\pi^5\zeta(3) - \frac{225}{512}\pi^3\zeta(5) + \frac{360045}{2048}\pi\zeta(7)$$

$$-\frac{7}{24576}\pi^4\psi^{(3)}\left(\frac{1}{4}\right) - \frac{5}{98304}\pi^2\psi^{(5)}\left(\frac{1}{4}\right) - \frac{1}{344064}\psi^{(7)}\left(\frac{1}{4}\right).$$

1.26 A Superb Integral with Logarithms and the Inverse Tangent Function, and a Surprisingly Beautiful Generalization of It

Show that

$$i) \int_0^1 \frac{\arctan(x)}{x} \log\left(\frac{1+x^2}{(1-x)^2}\right) \mathrm{d}x = \frac{\pi^3}{16}; \tag{1.49}$$

$$ii) \int_0^x \frac{\arctan(t)\log(1+t^2)}{t}\mathrm{d}t - 2\int_0^1 \frac{\arctan(xt)\log(1-t)}{t}\mathrm{d}t$$

$$= 2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)^3}, \ |x| \le 1. \tag{1.50}$$

1.27 A Kind of Deviant Pair of Integrals with Logarithms and Polylogarithms, Using Symmetry

Show that

$$i) \int_0^1 \frac{\log(1-x)\operatorname{Li}_2\left(\frac{x}{x-1}\right)}{1+x}\mathrm{d}x = \frac{29}{16}\zeta(4) + \frac{1}{4}\log^2(2)\zeta(2) - \frac{1}{8}\log^4(2); \tag{1.51}$$

$$ii) \int_0^1 \frac{\log^2(1-x)\operatorname{Li}_3\left(\frac{x}{x-1}\right)}{1+x}\mathrm{d}x$$

$$= \frac{1}{36}\log^6(2) - \frac{1}{6}\log^4(2)\zeta(2) + \frac{7}{24}\log^3(2)\zeta(3) + \frac{5}{8}\log^2(2)\zeta(4) - \frac{581}{48}\zeta(6)$$

$$-\frac{7}{8}\log(2)\zeta(2)\zeta(3) - \frac{79}{64}\zeta^2(3), \tag{1.52}$$

where ζ represents the Riemann zeta function and Li_n denotes the Polylogarithm function.

1.28 Wonderful Integrals Containing the Logarithm and the Polylogarithm, Involving Beautiful Ideas About Symmetry: The First Part

Prove that

$$i)\ \int_0^1 \frac{\log\left(\frac{1}{x}\right)\operatorname{Li}_2(x)}{1+x}\,\mathrm{d}x = \frac{3}{16}\zeta(4); \tag{1.53}$$

$$ii)\ \int_0^1 \frac{\log^n(x)\operatorname{Li}_{n+1}(x)}{1+x}\,\mathrm{d}x$$

$$= \frac{1}{2}(-1)^n n!\left(\zeta(2n+2) - \eta^2(n+1)\right)$$

$$= \frac{1}{2}(-1)^n n!\left(\zeta(2n+2) - \left(1-2^{-n}\right)^2\zeta^2(n+1)\right), \tag{1.54}$$

where ζ is the Riemann zeta function, η denotes the Dirichlet eta function, and Li_n represents the Polylogarithm function.

Examples:

For $n = 2$,

$$\int_0^1 \frac{\log^2(x)\operatorname{Li}_3(x)}{1+x}\,\mathrm{d}x = \zeta(6) - \frac{9}{16}\zeta^2(3);$$

For $n = 3$,

$$\int_0^1 \frac{\log^3(x)\operatorname{Li}_4(x)}{1+x}\,\mathrm{d}x = -\frac{41}{128}\zeta(8);$$

For $n = 4$,

$$\int_0^1 \frac{\log^4(x)\operatorname{Li}_5(x)}{1+x}\,\mathrm{d}x = 12\zeta(10) - \frac{675}{64}\zeta^2(5).$$

1.29 Wonderful Integrals Containing the Logarithm and the Polylogarithm, Involving Beautiful Ideas About Symmetry: The Second Part

Show that

$$\int_0^1 \frac{\log^n(x)\operatorname{Li}_{n+1}(-x)}{1+x^2}\mathrm{d}x = (-1)^{n-1}\frac{n!}{2^{n+1}}\eta(n+1)\beta(n+1)$$

$$= (-1)^{n-1}\frac{n!}{2^{n+1}}(1-2^{-n})\zeta(n+1)\beta(n+1), \tag{1.55}$$

where ζ is the Riemann zeta function, η denotes the Dirichlet eta function, Li_n represents the Polylogarithm function, and β designates the Dirichlet beta function.

Examples:

For $n = 1$,

$$\int_0^1 \frac{\log(x)\operatorname{Li}_2(-x)}{1+x^2}\mathrm{d}x = \frac{1}{8}\zeta(2)G;$$

For $n = 2$,

$$\int_0^1 \frac{\log^2(x)\operatorname{Li}_3(-x)}{1+x^2}\mathrm{d}x = -\frac{3}{512}\pi^3\zeta(3);$$

For $n = 3$,

$$\int_0^1 \frac{\log^3(x)\operatorname{Li}_4(-x)}{1+x^2}\mathrm{d}x = \frac{7}{2048}\left(\frac{1}{8}\zeta(4)\psi^{(3)}\left(\frac{1}{4}\right) - 105\zeta(8)\right).$$

1.30 Wonderful Integrals Containing the Logarithm and the Polylogarithm, Involving Beautiful Ideas About Symmetry: The Third Part

Show that

$$i)\ \int_0^1 \frac{x\log^n(x)\operatorname{Li}_{n+1}(x)}{1+x^2}\mathrm{d}x$$

(continued)

$$= (-1)^n \frac{n!}{2}\left(\zeta(2n+2) - (\beta(n+1))^2 - \frac{1}{4^{n+1}}(\eta(n+1))^2\right); \quad (1.56)$$

$$ii) \int_0^1 \frac{x \log^n(x) \operatorname{Li}_{n+1}(-x)}{1+x^2} \mathrm{d}x$$

$$= (-1)^n \frac{n!}{2}\left((\beta(n+1))^2 - \eta(2n+2) - \frac{1}{4^{n+1}}(\eta(n+1))^2\right), \quad (1.57)$$

where ζ is the Riemann zeta function, η represents the Dirichlet eta function, Li_n denotes the Polylogarithm function, and β designates the Dirichlet beta function.

Examples:

For the integral at $i)$, $n = 1$,

$$\int_0^1 \frac{x \log(x) \operatorname{Li}_2(x)}{1+x^2} \mathrm{d}x = \frac{1}{2}\left(G^2 - \frac{123}{128}\zeta(4)\right);$$

For the integral at $ii)$, $n = 1$,

$$\int_0^1 \frac{x \log(x) \operatorname{Li}_2(-x)}{1+x^2} \mathrm{d}x = \frac{1}{2}\left(\frac{117}{128}\zeta(4) - G^2\right);$$

For the integral at $i)$, $n = 2$,

$$\int_0^1 \frac{x \log^2(x) \operatorname{Li}_3(x)}{1+x^2} \mathrm{d}x = \frac{1}{1024}\left(79\zeta(6) - 9\zeta^2(3)\right);$$

For the integral at $ii)$, $n = 2$,

$$\int_0^1 \frac{x \log^2(x) \operatorname{Li}_3(-x)}{1+x^2} \mathrm{d}x = -\frac{1}{1024}\left(47\zeta(6) + 9\zeta^2(3)\right);$$

For the integral at $i)$, $n = 3$,

$$\int_0^1 \frac{x \log^3(x) \operatorname{Li}_4(x)}{1+x^2} \mathrm{d}x$$

$$= \frac{1}{2048}\left(\frac{2839}{16}\zeta(8) - 15\zeta(4)\psi^{(3)}\left(\frac{1}{4}\right) + \frac{1}{96}\left(\psi^{(3)}\left(\frac{1}{4}\right)\right)^2\right);$$

(continued)

For the integral at $ii)$, $n = 3$,

$$\int_0^1 \frac{x \log^3(x) \operatorname{Li}_4(-x)}{1+x^2} \mathrm{d}x$$

$$= \frac{1}{2048}\left(15\zeta(4)\psi^{(3)}\left(\frac{1}{4}\right) - \frac{1}{96}\left(\psi^{(3)}\left(\frac{1}{4}\right)\right)^2 - \frac{2921}{16}\zeta(8)\right).$$

1.31 Two Families of Special Polylogarithmic Integrals Expressed in Terms of Infinite Series with the Generalized Harmonic Number and the Tails of Some Functions

Prove that

$$i)\ \int_0^1 \operatorname{Li}_m\left(\frac{x^2}{1+x^2}\right) \mathrm{d}x = \sum_{n=1}^{\infty} \frac{n H_{n-1}^{(m)}}{(2n-1)2^{2n}} \binom{2n}{n} \left(\sum_{k=n}^{\infty} \frac{2^k}{k\binom{2k}{k}}\right); \tag{1.58}$$

$$ii)\ \int_0^1 \operatorname{Li}_m\left(\frac{x}{1+x}\right) \mathrm{d}x = \sum_{n=1}^{\infty} H_{n-1}^{(m)} \left(\log(2) - \sum_{k=1}^{n-1} \frac{1}{k2^k}\right), \tag{1.59}$$

where $H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m}$, $m \geq 1$, denotes the nth generalized harmonic number of order m and Li_n represents the Polylogarithm function.

1.32 A Generalized Integral Beautifully Connected to a Spectacular (and Simultaneously Strange) Series

Show that

$$I_n = \int_0^1 \frac{x^{2n}}{(1+x)(1+x^2)^n} \mathrm{d}x$$

(continued)

$$= \frac{1}{2}\log(2) + \frac{1}{2^{n+1}}\log(2) + \frac{H_n}{2^{n+1}} - \sum_{i=1}^{n}\frac{1}{i2^{i+1}} - \frac{\pi}{2^{n+2}}\sum_{j=0}^{n-1}\frac{1}{2^j}\binom{2j}{j}$$
$$+ \frac{1}{2^{n+1}}\sum_{j=1}^{n-1}\frac{1}{2^j}\binom{2j}{j}\sum_{i=1}^{j}\frac{2^i}{i\binom{2i}{i}}, \tag{1.60}$$

where $H_n = \sum_{k=1}^{n}\frac{1}{k}$ is the nth harmonic number.

1.33 A Special (and Possibly Slightly Daunting) Integral with Two Polylogarithms, $\mathrm{Li}_2\left(\frac{x}{x-1}\right)$ and $\mathrm{Li}_2\left(\frac{x}{x+1}\right)$

Prove that

$$\int_0^1 \frac{\mathrm{Li}_2\left(\frac{x}{x-1}\right)\mathrm{Li}_2\left(\frac{x}{x+1}\right)}{x}\mathrm{d}x$$
$$= \frac{1}{10}\log^5(2) - \frac{1}{2}\log^3(2)\zeta(2) + \frac{21}{16}\log^2(2)\zeta(3) - \frac{13}{8}\zeta(2)\zeta(3) - \frac{29}{32}\zeta(5)$$
$$+3\log(2)\,\mathrm{Li}_4\left(\frac{1}{2}\right) + 3\,\mathrm{Li}_5\left(\frac{1}{2}\right),$$

where ζ denotes the Riemann zeta function and Li_n represents the Polylogarithm function.

1.34 Exciting Challenging Triple Integrals with the Dilogarithm

Prove that

$$i)\ \int_0^1\int_0^1\int_0^1 \frac{\mathrm{Li}_2(x+y+z-xy-xz-yz+xyz)}{2-x-y-z+xy+xz+yz-xyz}\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

(continued)

$$= \frac{1}{4}\left(\frac{45}{2}\log(2)\zeta(4) + 3\zeta(2)\zeta(3) - \frac{139}{8}\zeta(5)\right);$$

$$ii)\ \int_0^1\int_0^1\int_0^1 \frac{(\operatorname{Li}_2(x+y+z-xy-xz-yz+xyz))^2}{x+y+z-xy-xz-yz+xyz}\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= \frac{65}{8}\zeta(7) - \frac{7}{2}\zeta(3)\zeta(4) - \zeta(2)\zeta(5),$$

where ζ denotes the Riemann zeta function and Li_2 represents the Dilogarithm function.

1.35 A Curious Integral with Polylogarithms Connected to a Double Integral with a Symmetrical Exponential Integrand

Let $n \geq 1$ be a positive integer. Then, prove that

$$\int_0^1 \sum_{k=1}^{n}(n-k)!(k-1)!\frac{\operatorname{Li}_{n-k+1}\left(\frac{x}{x-1}\right)\operatorname{Li}_k\left(\frac{x}{x-1}\right)}{x^2}\mathrm{d}x$$

$$= 2\cdot n!(\zeta(2)+\zeta(3)+\cdots+\zeta(n+1)),$$

where ζ denotes the Riemann zeta function and Li_n represents the Polylogarithm function.

Examples:

For $n = 3$,

$$\int_0^1 \frac{\left(\operatorname{Li}_2\left(\frac{x}{x-1}\right)\right)^2 + 4\log(1-x)\operatorname{Li}_3\left(\frac{x}{x-1}\right)}{x^2}\mathrm{d}x$$

$$= 12(\zeta(2)+\zeta(3)+\zeta(4));$$

(continued)

For $n = 4$,

$$\int_0^1 \frac{\operatorname{Li}_2\left(\frac{x}{x-1}\right)\operatorname{Li}_3\left(\frac{x}{x-1}\right) + 3\log(1-x)\operatorname{Li}_4\left(\frac{x}{x-1}\right)}{x^2}\mathrm{d}x$$

$$= 12(\zeta(2) + \zeta(3) + \zeta(4) + \zeta(5));$$

For $n = 5$,

$$\int_0^1 \frac{\left(\operatorname{Li}_3\left(\frac{x}{x-1}\right)\right)^2 + 3\operatorname{Li}_2\left(\frac{x}{x-1}\right)\operatorname{Li}_4\left(\frac{x}{x-1}\right) + 12\log(1-x)\operatorname{Li}_5\left(\frac{x}{x-1}\right)}{x^2}\mathrm{d}x$$

$$= 60(\zeta(2) + \zeta(3) + \zeta(4) + \zeta(5) + \zeta(6)).$$

1.36 Double Integrals Expressed in Terms of the Exponential Function and the Polylogarithm (of Orders 2, 3, 4, 5, and 6)

Prove that

$$i)\ \int_0^\infty \int_0^\infty \frac{\log\left(1-e^{-x}\right)\left(y\operatorname{Li}_2\left(e^{-x-y}\right) + \operatorname{Li}_3\left(e^{-x-y}\right)\right)}{1-e^{x+y}} e^{x+y}\mathrm{d}x\mathrm{d}y$$

$$= \frac{21}{8}\zeta(6) + \zeta^2(3); \tag{1.61}$$

$$ii)\ \int_0^\infty \int_0^\infty \frac{\log\left(1-e^{-x}\right)\left(y^2\operatorname{Li}_2\left(e^{-x-y}\right) - 2\operatorname{Li}_4\left(e^{-x-y}\right)\right)}{1-e^{-x-y}}\mathrm{d}x\mathrm{d}y$$

$$= 5\zeta(2)\zeta(5) - \frac{9}{2}\zeta(3)\zeta(4); \tag{1.62}$$

$$iii)\ \int_0^\infty \int_0^\infty \frac{\log\left(1-e^{-x}\right)\left(y^3\operatorname{Li}_2\left(e^{-x-y}\right) + 6\operatorname{Li}_5\left(e^{-x-y}\right)\right)}{1-e^{x+y}} e^{x+y}\mathrm{d}x\mathrm{d}y$$

(continued)

$$= 3\left(\frac{17}{36}\zeta(8) + 11\zeta(3)\zeta(5) - 3\zeta(2)\zeta^2(3)\right); \tag{1.63}$$

$$iv) \int_0^\infty \int_0^\infty \frac{\log\left(1-e^{-x}\right)\left(y^4 \operatorname{Li}_2\left(e^{-x-y}\right) - 24\operatorname{Li}_6\left(e^{-x-y}\right)\right)}{1-e^{-x-y}} \mathrm{d}x\mathrm{d}y$$

$$= 24\left(7\zeta(2)\zeta(7) - \frac{19}{3}\zeta(3)\zeta(6) - \frac{15}{4}\zeta(4)\zeta(5) + \zeta^3(3)\right), \tag{1.64}$$

where ζ denotes the Riemann zeta function and Li_n represents the Polylogarithm function.

1.37 Exponential Double Integrals with an Appealing Look

Calculate

$$i) \int_0^\infty \int_0^\infty \left(\frac{e^{-x}-e^{-y}}{x-y}\right)\left(\frac{1-e^{-x}}{x}\right)\left(\frac{1-e^{-y}}{y}\right) \mathrm{d}x\mathrm{d}y;$$

$$ii) \int_0^\infty \int_0^\infty \left(\frac{e^{-x^2}-e^{-y^2}}{x^2-y^2}\right)\left(\frac{1-e^{-x^2}}{x^2}\right)\left(\frac{1-e^{-y^2}}{y^2}\right) \mathrm{d}x\mathrm{d}y.$$

1.38 A Generalized Double Integral Involving a Symmetrical Exponential Integrand and a Limit Related to It

Let $n \geq 1$ be an integer. Evaluate

$$i) \int_0^\infty \int_0^\infty \frac{x^n - y^n}{e^x - e^y} \mathrm{d}x\mathrm{d}y,$$

and then calculate the limit

$$ii) \lim_{n\to\infty}\left(\frac{1}{n!}\int_0^\infty \int_0^\infty \frac{x^n - y^n}{e^x - e^y} \mathrm{d}x\mathrm{d}y - 2n\right).$$

1.39 A Special Multiple Integral and a Limit of It Involving the Euler–Mascheroni Constant γ, the Euler's Number e, and the Famous π All at Once

Evaluate the multiple integral

$$i)\ I(n)=\int_0^\infty\int_0^\infty\cdots\int_0^\infty \frac{e^{-x_{n+1}^{2n}-x_{n+2}^{2n}-\ldots-x_{2n}^{2n}}-e^{-x_1^{2n}-x_2^{2n}-\ldots-x_n^{2n}}}{x_1^{2n}+x_2^{2n}+\ldots+x_n^{2n}-x_{n+1}^{2n}-x_{n+2}^{2n}-\ldots-x_{2n}^{2n}} \times \mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_{2n},$$

and then calculate the limit

$$ii)\ \lim_{n\to\infty} I(n).$$

1.40 Some Curious Integrals Involving the Hyperbolic Tangent, Also Having Beautiful Connections with the Beta Function

Show that

$$i)\ \int_0^\infty \tanh(x)\left(\frac{1}{x}-\frac{x}{\pi^2+x^2}\right)\mathrm{d}x=2;$$

$$ii)\ \int_0^\infty \tanh(\pi x)\left(\frac{1}{8x^3+2x}+\frac{1}{64x^3+4x}+\frac{1}{512x^3+8x}+\cdots\right)\mathrm{d}x=1.$$

Let $n\ge 1$ be a positive integer. Prove the general case

$$iii)\ \int_0^\infty \tanh(\pi x)\left(\frac{1}{x}-\frac{x}{n^2+x^2}\right)\mathrm{d}x=2H_{2n}-H_n,$$

where $H_n=\sum_{k=1}^n \frac{1}{k}$ is the nth harmonic number.

(continued)

Then, show that

$$iv)\ \int_0^\infty \frac{\pi^2x^2\tanh(\pi x)+3\tanh(\pi x)-3\pi x}{x^5}\mathrm{d}x = 3(31\zeta(5)-14\zeta(2)\zeta(3))$$

and

$$v)\ \int_0^\infty \frac{2\pi^2x^2\tanh(\pi x)+12\tanh(\pi x)-6\pi^2x^2\operatorname{csch}(2\pi x)-9\pi x}{x^5}\mathrm{d}x$$

$$= 12(31\zeta(5)-16\zeta(2)\zeta(3)),$$

where ζ denotes the Riemann zeta function.

A challenging question: How would you calculate the integral from the point $iii)$ by using Beta function?

1.41 A Little Integral-Beast from *Inside Interesting Integrals* Together with a Similar Version of It Tamed by Real Methods

Show that

$$i)\ \int_0^\infty \frac{\sin(\sin(x))}{x}e^{\cos(x)}\mathrm{d}x = \frac{\pi}{2}(e-1);$$

$$ii)\ \int_0^\infty \frac{\sin(x)\sin(\sin(x))}{x^2}e^{\cos(x)}\mathrm{d}x = \frac{\pi}{2}(e-1).$$

1.42 Ramanujan's Integrals with Beautiful Connections with the Digamma Function and Frullani's Integral

Let $a, b, p, q > 0$. Prove that

$$i)\ \int_0^\infty \frac{(1+ax)^{-p}-(1+bx)^{-q}}{x}\mathrm{d}x = \psi(q)-\psi(p)+\log\left(\frac{b}{a}\right). \qquad (1.65)$$

(continued)

Then, if $a, b, c > 0$, show that

$$ii)\ \int_0^1 \left(\frac{x^{a-1}}{1-x} - \frac{cx^{b-1}}{1-x^c}\right) \mathrm{d}x = \psi\left(\frac{b}{c}\right) - \psi(a) + \log(c), \tag{1.66}$$

where $\psi(x)$ is Digamma function.

A challenging question: Could we reduce each integral to a sum of two Digamma integrals and one Frullani's integral?

1.43 The Complete Elliptic Integral of the First Kind Ramanujan Is Asked to Calculate in the Movie *The Man Who Knew Infinity* Together with Another Question Originating from His Work

Prove that

$$i)\ K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2(\theta)}}$$

$$= \frac{\pi}{2}\left\{1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 k^6 + \cdots\right\},$$

where $K(k)$ is the complete elliptic integral of the first kind.

Using $i)$, show that

$$ii)\ \frac{\Gamma^2\left(\frac{1}{4}\right)}{\sqrt{(2\pi)^3}} = 1 - \left(\frac{1}{2}\right)^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 + \cdots,$$

where Γ represents the Gamma function.

1.44 The First Double Integral I Published in *La Gaceta de la RSME*, Together with Another Integral Similar to It

Prove that

$$i)\ \int_{\sqrt{3}-\sqrt{2}}^{1}\int_{\sqrt{3}-\sqrt{2}}^{1}\frac{1}{(x+y)^2+(1+xy)^2}\mathrm{d}x\mathrm{d}y=\frac{1}{3}\left(G+\frac{\pi}{8}\log\left(2-\sqrt{3}\right)\right),$$

where G is the Catalan's constant.

Let c be equal to $\sqrt{2}+1-\sqrt{2\left(\sqrt{2}+1\right)}$. Then, show that

$$ii)\ \int_{c}^{1}\int_{c}^{1}\frac{1}{(x+y)(1+xy)}\mathrm{d}x\mathrm{d}y=\left(\frac{\pi}{4}\right)^2-\left(\frac{\log(\sqrt{2}+1)}{2}\right)^2.$$

1.45 An Out-of-Order Integral with an Integrand Expressed in Terms of an Infinite Series and a Generalization of It

Calculate

$$i)\ \int_0^{\infty}\frac{1}{x}\sum_{n=1}^{\infty}\left(\left(2^n x+1\right)^{-2^{n+1}}-\left(2^{n+1}x+1\right)^{-2^n}\right)\mathrm{d}x;$$

$$ii)\ \int_0^{\infty}\frac{1}{x}\sum_{n=1}^{\infty}\left(\left(\theta^n x+1\right)^{-\theta^{n+1}}-\left(\theta^{n+1}x+1\right)^{-\theta^n}\right)\mathrm{d}x,$$

where $\theta>1$ is a real number.

1.46 Pretty Charming Ramanujan-Like (Double) Integral Representations of the Riemann Zeta Function and Its Derivative

Let $s > 1/2$ be a real number. Prove that

$$i) \int_0^1 \int_0^1 \frac{1}{x(1-(1-x)y)} \sum_{n=1}^{\infty} (xy)^{n^s} \mathrm{d}x\mathrm{d}y = \zeta(2s);$$

$$ii) \int_0^1 \int_0^1 \frac{\log(xy)}{x(1-(1-x)y)} \sum_{n=1}^{\infty} n \log(n)(xy)^n \mathrm{d}x\mathrm{d}y = \frac{\pi^2}{3} \log\left(\frac{2\pi e^{\gamma}}{A^{12}}\right),$$

where ζ denotes the Riemann zeta function, γ denotes the Euler–Mascheroni constant, and A represents the Glaisher–Kinkelin constant.

1.47 The Elementary Calculation of a Fractional Part Integral Naturally Arising in an Exotic Triple Fractional Part Integral

Let p, q be positive integers with $p \geq q$. Prove that

$$\int_0^1 x^p \left\{\frac{1}{x}\right\}^q \mathrm{d}x$$

$$= \frac{q!}{(p+1)!} \left((p-q)! - \sum_{i=1}^{q} \frac{(p-i+1)!}{(q-i+1)!}(\zeta(p-i+2)-1)\right), \quad (1.67)$$

where $\{x\}$ is the fractional part of x, $\lfloor x \rfloor$ is the integer part of x, and ζ represents the Riemann zeta function.

1.48 The Calculation of a Beautiful Triple Fractional Part Integral with a Cubic Power

Prove that

$$\int_0^1 \int_0^1 \int_0^1 \left(\left\{\frac{x}{y}\right\}\left\{\frac{y}{z}\right\}\left\{\frac{z}{x}\right\}\right)^3 \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= 1-\frac{3}{8}\zeta(2)-\frac{3}{8}\zeta(3)-\frac{3}{8}\zeta(4)+\frac{21}{320}\zeta(6)+\frac{7}{160}\zeta(8)+\frac{1}{40}\zeta^2(3)+\frac{1}{40}\zeta(2)\zeta(3)$$

$$+\frac{1}{20}\zeta(2)\zeta(5)+\frac{1}{16}\zeta(3)\zeta(4)+\frac{1}{20}\zeta(3)\zeta(5)+\frac{1}{20}\zeta(4)\zeta(5),$$

where $\{x\}$ is the fractional part of x and ζ represents the Riemann zeta function.

1.49 The Calculation of a Generalized Triple Fractional Part Integral with Positive Integer Powers

Let $n \geq 1$ be a natural number. Prove that

$$\mathscr{I}_n = \int_0^1 \int_0^1 \int_0^1 \left(\left\{\frac{x}{y}\right\}\left\{\frac{y}{z}\right\}\left\{\frac{z}{x}\right\}\right)^n \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= 1-\frac{3}{2(n+1)}\sum_{i=1}^{n}\zeta(i+1)+\frac{1}{(n+1)^2(n+2)}$$

$$\times\left(\sum_{i=1}^{n}\zeta(i+1)\right)\left(\sum_{i=1}^{n}(i+1)\zeta(i+2)\right),$$

where $\{x\}$ is the fractional part of x and ζ represents the Riemann zeta function.

1.50 A Pair of Cute Fractional Part Integrals Involving the Cotangent Function

Calculate

$$i)\ \int_0^{\pi/2} \frac{\{\cot(x)\}}{\cot(x)} \mathrm{d}x;$$

$$ii)\ \int_0^{\pi/2} \{\cot(x)\} \mathrm{d}x,$$

where $\{x\}$ represents the fractional part of x.

1.51 Playing with a Resistant Classical Integral Family to the Real Methods That Responds to the Tricks Involving the Use of the Cauchy–Schlömilch Transformation

Let $a, b > 0$. Prove that

$$I = \int_0^{\infty} \frac{\cos(ax)}{b^2 + x^2} \mathrm{d}x = \frac{\pi}{2b} e^{-ab}; \quad J = \int_0^{\infty} \frac{x \sin(ax)}{b^2 + x^2} \mathrm{d}x = \frac{\pi}{2} e^{-ab};$$

$$K = \int_0^{\infty} \frac{\cos(ax)}{(b^2 + x^2)^2} \mathrm{d}x = \frac{\pi(1 + ab)}{4b^3} e^{-ab};$$

$$L = \int_0^{\infty} \frac{x \sin(ax)}{\left(b^2 + x^2\right)^2} \mathrm{d}x = \frac{\pi a}{4b} e^{-ab}.$$

A challenging question: Is it possible to calculate the integrals exclusively by real methods?

1.52 Calculating a Somewhat Strange-Looking Quartet of Integrals Involving the Trigonometric Functions

Prove that

$$i)\ I=\int_0^{\pi/2} x^2\tan(x)\log(\sin(x))\mathrm{d}x$$

$$=\frac{1}{24}\log^4(2)+\frac{1}{2}\log^2(2)\zeta(2)-\frac{17}{16}\zeta(4)+\operatorname{Li}_4\left(\frac{1}{2}\right); \tag{1.68}$$

$$ii)\ J=\int_0^{\pi/2} x^3\tan(x)\log(\sin(x))\mathrm{d}x=\frac{\pi^5}{960}+\frac{\pi^3}{16}\log^2(2)-\frac{9}{16}\pi\log(2)\zeta(3); \tag{1.69}$$

$$iii)\ K=\int_0^{\pi/2} x^2\log(\sin(x))\log(\cos(x))\mathrm{d}x$$

$$=\frac{\pi^5}{320}+\frac{\pi^3}{16}\log^2(2)-\frac{\pi}{48}\log^4(2)-\frac{3}{8}\pi\log(2)\zeta(3)-\frac{\pi}{2}\operatorname{Li}_4\left(\frac{1}{2}\right); \tag{1.70}$$

$$iv)\ L=\int_0^{\pi/2} x^3\log(\sin(x))\log(\cos(x))\mathrm{d}x$$

$$=\frac{1449}{512}\zeta(6)+\frac{45}{16}\log^2(2)\zeta(4)-\frac{3}{32}\log^4(2)\zeta(2)-\frac{27}{16}\log(2)\zeta(2)\zeta(3)$$
$$-\frac{9}{4}\zeta(2)\operatorname{Li}_4\left(\frac{1}{2}\right), \tag{1.71}$$

where ζ denotes the Riemann zeta function and Li_n represents the Polylogarithm function.

A challenging question: How about calculating the integrals without making use of the Fourier series of $\log(\sin(x))$, $\log(\cos(x))$?

1.53 Two Beautiful Representations of Catalan's Constant, $G = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \cdots$

Show that

$$i)\ G = \frac{7\zeta(3)}{4\pi} + \frac{1}{\pi}\int_0^{\pi/2}\int_0^{\pi/2} \frac{\log\left(\cos\left(\frac{x}{2}\right)\right) - \log\left(\cos\left(\frac{y}{2}\right)\right)}{\cos(x) - \cos(y)} \mathrm{d}x\mathrm{d}y;$$

$$ii)\ G = \sqrt{2}\left(1 + \frac{1}{3^2} - \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} - \frac{1}{13^2} - \frac{1}{15^2} + \cdots\right)$$

$$-\int_0^1 \frac{\log(1+\sqrt{1+x^2})}{1+x^2}\mathrm{d}x,$$

where G represents the Catalan's constant and ζ denotes the Riemann zeta function.

1.54 Proving Two Equalities with Tough Integrals Involving Logarithms and Polylogarithms

Prove, without calculating each integral, that

$$i)\ 3\int_0^{\pi/2} \tan(x)\log^2(\sin(x))\operatorname{Li}_4\left(-\cot^2(x)\right)\mathrm{d}x$$

$$= 2\int_0^{\pi/2} \cot(x)\log^3\left(\frac{1}{\cos(x)}\right)\operatorname{Li}_3\left(-\tan^2(x)\right)\mathrm{d}x; \tag{1.72}$$

$$ii)\ 3\int_0^{\pi/2} \tan(x)\log^2(\sin(x))\operatorname{Li}_5\left(-\cot^2(x)\right)\mathrm{d}x$$

$$= \int_0^{\pi/2} \cot(x)\log^4(\cos(x))\operatorname{Li}_3\left(-\tan^2(x)\right)\mathrm{d}x, \tag{1.73}$$

where Li_n denotes the Polylogarithm function.

1.55 Tough Integrals with Logarithms, Polylogarithms, and Trigonometric and Hyperbolic Functions

Prove that

$$i)\ \int_0^{\pi/2} \cot(x)\log(\cos(x))\log^2(\sin(x))\,\mathrm{Li}_3\left(-\tan^2(x)\right)\mathrm{d}x$$
$$= \frac{109}{128}\zeta(7) - \frac{23}{32}\zeta(3)\zeta(4) + \frac{1}{16}\zeta(2)\zeta(5); \tag{1.74}$$

$$ii)\ \int_0^{\log(1+\sqrt{2})} \coth(x)\log(\sinh(x))\log\left(2-\cosh^2(x)\right)\mathrm{Li}_2\left(\tanh^2(x)\right)\mathrm{d}x$$
$$= \frac{73}{128}\zeta(5) - \frac{17}{64}\zeta(2)\zeta(3), \tag{1.75}$$

where ζ denotes the Riemann zeta function and Li_n represents the Polylogarithm function.

1.56 A Double Integral Hiding a Beautiful Idea About the Symmetry and (Possibly) an Unexpected Closed-Form

Calculate

$$\int_0^{\pi/4}\int_0^{\pi/4} \arctan(\cos(x)\cot(y))\sec^3(x)\tan^2(y)\sec^2(y)\mathrm{d}x\mathrm{d}y.$$

1.57 An Exciting Representation of Catalan's Constant with Trigonometric Functions and Digamma Function

Show that

$$G = \int_0^{\pi/4} \tan(x)\sum_{n=1}^{\infty}(-1)^{n-1}\left(\psi\left(\frac{n}{2}\right) - \psi\left(\frac{n+1}{2}\right) + \frac{1}{n}\right)\sin(2nx)\mathrm{d}x$$

(continued)

$$+\int_0^{\pi/4}\cot(x)\sum_{n=1}^{\infty}\left(\psi\left(\frac{n+1}{2}\right)-\psi\left(\frac{n}{2}\right)-\frac{1}{n}\right)\sin(2nx)\mathrm{d}x,$$

where G denotes the Catalan's constant and ψ represents the Digamma function.

1.58 Evaluating an Enjoyable Trigonometric Integral Involving the Complete Elliptic Integral of the First Kind at Its Roots

Evaluate

$$\int_0^{\pi/2}\frac{1}{(1+\sin^4(x))\sqrt{1+\sin^2(x)}}\mathrm{d}x=\int_0^{\pi/2}\frac{1}{(1+\cos^4(x))\sqrt{1+\cos^2(x)}}\mathrm{d}x.$$

A challenging question: How would you calculate the integral without using identities involving the elliptic integrals/hypergeometric functions?

1.59 Integrating Over an Infinite Product with Factors Containing the Secant and the Hyperbolic Secant with Powers of 2

Calculate

$$\int_0^{\infty}\left(\frac{1+\operatorname{sech}\left(2^{-1}x\right)}{1+\sec\left(2^{-1}x\right)}\right)^{2^1}\cdot\left(\frac{1+\operatorname{sech}\left(2^{-2}x\right)}{1+\sec\left(2^{-2}x\right)}\right)^{2^2}\cdot\left(\frac{1+\operatorname{sech}\left(2^{-3}x\right)}{1+\sec\left(2^{-3}x\right)}\right)^{2^3}\cdots\mathrm{d}x.$$

1.60 Linking Two Generalized Integrals Involving the Polylogarithm Function to Seductive Series

Let m be a positive integer. Prove that

$$i)\ \sin(\theta)\sin\left(\frac{\theta}{2}\right)\int_0^1 \frac{x}{(1-x)\left(1-2x\cos(\theta)+x^2\right)}(\zeta(m+1)-\operatorname{Li}_{m+1}(x))\mathrm{d}x$$

$$= (-1)^{m-1}\sum_{k=1}^{\infty}\frac{H_{k+1}}{(k+1)^{m+1}}\sin\left(\frac{k\theta}{2}\right)\sin\left(\frac{(k+1)\theta}{2}\right)$$

$$+ (-1)^{m-1}\sum_{i=2}^{m}(-1)^{i-1}\zeta(i)\sum_{k=1}^{\infty}\frac{\sin\left(\frac{k\theta}{2}\right)\sin\left(\frac{(k+1)\theta}{2}\right)}{(k+1)^{m-i+2}}; \tag{1.76}$$

$$ii)\ \sin\left(\frac{\theta}{2}\right)\int_0^1 \frac{x(\cos(\theta)-x)}{(1-x)(1-2x\cos(\theta)+x^2)}(\zeta(m+1)-\operatorname{Li}_{m+1}(x))\mathrm{d}x$$

$$= (-1)^{m-1}\sum_{k=1}^{\infty}\frac{H_{k+1}}{(k+1)^{m+1}}\sin\left(\frac{k\theta}{2}\right)\cos\left(\frac{(k+1)\theta}{2}\right)$$

$$+ (-1)^{m-1}\sum_{i=2}^{m}(-1)^{i-1}\zeta(i)\sum_{k=1}^{\infty}\frac{\sin\left(\frac{k\theta}{2}\right)\cos\left(\frac{(k+1)\theta}{2}\right)}{(k+1)^{m-i+2}}, \tag{1.77}$$

where $H_n = \sum_{k=1}^{n}\frac{1}{k}$ is the nth harmonic number, ζ denotes the Riemann zeta function, and Li_n represents the Polylogarithm function.

Chapter 2
Hints

2.1 A Powerful Elementary Integral

To get the desired result, we use the substitution $x = \sin(t)$ combined with the trigonometric identity, $\arccos(\alpha) = 2\arctan\left(\sqrt{\frac{1-\alpha}{1+\alpha}}\right)$, $-1 < \alpha \leq 1$.

2.2 A Pair of Elementary Logarithmic Integrals We Might Find Very Useful for Solving the Problems in the Book

For the integral from point $i)$ make use of the integration by parts to obtain a recurrence relation, and for the integral from the point $ii)$ make the change of variable $x/a = y$ and then use the integral from the point $i)$.

2.3 Four Logarithmic Integrals Strongly Connected with the League of Harmonic Series

The four integrals are related to each other, and for deriving the squared log version we need the simple log version, then for deriving the cubic log version, we need the squared log version and so on. For an elementary approach, use wisely the integration by parts.

Also, for the resulting harmonic sums we might need Abel's summation (finite version) which states that if $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ are two sequences of real numbers with $A_n = \sum_{k=1}^{n} a_k$, then we have

C. I. Vălean, *(Almost) Impossible Integrals, Sums, and Series*, Problem Books in Mathematics, https://doi.org/10.1007/978-3-030-02462-8_2

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} + \sum_{k=1}^{n} A_k (b_k - b_{k+1}). \tag{2.1}$$

A short, elementary proof of the Abel's summation may be found in [1, Theorem 2.20, p. 55].

For a different strategy of calculating the resulting sums, we may wisely employ the relations between the complete homogeneous symmetric polynomials and the power sums.

2.4 Two Very Useful Classical Logarithmic Integrals That May Arise in the Calculation of Some Tough Integrals and Series

Simply make use of the integration by parts.

2.5 A Couple of Practical Definite Integrals Expressed in Terms of the Digamma Function

For the point $i)$ of the problem, use the series representation of Digamma function, $\psi(x) = -\gamma + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+x} \right)$. Then, for the point $ii)$, after making the *proper* change of variable, employ the result from the point $i)$.

2.6 A Useful Special Generalized Integral Expressed in Terms of the Polylogarithm Function

Without loss of generality, assume that $0 < y < 1$, and write the integral as $-\frac{1}{y-1} \int_0^1 \frac{\log^n(x)}{1 - \frac{y}{y-1}x} \mathrm{d}x$. Then, make the change of variable $\frac{y}{y-1}x = t$ and apply repetitively the integration by parts.

2.7 Two Little Tricky Classical Logarithmic Integrals

For a first solution to the point i), make use of the following integral representation of the Beta function, $\displaystyle\int_0^1 \frac{x^{a-1}+x^{b-1}}{(1+x)^{a+b}}\mathrm{d}x = \mathrm{B}(a,b)$. For a second solution to the point i), we may use the algebraic identity $\dfrac{1}{2}\left((A+B)^2+(A-B)^2-2A^2\right) = B^2$, where we set $A = \log(1-x)$ and $B = \log(1+x)$. Further, to get a third solution, consider a generalized integral of the type $\displaystyle\int_0^x \frac{\log^2(1+t)}{t}\mathrm{d}t$. To get a solution to the point ii), consider the identity $(A+B)^2 = A^2+2AB+B^2$, where we set $A = \log(1-x)$ and $B = \log(1+x)$, together with the value of the integral from the point i).

2.8 A Special Trio of Integrals with $\log^2(1-x)$ and $\log^2(1+x)$

For a first approach of the part i) of the problem make use of the following integral representation of the Beta function, $\displaystyle\int_{-1}^1 (1-t)^{x-1}(1+t)^{y-1}\mathrm{d}t = 2^{x+y-1}\,\mathrm{B}(x,y)$. Then, to approach the parts ii) and iii), consider the algebraic identity $A^2B^2 = \dfrac{1}{12}\left((A+B)^4+(A-B)^4-2A^4-2B^4\right)$, where we set $A = \log(1-x)$ and $B = \log(1+x)$.

2.9 *A Darn Integral in Disguise* (Possibly Harder Than It Seems to Be?), an Integral with Two Squared Logarithms on the Half of the Unit Interval

Exploit the logarithmic integral, $\displaystyle\int_{1/2}^1 \frac{\log^2(2x)\log^2(2(1-x))}{x}\mathrm{d}x$.

2.10 The Evaluation of a Class of Logarithmic Integrals Using a Slightly Modified Result from *Table of Integrals, Series, and Products* by I.S. Gradshteyn and I.M. Ryzhik Together with a Series Result Elementarily Proved by Guy Bastien

Employ the second equality in (4.3),

$$\log(1+x)\log(1-x) = -\sum_{k=1}^{\infty} x^{2k}\frac{H_{2k}-H_k}{k} - \frac{1}{2}\sum_{k=1}^{\infty}\frac{x^{2k}}{k^2}.$$

2.11 Logarithmic Integrals Containing an Infinite Series in the Integrand, Giving Values in Terms of Riemann Zeta Function

Make use of the identity in (1.10), $\displaystyle\int_0^1 \frac{x^{s-1}}{1+x}\mathrm{d}x = \psi(s) - \psi\left(\frac{s}{2}\right) - \log(2).$

2.12 Two Appealing Integral Representations of $\zeta(4)$ and $\zeta(2)G$

Use the integration by parts and reduce everything to more convenient integrals.

2.13 A Special Pair of Logarithmic Integrals with Connections in the Area of the Alternating Harmonic Series

Consider a system of relations with $I+J$ and $I-J$.

2.14 Another Special Pair of Logarithmic Integrals with Connections in the Area of the Alternating Harmonic Series

Act as in the previous section.

2.15 A Class of Tricky and Useful Integrals with Consecutive Positive Integer Powers for the Logarithms

Start with the integration by parts.

2.16 A Double Integral and a Triple Integral, Beautifully Connected with the Advanced Harmonic Series

Did you see the double trigonometric integral from my *Preface*?

2.17 Let's Take Two Double Logarithmic Integrals with Beautiful Values Expressed in Terms of the Riemann Zeta Function

For both integrals, make use of the fact that

$$\int_0^1 \log^t\left(\frac{1}{x}\right)\log^{1-t}\left(\frac{1}{y}\right)\mathrm{d}t = \frac{\log\left(\frac{1}{x}\right)-\log\left(\frac{1}{y}\right)}{\log\left(\log\left(\frac{1}{x}\right)\right)-\log\left(\log\left(\frac{1}{y}\right)\right)}.$$

2.18 Interesting Integrals Containing the Inverse Tangent Function and the Logarithmic Function

For a global solution, which means to attack both integrals at once, we consider calculating $I+J$ and $I-J$. Then, for a second solution to I and J, carefully integrate by parts and reduce the calculations to the integrals given in (1.24) and (1.25) from Sect. 1.13. In order to get a third solution to I and J, we transform the integrals into series, and then reduce all to the calculation of the alternating series, $\displaystyle\sum_{k=1}^{\infty}(-1)^{k-1}\frac{H_{2k}}{2k}=\frac{1}{8}\left(\frac{5}{2}\zeta(2)-\log^2(2)\right)$ and $\displaystyle\sum_{k=1}^{\infty}(-1)^{k-1}\frac{H_{2k-1}}{2k-1}=G-\frac{\pi}{8}\log(2)$.

2.19 Interesting Integrals Involving the Inverse Tangent Function and Dilogarithm Function

A route to follow boils down to starting with the integration by parts and then considering the possibility of using double integrals with a symmetrical integrand where to exploit the symmetry.

2.20 More Interesting Integrals Involving the Inverse Tangent Function and the Logarithmic Function: The First Part

Make up a system of relations with $I - J$ and $I + J$.

2.21 More Interesting Integrals Involving the Inverse Tangent Function and the Logarithmic Function: The Second Part

Start with the integral J, integrate by parts, and then make the change of variable $x = \frac{1-y}{1+y}$. To calculate the integral I, note and use that $\int_0^1 \frac{\arctan^2(x)\log(1-x)}{1+x^2}\mathrm{d}x$ $= \int_0^1 \frac{\arctan^2(x)\log(1-x^2)}{1+x^2}\,\mathrm{d}x - \underbrace{\int_0^1 \frac{\arctan^2(x)\log(1+x)}{1+x^2}\mathrm{d}x}_{J}.$

2.22 Challenging Integrals Involving $\arctan(x)$, $\log(x)$, $\log(1-x)$, $\operatorname{Li}_2(x)$, and $\operatorname{Li}_2(x^2)$

As a possible strategy to start with, for the results from the first two points of the problem, make use of the integration by parts, and for the result from the point iii), combine the use of the Dilogarithm function identity, $\operatorname{Li}_2(x) + \operatorname{Li}_2(-x) = \frac{1}{2}\operatorname{Li}_2(x^2)$, with the integration by parts.

2.23 Two More Special Challenging Integrals Involving $\arctan(x)$, $\log(x)$, $\log(1+x)$, and $\operatorname{Li}_2(-x)$

Use the integration by parts.

2.24 A Challenging Integral with the Inverse Tangent Function and an Excellent Generalization According to the Even Positive Powers of the Logarithm

For the first part of the problem, one may initially show that the hard core of the integral reduces to calculating $\displaystyle\int_0^1 \left(\int_0^y \frac{\log(z/y)}{(y+z)(1+z^2)}\mathrm{d}z\right)\mathrm{d}y$, and then exploit the symmetry. Next, for the second part of the problem prove and use that for $-1 < s < 0$, we have the following result,

$$\int_0^\infty \frac{x^s}{(1+x)(1+y^2x^2)}\mathrm{d}x = \frac{\pi}{2}\csc\left(\frac{\pi s}{2}\right)\frac{y^{-s}}{1+y^2}$$
$$+\frac{\pi}{2}\sec\left(\frac{\pi s}{2}\right)\frac{y^{1-s}}{1+y^2} - \frac{\pi\csc(\pi s)}{1+y^2}.$$

2.25 Let's Tango with an Exciting Integral Involving the Inverse Tangent Integral, the Lerch Transcendent Function, and the Logarithm with Odd Positive Powers

Make use of the following integral representation of the Inverse tangent integral, $\operatorname{Ti}_2(x) = -\displaystyle\int_0^1 \frac{x\log(y)}{1+x^2y^2}\mathrm{d}y$, combined with the relation in (3.131).

2.26 A Superb Integral with Logarithms and the Inverse Tangent Function, and a Surprisingly Beautiful Generalization of It

For a first solution to either of the two points, use the result in (4.4), and for a second solution use that $2\displaystyle\int_0^\infty \frac{\log(x)}{(x+1)^2+t^2}\mathrm{d}x = \frac{\arctan(t)\log(1+t^2)}{t},\ t\in\mathbb{R}.$

2.27 A Kind of Deviant Pair of Integrals with Logarithms and Polylogarithms, Using Symmetry

Employ the result in (1.12) and exploit the symmetry.

2.28 Wonderful Integrals Containing the Logarithm and the Polylogarithm, Involving Beautiful Ideas About Symmetry: The First Part

Reduce the main integral to a double integral by using the integral in Sect. 1.6, and then calculate the double integral by exploiting the symmetry of the integrand.

2.29 Wonderful Integrals Containing the Logarithm and the Polylogarithm, Involving Beautiful Ideas About Symmetry: The Second Part

A similar approach as in the previous section.

2.30 Wonderful Integrals Containing the Logarithm and the Polylogarithm, Involving Beautiful Ideas About Symmetry: The Third Part

A similar approach as in the previous two sections.

2.31 Two Families of Special Polylogarithmic Integrals Expressed in Terms of Infinite Series with the Generalized Harmonic Number and the Tails of Some Functions

To establish the results, we may use the identities in (3.182) and (3.179) in the next section.

2.32 A Generalized Integral Beautifully Connected to a Spectacular (and Simultaneously Strange) Series

Split wisely the integral and approach the resulting integrals by means of recurrence relations.

2.33 A Special (and Possibly Slightly Daunting) Integral with Two Polylogarithms, $\mathrm{Li}_2\left(\frac{x}{x-1}\right)$ and $\mathrm{Li}_2\left(\frac{x}{x+1}\right)$

Make use of the Landen's identity and/or the generating function in (4.10).

2.34 Exciting Challenging Triple Integrals with the Dilogarithm

Reduce both triple integrals to the calculation of harmonic series.

2.35 A Curious Integral with Polylogarithms Connected to a Double Integral with a Symmetrical Exponential Integrand

Make use of the result in (1.12).

2.36 Double Integrals Expressed in Terms of the Exponential Function and the Polylogarithm (of Orders 2, 3, 4, 5, and 6)

For all four integrals, make use of the result in (4.6) and reduce all to the calculation of the resulting harmonic series (where one doesn't need to know the values of each particular harmonic series involved).

2.37 Exponential Double Integrals with an Appealing Look

For both integrals use that

$$-e^{-y}\int_0^1 e^{-(x-y)t}\,\mathrm{d}t = \frac{e^{-x}-e^{-y}}{x-y}.$$

2.38 A Generalized Double Integral Involving a Symmetrical Exponential Integrand and a Limit Related to It

To get the generalization from the first part of the problem, start with splitting the inner integral as $\displaystyle\int_0^{\infty} = \int_0^{y} + \int_y^{\infty}$ and exploit the symmetry of the integrand.

2.39 A Special Multiple Integral and a Limit of It Involving the Euler–Mascheroni Constant γ, the Euler's Number e, and the Famous π All at Once

Make use of the result in (3.205).

2.40 Some Curious Integrals Involving the Hyperbolic Tangent, Also Having Beautiful Connections with the Beta Function

Start with the point $iii)$ of the problem (which represents the pivot for the rest of the points of the problem) that can be easily proved by using that $\tanh(\pi x) = \displaystyle\frac{8x}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2+(2x)^2}$.

2.41 A Little Integral-Beast from *Inside Interesting Integrals* Together with a Similar Version of It Tamed by Real Methods

Make use of the result, $\displaystyle\sum_{n=1}^{\infty} \frac{p^n \sin(nx)}{n!} = e^{p\cos(x)} \sin(p\sin(x))$.

2.42 Ramanujan's Integrals with Beautiful Connections with the Digamma Function and Frullani's Integral

For the integral from the point i), rearrange it and make use of the Digamma function integral representation, $\psi(s+1) = -\gamma + \displaystyle\int_0^1 \frac{x^s - 1}{x-1} \mathrm{d}x$.

Then, for the integral from the point ii), rearrange it and make use of the following integral representation of the Digamma function, $\psi(x) = \displaystyle\int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}}\right) \mathrm{d}t$.

2.43 The Complete Elliptic Integral of the First Kind Ramanujan Is Asked to Calculate in the Movie *The Man Who Knew Infinity* Together with Another Question Originating from His Work

For the point i) of the problem, start by using the auxiliary result, $\displaystyle\int_0^{\pi/2} \frac{\mathrm{d}x}{1 - a\sin^2(x)} = \frac{\pi}{2} \frac{1}{\sqrt{1-a}}$. As regards the point ii) of the problem, use the result from the point i).

2.44 The First Double Integral I Published in *La Gaceta de la RSME*, Together with Another Integral Similar to It

For both integrals, make the changes of variable $x = \dfrac{1-u}{1+u}$ and $y = \dfrac{1-v}{1+v}$.

2.45 An Out-of-Order Integral with an Integrand Expressed in Terms of an Infinite Series and a Generalization of It

Recall and use Ramanujan's result in (1.65).

2.46 Pretty Charming Ramanujan-Like (Double) Integral Representations of the Riemann Zeta Function and Its Derivative

For the beginning start with the fact that either $\dfrac{1}{(1-x)(1-xy)} = \displaystyle\sum_{k=1}^{\infty} x^{k-1}\frac{1-y^k}{1-y}$ or $\dfrac{1}{1-(1-x)y} = \displaystyle\sum_{k=1}^{\infty}((1-x)y)^{k-1}$.

2.47 The Elementary Calculation of a Fractional Part Integral Naturally Arising in an Exotic Triple Fractional Part Integral

Start out with the change of variable $x = 1/y$, and then integrate by parts to establish a recurrence relation.

2.48 The Calculation of a Beautiful Triple Fractional Part Integral with a Cubic Power

One of the starting ways is to make the proper variable changes such that one brings the integration limits of the initial integral to the form $\displaystyle\int_0^1 \int_0^{1/y} \int_0^{1/y}$ and then apply the integration by parts by means of the Leibniz integral rule.

One also needs particular cases of the fractional part integral from the previous section.

2.49 The Calculation of a Generalized Triple Fractional Part Integral with Positive Integer Powers

For a first solution, use the same strategy as the one described in the previous section. In the calculations one will need to use that $I_{n,n} = \int_0^1 x^n \left\{\frac{1}{x}\right\}^n \mathrm{d}x = 1 - \frac{1}{n+1}\sum_{i=1}^{n}\zeta(i+1)$ and $I_{n+1,n} = \int_0^1 x^{n+1}\left\{\frac{1}{x}\right\}^n \mathrm{d}x = \frac{1}{2} - \frac{1}{(n+2)(n+1)}\sum_{i=1}^{n}(i+1)\zeta(i+2)$, where $n \geq 1$, which can be derived from the generalization in (1.67). The first integral also appears in [2, 2.21, p. 103].

For another solution, exploit the symmetry to reduce the main integral to the calculation of two simpler integrals.

2.50 A Pair of Cute Fractional Part Integrals Involving the Cotangent Function

For both integrals use the fact that $\{x\} = x - \lfloor x \rfloor$, where $\{x\}$ denotes the fractional part of x and $\lfloor x \rfloor$ represents the integer part of x, and reduce all to the calculation of the corresponding series. The corresponding series from the point $i)$ of the problem might require the well-known Euler's infinite product for the sine, $\sin(x) = x\prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2}\right)$, and the corresponding series from the point $ii)$ of the problem might require the use of the classical result, $\sum_{k=2}^{\infty}\zeta(k)t^{k-1} = -\psi(1-t)-\gamma$.

2.51 Playing with a Resistant Classical Integral Family to the Real Methods That Responds to the Tricks Involving the Use of the Cauchy–Schlömilch Transformation

Start by using that $\frac{1}{b^2+x^2} = \int_0^{\infty} e^{-(b^2+x^2)y}\mathrm{d}y$, and later in the calculations use the integration by parts to calculate $\int_0^{\infty} x^{2n}e^{-yx^2}\mathrm{d}x$.

2.52 Calculating a Somewhat Strange-Looking Quartet of Integrals Involving the Trigonometric Functions

For all four points we might want to prove and use that

$$\sum_{n=1}^{\infty}\left(\psi\left(\frac{n+1}{2}\right)-\psi\left(\frac{n}{2}\right)-\frac{1}{n}\right)\sin(2nx)=-\tan(x)\log(\sin(x)).$$

2.53 Two Beautiful Representations of Catalan's Constant, $G=1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{9^2}-\cdots$

For the first part of the problem, rearrange the integral properly to make use of the fact that $\log(1+u)-\log(1+v)=\int_0^1\left(\frac{u}{1+tu}-\frac{v}{1+tv}\right)\mathrm{d}t$.

Then, for the second part of the problem, one might start with the following integral representation of the Catalan's constant, $-\int_0^1\frac{\log(x)}{1+x^2}\mathrm{d}x=G$.

2.54 Proving Two Equalities with Tough Integrals Involving Logarithms and Polylogarithms

Exploit the result in (1.12).

2.55 Tough Integrals with Logarithms, Polylogarithms, Trigonometric, and Hyperbolic Functions

For the point $i)$ of the problem let the variable change $\sin^2(x)=y$, and for the point $ii)$ of the problem make the change of variable $\sinh^2(x)=y$.

2.56 A Double Integral Hiding a Beautiful Idea About the Symmetry and (Possibly) an Unexpected Closed-Form

Properly arrange the integral and then switch from polar to Cartesian coordinates.

2.57 An Exciting Representation of Catalan's Constant with Trigonometric Functions and Digamma Function

Start with the calculation of the inner series.

2.58 Evaluating an Enjoyable Trigonometric Integral Involving the Complete Elliptic Integral of the First Kind at Its Roots

Check the possibility of using a double integral with a symmetrical integrand, and then exploit the symmetry to reduce the calculations to simpler integrals.

2.59 Integrating Over an Infinite Product with Factors Containing the Secant and the Hyperbolic Secant with Powers of 2

Prove and exploit the result, $\sum_{n=1}^{\infty}\left(\frac{1}{\sin\left(2^{-n}x\right)}-\frac{1}{\sinh\left(2^{-n}x\right)}\right) = \coth\left(\frac{x}{2}\right) - \cot\left(\frac{x}{2}\right)$.

2.60 Linking Two Generalized Integrals Involving the Polylogarithm Function to Seductive Series

Consider the relaxed version (the second version) of *The Master Theorem of Series*.

References

1. Bonar, D.D., Khoury, M.J.: Real Infinite Series, Classroom Resource Materials. The Mathematical Association of America, Washington, DC (2006)
2. Furdui, O.: Limits, Series and Fractional Part Integrals. Problems in Mathematical Analysis. Springer, New York (2013)

Chapter 3
Solutions

3.1 A Powerful Elementary Integral

Solution Let's start out with a powerful elementary integral that in the present book will allow us to solve the Basel problem (I mentioned a few things about it in the Preface), or calculate a beautiful double integral we'll meet in one of the next sections, and at the same time, if properly tweaked, it will also help us to derive a useful classical inverse sine series which appears in the fourth chapter.

Now, to calculate the integral and prove the stated result, we start with the change of variable $x = \sin(t)$, and we get

$$\int_0^1 \frac{1}{(1+yx)\sqrt{1-x^2}}\mathrm{d}x = \int_0^{\pi/2} \frac{1}{1+y\sin(t)}\mathrm{d}t$$

$$= \int_0^{\pi/2} \frac{1}{\sin^2\left(\dfrac{t}{2}\right) + \cos^2\left(\dfrac{t}{2}\right) + 2y\sin\left(\dfrac{t}{2}\right)\cos\left(\dfrac{t}{2}\right)}\mathrm{d}t$$

$$= 2\int_0^{\pi/2} \frac{\tan\left(\dfrac{t}{2}\right)'}{\tan^2\left(\dfrac{t}{2}\right) + 2y\tan\left(\dfrac{t}{2}\right) + 1}\mathrm{d}t \overset{\tan(t/2)=u}{=} 2\int_0^1 \frac{1}{u^2+2yu+1}\mathrm{d}u$$

$$= 2\int_0^1 \frac{1}{(u+y)^2 + (\sqrt{1-y^2})^2}\mathrm{d}u \overset{u+y=t}{=} 2\int_y^{y+1} \frac{1}{t^2 + (\sqrt{1-y^2})^2}\mathrm{d}t$$

C. I. Vălean, *(Almost) Impossible Integrals, Sums, and Series*, Problem Books in Mathematics, https://doi.org/10.1007/978-3-030-02462-8_3

$$= \frac{2}{\sqrt{1-y^2}}\left(\arctan\left(\frac{y+1}{\sqrt{1-y^2}}\right) - \arctan\left(\frac{y}{\sqrt{1-y^2}}\right)\right)$$

$$= \frac{2}{\sqrt{1-y^2}} \arctan\left(\sqrt{\frac{1-y}{1+y}}\right)$$

$$\left\{\text{use the trigonometric identity, } \arccos(\alpha) = 2\arctan\left(\sqrt{\frac{1-\alpha}{1+\alpha}}\right)\right\}$$

$$= \frac{\arccos(y)}{\sqrt{1-y^2}},$$

and the solution is complete.

The indefinite integral form of the main integral in sine may also be found in [68, p. 181].

To answer the proposed *challenging question* about solving the Basel problem, let's first integrate with respect to y, from -1 to 1, both sides of the identity above, and then we have

$$\int_{-1}^{1}\left(\int_0^1 \frac{1}{(1+yx)\sqrt{1-x^2}}\mathrm{d}x\right)\mathrm{d}y = \int_{-1}^{1} \frac{\arccos(y)}{\sqrt{1-y^2}}\mathrm{d}y$$
$$= -\frac{1}{2}\arccos^2(y)\Big|_{y=-1}^{y=1} = \frac{\pi^2}{2}. \quad (3.1)$$

On the other hand, reversing the integration order, the double integral in (3.1) can be written as

$$\int_{-1}^{1}\left(\int_0^1 \frac{1}{(1+yx)\sqrt{1-x^2}}\mathrm{d}x\right)\mathrm{d}y = \int_0^1\left(\int_{-1}^1 \frac{1}{(1+yx)\sqrt{1-x^2}}\mathrm{d}y\right)\mathrm{d}x$$

$$= 2\int_0^1 \frac{\operatorname{arctanh}(x)}{x\sqrt{1-x^2}}\mathrm{d}x = -\int_0^1 \frac{\log\left(\dfrac{1-x}{1+x}\right)}{x\sqrt{1-x^2}}\mathrm{d}x \overset{(1-x)/(1+x)=y^2}{=} -4\int_0^1 \frac{\log(y)}{1-y^2}\mathrm{d}y$$

$$= -4\int_0^1 \sum_{n=1}^{\infty} y^{2n-2}\log(y)\mathrm{d}y = -4\sum_{n=1}^{\infty}\int_0^1 y^{2n-2}\log(y)\mathrm{d}y = 4\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}. \quad (3.2)$$

Combining the results in (3.1) and (3.2), we get $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$, and since we can write that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\sum_{n=1}^{\infty}\frac{1}{(2n)^2}+\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}+\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$, we are immediately led to

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}=\zeta(2),$$

and the solution to the *challenging question* is complete.

If you're interested in more solutions to the Basel problem, then you may find an exciting collection of proofs in [17, Chapter 11, pp. 225–227] and [21]. Or if you enjoyed a creative way involving the Fourier series, you might also like to see [63, Chapter 4, pp. 150–153]. Lastly, one might also try to calculate the series by the summation techniques of the contour integration (see [12, pp. 151–153]).

In a more general view, the series in the Basel problem is a particular case of the celebrated Riemann zeta function (see [118], [18, Chapter 4, pp. 39–46], [26], [62, Chapter 2, pp. 15–24]), defined as

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\cdots,\ \Re(s)>1,$$

and particular values of it we'll find spread throughout the book since the major part of the integrals and series we'll meet have values expressed in terms of it.

I've considered this problem to be the first to add to the present book since my first 2015 book proposal at Springer.

3.2 A Pair of Elementary Logarithmic Integrals We Might Find Very Useful for Solving the Problems in the Book

Solution It's not hard to guess in this section we have to deal with two integrals of high school level, strongly related to each other, and (as you'll see) that are proving to be very useful in our journey.

We'll find particular cases of the given logarithmic integrals, and especially cases of the integral from the point *i*), in many places in the book.

For the part *i*) of the problem, we start by denoting $I_{(m,n)}=\int_0^1 x^m\log^n(x)\mathrm{d}x$, and upon integrating by parts, we get

$$\underbrace{\int_0^1 x^m\log^n(x)\mathrm{d}x}_{I_{(m,n)}}=\underbrace{\frac{1}{m+1}x^{m+1}\log^n(x)\Big|_{x=0}^{x=1}}_{0}-\frac{n}{m+1}\underbrace{\int_0^1 x^m\log^{n-1}(x)\mathrm{d}x}_{I_{(m,n-1)}},$$

which we can rewrite in terms of a recurrence relation, $\dfrac{I_{(m,n)}}{I_{(m,n-1)}} = -\dfrac{n}{m+1}$ or

$$\frac{I_{(m,k)}}{I_{(m,k-1)}} = -\frac{k}{m+1}. \tag{3.3}$$

Giving values to k in (3.3), from $k = 1$ to n, and then multiplying out all these relations, we obtain

$$\frac{I_{(m,1)}}{I_{(m,0)}} \cdot \frac{I_{(m,2)}}{I_{(m,1)}} \cdots \frac{I_{(m,n)}}{I_{(m,n-1)}} = (-1)^n \frac{n!}{(m+1)^n},$$

and after simplifications, we arrive at

$$I_{(m,n)} = \int_0^1 x^m \log^n(x)\mathrm{d}x = (-1)^n \frac{n!}{(m+1)^n} I_{(m,0)} = (-1)^n \frac{n!}{(m+1)^{n+1}},$$

and the part $i)$ of the problem is finalized.

Next, for the part $ii)$ of the problem, we make the change of variable $x/a = y$, and then we obtain

$$\int_0^a x^m \log^n(x)\mathrm{d}x = a^{m+1}\int_0^1 y^m \log^n(ay)\mathrm{d}y = a^{m+1}\int_0^1 y^m (\log(y)+\log(a))^n \mathrm{d}y$$

$$= a^{m+1}\int_0^1 y^m \sum_{k=0}^{n} \binom{n}{k} \log^k(y)\log^{n-k}(a)\mathrm{d}y$$

{reverse the order of summation and integration}

$$= a^{m+1}\sum_{k=0}^{n} \binom{n}{k} \log^{n-k}(a) \int_0^1 y^m \log^k(y)\mathrm{d}y$$

{make use of the first part of the problem}

$$= a^{m+1}\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{k!}{(m+1)^{k+1}} \log^{n-k}(a),$$

and the part $ii)$ of the problem is finalized.

Alternatively, both points can be approached with the variable change $x = e^{-t}$, and then it's easy to see that for the first point of the problem we recognize the Gamma function (see [10, 107], [78, Chapter 1, pp. 1–4]), which is the Paul's way to calculate this integral in *Inside Interesting Integrals* (see [64, p. 146, p. 380]).

The special case, $a = 1/2$, of the integral from the point $ii)$ plays an important part in the derivation of a bunch of key series relations from the chapter *Sums and Series* which will allow us to obtain the values of some harmonic series involving the powers of 2 in denominator.

The integral from the point $i)$ also appears in [64, Chapter 4, p. 146] and [66, p. 441].

3.3 Four Logarithmic Integrals Strongly Connected with the League of Harmonic Series

Solution It is worth mentioning that the origin of such integrals dates back to the time of the English mathematician Joseph Wolstenholme (1829–1891), and the first proposed integral appeared in his appreciated book[1] with mathematical problems. We'll want to keep the present logarithmic integrals, like the ones from the previous section, close to us in our toolbox as we advance through the book, and we'll find them particularly useful in the problems involving the harmonic numbers. The next solutions also aim to answer the proposed *challenging question*.

Considering the integration by parts, we have

$$I_n = \int_0^1 x^{n-1}\log(1-x)\mathrm{d}x = \frac{1}{n}\int_0^1 (x^n-1)'\log(1-x)\mathrm{d}x$$
$$= \underbrace{\frac{1}{n}(x^n-1)\log(1-x)\Big|_{x=0}^{x=1}}_{0}$$
$$-\frac{1}{n}\int_0^1 \frac{1-x^n}{1-x}\mathrm{d}x = -\frac{1}{n}\int_0^1 \sum_{k=1}^{n} x^{k-1}\mathrm{d}x = -\frac{1}{n}\sum_{k=1}^{n}\int_0^1 x^{k-1}\mathrm{d}x$$
$$= -\frac{1}{n}\sum_{k=1}^{n}\frac{1}{k} = -\frac{H_n}{n},$$

and the calculation to the first logarithmic integral is finalized.

[1] Amongst the wonderful collection of problems Joseph Wolstenholme gathered in his book *A Book of Mathematical Problems, on subjects included in Cambridge course* published in 1867, one can also find the integral I_n at the page 214, the problem 1031, and the original statement is: *Prove that, if n be a positive integer*, $\int_0^1 (1-x)^{n-1}\log\left(\frac{1}{x}\right)\mathrm{d}x = \frac{1}{n}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)$. It's easy to see that letting the variable change $1-x=y$, then multiplying both sides by (-1) and using the notation $H_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$, we obtain the form in which I_n is proposed in the present book.

Now, let's pass to the second integral, J_n, and using the integration by parts as in the first integral, we get

$$J_n = \int_0^1 x^{n-1}\log^2(1-x)\mathrm{d}x = \frac{1}{n}\int_0^1 (x^n-1)'\log^2(1-x)\mathrm{d}x$$
$$= \underbrace{\frac{1}{n}(x^n-1)\log^2(1-x)\Big|_{x=0}^{x=1}}_{0}$$
$$-\frac{2}{n}\int_0^1 \frac{1-x^n}{1-x}\log(1-x)\mathrm{d}x = -\frac{2}{n}\int_0^1 \sum_{k=1}^n x^{k-1}\log(1-x)\mathrm{d}x$$
$$= -\frac{2}{n}\sum_{k=1}^n \int_0^1 x^{k-1}\log(1-x)\mathrm{d}x = -\frac{2}{n}\sum_{k=1}^n I_k = \frac{2}{n}\sum_{k=1}^n \frac{H_k}{k} = \frac{H_n^2+H_n^{(2)}}{n}.$$

To see the last equality holds, we exploit the symmetry, and we write

$$H_n^2 = \sum_{i=1}^n\sum_{j=1}^n \frac{1}{ij} = \sum_{i=1}^n \left(\sum_{j=1}^i + \sum_{j=i}^n\right)\frac{1}{ij} - \sum_{i=1}^n \frac{1}{i^2} = 2\sum_{i=1}^n\sum_{j=1}^i \frac{1}{ij} - H_n^{(2)}$$
$$= 2\sum_{i=1}^n \frac{H_i}{i} - H_n^{(2)},$$

whence we obtain that $\displaystyle\sum_{i=1}^n \frac{H_i}{i} = \frac{H_n^2+H_n^{(2)}}{2}$ (see [31, p. 280]), and the calculation to the second logarithmic integral is finalized.

Next, for calculating the integral K_n, we use again the integration by parts, and we obtain

$$K_n = \int_0^1 x^{n-1}\log^3(1-x)\mathrm{d}x = \frac{1}{n}\int_0^1 (x^n-1)'\log^3(1-x)\mathrm{d}x$$
$$= \underbrace{\frac{1}{n}(x^n-1)\log^3(1-x)\Big|_{x=0}^{x=1}}_{0}$$
$$-\frac{3}{n}\int_0^1 \frac{1-x^n}{1-x}\log^2(1-x)\mathrm{d}x = -\frac{3}{n}\int_0^1 \sum_{k=1}^n x^{k-1}\log^2(1-x)\mathrm{d}x$$
$$= -\frac{3}{n}\sum_{k=1}^n \int_0^1 x^{k-1}\log^2(1-x)\mathrm{d}x = -\frac{3}{n}\sum_{k=1}^n J_k = -\frac{3}{n}\sum_{k=1}^n \frac{H_k^2+H_k^{(2)}}{k}. \tag{3.4}$$

Using Abel's summation (see (2.1)) with $a_k = 1/k$ and $b_k = H_k^2 + H_k^{(2)}$, we have

$$\sum_{k=1}^{n} \frac{H_k^2 + H_k^{(2)}}{k} = H_n(H_{n+1}^2 + H_{n+1}^{(2)}) - 2\sum_{k=1}^{n}\left(\frac{H_k^2}{k+1} + \frac{H_k}{(k+1)^2}\right)$$

$$= H_n(H_{n+1}^2 + H_{n+1}^{(2)}) - 2\sum_{k=1}^{n}\left(\frac{(H_{k+1} - 1/(k+1))^2}{k+1} + \frac{H_{k+1} - 1/(k+1)}{(k+1)^2}\right)$$

{reindex the sum, expand it and then leave out the term for $k = n+1$}

$$= H_n(H_{n+1}^2 + H_{n+1}^{(2)}) - 2\sum_{k=1}^{n}\frac{H_k^2}{k} + 2\sum_{k=1}^{n}\frac{H_k}{k^2} - 2\frac{H_{n+1}^2}{n+1} + 2\frac{H_{n+1}}{(n+1)^2}. \tag{3.5}$$

Then, we apply Abel's summation (see (2.1)) for $\sum_{k=1}^{n}\frac{H_k}{k^2}$, and we get

$$\sum_{k=1}^{n}\frac{H_k}{k^2} = H_{n+1}H_n^{(2)} - \sum_{k=1}^{n}\frac{H_k^{(2)}}{k+1} = H_{n+1}H_n^{(2)} - \sum_{k=1}^{n}\frac{H_{k+1}^{(2)} - \frac{1}{(k+1)^2}}{k+1}$$

{reindex the sum, expand it, and then leave out the term for $k = n+1$}

$$= H_{n+1}H_n^{(2)} - \sum_{k=1}^{n}\frac{H_k^{(2)}}{k} - \frac{H_{n+1}^{(2)}}{n+1} + H_n^{(3)} + \frac{1}{(n+1)^3}. \tag{3.6}$$

Now, we plug the result from (3.6) in (3.5) that gives

$$\sum_{k=1}^{n} \frac{H_k^2 + H_k^{(2)}}{k} = -2\sum_{k=1}^{n} \frac{H_k^2 + H_k^{(2)}}{k} + H_n(H_{n+1}^2 + H_{n+1}^{(2)}) + 2H_{n+1}H_n^{(2)} - 2\frac{H_{n+1}^{(2)}}{n+1}$$

$$+2H_n^{(3)} + \frac{2}{(n+1)^3} - 2\frac{H_{n+1}^2}{n+1} + 2\frac{H_{n+1}}{(n+1)^2}$$

$$\left\{\text{use that } H_{n+1}^{(m)} = H_n^{(m)} + \frac{1}{(n+1)^m} \text{ and then expand}\right\}$$

$$= -2\sum_{k=1}^{n} \frac{H_k^2 + H_k^{(2)}}{k} + H_n^3 + 3H_nH_n^{(2)} + 2H_n^{(3)},$$

from which we get

$$\sum_{k=1}^{n} \frac{H_k^2 + H_k^{(2)}}{k} = \frac{1}{3}(H_n^3 + 3H_n H_n^{(2)} + 2H_n^{(3)}). \tag{3.7}$$

Lastly, plugging the result from (3.7) in (3.4), we obtain that

$$K_n = \int_0^1 x^{n-1} \log^3(1-x)\mathrm{d}x = -\frac{3}{n}\sum_{k=1}^{n} \frac{H_k^2 + H_k^{(2)}}{k} = -\frac{H_n^3 + 3H_n H_n^{(2)} + 2H_n^{(3)}}{n},$$

and the calculation to the third logarithmic integral is finalized.

Finally, for the integral L_n, we proceed as in the previous cases and carefully apply the integration by parts that leads to

$$L_n = \int_0^1 x^{n-1} \log^4(1-x)\mathrm{d}x = \frac{1}{n}\int_0^1 (x^n - 1)' \log^4(1-x)\mathrm{d}x$$
$$= \underbrace{\frac{1}{n}(x^n - 1)\log^4(1-x)\Big|_{x=0}^{x=1}}_{0}$$
$$-\frac{4}{n}\int_0^1 \frac{1-x^n}{1-x}\log^3(1-x)\mathrm{d}x = -\frac{4}{n}\int_0^1 \sum_{i=1}^{n} x^{i-1}\log^3(1-x)\mathrm{d}x$$
$$= -\frac{4}{n}\sum_{i=1}^{n}\int_0^1 x^{i-1}\log^3(1-x)\mathrm{d}x = -\frac{4}{n}\sum_{i=1}^{n} K_i = \frac{4}{n}\sum_{i=1}^{n} \frac{H_i^3 + 3H_i H_i^{(2)} + 2H_i^{(3)}}{i}. \tag{3.8}$$

For the last sum in (3.8) we can use again Abel's summation to get the desired result, as at the previous points of the problem, but this time we might prefer to recall and use the complete homogeneous symmetric polynomial, $h_k(x_1, x_2, \ldots, x_n) = \sum_{1\le i_1\le i_2\le\ldots\le i_k\le n} x_{i_1}x_{i_2}\ldots x_{i_k}$, which we'll want to further express in terms of power sums, $p_k(x_1, x_2, \ldots, x_n) = x_1^k + x_2^k + \ldots + x_n^k$, using the recursive relation, $nh_n = \sum_{k=1}^{n} p_k h_{n-k}$ (see [45, p. 23]). Inspecting the small cases, we easily observe we have that $h_1 = p_1$, $h_2 = (p_1^2 + p_2)/2$, $h_3 = (p_1^3 + 3p_1p_2 + 2p_3)/6$, $h_4 = (p_1^4 + 6p_1^2p_2 + 8p_1p_3 + 3p_2^2 + 6p_4)/24$, $h_5 = (p_1^5 + 10p_1^3p_2 + 15p_1p_2^2 + 20p_1^2p_3 + 20p_2p_3 + 30p_1p_4 + 24p_5)/120$. Alternatively, to get h_k expressed in terms of p_k, we may inspect the coefficients of the generating function, $\sum_{k=0}^{\infty} t^k h_k = \exp\left(\sum_{k=1}^{\infty} t^k \frac{p_k}{k}\right)$, which can be derived by using the generating function, $\sum_{k=0}^{\infty} t^k h_k = \prod_{i=1}^{\infty} \frac{1}{1 - x_i t}$.

If we take a look at the sums from the previous points, we notice immediately we are interested in h_4 expressed in terms of power sums, and write that

$$h_4 = \sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=1}^{j} \sum_{l=1}^{k} x_i x_j x_k x_l = \frac{1}{24}\Bigg(\left(\sum_{k=1}^{n} x_k\right)^4 + 6\left(\sum_{k=1}^{n} x_k\right)^2 \left(\sum_{k=1}^{n} x_k^2\right)$$

$$+ 8\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} x_k^3\right) + 3\left(\sum_{k=1}^{n} x_k^2\right)^2 + 6\sum_{k=1}^{n} x_k^4\Bigg). \tag{3.9}$$

Now, it's easy to see (already knowing the sums from the previous points) that

$$\sum_{i=1}^{n} \frac{H_i^3 + 3H_i H_i^{(2)} + 2H_i^{(3)}}{i} = 3\sum_{i=1}^{n} \frac{1}{i} \sum_{j=1}^{i} \frac{H_j^2 + H_j^{(2)}}{j} = 6\sum_{i=1}^{n} \sum_{j=1}^{i} \frac{1}{ij} \sum_{k=1}^{j} \frac{H_k}{k}$$

$$= 6\sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=1}^{j} \sum_{l=1}^{k} \frac{1}{ijkl} = \frac{1}{4}(H_n^4 + 6H_n^2 H_n^{(2)} + 8H_n H_n^{(3)} + 3\left(H_n^{(2)}\right)^2 + 6H_n^{(4)}), \tag{3.10}$$

where for getting the last equality I set $x_k = 1/k$ in (3.9). Thus, by plugging the result from (3.10) in (3.8), we conclude that

$$L_n = \int_0^1 x^{n-1} \log^4(1-x) = \frac{4}{n} \sum_{k=1}^{n} \frac{H_k^3 + 3H_k H_k^{(2)} + 2H_k^{(3)}}{k}$$

$$= \frac{H_n^4 + 6H_n^2 H_n^{(2)} + 8H_n H_n^{(3)} + 3(H_n^{(2)})^2 + 6H_n^{(4)}}{n},$$

and the calculation to the fourth logarithmic integral is finalized.

More generally, after checking the small cases, the curious reader might observe the general case involving $\log^m(1-x)$, with m a positive integer, may be expressed in terms of the complete homogeneous symmetric polynomial,

$$\int_0^1 x^{n-1} \log^m(1-x)\mathrm{d}x = \frac{(-1)^m m!}{n} h_m\left(1, \frac{1}{2}, \ldots, \frac{1}{n}\right). \tag{3.11}$$

We want to prove the result in (3.11) by induction on m, and assuming that $\int_0^1 x^{n-1}\log^{m-1}(1-x)\mathrm{d}x = \frac{(-1)^{m-1}(m-1)!}{n}h_{m-1}\left(1,\frac{1}{2},\ldots,\frac{1}{n}\right)$, and using the integration by parts, we have

$$\int_0^1 x^{n-1}\log^m(1-x)\mathrm{d}x = \frac{1}{n}\int_0^1 (x^n-1)'\log^m(1-x)\mathrm{d}x$$

$$= -\frac{m}{n}\int_0^1 \frac{x^n-1}{x-1}\log^{m-1}(1-x)\mathrm{d}x = -\frac{m}{n}\sum_{k=1}^{n}\int_0^1 x^{k-1}\log^{m-1}(1-x)\mathrm{d}x$$

$$= \frac{(-1)^m m!}{n}\sum_{k=1}^{n}\frac{1}{k}h_{m-1}\left(1,\frac{1}{2},\ldots,\frac{1}{k}\right) = \frac{(-1)^m m!}{n}h_m\left(1,\frac{1}{2},\ldots,\frac{1}{n}\right),$$

and the result in (3.11) is proved. Also, we may observe that every integral can be viewed as a particular case of the Beta function, $\mathrm{B}(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1}\mathrm{d}u$, $\Re(x) > 0$, $\Re(y) > 0$ (see [94], [61, Chapter 6, pp. 53–70], and [78, Chapter 1, pp. 9–11]), which we can use to derive the results differently.

The first two integrals, I_n and J_n, also appear in two of the papers I published (in one as co-author) in [93] and [86], where they were successfully used for the derivation of two harmonic series, or in [28]. The integral I_n may also be found in **4.293.8** from [30].

3.4 Two Very Useful Classical Logarithmic Integrals That May Arise in the Calculation of Some Tough Integrals and Series

Solution Both integrals appear in the precious book *Polylogarithms and associated functions* by Leonard Lewin (pp. 159–160), so well described in the Foreword of his book prepared by the Professor of Mathematics, A.J. van der Poorten, Macquarie University, Australia, *This is a delightful book filled with extraordinary identities and wonderful formulas*. In general, you'll find in there all you need about the Polylogarithm function, from definitions to sophisticated identities, formulas of great help in the calculations with integrals and series. It's a truly extraordinary work on the Polylogarithms we'll want to return to in our journey through the book. Also, one may find both integrals in the famous series of books *Integrals and Series* by A.P. Prudnikov, Yu.A. Brychkov, and O.I. Marichev (see [68, Chapter 1, pp. 251–252]).

The solutions below follow the same simple idea presented in Lewin's book which is mainly based upon applications of the integration by parts. Then, for the integral from the point $i)$ we start with the integration by parts that yields

$$\int_0^x \frac{\log^2(1-t)}{t}\mathrm{d}t = \log(t)\log^2(1-t)\Big|_{t=0}^{t=x} + 2\int_0^x \frac{\log(t)\log(1-t)}{1-t}\mathrm{d}t$$

$$= \log(x)\log^2(1-x) + 2\int_0^x \frac{\log(t)\log(1-t)}{1-t}\mathrm{d}t$$

$$= \log(x)\log^2(1-x) + 2\int_0^x (\mathrm{Li}_2(1-t))'\log(1-t)\mathrm{d}t$$

$$= \log(x)\log^2(1-x) + 2\,\mathrm{Li}_2(1-t)\log(1-t)\Big|_{t=0}^{t=x} + 2\int_0^x \frac{\mathrm{Li}_2(1-t)}{1-t}\mathrm{d}t$$

$$= \log(x)\log^2(1-x) + 2\log(1-x)\,\mathrm{Li}_2(1-x) + 2\left(-\mathrm{Li}_3(1-t)\Big|_{t=0}^{t=x}\right)$$

$$= \log(x)\log^2(1-x) + 2\log(1-x)\,\mathrm{Li}_2(1-x) - 2\,\mathrm{Li}_3(1-x) + 2\zeta(3),$$

and the solution to the first logarithmic integral result is finalized.

To calculate the integral from the point $ii)$, we start with the integration by parts as we did in the previous integral, and we get

$$\int_0^x \frac{\log^2(1+t)}{t}\mathrm{d}t = \log(t)\log^2(1+t)\Big|_{t=0}^{t=x} - 2\int_0^x \frac{\log(t)\log(1+t)}{1+t}\mathrm{d}t$$

$$= \log(x)\log^2(1+x) - 2\int_0^x \frac{\log(t)\log(1+t)}{1+t}\mathrm{d}t$$

$$= \log(x)\log^2(1+x) - 2\int_0^x \frac{\left(\log(1+t) + \log\left(\frac{t}{1+t}\right)\right)\log(1+t)}{1+t}\mathrm{d}t$$

$$= \log(x)\log^2(1+x) - 2\int_0^x \frac{\log^2(1+t)}{1+t}\mathrm{d}t - 2\int_0^x \frac{\log\left(\frac{t}{1+t}\right)\log(1+t)}{1+t}\mathrm{d}t$$

$$= \log(x)\log^2(1+x) - \frac{2}{3}\log^3(1+x) - 2\int_0^x \left(\mathrm{Li}_2\left(\frac{1}{1+t}\right)\right)'\log(1+t)\mathrm{d}t$$

$$= \log(x)\log^2(1+x) - \frac{2}{3}\log^3(1+x) - 2\left(\operatorname{Li}_2\left(\frac{1}{1+t}\right)\log(1+t)\Bigg|_{t=0}^{t=x}\right)$$

$$+2\int_0^x \frac{1}{1+t}\operatorname{Li}_2\left(\frac{1}{1+t}\right)\mathrm{d}t$$

$$= \log(x)\log^2(1+x) - \frac{2}{3}\log^3(1+x) - 2\log(1+x)\operatorname{Li}_2\left(\frac{1}{1+x}\right)$$

$$+2\left(-\operatorname{Li}_3\left(\frac{1}{1+t}\right)\Bigg|_{t=0}^{t=x}\right)$$

$$= \log(x)\log^2(1+x) - \frac{2}{3}\log^3(1+x) - 2\log(1+x)\operatorname{Li}_2\left(\frac{1}{1+x}\right)$$

$$-2\operatorname{Li}_3\left(\frac{1}{1+x}\right) + 2\zeta(3),$$

and the solution to the second logarithmic integral result is finalized.

For instance, the integral result from the point $i)$ appears in the derivation of the last two generating functions from Sect. 4.10, and we will want to have it ready in hand at the right time.

3.5 A Couple of Practical Definite Integrals Expressed in Terms of the Digamma Function

Solution Both integrals prepare us for an introduction to the terrific Digamma function since they are expressed in terms of the mentioned function, and let's remember that Digamma function is defined as the logarithmic derivative of the Gamma function, $\psi(x) = \dfrac{d}{dx}\log(\Gamma(x)) = \dfrac{\Gamma'(x)}{\Gamma(x)}$. Since in the book we'll have to deal a lot with the harmonic numbers, one might also like to know that the extension of the harmonic number, $H_n = \displaystyle\sum_{k=1}^{n}\frac{1}{k}$, for the non-integer values of n, is achieved through the Digamma function relation, $H_n = \psi(n+1)+\gamma$, which is very useful in many cases. If you're interested to learn more about Digamma function, a beautiful introduction together with various, superb integral representations of it may be found in [61, Chapter 10, pp. 119–143]. Also, both the integral from the point i), with the first equality, and the integral from the point ii) appear in the same book in [61, Chapter 11, p. 146, p. 153].

As regards the point i) of the problem, to prove the first equality, we make use of the series representation of Digamma function, $\psi(x) = -\gamma + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+x}\right)$, and then we write

$$\int_0^1 \frac{x^{s-1}}{1+x}\mathrm{d}x = \int_0^1 \sum_{n=1}^{\infty}(-1)^{n-1}x^{s+n-2}\mathrm{d}x$$

{change the order of summation and integration}

$$= \sum_{n=1}^{\infty}(-1)^{n-1}\int_0^1 x^{s+n-2}\mathrm{d}x = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n+s-1}$$

{rearrange the sum considering n odd and even}

$$= \sum_{n=1}^{\infty}\left(\frac{1}{2n+s-2} - \frac{1}{2n+s-1}\right)$$

{add and subtract $1/(2n)$ under the sum}

$$= \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+(s-1)/2} - \left(\frac{1}{n} - \frac{1}{n+(s-2)/2}\right)\right)$$

$$= \frac{1}{2}\left(\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+(s-1)/2}\right) - \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+(s-2)/2}\right)\right)$$

{reindex both series and start from $n = 0$}

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+(1+s)/2}\right) - \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+s/2}\right)\right)$$

{make use of the series representation of the Digamma function stated above}

$$= \frac{1}{2}\left(\psi\left(\frac{1+s}{2}\right) - \psi\left(\frac{s}{2}\right)\right),$$

and the first equality is proved.

To prove the second equality,

$$\frac{1}{2}\left(\psi\left(\frac{1+s}{2}\right)-\psi\left(\frac{s}{2}\right)\right)=\psi(s)-\psi\left(\frac{s}{2}\right)-\log(2),$$

we make use of the formula in [1], pp. 258–259,

$$\psi(2x)=\frac{1}{2}\psi(x)+\frac{1}{2}\psi\left(x+\frac{1}{2}\right)+\log(2), \tag{3.12}$$

where upon replacing x by $s/2$, we obtain that

$$\psi(s)=\frac{1}{2}\psi\left(\frac{s}{2}\right)+\frac{1}{2}\psi\left(\frac{1+s}{2}\right)+\log(2),$$

and the second equality is proved.

A first fast way of proving the result in (3.12) relies on the use of the Legendre duplication formula in [1], p. 256, which states that $\Gamma(2z)=\dfrac{2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)}{\sqrt{\pi}}$ (e.g., it can be proved by combining Beta function and Gamma function as shown in [112]), where if we take log of both sides and differentiate, we obtain the desired formula.

Now, to prove the result in (3.12) differently, we start with $\psi(2x)$, and we write that

$$\psi(2x)=-\gamma+\sum_{n=0}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+2x}\right)$$

$$\{\text{split the series according to } n \text{ even and odd}\}$$

$$=-\gamma+\sum_{n=0}^{\infty}\left(\frac{1}{2n+1}-\frac{1}{2n+2x}\right)+\sum_{n=0}^{\infty}\left(\frac{1}{2n+2}-\frac{1}{2n+1+2x}\right)$$

$$=-\gamma+\sum_{n=0}^{\infty}\left(\frac{1}{2n+1}-\frac{1}{2n+2}+\frac{1}{2n+2}-\frac{1}{2n+2x}\right)$$
$$+\sum_{n=0}^{\infty}\left(\frac{1}{2n+2}-\frac{1}{2n+1+2x}\right)$$

$$\{\text{split the first series}\}$$

$$= \underbrace{\sum_{n=0}^{\infty}\left(\frac{1}{2n+1}-\frac{1}{2n+2}\right)}_{\log(2)}\underbrace{-\frac{\gamma}{2}+\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+x}\right)}_{1/2\psi(x)}$$

$$\underbrace{-\frac{\gamma}{2}+\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+x+1/2}\right)}_{1/2\psi(x+1/2)} = \log(2)+\frac{1}{2}\psi(x)+\frac{1}{2}\psi\left(x+\frac{1}{2}\right),$$

and the point $i)$ of the problem is finalized.

This second equality is not immediately obvious, but it is important to be aware of it, and we might like to keep it close to us when passing to the next sections (where we'll find it very useful for a particular section).

Next, for the point $ii)$ of the problem, letting the change of variable $x = -\log(\sqrt{t})$, we obtain

$$\int_0^{\infty}\tanh(x)e^{-yx}\mathrm{d}x = \frac{1}{2}\int_0^1\frac{1-t}{1+t}t^{y/2-1}\mathrm{d}t = \frac{1}{2}\left(\int_0^1\frac{t^{y/2-1}}{1+t}\mathrm{d}t-\int_0^1\frac{t^{y/2}}{1+t}\mathrm{d}t\right)$$

{both integrals are cases of the integral from the point $i)$}

$$= \frac{1}{2}\left(\psi\left(\frac{y+2}{4}\right)-\frac{1}{2}\psi\left(\frac{y}{4}\right)-\frac{1}{2}\psi\left(1+\frac{y}{4}\right)\right)$$

$$\left\{\text{for the last Digamma function, use the recurrence relation, } \psi(x+1)=\psi(x)+\frac{1}{x}\right\}$$

$$= \frac{1}{2}\left(\psi\left(\frac{y+2}{4}\right)-\psi\left(\frac{y}{4}\right)-\frac{2}{y}\right),$$

and the point $ii)$ of the problem is finalized.

Alternatively, to calculate the integral we may use the result[2] in **3.311.11** from [30],

[2]To point out an idea of proving the result, one might note that $r-s>0$, and writing that $\int_0^{\infty}\frac{e^{px}-e^{qx}}{e^{rx}-e^{sx}}\mathrm{d}x = \int_0^{\infty}\frac{e^{(p-r)x}-e^{(q-r)x}}{1-e^{(s-r)x}}\mathrm{d}x$, and then using the geometric series combined with the fact that $e^{(p-r)x}-e^{(q-r)x} = \int_{q-r}^{p-r}xe^{xu}\mathrm{d}u$, we arrive at $\int_0^{\infty}\frac{e^{px}-e^{qx}}{e^{rx}-e^{sx}}\mathrm{d}x = \int_0^{\infty}\sum_{n=0}^{\infty}e^{(s-r)xn}\left(\int_{q-r}^{p-r}xe^{xu}\mathrm{d}u\right)\mathrm{d}x = \sum_{n=0}^{\infty}\int_{q-r}^{p-r}\left(\int_0^{\infty}xe^{((s-r)n+u)x}\mathrm{d}x\right)\mathrm{d}u = \sum_{n=0}^{\infty}\left(\frac{1}{(s-r)n+q-r}-\frac{1}{(s-r)n+p-r}\right) = \frac{1}{r-s}\left(\psi\left(\frac{r-q}{r-s}\right)-\psi\left(\frac{r-p}{r-s}\right)\right),$

$$\int_0^\infty \frac{e^{px}-e^{qx}}{e^{rx}-e^{sx}}\mathrm{d}x = \frac{1}{r-s}\left(\psi\left(\frac{r-q}{r-s}\right)-\psi\left(\frac{r-p}{r-s}\right)\right),$$

where $r > s$, $r > p$, and $r > q$.

The integral from the point $ii)$ might be preferred instead of the integral from the point $i)$ in some calculations, and in one of the next sections we'll be convinced of its usefulness during the derivation of a curious helpful series.

3.6 A Useful Special Generalized Integral Expressed in Terms of the Polylogarithm Function

Solution *Why to take into consideration such an integral? Where is it useful?*, a couple of questions one might immediately ask. On the other hand, physicists might react saying *This looks like a disguised form of the integrals of the Bose–Einstein and Fermi–Dirac distributions*. And, yes, to see this we might simply let the variable change $x = e^{-t}$ in the main integral and then compare it with the integrals (3) and (4) that appear in [117]. In fact, we have in front of our eyes a formidable integral which plays an important part in the derivation of some very useful results we'll want to employ for some of the problems in the book.

Without loss of generality, we assume that $0 < y < 1$, and then we write that

$$\int_0^1 \frac{\log^n(x)}{1-y+yx}\mathrm{d}x = -\frac{1}{y-1}\int_0^1 \frac{\log^n(x)}{1-\dfrac{y}{y-1}x}\mathrm{d}x$$

$$\overset{\frac{y}{y-1}x=t}{=} -\frac{1}{y}\int_0^{y/(y-1)} \frac{\log^n\left(\dfrac{y-1}{y}t\right)}{1-t}\mathrm{d}t$$

$$= \frac{1}{y}\int_0^{y/(y-1)} (\log(1-t))' \log^n\left(\frac{y-1}{y}t\right)\mathrm{d}t$$

$$\{\text{apply the integration by parts}\}$$

where the last equality is obtained by using the Digamma function series definition, $\psi(z) = -\gamma + \sum_{n=0}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+z}\right)$.

$$= \underbrace{\frac{1}{y}\log(1-t)\log^n\left(\frac{y-1}{y}t\right)\Bigg|_{t=0}^{t=y/(y-1)}}_{0}$$

$$-\frac{n}{y}\int_0^{y/(y-1)}\frac{\log(1-t)}{t}\log^{n-1}\left(\frac{y-1}{y}t\right)\mathrm{d}t$$

$$= \frac{n}{y}\int_0^{y/(y-1)}(\mathrm{Li}_2(t))'\log^{n-1}\left(\frac{y-1}{y}t\right)\mathrm{d}t$$

{apply again the integration by parts}

$$= \underbrace{\frac{n}{y}\mathrm{Li}_2(t)\log^{n-1}\left(\frac{y-1}{y}t\right)\Bigg|_{t=0}^{t=y/(y-1)}}_{0}$$

$$-\frac{n(n-1)}{y}\int_0^{y/(y-1)}\frac{\mathrm{Li}_2(t)}{t}\log^{n-2}\left(\frac{y-1}{y}t\right)\mathrm{d}t$$

$$= -\frac{n(n-1)}{y}\int_0^{y/(y-1)}(\mathrm{Li}_3(t))'\log^{n-2}\left(\frac{y-1}{y}t\right)\mathrm{d}t$$

{apply the integration by parts for another $n-2$ times}

$$= (-1)^{n-1}\frac{n(n-1)(n-2)\cdots 1}{y}\int_0^{y/(y-1)}\frac{\mathrm{Li}_n(t)}{t}\mathrm{d}t$$

$$= (-1)^{n-1}\frac{n!}{y}\mathrm{Li}_{n+1}(t)\Bigg|_{t=0}^{t=y/(y-1)}$$

$$= (-1)^{n-1}n!\frac{\mathrm{Li}_{n+1}\left(\dfrac{y}{y-1}\right)}{y},$$

and the solution is complete.

Alternatively, when working with $y \in (-\infty, 0)$, we note that $0 < \dfrac{y}{y-1} < 1$, and we can use the geometric series for the calculation of $-\dfrac{1}{y-1}\displaystyle\int_0^1 \frac{\log^n(x)}{1-\dfrac{y}{y-1}x}\mathrm{d}x$.

The use of the generalized integral in the present book is essential for expressing the Polylogarithms of the type $\operatorname{Li}_n\left(\dfrac{x}{x-1}\right)$ in terms of series involving the harmonic numbers as we'll see in the chapter *Sums and Series*. To have an idea on how these series would look like, before arriving to the dedicated section from the next chapter where I treat such series representations, it is enough to recall the well-known Landen's identity (see [44, Chapter 1, p. 5], [78, Chapter 2, p. 107]), $\operatorname{Li}_2(x)+\operatorname{Li}_2\left(\dfrac{x}{x-1}\right)=-\dfrac{1}{2}\log^2(1-x)$, together with the series representations $\displaystyle\sum_{n=1}^{\infty}x^{n+1}\frac{H_n}{n+1}=\frac{1}{2}\log^2(1-x)$ and $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{n^2}=\operatorname{Li}_2(x)$.

3.7 Two Little Tricky Classical Logarithmic Integrals

Solution The first encounter of a newcomer, a less experienced one with such a pair of integrals could be a memorable one, and probably being very satisfied at the moment of finding a solution since they might appear pretty challenging. Then, once a solution has been found, one shouldn't be much surprised to see other challenges around the same integrals like *How would you calculate them without using Polylogarithms?* or *How would you calculate them without using special functions at all, no use of Polylogarithm, Beta function?* and so on.

For a first solution to the point i) of the problem, let's recollect the integral[3] representation of the Beta function given in **3.216.1** from [30],

$$\int_0^1 \frac{x^{a-1}+x^{b-1}}{(1+x)^{a+b}}\mathrm{d}x=\mathrm{B}(a,b). \tag{3.13}$$

Differentiating both sides of (3.13) with respect to a (once) and b (once) and then letting $a,b\to 0$, we get

[3]If we denote the integral by $I=\displaystyle\int_0^1\frac{x^{a-1}+x^{b-1}}{(1+x)^{a+b}}\mathrm{d}x=\mathrm{B}(a,b)$ and then make the change of variable $x=1/y$, we arrive at $I=\displaystyle\int_1^\infty\frac{x^{a-1}+x^{b-1}}{(1+x)^{a+b}}\mathrm{d}x$. Adding up the two integrals, we get $I=\dfrac{1}{2}\displaystyle\int_0^\infty\frac{x^{a-1}+x^{b-1}}{(1+x)^{a+b}}\mathrm{d}x=\frac{1}{2}\left(\int_0^\infty\frac{x^{a-1}}{(1+x)^{a+b}}\mathrm{d}x+\int_0^\infty\frac{x^{b-1}}{(1+x)^{a+b}}\mathrm{d}x\right)=$ $\dfrac{1}{2}(\mathrm{B}(a,b)+\mathrm{B}(b,a))=\mathrm{B}(a,b)$, where I used that $\mathrm{B}(a,b)=\mathrm{B}(b,a)$, due to symmetry, and the integral representation of the Beta function, $\displaystyle\int_0^\infty\frac{x^{a-1}}{(1+x)^{a+b}}\mathrm{d}x=\mathrm{B}(a,b)$. If we make the change of variable $\dfrac{x}{1+x}=y$ in the last integral, we obtain the definition of the Beta function, $\mathrm{B}(a,b)=\displaystyle\int_0^1 t^{a-1}(1-t)^{b-1}\mathrm{d}t$.

$$2\int_0^1 \left(\frac{\log^2(1+x)}{x} - \frac{\log(x)\log(1+x)}{x}\right) \mathrm{d}x$$

$$= 2\int_0^1 \frac{\log^2(1+x)}{x}\mathrm{d}x - 2\int_0^1 \frac{\log(x)\log(1+x)}{x}\mathrm{d}x = \lim_{\substack{a\to 0\\ b\to 0}} \frac{\partial^2}{\partial a\partial b}\,\mathrm{B}(a,b)$$

$$\{\text{use the integral representation of Beta function, } \mathrm{B}(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\mathrm{d}x\}$$

$$= \int_0^1 \frac{\log(x)\log(1-x)}{x(1-x)}\mathrm{d}x = \int_0^1 \frac{\log(x)\log(1-x)}{x}\mathrm{d}x + \int_0^1 \frac{\log(x)\log(1-x)}{1-x}\mathrm{d}x$$

{the two integrals in the right-hand side are equal, and to see}

{that use the change of variable $1 - x = y$ in either of them}

$$= 2\int_0^1 \frac{\log(x)\log(1-x)}{x}\mathrm{d}x,$$

from which we extract the value of the desired integral,

$$\int_0^1 \frac{\log^2(1+x)}{x}\mathrm{d}x = \int_0^1 \frac{\log(x)\log(1-x^2)}{x}\mathrm{d}x \overset{x^2=y}{=} \frac{1}{4}\int_0^1 \frac{\log(y)\log(1-y)}{y}\mathrm{d}y$$

$$= -\frac{1}{4}\int_0^1 \sum_{n=1}^{\infty} \frac{y^{n-1}}{n}\log(y)\mathrm{d}y = -\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1 y^{n-1}\log(y)\mathrm{d}y$$

$$= \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^3} = \frac{1}{4}\zeta(3),$$

which finalizes the first solution to the point *i*).

For a second solution, we might like to use a powerful (and easy to apply) technique based on the use of the elementary algebraic identities, exceptionally efficient when dealing with some logarithmic integrals.

Let's start with the well-known algebraic identities $(A+B)^2 = A^2 + 2AB + B^2$ and $(A-B)^2 = A^2 - 2AB + B^2$ that if we add up, we get that $(A+B)^2 + (A-B)^2 = 2A^2 + 2B^2$ or $\frac{1}{2}\left((A+B)^2 + (A-B)^2 - 2A^2\right) = B^2$. Setting in the last identity $A = \log(1-x)$ and $B = \log(1+x)$, dividing both side by x, and then considering the integration over both sides, from $x = 0$ to $x = 1$, we get

$$\int_0^1 \frac{\log^2(1+x)}{x}\mathrm{d}x$$

$$= \underbrace{\frac{1}{2}\int_0^1 \frac{\log^2(1-x^2)}{x}\mathrm{d}x}_{X} + \underbrace{\frac{1}{2}\int_0^1 \frac{1}{x}\log^2\left(\frac{1-x}{1+x}\right)\mathrm{d}x}_{Y} - \underbrace{\int_0^1 \frac{\log^2(1-x)}{x}\mathrm{d}x}_{Z}. \tag{3.14}$$

For the integral X in (3.14), we obtain

$$X = \int_0^1 \frac{\log^2(1-x^2)}{x}\mathrm{d}x \overset{1-x^2=y}{=} \frac{1}{2}\int_0^1 \frac{\log^2(y)}{1-y}\mathrm{d}y = \frac{1}{2}\int_0^1 \sum_{n=1}^{\infty} y^{n-1}\log^2(y)\mathrm{d}y$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\int_0^1 y^{n-1}\log^2(y)\mathrm{d}y = \sum_{n=1}^{\infty}\frac{1}{n^3} = \zeta(3). \tag{3.15}$$

Then, for the integral Y in (3.14), we get

$$Y = \int_0^1 \frac{1}{x}\log^2\left(\frac{1-x}{1+x}\right)\mathrm{d}x \overset{\frac{1-x}{1+x}=y}{=} 2\int_0^1 \frac{\log^2(y)}{1-y^2}\mathrm{d}y = 2\int_0^1 \sum_{n=1}^{\infty} y^{2(n-1)}\log^2(y)\mathrm{d}y$$

$$= 2\sum_{n=1}^{\infty}\int_0^1 y^{2(n-1)}\log^2(y)\mathrm{d}y = 4\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3} = \frac{7}{2}\zeta(3). \tag{3.16}$$

Lastly, for the integral Z in (3.14), we have

$$Z = \int_0^1 \frac{\log^2(1-x)}{x}\mathrm{d}x \overset{x=y^2}{=} 2\int_0^1 \frac{\log^2(1-y^2)}{y}\mathrm{d}y = 2\zeta(3), \tag{3.17}$$

since $Z = 2X$, and X is the integral in (3.15).

Collecting the values of the integrals X, Y, Z from (3.15), (3.16), and (3.17) in (3.14), we obtain that

$$\int_0^1 \frac{\log^2(1+x)}{x}\mathrm{d}x = \frac{1}{4}\zeta(3), \tag{3.18}$$

and the second solution to the point i) is finalized.

Historically, as I have recently found, we could say that such ways exploiting the algebraic identities have been used for successfully defeating the logarithmic integrals at least since P.J. De Doelder published the article *On some series containing $\psi(x)-\psi(y)$ and $(\psi(x)-\psi(y))^2$ for certain values of x and y* (1991), and we shouldn't be surprised if finding even older sources with such strategies, considering the simplicity of the method.

For a third solution, the value of the integral is obtained by considering the more general integral in (1.9) which leads to

$$\int_0^1 \frac{\log^2(1+x)}{x}\mathrm{d}x = 2\zeta(3) - \frac{2}{3}\log^3(2) - 2\log(2)\operatorname{Li}_2\left(\frac{1}{2}\right) - 2\operatorname{Li}_3\left(\frac{1}{2}\right) = \frac{1}{4}\zeta(3),$$

and the third solution to the point i) is finalized. In the calculations I used the special value of Dilogarithm function,

$$\operatorname{Li}_2\left(\frac{1}{2}\right) = \frac{1}{2}(\zeta(2) - \log^2(2)), \tag{3.19}$$

and the special value of Trilogarithm function,

$$\operatorname{Li}_3\left(\frac{1}{2}\right) = \frac{7}{8}\zeta(3) - \frac{1}{2}\zeta(2)\log(2) + \frac{1}{6}\log^3(2), \tag{3.20}$$

you may also find in [78, Chapter 2, p. 114]. How do we derive them? We need the relations,

$$\operatorname{Li}_2(x) + \operatorname{Li}_2(1-x) = \zeta(2) - \log(x)\log(1-x) \tag{3.21}$$

and

$$\operatorname{Li}_3\left(\frac{-x}{1-x}\right) + \operatorname{Li}_3(1-x) + \operatorname{Li}_3(x)$$

$$= \zeta(3) + \zeta(2)\log(1-x) - \frac{1}{2}\log(x)\log^2(1-x) + \frac{1}{6}\log^3(1-x), \tag{3.22}$$

we find in [44, Chapter 1, p. 5], [78, Chapter 2, p. 107], and [44, Chapter 6, p. 155], [78, Chapter 2, p. 113]. Setting $x = 1/2$ in (3.21) and (3.22), we get the special values mentioned above. In Lewin's book one can also find a derivation of the relations I used.

In order to calculate the value of the integral from the point ii), we may take benefit of the value of the integral from the point i), and returning to the identity $(A+B)^2 = A^2 + 2AB + B^2$ where we set $A = \log(1-x)$ and $B = \log(1+x)$, we get

$$\log^2(1-x^2) = \log^2(1-x) + 2\log(1-x)\log(1+x) + \log^2(1+x),$$

and if dividing both sides by x and then integrating from $x = 0$ to $x = 1$, we obtain

$$\int_0^1 \frac{\log(1-x)\log(1+x)}{x}\mathrm{d}x$$

$$= \frac{1}{2}\left(\int_0^1 \frac{\log^2(1-x^2)}{x}\mathrm{d}x - \int_0^1 \frac{\log^2(1-x)}{x}\mathrm{d}x - \int_0^1 \frac{\log^2(1+x)}{x}\mathrm{d}x\right)$$

$$\{\text{the value of the first two integrals are given in (3.15) and (3.17)}\}$$

$$= -\frac{5}{8}\zeta(3),$$

and the point $ii)$ of the problem is finalized.

Also, if we use algebraic identities in the last integral, it's easy to note we can write that $\displaystyle\int_0^1 \frac{\log(1-x)\log(1+x)}{x}\mathrm{d}x = \frac{1}{4}\int_0^1 \frac{\log^2(1-x^2)}{x}\mathrm{d}x - \frac{1}{4}\int_0^1 \frac{1}{x}\log^2\left(\frac{1-x}{1+x}\right)\mathrm{d}x$, which offers an easy alternative way of making the calculations without using the value of the integral from the point $i)$.

As a final note of the section, the integral calculation technique involving the manipulation of the algebraic identities is good to be kept in our toolbox before passing to the next sections. Also keep in mind that Beta function can be very useful in many calculations (if we are inspired to choose the right representation of it).

Other different ways of evaluating the integral from the point $i)$ may be found in [51] and [52].

3.8 A Special Trio of Integrals with $\log^2(1-x)$ and $\log^2(1+x)$

Solution If you followed the solutions from the previous section, I guess you might propose immediately the use of the algebraic identities technique employed there to deal with the integral from the point $i)$, and eventually this will lead you to the desired answer. In the integrand we have squared logarithms, but how about if having, say, 5 or higher integer values instead of 2 for both logarithms, like $\displaystyle\int_0^1 \log^5(1-x)\log^5(1+x)\mathrm{d}x$? Is there a shortcut?

For the part $i)$ of the problem, we can start with the following integral representation of the Beta function,

$$\int_{-1}^{1}(1-t)^{x-1}(1+t)^{y-1}\mathrm{d}t = 2^{x+y-1}\,\mathrm{B}(x,y), \tag{3.23}$$

which is derived from the Beta function definition, $\mathrm{B}(x,y)=\displaystyle\int_0^1 u^{x-1}(1-u)^{y-1}\mathrm{d}u$, by making the change of variable $1-2u=t$, and at the same time it can be viewed as a particular case of the result in **3.196.3** from [30], $\displaystyle\int_a^b (x-a)^{u-1}(b-x)^{v-1}\mathrm{d}x = (b-a)^{u+v-1}\,\mathrm{B}(u,v)$. Then, by differentiation, we obtain

$$\int_{-1}^{1}(1-t)^{x-1}\log^2(1-t)(1+t)^{y-1}\log^2(1+t)\mathrm{d}t = \frac{\partial^4}{\partial x^2\partial y^2}\left(2^{x+y-1}B(x,y)\right). \tag{3.24}$$

Therefore, considering the result in (3.24), we have

$$\int_{-1}^{1} \log^2(1-t)\log^2(1+t)\mathrm{d}t = \lim_{\substack{x\to 1\\ y\to 1}} \frac{\partial^4}{\partial x^2 \partial y^2}\left(2^{x+y-1}B(x,y)\right),$$

and if we use that $\int_{-1}^{1} \log^2(1-t)\log^2(1+t)\mathrm{d}t = 2\int_{0}^{1} \log^2(1-t)\log^2(1+t)\mathrm{d}t$, due to symmetry, we get

$$\int_{0}^{1} \log^2(1-t)\log^2(1+t)\mathrm{d}t = \frac{1}{4}\lim_{\substack{x\to 1\\ y\to 1}} \frac{\partial^4}{\partial x^2 \partial y^2}\left(2^{x+y}B(x,y)\right). \tag{3.25}$$

For the right-hand side of (3.25), we might prefer to use *Mathematica* to accomplish the calculations and conclude that

$$\int_{0}^{1} \log^2(1-x)\log^2(1+x)\mathrm{d}x$$

$$= 24 - 8\zeta(2) - 8\zeta(3) - \zeta(4) + 8\log(2)\zeta(2) - 4\log^2(2)\zeta(2) + 8\log(2)\zeta(3)$$

$$-24\log(2) + 12\log^2(2) - 4\log^3(2) + \log^4(2),$$

which is the desired result.

More generally, we can use the same strategy employed above to calculate the generalized integral, $\int_{0}^{1} \log^n(1-x)\log^n(1+x)\mathrm{d}x,\ n \geq 1$, that can be expressed as

$$\int_{0}^{1} \log^n(1-t)\log^n(1+t)\mathrm{d}t = \frac{1}{4}\lim_{\substack{x\to 1\\ y\to 1}} \frac{\partial^{2n}}{\partial x^n \partial y^n}\left(2^{x+y}B(x,y)\right).$$

Once again, we have seen that by a creative use of the Beta function, we manage to arrive at beautiful, (and often) simpler solutions.

Alternatively, we might start by considering the following algebraic identities $(A+B)^4 = A^4+4A^3B+6A^2B^2+4AB^3+B^4$ and $(A-B)^4 = A^4-4A^3B+6A^2B^2-4AB^3+B^4$ that if we add up lead to $(A+B)^4+(A-B)^4 = 2A^4+2B^4+12A^2B^2$ or $A^2B^2 = \frac{1}{12}\left((A+B)^4+(A-B)^4-2A^4-2B^4\right)$, where we set $A = \log(1-x)$ and $B = \log(1+x)$ and then consider the integration from $x = 0$ to $x = 1$.

For the point ii), we use the identity suggested for the alternative way above, where if setting $A = \log(1-x)$ and $B = \log(1+x)$, then dividing both sides of the identity above by x and integrating from $x = 0$ and $x = 1$, we get

$$\int_0^1 \frac{\log^2(1-x)\log^2(1+x)}{x}\mathrm{d}x$$

$$= \frac{1}{12}\underbrace{\int_0^1 \frac{\log^4(1-x^2)}{x}\mathrm{d}x}_{I_1} + \frac{1}{12}\underbrace{\int_0^1 \frac{1}{x}\log^4\left(\frac{1-x}{1+x}\right)\mathrm{d}x}_{I_2} - \frac{1}{6}\underbrace{\int_0^1 \frac{\log^4(1-x)}{x}\mathrm{d}x}_{I_3}$$

$$- \frac{1}{6}\underbrace{\int_0^1 \frac{\log^4(1+x)}{x}\mathrm{d}x}_{I_4}. \tag{3.26}$$

For the integral I_1 in (3.26), we have

$$I_1 = \int_0^1 \frac{\log^4(1-x^2)}{x}\mathrm{d}x \overset{1-x^2=y}{=} \frac{1}{2}\int_0^1 \frac{\log^4(y)}{1-y}\mathrm{d}y = \frac{1}{2}\int_0^1 \sum_{n=1}^{\infty} y^{n-1}\log^4(y)\mathrm{d}y$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\int_0^1 y^{n-1}\log^4(y)\mathrm{d}y = 12\sum_{n=1}^{\infty}\frac{1}{n^5} = 12\zeta(5). \tag{3.27}$$

Then, for the integral I_2 in (3.26), we get

$$I_2 = \int_0^1 \frac{1}{x}\log^4\left(\frac{1-x}{1+x}\right)\mathrm{d}x \overset{\frac{1-x}{1+x}=y}{=} 2\int_0^1 \frac{\log^4(y)}{1-y^2}\mathrm{d}y = 2\int_0^1 \sum_{n=1}^{\infty} y^{2(n-1)}\log^4(y)\mathrm{d}y$$

$$= 48\sum_{n=1}^{\infty}\frac{1}{(2n-1)^5} = 48\left(\sum_{n=1}^{\infty}\frac{1}{n^5} - \sum_{n=1}^{\infty}\frac{1}{(2n)^5}\right) = \frac{93}{2}\sum_{n=1}^{\infty}\frac{1}{n^5} = \frac{93}{2}\zeta(5). \tag{3.28}$$

Further, for the integral I_3 in (3.26), we have

$$I_3 = \int_0^1 \frac{\log^4(1-x)}{x}\mathrm{d}x \overset{x=y^2}{=} 2\int_0^1 \frac{\log^4(1-y^2)}{y}\mathrm{d}y = 24\zeta(5), \tag{3.29}$$

since $I_3 = 2I_1$, and I_1 is the integral in (3.27).

Lastly, for the integral I_4 in (3.26), we have

$$I_4 = \int_0^1 \frac{\log^4(1+x)}{x}\mathrm{d}x \overset{\frac{1}{1+x}=y}{=} \int_{1/2}^1 \frac{\log^4(y)}{y(1-y)}\mathrm{d}y = \int_{1/2}^1 \frac{\log^4(y)}{y}\mathrm{d}y + \int_{1/2}^1 \frac{\log^4(y)}{1-y}\mathrm{d}y$$

$$= \frac{1}{5}\log^5(2) + \int_{1/2}^1 \frac{\log^4(y)}{1-y}\mathrm{d}y = \frac{1}{2}\log^5(2) + \int_{1/2}^1 \sum_{n=1}^{\infty} y^{n-1}\log^4(y)\mathrm{d}y$$

$$= \frac{1}{5}\log^5(2) + \sum_{n=1}^{\infty}\int_{1/2}^{1} y^{n-1}\log^4(y)\mathrm{d}y$$

$$= \frac{1}{5}\log^5(2) + 24\sum_{n=1}^{\infty}\frac{1}{n^5} - \log^4(2)\sum_{n=1}^{\infty}\frac{1}{n2^n} - 4\log^3(2)\sum_{n=1}^{\infty}\frac{1}{n^2 2^n}$$

$$-12\log^2(2)\sum_{n=1}^{\infty}\frac{1}{n^3 2^n} - 24\log(2)\sum_{n=1}^{\infty}\frac{1}{n^4 2^n} - 24\sum_{n=1}^{\infty}\frac{1}{n^5 2^n}$$

$$= 24\zeta(5) - \frac{4}{5}\log^5(2) - 4\log^3(2)\operatorname{Li}_2\left(\frac{1}{2}\right) - 12\log^2(2)\operatorname{Li}_3\left(\frac{1}{2}\right)$$

$$-24\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right) - 24\operatorname{Li}_5\left(\frac{1}{2}\right)$$

$$= 4\log^3(2)\zeta(2) - \frac{21}{2}\log^2(2)\zeta(3) + 24\zeta(5) - \frac{4}{5}\log^5(2) - 24\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right)$$

$$- 24\operatorname{Li}_5\left(\frac{1}{2}\right), \tag{3.30}$$

where in the calculations I used the special values in (3.19) and (3.20).

Collecting the values of the integrals I_1, I_2, I_3, and I_4 from (3.27), (3.28), (3.29), and (3.30) in (3.26), we conclude that

$$\int_0^1 \frac{\log^2(1-x)\log^2(1+x)}{x}\mathrm{d}x$$

$$= \frac{2}{15}\log^5(2) - \frac{2}{3}\log^3(2)\zeta(2) + \frac{7}{4}\log^2(2)\zeta(3) - \frac{25}{8}\zeta(5) + 4\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right)$$

$$+4\operatorname{Li}_5\left(\frac{1}{2}\right),$$

and the point *ii*) of the problem is finalized.

A different approach based upon a reduction to the calculation of the harmonic series may be found in [122].

The integral from the point *iii*) is a special integral which naturally arises in the calculation strategy for what is called *A darn integral in disguise* from the next section. It seems to be a resistant integral, it also appeared in [47], and at the moment of finalizing the book was still without a full solution for some years.

Using the same algebraic identity, $A^2B^2 = \frac{1}{12}\Big((A+B)^4 + (A-B)^4 - 2A^4 - 2B^4\Big)$, setting $A = \log(1-x)$ and $B = \log(1+x)$, then dividing both sides of the result above by $1+x$ and integrating from $x = 0$ to $x = 1$, we get

$$\int_0^1 \frac{\log^2(1-x)\log^2(1+x)}{1+x}\mathrm{d}x$$

$$= \frac{1}{12}\underbrace{\int_0^1 \frac{\log^4(1-x^2)}{1+x}\mathrm{d}x}_{J_1} + \frac{1}{12}\underbrace{\int_0^1 \frac{1}{1+x}\log^4\left(\frac{1-x}{1+x}\right)\mathrm{d}x}_{J_2} - \frac{1}{6}\underbrace{\int_0^1 \frac{\log^4(1-x)}{1+x}\mathrm{d}x}_{J_3}$$

$$- \frac{1}{6}\underbrace{\int_0^1 \frac{\log^4(1+x)}{1+x}\mathrm{d}x}_{J_4}. \tag{3.31}$$

From all the resulting integrals, the integral J_1 is by far the trickiest one, and what we would like to do is to reduce it to a Beta function form, and then we write

$$J_1 = \int_0^1 \frac{\log^4(1-x^2)}{1+x}\mathrm{d}x = \int_0^1 (1-x)\frac{\log^4(1-x^2)}{1-x^2}\mathrm{d}x$$

$$\overset{x^2=y}{=} \frac{1}{2}\int_0^1 \frac{1-\sqrt{y}}{\sqrt{y}} \cdot \frac{\log^4(1-y)}{1-y}\mathrm{d}y$$

$$= -\frac{1}{10}\int_0^1 \frac{1-\sqrt{y}}{\sqrt{y}}\left(\log^5(1-y)\right)' \mathrm{d}y$$

$$\{\text{apply the integration by parts}\}$$

$$= \underbrace{-\frac{1}{10}\frac{1-\sqrt{y}}{\sqrt{y}}\log^5(1-y)\Bigg|_{y=0}^{y=1}}_{0} - \frac{1}{20}\int_0^1 \frac{\log^5(1-y)}{y^{3/2}}\mathrm{d}y$$

$$= -\frac{1}{20}\lim_{\substack{x\to -1/2\\ y\to 1}} \frac{\partial}{\partial y^5}\mathrm{B}(x,y)$$

$$= \frac{16}{5}\log^5(2) - 16\log^3(2)\zeta(2) + 48\log^2(2)\zeta(3) - 54\log(2)\zeta(4) - 24\zeta(2)\zeta(3)$$

$$+72\zeta(5), \tag{3.32}$$

where we might prefer to use *Mathematica* to accomplish the calculations.

At this point, the careful reader may worry that the Beta function,

$$\mathrm{B}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

is not defined for $x \le 0$. However, its derivative with respect to y,

$$\frac{\partial}{\partial y}\mathrm{B}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\log(1-t)\mathrm{d}t = \mathrm{B}(x, y)(\psi(y) - \psi(x+y)),$$

is defined for $x > -1$, so this is not an issue.

Further, for the integral J_2, we get

$$J_2 = \int_0^1 \frac{1}{1+x}\log^4\left(\frac{1-x}{1+x}\right)\mathrm{d}x \overset{(1-x)/(1+x)=y}{=} \int_0^1 \frac{\log^4(y)}{1+y}\mathrm{d}y$$

$$= \int_0^1 \sum_{n=1}^{\infty}(-1)^{n-1}y^{n-1}\log^4(y)\mathrm{d}y = \sum_{n=1}^{\infty}(-1)^{n-1}\int_0^1 y^{n-1}\log^4(y)\mathrm{d}y$$

$$= 24\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^5} = 24\left(\sum_{n=1}^{\infty}\frac{1}{n^5} - 2\sum_{n=1}^{\infty}\frac{1}{(2n)^5}\right) = \frac{45}{2}\sum_{n=1}^{\infty}\frac{1}{n^5} = \frac{45}{2}\zeta(5). \tag{3.33}$$

Then, for the integral J_3, we obtain

$$J_3 = \int_0^1 \frac{\log^4(1-x)}{1+x}\mathrm{d}x \overset{1-x=y}{=} \int_0^1 \frac{\log^4(y)}{2-y}\mathrm{d}y = \frac{1}{2}\int_0^1 \frac{\log^4(y)}{1-y/2}\mathrm{d}y$$

$$= \frac{1}{2}\int_0^1 \sum_{n=1}^{\infty}\left(\frac{y}{2}\right)^{n-1}\log^4(y)\mathrm{d}y$$

{reverse the order of summation and integration}

$$= \sum_{n=1}^{\infty}\frac{1}{2^n}\int_0^1 y^{n-1}\log^4(y)\mathrm{d}y = 24\sum_{n=1}^{\infty}\frac{1}{n^5 2^n} = 24\,\mathrm{Li}_5\left(\frac{1}{2}\right). \tag{3.34}$$

For the last integral, J_4, since we have an elementary antiderivative, we get

$$J_4 = \int_0^1 \frac{\log^4(1+x)}{1+x}\mathrm{d}x = \frac{1}{5}\log^5(2). \tag{3.35}$$

Upon collecting the values of the integrals J_1, J_2, J_3, and J_4 from (3.32), (3.33), (3.34), and (3.35) in (3.31), we conclude that

$$\int_0^1 \frac{\log^2(1-x)\log^2(1+x)}{1+x}\mathrm{d}x$$

$$= \frac{63}{8}\zeta(5) - \frac{9}{2}\log(2)\zeta(4) + 4\log^2(2)\zeta(3) - \frac{4}{3}\log^3(2)\zeta(2) - 2\zeta(2)\zeta(3)$$

$$+\frac{7}{30}\log^5(2) - 4\operatorname{Li}_5\left(\frac{1}{2}\right),$$

and the solution is finalized.

With the last result in hand, we are ready to confront a truly challenging logarithmic integral, the integral from the next section.

3.9 *A Darn Integral in Disguise* (Possibly Harder Than It Seems to Be?), an Integral with Two Squared Logarithms on the Half of the Unit Interval

Solution If it happened to play with the series $\sum_{n=1}^{\infty} \frac{H_n}{n^4 2^n}$, you possibly arrived at some point at the integral we want to calculate, and maybe you also realized soon it's a hard nut to crack. Now, it's worth mentioning (especially if you didn't meet them before) that the series of the type $\sum_{n=1}^{\infty} \frac{H_n}{n 2^n}$, $\sum_{n=1}^{\infty} \frac{H_n}{n^2 2^n}$, $\sum_{n=1}^{\infty} \frac{H_n}{n^3 2^n}$ together with the one mentioned above (and there are also other variations) are very useful in the calculation of many integrals. In fact, the integral we want to calculate now will help us to obtain a critical relation that if combined with other relations, we'll manage to extract the first series mentioned above.

The first step is to reduce the integral to the calculation of the integral version from $x = 1/2$ to $x = 1$, and we write

$$\int_0^{1/2} \frac{\log^2(x)\log^2(1-x)}{x}\mathrm{d}x = \left(\int_0^1 - \int_{1/2}^1\right) \frac{\log^2(x)\log^2(1-x)}{x}\mathrm{d}x$$

$$= \int_0^1 \frac{\log^2(x)\log^2(1-x)}{x}\mathrm{d}x - \underbrace{\int_{1/2}^1 \frac{\log^2(x)\log^2(1-x)}{x}\mathrm{d}x}_{I}$$

$$= 2\int_0^1 \sum_{n=1}^{\infty} x^n \frac{H_n}{n+1} \log^2(x)\mathrm{d}x - I = 2\sum_{n=1}^{\infty} \frac{H_n}{n+1} \int_0^1 x^n \log^2(x)\mathrm{d}x - I$$

$$= 4\sum_{n=1}^{\infty} \frac{H_n}{(n+1)^4} - I = 4\sum_{n=1}^{\infty} \frac{H_{n+1} - 1/(n+1)}{(n+1)^4} - I$$

$$= 4\sum_{n=1}^{\infty} \frac{H_{n+1}}{(n+1)^4} - 4\sum_{n=1}^{\infty} \frac{1}{(n+1)^5} - I$$

$$\{\text{reindex both series}\}$$

$$= 4\sum_{n=1}^{\infty} \frac{H_n}{n^4} - 4\sum_{n=1}^{\infty} \frac{1}{n^5} - I = 8\zeta(5) - 4\zeta(2)\zeta(3) - I. \tag{3.36}$$

The second step is to express the integral I using $J = \int_{1/2}^1 \frac{\log^2(2x)\log^2(2(1-x))}{x}\mathrm{d}x$. *But why?* you might curiously wonder. It may not be immediately obvious, but with the change of variable $x = \frac{1+y}{2}$, the integral J becomes

$$J = \int_{1/2}^1 \frac{\log^2(2x)\log^2(2(1-x))}{x}\mathrm{d}x = \int_0^1 \frac{\log^2(1+y)\log^2(1-y)}{1+y}\mathrm{d}y, \tag{3.37}$$

which is precisely the integral from the previous section I already calculated.

By expanding the first integral in (3.37), we obtain

$$\int_{1/2}^1 \frac{\log^2(2(1-x))\log^2(2x)}{x}\mathrm{d}x = \int_{1/2}^1 \frac{(\log(2)+\log(1-x))^2(\log(2)+\log(x))^2}{x}\mathrm{d}x$$

$$= \log^4(2)\int_{1/2}^1 \frac{1}{x}\mathrm{d}x + 2\log^3(2)\int_{1/2}^1 \frac{\log(x)}{x}\mathrm{d}x + \log^2(2)\int_{1/2}^1 \frac{\log^2(x)}{x}\mathrm{d}x$$

$$+2\log^3(2)\int_{1/2}^1 \frac{\log(1-x)}{x}\mathrm{d}x + 4\log^2(2)\int_{1/2}^1 \frac{\log(x)\log(1-x)}{x}\mathrm{d}x$$

$$+2\log(2)\int_{1/2}^1 \frac{\log^2(x)\log(1-x)}{x}\mathrm{d}x + \log^2(2)\int_{1/2}^1 \frac{\log^2(1-x)}{x}\mathrm{d}x$$

$$+2\log(2)\int_{1/2}^1 \frac{\log(x)\log^2(1-x)}{x}\mathrm{d}x + \int_{1/2}^1 \frac{\log^2(x)\log^2(1-x)}{x}\mathrm{d}x$$

$$= \frac{1}{3}\log^5(2) + 2\log^3(2)\underbrace{\int_{1/2}^{1}\frac{\log(1-x)}{x}\mathrm{d}x}_{J_1} + 4\log^2(2)\underbrace{\int_{1/2}^{1}\frac{\log(x)\log(1-x)}{x}\mathrm{d}x}_{J_2}$$

$$+2\log(2)\underbrace{\int_{1/2}^{1}\frac{\log^2(x)\log(1-x)}{x}\mathrm{d}x}_{J_3} + \log^2(2)\underbrace{\int_{1/2}^{1}\frac{\log^2(1-x)}{x}\mathrm{d}x}_{J_4}$$

$$+ 2\log(2)\underbrace{\int_{1/2}^{1}\frac{\log(x)\log^2(1-x)}{x}\mathrm{d}x}_{J_5} + \underbrace{\int_{1/2}^{1}\frac{\log^2(x)\log^2(1-x)}{x}\mathrm{d}x}_{I}. \tag{3.38}$$

For the integral J_1 in (3.38), by using the Dilogarithm function, we have

$$J_1 = \int_{1/2}^{1}\frac{\log(1-x)}{x}\mathrm{d}x = -\operatorname{Li}_2(x)\Big|_{x=1/2}^{x=1} = \operatorname{Li}_2\left(\frac{1}{2}\right) - \operatorname{Li}_2(1)$$

{use the special value of Dilogarithm function in (3.19)}

$$= -\frac{1}{2}\left(\zeta(2) + \log^2(2)\right). \tag{3.39}$$

Then, for the integral J_2 in (3.38) we apply the integration by parts, and we get

$$J_2 = \int_{1/2}^{1}\frac{\log(x)\log(1-x)}{x}\mathrm{d}x = -\int_{1/2}^{1}\log(x)(\operatorname{Li}_2(x))'\mathrm{d}x$$

$$= -\log(x)\operatorname{Li}_2(x)\Big|_{x=1/2}^{x=1} + \int_{1/2}^{1}\frac{\operatorname{Li}_2(x)}{x}\,\mathrm{d}\,x = -\log(2)\operatorname{Li}_2\left(\frac{1}{2}\right) + \operatorname{Li}_3(x)\Big|_{x=1/2}^{x=1}$$

$$= -\log(2)\operatorname{Li}_2\left(\frac{1}{2}\right) - \operatorname{Li}_3\left(\frac{1}{2}\right) + \operatorname{Li}_3(1)$$

{use the special values of Dilogarithm and Trilogarithm functions in (3.19) and (3.20)}

$$= \frac{1}{8}\zeta(3) + \frac{1}{3}\log^3(2). \tag{3.40}$$

For the integral J_3 in (3.38), we have

$$J_3 = \int_{1/2}^{1} \frac{\log^2(x)\log(1-x)}{x}\mathrm{d}x = -\int_{1/2}^{1} (\mathrm{Li}_2(x))' \log^2(x)\mathrm{d}x$$

{make use of the integration by parts}

$$= -\mathrm{Li}_2(x)\log^2(x)\Big|_{x=1/2}^{x=1} + 2\int_{1/2}^{1} \frac{\mathrm{Li}_2(x)\log(x)}{x}\mathrm{d}x$$

$$= \log^2(2)\,\mathrm{Li}_2\left(\frac{1}{2}\right) + 2\int_{1/2}^{1} (\mathrm{Li}_3(x))' \log(x)\mathrm{d}x$$

$$= \log^2(2)\,\mathrm{Li}_2\left(\frac{1}{2}\right) + 2\,\mathrm{Li}_3(x)\log(x)\Big|_{x=1/2}^{x=1} - 2\int_{1/2}^{1} \frac{\mathrm{Li}_3(x)}{x}\mathrm{d}x = \log^2(2)\,\mathrm{Li}_2\left(\frac{1}{2}\right)$$

$$+2\log(2)\,\mathrm{Li}_3\left(\frac{1}{2}\right) - 2\int_{1/2}^{1} (\mathrm{Li}_4(x))'\mathrm{d}x = \log^2(2)\,\mathrm{Li}_2\left(\frac{1}{2}\right) + 2\log(2)\,\mathrm{Li}_3\left(\frac{1}{2}\right)$$

$$+2\,\mathrm{Li}_4\left(\frac{1}{2}\right) - 2\zeta(4)$$

$$= \frac{7}{4}\log(2)\zeta(3) - \frac{1}{2}\log^2(2)\zeta(2) - 2\zeta(4) - \frac{1}{6}\log^4(2) + 2\,\mathrm{Li}_4\left(\frac{1}{2}\right). \tag{3.41}$$

Further, for the integral J_4 in (3.38), we write

$$J_4 = \int_{1/2}^{1} \frac{\log^2(1-x)}{x}\mathrm{d}x = \left(\int_0^1 - \int_0^{1/2}\right) \frac{\log^2(1-x)}{x}\mathrm{d}x$$

$$= \int_0^1 \frac{\log^2(1-x)}{x}\mathrm{d}x - \int_0^{1/2} \frac{\log^2(1-x)}{x}\mathrm{d}x$$

{for the remaining integrals, make use of the result in (1.8) with $x = 1$ and $x = 1/2$}

$$= \log^3(2) + 2\log(2)\,\mathrm{Li}_2\left(\frac{1}{2}\right) + 2\,\mathrm{Li}_3\left(\frac{1}{2}\right) = \frac{7}{4}\zeta(3) + \frac{1}{3}\log^3(2). \tag{3.42}$$

Next, we calculate the integral J_5 in (3.38), and we have

$$J_5 = \int_{1/2}^{1} \frac{\log(x)\log^2(1-x)}{x} \mathrm{d}x \overset{1-x=y}{=} \int_0^{1/2} \frac{\log(1-y)\log^2(y)}{1-y} \mathrm{d}y$$

{employ the result in (1.28), the case $p = 1$}

$$= -\frac{1}{4}\left(\zeta(4) + \log^4(2)\right). \tag{3.43}$$

Collecting the values of the integrals J_1, J_2, J_3, J_4, and J_5 from (3.39), (3.40), (3.41), (3.42), and (3.43) in (3.38), we get

$$\int_{1/2}^{1} \frac{\log^2(2(1-x))\log^2(2x)}{x} \mathrm{d}x = \frac{1}{6}\log^5(2) - 2\log^3(2)\zeta(2) + \frac{23}{4}\log^2(2)\zeta(3)$$

$$-\frac{9}{2}\log(2)\zeta(4) + 4\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right) + I,$$

whence we arrive at

$$I = \int_{1/2}^{1} \frac{\log^2(x)\log^2(1-x)}{x} \mathrm{d}x = \int_{1/2}^{1} \frac{\log^2(2(1-x))\log^2(2x)}{x} \mathrm{d}x$$

$$-\frac{1}{6}\log^5(2) + 2\log^3(2)\zeta(2) - \frac{23}{4}\log^2(2)\zeta(3) + \frac{9}{2}\log(2)\zeta(4) - 4\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right)$$

{combine and employ the results in (3.37) and (1.17)}

$$= \frac{1}{15}\log^5(2) + \frac{2}{3}\log^3(2)\zeta(2) - \frac{7}{4}\log^2(2)\zeta(3) - 2\zeta(2)\zeta(3) + \frac{63}{8}\zeta(5)$$

$$-4\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right) - 4\operatorname{Li}_5\left(\frac{1}{2}\right). \tag{3.44}$$

Finally, by combining the results in (3.36) and (3.44), we obtain that

$$\int_0^{1/2} \frac{\log^2(x)\log^2(1-x)}{x} \mathrm{d}x$$

$$= \frac{1}{8}\zeta(5) - 2\zeta(2)\zeta(3) - \frac{2}{3}\log^3(2)\zeta(2) + \frac{7}{4}\log^2(2)\zeta(3) - \frac{1}{15}\log^5(2)$$

$$+4\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right) + 4\operatorname{Li}_5\left(\frac{1}{2}\right),$$

and the solution is complete.

The present solution also answers the proposed *challenging question* of calculating the integral by real methods. In a more recent paper, in [122], you may find a different derivation where one needs first the value of $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^4}$ to get the value of $\sum_{n=1}^{\infty}\frac{H_n}{n^4 2^n}$, and then, with this, to extract the value of the integral. Note that in my solution one needs none of the mentioned challenging series.

3.10 The Evaluation of a Class of Logarithmic Integrals Using a Slightly Modified Result from *Table of Integrals, Series and Products* by I.S. Gradshteyn and I.M. Ryzhik Together with a Series Result Elementarily Proved by Guy Bastien

Solution In the previous section we have dealt with an integral about I said it would help us to derive a certain harmonic series, but this time we'll use the harmonic series to calculate the given integral. The calculation of the series involving the generalized harmonic number of order m, $H_n^{(m)} = \sum_{k=1}^{n}\frac{1}{k^m}$, $m \geq 1$, $m \in \mathbb{N}$, has old roots in the history of mathematics, and I may mention here at least the exchange of letters between the famous mathematicians Christian Goldbach (1690–1764) and Leonhard Euler (1707 -1783), back in 1742–1743, on the series of the type $\sum_{k=1}^{\infty}\frac{1}{k^n}\sum_{j=1}^{k}\frac{1}{j^m}$, where m, n are positive integers with $n \geq 2$, and then the success of Euler in proving that

$$2\sum_{k=1}^{\infty}\frac{H_k}{k^n} = (n+2)\zeta(n+1) - \sum_{k=1}^{n-2}\zeta(n-k)\zeta(k+1),\ n \in \mathbb{N},\ n \geq 2, \tag{3.45}$$

where $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$ denotes the Riemann zeta function, a result rediscovered many times in the mathematical literature by (new) various techniques, and you may find such examples in [22, 37], [78, Chapter 2, pp. 103–105] or in [77] where various extensions of it are given.

Moreover, the Euler sum in (3.45) is a kind of cornerstone in the problems with the harmonic series, since in the calculation process we'll usually arrive at such series.

For a fast solution to the proposed integral, we need the classical linear Euler sum in (3.45), where we replace n by $2n + 2$,

$$\sum_{k=1}^{\infty} \frac{H_k}{k^{2n+2}} = (n+2)\zeta(2n+3) - \frac{1}{2}\sum_{k=1}^{2n} \zeta(2n-k+2)\zeta(k+1), \tag{3.46}$$

and then we also need the series result beautifully proved by G. Bastien in [13],

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{H_k}{k^{2n}} = \left(n+\frac{1}{2}\right)\eta(2n+1) - \frac{1}{2}\zeta(2n+1) - \sum_{i=1}^{n-1}\eta(2i)\zeta(2n-2i+1), \tag{3.47}$$

where we replace n by $n + 1$,

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{H_k}{k^{2n+2}} = \left(n+\frac{3}{2}\right)\eta(2n+3) - \frac{1}{2}\zeta(2n+3) - \sum_{i=1}^{n}\eta(2i)\zeta(2n-2i+3),$$

that may be rewritten as

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{H_k}{k^{2n+2}} = \left(n+\frac{3}{2}\right)\left(1-\frac{1}{2^{2n+2}}\right)\zeta(2n+3) - \frac{1}{2}\zeta(2n+3)$$
$$-\sum_{i=1}^{n}\left(1-\frac{1}{2^{2i-1}}\right)\zeta(2i)\zeta(2n-2i+3), \tag{3.48}$$

where I used the fact that the Dirichlet eta function (see [102], [1, pp. 807–808]), $\eta(s) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^s}$, can be expressed in terms of the Riemann zeta function by the relation $\eta(s) = (1-2^{1-s})\zeta(s)$.

Now, to calculate the main integral, we'll employ the second equality in (4.3),

$$\log(1+x)\log(1-x) = -\sum_{k=1}^{\infty} x^{2k}\frac{H_{2k}-H_k}{k} - \frac{1}{2}\sum_{k=1}^{\infty}\frac{x^{2k}}{k^2},$$

where if multiplying both side by $\log^{2n}(x)/x$ and then integrating from $x = 0$ to $x = 1$, we get

$$\int_0^1 \frac{\log(1-x)\log^{2n}(x)\log(1+x)}{x}\mathrm{d}x$$

$$= -\int_0^1 \sum_{k=1}^{\infty} x^{2k-1} \log^{2n}(x) \frac{H_{2k} - H_k}{k} \mathrm{d}x - \frac{1}{2} \int_0^1 \sum_{k=1}^{\infty} \frac{x^{2k-1}}{k^2} \log^{2n}(x) \mathrm{d}x$$

{reverse the order of summation and integration}

$$= -\sum_{k=1}^{\infty} \frac{H_{2k} - H_k}{k} \int_0^1 x^{2k-1} \log^{2n}(x) \mathrm{d}x - \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k^2} \int_0^1 x^{2k-1} \log^{2n}(x) \mathrm{d}x$$

{make use of the result in (1.2)}

$$= -2(2n)! \sum_{k=1}^{\infty} \frac{H_{2k}}{(2k)^{2n+2}} + \frac{(2n)!}{2^{2n+1}} \sum_{k=1}^{\infty} \frac{H_k}{k^{2n+2}} - \frac{(2n)!}{2^{2n+2}} \sum_{k=1}^{\infty} \frac{1}{k^{2n+3}}$$

{split the first series and write it as a sum of two series}

$$= -(2n)! \sum_{k=1}^{\infty} \frac{H_k}{k^{2n+2}} - (2n)! \sum_{k=1}^{\infty} (-1)^k \frac{H_k}{k^{2n+2}} + \frac{(2n)!}{2^{2n+1}} \sum_{k=1}^{\infty} \frac{H_k}{k^{2n+2}} - \frac{(2n)!}{2^{2n+2}} \zeta(2n+3)$$

$$= (2n)! \left(\frac{1}{2^{2n+1}} - 1 \right) \sum_{k=1}^{\infty} \frac{H_k}{k^{2n+2}} + (2n)! \sum_{k=1}^{\infty} (-1)^{k-1} \frac{H_k}{k^{2n+2}} - \frac{(2n)!}{2^{2n+2}} \zeta(2n+3)$$

{make use of the series results in (3.46) and (3.48)}

$$= \frac{1}{2}(2n)! \left(1 - \frac{1}{2^{2n+1}} \right) \sum_{k=1}^{2n} \zeta(k+1)\zeta(2n-k+2)$$

$$-(2n)! \sum_{k=1}^{n} \left(1 - \frac{1}{2^{2k-1}} \right) \zeta(2k)\zeta(2n-2k+3)$$

$$+ \frac{1}{2^{2n+3}} (2n+3-2^{2n+3})(2n)!\zeta(2n+3),$$

and the solution is finalized.

One might naturally ask, *How do we arrive at considering such integrals?* In this case the key observation for obtaining such a generalized integral has been represented by the study of the second equality in (4.3). Curiously, in *Table of Integrals, Series and Products* by I.S. Gradshteyn and I.M. Ryzhik (8th Edition), in **1.516.3**, we may only find the first equality presented in (4.3). However, the second equality is easy to establish if we use Botez–Catalan identity, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{2n} = H_{2n} - H_n$.

The particular case $n = 1$ together with another solution involving the Beta function and its derivatives may be found in [54].

3.11 Logarithmic Integrals Containing an Infinite Series in the Integrand, Giving Values in Terms of Riemann Zeta Function

Solution From the very beginning, I would ask you if you managed to go through Sect. 1.5. If you didn't yet, I might suggest you to return back (now or later) to the mentioned section to find out more about the result we want to employ here.

Let's consider the result in (1.10), where we want to focus on the second equality (the less obvious one), and then we write

$$\int_0^1 \frac{x^{s-1}}{1+x}\mathrm{d}x = \psi(s) - \psi\left(\frac{s}{2}\right) - \log(2), \tag{3.49}$$

where differentiating two times both sides above, we get

$$\int_0^1 \frac{x^{s-1}\log^2(x)}{1+x}\mathrm{d}x = \psi^{(2)}(s) - \frac{1}{4}\psi^{(2)}\left(\frac{s}{2}\right). \tag{3.50}$$

Upon replacing s by 2^n in (3.50) and then multiplying both sides by 4^n, we obtain

$$\int_0^1 \frac{4^n x^{2^n-1}\log^2(x)}{1+x}\mathrm{d}x = 4^n\psi^{(2)}(2^n) - 4^{n-1}\psi^{(2)}\left(2^{n-1}\right). \tag{3.51}$$

Returning to our initial integral, we write

$$\int_0^1 \left(4x^2 + 4^2x^{2^2} + 4^3x^{2^3} + 4^4x^{2^4} + 4^5x^{2^5} + \cdots\right)\frac{\log^2(x)}{x(1+x)}\mathrm{d}x$$

$$= \int_0^1 \frac{\log^2(x)}{1+x}\sum_{n=1}^{\infty} 4^n x^{2^n-1}\mathrm{d}x = \sum_{n=1}^{\infty}\int_0^1 \frac{4^n x^{2^n-1}\log^2(x)}{1+x}\mathrm{d}x$$

{make use of the result in (3.51)}

$$= \sum_{n=1}^{\infty}(4^n\psi^{(2)}(2^n) - 4^{n-1}\psi^{(2)}\left(2^{n-1}\right))$$

$$= \lim_{N\to\infty}\sum_{n=1}^{N}(4^n\psi^{(2)}(2^n) - 4^{n-1}\psi^{(2)}\left(2^{n-1}\right)) = \lim_{N\to\infty}(4^N\psi^{(2)}\left(2^N\right) - \psi^{(2)}(1))$$

$$= 2\zeta(3) - 1,$$

where I used the asymptotic expansion of Digamma function (see [1, p. 259], [78, Chapter 1, p. 22]), $\psi(x) = \log(x) + O\left(\frac{1}{x}\right)$, as $x \to \infty$, that by differentiation led to $x^2\psi^{(2)}(x) = -1 + O\left(\frac{1}{x}\right)$, as $x \to \infty$, for which I also considered the big-O notation in [95] and the series representation of the Polygamma function,[4] $\psi^{(m)}(n) = (-1)^{m-1}m!\sum_{k=n}^{\infty}\frac{1}{k^{m+1}}$, $m \geq 1$, and the solution to the point i) of the problem is complete.

In Sect. 3.5 I briefly talked about the definition of Digamma function, and starting from that definition and considering the mth derivative, $m \geq 1$, $m \in \mathbb{N}$, $\psi^{(m)}(x) = \frac{d^{m+1}}{dx^{m+1}}\log(\Gamma(x)) = \frac{d^m}{dx^m}\frac{\Gamma'(x)}{\Gamma(x)} = \frac{d^m}{dx^m}\psi(x)$, we have arrived at what is called the Polygamma function of order m. More details about Polygamma function may be found in [116], [79, pp. 33–37].

Now, to deal with the general case, we proceed in the same fashion as we did for the point i), where if differentiating both sides of the result in (3.49) k times with respect to s, we get

$$\int_0^1 \frac{x^{s-1}\log^k(x)}{1+x}\mathrm{d}x = \psi^{(k)}(s) - \frac{1}{2^k}\psi^{(k)}\left(\frac{s}{2}\right). \tag{3.52}$$

Upon replacing s by 2^n in (3.52) and then multiplying both sides by 2^{kn}, we obtain

$$\int_0^1 \frac{2^{kn}x^{2^n-1}\log^k(x)}{1+x}\mathrm{d}x = 2^{kn}\psi^{(k)}(2^n) - 2^{k(n-1)}\psi^{(k)}\left(2^{n-1}\right). \tag{3.53}$$

Now, considering the generalization we want to calculate, we have

$$\int_0^1 \left(2^k x + 2^{2k}x^{2^2-1} + 2^{3k}x^{2^3-1} + 2^{4k}x^{2^4-1} + 2^{5k}x^{2^5-1} + \cdots\right)\frac{\log^k(x)}{1+x}\mathrm{d}x$$

$$= \int_0^1 \frac{\log^k(x)}{1+x}\sum_{n=1}^{\infty}2^{kn}x^{2^n-1}\mathrm{d}x = \sum_{n=1}^{\infty}\int_0^1 \frac{2^{kn}x^{2^n-1}\log^k(x)}{1+x}\mathrm{d}x$$

{make use of the result in (3.53)}

[4]The mentioned representation is obtained based upon the recurrence relation of the Polygamma function, $\psi^{(m)}(x+1) = \psi^{(m)}(x) + (-1)^m\frac{m!}{x^{m+1}}$, which is derived by differentiation from the recurrence relation of the Digamma function, $\psi(x+1) = \psi(x) + \frac{1}{x}$.

$$= \sum_{n=1}^{\infty}\left(2^{kn}\psi^{(k)}(2^n) - 2^{k(n-1)}\psi^{(k)}\left(2^{n-1}\right)\right)$$

$$= \lim_{N\to\infty}\sum_{n=1}^{N}\left(2^{kn}\psi^{(k)}(2^n) - 2^{k(n-1)}\psi^{(k)}\left(2^{n-1}\right)\right)$$

$$= \lim_{N\to\infty}(2^{kN}\psi^{(k)}\left(2^N\right) - \psi^{(k)}(1)). \tag{3.54}$$

In order to calculate the limit in (3.11), we use that $\psi^{(m)}(n) = (-1)^{m-1}m! \sum_{k=n}^{\infty}\frac{1}{k^{m+1}}$, and upon setting $n = 1$, we get

$$\psi^{(m)}(1) = (-1)^{m-1}m!\sum_{k=1}^{\infty}\frac{1}{k^{m+1}} = (-1)^{m-1}m!\zeta(m+1). \tag{3.55}$$

On the other hand, using the asymptotic expansion of Digamma function (see [1, p. 259], [78, Chapter 1, p. 22]), $\psi(x) = \log(x) + O\left(\frac{1}{x}\right)$, and differentiating it k times, we get $\psi^{(k)}(x) = \frac{(-1)^{k-1}(k-1)!}{x^k} + O\left(\frac{1}{x^{k+1}}\right)$. Next, replacing x by 2^N, $\psi^{(k)}\left(2^N\right) = \frac{(-1)^{k-1}(k-1)!}{2^{kN}} + O\left(\frac{1}{2^{(k+1)N}}\right)$, multiplying both sides of the last equality by 2^{kN}, and then taking the limit as $N \to \infty$, we have

$$\lim_{N\to\infty} 2^{kN}\psi^{(k)}\left(2^N\right) = \lim_{N\to\infty}\left((-1)^{k-1}(k-1)! + O\left(\frac{1}{2^N}\right)\right) = (-1)^{k-1}(k-1)!. \tag{3.56}$$

Finally, collecting the results from (3.55) and (3.56) in (3.11), we conclude that

$$\int_0^1 \left(2^k x + 2^{2k}x^{2^2-1} + 2^{3k}x^{2^3-1} + 2^{4k}x^{2^4-1} + 2^{5k}x^{2^5-1} + \cdots\right)\frac{\log^k(x)}{1+x}\mathrm{d}x$$

$$= (-1)^{k-1}(k-1)! - (-1)^{k-1}k!\zeta(k+1) = (-1)^k(k-1)!(k\zeta(k+1)-1),$$

and the solution to the general case from the point $ii)$ is complete.

In this section the strategy was to reduce everything to the calculation of a telescoping sum, and it's not the only place where we would want to make benefit of the telescoping sums. Sometimes, the possibility of reducing the calculations to a telescoping sum is not *that visible* (as it happens in the current problem).

The literature is rich with various problems that eventually beautifully get reduced to the calculations of some telescoping sums (see [35, 69–72], [11, pp. 252–253], [40, pp. 58–59], [76, 82, 83, 89]).

3.12 Two Appealing Integral Representations of $\zeta(4)$ and $\zeta(2)G$

Solution If for a moment there was the temptation to think we had in front of our eyes a bunch of random integrals, then as soon as we noticed well the simple, delightful values of the integrals, that thought was probably swept off instantly. *How to approach the integrals? Individually? To bring them to another form?*, some of the questions that would immediately arise. The truly intriguing part is the *challenging question*, which asks to prove the results without using harmonic series, a challenge I suspect the experienced ones in such calculations will enjoy much.

Now, for the first part of the problem, we write

$$I = 2\int_0^1 \frac{\operatorname{arctanh}(x)\operatorname{Li}_2(x)}{x}\mathrm{d}x - \int_0^1 \frac{\log(1-x)\log(x)\log(1+x)}{x}\mathrm{d}x$$
$$= \int_0^1 \frac{\log(1+x)\operatorname{Li}_2(x)}{x}\mathrm{d}x - \int_0^1 \frac{\log(1-x)\operatorname{Li}_2(x)}{x}\mathrm{d}x$$
$$- \int_0^1 \frac{\log(1-x)\log(x)\log(1+x)}{x}\mathrm{d}x. \tag{3.57}$$

If we consider the last integral in (3.57) and then integrate by parts, we have

$$\int_0^1 \frac{\log(1-x)\log(x)\log(1+x)}{x}\mathrm{d}x = \int_0^1 (-\operatorname{Li}_2(x))'\log(x)\log(1+x)\mathrm{d}x$$
$$= \underbrace{-\operatorname{Li}_2(x)\log(x)\log(1+x)\Big|_{x=0}^{x=1}}_{0} + \int_0^1 \frac{\log(1+x)\operatorname{Li}_2(x)}{x}\mathrm{d}x$$
$$+ \int_0^1 \frac{\log(x)\operatorname{Li}_2(x)}{1+x}\mathrm{d}x$$
$$= \int_0^1 \frac{\log(1+x)\operatorname{Li}_2(x)}{x}\mathrm{d}x + \int_0^1 \frac{\log(x)\operatorname{Li}_2(x)}{1+x}\mathrm{d}x. \tag{3.58}$$

If we plug the result from (3.58) in (3.57), we obtain

$$I = -\int_0^1 \frac{\log(x)\operatorname{Li}_2(x)}{1+x}\mathrm{d}x - \int_0^1 \frac{\log(1-x)\operatorname{Li}_2(x)}{x}\mathrm{d}x$$

$$\{\text{make use of the result in (1.53)}\}$$

$$= \frac{3}{16}\zeta(4) + \frac{1}{2}\int_0^1 \left((\operatorname{Li}_2(x))^2\right)' \mathrm{d}x = \frac{3}{16}\zeta(4) + \frac{1}{2}(\operatorname{Li}_2(x))^2\Big|_{x=0}^{x=1} = \frac{23}{16}\zeta(4),$$

and the first part of the problem is finalized.

As a final note to this part of the problem, if we check the solution provided for the result in (1.53) I used above, we'll see there we exploited the symmetry in double integrals, and the calculations have been finalized without using harmonic series, which follows the requirement of the *challenging question*.

The integral $\displaystyle\int_0^1 \frac{\log(1-x)\log(x)\log(1+x)}{x}\mathrm{d}x$ also appears in [50], and a suggested approach is based upon the use of the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^3}$ and $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(2)}}{n^2}$, and alternatively, one can use the approach in [54] for a similar integral.

Further, for the second part of the problem, we start with writing that

$$\int_0^1 \frac{\arctan(x)\log(x)\log(1+x)}{x}\mathrm{d}x = -\int_0^1 \arctan(x)\log(x)(\operatorname{Li}_2(-x))'\mathrm{d}x$$

$$= \underbrace{-\arctan(x)\log(x)\operatorname{Li}_2(-x)\Big|_{x=0}^{x=1}}_{0} + \int_0^1 \frac{\arctan(x)\operatorname{Li}_2(-x)}{x}\mathrm{d}x$$

$$+ \int_0^1 \frac{\log(x)\operatorname{Li}_2(-x)}{1+x^2}\mathrm{d}x$$

$$= \int_0^1 \frac{\arctan(x)\operatorname{Li}_2(-x)}{x}\mathrm{d}x + \int_0^1 \frac{\log(x)\operatorname{Li}_2(-x)}{1+x^2}\mathrm{d}x,$$

which if we plug in

$$J = \int_0^1 \frac{\arctan(x)\log(x)\log(1+x)}{x}\mathrm{d}x - \int_0^1 \frac{\arctan(x)\operatorname{Li}_2(-x)}{x}\mathrm{d}x,$$

we obtain that

$$J = \int_0^1 \frac{\log(x)\operatorname{Li}_2(-x)}{1+x^2}\mathrm{d}x = \frac{1}{8}\zeta(2)G,$$

where I made use of the result in (1.55), the case $n = 1$, and the second part of the problem is finalized. Note that $G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$ is the Catalan's constant (see [96], [27, pp. 53–57])

Like the result in (1.53), the result in (1.55) is obtained by exploiting the symmetry as well, without using the harmonic series, and therefore also this solution answers the *challenging question*.

In fact, the *challenging question* has introduced us to the realm of the powerful integration technique based on the use of the symmetry, which we will use in some of the next sections.

3.13 A Special Pair of Logarithmic Integrals with Connections in the Area of the Alternating Harmonic Series

Solution I didn't intend to add a *challenging question* to this section until the approaching moment of finalizing the book, when I remembered the people that tried the integral J, I submitted to *The American Mathematical Monthly*, the problem **11966** (see [90]), and shared their solutions with me, had all more or less the same strategy of attacking the problem, the differentiation under the integral sign. *How about coming up with a different approach, without using the differentiation under the integral sign, double integrals?*

Let's approach both integrals at once and start out by making up the following system of relations involving the integrals I and J,

$$S : \begin{cases} I + J = \underbrace{\int_0^1 \frac{x \log(1-x^2)}{1+x^2} \mathrm{d}x}_{A}; \\ I - J = \underbrace{\int_0^1 \frac{x}{1+x^2} \log\left(\frac{1-x}{1+x}\right) \mathrm{d}x}_{B}. \end{cases} \tag{3.59}$$

Now, for the integral A in (3.59), we have

$$A = \int_0^1 \frac{x \log(1-x^2)}{1+x^2} \mathrm{d}x \overset{x^2 = \frac{1-t}{1+t}}{=} \frac{1}{2} \int_0^1 \frac{1}{1+t} \log\left(\frac{2t}{1+t}\right) \mathrm{d}t$$

$$= \frac{\log(2)}{2} \int_0^1 \frac{1}{1+t} \mathrm{d}t + \frac{1}{2} \int_0^1 \frac{\log(t)}{1+t} \mathrm{d}t - \frac{1}{2} \int_0^1 \frac{\log(1+t)}{1+t} \mathrm{d}t$$

$$= \frac{1}{2}\int_0^1 \frac{\log(t)}{1+t}\mathrm{d}t + \frac{\log^2(2)}{4} = \frac{1}{2}\int_0^1 \sum_{n=1}^{\infty}(-1)^{n-1}t^{n-1}\log(t)\mathrm{d}t + \frac{\log^2(2)}{4}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\int_0^1 t^{n-1}\log(t)\mathrm{d}t + \frac{\log^2(2)}{4}$$

$$= -\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2} + \frac{\log^2(2)}{4} = \frac{1}{4}\left(\log^2(2) - \zeta(2)\right). \tag{3.60}$$

Then, for the integral B in (3.59), we get

$$B = \int_0^1 \frac{x}{1+x^2}\log\left(\frac{1-x}{1+x}\right)\mathrm{d}x \overset{\frac{1-x}{1+x}=t}{=} \int_0^1 \frac{(1-t)\log(t)}{(1+t)(1+t^2)}\mathrm{d}t = \int_0^1 \frac{\log(t)}{1+t}\mathrm{d}t$$

$$-\int_0^1 \frac{t\log(t)}{1+t^2}\mathrm{d}t = \int_0^1 \sum_{n=1}^{\infty}(-1)^{n-1}t^{n-1}\log(t)\mathrm{d}t - \int_0^1 \sum_{n=1}^{\infty}(-1)^{n-1}t^{2n-1}\log(t)\mathrm{d}t$$

$$= \sum_{n=1}^{\infty}(-1)^{n-1}\int_0^1 t^{n-1}\log(t)\mathrm{d}t - \sum_{n=1}^{\infty}(-1)^{n-1}\int_0^1 t^{2n-1}\log(t)\mathrm{d}t$$

$$= \sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2} - \frac{1}{4}\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2} = -\frac{3}{4}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2} = -\frac{3}{8}\zeta(2). \tag{3.61}$$

Collecting the results from (3.60) and (3.61) in the initial system of relations from (3.59), we obtain

$$S : \begin{cases} I + J = \frac{1}{4}\left(\log^2(2) - \zeta(2)\right); \\ I - J = -\frac{3}{8}\zeta(2), \end{cases}$$

whence we conclude that

$$I = \int_0^1 \frac{x\log(1-x)}{1+x^2}\mathrm{d}x = \frac{1}{8}\left(\log^2(2) - \frac{5}{2}\zeta(2)\right)$$

and

$$J = \int_0^1 \frac{x\log(1+x)}{1+x^2}\mathrm{d}x = \frac{1}{8}\left(\log^2(2) + \frac{1}{2}\zeta(2)\right),$$

and the solution is complete.

It's good to keep in mind that sometimes approaching the integrals by making up such systems of relations leads to simpler, more elegant solutions, and we want to remember this strategy as we continue the journey through the next sections.

For a first alternative way, exploit the symmetry in the following double integrals

$$I=\int_0^1 \frac{x\log(1-x)}{1+x^2}\mathrm{d}x=-\int_0^1\left(\int_0^1 \frac{x^2}{(1+x^2)(1-xy)}\mathrm{d}y\right)\mathrm{d}x$$

and

$$J=\int_0^1 \frac{x\log(1+x)}{1+x^2}\mathrm{d}x=\int_0^1\left(\int_0^1 \frac{x^2}{(1+x^2)(1+xy)}\mathrm{d}y\right)\mathrm{d}x.$$

For a second alternative way, I would like to bring to the attention of the curious reader a fast strategy with dilogarithms. If we consider $\operatorname{Li}_2\left(\frac{2x}{1+x^2}\right)$ which we differentiate and then integrate back, we can put all in the convenient form

$$\int_0^x \frac{t\log(1-t)}{1+t^2}\mathrm{d}t=\frac{1}{4}\left(\frac{1}{2}\log^2(1+x^2)-2\operatorname{Li}_2(x)+\frac{1}{2}\operatorname{Li}_2\left(-x^2\right)+\operatorname{Li}_2\left(\frac{2x}{1+x^2}\right)\right),\tag{3.62}$$

where we get the desired integrals by letting $x\to 1^-$, and $x\to -1$ respectively.

For another solution of approaching the integrals, one using the differentiation under the integral sign, see [84].

3.14 Another Special Pair of Logarithmic Integrals with Connections in the Area of the Alternating Harmonic Series

Solution It's easy to guess, if you didn't skip the previous section, we want to use here the same idea of making up a system of relations with the given integrals (note the similarity with the integrands from the previous section—we also have a $\log(x)$ added in the numerator of the integrands),

$$S:\begin{cases} I+J=\underbrace{\int_0^1 \frac{x\log(x)\log(1-x^2)}{1+x^2}\mathrm{d}x}_{A};\\ I-J=\underbrace{\int_0^1 \frac{x\log(x)}{1+x^2}\log\left(\frac{1-x}{1+x}\right)\mathrm{d}x}_{B}. \end{cases}\tag{3.63}$$

Now, for the integral A in (3.63), we have

$$A=\int_0^1 \frac{x\log(x)\log(1-x^2)}{1+x^2}\mathrm{d}x \overset{x^2=t}{=} \frac{1}{4}\int_0^1 \frac{\log(t)\log(1-t)}{1+t}\mathrm{d}t$$

$$=\frac{1}{4}\int_0^1 \frac{\log(t)\log\left(\dfrac{1-t^2}{1+t}\right)}{1+t}\mathrm{d}t$$

$$=\frac{1}{4}\int_0^1 \frac{(1-t)\log(t)\log(1-t^2)}{1-t^2}\mathrm{d}t-\frac{1}{4}\int_0^1 \frac{\log(t)\log(1+t)}{1+t}\mathrm{d}t$$

$$=\frac{1}{4}\underbrace{\int_0^1 \frac{\log(t)\log(1-t^2)}{1-t^2}\mathrm{d}t}_{C}-\frac{1}{4}\underbrace{\int_0^1 \frac{t\log(t)\log(1-t^2)}{1-t^2}\mathrm{d}t}_{D}$$

$$-\frac{1}{4}\underbrace{\int_0^1 \frac{\log(t)\log(1+t)}{1+t}\mathrm{d}t}_{E}. \tag{3.64}$$

For the integral C in (3.14), we make the change of variable $t^2=u$ that leads to

$$C=\int_0^1 \frac{\log(t)\log(1-t^2)}{1-t^2}\mathrm{d}t=\frac{1}{4}\int_0^1 \frac{\log(u)\log(1-u)}{\sqrt{u}(1-u)}\mathrm{d}u$$

$$=\frac{1}{4}\lim_{\substack{x\to 1/2\\ y\to 0}}\frac{\partial^2}{\partial x\partial y}\mathrm{B}(x,y)$$

$$=\frac{1}{4}\left(7\zeta(3)-6\log(2)\zeta(2)\right), \tag{3.65}$$

where to get the value of the limit, we might prefer to use *Mathematica*.

Then, for the integral D in (3.14), we proceed as above and make the change of variable $t^2=u$ that gives

$$D=\int_0^1 \frac{t\log(t)\log(1-t^2)}{1-t^2}\mathrm{d}t=\frac{1}{4}\int_0^1 \frac{\log(u)\log(1-u)}{1-u}\mathrm{d}t$$

$$=-\frac{1}{4}\int_0^1 \sum_{n=1}^{\infty} u^n H_n\log(u)\mathrm{d}u$$

{change the order of summation and integration}

$$= -\frac{1}{4}\sum_{n=1}^{\infty} H_n \int_0^1 u^n \log(u)\mathrm{d}u = \frac{1}{4}\sum_{n=1}^{\infty} \frac{H_n}{(n+1)^2} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{H_{n+1} - 1/(n+1)}{(n+1)^2}$$

{reindex the series}

$$= \frac{1}{4}\sum_{n=1}^{\infty} \frac{H_n - 1/n}{n^2} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{H_n}{n^2} - \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{4}\zeta(3), \tag{3.66}$$

where I used the classical Euler sum in (3.45), the case $n = 2$.

Next, for the integral E in (3.14), we integrate by parts, and then we have

$$E = \int_0^1 \frac{\log(t)\log(1+t)}{1+t}\mathrm{d}t = \frac{1}{2}\int_0^1 \log(t)\left(\log^2(1+t)\right)' \mathrm{d}t$$

$$= \underbrace{\frac{1}{2}\log(t)\log^2(1+t)\bigg|_{t=0}^{t=1}}_{0} - \frac{1}{2}\int_0^1 \frac{\log^2(1+t)}{t}\mathrm{d}t = -\frac{1}{8}\zeta(3), \tag{3.67}$$

where the value of the last integral is given in (1.13).

By collecting the values of the integrals C, D, and E from (3.65), (3.66), and (3.67) in (3.14), we obtain

$$A = \int_0^1 \frac{x\log(x)\log(1-x^2)}{1+x^2}\mathrm{d}x = \frac{13}{32}\zeta(3) - \frac{3}{8}\log(2)\zeta(2). \tag{3.68}$$

Then, for the integral B in (3.63), we have

$$B = \int_0^1 \frac{x\log(x)\log\left(\frac{1-x}{1+x}\right)}{1+x^2}\mathrm{d}x \overset{\frac{1-x}{1+x}=t}{=} \int_0^1 \frac{(1-t)\log(t)\log\left(\frac{1-t}{1+t}\right)}{(1+t)(1+t^2)}\mathrm{d}t$$

$$= \int_0^1 \frac{\log(t)\log(1-t)}{1+t}\mathrm{d}t - \int_0^1 \frac{\log(t)\log(1+t)}{1+t}\mathrm{d}t - \underbrace{\int_0^1 \frac{t\log(t)\log\left(\frac{1-t}{1+t}\right)}{1+t^2}\mathrm{d}t}_{B},$$

whence we get

$$B = \frac{1}{2}\int_0^1 \frac{\log(t)\log(1-t)}{1+t}\mathrm{d}t - \frac{1}{2}\int_0^1 \frac{\log(t)\log(1+t)}{1+t}\mathrm{d}t = \frac{7}{8}\zeta(3) - \frac{3}{4}\log(2)\zeta(2), \tag{3.69}$$

where in (3.69) the first integral is $4A$, and I used the value in (3.68), and the second integral is the integral E, and then I made use of the value obtained in (3.67).

Collecting the results from (3.68) and (3.69) in the initial system of relations from (3.63), we get

$$S:\begin{cases} I+J=\dfrac{13}{32}\zeta(3)-\dfrac{3}{8}\log(2)\zeta(2); \\ \\ I-J=\dfrac{7}{8}\zeta(3)-\dfrac{3}{4}\log(2)\zeta(2), \end{cases}$$

whence we conclude that

$$I=\int_0^1 \frac{x\log(x)\log(1-x)}{1+x^2}\mathrm{d}x=\frac{1}{16}\left(\frac{41}{4}\zeta(3)-9\log(2)\zeta(2)\right)$$

and

$$J=\int_0^1 \frac{x\log(x)\log(1+x)}{1+x^2}\mathrm{d}x=\frac{1}{16}\left(3\log(2)\zeta(2)-\frac{15}{4}\zeta(3)\right),$$

and the solution is complete.

Both integrals play an important part in the derivation of some more advanced integrals in the book. For example, one could create a strategy to attack the integrals from Sect. 1.19 by using the present integrals too.

The curious reader, who also wants to know how to attack the integrals differently, might choose to employ the result in (3.62). For example, if we multiply both sides of (3.62) by $1/x$ and integrate from $x=0$ to $x=1$, we get for the left-hand side, after an application of the integration by parts, the integral from the point i) with a minus sign in front. On the other hand, in the right-hand side we obtain

$$\underbrace{\frac{1}{8}\int_0^1 \frac{\log^2(1+x^2)}{x}\mathrm{d}x-\frac{1}{2}\int_0^1 \frac{\mathrm{Li}_2(x)}{x}\mathrm{d}x+\frac{1}{8}\int_0^1 \frac{\mathrm{Li}_2(-x^2)}{x}\mathrm{d}x}_{-17/32\zeta(3)}$$

$$+\underbrace{\frac{1}{4}\int_0^1 \frac{1}{x}\mathrm{Li}_2\left(\frac{2x}{1+x^2}\right)\mathrm{d}x}_{1/16(9\log(2)\zeta(2)-7/4\zeta(3))}=\frac{1}{16}\left(9\log(2)\zeta(2)-\frac{41}{4}\zeta(3)\right),$$

where the first three integrals are straightforward, and for the fourth integral we use the Polylogarithm function representation in Sect. 1.6, and after reversing the order of integration, it's easy to show, using the Fourier series of $\log(\cos(x))$, that

$$\int_0^1 \frac{1}{x} \operatorname{Li}_2\left(\frac{2x}{1+x^2}\right) dx = \int_0^{\pi/2} (x-\pi)\log(\cos(x))dx$$
$$= \frac{1}{4}\left(9\log(2)\zeta(2) - \frac{7}{4}\zeta(3)\right).$$

3.15 A Class of Tricky and Useful Integrals with Consecutive Positive Integer Powers for the Logarithms

Solution In this section allow me to introduce you to the proposed generalized integral by considering first the following result involving a sum of series,

$$\frac{\log^2(2)}{2}\sum_{n=1}^{\infty}\frac{\varphi(n)}{(n+1)^2 2^n} + \log(2)\sum_{n=1}^{\infty}\frac{\varphi(n)}{(n+1)^3 2^n} + \sum_{n=1}^{\infty}\frac{\varphi(n)}{(n+1)^4 2^n}$$
$$= \frac{23}{8}\zeta(6) - 2\zeta^2(3) - \frac{1}{18}\log^6(2),$$

where $\varphi(n) = H_n^2 - H_n^{(2)}$. How would you like to approach such a problem? I would only tell you as a hint, for the series result above, that you might like to consider the case $p = 2$ of the generalized integral we want to calculate together with the use of the series, $3\sum_{n=1}^{\infty}\frac{x^{n+1}}{n+1}\left(H_n^2 - H_n^{(2)}\right) = -\log^3(1-x)$, and then let you to discover alone the whole solution (and if you don't succeed, no worry, you can put the problem aside and return back to it after gaining more experience from the chapter *Sums and Series*). In the next chapter, when dealing with the harmonic series containing powers of 2 in denominator, we'll want to know how to evaluate such an integral.

Let's start with the integration by parts, and we write

$$I = \int_0^{1/2} \frac{\log^p(1-x)\log^{p+1}(x)}{1-x} dx$$
$$= \int_0^{1/2}\left(-\frac{1}{1+p}\log^{p+1}(1-x)\right)' \log^{p+1}(x)dx$$
$$= -\frac{1}{1+p}(\log(1-x)\log(x))^{p+1}\Big|_{x=0}^{x=1/2} + \int_0^{1/2}\frac{\log^{p+1}(1-x)\log^p(x)}{x}dx$$
$$= -\frac{1}{1+p}\log^{2(p+1)}(2) + \left(\int_0^1 - \int_{1/2}^1\right)\frac{\log^{p+1}(1-x)\log^p(x)}{x}dx$$

$$= -\frac{1}{1+p}\log^{2(p+1)}(2) + \int_0^1 \frac{\log^{p+1}(1-x)\log^p(x)}{x}\mathrm{d}x$$
$$-\int_{1/2}^1 \frac{\log^{p+1}(1-x)\log^p(x)}{x}\mathrm{d}x$$

{make use of the Beta function definition, $\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t$, for}

{the first integral, and in the second integral let the change of variable $1-x=y$}

$$= -\frac{1}{1+p}\log^{2(p+1)}(2) + \lim_{\substack{x\to 0\\ y\to 1}}\frac{\partial^{2p+1}}{\partial x^p \partial y^{p+1}}\mathrm{B}(x,y) - \underbrace{\int_0^{1/2}\frac{\log^p(1-y)\log^{p+1}(y)}{1-y}\mathrm{d}y}_{I},$$

whence we obtain that

$$I = \int_0^{1/2}\frac{\log^p(1-x)\log^{p+1}(x)}{1-x}\mathrm{d}x = -\frac{1}{2(1+p)}\log^{2(p+1)}(2)$$
$$+\frac{1}{2}\lim_{\substack{x\to 0\\ y\to 1}}\frac{\partial^{2p+1}}{\partial x^p \partial y^{p+1}}\mathrm{B}(x,y),$$

and the solution is finalized.

The case $p=1$ may also be found mentioned and evaluated in [24, p. 128].

3.16 A Double Integral and a Triple Integral, Beautifully Connected with the Advanced Harmonic Series

Solution If you didn't skip my *Preface*, you might figure out how to deal with such integrals. The first phase of the solutions is pretty straightforward, and that refers to the transformation of the integrals into (beautiful classical) harmonic series, and then we have to calculate the series which represent the challenging part of the section.

For the first point of the problem, let's start with the generating function in (4.5), and then we write

$$\int_0^1\left(\int_0^1 \frac{\log(1-x)\log(1-y)\log(1+xy)}{xy(1+xy)}\mathrm{d}x\right)\mathrm{d}y$$

$$= \int_0^1 \left(\int_0^1 \log(1-x)\log(1-y) \sum_{n=1}^{\infty} (-1)^{n-1} (xy)^{n-1} H_n \mathrm{d}x \right) \mathrm{d}y$$

{reverse the order of summation and integration}

$$= \sum_{n=1}^{\infty} (-1)^{n-1} H_n \int_0^1 y^{n-1} \log(1-y) \left(\int_0^1 x^{n-1} \log(1-x) \mathrm{d}x \right) \mathrm{d}y$$

{make use of the identity in (1.4)}

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n^3}{n^2}$$

$$= \frac{1}{5}\log^5(2) - \log^3(2)\zeta(2) + \frac{21}{8}\log^2(2)\zeta(3) - \frac{27}{16}\zeta(2)\zeta(3) - \frac{9}{4}\zeta(5)$$

$$+6\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right) + 6\operatorname{Li}_5\left(\frac{1}{2}\right),$$

where I used the result in (4.94) and the solution to the point $i)$ of the problem is finalized.

For the second part of the problem, we employ the geometric series, $\sum_{n=1}^{\infty} \frac{1-z^n}{1-z}(xy)^{n-1} = \frac{1}{(1-xy)(1-xyz)}$, and write that

$$\int_0^1 \left(\int_0^1 \left(\int_0^1 \frac{\log(1-x)\log(1-y)\log(1-z)}{(1-xy)(1-xyz)} \mathrm{d}x \right) \mathrm{d}y \right) \mathrm{d}z$$

$$= \int_0^1 \left(\int_0^1 \left(\int_0^1 \sum_{n=1}^{\infty} \frac{1-z^n}{1-z} (xy)^{n-1} \log(1-x)\log(1-y)\log(1-z) \mathrm{d}x \right) \mathrm{d}y \right) \mathrm{d}z$$

{reverse the order of summation and integration}

$$= \sum_{n=1}^{\infty} \int_0^1 \frac{1-z^n}{1-z} \log(1-z) \left(\int_0^1 y^{n-1} \log(1-y) \left(\int_0^1 x^{n-1} \log(1-x) \mathrm{d}x \right) \mathrm{d}y \right) \mathrm{d}z$$

{use the integral in (1.4), and note that the integral in z may be calculated by}

{combining the integration by parts and the result in (1.5) that immediately}

$$\left\{\text{lead to the value of the integral in } z, \int_0^1 \frac{1-z^n}{1-z}\log(1-z)\mathrm{d}z = -\frac{1}{2}(H_n^2 + H_n^{(2)})\right\}$$

$$= -\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n^4}{n^2} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n^2 H_n^{(2)}}{n^2} = -\frac{1}{2}\left(5\zeta^2(3) + \frac{1061}{24}\zeta(6)\right),$$

where for getting the last equality I made use of the values of the harmonic series in (4.39) and (4.38), and the solution to the point ii) of the problem is finalized.

Now, if it happens to have a hard time with the resulting harmonic series from both points of the problem, no worry, they are calculated in the sixth chapter where I'll treat and calculate many other harmonic series.

3.17 Let's Take Two Double Logarithmic Integrals with Beautiful Values Expressed in Terms of the Riemann Zeta Function

Let's prepare now for an exciting experience with two double logarithmic integrals. Although at the surface they might look scary, the double integrals actually hide a simple, beautiful idea, and by some steps we'll be able to reduce everything to the calculation of easy to do integrals in one variable. Also, a short review of the Gamma function properties could be pretty useful to accomplish the needed calculations (resources for such a review may be found in [10, 107], [9, pp. 1–60]).

For both integrals we might like to start with the elementary integral, $\int_0^1 x^t \mathrm{d}t = \left.\frac{x^t}{\log(x)}\right|_{t=0}^{t=1} = \frac{x-1}{\log(x)}$, where if we replace x by x/y and then multiply both sides by y, we obtain

$$y\int_0^1 \left(\frac{x}{y}\right)^t \mathrm{d}t = \frac{x-y}{\log(x)-\log(y)}. \tag{3.70}$$

Further, if we replace x by $\log\left(\frac{1}{x}\right)$ and y by $\log\left(\frac{1}{y}\right)$ in (3.70), we get

$$\int_0^1 \log^t\left(\frac{1}{x}\right)\log^{1-t}\left(\frac{1}{y}\right)\mathrm{d}t = \frac{\log\left(\frac{1}{x}\right) - \log\left(\frac{1}{y}\right)}{\log\left(\log\left(\frac{1}{x}\right)\right) - \log\left(\log\left(\frac{1}{y}\right)\right)}. \tag{3.71}$$

Integrating both sides of (3.71) with respect to x and y over the unit interval, we have

$$\int_0^1 \left(\int_0^1 \frac{\log\left(\frac{1}{x}\right) - \log\left(\frac{1}{y}\right)}{\log\left(\log\left(\frac{1}{x}\right)\right) - \log\left(\log\left(\frac{1}{y}\right)\right)} \mathrm{d}x \right) \mathrm{d}y$$

$$= \int_0^1 \left(\int_0^1 \left(\int_0^1 \log^t\left(\frac{1}{x}\right) \log^{1-t}\left(\frac{1}{y}\right) \mathrm{d}t \right) \mathrm{d}x \right) \mathrm{d}y$$

{reverse the order of integration}

$$= \int_0^1 \left(\int_0^1 \left(\int_0^1 \log^t\left(\frac{1}{x}\right) \log^{1-t}\left(\frac{1}{y}\right) \mathrm{d}x \right) \mathrm{d}y \right) \mathrm{d}t$$

{make the changes of variable $x = e^{-u}$ and $y = e^{-v}$}

$$= \int_0^1 \left(\int_0^\infty v^{1-t} e^{-v} \left(\int_0^\infty u^t e^{-u} \mathrm{d}u \right) \mathrm{d}v \right) \mathrm{d}t = \int_0^1 \Gamma(2-t)\Gamma(1+t)\mathrm{d}t$$

{make use of the Gamma function property, $\Gamma(x+1) = x\Gamma(x)$}

$$= \int_0^1 t(1-t)\Gamma(1-t)\Gamma(t)\mathrm{d}t$$

$\left\{\text{make use of Euler's reflection formula, } \Gamma(t)\Gamma(1-t) = \frac{\pi}{\sin(\pi t)}\right\}$

$$= \pi \int_0^1 \frac{t(1-t)}{\sin(\pi t)} \mathrm{d}t \overset{\pi t = z}{=} \frac{1}{\pi^2} \int_0^\pi \frac{(\pi - z)z}{\sin(z)} \mathrm{d}z. \tag{3.72}$$

Using that $\int \frac{1}{\sin(x)} \mathrm{d}x = \frac{1}{2} \int \frac{1}{\sin(x/2)\cos(x/2)} \mathrm{d}x = \frac{1}{2} \int \left(\frac{\cos(x/2)}{\sin(x/2)} + \frac{\sin(x/2)}{\cos(x/2)} \right) \mathrm{d}x = \int \frac{(\sin(x/2))'}{\sin(x/2)} \mathrm{d}x - \int \frac{(\cos(x/2))'}{\cos(x/2)} \mathrm{d}x = \log(\tan(x/2)) + C$, we apply the integration by parts for the remaining integral in (3.72), and then we write

$$\int_0^\pi \frac{(\pi - z)z}{\sin(z)} \mathrm{d}z = \int_0^\pi \left(\log\left(\tan\left(\frac{z}{2}\right)\right)\right)' (\pi - z)z\mathrm{d}z = \underbrace{\log\left(\tan\left(\frac{z}{2}\right)\right)(\pi - z)z\Big|_{z=0}^{z=\pi}}_{0}$$

$$
\begin{aligned}
-\int_0^{\pi}(\pi-2z)\log\left(\tan\left(\frac{z}{2}\right)\right)\mathrm{d}z &= -\pi\underbrace{\int_0^{\pi}\log\left(\tan\left(\frac{z}{2}\right)\right)\mathrm{d}z}_{0}\\
&+2\int_0^{\pi}z\log\left(\tan\left(\frac{z}{2}\right)\right)\mathrm{d}z\\
=2\int_0^{\pi}z\log\left(\tan\left(\frac{z}{2}\right)\right)\mathrm{d}z &\overset{z/2=w}{=} 8\int_0^{\pi/2}w\log\left(\tan(w)\right)\mathrm{d}w\\
&= -16\int_0^{\pi/2}\sum_{k=1}^{\infty}\frac{w\cos(2(2k-1)w)}{2k-1}\mathrm{d}w
\end{aligned}
$$

{reverse the order of summation and integration}

$$
= -16\sum_{k=1}^{\infty}\int_0^{\pi/2}\frac{w\cos(2(2k-1)w)}{2k-1}\mathrm{d}w
$$

$$
= -16\sum_{k=1}^{\infty}\left(\frac{\pi\sin(2k\pi)}{4(2k-1)^3}-\frac{k\pi\sin(2k\pi)}{2(2k-1)^3}-\frac{\cos^2(k\pi)}{2(2k-1)^3}\right)
$$

$$
= -4\pi\underbrace{\sum_{k=1}^{\infty}\frac{\sin(2k\pi)}{(2k-1)^3}}_{0}+8\pi\underbrace{\sum_{k=1}^{\infty}\frac{k\sin(2k\pi)}{(2k-1)^3}}_{0}+8\sum_{k=1}^{\infty}\frac{\cos^2(k\pi)}{(2k-1)^3}=8\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3}
$$

$$
= 7\zeta(3), \tag{3.73}
$$

and in the calculations I used the Fourier series of $\log(\tan(x))$,

$$
\log(\tan(x)) = -2\sum_{k=1}^{\infty}\frac{\cos(2(2k-1)x)}{2k-1},\ 0<x<\frac{\pi}{2}, \tag{3.74}
$$

that appears in the form $\dfrac{1}{2}\log\left(\cot\left(\dfrac{x}{2}\right)\right)$ in **1.442.2** from [30], where, when needed, we use the simple fact that $\dfrac{1}{2}\log\left(\cot\left(\dfrac{x}{2}\right)\right) = -\dfrac{1}{2}\log\left(\tan\left(\dfrac{x}{2}\right)\right)$. An excellent work on the Fourier series may be found in [85].

Hence, if we combine the results from (3.72) and (3.73), we conclude that

$$\int_0^1 \int_0^1 \frac{\log\left(\frac{1}{x}\right) - \log\left(\frac{1}{y}\right)}{\log\left(\log\left(\frac{1}{x}\right)\right) - \log\left(\log\left(\frac{1}{y}\right)\right)} \mathrm{d}x\mathrm{d}y = \frac{7\zeta(3)}{6\zeta(2)},$$

and the part i) of the problem is complete.

As regards the auxiliary integral $\int \frac{1}{\sin(x)}\mathrm{d}x$, we may also use the Weierstrass substitution (see [120]) as proposed in [66, p. 379].

For the second point of the problem, we act similarly and make use again of the result in (3.71) that leads immediately to

$$\int_0^1 \left(\int_0^1 \log^{t+u}\left(\frac{1}{x}\right)\log^{2-t-u}\left(\frac{1}{y}\right)\mathrm{d}t\right)\mathrm{d}u$$

$$= \left(\frac{\log\left(\frac{1}{x}\right) - \log\left(\frac{1}{y}\right)}{\log\left(\log\left(\frac{1}{x}\right)\right) - \log\left(\log\left(\frac{1}{y}\right)\right)}\right)^2. \tag{3.75}$$

Integrating both sides of (3.75) with respect to x and y over the unit interval, we obtain

$$\int_0^1 \left(\int_0^1 \left(\frac{\log\left(\frac{1}{x}\right) - \log\left(\frac{1}{y}\right)}{\log\left(\log\left(\frac{1}{x}\right)\right) - \log\left(\log\left(\frac{1}{y}\right)\right)}\right)^2 \mathrm{d}x\right)\mathrm{d}y$$

$$= \int_0^1 \left(\int_0^1 \left(\int_0^1 \left(\int_0^1 \log^{t+u}\left(\frac{1}{x}\right)\log^{2-t-u}\left(\frac{1}{y}\right)\mathrm{d}t\right)\mathrm{d}u\right)\mathrm{d}x\right)\mathrm{d}y$$

$$\{\text{reverse the order of integration}\}$$

$$= \int_0^1 \left(\int_0^1 \left(\int_0^1 \left(\int_0^1 \log^{t+u}\left(\frac{1}{x}\right)\log^{2-t-u}\left(\frac{1}{y}\right)\mathrm{d}x\right)\mathrm{d}y\right)\mathrm{d}t\right)\mathrm{d}u$$

$$\{\text{make the changes of variable } x = e^{-v} \text{ and } y = e^{-s}\}$$

$$= \int_0^1 \left(\int_0^1 \left(\int_0^1 s^{2-t-u}e^{-s}\left(\int_0^1 v^{t+u}e^{-v}\mathrm{d}v\right)\mathrm{d}s\right)\mathrm{d}t\right)\mathrm{d}u$$

$$= \int_0^1 \left(\int_0^1 \Gamma(3-t-u)\Gamma(1+t+u)\mathrm{d}t \right) \mathrm{d}u$$

$$\overset{u+t=z}{=} \int_0^1 \left(\int_u^{u+1} \Gamma(3-z)\Gamma(1+z)\mathrm{d}z \right) \mathrm{d}u. \tag{3.76}$$

Integrating by parts in (3.76), we get

$$\int_0^1 u' \left(\int_u^{u+1} \Gamma(3-z)\Gamma(1+z)\mathrm{d}z \right) \mathrm{d}u = u \int_u^{u+1} \Gamma(3-z)\Gamma(1+z)\mathrm{d}z \Big|_{u=0}^{u=1}$$

$$-\int_0^1 u\Gamma(2-u)\Gamma(2+u)\mathrm{d}u + \int_0^1 u\Gamma(3-u)\Gamma(1+u)\mathrm{d}u = \underbrace{\int_1^2 \Gamma(3-z)\Gamma(1+z)\mathrm{d}z}_{I_1}$$

$$-\underbrace{\int_0^1 u\Gamma(2-u)\Gamma(2+u)\mathrm{d}u}_{I_2} + \underbrace{\int_0^1 u\Gamma(3-u)\Gamma(1+u)\mathrm{d}u}_{I_3}. \tag{3.77}$$

For the integral I_1 in (3.77), we have

$$I_1 = \int_1^2 \Gamma(3-z)\Gamma(1+z)\mathrm{d}z \overset{z-1=s}{=} \int_0^1 \Gamma(2-s)\Gamma(2+s)\mathrm{d}s$$

$$\{\text{make use of the Gamma function property, } \Gamma(x+1) = x\Gamma(x)\}$$

$$= \int_0^1 s(1-s^2)\Gamma(1-s)\Gamma(s)\mathrm{d}s$$

$$\left\{\text{make use of Euler's reflection formula, } \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}\right\}$$

$$= \pi \int_0^1 \frac{s(1-s^2)}{\sin(\pi s)}\mathrm{d}s \overset{s\pi=w}{=} \int_0^\pi \frac{w/\pi(1-(w/\pi)^2)}{\sin(w)}\mathrm{d}w$$

$$= \frac{1}{\pi} \int_0^\pi \left(\log\left(\tan\left(\frac{w}{2} \right) \right) \right)' w \left(1 - \left(\frac{w}{\pi} \right)^2 \right) \mathrm{d}w$$

$$\{\text{apply the integration by parts}\}$$

$$= \underbrace{\frac{1}{\pi} \log\left(\tan\left(\frac{w}{2} \right) \right) w \left(1 - \left(\frac{w}{\pi} \right)^2 \right) \Big|_{w=0}^{w=\pi}}_{0} - \frac{1}{\pi} \underbrace{\int_0^\pi \log\left(\tan\left(\frac{w}{2} \right) \right) \mathrm{d}w}_{0}$$

$$+\frac{3}{\pi^3}\int_0^{\pi} w^2 \log\left(\tan\left(\frac{w}{2}\right)\right) \mathrm{d}w = \frac{3}{\pi^3}\int_0^{\pi} w^2 \log\left(\tan\left(\frac{w}{2}\right)\right) \mathrm{d}w$$

$$\overset{w/2=r}{=} \frac{24}{\pi^3}\int_0^{\pi/2} r^2 \log\left(\tan(r)\right) \mathrm{d}r$$

{use the Fourier series in (3.74)}

$$= -\frac{48}{\pi^3}\int_0^{\pi/2} \sum_{k=1}^{\infty} \frac{r^2\cos(2(2k-1)r)}{2k-1}\mathrm{d}r$$

{reverse the order of summation and integration}

$$= -\frac{48}{\pi^3}\sum_{k=1}^{\infty}\int_0^{\pi/2} \frac{r^2\cos(2(2k-1)r)}{2k-1}\mathrm{d}r = \frac{6}{\pi^3}\underbrace{\sum_{k=1}^{\infty}\frac{\left(\pi^2(2k-1)^2-2\right)\sin(2\pi k)}{(2k-1)^4}}_{0}$$

$$+\frac{12}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos(2\pi k)}{(2k-1)^3} = \frac{12}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3} = \frac{7\zeta(3)}{4\zeta(2)}. \tag{3.78}$$

Next, for the integral I_2 in (3.77), we have

$$I_2 = \int_0^1 u\Gamma(2-u)\Gamma(2+u)\mathrm{d}u$$

{make use of the Gamma function property, $\Gamma(x+1) = x\Gamma(x)$}

$$= \int_0^1 u^2(1-u^2)\Gamma(1-u)\Gamma(u)\mathrm{d}u$$

$$\left\{\text{make use of Euler's reflection formula, } \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}\right\}$$

$$= \pi\int_0^1 \frac{u^2(1-u^2)}{\sin(\pi u)}\mathrm{d}u \overset{\pi u=s}{=} \int_0^{\pi} \frac{(s/\pi)^2(1-(s/\pi)^2)}{\sin(s)}\mathrm{d}s$$

$$= \int_0^{\pi} \left(\log\left(\tan\left(\frac{s}{2}\right)\right)\right)' \left(\frac{s}{\pi}\right)^2 \left(1-\left(\frac{s}{\pi}\right)^2\right) \mathrm{d}s$$

{apply the integration by parts}

$$= \underbrace{\log\left(\tan\left(\frac{s}{2}\right)\right)\left(\frac{s}{\pi}\right)^2\left(1-\left(\frac{s}{\pi}\right)^2\right)\Bigg|_{s=0}^{s=\pi}}_{0} - \frac{2}{\pi^4}\int_0^{\pi}\log\left(\tan\left(\frac{s}{2}\right)\right)s(\pi^2-2s^2)\mathrm{d}s$$

$$= -\frac{2}{\pi^2}\int_0^{\pi}s\log\left(\tan\left(\frac{s}{2}\right)\right)\mathrm{d}s + \frac{4}{\pi^4}\int_0^{\pi}s^3\log\left(\tan\left(\frac{s}{2}\right)\right)\mathrm{d}s$$

{note the first integral can be extracted from (3.73)}

$$= -\frac{7\zeta(3)}{6\zeta(2)} + \frac{4}{\pi^4}\int_0^{\pi}s^3\log\left(\tan\left(\frac{s}{2}\right)\right)\mathrm{d}s \overset{s/2=t}{=} -\frac{7\zeta(3)}{6\zeta(2)} + \frac{64}{\pi^4}\int_0^{\pi/2}t^3\log(\tan(t))\,\mathrm{d}t$$

{use the Fourier series in (3.74)}

$$= -\frac{7\zeta(3)}{6\zeta(2)} - \frac{128}{\pi^4}\int_0^{\pi/2}\sum_{k=1}^{\infty}\frac{t^3\cos(2(2k-1)t)}{2k-1}\mathrm{d}t$$

{reverse the order of summation and integration}

$$= -\frac{7\zeta(3)}{6\zeta(2)} - \frac{128}{\pi^4}\sum_{k=1}^{\infty}\frac{1}{2k-1}\int_0^{\pi/2}t^3\cos(2(2k-1)t)\mathrm{d}t$$

$$= -\frac{7\zeta(3)}{6\zeta(2)} + \frac{8}{\pi^3}\underbrace{\sum_{k=1}^{\infty}\frac{(2k-1)^2\pi^2-6}{(2k-1)^4}\sin(2k\pi)}_{0} - \frac{48}{\pi^4}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^5}$$

$$-\frac{48}{\pi^4}\sum_{k=1}^{\infty}\frac{\cos(2k\pi)}{(2k-1)^5} + \frac{24}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos(2k\pi)}{(2k-1)^3} = -\frac{7\zeta(3)}{6\zeta(2)} - \frac{96}{\pi^4}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^5}$$

$$+\frac{24}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3} = \frac{7\zeta(3)}{3\zeta(2)} - \frac{31\zeta(5)}{30\zeta(4)}. \tag{3.79}$$

Lastly, for the integral I_3 in (3.77), we have

$$I_3 = \int_0^1 u\Gamma(3-u)\Gamma(1+u)\mathrm{d}u \overset{1-u=t}{=} \int_0^1 (1-t)\Gamma(2+t)\Gamma(2-t)\mathrm{d}t$$

$$= \underbrace{\int_0^1 \Gamma(2+t)\Gamma(2-t)\mathrm{d}t}_{I_1} - \underbrace{\int_0^1 t\Gamma(2+t)\Gamma(2-t)\mathrm{d}t}_{I_2} = \frac{31\zeta(5)}{30\zeta(4)} - \frac{7\zeta(3)}{12\zeta(2)}. \tag{3.80}$$

Collecting the results from (3.78), (3.79) and (3.80) in (3.77), we conclude that

$$\int_0^1\int_0^1 \left(\frac{\log\left(\frac{1}{x}\right)-\log\left(\frac{1}{y}\right)}{\log\left(\log\left(\frac{1}{x}\right)\right)-\log\left(\log\left(\frac{1}{y}\right)\right)}\right)^2 \mathrm{d}x\mathrm{d}y=\frac{31\zeta(5)}{15\zeta(4)}-\frac{7\zeta(3)}{6\zeta(2)},$$

and the part $ii)$ of the problem is complete.

These double integrals are not the only ones from the book where we need to act similarly. The study of *integrals with symmetrical integrands* idea that led me to the creation of the actual integrals also led me to the derivation of other such appealing integrals you may find in the next sections. One good lesson to learn from this section is that sometimes finding a simple integral representation of the integrand might be extremely useful. Let's add this to our portofolio of strategies.

3.18 Interesting Integrals Containing the Inverse Tangent Function and the Logarithmic Function

Solution One of the things I particularly love about calculating integrals, sums, and series is to approach them from more *directions*, using different strategies, ideas. In this section *I'll take the liberty* of providing three full solutions to the given integrals, which I usually don't do elsewhere in this book for well-known reasons (trying to add more problems to the book while following a certain pages limit).

For a global solution that approaches both integrals, we consider the system of relations involving the following integrals I and J (we are already familiar with such an approach from the previous sections),

$$S:\begin{cases} I+J=\underbrace{\displaystyle\int_0^1 \arctan(x)\log(1-x^2)\mathrm{d}x}_{U}; \\ I-J=\underbrace{\displaystyle\int_0^1 \arctan(x)\log\left(\frac{1-x}{1+x}\right)\mathrm{d}x}_{V}. \end{cases} \tag{3.81}$$

We start with the calculation of the integral U in (3.81) and proceed as follows

$$U=\int_0^1 \arctan(x)\log(1-x^2)\mathrm{d}x=\int_0^1\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{(-1)^n x^{2k+2n-1}}{k(2n-1)}\right)\mathrm{d}x$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\left(\sum_{k=1}^{\infty}\frac{1}{k}\int_0^1 x^{2k+2n-1}\mathrm{d}x\right)=\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\left(\sum_{k=1}^{\infty}\frac{1}{k(k+n)}\right)$$

$$= -\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}H_n}{n(2n-1)} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{2n} - \sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{2n-1}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}H_n\int_0^1 x^{n-1}\mathrm{d}x - \sum_{n=1}^{\infty}(-1)^{n-1}H_n\int_0^1 x^{2n-2}\mathrm{d}x$$

{reverse the order of integration and summation}

$$= \frac{1}{2}\int_0^1\sum_{n=1}^{\infty}(-1)^{n-1}H_n x^{n-1}\mathrm{d}x - \int_0^1\sum_{n=1}^{\infty}(-1)^{n-1}H_n x^{2n-2}\mathrm{d}x$$

{consider the result in (4.5)}

$$= \frac{1}{2}\int_0^1\frac{\log(1+x)}{x(1+x)}\mathrm{d}x - \int_0^1\frac{\log(1+x^2)}{x^2(1+x^2)}\mathrm{d}x$$

$$= \underbrace{\frac{1}{2}\int_0^1\frac{\log(1+x)}{x}\mathrm{d}x}_{U_1} - \underbrace{\frac{1}{2}\int_0^1\frac{\log(1+x)}{1+x}\mathrm{d}x}_{U_2} - \underbrace{\int_0^1\frac{\log(1+x^2)}{x^2}\mathrm{d}x}_{U_3}$$

$$+\underbrace{\int_0^1\frac{\log(1+x^2)}{1+x^2}\mathrm{d}x}_{U_4}. \tag{3.82}$$

For the integral U_1 in (3.18), we have

$$U_1 = \int_0^1\frac{\log(1+x)}{x}\mathrm{d}x = \int_0^1\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{n-1}}{n}\mathrm{d}x = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^1 x^{n-1}\mathrm{d}x$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \frac{1}{2}\zeta(2). \tag{3.83}$$

Further, for the integral U_2 in (3.18), we get

$$U_2 = \int_0^1\frac{\log(1+x)}{1+x}\mathrm{d}x = \frac{1}{2}\log^2(1+x)\Big|_{x=0}^{x=1} = \frac{1}{2}\log^2(2). \tag{3.84}$$

Then, for the integral U_3 in (3.18), we apply the integration by parts that yields

$$U_3 = \int_0^1 \frac{\log(1+x^2)}{x^2} \mathrm{d}x = -\int_0^1 \left(\frac{1}{x}\right)' \log(1+x^2)\mathrm{d}x = -\frac{\log(1+x^2)}{x}\bigg|_{x=0}^{x=1}$$

$$+ 2\int_0^1 \frac{1}{1+x^2}\mathrm{d}x = -\log(2) + \left(2\arctan(x)\bigg|_{x=0}^{x=1}\right) = \frac{\pi}{2} - \log(2). \tag{3.85}$$

Lastly, for the integral U_4 in (3.18), we make the change of variable $x = \tan(y)$ and employ the Fourier series in **1.441.4** from [30],

$$\log(\cos(x)) = -\log(2) + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\cos(2nx)}{n}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}, \tag{3.86}$$

that leads to

$$U_4 = \int_0^1 \frac{\log(1+x^2)}{1+x^2} \mathrm{d}x = -2\int_0^{\pi/4} \log(\cos(y))\mathrm{d}y$$

$$= -2\int_0^{\pi/4} \left(-\log(2) + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\cos(2ny)}{n}\right) \mathrm{d}y$$

$$= 2\log(2)\int_0^{\pi/4} \mathrm{d}y - 2\int_0^{\pi/4} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\cos(2ny)}{n} \mathrm{d}y$$

{reverse the order of summation and integration}

$$= \frac{\pi}{2}\log(2) - 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \int_0^{\pi/4} \cos(2ny)\mathrm{d}y$$

$$= \frac{\pi}{2}\log(2) - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin(n\pi/2)}{n^2}$$

{note the pattern of the numerator, that is $1, 0, -1, 0, 1, 0, -1, 0, \ldots$}

$$= \frac{\pi}{2}\log(2) - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2} = \frac{\pi}{2}\log(2) - G. \tag{3.87}$$

Collecting the values of the integrals U_1, U_2, U_3, and U_4 given in (3.83), (3.84), (3.85), and (3.87) in (3.18), we obtain that

$$U = \int_0^1 \arctan(x)\log(1-x^2)\mathrm{d}x$$

$$= \frac{1}{4}\zeta(2) + \frac{\pi}{2}\log(2) - \frac{\pi}{2} + \log(2) - \frac{1}{4}\log^2(2) - G. \tag{3.88}$$

Further, for the integral V in (3.81), we have

$$V = \int_0^1 \arctan(x)\log\left(\frac{1-x}{1+x}\right)\mathrm{d}x \overset{\frac{1-x}{1+x}=y}{=} 2\int_0^1 \arctan\left(\frac{1-y}{1+y}\right)\frac{\log(y)}{(1+y)^2}\mathrm{d}y$$

$$= 2\int_0^1 \left(\frac{\pi}{4} - \arctan(y)\right)\frac{\log(y)}{(1+y)^2}\mathrm{d}y$$

$$= \frac{\pi}{2}\int_0^1 \frac{\log(y)}{(1+y)^2}\mathrm{d}y - 2\int_0^1 \arctan(y)\frac{\log(y)}{(1+y)^2}\mathrm{d}y$$

$$= -\frac{\pi}{2}\log(2) - 2\int_0^1 \arctan(y)\frac{\log(y)}{(1+y)^2}\mathrm{d}y. \tag{3.89}$$

For the remaining integral in (3.89) we apply the integration by parts, and then we get

$$\int_0^1 \arctan(y)\frac{\log(y)}{(1+y)^2}\mathrm{d}y = \int_0^1 \arctan(y)\left(\frac{y\log(y)}{1+y} - \log(1+y)\right)'\mathrm{d}y$$

$$= \arctan(y)\left(\frac{y\log(y)}{1+y} - \log(1+y)\right)\Bigg|_{y=0}^{y=1}$$

$$+ \int_0^1 \left(\frac{\log(y)}{2(1+y)} + \frac{\log(1+y)}{1+y^2} - \frac{\log(y)}{2(1+y^2)} - \frac{y\log(y)}{2(1+y^2)}\right)\mathrm{d}y$$

$$= -\frac{\pi}{4}\log(2) + \frac{1}{2}\underbrace{\int_0^1 \frac{\log(y)}{1+y}\mathrm{d}y}_{V_1} + \underbrace{\int_0^1 \frac{\log(1+y)}{1+y^2}\mathrm{d}y}_{V_2} - \frac{1}{2}\underbrace{\int_0^1 \frac{\log(y)}{1+y^2}\mathrm{d}y}_{V_3}$$

$$-\frac{1}{2}\underbrace{\int_0^1 \frac{y\log(y)}{1+y^2}\mathrm{d}y}_{V_4}. \tag{3.90}$$

So, for the integral V_1 in (3.18), we have

$$V_1 = \int_0^1 \frac{\log(y)}{1+y} \mathrm{d}y = \int_0^1 \sum_{n=1}^{\infty} (-1)^{n-1} y^{n-1} \log(y) \mathrm{d}y$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \int_0^1 y^{n-1} \log(y) \mathrm{d}y$$

$$= -\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2} = -\frac{1}{2}\zeta(2). \tag{3.91}$$

Next, for the integral V_2 in (3.18), we get

$$V_2 = \int_0^1 \frac{\log(1+y)}{1+y^2} \mathrm{d}y \overset{\frac{1-z}{1+z}=y}{=} \int_0^1 \frac{\log\left(\frac{2}{1+z}\right)}{1+z^2} \mathrm{d}z = \log(2) \int_0^1 \frac{1}{1+z^2} \mathrm{d}z -$$

$$\underbrace{\int_0^1 \frac{\log(1+z)}{1+z^2} \mathrm{d}z}_{V_2} = \frac{\pi}{4} \log(2) - V_2,$$

whence we get that

$$V_2 = \int_0^1 \frac{\log(1+y)}{1+y^2} \mathrm{d}y = \frac{\pi}{8} \log(2). \tag{3.92}$$

The same strategy of calculating the integral V_2 is also considered in [49, 74].

Then, for the integral V_3 in (3.18), we obtain

$$V_3 = \int_0^1 \frac{\log(y)}{1+y^2} \mathrm{d}y = \int_0^1 \sum_{k=1}^{\infty} (-1)^{k-1} y^{2k-2} \log(y) \mathrm{d}y$$

{reverse the order of summation and integration}

$$= \sum_{k=1}^{\infty} (-1)^{k-1} \int_0^1 y^{2k-2} \log(y) \mathrm{d}y = -\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^2} = -G. \tag{3.93}$$

Lastly, for the integral V_4 in (3.18), we get

$$V_4 = \int_0^1 \frac{y \log(y)}{1+y^2} \mathrm{d}y = \int_0^1 \sum_{k=1}^{\infty} (-1)^{k-1} y^{2k-1} \log(y) \mathrm{d}y$$

$$\{\text{reverse the order of summation and integration}\}$$

$$= \sum_{k=1}^{\infty} (-1)^{k-1} \int_0^1 y^{2k-1} \log(y) \mathrm{d}y = -\frac{1}{4} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} = -\frac{1}{8}\zeta(2). \tag{3.94}$$

Collecting the values of the integrals V_1, V_2, V_3, and V_4 from (3.91), (3.92), (3.93), and (3.94) in (3.18), we have

$$\int_0^1 \arctan(y) \frac{\log(y)}{(1+y)^2} \mathrm{d}y = \frac{1}{2}G - \frac{\pi}{8}\log(2) - \frac{3}{16}\zeta(2). \tag{3.95}$$

Therefore, if we plug the result from (3.95) in (3.89), we get

$$V = \int_0^1 \arctan(x) \log\left(\frac{1-x}{1+x}\right) \mathrm{d}x = \frac{3}{8}\zeta(2) - \frac{\pi}{4}\log(2) - G. \tag{3.96}$$

Returning to our initial system of relations, based upon the results in (3.88) and (3.96), we have

$$S: \begin{cases} I + J = \dfrac{1}{4}\zeta(2) + \dfrac{\pi}{2}\log(2) - \dfrac{\pi}{2} + \log(2) - \dfrac{1}{4}\log^2(2) - G; \\ \\ I - J = \dfrac{3}{8}\zeta(2) - \dfrac{\pi}{4}\log(2) - G, \end{cases}$$

whence we obtain that

$$I = \int_0^1 \arctan(x) \log(1-x) \mathrm{d}x$$

$$= \frac{1}{8}\left(4\log(2) - \log^2(2) + \frac{5}{2}\zeta(2) - 2\pi + \pi\log(2) - 8G\right)$$

and

$$J = \int_0^1 \arctan(x) \log(1+x) \mathrm{d}x$$

$$= \frac{1}{8}\left(3\pi\log(2) + 4\log(2) - \frac{1}{2}\zeta(2) - \log^2(2) - 2\pi\right),$$

and the first solution is finalized.

For a second solution to the integral I (which is going to be pretty fast since we also use results previously obtained), we apply the integration by parts, and then we write

$$I = \int_0^1 \arctan(x)\log(1-x)\mathrm{d}x = \int_0^1 (-(1-x)\log(1-x)-x)' \arctan(x)\mathrm{d}x$$

$$= \underbrace{(-(1-x)\log(1-x)-x)\arctan(x)\Big|_{x=0}^{x=1}}_{-\pi/4} + \int_0^1 \frac{x}{1+x^2}\mathrm{d}x + \int_0^1 \frac{\log(1-x)}{1+x^2}\mathrm{d}x$$

$$- \int_0^1 \frac{x\log(1-x)}{1+x^2}\mathrm{d}x$$

{the first integral is straightforward, for the middle integral make the change}

$$\left\{\text{of variable } x = \frac{1-y}{1+y}, \text{ and the last integral is already calculated in (1.24)}\right\}$$

$$= \frac{1}{8}\left(4\log(2) - \log^2(2) + \frac{5}{2}\zeta(2) - 2\pi\right) + \int_0^1 \frac{\log\left(\dfrac{2x}{1+x}\right)}{1+x^2}\mathrm{d}x$$

$$= \frac{1}{8}\left(4\log(2) - \log^2(2) + \frac{5}{2}\zeta(2) - 2\pi\right) + \int_0^1 \frac{\log(2)}{1+x^2}\mathrm{d}x + \int_0^1 \frac{\log(x)}{1+x^2}\mathrm{d}x$$

$$- \int_0^1 \frac{\log(1+x)}{1+x^2}\mathrm{d}x$$

{the last two integrals are V_3, V_2 given in (3.93), (3.92) from the previous solution}

$$= \frac{1}{8}\left(4\log(2) - \log^2(2) + \frac{5}{2}\zeta(2) - 2\pi + \pi\log(2) - 8G\right),$$

and the second solution to the integral I is finalized.

For a second solution to the integral J, we apply the integration by parts (as we previously did for the integral I), and then we have

$$J = \int_0^1 \arctan(x)\log(1+x)\mathrm{d}x = \int_0^1 x' \arctan(x)\log(1+x)\mathrm{d}x$$

$$= x\arctan(x)\log(1+x)\Big|_{x=0}^{x=1} - \int_0^1 \left(\frac{x\arctan(x)}{1+x} - \frac{x\log(1+x)}{1+x^2}\right)\mathrm{d}x$$

$$= \frac{\pi}{4}\log(2) - \int_0^1 \arctan(x)\mathrm{d}x + \int_0^1 \frac{\arctan(x)}{1+x}\mathrm{d}x - \int_0^1 \frac{x\log(1+x)}{1+x^2}\mathrm{d}x$$

$$\{\text{make use of the result in (1.25)}\}$$

$$= \frac{\pi}{4}\log(2) - \frac{1}{16}\zeta(2) + \frac{\log(2)}{2} - \frac{\log^2(2)}{8} - \frac{\pi}{4} + \int_0^1 \frac{\arctan(x)}{1+x}\mathrm{d}x. \tag{3.97}$$

For the last integral, we apply the integration by parts, and then we get

$$\int_0^1 \frac{\arctan(x)}{1+x}\mathrm{d}x = \log(1+x)\arctan(x)\Big|_{x=0}^{x=1} - \int_0^1 \frac{\log(1+x)}{1+x^2}\mathrm{d}x$$

$$= \frac{\pi}{4}\log(2) - \int_0^1 \frac{\log(1+x)}{1+x^2}\mathrm{d}x = \frac{\pi}{8}\log(2), \tag{3.98}$$

where the last integral is the integral V_2 given in (3.92).

Thus, if we plug the result from (3.98) in (3.97), we conclude that

$$J = \int_0^1 \arctan(x)\log(1+x)\mathrm{d}x = \frac{3}{8}\pi\log(2) + \frac{\log(2)}{2} - \frac{1}{16}\zeta(2) - \frac{\log^2(2)}{8} - \frac{\pi}{4}$$

$$= \frac{1}{8}\left(3\pi\log(2) + 4\log(2) - \frac{1}{2}\zeta(2) - \log^2(2) - 2\pi\right),$$

and the second solution to the integral J is finalized.

Next, for a third solution to the integral I, let's try an approach based mainly on the series manipulations, where we'll also need two alternating series, $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{H_{2k}}{2k} = \frac{1}{8}\left(\frac{5}{2}\zeta(2) - \log^2(2)\right)$ and $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{H_{2k-1}}{2k-1} = G - \frac{\pi}{8}\log(2)$, which are easy to get[5] if we use the generating function in (4.5), and then we write

[5]Based on (4.5), we obtain by integration that $\sum_{n=1}^{\infty}x^n\frac{H_n}{n} = \frac{1}{2}\log^2(1-x)+\operatorname{Li}_2(x)$, where if we set $x=i$, we are led to $\sum_{n=1}^{\infty}i^n\frac{H_n}{n} = \sum_{n=1}^{\infty}(-1)^n\frac{H_{2n}}{2n} + i\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_{2n-1}}{2n-1} = \frac{1}{2}\log^2(1-i)+\operatorname{Li}_2(i)$. Since $\log^2(1-i) = \frac{1}{4}\log^2(2) - \frac{\pi^2}{16} - i\frac{\pi}{4}\log(2)$, using the fact that $\log(x+iy) = \log(\sqrt{x^2+y^2}) + i\arctan\left(\frac{y}{x}\right)$, $x > 0$, and $\operatorname{Li}_2(i) = \sum_{n=1}^{\infty}\frac{i^n}{n^2} = -\frac{1}{4}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2} + i\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(2n-1)^2} = -\frac{\pi^2}{48} + iG$, we obtain that $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{H_{2k}}{2k} = \frac{5}{16}\zeta(2) - \frac{1}{8}\log^2(2)$ and $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{H_{2k-1}}{2k-1} = G - \frac{\pi}{8}\log(2)$.

$$I=\int_0^1 \arctan(x)\log(1-x)\mathrm{d}x=-\int_0^1 \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}(-1)^{k-1}\frac{x^{2k+n-1}}{(2k-1)n}\right)\mathrm{d}x$$

{reverse the order of summation and integration}

$$=-\sum_{k=1}^{\infty}(-1)^{k-1}\left(\sum_{n=1}^{\infty}\int_0^1 \frac{x^{2k+n-1}}{(2k-1)n}\mathrm{d}x\right)=-\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k-1}\left(\sum_{n=1}^{\infty}\frac{1}{(2k+n)n}\right)$$

$$=-\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k(2k-1)}\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+2k}\right)\right)$$

$$\left\{\text{use that } \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+k}\right)=H_k\right\}$$

$$=-\sum_{k=1}^{\infty}(-1)^{k-1}\frac{H_{2k}}{2k(2k-1)}=\sum_{k=1}^{\infty}\left((-1)^{k-1}\frac{H_{2k}}{2k}-(-1)^{k-1}\frac{H_{2k}}{2k-1}\right)$$

$$=\sum_{k=1}^{\infty}\left((-1)^{k-1}\frac{H_{2k}}{2k}-(-1)^{k-1}\frac{H_{2k-1}+1/(2k)}{2k-1}\right)$$

$$=\sum_{k=1}^{\infty}(-1)^{k-1}\frac{H_{2k}}{2k}-\sum_{k=1}^{\infty}(-1)^{k-1}\frac{H_{2k-1}}{2k-1}-\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k(2k-1)}$$

$$=\frac{5}{16}\zeta(2)+\frac{\pi}{8}\log(2)-G-\frac{1}{8}\log^2(2)+\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}-\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k-1}$$

$$=\frac{1}{8}\left(4\log(2)-\log^2(2)+\frac{5}{2}\zeta(2)-2\pi+\pi\log(2)-8G\right),$$

and the third solution to the integral I is finalized.

Then, to get a third solution to the integral J using the strategy in the previous solution, we write

$$J=\int_0^1 \arctan(x)\log(1+x)\mathrm{d}x=\int_0^1 \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}(-1)^{k+n}\frac{x^{2k+n-1}}{(2k-1)n}\right)\mathrm{d}x$$

{reverse the order of summation and integration}

$$= \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{(-1)^{k+n}}{(2k-1)n}\int_0^1 x^{2k+n-1}\mathrm{d}x\right) = \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{(-1)^{k+n}}{(2k-1)(2k+n)n}\right)$$

{split the double series according to n even and odd}

$$= \underbrace{\frac{1}{4}\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{(-1)^{k}}{(2k-1)(k+n)n}\right)}_{S_1} - \underbrace{\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{(-1)^{k}}{(2k-1)(2k+2n-1)(2n-1)}\right)}_{S_2}. \tag{3.99}$$

Now, for the double series S_1 in (3.99), we have

$$S_1 = \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{(-1)^{k}}{(2k-1)(k+n)n}\right) = \sum_{k=1}^{\infty}\frac{(-1)^{k}}{2k-1}\left(\sum_{n=1}^{\infty}\frac{1}{(k+n)n}\right)$$

$$\left\{\text{use that } \sum_{n=1}^{\infty}\frac{1}{n(n+k)} = \frac{H_k}{k}\right\}$$

$$= \sum_{k=1}^{\infty}(-1)^k\frac{H_k}{(2k-1)k} = 2\sum_{k=1}^{\infty}(-1)^k\frac{H_k}{(2k-1)2k}$$

$$= 2\sum_{k=1}^{\infty}(-1)^k H_k\left(\frac{1}{2k-1}-\frac{1}{2k}\right)$$

$$= 2\sum_{k=1}^{\infty}(-1)^k\frac{H_k}{2k-1} - \sum_{k=1}^{\infty}(-1)^k\frac{H_k}{k}$$

$$= 2\left(\underbrace{\frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{H_k}{k} - \sum_{k=1}^{\infty}(-1)^{k-1}\frac{H_k}{2k-1}}_{S_3}\right)$$

{S_3 appears in the first solution, while getting (3.18), and its value is given in (3.88)}

$$= \frac{1}{2}\zeta(2) + \log(2)\pi - \pi + 2\log(2) - \frac{1}{2}\log^2(2) - 2G. \tag{3.100}$$

Further, for the double series S_2 in (3.99), we get

$$S_2=\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{(-1)^k}{(2k-1)(2k+2n-1)(2n-1)}\right)$$

$$=\sum_{k=1}^{\infty}\frac{(-1)^k}{2k-1}\left(\sum_{n=1}^{\infty}\frac{1}{(2k+2n-1)(2n-1)}\right)$$

$$=\sum_{k=1}^{\infty}\frac{(-1)^k}{2k-1}\left(\sum_{n=1}^{\infty}\frac{1}{n(n+2k)}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n(n+k)}\right)$$

$$\left\{\text{use that }\sum_{n=1}^{\infty}\frac{1}{n(n+k)}=\frac{H_k}{k}\right\}$$

$$=\sum_{k=1}^{\infty}\frac{(-1)^k}{2k-1}\left(\frac{H_{2k}}{2k}-\frac{1}{4}\frac{H_k}{k}\right)=\sum_{k=1}^{\infty}(-1)^k\frac{H_{2k}}{2k(2k-1)}-\frac{1}{2}\sum_{k=1}^{\infty}(-1)^k\frac{H_k}{2k(2k-1)}$$

$$=\frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{H_k}{2k-1}-\frac{1}{4}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{H_k}{k}-\sum_{k=1}^{\infty}(-1)^{k-1}\frac{H_{2k}}{2k-1}$$
$$+\frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{H_{2k}}{k}$$

$$=-\frac{1}{2}\left(\underbrace{\frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{H_k}{k}-\sum_{k=1}^{\infty}(-1)^{k-1}\frac{H_k}{2k-1}}_{S_3}\right)+\sum_{k=1}^{\infty}(-1)^{k-1}\frac{H_{2k}}{2k}$$

$$-\sum_{k=1}^{\infty}(-1)^{k-1}\frac{H_{2k-1}}{2k-1}+\frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k}-\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{2k-1}$$

$$=\frac{3}{16}\zeta(2)-\frac{\pi}{8}\log(2)-\frac{1}{2}G. \tag{3.101}$$

Collecting the values of the series S_1 and S_2 from (3.100) and (3.101) in (3.99), we conclude that

$$J=\int_0^1 \arctan(x)\log(1+x)\mathrm{d}x=\frac{3}{8}\pi\log(2)+\frac{\log(2)}{2}-\frac{1}{16}\zeta(2)-\frac{\log^2(2)}{8}-\frac{\pi}{4}$$

$$=\frac{1}{8}(3\pi\log(2)+4\log(2)-\frac{1}{2}\zeta(2)-\log^2(2)-2\pi),$$

and the calculations are finalized.

All in all, approaching the problems in different ways, establishing connections with various results (sometimes pretty subtle), I think is one of the important steps to gain insight in the art of solving mathematical problems. I would highly recommend not to get limited to the solutions provided in this book, and if possible to try to find new ones.

Also, from the second solution one may clearly observe the connections with the integrals in Sect. 1.13 and the Putnam integral in [73].

3.19 Interesting Integrals Involving the Inverse Tangent Function and Dilogarithm Function

Solution From the very beginning, we note the similarity with the previous pair of integrals, where now we have Dilogarithms (see [100], [9, pp. 102–107], [44, pp. 1–37]) in the place of logarithms, and at the same time we might expect an increased difficulty level in comparison with the integrals from the previous section. We'll also find these integrals particularly useful in other sections where we'll try to calculate more advanced integrals.

The proposed *challenging question* is pretty enjoyable, and the present solution will try to answer it.

Let's start with the integration by parts for the integral I, and then we write

$$I=\int_0^1 \arctan(x)\operatorname{Li}_2(x)\mathrm{d}x=\underbrace{x\arctan(x)\operatorname{Li}_2(x)\mathrm{d}x\Big|_{x=0}^{x=1}}_{\pi^3/24}$$

$$+\int_0^1 \arctan(x)\log(1-x)\mathrm{d}x-\int_0^1 \frac{x\operatorname{Li}_2(x)}{1+x^2}\mathrm{d}x$$

$$=\frac{\pi^3}{24}+\int_0^1 \arctan(x)\log(1-x)\mathrm{d}x-\int_0^1 \frac{x\operatorname{Li}_2(x)}{1+x^2}\mathrm{d}x$$

{the value of the first integral is given in (1.33)}

$$=\frac{1}{2}\log(2)-\frac{1}{8}\log^2(2)+\frac{\pi^3}{24}+\frac{5}{96}\pi^2-\frac{\pi}{4}+\frac{\pi}{8}\log(2)-G-\int_0^1 \frac{x\operatorname{Li}_2(x)}{1+x^2}\mathrm{d}x. \tag{3.102}$$

For the remaining integral in (3.102), we write it as a useful double integral, and then we have

$$\int_0^1 \frac{x\,\mathrm{Li}_2(x)}{1+x^2}\mathrm{d}x = -\int_0^1 \left(\int_0^1 \frac{x^2\log(y)}{(1+x^2)(1-xy)}\mathrm{d}y\right)\mathrm{d}x$$

$$\{\text{reverse the integration order}\}$$

$$= -\int_0^1 \left(\int_0^1 \frac{x^2\log(y)}{(1+x^2)(1-xy)}\mathrm{d}x\right)\mathrm{d}y = \int_0^1 \left(\int_0^1 \frac{x^2(\log(x)-\log(xy))}{(1+x^2)(1-xy)}\mathrm{d}x\right)\mathrm{d}y$$

$$= \int_0^1 \left(\int_0^1 \frac{x^2\log(x)}{(1+x^2)(1-xy)}\mathrm{d}x\right)\mathrm{d}y - \underbrace{\int_0^1 \left(\int_0^1 \frac{x^2\log(xy)}{(1+x^2)(1-xy)}\mathrm{d}x\right)\mathrm{d}y}_{U}$$

$$\{\text{reverse the integration order in the first double integral}\}$$

$$= \int_0^1 \left(\int_0^1 \frac{x^2\log(x)}{(1+x^2)(1-xy)}\mathrm{d}y\right)\mathrm{d}x - U = -\int_0^1 \frac{x\log(x)\log(1-x)}{1+x^2}\mathrm{d}x - U$$

$$\{\text{the value of the first integral is calculated in (1.26)}\}$$

$$= \frac{3}{32}\log(2)\pi^2 - \frac{41}{64}\zeta(3) - U. \tag{3.103}$$

To calculate the remaining U integral in (3.103), we use the symmetry of the integrand that gives

$$2U = \int_0^1 \left(\int_0^1 \frac{x^2\log(xy)}{(1+x^2)(1-xy)}\mathrm{d}x\right)\mathrm{d}y + \int_0^1 \left(\int_0^1 \frac{y^2\log(xy)}{(1+y^2)(1-xy)}\mathrm{d}y\right)\mathrm{d}x$$

$$\{\text{reverse the integration order in the second double integral}\}$$

$$= \int_0^1 \left(\int_0^1 \frac{x^2\log(xy)}{(1+x^2)(1-xy)}\mathrm{d}x\right)\mathrm{d}y + \int_0^1 \left(\int_0^1 \frac{y^2\log(xy)}{(1+y^2)(1-xy)}\mathrm{d}x\right)\mathrm{d}y$$

$$= \int_0^1 \left(\int_0^1 \frac{(x^2+y^2+2x^2y^2)\log(xy)}{(1+x^2)(1+y^2)(1-xy)}\mathrm{d}x\right)\mathrm{d}y$$

$$= \int_0^1 \left(\int_0^1 \frac{((1+x^2)(1+y^2)-(1-(xy)^2))\log(xy)}{(1+x^2)(1+y^2)(1-xy)}\mathrm{d}x\right)\mathrm{d}y$$

$$= \int_0^1 \left(\int_0^1 \frac{\log(xy)}{1-xy}\mathrm{d}x\right)\mathrm{d}y - \int_0^1 \left(\int_0^1 \frac{(1+xy)\log(xy)}{(1+x^2)(1+y^2)}\mathrm{d}x\right)\mathrm{d}y$$

$$\{\text{exploit the symmetry of the integrands}\}$$

$$= 2\int_0^1 \left(\int_0^1 \frac{\log(x)}{1-xy}\mathrm{d}x\right)\mathrm{d}y - 2\int_0^1 \frac{1}{1+y^2}\left(\int_0^1 \frac{\log(x)}{1+x^2}\mathrm{d}x\right)\mathrm{d}y$$

$$-2\int_0^1 \frac{y}{1+y^2}\left(\int_0^1 \frac{x\log(x)}{1+x^2}\mathrm{d}x\right)\mathrm{d}y = \frac{\pi}{2}G + \frac{\pi^2}{48}\log(2) - 2\zeta(3),$$

whence we get that

$$U = \int_0^1 \left(\int_0^1 \frac{x^2\log(xy)}{(1+x^2)(1-xy)}\mathrm{d}x\right)\mathrm{d}y = \frac{\pi}{4}G + \frac{\pi^2}{96}\log(2) - \zeta(3), \tag{3.104}$$

where I used simple results with integrals.[6]

Now, if we plug the result from (3.104) in (3.107), we obtain

$$\int_0^1 \frac{x\,\mathrm{Li}_2(x)}{1+x^2}\mathrm{d}x = \frac{23}{64}\zeta(3) + \frac{1}{2}\log(2)\zeta(2) - \frac{\pi}{4}G. \tag{3.105}$$

Lastly, by plugging the result from (3.105) in (3.103), we conclude that

$$I = \int_0^1 \arctan(x)\,\mathrm{Li}_2(x)\mathrm{d}x$$

$$= \frac{\pi}{4}G - G - \frac{\pi}{4} + \frac{5}{96}\pi^2 + \frac{\pi^3}{24} + \frac{\pi}{8}\log(2) + \frac{1}{2}\log(2) - \frac{1}{8}\log^2(2)$$

$$-\frac{\pi^2}{12}\log(2) - \frac{23}{64}\zeta(3),$$

and the calculation to the integral I is finalized.

Now, for the integral J, we proceed as in the case of the integral I. Starting with the integration by parts, we get

[6]Note that $\displaystyle\int_0^1\left(\int_0^1 \frac{\log(x)}{1-xy}\mathrm{d}x\right)\mathrm{d}y = \int_0^1\left(\int_0^1 \sum_{n=1}^{\infty}(xy)^{n-1}\log(x)\mathrm{d}x\right)\mathrm{d}y = \sum_{n=1}^{\infty}\int_0^1\left(\int_0^1 (xy)^{n-1}\log(x)\mathrm{d}x\right)\mathrm{d}y = -\sum_{n=1}^{\infty}\frac{1}{n^3} = -\zeta(3)$. Then, $\displaystyle\int_0^1 \frac{\log(x)}{1+x^2}\mathrm{d}x = \int_0^1 \sum_{n=1}^{\infty}(-1)^{n-1}x^{2n-2}\log(x)\mathrm{d}x = \sum_{n=1}^{\infty}(-1)^{n-1}\int_0^1 x^{2n-2}\log(x)\mathrm{d}x = \sum_{n=1}^{\infty}(-1)^n\frac{1}{(2n-1)^2} = -G$. Finally, we have that $\displaystyle\int_0^1 \frac{x\log(x)}{1+x^2}\mathrm{d}x = \int_0^1 \sum_{n=1}^{\infty}(-1)^{n-1}x^{2n-1}\log(x)\,\mathrm{d}x = \sum_{n=1}^{\infty}(-1)^{n-1}\int_0^1 x^{2n-1}\log(x)\mathrm{d}x = \frac{1}{4}\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2} = -\frac{\pi^2}{48}$.

$$J = \int_0^1 \arctan(x)\,\mathrm{Li}_2(-x)\mathrm{d}x = \underbrace{x\arctan(x)\,\mathrm{Li}_2(-x)\Big|_{x=0}^{x=1}}_{-\pi^3/48}$$

$$+\int_0^1 \arctan(x)\log(1+x)\mathrm{d}x - \int_0^1 \frac{x\,\mathrm{Li}_2(-x)}{1+x^2}\mathrm{d}x$$

$$= -\frac{\pi^3}{48} + \int_0^1 \arctan(x)\log(1+x)\mathrm{d}x - \int_0^1 \frac{x\,\mathrm{Li}_2(-x)}{1+x^2}\mathrm{d}x$$

{the value of the first integral is given in (1.34)}

$$= \frac{1}{2}\log(2) - \frac{1}{8}\log^2(2) + \frac{3}{8}\log(2)\pi - \frac{\pi}{4} - \frac{\pi^2}{96} - \frac{\pi^3}{48} - \int_0^1 \frac{x\,\mathrm{Li}_2(-x)}{1+x^2}\mathrm{d}x. \tag{3.106}$$

Rewriting the remaining integral in (3.106) as a double integral, we have

$$\int_0^1 \frac{x\,\mathrm{Li}_2(-x)}{1+x^2}\mathrm{d}x = \int_0^1 \left(\int_0^1 \frac{x^2\log(y)}{(1+x^2)(1+xy)}\mathrm{d}y\right)\mathrm{d}x$$

{reverse the integration order}

$$= \int_0^1 \left(\int_0^1 \frac{x^2\log(y)}{(1+x^2)(1+xy)}\mathrm{d}x\right)\mathrm{d}y = \int_0^1 \left(\int_0^1 \frac{x^2(\log(xy)-\log(x))}{(1+x^2)(1+xy)}\mathrm{d}x\right)\mathrm{d}y$$

$$= \underbrace{\int_0^1 \left(\int_0^1 \frac{x^2\log(xy)}{(1+x^2)(1+xy)}\mathrm{d}x\right)\mathrm{d}y}_{V} - \int_0^1 \left(\int_0^1 \frac{x^2\log(x)}{(1+x^2)(1+xy)}\mathrm{d}x\right)\mathrm{d}y$$

{reverse the integration order in the second double integral}

$$= V - \int_0^1 \left(\int_0^1 \frac{x^2\log(x)}{(1+x^2)(1+xy)}\mathrm{d}y\right)\mathrm{d}x = V - \int_0^1 \frac{x\log(x)\log(1+x)}{1+x^2}\mathrm{d}x$$

{the value of the second integral is given in (1.27)}

$$= V + \frac{15}{64}\zeta(3) - \frac{\pi^2}{32}\log(2). \tag{3.107}$$

Further, for the integral V in (3.107), we exploit the symmetry of the integrand, and we get

$$2V = \int_0^1 \left(\int_0^1 \frac{x^2\log(xy)}{(1+x^2)(1+xy)}\mathrm{d}x\right)\mathrm{d}y + \int_0^1 \left(\int_0^1 \frac{y^2\log(xy)}{(1+y^2)(1+xy)}\mathrm{d}y\right)\mathrm{d}x$$

$$\{\text{reverse the integration order in the second double integral}\}$$

$$= \int_0^1 \left(\int_0^1 \frac{x^2 \log(xy)}{(1+x^2)(1+xy)} \mathrm{d}x \right) \mathrm{d}y + \int_0^1 \left(\int_0^1 \frac{y^2 \log(xy)}{(1+y^2)(1+xy)} \mathrm{d}x \right) \mathrm{d}y$$

$$= \int_0^1 \left(\int_0^1 \frac{(x^2+y^2+2x^2y^2) \log(xy)}{(1+x^2)(1+y^2)(1+xy)} \mathrm{d}x \right) \mathrm{d}y$$

$$= \int_0^1 \left(\int_0^1 \frac{((1+x^2)(1+y^2)-(1-(xy)^2)) \log(xy)}{(1+x^2)(1+y^2)(1+xy)} \mathrm{d}x \right) \mathrm{d}y$$

$$= \int_0^1 \left(\int_0^1 \frac{\log(xy)}{1+xy} \mathrm{d}x \right) \mathrm{d}y - \int_0^1 \left(\int_0^1 \frac{(1-xy) \log(xy)}{(1+x^2)(1+y^2)} \mathrm{d}x \right) \mathrm{d}y$$

$$= \int_0^1 \left(\int_0^1 \frac{\log(xy)}{1+xy} \mathrm{d}x \right) \mathrm{d}y - \int_0^1 \left(\int_0^1 \frac{\log(xy)}{(1+x^2)(1+y^2)} \mathrm{d}x \right) \mathrm{d}y$$

$$+ \int_0^1 \left(\int_0^1 \frac{xy \log(xy)}{(1+x^2)(1+y^2)} \mathrm{d}x \right) \mathrm{d}y = 2 \int_0^1 \left(\int_0^1 \frac{\log(x)}{1+xy} \mathrm{d}x \right) \mathrm{d}y$$

$$-2 \int_0^1 \frac{\log(x)}{1+x^2} \left(\int_0^1 \frac{1}{1+y^2} \mathrm{d}x \right) \mathrm{d}y + 2 \int_0^1 \frac{x \log(x)}{1+x^2} \left(\int_0^1 \frac{y}{1+y^2} \mathrm{d}x \right) \mathrm{d}y$$

$$= \frac{\pi}{2} G - \frac{\pi^2}{48} \log(2) - \frac{3}{2} \zeta(3),$$

whence we obtain

$$V = \int_0^1 \left(\int_0^1 \frac{x^2 \log(xy)}{(1+x^2)(1+xy)} \mathrm{d}x \right) \mathrm{d}y = \frac{\pi}{4} G - \frac{\pi^2}{96} \log(2) - \frac{3}{4} \zeta(3), \tag{3.108}$$

where I used the values of a few easy integrals.[7]

[7]Note that we already met the integrals $\displaystyle\int_0^1 \frac{\log(x)}{1+x^2} \mathrm{d}x$ and $\displaystyle\int_0^1 \frac{x \log(x)}{1+x^2} \mathrm{d}x$ during the calculations of the integral I which are straightforward if we use the geometric series. Next, it's easy to see that $\displaystyle\int_0^1 \left(\int_0^1 \frac{\log(x)}{1+xy} \mathrm{d}x \right) \mathrm{d}y = \int_0^1 \left(\int_0^1 \sum_{n=1}^{\infty} (-1)^{n-1} (xy)^{n-1} \log(x) \mathrm{d}x \right) \mathrm{d}y = \sum_{n=1}^{\infty} (-1)^{n-1} \int_0^1 \left(\int_0^1 (xy)^{n-1} \log(x) \mathrm{d}x \right) \mathrm{d}y = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^3} = -\frac{3}{4} \zeta(3).$

Now, if we plug the result from (3.108) in (3.107), we get

$$\int_0^1 \frac{x\,\mathrm{Li}_2(-x)}{1+x^2}\mathrm{d}x = \frac{\pi}{4}G - \frac{\pi^2}{24}\log(2) - \frac{33}{64}\zeta(3). \tag{3.109}$$

Finally, if we plug the result from (3.109) in (3.106), we conclude that

$$J = \int_0^1 \arctan(x)\,\mathrm{Li}_2(-x)\mathrm{d}x$$

$$= \frac{33}{64}\zeta(3) + \frac{\pi^2}{24}\log(2) - \frac{\pi^3}{48} - \frac{\pi^2}{96} - \frac{\pi}{4} - \frac{\pi}{4}G + \frac{3}{8}\log(2)\pi$$

$$+\frac{1}{2}\log(2) - \frac{1}{8}\log^2(2),$$

and the calculation to the integral J is finalized.

Often, we need to combine more integration techniques, all depending on the form we reduce the main integral to. In this case, I made use of the symmetry in double integrals, but at the same time I had to play with the integrals from the previous section and the ones from (1.26) and (1.27) which in Sect. 1.14 are approached by creating a system of relations with both of them. As seen, the present approach also answers the proposed *challenging question*.

3.20 More Interesting Integrals Involving the Inverse Tangent Function and the Logarithmic Function: The First Part

Solution In the second section back I calculated the integrals, $\displaystyle\int_0^1 \arctan(x)\log(1-x)\mathrm{d}x$ and $\displaystyle\int_0^1 \arctan(x)\log(1+x)\mathrm{d}x$, and now I want to consider the versions with the integrands divided by $1+x^2$. If you already went through the previous sections, it's not hard to guess what we could do here, what (possible) strategy to adopt.

Let's denote $I = \displaystyle\int_0^1 \frac{\arctan(x)\log(1-x)}{1+x^2}\mathrm{d}x$ and $J = \displaystyle\int_0^1 \frac{\arctan(x)\log(1+x)}{1+x^2}\mathrm{d}x$, and then we make up the system of relations with the integrals I and J,

$$S: \begin{cases} I+J = \underbrace{\int_0^1 \frac{\arctan(x)\log(1-x^2)}{1+x^2}\mathrm{d}x}_{U}; \\ I-J = \underbrace{\int_0^1 \frac{\arctan(x)}{1+x^2}\log\left(\frac{1-x}{1+x}\right)\mathrm{d}x}_{V}. \end{cases} \tag{3.110}$$

For the integral U in (3.110), we make the change of variable $x = \tan(y)$, and then we have

$$U = \int_0^1 \frac{\arctan(x)\log(1-x^2)}{1+x^2}\mathrm{d}x = \int_0^{\pi/4} y\log\left(\frac{\cos(2y)}{\cos^2(y)}\right)\mathrm{d}y$$

$$= \underbrace{\int_0^{\pi/4} x\log(\cos(2x))\mathrm{d}x}_{U_1} - 2\underbrace{\int_0^{\pi/4} x\log(\cos(x))\mathrm{d}x}_{U_2}. \tag{3.111}$$

Now, for both integrals in (3.111) we make use of the Fourier series in (3.86) (see **1.441.4** from [30]). Then, for the first integral in (3.111), we have

$$U_1 = \int_0^{\pi/4} x\log(\cos(2x))\mathrm{d}x = \int_0^{\pi/4} x\left(-\log(2) + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos(4nx)}{n}\right)\mathrm{d}x$$

$$= -\frac{3}{16}\log(2)\zeta(2) + \int_0^{\pi/4} x\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos(4nx)}{n}\mathrm{d}x$$

{reverse the order of summation and integration}

$$= -\frac{3}{16}\log(2)\zeta(2) + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^{\pi/4} x\cos(4nx)\mathrm{d}x$$

$$= -\frac{3}{16}\log(2)\zeta(2) + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{\cos(\pi n)}{16n^2} + \frac{\pi\sin(\pi n)}{16n} - \frac{1}{16n^2}\right)$$

$$= -\frac{3}{16}\log(2)\zeta(2) - \frac{1}{16}\sum_{n=1}^{\infty}\frac{1}{n^3} - \frac{1}{16}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}$$

$$= -\frac{3}{16}\log(2)\zeta(2) - \frac{7}{64}\zeta(3). \tag{3.112}$$

Further, for the integral U_2 in (3.111), we get

$$U_2 = \int_0^{\pi/4} x \log(\cos(x)) \mathrm{d}x = \int_0^{\pi/4} x \left(-\log(2) + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\cos(2nx)}{n} \right) \mathrm{d}x$$

$$= -\frac{3}{16} \log(2)\zeta(2) + \int_0^{\pi/4} x \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\cos(2nx)}{n} \mathrm{d}x$$

{reverse the order of summation and integration}

$$= -\frac{3}{16} \log(2)\zeta(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \int_0^{\pi/4} x \cos(2nx) \mathrm{d}x$$

$$= -\frac{3}{16} \log(2)\zeta(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{\pi \sin\left(\frac{\pi n}{2}\right)}{8n} + \frac{\cos\left(\frac{\pi n}{2}\right)}{4n^2} - \frac{1}{4n^2} \right)$$

$$= -\frac{3}{16} \log(2)\zeta(2) + \frac{\pi}{8} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2} - \frac{7}{32} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$$

$$= \frac{\pi}{8} G - \frac{3}{16} \log(2)\zeta(2) - \frac{21}{128} \zeta(3). \tag{3.113}$$

If we plug the values of the integrals U_1 and U_2 from (3.112) and (3.113) in (3.111), we get

$$U = \int_0^1 \frac{\arctan(x) \log(1-x^2)}{1+x^2} \mathrm{d}x = \frac{7}{32}\zeta(3) + \frac{3}{16} \log(2)\zeta(2) - \frac{\pi}{4} G. \tag{3.114}$$

Lastly, to calculate the integral V, we make the change of variable $\frac{1-x}{1+x} = y$ that gives

$$V = \int_0^1 \frac{\arctan(x) \log\left(\frac{1-x}{1+x}\right)}{1+x^2} \mathrm{d}x = \int_0^1 \frac{(\pi/4 - \arctan(y)) \log(y)}{1+y^2} \mathrm{d}y$$

$$= \frac{\pi}{4} \int_0^1 \frac{\log(y)}{1+y^2} \mathrm{d}y - \int_0^1 \frac{\arctan(y) \log(y)}{1+y^2} \mathrm{d}y$$

{make the change of variable $y = \tan(x)$ in the second integral}

$$= \frac{\pi}{4}\int_0^1 \sum_{n=1}^{\infty}(-1)^{n-1}x^{2n-2}\log(x)\mathrm{d}x - \int_0^{\pi/4} x\log(\tan(x))\mathrm{d}x$$

{for the second integral use the Fourier series in (3.74) (see **1.442.2** from [30])}

$$= \frac{\pi}{4}\sum_{n=1}^{\infty}(-1)^{n-1}\int_0^1 x^{2n-2}\log(x)\mathrm{d}x + 2\int_0^{\pi/4} x\sum_{n=1}^{\infty}\frac{\cos(2(2n-1)x)}{2n-1}\mathrm{d}x$$

$$= -\frac{\pi}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2} + 2\sum_{n=1}^{\infty}\frac{1}{2n-1}\int_0^{\pi/4} x\cos(2(2n-1)x)\mathrm{d}x$$

$$= -\frac{\pi}{4}G + 2\sum_{n=1}^{\infty}\frac{1}{2n-1}\left(-\frac{1}{4(2n-1)^2} - \frac{\pi}{8}\frac{\cos(\pi n)}{2n-1} + \frac{\sin(\pi n)}{4(2n-1)^2}\right)$$

$$= -\frac{\pi}{4}G - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3} + \frac{\pi}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{\sin(n\pi)}{(2n-1)^3} = -\frac{7}{16}\zeta(3). \tag{3.115}$$

Collecting the values of the integrals U and V from (3.114) and (3.115) in (3.110), we have

$$S: \begin{cases} I + J = \dfrac{7}{32}\zeta(3) + \dfrac{3}{16}\log(2)\zeta(2) - \dfrac{\pi}{4}G; \\ I - J = -\dfrac{7}{16}\zeta(3), \end{cases}$$

whence we obtain that

$$I = \int_0^1 \frac{\arctan(x)\log(1-x)}{1+x^2}\mathrm{d}x = \frac{1}{8}\left(\frac{3}{4}\log(2)\zeta(2) - \frac{7}{8}\zeta(3) - \pi G\right)$$

and

$$J = \int_0^1 \frac{\arctan(x)\log(1+x)}{1+x^2}\mathrm{d}x = \frac{1}{8}\left(\frac{21}{8}\zeta(3) + \frac{3}{4}\log(2)\zeta(2) - \pi G\right),$$

and the solution is finalized.

Again, we took benefit of the powerful approach involving the use of a system of relations with integrals. Undoubtedly, we have the advantage of having both integrals in the same section, which could easily make us think of such an approach.

For an alternative solution, make the change of variable $x = \tan(y)$ in both integrals, and then exploit that $1 - \tan(x) = \sqrt{2}\sec(x)\sin\left(\frac{\pi}{4} - x\right)$, $1 + \tan(x) = \sqrt{2}\sec(x)\sin\left(x + \frac{\pi}{4}\right)$ and the Fourier series of $\log(\sin(x))$, $\log(\cos(x))$.

3.21 More Interesting Integrals Involving the Inverse Tangent Function and the Logarithmic Function: The Second Part

Solution We continue now with other similar versions of the integrals from the previous section, and this time we consider the squared inverse tangent instead of a simple inverse tangent. Surely, we're tempted to proceed as in the previous section, but I think we might like to make some little adjustments to such a strategy (and we could start with making a nice symmetry-related observation on the integral J).

Let's start with the integral J we integrate by parts, and then we have

$$J = \int_0^1 \frac{\arctan^2(x)\log(1+x)}{1+x^2}\,\mathrm{d}x = \underbrace{\frac{1}{3}\arctan^3(x)\log(1+x)\Big|_{x=0}^{x=1}}_{\log(2)\pi^3/192} - \frac{1}{3}\int_0^1 \frac{\arctan^3(x)}{1+x}\,\mathrm{d}x$$

$$= \log(2)\frac{\pi^3}{192} - \frac{1}{3}\underbrace{\int_0^1 \frac{\arctan^3(x)}{1+x}\,\mathrm{d}x}_{J_1}. \tag{3.116}$$

For the integral J_1 in (3.116), make the change of variable $x = \dfrac{1-y}{1+y}$ that beautifully gives

$$J_1 = \int_0^1 \frac{\arctan^3(x)}{1+x}\,\mathrm{d}x = \int_0^1 \frac{(\pi/4 - \arctan(y))^3}{1+y}\,\mathrm{d}y$$

$$= \frac{\pi^3}{64}\int_0^1 \frac{1}{1+x}\,\mathrm{d}x - \frac{3}{16}\pi^2\int_0^1 \frac{\arctan(x)}{1+x}\,\mathrm{d}x + \frac{3}{4}\pi\int_0^1 \frac{\arctan^2(x)}{1+x}\,\mathrm{d}x - \underbrace{\int_0^1 \frac{\arctan^3(x)}{1+x}\,\mathrm{d}x}_{J_1},$$

whence we get that

$$J_1 = \int_0^1 \frac{\arctan^3(x)}{1+x}\mathrm{d}x = \frac{\log(2)}{128}\pi^3 - \frac{3}{32}\pi^2 \underbrace{\int_0^1 \frac{\arctan(x)}{1+x}\mathrm{d}x}_{J_2}$$
$$+\frac{3}{8}\pi \underbrace{\int_0^1 \frac{\arctan^2(x)}{1+x}\mathrm{d}x}_{J_3}, \tag{3.117}$$

and happily we were able to express J_1 in terms of simpler integrals.

Note the integral J_2 from (3.21) is calculated in (3.98), and we have

$$J_2 = \int_0^1 \frac{\arctan(x)}{1+x}\mathrm{d}x = \frac{\pi}{8}\log(2). \tag{3.118}$$

Further, for the integral J_3 in (3.21), we integrate by parts that gives

$$J_3 = \int_0^1 \frac{\arctan^2(x)}{1+x}\mathrm{d}x = \underbrace{\log(1+x)\arctan^2(x)\Big|_{x=0}^{x=1}}_{\log(2)\pi^2/16}$$
$$-2\int_0^1 \frac{\arctan(x)\log(1+x)}{1+x^2}\mathrm{d}x$$
$$= \frac{3}{8}\log(2)\zeta(2) - 2\int_0^1 \frac{\arctan(x)\log(1+x)}{1+x^2}\mathrm{d}x$$

{make use of the result in (1.38)}

$$= \frac{\pi}{4}G + \frac{3}{16}\log(2)\zeta(2) - \frac{21}{32}\zeta(3). \tag{3.119}$$

If we plug the values of the integrals J_2 and J_3 from (3.118) and (3.119) in (3.21), we get

$$J_1 = \int_0^1 \frac{\arctan^3(x)}{1+x}\mathrm{d}x = \log(2)\frac{\pi^3}{128} + \frac{9}{16}\zeta(2)G - \frac{63}{256}\pi\zeta(3). \tag{3.120}$$

Collecting the result from (3.120) in (3.116), we obtain that

$$J = \int_0^1 \frac{\arctan^2(x)\log(1+x)}{1+x^2}\mathrm{d}x = \log(2)\frac{\pi^3}{384} + \frac{21}{256}\pi\zeta(3) - \frac{3}{16}\zeta(2)G. \tag{3.121}$$

Next, to calculate the integral I, we notice that

$$I = \int_0^1 \frac{\arctan^2(x)\log(1-x)}{1+x^2}\mathrm{d}x$$

$$= \int_0^1 \frac{\arctan^2(x)\log(1-x^2)}{1+x^2}\mathrm{d}x - \underbrace{\int_0^1 \frac{\arctan^2(x)\log(1+x)}{1+x^2}\mathrm{d}x}_{J}$$

$$= \underbrace{\int_0^1 \frac{\arctan^2(x)\log(1-x^2)}{1+x^2}\mathrm{d}x}_{I_1} - \log(2)\frac{\pi^3}{384} - \frac{21}{256}\pi\zeta(3) + \frac{3}{16}\zeta(2)G. \tag{3.122}$$

For the integral I_1, we make the change of variable $x = \tan(y)$ that gives

$$I_1 = \int_0^1 \frac{\arctan^2(x)\log(1-x^2)}{1+x^2}\mathrm{d}x = \int_0^{\pi/4} y^2 \log\left(\frac{\cos(2y)}{\cos^2(y)}\right)\mathrm{d}y$$

$$= \underbrace{\int_0^{\pi/4} y^2\log(\cos(2y))\mathrm{d}y}_{I_2} - 2\underbrace{\int_0^{\pi/4} y^2\log(\cos(y))\mathrm{d}y}_{I_3}. \tag{3.123}$$

Now, for both integrals in (3.123) we make use of the Fourier series in (3.86) (see **1.441.4** from [30]). Therefore, for the integral I_2 we have

$$I_2 = \int_0^{\pi/4} y^2\log(\cos(2y))\mathrm{d}y = \int_0^{\pi/4} y^2\left(-\log(2) + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos(4ny)}{n}\right)\mathrm{d}y$$

$$= -\log(2)\frac{\pi^3}{192} + \int_0^{\pi/4} y^2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos(4ny)}{n}\mathrm{d}y$$

{reverse the order of summation and integration}

$$= -\log(2)\frac{\pi^3}{192} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^{\pi/4} y^2\cos(4ny)\mathrm{d}y$$

$$= -\log(2)\frac{\pi^3}{192} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{\pi^2\sin(\pi n)}{64n} + \frac{\pi\cos(\pi n)}{32n^2} - \frac{\sin(\pi n)}{32n^3}\right)$$

$$= -\log(2)\frac{\pi^3}{192} + \frac{\pi^2}{64}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin(\pi n)}{n^2} + \frac{\pi}{32}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos(\pi n)}{n^3}$$

$$-\frac{1}{32}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin(\pi n)}{n^4} = -\log(2)\frac{\pi^3}{192} - \frac{\pi}{32}\sum_{n=1}^{\infty}\frac{1}{n^3}$$

$$= -\log(2)\frac{\pi^3}{192} - \frac{\pi}{32}\zeta(3). \tag{3.124}$$

Further, for the integral I_3 in (3.123), we have

$$I_3 = \int_0^{\pi/4} y^2 \log(\cos(y))\mathrm{d}y = \int_0^{\pi/4} y^2\left(-\log(2) + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos(2ny)}{n}\right)\mathrm{d}y$$

$$= -\log(2)\frac{\pi^3}{192} + \int_0^{\pi/4} y^2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos(2ny)}{n}\mathrm{d}y$$

{reverse the order of summation and integration}

$$= -\log(2)\frac{\pi^3}{192} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^{\pi/4} y^2\cos(2ny)\mathrm{d}y$$

$$= -\log(2)\frac{\pi^3}{192} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{\pi^2\sin\left(\frac{\pi n}{2}\right)}{32n} + \frac{\pi\cos\left(\frac{\pi n}{2}\right)}{8n^2} - \frac{\sin\left(\frac{\pi n}{2}\right)}{4n^3}\right)$$

$$= -\log(2)\frac{\pi^3}{192} + \frac{\pi^2}{32}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin\left(\frac{\pi n}{2}\right)}{n^2} + \frac{\pi}{8}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos\left(\frac{\pi n}{2}\right)}{n^3}$$

$$-\frac{1}{4}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin\left(\frac{\pi n}{2}\right)}{n^4}$$

$$= -\log(2)\frac{\pi^3}{192} + \frac{\pi^2}{32}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2} + \frac{\pi}{64}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3} - \frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^4}$$

{split the last series according to n odd and even}

$$= -\log(2)\frac{\pi^3}{192} + \frac{\pi^2}{32}G + \frac{3}{256}\pi\zeta(3) - \frac{1}{1024}\sum_{n=1}^{\infty}\left(\frac{1}{(n-3/4)^4} - \frac{1}{(n-1/4)^4}\right)$$

$$= -\log(2)\frac{\pi^3}{192}+\frac{\pi^2}{32}G+\frac{3}{256}\pi\zeta(3)-\frac{1}{1024}\sum_{n=1}^{\infty}\frac{1}{(n-3/4)^4}+\frac{1}{1024}\sum_{n=1}^{\infty}\frac{1}{(n-1/4)^4}$$

$$\left\{\text{use Polygamma function series representation, } \psi^{(m)}(z)=(-1)^{m-1}m!\sum_{k=0}^{\infty}\frac{1}{(z+k)^{m+1}}\right\}$$

$$= -\log(2)\frac{\pi^3}{192}+\frac{\pi^2}{32}G+\frac{3}{256}\pi\zeta(3)+\frac{1}{6144}\psi^{(3)}\left(\frac{3}{4}\right)-\frac{1}{6144}\psi^{(3)}\left(\frac{1}{4}\right)$$

$$\left\{\text{use the reflection formula, } (-1)^m\psi^{(m)}(1-z)-\psi^{(m)}(z)=\pi\frac{d^m}{dz^m}\cot(\pi z),\ \text{with } m=3,\right\}$$

$$\left\{z=1/4, \text{ that gives } \psi^{(3)}\left(\frac{3}{4}\right)+\psi^{(3)}\left(\frac{1}{4}\right)=16\pi^4, \text{ from which we extract } \psi^{(3)}\left(\frac{3}{4}\right)\right\}$$

$$= \frac{\pi^4}{384}-\log(2)\frac{\pi^3}{192}+\frac{\pi^2}{32}G+\frac{3}{256}\pi\zeta(3)-\frac{1}{3072}\psi^{(3)}\left(\frac{1}{4}\right). \tag{3.125}$$

If we plug the values of the integrals I_2 and I_3 from (3.124) and (3.125) in (3.123), we have

$$I_1=\int_0^1\frac{\arctan^2(x)\log(1-x^2)}{1+x^2}\mathrm{d}x$$

$$= \frac{1}{1536}\psi^{(3)}\left(\frac{1}{4}\right)-\frac{7}{128}\pi\zeta(3)-\frac{\pi^2}{16}G+\log(2)\frac{\pi^3}{192}-\frac{\pi^4}{192}. \tag{3.126}$$

Finally, collecting the result from (3.126) in (3.122), we obtain that

$$I=\int_0^1\frac{\arctan^2(x)\log(1-x)}{1+x^2}\mathrm{d}x$$

$$= \frac{1}{1536}\psi^{(3)}\left(\frac{1}{4}\right)-\frac{\pi^4}{192}+\log(2)\frac{\pi^3}{384}-\frac{\pi^2}{32}G-\frac{35}{256}\pi\zeta(3),$$

and the solution is complete.

The Polygamma function reflection formula I used in the calculations above may also be found in [116], [79, p. 33].

Again, for an alternative solution focused on an individual approach of the integrals, make the change of variable $x=\tan(y)$ in both integrals, and then exploit that $1-\tan(x)=\sqrt{2}\sec(x)\sin\left(\frac{\pi}{4}-x\right)$, $1+\tan(x)=\sqrt{2}\sec(x)\sin\left(x+\frac{\pi}{4}\right)$ and the Fourier series of $\log(\sin(x))$, $\log(\cos(x))$.

3.22 Challenging Integrals Involving $\arctan(x)$, $\log(x)$, $\log(1-x)$, $\operatorname{Li}_2(x)$, and $\operatorname{Li}_2(x^2)$

Solution When encountering the results with the tough integrals in this section, one impulse might be *Let's try to calculate the integrals separately and then put the results together*, and if we run *Mathematica*, say, for either of the two integrals from the point $i)$, we arrive at some results involving the Trilogarithm function ([119], [44, Chapter 6, pp. 153–187], [78, Chapter 2, pp. 113–114]) with a complex argument, $i \operatorname{Li}_3\left(\frac{1+i}{2}\right)$, that doesn't seem to reduce to simpler forms in well-known constants only. Depending on the way we choose in our calculations, sometimes things may become rather difficult, unfortunately. Therefore, one concern is about the starting point of the calculations, and our lucky card here is to try to rearrange the initial integrals and bring them to more convenient forms (that often can be achieved with the integration by parts). Happily, all the resulting integrals we'll get in this process are calculated in the present book.

Starting with the integration by parts at the point $i)$, we write

$$\int_0^1 \arctan(x)\log(x)\log(1-x)\mathrm{d}x = \int_0^1 x'\arctan(x)\log(x)\log(1-x)\mathrm{d}x$$

$$= \underbrace{x\arctan(x)\log(x)\log(1-x)\Big|_{x=0}^{x=1}}_{0} + \int_0^1 \frac{x\arctan(x)\log(x)}{1-x}\mathrm{d}x$$

$$-\int_0^1 \arctan(x)\log(1-x)\mathrm{d}x - \int_0^1 \frac{x\log(1-x)\log(x)}{1+x^2}\mathrm{d}x,$$

from which we obtain immediately that

$$\int_0^1 \arctan(x)\log(x)\log(1-x)\mathrm{d}x - \int_0^1 \frac{x\log(x)\arctan(x)}{1-x}\mathrm{d}x$$

$$= -\int_0^1 \arctan(x)\log(1-x)\mathrm{d}x - \int_0^1 \frac{x\log(1-x)\log(x)}{1+x^2}\mathrm{d}x$$

{the values of the integrals are given in (1.33) and (1.26)}

$$= G - \frac{41}{64}\zeta(3) + \frac{3}{32}\log(2)\pi^2 - \frac{5}{96}\pi^2 - \frac{1}{8}\log(2)\pi + \frac{\pi}{4} - \frac{1}{2}\log(2) + \frac{1}{8}\log^2(2),$$

and the solution to the point $i)$ is finalized.

As regards the second part of the problem, we apply the integration by parts for the first integral, and then we write

$$\int_0^1 \arctan(x)\log(x)\operatorname{Li}_2(x)\mathrm{d}x = \int_0^1 x' \arctan(x)\log(x)\operatorname{Li}_2(x)\mathrm{d}x$$

$$= \underbrace{x\arctan(x)\log(x)\operatorname{Li}_2(x)\Big|_{x=0}^{x=1}}_{0} + \int_0^1 \arctan(x)\log(1-x)\log(x)\mathrm{d}x$$

$$- \int_0^1 \arctan(x)\operatorname{Li}_2(x)\mathrm{d}x - \int_0^1 \frac{x\log(x)\operatorname{Li}_2(x)}{1+x^2}\mathrm{d}x. \tag{3.127}$$

Returning to the main question and using the result in (3.127), we have

$$\int_0^1 \arctan(x)\log(x)\operatorname{Li}_2(x)\mathrm{d}x - \int_0^1 \frac{x\log(x)\arctan(x)}{1-x}\mathrm{d}x$$

$$= \int_0^1 \arctan(x)\log(1-x)\log(x)\mathrm{d}x - \int_0^1 \frac{x\log(x)\arctan(x)}{1-x}\mathrm{d}x$$

$$- \int_0^1 \arctan(x)\operatorname{Li}_2(x)\mathrm{d}x - \int_0^1 \frac{x\log(x)\operatorname{Li}_2(x)}{1+x^2}\mathrm{d}x$$

$$\{\text{make use of the results in (1.41), (1.35) and (1.56), with } n=1\}$$

$$= 2G - \frac{1}{2}G^2 - \frac{\pi}{4}G + \frac{41}{7680}\pi^4 - \frac{\pi^3}{24} - \frac{9}{32}\zeta(3) - \frac{5\pi^2}{48} + \frac{\pi}{2} + \frac{\log^2(2)}{4} + \frac{17}{96}\log(2)\pi^2$$

$$-\frac{\pi}{4}\log(2) - \log(2),$$

and the solution to the point $ii)$ is finalized.

For the third part of the problem, we might like to start with the Dilogarithm function identity, $\operatorname{Li}_2(x) + \operatorname{Li}_2(-x) = \frac{1}{2}\operatorname{Li}_2(x^2)$, and then we write

$$\int_0^1 \arctan(x)\log(x)\operatorname{Li}_2(x^2)\mathrm{d}x - 2\int_0^1 \frac{x\log(x)\arctan(x)}{1-x}\mathrm{d}x$$

$$= 2\int_0^1 \arctan(x)\log(x)\operatorname{Li}_2(x)\mathrm{d}x - 2\int_0^1 \frac{x\log(x)\arctan(x)}{1-x}\mathrm{d}x$$

$$+2\int_0^1 \arctan(x)\log(x)\operatorname{Li}_2(-x)\mathrm{d}x$$

{make use of the results from the point ii) and (1.45)}

$$= 4G + \log(2)G + \frac{\pi^4}{1920} - \frac{7}{96}\pi^3 - \frac{9}{8}\zeta(3) - \frac{5}{24}\log(2)\pi^2 - 2\log(2)\pi - \frac{5}{24}\pi^2$$

$$+\frac{5}{2}\pi - 5\log(2) + \log^2(2),$$

and the solution to the point iii) is finalized.

How do we know, guess the advantageous forms we should start with when dealing with such problems? I think there is no magical way to tell it. However, what we can do is to make investigations combined with the experience gained, which is a key point, and then we may arrive at solutions like the ones presented above.

3.23 Two More Special Challenging Integrals Involving $\arctan(x)$, $\log(x)$, $\log(1+x)$, and $\operatorname{Li}_2(-x)$

Solution I wonder if you jumped here while working on the point iii) from the previous section. The integral at the point ii) in this section appeared in the solution of the result at the point iii) from the previous section. We have to cope again with some integrals for which I want to emphasize the adjective *challenging*.

The strategy we want to employ is to break down the integrals into simpler integrals, and applying the integration by parts, we get

$$\int_0^1 \arctan(x)\log(x)\log(1+x)\mathrm{d}x = \int_0^1 x'\arctan(x)\log(x)\log(1+x)\mathrm{d}x$$

$$= \underbrace{x\arctan(x)\log(x)\log(1+x)\Big|_{x=0}^{x=1}}_{0} - \int_0^1 \arctan(x)\log(x)\mathrm{d}x$$

$$+\int_0^1 \frac{\arctan(x)\log(x)}{1+x}\mathrm{d}x - \int_0^1 \arctan(x)\log(1+x)\mathrm{d}x$$

$$-\int_0^1 \frac{x\log(x)\log(1+x)}{1+x^2}\mathrm{d}x$$

where if we use that $\int_0^1 \arctan(x)\log(x)\mathrm{d}x = \int_0^1 \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}\log(x)\mathrm{d}x$

$$= \sum_{n=1}^{\infty}(-1)^{n-1}\int_0^1 \frac{x^{2n-1}}{2n-1}\log(x)\mathrm{d}x = \sum_{n=1}^{\infty}(-1)^n\frac{1}{4n^2(2n-1)} = \frac{1}{4}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}$$

$$+\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} - \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{2n-1} = \frac{\pi^2}{48} - \frac{\pi}{4} + \frac{\log(2)}{2},$$

which we combine with the values of the integrals given in (1.46), (1.34), and (1.27), we conclude that

$$\int_0^1 \arctan(x)\log(x)\log(1+x)\mathrm{d}x$$

$$= \frac{\log(2)}{2}G - \frac{\pi^3}{64} + \frac{15}{64}\zeta(3) - \log(2)\frac{\pi^2}{32} - \frac{\pi^2}{96} - \frac{3}{8}\log(2)\pi + \frac{\pi}{2} + \frac{1}{8}\log^2(2) - \log(2),$$

and the part i) of the problem is finalized.

Proceeding similarly as we did for the part i) of the problem, we integrate by parts, and then we have

$$\int_0^1 \arctan(x)\log(x)\operatorname{Li}_2(-x)\mathrm{d}x = \int_0^1 x'\arctan(x)\log(x)\operatorname{Li}_2(-x)\mathrm{d}x$$

$$= \underbrace{x\arctan(x)\log(x)\operatorname{Li}_2(-x)\Big|_{x=0}^{x=1}}_{0} + \int_0^1 \arctan(x)\log(x)\log(1+x)\mathrm{d}x$$

$$-\int_0^1 \arctan(x)\operatorname{Li}_2(-x)\mathrm{d}x - \int_0^1 \frac{x\log(x)\operatorname{Li}_2(-x)}{1+x^2}\mathrm{d}x$$

{the values of the integrals are given in (1.44), (1.36), and (1.57), with $n = 1$}

$$= \frac{1}{2}G^2 + \frac{\pi}{4}G + \frac{\log(2)}{2}G - \frac{13}{2560}\pi^4 + \frac{\pi^3}{192} - \frac{9}{32}\zeta(3) - \frac{7}{96}\log(2)\pi^2 - \frac{3}{4}\log(2)\pi$$

$$+\frac{3}{4}\pi - \frac{3}{2}\log(2) + \frac{1}{4}\log^2(2),$$

and the part ii) of the problem is finalized.

As seen, I tried to reduce both proposed integrals to simpler integrals. All the resulting integrals were calculated in the previous sections, excepting two, $\displaystyle\int_0^1 \frac{\arctan(x)\log(x)}{1+x}\mathrm{d}x$ and $\displaystyle\int_0^1 \frac{x\log(x)\operatorname{Li}_2(-x)}{1+x^2}\mathrm{d}x$, which represent the hard nuts of the problem and give the real substance to the word *challenging* used in the title of the section. The former one we'll meet right in the next section together with a beautiful generalization of it.

3.24 A Challenging Integral with the Inverse Tangent Function and an Excellent Generalization According to the Even Positive Powers of the Logarithm

Solution The integral from the point i) naturally arises in the calculation process of the challenging integral i) from the previous section, and we want to know how to calculate it. Further, using a different idea, we'll also make the generalization of a similar version of the integral from the point i), the version where we consider $\log^{2n}(x)$ instead of $\log(x)$, with $n \geq 1,\ n \in \mathbb{N}$.

Let's start with the point i) of the problem, and noting and using that $\arctan(x) = \displaystyle\int_0^1 \frac{x}{1+x^2y^2}\mathrm{d}y$, we get

$$\int_0^1 \frac{\arctan(x)\log(x)}{1+x}\mathrm{d}x = \int_0^1 \left(\int_0^1 \frac{x\log(x)}{(1+x)(1+x^2y^2)}\mathrm{d}y\right)\mathrm{d}x$$

$$= \int_0^1 \left(\int_0^1 \frac{\log(x)}{1+x^2y^2}\mathrm{d}y\right)\mathrm{d}x - \int_0^1 \left(\int_0^1 \frac{\log(x)}{(1+x)(1+x^2y^2)}\mathrm{d}y\right)\mathrm{d}x$$

{reverse the order of integration in the second double integral}

$$= \int_0^1 \frac{\arctan(x)\log(x)}{x}\mathrm{d}x - \underbrace{\int_0^1 \left(\int_0^1 \frac{\log(x)}{(1+x)(1+x^2y^2)}\mathrm{d}x\right)\mathrm{d}y}_{I}$$

$$= \int_0^1 \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^{2k-2}}{2k-1}\log(x)\mathrm{d}x - I$$

{reverse the order of summation and integration}

$$= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k-1}\int_0^1 x^{2k-2}\log(x)\mathrm{d}x - I = -\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^3} - I = -\frac{\pi^3}{32} - I, \tag{3.128}$$

where the last series is a known particular case[8] of the Dirichlet beta function (see [101]), $\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}$, i.e., $\beta(3) = \frac{\pi^3}{32}$.

For the remaining integral in (3.128), we make the change of variable $yx = z$, and at the same time we want to exploit the idea[9] of symmetrical double integrals, that gives

$$I = \int_0^1 \left(\int_0^y \frac{\log(z/y)}{(y+z)(1+z^2)} \mathrm{d}z \right) \mathrm{d}y$$

$$= \int_0^1 \left(\int_0^y \frac{\log(z/y)}{(y+z)(1+z^2)} + \frac{\log(y/z)}{(y+z)(1+y^2)} \mathrm{d}z \right) \mathrm{d}y$$

$$- \int_0^1 \left(\int_0^y \frac{\log(y/z)}{(y+z)(1+y^2)} \mathrm{d}z \right) \mathrm{d}y$$

{exploit the symmetry of the integrand in the first double integral and}

{in the second double integral make the change of variable $z/y = x$}

[8]Splitting according to n odd and even and using the Polygamma function reflection formula, $(-1)^m \psi^{(m)}(1-z) - \psi^{(m)}(z) = \pi \frac{d^m}{dz^m} \cot(\pi z)$, we get $\beta(3) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3} = \sum_{n=1}^{\infty} \frac{1}{(4n-3)^3} - \sum_{n=1}^{\infty} \frac{1}{(4n-1)^3} = \frac{1}{64} \left(\sum_{n=1}^{\infty} \frac{1}{(n-3/4)^3} - \sum_{n=1}^{\infty} \frac{1}{(n-1/4)^3} \right) = \frac{1}{64} \lim_{x \to 1/4} \left(\sum_{n=1}^{\infty} \frac{1}{(n-1+x)^3} - \sum_{n=1}^{\infty} \frac{1}{(n-x)^3} \right) = \frac{1}{128} \lim_{x \to 1/4} \left(\psi^{(2)}(1-x) - \psi^{(2)}(x) \right) = \frac{\pi^3}{32}$, where I also used the Polygamma function series representation, $\psi^{(m)}(z) = (-1)^{m-1} m! \sum_{k=0}^{\infty} \frac{1}{(z+k)^{m+1}}$.

[9]Usually, if having a symmetrical function in two variables, $f(x,y) = f(y,x)$, then, upon reversing the order of integration, we may write $I = \int_0^1 \left(\int_0^y f(x,y) \mathrm{d}x \right) \mathrm{d}y = \int_0^1 \left(\int_x^1 f(x,y) \mathrm{d}y \right) \mathrm{d}x = \int_0^1 \left(\left(\int_0^1 - \int_0^x \right) f(x,y) \mathrm{d}y \right) \mathrm{d}x = \int_0^1 \left(\int_0^1 f(x,y) \mathrm{d}y \right) \mathrm{d}x - I$, from which we obtain that $I = \int_0^1 \left(\int_0^y f(x,y) \mathrm{d}x \right) \mathrm{d}y = \frac{1}{2} \int_0^1 \left(\int_0^1 f(x,y) \mathrm{d}y \right) \mathrm{d}x = \frac{1}{2} \int_0^1 \left(\int_0^1 f(x,y) \mathrm{d}x \right) \mathrm{d}y$.

$$= \frac{1}{2}\int_0^1 \left(\int_0^1 \frac{\log(z/y)}{(y+z)(1+z^2)} + \frac{\log(y/z)}{(y+z)(1+y^2)}\mathrm{d}z\right)\mathrm{d}y$$
$$+\int_0^1 \frac{\log(x)}{1+x}\mathrm{d}x \int_0^1 \frac{1}{1+y^2}\mathrm{d}y$$
$$= \frac{1}{2}\int_0^1 \left(\int_0^1 \frac{(y^2-z^2)\log(z/y)}{(y+z)(1+y^2)(1+z^2)}\mathrm{d}z\right)\mathrm{d}y + \frac{\pi}{4}\int_0^1 \frac{\log(x)}{1+x}\mathrm{d}x$$
$$= \frac{1}{2}\int_0^1 \left(\int_0^1 \frac{y\log(z)+z\log(y)-y\log(y)-z\log(z)}{(1+y^2)(1+z^2)}\mathrm{d}z\right)\mathrm{d}y$$
$$+\frac{\pi}{4}\int_0^1 \sum_{k=1}^{\infty}(-1)^{k-1}x^{k-1}\log(x)\mathrm{d}x$$
$$= \int_0^1 \frac{y}{1+y^2}\mathrm{d}y \int_0^1 \frac{\log(z)}{1+z^2}\mathrm{d}z - \int_0^1 \frac{y\log(y)}{1+y^2}\mathrm{d}y \int_0^1 \frac{1}{1+z^2}\mathrm{d}z - \frac{\pi}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}$$
$$= \frac{\log(2)}{2}\int_0^1 \sum_{k=1}^{\infty}(-1)^{k-1}z^{2k-2}\log(z)\mathrm{d}z - \frac{\pi}{4}\int_0^1 \sum_{k=1}^{\infty}(-1)^{k-1}y^{2k-1}\log(y)\mathrm{d}y - \frac{\pi^3}{48}$$
$$= \frac{\log(2)}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\int_0^1 z^{2k-2}\log(z)\mathrm{d}z - \frac{\pi}{4}\sum_{k=1}^{\infty}(-1)^{k-1}\int_0^1 y^{2k-1}\log(y)\mathrm{d}y - \frac{\pi^3}{48}$$
$$= \frac{\log(2)}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k-1)^2} - \frac{\pi}{16}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2} - \frac{\pi^3}{48} = -\frac{\log(2)}{2}G - \frac{\pi^3}{64}. \tag{3.129}$$

If we plug the result from (3.129) in (3.128), we conclude that

$$\int_0^1 \frac{\arctan(x)\log(x)}{1+x}\mathrm{d}x = \frac{\log(2)}{2}G - \frac{\pi^3}{64}, \tag{3.130}$$

and the calculations to the point $i)$ are finalized.

This integral version also appears in [55] with different approaches.

To tackle the point $ii)$ of the problem, we first prove that for $-1 < s < 0,\ s \in \mathbb{R}$, we have the following result,

$$\int_0^{\infty} \frac{x^s}{(1+x)(1+y^2x^2)}\mathrm{d}x$$
$$= \frac{\pi}{2}\csc\left(\frac{\pi s}{2}\right)\frac{y^{-s}}{1+y^2} + \frac{\pi}{2}\sec\left(\frac{\pi s}{2}\right)\frac{y^{1-s}}{1+y^2} - \frac{\pi\csc(\pi s)}{1+y^2}. \tag{3.131}$$

Proof We start with the partial fraction decomposition, and then we write

$$\int_0^\infty \frac{x^s}{(1+x)(1+y^2x^2)}\mathrm{d}x$$

$$= \frac{1}{1+y^2}\int_0^\infty \frac{x^s}{1+x}\mathrm{d}x + \frac{y^2}{1+y^2}\int_0^\infty \frac{x^s}{1+y^2x^2}\mathrm{d}x - \frac{y^2}{1+y^2}\int_0^\infty \frac{x^{s+1}}{1+y^2x^2}\mathrm{d}x$$

{for the second and third integrals apply the variable change $y^2x^2 = t$}

$$= \frac{1}{2}\frac{y^{1-s}}{1+y^2}\int_0^\infty \frac{t^{(s-1)/2}}{1+t}\mathrm{d}t - \frac{1}{2}\frac{y^{-s}}{1+y^2}\int_0^\infty \frac{t^{s/2}}{1+t}\mathrm{d}t - \frac{\pi\csc(\pi s)}{1+y^2}$$

$$= \frac{\pi}{2}\csc\left(\frac{\pi s}{2}\right)\frac{y^{-s}}{1+y^2} + \frac{\pi}{2}\sec\left(\frac{\pi s}{2}\right)\frac{y^{1-s}}{1+y^2} - \frac{\pi\csc(\pi s)}{1+y^2},$$

and the proof of the result is complete. In the calculations I also made use of the classical result,[10] $\displaystyle\int_0^\infty \frac{t^{x-1}}{1+t}\mathrm{d}t = \frac{\pi}{\sin(\pi x)}$.

If integrating both sides of the result in (3.131) from $y=0$ to $y=1$, we obtain

$$\int_0^1 \left(\int_0^\infty \frac{x^s}{(1+x)(1+y^2x^2)}\mathrm{d}x\right)\mathrm{d}y = \int_0^\infty \left(\int_0^1 \frac{x^s}{(1+x)(1+y^2x^2)}\mathrm{d}y\right)\mathrm{d}x$$

$$= \int_0^\infty \frac{x^{s-1}\arctan(x)}{1+x}\mathrm{d}x$$

$$= \frac{\pi}{2}\csc\left(\frac{\pi s}{2}\right)\int_0^1 \frac{y^{-s}}{1+y^2}\mathrm{d}y + \frac{\pi}{2}\sec\left(\frac{\pi s}{2}\right)\int_0^1 \frac{y^{1-s}}{1+y^2}\mathrm{d}y$$

$$-\pi\csc(\pi s)\int_0^1 \frac{1}{1+y^2}\mathrm{d}y$$

{in the first two integrals make the change of variable $y^2 = z$}

[10] Using the Beta function in the form $\mathrm{B}(x,y) = \displaystyle\int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\mathrm{d}t$, where if we set $x+y=1$, combined with the identity connecting the Beta function and Gamma function, $\mathrm{B}(x,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, and the Euler's reflection formula, $\Gamma(a)\Gamma(1-a) = \dfrac{\pi}{\sin(\pi a)}$, we obtain that $\displaystyle\int_0^\infty \frac{t^{x-1}}{1+t}\mathrm{d}t = \frac{\pi}{\sin(\pi x)}$.

$$= \frac{\pi}{4}\csc\left(\frac{\pi s}{2}\right)\int_0^1 \frac{z^{-(1/2+s/2)}}{1+z}\mathrm{d}z + \frac{\pi}{4}\sec\left(\frac{\pi s}{2}\right)\int_0^1 \frac{z^{-s/2}}{1+z}\mathrm{d}z - \frac{\pi^2}{4}\csc(\pi s)$$

{e.g., make use of the result in (1.10), the first equality}

$$= \frac{\pi}{8}\csc\left(\frac{\pi s}{2}\right)\left(\psi\left(\frac{3}{4}-\frac{s}{4}\right)-\psi\left(\frac{1}{4}-\frac{s}{4}\right)\right)$$

$$+\frac{\pi}{8}\sec\left(\frac{\pi s}{2}\right)\left(\psi\left(1-\frac{s}{4}\right)-\psi\left(\frac{1}{2}-\frac{s}{4}\right)\right)-\frac{\pi^2}{4}\csc(\pi s). \tag{3.132}$$

Returning to the main integral of the problem, we write

$$\int_0^1 \frac{\arctan(x)\log^{2n}(x)}{1+x}\mathrm{d}x = \underbrace{\int_0^1 \frac{\arctan(x)\log^{2n}(x)}{x}\mathrm{d}x}_{J_1} - \underbrace{\int_0^1 \frac{\arctan(x)\log^{2n}(x)}{x(1+x)}\mathrm{d}x}_{J_2}. \tag{3.133}$$

The first integral in the right-hand side of (3.133) is straightforward, and we have

$$J_1 = \int_0^1 \frac{\arctan(x)\log^{2n}(x)}{x}\mathrm{d}x = \int_0^1 \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^{2k-2}}{2k-1}\log^{2n}(x)\mathrm{d}x$$

{reverse the order of summation and integration}

$$= \sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{2k-1}\int_0^1 x^{2k-2}\log^{2n}(x)\mathrm{d}x = (2n)!\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{(2k-1)^{2n+2}}$$

$$= \beta(2n+2)(2n)!. \tag{3.134}$$

Next, for the second integral in the right-hand side of (3.133), we write

$$J_2 = \int_0^1 \frac{\arctan(x)\log^{2n}(x)}{x(1+x)}\mathrm{d}x = \int_0^{\infty} \frac{\arctan(x)\log^{2n}(x)}{x(1+x)}\mathrm{d}x$$

$$-\int_1^{\infty} \frac{\arctan(x)\log^{2n}(x)}{x(1+x)}\mathrm{d}x$$

{make the change of variable $x = 1/y$ in the second integral}

$$= \int_0^{\infty} \frac{\arctan(x)\log^{2n}(x)}{x(1+x)}\mathrm{d}x - \int_0^1 \frac{\arctan(1/y)\log^{2n}(y)}{1+y}\mathrm{d}x$$

{employ the identity $\arctan(x)+\arctan(1/x)=\pi/2,\ x>0$}

$$= \int_0^\infty \frac{\arctan(x)\log^{2n}(x)}{x(1+x)}\mathrm{d}x + \int_0^1 \frac{\arctan(x)\log^{2n}(x)}{1+x}\mathrm{d}x - \frac{\pi}{2}\int_0^1 \frac{\log^{2n}(x)}{1+x}\mathrm{d}x$$

$$= \int_0^\infty \frac{\arctan(x)\log^{2n}(x)}{x(1+x)}\mathrm{d}x + \int_0^1 \frac{\arctan(x)\log^{2n}(x)}{1+x}\mathrm{d}x$$

$$- \frac{\pi}{2}(1-2^{-2n})\zeta(2n+1)(2n)!, \tag{3.135}$$

where I used that $\int_0^1 \frac{\log^{2n}(x)}{1+x}\mathrm{d}x = \int_0^1 \sum_{k=1}^{\infty}(-1)^{k-1}x^{k-1}\log^{2n}(x)\mathrm{d}x$ $= \sum_{k=1}^{\infty}(-1)^{k-1}\int_0^1 x^{k-1}\log^{2n}(x)\mathrm{d}x = (2n)!\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k^{2n+1}} = \eta(2n+1)(2n)! = (1-2^{-2n})\zeta(2n+1)(2n)!$.

Collecting the integrals J_1 and J_2 from (3.134) and (3.135) in (3.133), we obtain

$$\int_0^1 \frac{\arctan(x)\log^{2n}(x)}{1+x}\mathrm{d}x = (2n)!\beta(2n+2) + \frac{\pi}{2}(1-2^{-2n})\zeta(2n+1)(2n)!$$
$$- \int_0^\infty \frac{\arctan(x)\log^{2n}(x)}{x(1+x)}\mathrm{d}x - \int_0^1 \frac{\arctan(x)\log^{2n}(x)}{1+x}\mathrm{d}x,$$

and using the identity in (3.132) to prove the result,

$$\int_0^\infty \frac{\arctan(x)\log^{2n}(x)}{x(1+x)}\mathrm{d}x$$
$$= \frac{\pi}{8}\lim_{s\to 0}\left(\frac{d^{2n}}{ds^{2n}}\left(\csc\left(\frac{\pi s}{2}\right)\left(\psi\left(\frac{3}{4}-\frac{s}{4}\right)-\psi\left(\frac{1}{4}-\frac{s}{4}\right)\right)\right.\right.$$
$$\left.\left.+\sec\left(\frac{\pi s}{2}\right)\left(\psi\left(1-\frac{s}{4}\right)-\psi\left(\frac{1}{2}-\frac{s}{4}\right)\right)-2\pi\csc(\pi s)\right)\right),$$

we conclude that

$$\int_0^1 \frac{\arctan(x)\log^{2n}(x)}{1+x}\mathrm{d}x = \frac{\pi}{4}(1-2^{-2n})\zeta(2n+1)(2n)!$$

$$+\frac{1}{2}\beta(2n+2)(2n)! - \frac{\pi}{16}\lim_{s\to 0}\left(\frac{d^{2n}}{ds^{2n}}\left(\csc\left(\frac{\pi s}{2}\right)\left(\psi\left(\frac{3}{4}-\frac{s}{4}\right)-\psi\left(\frac{1}{4}-\frac{s}{4}\right)\right)\right.\right.$$

$$\left.\left.+\sec\left(\frac{\pi s}{2}\right)\left(\psi\left(1-\frac{s}{4}\right)-\psi\left(\frac{1}{2}-\frac{s}{4}\right)\right)-2\pi\csc(\pi s)\right)\right), \tag{3.136}$$

and the calculations to the point $ii)$ are finalized.

In this section, for the first part of the problem, we have convinced ourselves of the incredible power of the technique that exploits the symmetry in double integrals, and we'll continue to meet such approaches in the next sections.

The idea used for the generalization from the second point of the problem can also be successfully employed for more advanced integrals, and such an example (which is pretty delightful) may be found in the next section.

The curious reader interested in a larger generalization, with the logarithm raised at any positive integer, may use the strategy in [56] (and the same idea to try in the next section for a larger generalization).

3.25 Let's Tango with an Exciting Integral Involving the Inverse Tangent Integral, the Lerch Transcendent Function, and the Logarithm with Odd Positive Powers

Solution First of all, let's prepare ourselves for the encounter with what is called the Inverse tangent integral. In Lewin's book in [44], there is a whole chapter dedicated to this function, named *The inverse tangent integral*, and then another chapter dedicated to the more general version of the mentioned function, named *The generalized inverse tangent integral*. If you didn't have a chance yet to take a look on these chapters, and plan to do it, be ready to meet absolutely extraordinary and very useful formulae, identities with these functions. The Inverse tangent integral is defined in terms of Dilogarithm function with a complex argument, $\operatorname{Li}_2(ix) = \frac{1}{4}\operatorname{Li}_2(-x^2) + i\operatorname{Ti}_2(x)$, and there are various known representations of it like the series representation, $\operatorname{Ti}_2(x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)^2}$, the integral representation, $\operatorname{Ti}_2(x) = \int_0^x \frac{\arctan(t)}{t}\mathrm{d}t$, and last but not least, I point out the integral representation in (3.138) (below), we'll find very useful in this section.

During the calculations it won't be necessary to resort to the *heavy artillery* of these chapters I already mentioned. In fact, we need one more thing, the inverse relation,

$$\operatorname{Ti}_2(x) - \operatorname{Ti}_2\left(\frac{1}{x}\right) = \frac{\pi}{2}\operatorname{sgn}(x)\log(|x|), \tag{3.137}$$

we may also find in [78, Chapter 2, p. 110].

To start the solution, we need in the calculations the fact that

$$\operatorname{Ti}_2(x) = -\int_0^1 \frac{x\log(y)}{1+x^2y^2}\mathrm{d}y, \tag{3.138}$$

which is straightforward to show[11] by using the geometric series.

Now, multiplying both sides of the relation in (3.131) by $-\log(y)$ and then integrating from $y=0$ to $y=1$, we get

$$\int_0^1 \left(\int_0^\infty -\frac{x^s\log(y)}{(1+x)(1+y^2x^2)}\mathrm{d}x\right)\mathrm{d}y = \int_0^\infty \left(\int_0^1 -\frac{x^s\log(y)}{(1+x)(1+x^2y^2)}\mathrm{d}y\right)\mathrm{d}x$$

$$= \int_0^\infty \frac{x^{s-1}\operatorname{Ti}_2(x)}{1+x}\mathrm{d}x = -\frac{\pi}{2}\csc\left(\frac{\pi s}{2}\right)\int_0^1 \frac{y^{-s}\log(y)}{1+y^2}\mathrm{d}y$$

$$-\frac{\pi}{2}\sec\left(\frac{\pi s}{2}\right)\int_0^1 \frac{y^{1-s}\log(y)}{1+y^2}\mathrm{d}y$$

$$+\pi\csc(\pi s)\int_0^1 \frac{\log(y)}{1+y^2}\mathrm{d}y$$

{make the change of variable $y^2=z$ in the first two integrals}

$$= -\frac{\pi}{8}\csc\left(\frac{\pi s}{2}\right)\underbrace{\int_0^1 \frac{z^{-(s+1)/2}\log(z)}{1+z}\mathrm{d}z}_{I_1} - \frac{\pi}{8}\sec\left(\frac{\pi s}{2}\right)\underbrace{\int_0^1 \frac{z^{-s/2}\log(z)}{1+z}\mathrm{d}z}_{I_2}$$

$$+\pi\csc(\pi s)\underbrace{\int_0^1 \frac{\log(y)}{1+y^2}\mathrm{d}y}_{I_3}. \tag{3.139}$$

Recalling and using the first equality in (1.10), then the first integral in (3.139) leads to

$$I_1 = \int_0^1 \frac{z^{-(s+1)/2}\log(z)}{1+z}\mathrm{d}z = \frac{1}{4}\left(\psi^{(1)}\left(\frac{3}{4}-\frac{s}{4}\right) - \psi^{(1)}\left(\frac{1}{4}-\frac{s}{4}\right)\right), \tag{3.140}$$

and similarly, for the second integral, we obtain

$$I_2 = \int_0^1 \frac{z^{-s/2}\log(z)}{1+z}\mathrm{d}z = \frac{1}{4}\left(\psi^{(1)}\left(1-\frac{s}{4}\right) - \psi^{(1)}\left(\frac{1}{2}-\frac{s}{4}\right)\right), \tag{3.141}$$

[11] It's easy to see, if we use the geometric series, that $-\int_0^1 \frac{x\log(y)}{1+x^2y^2}\mathrm{d}y = \int_0^1 x\sum_{n=1}^\infty(-1)^n (xy)^{2n-2}\log(y)\mathrm{d}y = \sum_{n=1}^\infty (-1)^n x^{2n-1}\int_0^1 y^{2n-2}\log(y)\mathrm{d}y = \sum_{n=1}^\infty(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)^2} = \operatorname{Ti}_2(x)$.

where we first differentiated (1.10) with respect to s and then plugged in the needed values.

The last integral in (3.139) is a known one, and we write that

$$I_3 = \int_0^1 \frac{\log(y)}{1+y^2}\mathrm{d}y = \int_0^1 \sum_{n=1}^{\infty}(-1)^{n-1} y^{2n-2}\log(y)\mathrm{d}y$$

$$= \sum_{n=1}^{\infty}(-1)^{n-1}\int_0^1 y^{2n-2}\log(y)\mathrm{d}y$$

$$= \sum_{n=1}^{\infty}(-1)^n \frac{1}{(2n-1)^2} = -G. \tag{3.142}$$

Collecting the values of the integrals I_1, I_2, and I_3 from (3.140), (3.141), and (3.142) in (3.139), we obtain

$$\int_0^{\infty} \frac{x^{s-1}\operatorname{Ti}_2(x)}{1+x}\mathrm{d}x = \frac{\pi}{32}\csc\left(\frac{\pi s}{2}\right)\left(\psi^{(1)}\left(\frac{1}{4}-\frac{s}{4}\right)-\psi^{(1)}\left(\frac{3}{4}-\frac{s}{4}\right)\right)$$
$$+\frac{\pi}{32}\sec\left(\frac{\pi s}{2}\right)\left(\psi^{(1)}\left(\frac{1}{2}-\frac{s}{4}\right)-\psi^{(1)}\left(1-\frac{s}{4}\right)\right)$$
$$-\pi\csc(\pi s)G. \tag{3.143}$$

Upon returning to the main question, we write

$$\int_0^1 \frac{\operatorname{Ti}_2(x)\log^{2n-1}(x)}{1+x}\mathrm{d}x = \underbrace{\int_0^1 \operatorname{Ti}_2(x)\frac{\log^{2n-1}(x)}{x}\mathrm{d}x}_{I_4} - \underbrace{\int_0^1 \operatorname{Ti}_2(x)\frac{\log^{2n-1}(x)}{x(1+x)}\mathrm{d}x}_{I_5}, \tag{3.144}$$

where you may note I used the same strategy as in the previous section.

Now, for the integral I_4 in (3.144), we write

$$I_4 = \int_0^1 \operatorname{Ti}_2(x)\frac{\log^{2n-1}(x)}{x}\mathrm{d}x = \int_0^1 \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^2}x^{2k-2}\log^{2n-1}(x)\mathrm{d}x$$

{reverse the order of summation and integration}

$$= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^2}\int_0^1 x^{2k-2}\log^{2n-1}(x)\mathrm{d}x$$

$$\{\text{make use of the result in (1.2)}\}$$

$$= -(2n-1)!\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^{2n+2}} = -\beta(2n+2)(2n-1)!. \tag{3.145}$$

For the second integral in (3.144), we need the inverse relation in (3.137), and we write

$$I_5 = \int_0^1 \mathrm{Ti}_2(x)\frac{\log^{2n-1}(x)}{x(1+x)}\mathrm{d}x = \int_0^{\infty} \mathrm{Ti}_2(x)\frac{\log^{2n-1}(x)}{x(1+x)}\mathrm{d}x$$
$$-\int_1^{\infty} \mathrm{Ti}_2(x)\frac{\log^{2n-1}(x)}{x(1+x)}\mathrm{d}x$$

$$\{\text{make the change of variable } x = 1/y \text{ in the second integral}\}$$

$$= \int_0^{\infty} \mathrm{Ti}_2(x)\frac{\log^{2n-1}(x)}{x(1+x)}\mathrm{d}x + \int_0^1 \mathrm{Ti}_2\left(\frac{1}{y}\right)\frac{\log^{2n-1}(y)}{1+y}\mathrm{d}y$$

$$= \int_0^{\infty} \mathrm{Ti}_2(x)\frac{\log^{2n-1}(x)}{x(1+x)}\mathrm{d}x + \int_0^1 \mathrm{Ti}_2(x)\frac{\log^{2n-1}(x)}{1+x}\mathrm{d}x - \frac{\pi}{2}\int_0^1 \frac{\log^{2n}(x)}{1+x}\mathrm{d}x$$

$$= \int_0^{\infty} \mathrm{Ti}_2(x)\frac{\log^{2n-1}(x)}{x(1+x)}\mathrm{d}x + \int_0^1 \frac{\mathrm{Ti}_2(x)\log^{2n-1}(x)}{1+x}\mathrm{d}x$$
$$-\frac{\pi}{2}(1-2^{-2n})\zeta(2n+1)(2n)!, \tag{3.146}$$

where I used that $\int_0^1 \frac{\log^{2n}(x)}{1+x}\mathrm{d}x = \int_0^1 \sum_{k=1}^{\infty}(-1)^{k-1}x^{k-1}\log^{2n}(x)\mathrm{d}x$
$= \sum_{k=1}^{\infty}(-1)^{k-1}\int_0^1 x^{k-1}\log^{2n}(x)\mathrm{d}x = (2n)!\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^{2n+1}} = \eta(2n+1)(2n)!$
$= (1-2^{-2n})\zeta(2n+1)(2n)!.$

Collecting the results from (3.145) and (3.146) in (3.144), we obtain

$$\int_0^1 \mathrm{Ti}_2(x)\frac{\log^{2n-1}(x)}{1+x}\mathrm{d}x = \frac{\pi}{2}(1-2^{-2n})\zeta(2n+1)(2n)! - \beta(2n+2)(2n-1)!$$
$$-\int_0^{\infty} \mathrm{Ti}_2(x)\frac{\log^{2n-1}(x)}{x(1+x)}\mathrm{d}x - \int_0^1 \frac{\mathrm{Ti}_2(x)\log^{2n-1}(x)}{1+x}\mathrm{d}x,$$

and using the identity in (3.143) to prove the result,

$$\int_0^{\infty} \operatorname{Ti}_2(x)\frac{\log^{2n-1}(x)}{x(1+x)}\mathrm{d}x$$

$$= \frac{\pi}{32}\lim_{s\to 0}\left(\frac{d^{2n-1}}{ds^{2n-1}}\left(\csc\left(\frac{\pi s}{2}\right)\left(\psi^{(1)}\left(\frac{1}{4}-\frac{s}{4}\right)-\psi^{(1)}\left(\frac{3}{4}-\frac{s}{4}\right)\right)\right.\right.$$

$$\left.\left.+\sec\left(\frac{\pi s}{2}\right)\left(\psi^{(1)}\left(\frac{1}{2}-\frac{s}{4}\right)-\psi^{(1)}\left(1-\frac{s}{4}\right)\right)-32G\csc(\pi s)\right)\right),$$

we conclude that

$$\int_0^1 \operatorname{Ti}_2(x)\frac{\log^{2n-1}(x)}{1+x}\mathrm{d}x = \frac{\pi}{4}(1-2^{-2n})\zeta(2n+1)(2n)! - \frac{1}{2}\beta(2n+2)(2n-1)!$$

$$-\frac{\pi}{64}\lim_{s\to 0}\left(\frac{d^{2n-1}}{ds^{2n-1}}\left(\csc\left(\frac{\pi s}{2}\right)\left(\psi^{(1)}\left(\frac{1}{4}-\frac{s}{4}\right)-\psi^{(1)}\left(\frac{3}{4}-\frac{s}{4}\right)\right)\right.\right.$$

$$\left.\left.+\sec\left(\frac{\pi s}{2}\right)\left(\psi^{(1)}\left(\frac{1}{2}-\frac{s}{4}\right)-\psi^{(1)}\left(1-\frac{s}{4}\right)\right)-32G\csc(\pi s)\right)\right),$$

and the solution is finalized.

In formulating the main question I also used the Lerch transcendent function (see [113]), $\Phi(z,s,a) = \sum_{k=0}^{\infty}\frac{z^k}{(a+k)^s}$, but for the calculations it was enough to only recognize the connection with the Inverse tangent integral which is easy to note if comparing their series representations.

3.26 A Superb Integral with Logarithms and the Inverse Tangent Function, and a Surprisingly Beautiful Generalization of It

Solution Just a few sections back from here I presented sums of integrals where when treated separately they lead to results involving the Trilogarithm function ([119], [44, Chapter 6, pp. 153–187]) with a complex argument, that is in Sect. 1.22.

If we consider the integral from the first point where we write that $\log\left(\frac{1+x^2}{(1-x)^2}\right) = \log(1+x^2) - 2\log(1-x)$, then split it accordingly, and give it a try with *Mathematica*, we realize at once we fall under the same category of integrals like the ones from the section mentioned above. So, looks like we need some ingenious ways!

I'll treat first the generalization from the second point, and using the result in (4.4), we write that

$$\int_0^x \frac{\arctan(t)\log(1+t^2)}{t}\mathrm{d}t = 2\int_0^x \sum_{k=1}^{\infty}(-1)^{k-1} t^{2k}\frac{H_{2k}}{2k+1}\mathrm{d}t$$

{reverse the order of summation and integration}

$$= 2\sum_{k=1}^{\infty}(-1)^{k-1}\int_0^x t^{2k}\frac{H_{2k}}{2k+1}\mathrm{d}t = 2\sum_{k=1}^{\infty}(-1)^{k-1}x^{2k+1}\frac{H_{2k}}{(2k+1)^2}. \tag{3.147}$$

Then, for the other integral of the generalization, we have

$$\int_0^1 \frac{\arctan(xt)\log(1-t)}{t}\mathrm{d}t = \int_0^1 \sum_{k=1}^{\infty}(-1)^{k-1}x^{2k-1}\frac{t^{2k-2}}{2k-1}\log(1-t)\mathrm{d}t$$

{reverse the order of summation and integration}

$$= \sum_{k=1}^{\infty}(-1)^{k-1}x^{2k-1}\frac{1}{2k-1}\int_0^1 t^{2k-2}\log(1-t)\mathrm{d}t$$

{make use of the result in (1.4)}

$$= \sum_{k=1}^{\infty}(-1)^{k}x^{2k-1}\frac{H_{2k-1}}{(2k-1)^2} = \sum_{k=0}^{\infty}(-1)^{k-1}x^{2k+1}\frac{H_{2k}+1/(2k+1)}{(2k+1)^2}$$

$$= \sum_{k=0}^{\infty}(-1)^{k-1}x^{2k+1}\frac{H_{2k}}{(2k+1)^2} + \sum_{k=0}^{\infty}(-1)^{k-1}\frac{x^{2k+1}}{(2k+1)^3}$$

{start in the first series from $k=1$, and reindex the second series}

$$= \sum_{k=1}^{\infty}(-1)^{k-1}x^{2k+1}\frac{H_{2k}}{(2k+1)^2} - \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^{2k-1}}{(2k-1)^3}. \tag{3.148}$$

By combining the results from (3.147) and (3.148), we obtain that

$$\int_0^x \frac{\arctan(t)\log(1+t^2)}{t}\mathrm{d}t - 2\int_0^1 \frac{\arctan(xt)\log(1-t)}{t}\mathrm{d}t$$
$$= 2\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^{2k-1}}{(2k-1)^3},$$

and the first solution to the point $ii)$ of the problem is finalized.

For a second solution to the point $ii)$ of the problem, we need the following result

$$2\int_0^{\infty} \frac{\log(x)}{(x+1)^2+t^2}\mathrm{d}x = \frac{\arctan(t)\log(1+t^2)}{t},\ t \in \mathbb{R}. \tag{3.149}$$

Proof In order to prove the result in (3.149), we want to first show that

$$\int_0^{\infty} \frac{\log(x)}{(x+a)(x+b)}\mathrm{d}x = \frac{1}{2}\left(\frac{\log^2(a)-\log^2(b)}{a-b}\right),\ a,b>0. \tag{3.150}$$

If we let the variable change $x = ab/y$, we get

$$I = \int_0^{\infty} \frac{\log(x)}{(x+a)(x+b)}\mathrm{d}x = \int_0^{\infty} \frac{\log(ab/y)}{(y+a)(y+b)}\mathrm{d}y$$
$$= \log(ab)\int_0^{\infty} \frac{1}{(x+a)(x+b)}\mathrm{d}x - I,$$

from which we obtain that

$$I = \frac{1}{2}\log(ab)\int_0^{\infty} \frac{1}{(x+a)(x+b)}\mathrm{d}x = \frac{1}{2}\frac{\log(ab)}{a-b}\log\left(\frac{x+b}{x+a}\right)\Bigg|_{x=0}^{x=\infty}$$
$$= \frac{1}{2}\left(\frac{\log^2(a)-\log^2(b)}{a-b}\right),$$

and the auxiliary result is proved.

Alternatively, if we start with the fact that, $\displaystyle\int_0^{\infty} \frac{x^{s-1}}{x+a}\mathrm{d}x = \pi a^{s-1}\csc(\pi s)$, which comes immediately from the well-known classical result, $\displaystyle\int_0^{\infty} \frac{y^{s-1}}{1+y}\mathrm{d}y = \pi\csc(\pi s)$, that we met and proved in a footnote of Sect. 3.24, and then differentiate once with respect to s, we get

$$\int_0^\infty \frac{x^{s-1}\log(x)}{x+a}\mathrm{d}x = \pi a^{s-1}\csc(\pi s)(\log(a) - \pi\cot(\pi s)). \tag{3.151}$$

Based upon the result in (3.151), we obtain that

$$\int_0^\infty \frac{x^{s-1}\log(x)}{(x+a)(x+b)}\mathrm{d}x$$

$$= \frac{\pi\csc(\pi s)}{b-a}(a^{s-1}\log(a) - b^{s-1}\log(b) + \pi b^{s-1}\cot(\pi s) - \pi a^{s-1}\cot(\pi s)). \tag{3.152}$$

If we let $s \to 1$ in (3.152), we get again the result in (3.150),

$$\int_0^\infty \frac{\log(x)}{(x+a)(x+b)}\mathrm{d}x = \frac{1}{2}\left(\frac{\log^2(a) - \log^2(b)}{a-b}\right).$$

Therefore, by setting $a = 1 + it$ and $b = 1 - it$ in (3.150), we obtain

$$\int_0^\infty \frac{\log(x)}{(x+1)^2+t^2}\mathrm{d}x = \frac{i}{4t}\left(\log^2(1-it) - \log^2(1+it)\right) = \frac{\arctan(t)\log(1+t^2)}{2t},$$

where I used that $\log(a+ib) = \log(\sqrt{a^2+b^2}) + i\arctan\left(\frac{b}{a}\right)$, $a > 0$, and the proof of the result in (3.149) is finalized.

Next, if we replace x by y in the left-hand side of (3.149) and integrate with respect to t, from $t = 0$ to $t = x$, we have

$$\int_0^x \frac{\arctan(t)\log(1+t^2)}{t}\mathrm{d}t = 2\int_0^x \left(\int_0^\infty \frac{\log(y)}{(y+1)^2+t^2}\mathrm{d}y\right)\mathrm{d}t$$

$$\{\text{reverse the order of integration}\}$$

$$= 2\int_0^\infty \left(\int_0^x \frac{\log(y)}{(y+1)^2+t^2}\mathrm{d}t\right)\mathrm{d}y = 2\int_0^\infty \frac{1}{1+y}\arctan\left(\frac{x}{1+y}\right)\log(y)\mathrm{d}y$$

$$\left\{\text{make the change of variable } \frac{1}{1+y} = t\right\}$$

$$= 2\int_0^1 \frac{\arctan(xt)\log((1-t)/t)}{t}\mathrm{d}t$$

$$= 2\int_0^1 \frac{\arctan(xt)\log(1-t)}{t}\mathrm{d}t - 2\int_0^1 \frac{\arctan(xt)\log(t)}{t}\mathrm{d}t$$

$$= 2\int_0^1 \frac{\arctan(xt)\log(1-t)}{t}\mathrm{d}t - 2\int_0^1 \sum_{k=1}^{\infty}(-1)^{k-1}\frac{(xt)^{2k-1}}{2k-1}\frac{\log(t)}{t}\mathrm{d}t$$

$$\{\text{reverse the order of summation and integration}\}$$

$$= 2\int_0^1 \frac{\arctan(xt)\log(1-t)}{t}\mathrm{d}t - 2\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^{2k-1}}{2k-1}\int_0^1 t^{2k-2}\log(t)\mathrm{d}t$$

$$= 2\int_0^1 \frac{\arctan(xt)\log(1-t)}{t}\mathrm{d}t + 2\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^{2k-1}}{(2k-1)^3},$$

from which we obtain that

$$\int_0^x \frac{\arctan(t)\log(1+t^2)}{t}\mathrm{d}t - 2\int_0^1 \frac{\arctan(xt)\log(1-t)}{t}\mathrm{d}t$$
$$= 2\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^{2k-1}}{(2k-1)^3},$$

and the second solution to the point ii) of the problem is finalized.

I submitted the integral from the point i) to *The American Mathematical Monthly*, the problem **12054** (see [92]). Now, by setting $x = 1$ in the generalization from the point ii), we obtain

$$\int_0^1 \frac{\arctan(t)}{t}\log\left(\frac{1+t^2}{(1-t)^2}\right)\mathrm{d}t = 2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^3} = \frac{\pi^3}{16},$$

where the last series we already met and calculated in a footnote of Sect. 3.24, and the solution to the point i) of the problem is finalized.

The curious reader might also find a generalization of the result from (3.149) in [64, Chapter 8, p. 333], obtained by means of contour integration.

3.27 A Kind of Deviant Pair of Integrals with Logarithms and Polylogarithms, Using Symmetry

Solution *These integrals look awesome!*, a *natural* reaction I might expect from you! But how about their level of difficulty? Well, if we use the *right tool* (of course, we'll want to use it!) and combine it with exploiting the symmetry in double integrals, then we'll find the derivation of both results pretty enjoyable. Let's see now how to do it exactly!

If we employ the result in (1.12), with $n = 1$, $\operatorname{Li}_2\left(\dfrac{x}{x-1}\right) = \displaystyle\int_0^1 \frac{x\log(t)}{1-x(1-t)}\mathrm{d}t$, and then make the change of variable $1 - t = y$, we get $\operatorname{Li}_2\left(\dfrac{x}{x-1}\right) = \displaystyle\int_0^1 \frac{x\log(1-y)}{1-xy}\mathrm{d}y$, and then, for the point $i)$ of the problem, we write

$$\int_0^1 \frac{\log(1-x)\operatorname{Li}_2\left(\dfrac{x}{x-1}\right)}{1+x}\mathrm{d}x = \int_0^1 \left(\int_0^1 \frac{x\log(1-x)\log(1-y)}{(1+x)(1-xy)}\mathrm{d}y\right)\mathrm{d}x$$

$$\{\text{use the symmetry of the integrand}\}$$

$$= \frac{1}{2}\left(\int_0^1 \left(\int_0^1 \frac{x\log(1-x)\log(1-y)}{(1+x)(1-xy)}\mathrm{d}y\right)\mathrm{d}x + \int_0^1 \left(\int_0^1 \frac{y\log(1-x)\log(1-y)}{(1+y)(1-xy)}\mathrm{d}x\right)\mathrm{d}y\right)$$

$$\{\text{for the first double integral, reverse the order of integration}\}$$

$$= \frac{1}{2}\left(\int_0^1 \left(\int_0^1 \frac{x\log(1-x)\log(1-y)}{(1+x)(1-xy)}\mathrm{d}x\right)\mathrm{d}y + \int_0^1 \left(\int_0^1 \frac{y\log(1-x)\log(1-y)}{(1+y)(1-xy)}\mathrm{d}x\right)\mathrm{d}y\right)$$

$$= \frac{1}{2}\int_0^1 \left(\int_0^1 \frac{(x+y+2xy)\log(1-x)\log(1-y)}{(1+x)(1+y)(1-xy)}\mathrm{d}x\right)\mathrm{d}y$$

$$= \frac{1}{2}\int_0^1 \left(\int_0^1 \frac{((1+x)(1+y)-(1-xy))\log(1-x)\log(1-y)}{(1+x)(1+y)(1-xy)}\mathrm{d}x\right)\mathrm{d}y$$

$$= \frac{1}{2}\underbrace{\int_0^1 \left(\int_0^1 \frac{\log(1-x)\log(1-y)}{1-xy}\mathrm{d}x\right)\mathrm{d}y}_{I} - \frac{1}{2}\underbrace{\int_0^1 \left(\int_0^1 \frac{\log(1-x)\log(1-y)}{(1+x)(1+y)}\mathrm{d}x\right)\mathrm{d}y}_{J}. \tag{3.153}$$

Now, for the integral I in (3.153), we write that

$$I = \int_0^1 \left(\int_0^1 \frac{\log(1-x)\log(1-y)}{1-xy} \mathrm{d}y \right) \mathrm{d}x$$

$$= \int_0^1 \left(\int_0^1 \sum_{n=1}^{\infty} (xy)^{n-1} \log(1-x)\log(1-y)\mathrm{d}y \right) \mathrm{d}x$$

$$\{\text{reverse the order of summation and integration}\}$$

$$= \sum_{n=1}^{\infty} \int_0^1 x^{n-1}\log(1-x)\mathrm{d}x \int_0^1 y^{n-1}\log(1-y)\mathrm{d}y = \sum_{n=1}^{\infty} \left(\frac{H_n}{n}\right)^2 = \frac{17}{4}\zeta(4), \tag{3.154}$$

where for the penultimate equality I made use of the result in (1.4), and the last series is the quadratic series of Au-Yeung calculated in (4.29).

Further, to calculate the integral J in (3.153) it is enough to calculate $\displaystyle\int_0^1 \frac{\log(1-x)}{1+x}\mathrm{d}x$,

$$\int_0^1 \frac{\log(1-x)}{1+x}\mathrm{d}x \overset{(1-x)/(1+x)=y}{=} \int_0^1 \frac{\log(y)}{1+y}\mathrm{d}y + \log(2)\int_0^1 \frac{1}{1+y}\mathrm{d}y - \int_0^1 \frac{\log(1+y)}{1+y}\mathrm{d}y$$

$$= \frac{\log^2(2)}{2} + \int_0^1 \frac{\log(y)}{1+y}\mathrm{d}y = \frac{\log^2(2)}{2} + \int_0^1 \sum_{k=1}^{\infty} (-1)^{k-1} y^{k-1}\log(y)\mathrm{d}y$$

$$= \frac{\log^2(2)}{2} + \sum_{k=1}^{\infty} (-1)^{k-1} \int_0^1 y^{k-1}\log(y)\mathrm{d}y = \frac{\log^2(2)}{2} - \sum_{k=1}^{\infty} (-1)^{k-1}\frac{1}{k^2}$$

$$= \frac{1}{2}\left(\log^2(2) - \zeta(2)\right). \tag{3.155}$$

Using the result from (3.155) in the integral J from (3.153), we have

$$J = \int_0^1 \left(\int_0^1 \frac{\log(1-x)\log(1-y)}{(1+x)(1+y)}\mathrm{d}y \right)\mathrm{d}x = \frac{1}{4}\left(\log^2(2) - \zeta(2)\right)^2. \tag{3.156}$$

By collecting the values from (3.154) and (3.156) in (3.153), we conclude that

$$\int_0^1 \frac{\log(1-x)\operatorname{Li}_2\left(\dfrac{x}{x-1}\right)}{1+x}\mathrm{d}x = \frac{29}{16}\zeta(4) + \frac{1}{4}\log^2(2)\zeta(2) - \frac{1}{8}\log^4(2),$$

and the part $i)$ of the problem is finalized.

For an alternative solution, the curious reader might also think of using the Landen's identity (see [44, Chapter 1, p. 5], [78, Chapter 2, p. 107]).

As in the first part, for calculating the integrals from the point $ii)$ we make use of the result in (1.12), with $n = 2$, to get that $-\dfrac{1}{2}\displaystyle\int_0^1 \frac{x\log^2(1-y)}{1-xy}\mathrm{d}y = \operatorname{Li}_3\left(\dfrac{x}{x-1}\right)$, and then we write

$$\int_0^1 \frac{\log^2(1-x)\operatorname{Li}_3\left(\dfrac{x}{x-1}\right)}{1+x}\mathrm{d}x = -\frac{1}{2}\int_0^1\left(\int_0^1 \frac{x\log^2(1-x)\log^2(1-y)}{(1+x)(1-xy)}\mathrm{d}y\right)\mathrm{d}x$$

$$\{\text{use the symmetry of the integrand}\}$$

$$= -\frac{1}{4}\left(\int_0^1\left(\int_0^1 \frac{x\log^2(1-x)\log^2(1-y)}{(1+x)(1-xy)}\mathrm{d}y\right)\mathrm{d}x\right.$$

$$\left.+\int_0^1\left(\int_0^1 \frac{y\log^2(1-x)\log^2(1-y)}{(1+y)(1-xy)}\mathrm{d}x\right)\mathrm{d}y\right)$$

$$\{\text{for the first double integral, reverse the order of integration}\}$$

$$= -\frac{1}{4}\left(\int_0^1\left(\int_0^1 \frac{x\log^2(1-x)\log^2(1-y)}{(1+x)(1-xy)}\mathrm{d}x\right)\mathrm{d}y\right.$$

$$\left.+\int_0^1\left(\int_0^1 \frac{y\log^2(1-x)\log^2(1-y)}{(1+y)(1-xy)}\mathrm{d}x\right)\mathrm{d}y\right)$$

$$= -\frac{1}{4}\int_0^1\left(\int_0^1 \frac{(x+y+2xy)\log^2(1-x)\log^2(1-y)}{(1+x)(1+y)(1-xy)}\mathrm{d}x\right)\mathrm{d}y$$

$$= -\frac{1}{4}\int_0^1\left(\int_0^1 \frac{((1+x)(1+y)-(1-xy))\log^2(1-x)\log^2(1-y)}{(1+x)(1+y)(1-xy)}\mathrm{d}x\right)\mathrm{d}y$$

$$= -\frac{1}{4}\underbrace{\int_0^1 \left(\int_0^1 \frac{\log^2(1-x)\log^2(1-y)}{1-xy}\mathrm{d}x\right)\mathrm{d}y}_{K}$$

$$+\frac{1}{4}\underbrace{\int_0^1 \left(\int_0^1 \frac{\log^2(1-x)\log^2(1-y)}{(1+x)(1+y)}\mathrm{d}x\right)\mathrm{d}y}_{L}. \tag{3.157}$$

For the integral K in (3.157), we have that

$$K = \int_0^1 \left(\int_0^1 \frac{\log^2(1-x)\log^2(1-y)}{1-xy}\mathrm{d}x\right)\mathrm{d}y$$

$$= \int_0^1 \left(\int_0^1 \sum_{n=1}^{\infty}(xy)^{n-1}\log^2(1-x)\log^2(1-y)\mathrm{d}x\right)\mathrm{d}y$$

{reverse the order of summation and integration}

$$= \sum_{n=1}^{\infty}\int_0^1 x^{n-1}\log^2(1-x)\mathrm{d}x\int_0^1 y^{n-1}\log^2(1-y)\mathrm{d}y$$

{make use of the result in (1.5)}

$$= \sum_{n=1}^{\infty}\left(\frac{H_n^2+H_n^{(2)}}{n}\right)^2 = \sum_{n=1}^{\infty}\frac{H_n^4}{n^2} + 2\sum_{n=1}^{\infty}\frac{H_n^2H_n^{(2)}}{n^2} + \sum_{n=1}^{\infty}\frac{(H_n^{(2)})^2}{n^2}$$

{the first two series are given in (4.39) and (4.38), and for the third series let $n \to \infty$}

{in the result from (4.15), with $p = 2$, and combine it with the series from (6.71)}

$$= \frac{581}{12}\zeta(6) + 8\zeta^2(3). \tag{3.158}$$

Further, to calculate the integral L in (3.157), it is enough to calculate $\displaystyle\int_0^1 \frac{\log^2(1-x)}{1+x}\mathrm{d}x$,

$$\int_0^1 \frac{\log^2(1-x)}{1+x}\mathrm{d}x \overset{1-x=y}{=} \int_0^1 \frac{\log^2(y)}{2-y}\mathrm{d}y = \frac{1}{2}\int_0^1 \frac{\log^2(y)}{1-y/2}\mathrm{d}y$$

$$= \frac{1}{2}\int_0^1 \sum_{n=1}^{\infty}\left(\frac{y}{2}\right)^{n-1}\log^2(y)\mathrm{d}y = \sum_{n=1}^{\infty}\frac{1}{2^n}\int_0^1 y^{n-1}\log^2(y)\mathrm{d}y = 2\sum_{n=1}^{\infty}\frac{1}{2^n n^3}$$

$$= 2\operatorname{Li}_3\left(\frac{1}{2}\right)$$

{use the special value in (3.20)}

$$= \frac{7}{4}\zeta(3) - \log(2)\zeta(2) + \frac{1}{3}\log^3(2). \tag{3.159}$$

Thus, using the result from (3.159) in the integral L from (3.157), we get

$$L = \int_0^1 \left(\int_0^1 \frac{\log^2(1-x)\log^2(1-y)}{(1+x)(1+y)}\mathrm{d}x\right)\mathrm{d}y$$

$$= \left(\frac{7}{4}\zeta(3) - \log(2)\zeta(2) + \frac{1}{3}\log^3(2)\right)^2. \tag{3.160}$$

By collecting the results from (3.158) and (3.160) in (3.157), we conclude that

$$\int_0^1 \frac{\log^2(1-x)\operatorname{Li}_3\left(\frac{x}{x-1}\right)}{1+x}\mathrm{d}x$$

$$= \frac{1}{36}\log^6(2) - \frac{1}{6}\log^4(2)\zeta(2) + \frac{7}{24}\log^3(2)\zeta(3) + \frac{5}{8}\log^2(2)\zeta(4) - \frac{581}{48}\zeta(6)$$

$$-\frac{7}{8}\log(2)\zeta(2)\zeta(3) - \frac{79}{64}\zeta^2(3),$$

and the part $ii)$ of the problem is finalized.

We'll continue to see the power of the symmetry in the next few sections, which will allow us to evaluate beautiful integrals with logarithms and Polylogarithms.

3.28 Wonderful Integrals Containing the Logarithm and the Polylogarithm, Involving Beautiful Ideas About Symmetry: The First Part

Solution Do you remember the integral from the point $i)$ we had to use in Sect. 1.12? I might guess you probably arrived here if you went through the previously mentioned section (at least out of curiosity). In general, noticing the possibility of successfully exploiting the symmetry of a multiple integral is nothing but *a wonderful moment*, and the usual consequence is that everything reduces to simpler calculations. This section (and some of the later ones, as you'll see) could be a nice opportunity to train your *symmetry eye*. When it's possible, just use that! It's awesome!

Now, it's enough to prove the generalization from the point ii), and using the integral in Sect. 1.6, we have

$$\int_0^1 \frac{\log^n(x)\operatorname{Li}_{n+1}(x)}{1+x}\mathrm{d}x = \frac{(-1)^n}{n!}\int_0^1\left(\int_0^1 \frac{x\log^n(x)\log^n(y)}{(1+x)(1-xy)}\mathrm{d}y\right)\mathrm{d}x. \tag{3.161}$$

Then, for the last double integral in (3.161), we exploit the symmetry, and then it's easy to note that

$$\int_0^1\left(\int_0^1 \frac{x\log^n(x)\log^n(y)}{(1+x)(1-xy)}\mathrm{d}y\right)\mathrm{d}x$$

$$= \frac{1}{2}\left(\int_0^1\left(\int_0^1 \frac{x\log^n(x)\log^n(y)}{(1+x)(1-xy)}\mathrm{d}y\right)\mathrm{d}x + \int_0^1\left(\int_0^1 \frac{y\log^n(x)\log^n(y)}{(1+y)(1-xy)}\mathrm{d}x\right)\mathrm{d}y\right)$$

$$\{\text{for the first double integral, reverse the order of integration}\}$$

$$= \frac{1}{2}\left(\int_0^1\left(\int_0^1 \frac{x\log^n(x)\log^n(y)}{(1+x)(1-xy)}\mathrm{d}x\right)\mathrm{d}y + \int_0^1\left(\int_0^1 \frac{y\log^n(x)\log^n(y)}{(1+y)(1-xy)}\mathrm{d}x\right)\mathrm{d}y\right)$$

$$= \frac{1}{2}\int_0^1\left(\int_0^1\left(\frac{x}{(1+x)(1-xy)} + \frac{y}{(1+y)(1-xy)}\right)\log^n(x)\log^n(y)\mathrm{d}x\right)\mathrm{d}y$$

$$= \frac{1}{2}\int_0^1\left(\int_0^1 \frac{x+y+2xy}{(1+x)(1+y)(1-xy)}\log^n(x)\log^n(y)\mathrm{d}x\right)\mathrm{d}y$$

$$= \frac{1}{2}\int_0^1\left(\int_0^1 \frac{(1+x)(1+y)-(1-xy)}{(1+x)(1+y)(1-xy)}\log^n(x)\log^n(y)\mathrm{d}x\right)\mathrm{d}y$$

$$= \frac{1}{2}\underbrace{\int_0^1\left(\int_0^1 \frac{\log^n(x)\log^n(y)}{1-xy}\mathrm{d}x\right)\mathrm{d}y}_{I_1} - \frac{1}{2}\underbrace{\int_0^1 \frac{\log^n(x)}{1+x}\mathrm{d}x}_{I_2}\underbrace{\int_0^1 \frac{\log^n(y)}{1+y}\mathrm{d}y}_{I_2}. \tag{3.162}$$

For the integral I_1 in (3.162), we write

$$I_1 = \int_0^1\left(\int_0^1 \frac{\log^n(x)\log^n(y)}{1-xy}\mathrm{d}x\right)\mathrm{d}y$$

$$= \int_0^1 \left(\int_0^1 \sum_{k=1}^{\infty} (xy)^{k-1} \log^n(x) \log^n(y) \mathrm{d}x \right) \mathrm{d}y$$

{reverse the order of summation and integration}

$$= \sum_{k=1}^{\infty} \int_0^1 x^{k-1} \log^n(x) \mathrm{d}x \int_0^1 y^{k-1} \log^n(y) \mathrm{d}y$$

$$\left\{ \text{use the result in (1.2) to write that } \int_0^1 x^{k-1} \log^n(x) \mathrm{d}x = (-1)^n \frac{n!}{k^{n+1}} \right\}$$

$$= (n!)^2 \sum_{k=1}^{\infty} \frac{1}{k^{2n+2}} = (n!)^2 \zeta(2n+2). \tag{3.163}$$

Further, for the integral I_2 in (3.162), we have

$$I_2 = \int_0^1 \frac{\log^n(x)}{1+x} \mathrm{d}x = \int_0^1 \sum_{k=1}^{\infty} (-1)^{k-1} x^{k-1} \log^n(x) \mathrm{d}x$$

{reverse the order of summation and integration}

$$= \sum_{k=1}^{\infty} (-1)^{k-1} \int_0^1 x^{k-1} \log^n(x) \mathrm{d}x$$

$$\left\{ \text{use the result in (1.2) to write that } \int_0^1 x^{k-1} \log^n(x) \mathrm{d}x = (-1)^n \frac{n!}{k^{n+1}} \right\}$$

$$= (-1)^n n! \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^{n+1}} = (-1)^n n! \eta(n+1) = (-1)^n (1 - 2^{-n}) n! \zeta(n+1). \tag{3.164}$$

Collecting the values of the integrals from (3.163) and (3.164) in (3.162), we obtain

$$\int_0^1 \left(\int_0^1 \frac{x \log^n(x) \log^n(y)}{(1+x)(1-xy)} \mathrm{d}y \right) \mathrm{d}x = \frac{1}{2} (n!)^2 \left(\zeta(2n+2) - \left(1 - 2^{-n}\right)^2 \zeta^2(n+1) \right). \tag{3.165}$$

Hence, if we plug the result from (3.165) in (3.161), we conclude that

$$\int_0^1 \frac{\log^n(x) \operatorname{Li}_{n+1}(x)}{1+x} \mathrm{d}x = \frac{1}{2} (-1)^n n! \left(\zeta(2n+2) - (1 - 2^{-n})^2 \zeta^2(n+1) \right),$$

and the solution is complete. The integral from the point $i)$ is obtained by considering $n = 1$ in the generalized integral.

For instance, in the chapter *Sums and Series* we'll meet a problem where using such an integral is one of the possible keys to solve it (and I'll let you discover on your own which this problem is).

3.29 Wonderful Integrals Containing the Logarithm and the Polylogarithm, Involving Beautiful Ideas About Symmetry: The Second Part

Solution In this section we continue with a similar strategy as in the previous section, and we'll want to exploit again the power of the symmetry in double integrals. Then, we could also remember that the case $n = 1$ appeared in the calculations of the integral J from Sect. 1.12. Using the integral in Sect. 1.6, we have

$$\int_0^1 \frac{\log^n(x)\operatorname{Li}_{n+1}(-x)}{1+x^2}\mathrm{d}x = \frac{(-1)^{n-1}}{n!}\int_0^1\left(\int_0^1 \frac{x\log^n(x)\log^n(y)}{(1+x^2)(1+xy)}\mathrm{d}y\right)\mathrm{d}x. \tag{3.166}$$

Exploiting the symmetry of the double integral in (3.166), we write that

$$\int_0^1\left(\int_0^1 \frac{x\log^n(x)\log^n(y)}{(1+x^2)(1+xy)}\mathrm{d}y\right)\mathrm{d}x$$

$$= \frac{1}{2}\left(\int_0^1\left(\int_0^1 \frac{x\log^n(x)\log^n(y)}{(1+x^2)(1+xy)}\mathrm{d}y\right)\mathrm{d}x + \int_0^1\left(\int_0^1 \frac{y\log^n(x)\log^n(y)}{(1+y^2)(1+xy)}\mathrm{d}y\right)\mathrm{d}x\right)$$

$$= \frac{1}{2}\left(\int_0^1\left(\int_0^1\left(\frac{x}{(1+x^2)(1+xy)}+\frac{y}{(1+y^2)(1+xy)}\right)\log^n(x)\log^n(y)\mathrm{d}y\right)\mathrm{d}x\right)$$

$$= \frac{1}{2}\int_0^1\left(\int_0^1 \frac{x+y+x^2y+xy^2}{(1+x^2)(1+y^2)(1+xy)}\log^n(x)\log^n(y)\mathrm{d}y\right)\mathrm{d}x$$

$$= \frac{1}{2}\int_0^1\left(\int_0^1 \frac{(x+y)(1+xy)}{(1+x^2)(1+y^2)(1+xy)}\log^n(x)\log^n(y)\mathrm{d}y\right)\mathrm{d}x$$

$$= \frac{1}{2}\int_0^1\left(\int_0^1 \frac{x+y}{(1+x^2)(1+y^2)}\log^n(x)\log^n(y)\mathrm{d}y\right)\mathrm{d}x$$

$$= \int_0^1 \frac{x\log^n(x)}{1+x^2}\mathrm{d}x\int_0^1 \frac{\log^n(y)}{1+y^2}\mathrm{d}y$$

$$= \left(\int_0^1 \sum_{k=1}^{\infty} (-1)^{k-1} x^{2k-1} \log^n(x) \mathrm{d}x \right) \left(\int_0^1 \sum_{k=1}^{\infty} (-1)^{k-1} y^{2k-2} \log^n(y) \mathrm{d}y \right)$$

{reverse the order of summation and integration}

$$= \left(\sum_{k=1}^{\infty} (-1)^{k-1} \int_0^1 x^{2k-1} \log^n(x) \mathrm{d}x \right) \left(\sum_{k=1}^{\infty} (-1)^{k-1} \int_0^1 y^{2k-2} \log^n(y) \mathrm{d}y \right)$$

{make use of the result in (1.2)}

$$= \frac{n!^2}{2^{n+1}} \left(\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k^{n+1}} \right) \left(\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{(2k-1)^{n+1}} \right) = \frac{n!^2}{2^{n+1}} \eta(n+1)\beta(n+1)$$

$$= \frac{n!^2}{2^{n+1}} (1 - 2^{-n}) \zeta(n+1)\beta(n+1). \tag{3.167}$$

Finally, by plugging the result from (3.167) in (3.166), we conclude that

$$\int_0^1 \frac{\log^n(x) \operatorname{Li}_{n+1}(-x)}{1+x^2} \mathrm{d}x = (-1)^{n-1} \frac{n!}{2^{n+1}} \eta(n+1)\beta(n+1)$$

$$= (-1)^{n-1} \frac{n!}{2^{n+1}} (1 - 2^{-n}) \zeta(n+1)\beta(n+1),$$

and the solution is complete.

As you may see, there are various, beautiful ways of exploiting the symmetry, and to develop a good sense around such approaches, much practice is needed.

3.30 Wonderful Integrals Containing the Logarithm and the Polylogarithm, Involving Beautiful Ideas About Symmetry: The Third Part

Solution I don't plan to move away yet from the symmetry idea I previously used, and we'll continue evaluating another two integrals by constructing a strategy based upon exploiting the symmetry. Also, in the chapter *Sums and Series* we'll meet a problem where one of these integrals may be a possible key to solve it.

For the generalization from the point $i)$, we need the simple fact that

$$\frac{X^2}{(1+X^2)(1-YX)} + \frac{Y^2}{(1+Y^2)(1-YX)} = \frac{X^2 + X^2Y^2 + Y^2 + X^2Y^2}{(1+X^2)(1+Y^2)(1-YX)}$$

$$= \frac{(1+X^2)(1+Y^2)-(1-(XY)^2)}{(1+X^2)(1+Y^2)(1-YX)} = \frac{1}{1-YX} - \frac{1}{(1+X^2)(1+Y^2)} - \frac{XY}{(1+X^2)(1+Y^2)}. \tag{3.168}$$

Now, based on the integral in Sect. 1.6, we write

$$\int_0^1 \frac{x\log^n(x)\operatorname{Li}_{n+1}(x)}{1+x^2}\mathrm{d}x = \frac{(-1)^n}{n!}\int_0^1\left(\int_0^1 \frac{x^2\log^n(x)\log^n(y)}{(1+x^2)(1-yx)}\mathrm{d}y\right)\mathrm{d}x$$

{make use of the result in (3.168)}

$$= \frac{(-1)^n}{2\cdot n!}\left(\int_0^1\left(\int_0^1 \frac{\log^n(x)\log^n(y)}{1-yx}\mathrm{d}y\right)\mathrm{d}x - \int_0^1\left(\int_0^1 \frac{\log^n(x)\log^n(y)}{(1+x^2)(1+y^2)}\mathrm{d}y\right)\mathrm{d}x - \int_0^1\left(\int_0^1 \frac{xy\log^n(x)\log^n(y)}{(1+x^2)(1+y^2)}\mathrm{d}y\right)\mathrm{d}x\right)$$

$$= \frac{(-1)^n}{2\cdot n!}\left(\int_0^1\left(\int_0^1 \frac{\log^n(x)\log^n(y)}{1-yx}\mathrm{d}y\right)\mathrm{d}x - \left(\int_0^1 \frac{\log^n(x)}{1+x^2}\mathrm{d}x\right)^2 - \left(\int_0^1 \frac{x\log^n(x)}{1+x^2}\mathrm{d}x\right)^2\right)$$

$$= \frac{(-1)^n}{2\cdot n!}\left(\int_0^1\left(\int_0^1 \sum_{k=1}^{\infty}(xy)^{k-1}\log^n(x)\log^n(y)\mathrm{d}y\right)\mathrm{d}x - \left(\int_0^1 \sum_{k=1}^{\infty}(-1)^{k-1}x^{2k-2}\log^n(x)\mathrm{d}x\right)^2 - \left(\int_0^1 \sum_{k=1}^{\infty}(-1)^{k-1}x^{2k-1}\log^n(x)\mathrm{d}x\right)^2\right)$$

{reverse the order of summation and integration}

$$= \frac{(-1)^n}{2\cdot n!}\left(\sum_{k=1}^{\infty}\int_0^1\left(\int_0^1 (xy)^{k-1}\log^n(x)\log^n(y)\mathrm{d}y\right)\mathrm{d}x - \left(\sum_{k=1}^{\infty}(-1)^{k-1}\int_0^1 x^{2k-2}\log^n(x)\mathrm{d}x\right)^2 - \left(\sum_{k=1}^{\infty}(-1)^{k-1}\int_0^1 x^{2k-1}\log^n(x)\mathrm{d}x\right)^2\right)$$

{make use of the result in (1.2)}

$$
\begin{aligned}
&= \frac{1}{2}(-1)^n n!\Bigg(\sum_{k=1}^{\infty}\frac{1}{k^{2n+2}} \\
&\quad -\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^{n+1}}\right)^2-\left(\frac{1}{2^{n+1}}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^{n+1}}\right)^2\Bigg) \\
&= (-1)^n\frac{n!}{2}\left(\zeta(2n+2)-(\beta(n+1))^2-\frac{1}{4^{n+1}}(\eta(n+1))^2\right),
\end{aligned}
$$

and the point $i)$ of the problem is finalized.

To calculate the integral from the point $ii)$, we need the result in (3.168), where we replace Y by $-Y$, and then we have

$$
\begin{aligned}
&\frac{X^2}{(1+X^2)(1+YX)}+\frac{Y^2}{(1+Y^2)(1+YX)} \\
&= \frac{1}{1+YX}-\frac{1}{(1+X^2)(1+Y^2)}+\frac{XY}{(1+X^2)(1+Y^2)}.
\end{aligned}
\tag{3.169}
$$

Then, based on the integral in Sect. 1.6, we write

$$
\begin{aligned}
\int_0^1 \frac{x\log^n(x)\operatorname{Li}_{n+1}(-x)}{1+x^2}\mathrm{d}x &= \frac{(-1)^{n-1}}{n!}\int_0^1\left(\int_0^1\frac{x^2\log^n(x)\log^n(y)}{(1+x^2)(1+yx)}\mathrm{d}y\right)\mathrm{d}x \\
&\qquad \{\text{make use of the result in (3.169)}\} \\
&= \frac{(-1)^{n-1}}{2\cdot n!}\Bigg(\int_0^1\left(\int_0^1\frac{\log^n(x)\log^n(y)}{1+xy}\mathrm{d}y\right)\mathrm{d}x \\
&\quad -\int_0^1\left(\int_0^1\frac{\log^n(x)\log^n(y)}{(1+x^2)(1+y^2)}\mathrm{d}y\right)\mathrm{d}x \\
&\quad +\int_0^1\left(\int_0^1\frac{xy\log^n(x)\log^n(y)}{(1+x^2)(1+y^2)}\mathrm{d}y\right)\mathrm{d}x\Bigg) \\
&= \frac{(-1)^{n-1}}{2\cdot n!}\Bigg(\int_0^1\left(\int_0^1\frac{\log^n(x)\log^n(y)}{1+xy}\mathrm{d}y\right)\mathrm{d}x \\
&\quad -\left(\int_0^1\frac{\log^n(x)}{1+x^2}\mathrm{d}x\right)^2+\left(\int_0^1\frac{x\log^n(x)}{1+x^2}\mathrm{d}x\right)^2\Bigg) \\
&= \frac{(-1)^{n-1}}{2\cdot n!}\Bigg(\int_0^1\left(\int_0^1\sum_{k=1}^{\infty}(-1)^{k-1}(xy)^{k-1}\log^n(x)\log^n(y)\mathrm{d}y\right)\mathrm{d}x
\end{aligned}
$$

$$-\left(\int_0^1 \sum_{k=1}^{\infty}(-1)^{k-1}x^{2k-2}\log^n(x)\mathrm{d}x\right)^2+\left(\int_0^1 \sum_{k=1}^{\infty}(-1)^{k-1}x^{2k-1}\log^n(x)\mathrm{d}x\right)^2\Bigg)$$

{reverse the order of summation and integration}

$$=\frac{(-1)^{n-1}}{2\cdot n!}\left(\sum_{k=1}^{\infty}(-1)^{k-1}\int_0^1\left(\int_0^1 (xy)^{k-1}\log^n(x)\log^n(y)\mathrm{d}y\right)\mathrm{d}x\right.$$

$$\left.-\left(\sum_{k=1}^{\infty}(-1)^{k-1}\int_0^1 x^{2k-2}\log^n(x)\mathrm{d}x\right)^2+\left(\sum_{k=1}^{\infty}(-1)^{k-1}\int_0^1 x^{2k-1}\log^n(x)\mathrm{d}x\right)^2\right)$$

{make use of the result in (1.2)}

$$=(-1)^{n-1}\frac{n!}{2}\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^{2n+2}}-\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^{n+1}}\right)^2+\left(\frac{1}{2^{n+1}}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^{n+1}}\right)^2\right)$$

$$=(-1)^n\frac{n!}{2}\left((\beta(n+1))^2-\eta(2n+2)-\frac{1}{4^{n+1}}(\eta(n+1))^2\right),$$

and the part ii) of the problem is finalized.

One will find useful the special values of the Dirichlet beta function given in [101] when trying to get some particular integrals for some values of n. Also, it's good to know the specialized paper in [39] on the Polygamma function values for the arguments $x = 1/4$ and $x = 3/4$, where connections between some Polygamma function values and Dirichlet beta function values are shown.

The story of the solutions involving the symmetry won't end here, and in some of the next sections we might want to use again the symmetry-related ideas. So, let's always be prepared for exploiting the symmetry!

3.31 Two Families of Special Polylogarithmic Integrals Expressed in Terms of Infinite Series with the Generalized Harmonic Number and the Tails of Some Functions

Solution Sometimes we might prefer to establish relations between integrals and series instead of trying to find the precise values of some integrals and series. It is exactly what we will do in this section, and will try to establish beautiful series representations for the two polylogarithmic integrals.

For both integrals, we may exploit some simple results with integrals we'll meet in the next section. Thus, for the first point of the problem we'll exploit the integral result in (3.182),

$$\int_0^1 \frac{x^{2n-2}}{(1+x^2)^n} \mathrm{d}x = \frac{(2n-3)!!}{(2n-2)!!} \left(\frac{\pi}{4} - \sum_{i=1}^{n-1} \frac{(2i-2)!!}{2^i (2i-1)!!} \right), \tag{3.170}$$

where $n!!$ is the double factorial (see [103]), $n!! = \begin{cases} n \cdot (n-2) \cdots 5 \cdot 3 \cdot 1, & n > 0 \text{ odd;} \\ n \cdot (n-2) \cdots 6 \cdot 4 \cdot 2, & n > 0 \text{ even;} \\ 1, & n = -1, 0. \end{cases}$

A first observation based upon the result in (3.170) is that $\displaystyle\sum_{n=1}^{\infty} \frac{(2n-2)!!}{2^n (2n-1)!!} = \frac{\pi}{4}$. To see this is true, we may consider $n \geq 2$, and we write

$$0 \leq \frac{\pi}{4} - \sum_{i=1}^{n-1} \frac{(2i-2)!!}{2^i (2i-1)!!} = \frac{(2n-2)!!}{(2n-3)!!} \int_0^1 \frac{x^{2n-2}}{(1+x^2)^n} \mathrm{d}x$$

$$= \frac{2^{2n}}{\binom{2n}{n}} \frac{2n-1}{2n} \int_0^1 \frac{x^{2n-2}}{(1+x^2)^n} \mathrm{d}x \leq \frac{2^{2n}}{\binom{2n}{n}} \frac{2n-1}{2n} \int_0^1 \frac{x}{(1+x^2)^n} \mathrm{d}x$$

$$= \frac{2^{2n}}{\binom{2n}{n}} \frac{2n-1}{2n} \left(\frac{1}{2(n-1)} - \frac{1}{2^n (n-1)} \right),$$

where if we consider the asymptotic expansion behavior of the central binomial coefficient, $\binom{2n}{n} \approx \dfrac{4^n}{\sqrt{\pi n}}$, as $n \to \infty$, we obtain the desired value,

$$\sum_{n=1}^{\infty} \frac{(2n-2)!!}{2^n (2n-1)!!} = \frac{\pi}{4}. \tag{3.171}$$

But how could we derive the stated asymptotical behavior of the central binomial coefficient? At this point we want to recollect the Scottish mathematician James Stirling (1692–1770) and his incredibly useful approximation known today as Stirling's approximation or Stirling's formula (see [1, p. 257], [9, p. 21], [34, p. 86–88], [33, Chapter 2, pp. 165–167], [78, Chapter 1, p. 8]),

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e} \right)^n. \tag{3.172}$$

It's straightforward to see that writing $\displaystyle\binom{2n}{n}=\frac{(2n)!}{(n!)^2}$ and then applying the Stirling's approximation in (3.172), we arrive immediately at $\displaystyle\binom{2n}{n}\approx\frac{4^n}{\sqrt{\pi n}}$ (see [40, p. 110]).

Alternatively, it's easy to see that the value of the series in (3.171) is obtained directly by employing the result in (4.1).

Now, using the result in (3.171), we may write the initial result in (3.170) as

$$\begin{aligned}\int_0^1 \frac{x^{2n-2}}{(1+x^2)^n}\mathrm{d}x &= \frac{(2n-3)!!}{(2n-2)!!}\left(\frac{\pi}{4}-\sum_{i=1}^{n-1}\frac{(2i-2)!!}{2^i(2i-1)!!}\right)\\ &= \frac{(2n-3)!!}{(2n-2)!!}\left(\sum_{i=n}^{\infty}\frac{(2i-2)!!}{2^i(2i-1)!!}\right),\end{aligned}$$

and if we use $\displaystyle\frac{(2n)!!}{(2n+1)!!}=\frac{2^{2n}}{(2n+1)\binom{2n}{n}}$ and $\displaystyle\frac{(2n-1)!!}{(2n)!!}=\frac{1}{2^{2n}}\binom{2n}{n}$, we get

$$\int_0^1 \frac{x^{2n-2}}{(1+x^2)^n}\mathrm{d}x = \frac{n}{(2n-1)2^{2n}}\binom{2n}{n}\left(\sum_{k=n}^{\infty}\frac{2^k}{k\binom{2k}{k}}\right). \tag{3.173}$$

Further, upon multiplying both sides of (3.173) by $H_{n-1}^{(m)}$ and then considering the summation from $n=1$ to ∞, we obtain that

$$\sum_{n=1}^{\infty}\frac{nH_{n-1}^{(m)}}{(2n-1)2^{2n}}\binom{2n}{n}\left(\sum_{k=n}^{\infty}\frac{2^k}{k\binom{2k}{k}}\right)=\sum_{n=1}^{\infty}\int_0^1\frac{x^{2n-2}H_{n-1}^{(m)}}{(1+x^2)^n}\mathrm{d}x$$

$$\{\text{reverse the order of integration and summation}\}$$

$$=\int_0^1\sum_{n=1}^{\infty}\frac{x^{2n-2}H_{n-1}^{(m)}}{(1+x^2)^n}\mathrm{d}x=\int_0^1\operatorname{Li}_m\left(\frac{x^2}{1+x^2}\right)\mathrm{d}x,$$

where for getting the last equality, I made use of the generating function in (4.6), and the part i) of the problem is finalized.

For the second point of the problem we use that

$$\int_0^1 \frac{x^{n-1}}{(1+x)^n}\mathrm{d}x \overset{t=x/(1+x)}{=} \int_0^{1/2} \frac{t^{n-1}}{1-t}\mathrm{d}t = \int_0^{1/2}\left(\frac{1}{1-t} - \sum_{k=1}^{n-1} t^{k-1}\right)\mathrm{d}t$$

$$= \log(2) - \sum_{k=1}^{n-1}\frac{1}{k2^k}. \tag{3.174}$$

Multiplying both sides of (3.174) by $H_{n-1}^{(m)}$ and then considering the sum from $n = 1$ to ∞, we get

$$\sum_{n=1}^{\infty} H_{n-1}^{(m)}\left(\log(2) - \sum_{k=1}^{n-1}\frac{1}{k2^k}\right) = \sum_{n=1}^{\infty} H_{n-1}^{(m)} \int_0^1 \frac{x^{n-1}}{(1+x)^n}\mathrm{d}x$$

{reverse the order of integration and summation}

$$= \int_0^1 \sum_{n=1}^{\infty} H_{n-1}^{(m)} \frac{x^{n-1}}{(1+x)^n}\mathrm{d}x$$

{make use of the generating function in (4.6)}

$$= \int_0^1 \mathrm{Li}_m\left(\frac{x}{1+x}\right)\mathrm{d}x,$$

and the part ii) of the problem is finalized.

For example, in the chapter *Sums and Series* there is a section where one (you have to discover on your own which one) of these identities is proving to be very useful at expressing some polylogarithmic values in terms of fascinating sums of infinite series. If you enjoy long sums of series, I guess you'll have a nice time in there!

3.32 A Generalized Integral Beautifully Connected to a Spectacular (and Simultaneously Strange) Series

Solution A natural, immediate question will concern the origin of such an integral. *Why in this form? Is it just an example off the top of the author's head maybe?* In fact, the integral stems from the work on a multiple series we'll meet in the chapter *Sums and Series*, which can be brought to this form. We'll want to attack it in a simple way, by rearranging the integral and using the recurrence relations.

For the beginning, we write

$$I_n = \int_0^1 \frac{x^{2n}}{(1+x)(1+x^2)^n}\mathrm{d}x = \int_0^1 \frac{x^{2n}+x^{2n-1}-x^{2n-1}-x^{2n-2}+x^{2n-2}}{(1+x)(1+x^2)^n}\mathrm{d}x$$

$$= \int_0^1 \frac{x^{2n}+x^{2n-1}}{(1+x)(1+x^2)^n}\mathrm{d}x - \int_0^1 \frac{x^{2n-1}+x^{2n-2}}{(1+x)(1+x^2)^n}\mathrm{d}x + \int_0^1 \frac{x^{2n-2}}{(1+x)(1+x^2)^n}\mathrm{d}x$$

$$= \underbrace{\int_0^1 \frac{x^{2n-1}}{(1+x^2)^n}\mathrm{d}x}_{J_n} - \underbrace{\int_0^1 \frac{x^{2n-2}}{(1+x^2)^n}\mathrm{d}x}_{K_n} + \int_0^1 \frac{x^{2n-2}}{(1+x)(1+x^2)^n}\mathrm{d}x. \tag{3.175}$$

Since we have that $\dfrac{1}{(1+x)(1+x^2)} = \dfrac{1}{2(1+x)} + \dfrac{1-x}{2\left(1+x^2\right)}$, then $\dfrac{1}{(1+x)(1+x^2)^n} = \dfrac{1}{2(1+x)(1+x^2)^{n-1}} + \dfrac{1-x}{2(1+x^2)^n}$, which gives for the last integral in (3.175) that

$$\int_0^1 \frac{x^{2n-2}}{(1+x)(1+x^2)^n}\mathrm{d}x$$

$$= \frac{1}{2}\underbrace{\int_0^1 \frac{x^{2n-2}}{(1+x)(1+x^2)^{n-1}}\mathrm{d}x}_{I_{n-1}} + \frac{1}{2}\underbrace{\int_0^1 \frac{x^{2n-2}}{(1+x^2)^n}\mathrm{d}x}_{K_n} - \frac{1}{2}\underbrace{\int_0^1 \frac{x^{2n-1}}{(1+x^2)^n}\mathrm{d}x}_{J_n}. \tag{3.176}$$

If we plug the result from (3.176) in (3.175), we get

$$I_n = \frac{1}{2}J_n - \frac{1}{2}K_n + \frac{1}{2}I_{n-1}. \tag{3.177}$$

For the integral J_n in (3.177), we integrate by parts, and we write

$$J_n = \int_0^1 \frac{x^{2n-1}}{(1+x^2)^n}\mathrm{d}x = \int_0^1 \left(-\frac{1}{2(n-1)(1+x^2)^{n-1}}\right)' x^{2n-2}\mathrm{d}x$$

$$= \underbrace{-\frac{x^{2n-2}}{2(n-1)(1+x^2)^{n-1}}\bigg|_{x=0}^{x=1}}_{-1/((n-1)2^n)} + \underbrace{\int_0^1 \frac{x^{2n-3}}{(1+x^2)^{n-1}}\mathrm{d}x}_{J_{n-1}},$$

whence we get

$$J_n - J_{n-1} = -\frac{1}{(n-1)2^n}. \tag{3.178}$$

Further, replacing n by k in (3.178), then giving values to k from $k = 2$ to n and summing up the resulting relations, we obtain $J_n - J_1 = -\sum_{k=2}^{n} \frac{1}{(k-1)2^k} = -\sum_{k=1}^{n-1} \frac{1}{k2^{k+1}}$, and since $J_1 = \int_0^1 \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\log(2)$, we get

$$J_n = \frac{1}{2}\log(2) - \sum_{k=1}^{n-1} \frac{1}{k2^{k+1}}. \tag{3.179}$$

Then, for the integral K_n in (3.177), we integrate by parts, and we have

$$K_n = \int_0^1 \frac{x^{2n-2}}{(1+x^2)^n}\mathrm{d}x = \int_0^1 \left(-\frac{1}{2(n-1)(1+x^2)^{n-1}}\right)' x^{2n-3}\mathrm{d}x$$

$$= \underbrace{-\frac{x^{2n-3}}{2(n-1)(1+x^2)^{n-1}}\bigg|_{x=0}^{x=1}}_{-1/((n-1)2^n)} + \frac{2n-3}{2n-2}\underbrace{\int_0^1 \frac{x^{2n-4}}{(1+x^2)^{n-1}}\mathrm{d}x}_{K_{n-1}},$$

whence we get that $K_n - \frac{2n-3}{2n-2}K_{n-1} = -\frac{1}{(n-1)2^n}$, and if rearranged, then

$$\frac{2n-2}{2n-3}K_n - K_{n-1} = -\frac{1}{2^{n-1}(2n-3)}. \tag{3.180}$$

Multiplying both sides of (3.180) by $\frac{(2n-4)!!}{(2n-5)!!}$, we have

$$\frac{(2n-2)!!}{(2n-3)!!}K_n - \frac{(2n-4)!!}{(2n-5)!!}K_{n-1} = -\frac{(2n-4)!!}{2^{n-1}(2n-3)!!}. \tag{3.181}$$

Replacing n by i in (3.181), giving values to i from $i = 2$ to n and then summing the resulting relations, we get

$$\frac{(2n-2)!!}{(2n-3)!!}K_n - K_1 = -\sum_{i=2}^{n} \frac{(2i-4)!!}{2^{i-1}(2i-3)!!},$$

or if we consider that $K_1 = \int_0^1 \frac{1}{1+x^2}\mathrm{d}x = \frac{\pi}{4}$, and then reindex the sum, we have

$$K_n = \frac{(2n-3)!!}{(2n-2)!!}\left(\frac{\pi}{4} - \sum_{i=1}^{n-1} \frac{(2i-2)!!}{2^i(2i-1)!!}\right). \tag{3.182}$$

Returning to (3.177), rearranging it as $I_n - \dfrac{1}{2}I_{n-1} = \dfrac{1}{2}J_n - \dfrac{1}{2}K_n$, and then multiplying both sides by 2^n, we obtain

$$2^n I_n - 2^{n-1}I_{n-1} = 2^{n-1}J_n - 2^{n-1}K_n. \tag{3.183}$$

Replacing n by j in (3.183), giving values to j from $j = 2$ to n and then summing all the resulting relations, we get

$$\sum_{j=2}^{n}(2^j I_j - 2^{j-1}I_{j-1}) = 2^n I_n - 2I_1 = \sum_{j=2}^{n} 2^{j-1}J_j - \sum_{j=2}^{n} 2^{j-1}K_j,$$

or, if we use that $I_1 = \displaystyle\int_0^1 \frac{x^2}{(1+x)(1+x^2)}\mathrm{d}x = \frac{1}{4}\left(3\log(2) - \frac{\pi}{2}\right)$, then

$$I_n = \frac{1}{2^{n+1}}\left(3\log(2) - \frac{\pi}{2}\right) + \frac{1}{2^n}\sum_{j=2}^{n} 2^{j-1}J_j - \frac{1}{2^n}\sum_{j=2}^{n} 2^{j-1}K_j. \tag{3.184}$$

If we plug the values from (3.179) and (3.182) in (3.184), we get

$$I_n = \frac{1}{2^{n+1}}\left(3\log(2) - \frac{\pi}{2}\right) + \frac{1}{2^n}\sum_{j=2}^{n} 2^{j-1}\left(\frac{1}{2}\log(2) - \sum_{k=1}^{j-1}\frac{1}{k2^{k+1}}\right)$$

$$-\frac{1}{2^n}\sum_{j=2}^{n} 2^{j-1}\frac{(2j-3)!!}{(2j-2)!!}\left(\frac{\pi}{4} - \sum_{i=1}^{j-1}\frac{(2i-2)!!}{2^i(2i-1)!!}\right)$$

$$\{\text{reindex the 2 outer sums}\}$$

$$= \frac{1}{2^{n+1}}\left(3\log(2) - \frac{\pi}{2}\right) + \frac{1}{2^n}\sum_{j=1}^{n-1} 2^{j}\left(\frac{1}{2}\log(2) - \sum_{i=1}^{j}\frac{1}{i2^{i+1}}\right)$$

$$-\frac{1}{2^n}\sum_{j=1}^{n-1} 2^{j}\frac{(2j-1)!!}{(2j)!!}\left(\frac{\pi}{4} - \sum_{i=1}^{j}\frac{(2i-2)!!}{2^i(2i-1)!!}\right)$$

$$= \frac{2^{n+1}\log(2) + 2\log(2) - \pi}{2^{n+2}} - \frac{1}{2^n}\sum_{j=1}^{n-1}\sum_{i=1}^{j}\frac{2^j}{i2^{i+1}} - \frac{\pi}{2^{n+2}}\sum_{j=1}^{n-1} 2^j\frac{(2j-1)!!}{(2j)!!}$$

$$+\frac{1}{2^n}\sum_{j=1}^{n-1}\sum_{i=1}^{j}\frac{(2i-2)!!(2j-1)!!}{2^{i-j}(2i-1)!!(2j)!!},$$

and since by reversing the summation order[12](see [17, Exercise **1.6.2**, p. 22], [31, p. 36]) we have $\displaystyle\sum_{j=1}^{n-1}\sum_{i=1}^{j}\frac{2^j}{i2^{i+1}}=\sum_{i=1}^{n-1}\sum_{j=i}^{n-1}\frac{2^j}{i2^{i+1}}=\sum_{i=1}^{n-1}\frac{2^n-2^i}{i2^{i+1}}=2^n\sum_{i=1}^{n-1}\frac{1}{i2^{i+1}}-\frac{1}{2}H_{n-1}$, we conclude that

$$I_n=\int_0^1\frac{x^{2n}}{(1+x)(1+x^2)^n}\mathrm{d}x$$

$$=\frac{1}{2}\log(2)+\frac{1}{2^{n+1}}\log(2)+\frac{H_n}{2^{n+1}}-\sum_{i=1}^{n}\frac{1}{i2^{i+1}}-\frac{\pi}{2^{n+2}}\sum_{j=0}^{n-1}\frac{1}{2^j}\binom{2j}{j}$$

$$+\frac{1}{2^{n+1}}\sum_{j=1}^{n-1}\frac{1}{2^j}\binom{2j}{j}\sum_{i=1}^{j}\frac{2^i}{i\binom{2i}{i}},$$

and the solution is complete.

The technique involving the use of the recurrence relations is often proving to be powerful, and therefore we want to be always prepared to use it when needed.

The power of the recurrence relations, while playing with integrals, is known from old times, and you may find some classical examples (like Wallis' integral) in [17, p. 113], [61, pp. 80–82].

The curious reader, who would also enjoy exploring other ways, might try to use what is known in the literature as *Snake Oil Method*, you may find described in [121, pp. 126–138], which means to calculate $\sum_{n=0}^{\infty}t^nI_n$ and get $G(t)$, and then identify the coefficients of the generating function $G(t)$ in order to extract the values of I_n.

---

[12]Given the sum $\displaystyle\sum_{j=1}^{n}\sum_{i=1}^{j}a_{(i,j)}$, by changing the summation order we get $\displaystyle\sum_{j=1}^{n}\sum_{i=1}^{j}a_{(i,j)}=\sum_{i=1}^{n}\sum_{j=i}^{n}a_{(i,j)}$. To easily understand what happens, suppose $n=3$ and put all the resulting $a_{(i,j)}$s on a grid. Then, we see immediately that in one case we first sum the terms on columns and then add all together, and in the other case we first add the terms on rows and then add all together. So, we talk about the same terms for both sums which we sum in two different ways. In the chapter *Sums and Series* we'll find particularly useful this kind of change of summation order, both in the finite form as in the case above, and also in the infinite form with $n\to\infty$, $\displaystyle\sum_{j=1}^{\infty}\left(\sum_{i=1}^{j}a_{(i,j)}\right)=\sum_{i=1}^{\infty}\left(\sum_{j=i}^{\infty}a_{(i,j)}\right)$.

3.33 A Special (and Possibly Slightly Daunting) Integral with Two Polylogarithms, $\operatorname{Li}_2\left(\frac{x}{x-1}\right)$ and $\operatorname{Li}_2\left(\frac{x}{x+1}\right)$

Solution This is one of the integrals that will bring us on the realm of the advanced alternating harmonic series, which is an exciting experience. Despite its daunting appearance, with the right tools in hands, everything will go smoothly and we will get the desired value of the integral.

For this solution we might like to combine the Landen's identity (see [44, Chapter 1, p. 5], [78, Chapter 2, p. 107]), $\operatorname{Li}_2(x)+\operatorname{Li}_2\left(\frac{x}{x-1}\right)=-\frac{1}{2}\log^2(1-x)$, the generating function in (4.10), the identity in (1.5) and the fact that $\int_0^1 x^{n-1}\operatorname{Li}_2(x)\mathrm{d}x=\int_0^1\sum_{k=1}^{\infty}\frac{x^{k+n-1}}{k^2}\mathrm{d}x=\sum_{k=1}^{\infty}\int_0^1\frac{x^{k+n-1}}{k^2}\mathrm{d}x=\sum_{k=1}^{\infty}\frac{1}{k^2(k+n)}=\frac{1}{n}\sum_{k=1}^{\infty}\frac{1}{k^2}-\frac{1}{n}\sum_{k=1}^{\infty}\frac{1}{k(k+n)}=\frac{\zeta(2)}{n}-\frac{H_n}{n^2}$, which is obtained immediately if we use the series representation of $\operatorname{Li}_2(x)$.

Then, we proceed as follows

$$\int_0^1\frac{\operatorname{Li}_2\left(\dfrac{x}{x-1}\right)\operatorname{Li}_2\left(\dfrac{x}{x+1}\right)}{x}\mathrm{d}x$$

$$=-\int_0^1\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n}x^{n-1}\left(\frac{1}{2}\log^2(1-x)+\operatorname{Li}_2(x)\right)\mathrm{d}x$$

{reverse the order of summation and integration}

$$=-\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n}\left(\frac{1}{2}\int_0^1 x^{n-1}\log^2(1-x)\mathrm{d}x+\int_0^1 x^{n-1}\operatorname{Li}_2(x)\mathrm{d}x\right)$$

$$=-\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n}\left(\frac{\zeta(2)}{n}-\frac{H_n}{n^2}+\frac{H_n^2}{2n}+\frac{H_n^{(2)}}{2n}\right)$$

$$=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2}{n^3}-\zeta(2)\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^2}-\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^3}{n^2}$$
$$-\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_nH_n^{(2)}}{n^2}$$

{the values of the series are given in (4.93), (4.88), (4.94), and (4.95)}

$$= \frac{1}{10}\log^5(2) - \frac{1}{2}\log^3(2)\zeta(2) + \frac{21}{16}\log^2(2)\zeta(3) - \frac{13}{8}\zeta(2)\zeta(3) - \frac{29}{32}\zeta(5)$$

$$+3\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right) + 3\operatorname{Li}_5\left(\frac{1}{2}\right),$$

and the solution is finalized.

Alternatively, we could use Landen's identity two times, or the generating function in (4.10) two times.

3.34 Exciting Challenging Triple Integrals with the Dilogarithm

Solution To be known from the very beginning, we play now with tough integrals. Moreover, if we want to remain in the coordinates of the simple real methods, the integrals might be more challenging, and this is the way chosen for these solutions. The strategy to *conquer* such integrals will be developed in two phases: in the first phase we want to turn each integral into a sum of series, and in the second phase, of course, we want to calculate the resulting series.

Let's observe first that $x+y+z-xy-xz-yz+xyz = 1-(1-x)(1-y)(1-z)$ and $2-x-y-z+xy+xz+yz-xyz = 1+(1-x)(1-y)(1-z)$, and then we have, for the point i) of the problem, that

$$\int_0^1 \left(\int_0^1 \left(\int_0^1 \frac{\operatorname{Li}_2(x+y+z-xy-xz-yz+xyz)}{2-x-y-z+xy+xz+yz-xyz}\mathrm{d}x\right)\mathrm{d}y\right)\mathrm{d}z$$

$$= \int_0^1 \left(\int_0^1 \left(\int_0^1 \frac{\operatorname{Li}_2(1-(1-x)(1-y)(1-z))}{1+(1-x)(1-y)(1-z)}\mathrm{d}x\right)\mathrm{d}y\right)\mathrm{d}z$$

{make the changes of variable $x = 1-u$, $y = 1-v$ and $z = 1-t$, }

{and then return to the initial notation by replacing u, v, t by x, y, z}

$$= \int_0^1 \left(\int_0^1 \left(\int_0^1 \frac{\operatorname{Li}_2(1-xyz)}{1+xyz}\mathrm{d}x\right)\mathrm{d}y\right)\mathrm{d}z$$

$$\overset{yx=u}{=} \int_0^1 \left(\int_0^1 \left(\int_0^y \frac{\operatorname{Li}_2(1-uz)}{y(1+uz)}\mathrm{d}u\right)\mathrm{d}y\right)\mathrm{d}z$$

{reverse the order of integration in the inner double integrals}

$$= \int_0^1 \left(\int_0^1 \left(\int_u^1 \frac{\mathrm{Li}_2(1-uz)}{y(1+uz)} \mathrm{d}y \right) \mathrm{d}u \right) \mathrm{d}z = - \int_0^1 \left(\int_0^1 \frac{\log(u)\, \mathrm{Li}_2(1-uz)}{1+uz} \mathrm{d}u \right) \mathrm{d}z$$

$$\{\text{reverse the order of integration}\}$$

$$= - \int_0^1 \left(\int_0^1 \frac{\log(u)\, \mathrm{Li}_2(1-uz)}{1+uz} \mathrm{d}z \right) \mathrm{d}u \overset{uz=v}{=} - \int_0^1 \left(\int_0^u \frac{\log(u)\, \mathrm{Li}_2(1-v)}{u(1+v)} \mathrm{d}v \right) \mathrm{d}u$$

$$\{\text{reverse the order of integration}\}$$

$$= - \int_0^1 \left(\int_v^1 \frac{\log(u)\, \mathrm{Li}_2(1-v)}{u(1+v)} \mathrm{d}u \right) \mathrm{d}v = \frac{1}{2} \int_0^1 \frac{\log^2(v)\, \mathrm{Li}_2(1-v)}{1+v} \mathrm{d}v. \tag{3.185}$$

Based upon the use of the Dilogarithm function reflection formula (see [44, Chapter 1, p. 5], [100, (5)], [78, Chapter 2, p. 107]), $\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \zeta(2) - \log(x)\log(1-x)$, we get

$$\int_0^1 \frac{\log^2(v)\, \mathrm{Li}_2(1-v)}{1+v} \mathrm{d}v$$

$$= \zeta(2) \underbrace{\int_0^1 \frac{\log^2(v)}{1+v} \mathrm{d}v}_{I_1} - \underbrace{\int_0^1 \frac{\log^2(v)\, \mathrm{Li}_2(v)}{1+v} \mathrm{d}v}_{I_2} - \underbrace{\int_0^1 \frac{\log^3(v) \log(1-v)}{1+v} \mathrm{d}v}_{I_3}. \tag{3.186}$$

For the first integral in (3.186), we have

$$I_1 = \int_0^1 \frac{\log^2(v)}{1+v} \mathrm{d}v = \int_0^1 \sum_{n=1}^{\infty} (-1)^{n-1} v^{n-1} \log^2(v) \mathrm{d}v$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \int_0^1 v^{n-1} \log^2(v) \mathrm{d}v$$

$$= 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} = \frac{3}{2} \zeta(3). \tag{3.187}$$

Next, for the second integral in (3.186), we write

$$I_2 = \int_0^1 \frac{\log^2(v)\, \mathrm{Li}_2(v)}{1+v} \mathrm{d}v = \int_0^1 \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} (-1)^{n-1} \frac{v^{k+n-1}}{k^2} \right) \log^2(v) \mathrm{d}v$$

$$\{\text{reverse the order of summation and integration}\}$$

$$= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^{n-1}\frac{1}{k^2}\int_0^1 v^{k+n-1}\log^2(v)\mathrm{d}v\right) = 2\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^{n-1}\frac{1}{k^2(k+n)^3}\right)$$

$$= 2\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^{n-1}\frac{1}{k^2n^3}\right) - 6\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^{n-1}\frac{1}{n^4}\left(\frac{1}{k}-\frac{1}{k+n}\right)\right)$$

$$+2\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^{n-1}\frac{1}{n^2(k+n)^3}\right) + 4\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^{n-1}\frac{1}{n^3(k+n)^2}\right)$$

$$= \frac{3}{2}\zeta(2)\zeta(3) - 6\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^4} + 2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}(\zeta(3)-H_n^{(3)})$$

$$+4\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^3}(\zeta(2)-H_n^{(2)}) = \frac{3}{2}\zeta(2)\zeta(3) + 2\zeta(3)\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$$

$$+4\zeta(2)\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3} - 6\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^4} - 2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(3)}}{n^2}$$

$$-4\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(2)}}{n^3}$$

{the last three series are given in (4.89), (4.91), and (4.90)}

$$= \frac{9}{2}\zeta(2)\zeta(3) - \frac{67}{8}\zeta(5). \tag{3.188}$$

Further, for the third integral in (3.186), we have

$$I_3 = \int_0^1 \frac{\log^3(v)\log(1-v)}{1+v}\mathrm{d}v = -\int_0^1 \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^{n-1}\frac{v^{k+n-1}}{k}\right)\log^3(v)\mathrm{d}v$$

{reverse the order of summation and integration}

$$= -\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^{n-1}\frac{1}{k}\int_0^1 v^{k+n-1}\log^3(v)\mathrm{d}v\right) = 6\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^{n-1}\frac{1}{k(k+n)^4}\right)$$

$$= 6\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^{n-1}\frac{1}{n^4}\left(\frac{1}{k}-\frac{1}{k+n}\right)\right) - 6\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^{n-1}\frac{1}{n(k+n)^4}\right)$$

$$-6\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^{n-1}\frac{1}{n^2(k+n)^3}\right)-6\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^{n-1}\frac{1}{n^3(k+n)^2}\right)$$

$$=6\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^4}-6\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}(\zeta(4)-H_n^{(4)})$$

$$-6\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}(\zeta(3)-H_n^{(3)})-6\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^3}(\zeta(2)-H_n^{(2)})$$

$$=6\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^4}+6\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(4)}}{n}+6\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(3)}}{n^2}+6\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(2)}}{n^3}$$

$$-6\zeta(4)\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}-6\zeta(3)\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}-6\zeta(2)\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}$$

{the first four series are given in (4.89), (4.92), (4.91), and (4.90)}

$$=\frac{273}{16}\zeta(5)-\frac{45}{4}\log(2)\zeta(4)-\frac{9}{2}\zeta(2)\zeta(3). \tag{3.189}$$

If we plug the results from (3.187), (3.188), and (3.189) in (3.186), we obtain

$$\int_0^1\frac{\log^2(v)\operatorname{Li}_2(1-v)}{1+v}\mathrm{d}v=\frac{45}{4}\log(2)\zeta(4)+\frac{3}{2}\zeta(2)\zeta(3)-\frac{139}{16}\zeta(5), \tag{3.190}$$

and by plugging the result from (3.190) in (3.185), we conclude that

$$\int_0^1\int_0^1\int_0^1\frac{\operatorname{Li}_2(x+y+z-xy-xz-yz+xyz)}{2-x-y-z+xy+xz+yz-xyz}\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$=\frac{1}{4}\left(\frac{45}{2}\log(2)\zeta(4)+3\zeta(2)\zeta(3)-\frac{139}{8}\zeta(5)\right),$$

and the part i) of the problem is finalized.

For the part ii) of the problem we proceed as before, and then we write

$$\int_0^1\left(\int_0^1\left(\int_0^1\frac{(\operatorname{Li}_2(x+y+z-xy-xz-yz+xyz))^2}{x+y+z-xy-xz-yz+xyz}\mathrm{d}x\right)\mathrm{d}y\right)\mathrm{d}z$$

{make the changes of variable $x=1-u$, $y=1-v$ and $z=1-t$}

{and then return to the initial notation by replacing u, v, t by x, y, z}

$$= \int_0^1 \left(\int_0^1 \left(\int_0^1 \frac{(\operatorname{Li}_2(1 - xyz))^2}{1 - xyz} \mathrm{d}x \right) \mathrm{d}y \right) \mathrm{d}z$$

{make the change of variable $yx = u$}

$$= \int_0^1 \left(\int_0^1 \left(\int_0^y \frac{(\operatorname{Li}_2(1 - uz))^2}{y(1 - uz)} \mathrm{d}u \right) \mathrm{d}y \right) \mathrm{d}z$$

{reverse the order of integration in the inner double integrals}

$$= \int_0^1 \left(\int_0^1 \left(\int_u^1 \frac{(\operatorname{Li}_2(1 - uz))^2}{y(1 - uz)} \mathrm{d}y \right) \mathrm{d}u \right) \mathrm{d}z$$

$$= - \int_0^1 \left(\int_0^1 \frac{\log(u)\,(\operatorname{Li}_2(1 - uz))^2}{1 - uz} \mathrm{d}u \right) \mathrm{d}z$$

{reverse the order of integration}

$$= \ - \int_0^1 \left(\int_0^1 \frac{\log(u)\,(\operatorname{Li}_2(1 - uz))^2}{1 - uz} \mathrm{d}z \right) \mathrm{d}u$$

$$\overset{uz=t}{=} - \int_0^1 \left(\int_0^u \frac{\log(u)\,(\operatorname{Li}_2(1 - t))^2}{u(1 - t)} \mathrm{d}t \right) \mathrm{d}u$$

{reverse the order of integration}

$$= - \int_0^1 \left(\int_t^1 \frac{\log(u)\,(\operatorname{Li}_2(1 - t))^2}{u(1 - t)} \mathrm{d}u \right) \mathrm{d}t = \frac{1}{2} \int_0^1 \frac{\log^2(t)\,(\operatorname{Li}_2(1 - t))^2}{1 - t} \mathrm{d}t$$

{make the change of variable $1 - t = s$}

$$= \frac{1}{2} \int_0^1 \frac{\log^2(1 - s)\,(\operatorname{Li}_2(s))^2}{s} \mathrm{d}s. \tag{3.191}$$

Now, we want to turn the remaining integral in (3.191) into a sum of harmonic series, and we write

$$\int_0^1 \frac{\log^2(1 - s)\,(\operatorname{Li}_2(s))^2}{s} \mathrm{d}s = 2 \int_0^1 \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} s^{i+j+k} \frac{H_i}{(i+1) j^2 k^2} \right) \right) \mathrm{d}s$$

$$\{\text{reverse the order of summation and integration}\}$$

$$= 2\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty}\frac{H_i}{(i+1)j^2k^2}\int_0^1 s^{i+j+k}\mathrm{d}s\right)\right)$$

$$= 2\sum_{j=1}^{\infty}\frac{1}{j^2}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\left(\sum_{i=1}^{\infty}\frac{H_i}{(i+1)(i+j+k+1)}\right)\right)$$

$$\{\text{make use of the application in (4.21), the case } m=1\}$$

$$= \sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{H_{j+k}^2+H_{j+k}^{(2)}}{j^2k^2(j+k)}\right)$$

$$\{\text{make the change of variable } j+k=l\}$$

$$= \sum_{l=2}^{\infty}\left(\sum_{k=1}^{l-1}\frac{H_l^2+H_l^{(2)}}{(l-k)^2k^2l}\right) = \sum_{l=1}^{\infty}\frac{H_l^2+H_l^{(2)}}{l}\left(\sum_{k=1}^{l-1}\frac{1}{(l-k)^2k^2}\right)$$

$$= \sum_{l=1}^{\infty}\frac{H_l^2+H_l^{(2)}}{l}\left(\frac{2}{l^3}\sum_{k=1}^{l-1}\frac{1}{k}+\frac{2}{l^3}\sum_{k=1}^{l-1}\frac{1}{l-k}+\frac{1}{l^2}\sum_{k=1}^{l-1}\frac{1}{k^2}+\frac{1}{l^2}\sum_{k=1}^{l-1}\frac{1}{(l-k)^2}\right)$$

$$= \sum_{l=1}^{\infty}\frac{H_l^2+H_l^{(2)}}{l}\left(4\frac{H_{l-1}}{l^3}+2\frac{H_{l-1}^{(2)}}{l^2}\right) = \sum_{l=1}^{\infty}\frac{H_l^2+H_l^{(2)}}{l}\left(4\frac{H_l}{l^3}+2\frac{H_l^{(2)}}{l^2}-\frac{6}{l^4}\right)$$

$$= 4\sum_{l=1}^{\infty}\frac{H_l^3}{l^4}+4\sum_{l=1}^{\infty}\frac{H_lH_l^{(2)}}{l^4}+2\sum_{l=1}^{\infty}\frac{H_l^2H_l^{(2)}}{l^3}+2\sum_{l=1}^{\infty}\frac{(H_l^{(2)})^2}{l^3}-6\sum_{l=1}^{\infty}\frac{H_l^2}{l^5}$$
$$-6\sum_{l=1}^{\infty}\frac{H_l^{(2)}}{l^5}$$

{the values of the series are given in (4.61), (4.58), (4.53), (4.65), (4.32), and (6.75)}

$$= \frac{65}{4}\zeta(7)-7\zeta(3)\zeta(4)-2\zeta(2)\zeta(5). \tag{3.192}$$

Finally, by plugging the result from (3.192) in (3.191), we conclude that

$$\int_0^1\int_0^1\int_0^1 \frac{(\mathrm{Li}_2(x+y+z-xy-xz-yz+xyz))^2}{x+y+z-xy-xz-yz+xyz}\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= \frac{65}{8}\zeta(7)-\frac{7}{2}\zeta(3)\zeta(4)-\zeta(2)\zeta(5),$$

and the part $ii)$ of the problem is finalized.

Alternatively, once we have arrived at the integral in (3.191), we may employ the Cauchy product of two series (see [33, Chapter III, pp. 197–199]) for $(\operatorname{Li}_2(x))^2$, that is

$$(\operatorname{Li}_2(x))^2 = 4\sum_{n=1}^{\infty} x^n \frac{H_n}{n^3} + 2\sum_{n=1}^{\infty} x^n \frac{H_n^{(2)}}{n^2} - 6\sum_{n=1}^{\infty} \frac{x^n}{n^4},$$

which we further combine with the identity in (1.5) to reduce everything to the calculation of a sum of harmonic series as above.

As in the previous section, we easily observe the greatest difficulty is represented by the evaluation of the resulting harmonic series. All the series are to be found in the next chapter, and if you have trouble with evaluating them, no worry, there is some good news: all of them are calculated! Even more than that, I provided for them solutions by series manipulations only (I hope you'll enjoy).

3.35 A Curious Integral with Polylogarithms Connected to a Double Integral with a Symmetrical Exponential Integrand

Solution Here we'll play with an integral which involves an apparently weird integrand, but despite of this, we see it actually has a nice closed-form expressed in terms of factorial and Riemann zeta function values as appeared in the statement of the question. In fact, I'll show a beautiful connection with a double integral with a symmetrical exponential integrand which is proposed in one of the next sections.

Based upon the result in (1.12), we can write that $\displaystyle\int_0^1 \frac{\log^{n-k}(u)}{1-t+tu}\mathrm{d}u = (-1)^{n-k-1}(n-k)!\frac{\operatorname{Li}_{n-k+1}\left(\frac{t}{t-1}\right)}{t}$ and $\displaystyle\int_0^1 \frac{\log^{k-1}(v)}{1-t+tv}\mathrm{d}v = (-1)^k(k-1)!\frac{\operatorname{Li}_k\left(\frac{t}{t-1}\right)}{t}$ that gives

$$\int_0^1 \sum_{k=1}^{n}(n-k)!(k-1)!\frac{\operatorname{Li}_{n-k+1}\left(\frac{t}{t-1}\right)\operatorname{Li}_k\left(\frac{t}{t-1}\right)}{t^2}\mathrm{d}t$$

$$= \sum_{k=1}^{n}\int_0^1\left(\int_0^1\left(\int_0^1 (-1)^{n-1}\frac{(\log(u))^{n-k}(\log(v))^{k-1}}{(1-t+tu)(1-t+tv)}\mathrm{d}u\right)\mathrm{d}v\right)\mathrm{d}t$$

$$\{\text{make the changes of variable } \log(u) = -x \text{ and } \log(v) = -y\}$$

$$= \sum_{k=1}^{n} \int_0^1 \left(\int_0^\infty \left(\int_0^\infty \frac{x^{n-k} y^{k-1} e^{-x} e^{-y}}{(1 - t + te^{-x})(1 - t + te^{-y})} \mathrm{d}x \right) \mathrm{d}y \right) \mathrm{d}t$$

$$\{\text{reverse the order of integration}\}$$

$$= \int_0^\infty \left(\int_0^\infty \left(\sum_{k=1}^{n} \int_0^1 \frac{x^{n-k} y^{k-1}}{(1 + (e^x - 1)(1 - t))(1 + (e^y - 1)(1 - t))} \mathrm{d}t \right) \mathrm{d}x \right) \mathrm{d}y$$

$$\{\text{make the change of variable } 1 - t = s\}$$

$$= \int_0^\infty \left(\int_0^\infty \left(\sum_{k=1}^{n} \int_0^1 \frac{x^{n-k} y^{k-1}}{(1 + (e^x - 1)s)(1 + (e^y - 1)s)} \mathrm{d}s \right) \mathrm{d}x \right) \mathrm{d}y$$

$$= \int_0^\infty \left(\int_0^\infty \sum_{k=1}^{n} x^{n-k} y^{k-1} \right.$$

$$\left. \times \left(\left. \frac{\log(1 + (e^x - 1)s) - \log(1 + (e^y - 1)s)}{e^x - e^y} \right|_{s=0}^{s=1} \right) \mathrm{d}x \right) \mathrm{d}y$$

$$= \int_0^\infty \left(\int_0^\infty \frac{x - y}{e^x - e^y} \sum_{k=1}^{n} x^{n-k} y^{k-1} \mathrm{d}x \right) \mathrm{d}y = \int_0^\infty \left(\int_0^\infty \frac{x^n - y^n}{e^x - e^y} \mathrm{d}x \right) \mathrm{d}y$$

$$\{\text{make use of the result in (3.217)}\}$$

$$= 2 \cdot n!(\zeta(2) + \zeta(3) + \cdots + \zeta(n + 1)),$$

and the solution is complete.

Once again we gladly took benefit of the usefulness of the result in (1.12) which we want to keep close to us and use it when needed. The final double integral is an already *privileged* one since it appeared on the pages of the famous journal, *The American Mathematical Monthly*, more exactly it appears inside of a limit I proposed in the mentioned journal (you'll meet the limit in one of the next sections).

Integrals involving squared Polylogarithms or a product of two Polylogarithms are known in the classical literature as you may see in [3, 28].

3.36 Double Integrals Expressed in Terms of the Exponential Function and the Polylogarithm (of Orders 2, 3, 4, 5, and 6)

Solution The integrals we treat in this section are pretty challenging without a careful approach with respect to the series behind them. One of the tools we want to use while trying to turn the integrals into series (because we prefer the series form) is the generating function of the generalized harmonic numbers in (4.6). *But how would we calculate the harmonic series behind the integrals?*

For the part i) of the problem, we have

$$\int_0^\infty \left(\int_0^\infty \frac{\log\left(1-e^{-x}\right)\left(y\operatorname{Li}_2\left(e^{-x-y}\right)+\operatorname{Li}_3\left(e^{-x-y}\right)\right)}{1-e^{x+y}}e^{x+y}\mathrm{d}x\right)\mathrm{d}y$$

$$\{\text{make the changes of variable } e^{-x}=u \text{ and } e^{-y}=v\}$$

$$=\int_0^1\left(\int_0^1 \frac{\log(1-u)\left(\log(v)\operatorname{Li}_2(uv)-\operatorname{Li}_3(uv)\right)}{uv(1-uv)}\mathrm{d}u\right)\mathrm{d}v$$

$$=\underbrace{\int_0^1\left(\int_0^1 \frac{\log(1-u)\log(v)\operatorname{Li}_2(uv)}{uv(1-uv)}\mathrm{d}u\right)\mathrm{d}v}_{I_1}$$

$$-\underbrace{\int_0^1\left(\int_0^1 \frac{\log(1-u)\operatorname{Li}_3(uv)}{uv(1-uv)}\mathrm{d}u\right)\mathrm{d}v}_{I_2}. \tag{3.193}$$

If we make use of the result in (4.6), we can write that $\displaystyle\sum_{n=1}^{\infty}(uv)^{n-1}H_n^{(2)} = \frac{\operatorname{Li}_2(uv)}{uv(1-uv)}$, and then the first integral in (3.193) becomes

$$I_1=\int_0^1\left(\int_0^1 \frac{\log(1-u)\log(v)\operatorname{Li}_2(uv)}{uv(1-uv)}\mathrm{d}u\right)\mathrm{d}v$$

$$=\int_0^1 \log(v)\left(\int_0^1 \log(1-u)\sum_{n=1}^{\infty}(uv)^{n-1}H_n^{(2)}\mathrm{d}u\right)\mathrm{d}v$$

$$\{\text{reverse the order of summation and integration}\}$$

$$= \sum_{n=1}^{\infty} H_n^{(2)} \int_0^1 v^{n-1} \log(v) \left(\int_0^1 u^{n-1} \log(1-u) \mathrm{d}u \right) \mathrm{d}v$$

{employ the result in (1.4)}

$$= \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^3}. \tag{3.194}$$

Further, for the second integral in (3.193), we make use of the result in (4.6) that gives $\sum_{n=1}^{\infty} (uv)^{n-1} H_n^{(3)} = \dfrac{\mathrm{Li}_3(uv)}{uv(1-uv)}$, and then we have

$$I_2 = \int_0^1 \left(\int_0^1 \frac{\log(1-u)\,\mathrm{Li}_3(uv)}{uv(1-uv)} \mathrm{d}u \right) \mathrm{d}v$$

$$= \int_0^1 \left(\int_0^1 \log(1-u) \sum_{n=1}^{\infty} (vu)^{n-1} H_n^{(3)} \mathrm{d}u \right) \mathrm{d}v$$

{reverse the order of summation and integration}

$$= \sum_{n=1}^{\infty} H_n^{(3)} \int_0^1 v^{n-1} \left(\int_0^1 u^{n-1} \log(1-u) \mathrm{d}u \right) \mathrm{d}v$$

{employ the result in (1.4)}

$$= -\sum_{n=1}^{\infty} \frac{H_n H_n^{(3)}}{n^2}. \tag{3.195}$$

Collecting the results from (3.194) and (3.195) in (3.193), we conclude that

$$\int_0^{\infty} \left(\int_0^{\infty} \frac{\log\left(1-e^{-x}\right)\left(y\,\mathrm{Li}_2\left(e^{-x-y}\right) + \mathrm{Li}_3\left(e^{-x-y}\right)\right)}{1-e^{x+y}} e^{x+y} \mathrm{d}x \right) \mathrm{d}y$$

$$= \sum_{n=1}^{\infty} \frac{H_n H_n^{(3)}}{n^2} + \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^3}$$

{consider the identity in (4.40)}

$$= \frac{21}{8}\zeta(6) + \zeta^2(3),$$

and the part i) of the problem is finalized.

Then, for the point $ii)$ of the problem, we write

$$\int_0^{\infty}\left(\int_0^{\infty}\frac{\log\left(1-e^{-x}\right)\left(y^2\operatorname{Li}_2\left(e^{-x-y}\right)-2\operatorname{Li}_4\left(e^{-x-y}\right)\right)}{1-e^{-x-y}}\mathrm{d}x\right)\mathrm{d}y$$

$$\{\text{make the changes of variable } e^{-x}=u \text{ and } e^{-y}=v\}$$

$$=\int_0^1\left(\int_0^1\frac{\log(1-u)\left(\log^2(v)\operatorname{Li}_2(uv)-2\operatorname{Li}_4(uv)\right)}{uv(1-uv)}\mathrm{d}u\right)\mathrm{d}v$$

$$=\underbrace{\int_0^1\left(\int_0^1\frac{\log(1-u)\log^2(v)\operatorname{Li}_2(uv)}{uv(1-uv)}\mathrm{d}u\right)\mathrm{d}v}_{J_1}$$

$$-2\underbrace{\int_0^1\left(\int_0^1\frac{\log(1-u)\operatorname{Li}_4(uv)}{uv(1-uv)}\mathrm{d}u\right)\mathrm{d}v}_{J_2}. \tag{3.196}$$

For the integral J_1 in (3.196), we use that $\displaystyle\sum_{n=1}^{\infty}(uv)^{n-1}H_n^{(2)}=\frac{\operatorname{Li}_2(uv)}{uv(1-uv)}$, which is obtained based upon the result in (4.6), and then we have

$$J_1=\int_0^1\left(\int_0^1\frac{\log(1-u)\log^2(v)\operatorname{Li}_2(uv)}{uv(1-uv)}\mathrm{d}u\right)\mathrm{d}v$$

$$=\int_0^1\log^2(v)\left(\int_0^1\log(1-u)\sum_{n=1}^{\infty}(uv)^{n-1}H_n^{(2)}\mathrm{d}u\right)\mathrm{d}v$$

$$\{\text{reverse the order of summation and integration}\}$$

$$=\sum_{n=1}^{\infty}H_n^{(2)}\int_0^1 v^{n-1}\log^2(v)\left(\int_0^1 u^{n-1}\log(1-u)\mathrm{d}u\right)\mathrm{d}v$$

$$\{\text{make use of the result in (1.4)}\}$$

$$=-2\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^4}. \tag{3.197}$$

Further, for the integral J_2 in (3.196) using that $\sum_{n=1}^{\infty}(uv)^{n-1}H_n^{(4)} = \frac{\text{Li}_4(uv)}{uv(1-uv)}$, which is obtained based upon the result in (4.6), we write

$$J_2 = \int_0^1 \left(\int_0^1 \frac{\log(1-u)\,\text{Li}_4(uv)}{uv(1-uv)}\mathrm{d}u\right)\mathrm{d}v$$

$$= \int_0^1 \left(\int_0^1 \log(1-u)\sum_{n=1}^{\infty}(uv)^{n-1}H_n^{(4)}\mathrm{d}u\right)\mathrm{d}v$$

{reverse the order of summation and integration}

$$= \sum_{n=1}^{\infty} H_n^{(4)} \int_0^1 v^{n-1}\left(\int_0^1 u^{n-1}\log(1-u)\mathrm{d}u\right)\mathrm{d}v$$

{make use of the result in (1.4)}

$$= -\sum_{n=1}^{\infty}\frac{H_n H_n^{(4)}}{n^2}. \tag{3.198}$$

Collecting the results from (3.197) and (3.198) in (3.196), we obtain that

$$\int_0^{\infty}\left(\int_0^{\infty}\frac{\log\left(1-e^{-x}\right)\left(y^2\,\text{Li}_2\left(e^{-x-y}\right)-2\,\text{Li}_4\left(e^{-x-y}\right)\right)}{1-e^{-x-y}}\mathrm{d}x\right)\mathrm{d}y$$

$$= 2\left(\sum_{n=1}^{\infty}\frac{H_n H_n^{(4)}}{n^2} - \sum_{n=1}^{\infty}\frac{H_n H_n^{(2)}}{n^4}\right)$$

{consider the identity in (4.41)}

$$= 5\zeta(2)\zeta(5) - \frac{9}{2}\zeta(3)\zeta(4),$$

and the part $ii)$ of the problem is finalized.

Passing to the point $iii)$ of the problem, we have

$$\int_0^{\infty}\left(\int_0^{\infty}\frac{\log\left(1-e^{-x}\right)\left(y^3\,\text{Li}_2\left(e^{-x-y}\right)+6\,\text{Li}_5\left(e^{-x-y}\right)\right)}{1-e^{x+y}}e^{x+y}\mathrm{d}x\right)\mathrm{d}y$$

{make the changes of variable $e^{-x} = u$ and $e^{-y} = v$}

$$= \int_0^1 \left(\int_0^1 \frac{\log(1-u)(\log^3(v)\operatorname{Li}_2(uv) - 6\operatorname{Li}_5(uv))}{uv(1-uv)} \mathrm{d}u \right) \mathrm{d}v$$

$$= \underbrace{\int_0^1 \left(\int_0^1 \frac{\log(1-u)\log^3(v)\operatorname{Li}_2(uv)}{uv(1-uv)} \mathrm{d}u \right) \mathrm{d}v}_{K_1}$$

$$-6 \underbrace{\int_0^1 \left(\int_0^1 \frac{\log(1-u)\operatorname{Li}_5(uv)}{uv(1-uv)} \mathrm{d}u \right) \mathrm{d}v}_{K_2}. \tag{3.199}$$

Now, considering the first integral in (3.199), where we use that $\sum_{n=1}^{\infty}(uv)^{n-1}H_n^{(2)} = \frac{\operatorname{Li}_2(uv)}{uv(1-uv)}$, based upon the result in (4.6), we write

$$K_1 = \int_0^1 \left(\int_0^1 \frac{\log(1-u)\log^3(v)\operatorname{Li}_2(uv)}{uv(1-uv)} \mathrm{d}u \right) \mathrm{d}v$$

$$= \int_0^1 \log^3(v) \left(\int_0^1 \log(1-u) \sum_{n=1}^{\infty}(uv)^{n-1} H_n^{(2)} \mathrm{d}u \right) \mathrm{d}v$$

{reverse the order of summation and integration}

$$= \sum_{n=1}^{\infty} H_n^{(2)} \int_0^1 v^{n-1}\log^3(v) \left(\int_0^1 u^{n-1}\log(1-u)\mathrm{d}u \right) \mathrm{d}v$$

{make use of the result in (1.4)}

$$= 6\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^5}. \tag{3.200}$$

Further, for the integral K_2 in (3.199), we use that $\sum_{n=1}^{\infty}(uv)^{n-1}H_n^{(5)} = \frac{\operatorname{Li}_5(uv)}{uv(1-uv)}$, which is obtained based upon the result in (4.6), and then we have

$$K_2 = \int_0^1 \left(\int_0^1 \frac{\log(1-u)\operatorname{Li}_5(uv)}{uv(1-uv)} \mathrm{d}u \right) \mathrm{d}v$$

$$= \int_0^1 \left(\int_0^1 \log(1-u) \sum_{n=1}^{\infty} (uv)^{n-1} H_n^{(5)} \mathrm{d}u \right) \mathrm{d}v$$

{reverse the order of summation and integration}

$$= \sum_{n=1}^{\infty} H_n^{(5)} \int_0^1 v^{n-1} \left(\int_0^1 u^{n-1} \log(1-u) \mathrm{d}u \right) \mathrm{d}v$$

{make use of the result in (1.4)}

$$= -\sum_{n=1}^{\infty} \frac{H_n H_n^{(5)}}{n^2}. \tag{3.201}$$

Collecting the results from (3.200) and (3.201) in (3.199), we get that

$$\int_0^{\infty} \left(\int_0^{\infty} \frac{\log\left(1-e^{-x}\right)\left(y^3 \operatorname{Li}_2\left(e^{-x-y}\right) + 6 \operatorname{Li}_5\left(e^{-x-y}\right)\right)}{1-e^{x+y}} e^{x+y} \mathrm{d}x \right) \mathrm{d}y$$

$$= 6 \left(\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^5} + \sum_{n=1}^{\infty} \frac{H_n H_n^{(5)}}{n^2} \right)$$

{employ the identity in (4.42)}

$$= 3 \left(\frac{17}{36} \zeta(8) + 11 \zeta(3)\zeta(5) - 3\zeta(2)\zeta^2(3) \right),$$

and the part $iii)$ of the problem is finalized.

Lastly, for the part $iv)$ of the problem, we have

$$\int_0^{\infty} \left(\int_0^{\infty} \frac{\log\left(1-e^{-x}\right)\left(y^4 \operatorname{Li}_2\left(e^{-x-y}\right) - 24 \operatorname{Li}_6\left(e^{-x-y}\right)\right)}{1-e^{-x-y}} \mathrm{d}x \right) \mathrm{d}y$$

{make the changes of variable $e^{-x} = u$ and $e^{-y} = v$}

$$= \int_0^1 \left(\int_0^1 \frac{\log(1-u)\left(\log^4(v) \operatorname{Li}_2(uv) - 24 \operatorname{Li}_6(uv)\right)}{uv(1-uv)} \mathrm{d}u \right) \mathrm{d}v$$

$$= \underbrace{\int_0^1 \left(\int_0^1 \frac{\log(1-u) \log^4(v) \operatorname{Li}_2(uv)}{uv(1-uv)} \mathrm{d}u \right) \mathrm{d}v}_{L_1}$$

$$\underbrace{-24\int_0^1\left(\int_0^1\frac{\log(1-u)\operatorname{Li}_6(uv)}{uv(1-uv)}\mathrm{d}u\right)\mathrm{d}v}_{L_2}. \tag{3.202}$$

If we consider the first integral in (3.202), where we use that $\sum_{n=1}^{\infty}(uv)^{n-1}H_n^{(2)} = \frac{\operatorname{Li}_2(uv)}{uv(1-uv)}$, which comes from the result in (4.6), we write

$$L_1 = \int_0^1\left(\int_0^1\frac{\log(1-u)\log^4(v)\operatorname{Li}_2(uv)}{uv(1-uv)}\mathrm{d}u\right)\mathrm{d}v$$

$$= \int_0^1 \log^4(v)\left(\int_0^1 \log(1-u)\sum_{n=1}^{\infty}(uv)^{n-1}H_n^{(2)}\mathrm{d}u\right)\mathrm{d}v$$

{reverse the order of summation and integration}

$$= \sum_{n=1}^{\infty} H_n^{(2)}\int_0^1 v^{n-1}\log^4(v)\left(\int_0^1 u^{n-1}\log(1-u)\mathrm{d}u\right)\mathrm{d}v$$

{make use of the result in (1.4)}

$$= -24\sum_{n=1}^{\infty}\frac{H_n H_n^{(2)}}{n^6}. \tag{3.203}$$

As regards the second integral in (3.202), we use that $\sum_{n=1}^{\infty}(uv)^{n-1}H_n^{(6)} = \frac{\operatorname{Li}_6(uv)}{uv(1-uv)}$, based upon the result in (4.6), and then we write

$$L_2 = \int_0^1\left(\int_0^1\frac{\log(1-u)\operatorname{Li}_6(uv)}{uv(1-uv)}\mathrm{d}u\right)\mathrm{d}v$$

$$= \int_0^1\left(\int_0^1 \log(1-u)\sum_{n=1}^{\infty}(uv)^{n-1}H_n^{(6)}\mathrm{d}u\right)\mathrm{d}v$$

{reverse the order of summation and integration}

$$= \sum_{n=1}^{\infty} H_n^{(6)}\int_0^1 v^{n-1}\left(\int_0^1 u^{n-1}\log(1-u)\mathrm{d}u\right)\mathrm{d}v$$

$$\{\text{make use of the result in (1.4)}\}$$

$$= -\sum_{n=1}^{\infty} \frac{H_n H_n^{(6)}}{n^2}. \tag{3.204}$$

Collecting the results from (3.203) and (3.204) in (3.202), we obtain that

$$\int_0^{\infty} \left(\int_0^{\infty} \frac{\log\left(1-e^{-x}\right)\left(y^4 \operatorname{Li}_2\left(e^{-x-y}\right) - 24\operatorname{Li}_6\left(e^{-x-y}\right)\right)}{1-e^{-x-y}} \mathrm{d}x \right) \mathrm{d}y$$

$$= 24 \left(\sum_{n=1}^{\infty} \frac{H_n H_n^{(6)}}{n^2} - \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^6} \right)$$

$$\{\text{employ the identity in (4.43)}\}$$

$$= 24 \left(7\zeta(2)\zeta(7) - \frac{19}{3}\zeta(3)\zeta(6) - \frac{15}{4}\zeta(4)\zeta(5) + \zeta^3(3) \right),$$

and the part $iv)$ of the problem is finalized.

Now we can answer the question from the beginning of the section. During the calculations we observed another important strategy, that is I preferred to calculate both series at once, for each point of the problem, rather than trying to calculate them separately, and this move made a huge difference since it would have been much harder to try to calculate each series separately. Establishing relations amongst harmonic series and using them strategically may ease substantially the work to do for various problems, which is a very good lesson to learn (and you'll also be convinced of that while going through the next problems from the book).

3.37 Exponential Double Integrals with an Appealing Look

Solution I think you might have an *advantage* (in the sense of guessing the way to go) if you already passed through Sect. 3.17 since we'll use a similar idea. In this section we'll also try something nice for the integral from the second point, that is to calculate it *almost entirely* by geometrical interpretation! (well, not directly, but first we might like to turn it into simpler integrals).

For both points of the problem, we might like to start with a classical elementary integral, $\displaystyle\int_0^1 e^{-xt}\mathrm{d}t = \frac{1-e^{-x}}{x}$. Now, if we replace x by $x-y$ and then multiply both sides by $-e^{-y}$, we get

$$-e^{-y}\int_0^1 e^{-(x-y)t}\mathrm{d}t = \frac{e^{-x}-e^{-y}}{x-y}. \tag{3.205}$$

Therefore, for the first double integral, we write

$$\int_0^\infty \left(\int_0^\infty \left(\frac{e^{-x}-e^{-y}}{x-y}\right)\left(\frac{1-e^{-x}}{x}\right)\left(\frac{1-e^{-y}}{y}\right)\mathrm{d}x\right)\mathrm{d}y$$

$$= -\int_0^\infty \left(\int_0^\infty \left(\int_0^1 \left(\int_0^1 \left(\int_0^1 e^{-(t+u)x-(1-t+v)y}\mathrm{d}t\right)\mathrm{d}u\right)\mathrm{d}v\right)\mathrm{d}x\right)\mathrm{d}y$$

$$\{\text{reverse the order of integration}\}$$

$$= -\int_0^1 \left(\int_0^1 \left(\int_0^1 \left(\int_0^\infty \left(\int_0^\infty e^{-(t+u)x-(1-t+v)y}\mathrm{d}x\right)\mathrm{d}y\right)\mathrm{d}u\right)\mathrm{d}v\right)\mathrm{d}t$$

$$= -\int_0^1 \left(\int_0^1 \left(\int_0^1 \frac{1}{(t+u)(1-t+v)}\mathrm{d}u\right)\mathrm{d}v\right)\mathrm{d}t$$

$$= -\int_0^1 \left(\int_0^1 \frac{\log(1+t)-\log(t)}{1-t+v}\mathrm{d}v\right)\mathrm{d}t$$

$$= -\int_0^1 (\log(1+t)-\log(t))(\log(2-t)-\log(1-t))\mathrm{d}t$$

$$= -\underbrace{\int_0^1 \log(2-t)\log(1+t)\mathrm{d}t}_{I_1} + \underbrace{\int_0^1 \log(1-t)\log(1+t)\mathrm{d}t}_{I_2}$$

$$+\underbrace{\int_0^1 \log(2-t)\log(t)\mathrm{d}t}_{I_3} - \underbrace{\int_0^1 \log(t)\log(1-t)\mathrm{d}t}_{I_4}. \qquad (3.206)$$

For the integral I_1 from (3.206), we make the change of variable $\frac{1}{1+t}=u$, and then we write

$$I_1 = \int_0^1 \log(2-t)\log(1+t)\mathrm{d}t = -\int_{1/2}^1 \log\left(3-\frac{1}{u}\right)\frac{\log(u)}{u^2}\mathrm{d}u$$

$$= -\log(3)\int_{1/2}^1 \frac{\log(u)}{u^2}\mathrm{d}u - \int_{1/2}^1 \log\left(1-\frac{1}{3u}\right)\frac{\log(u)}{u^2}\mathrm{d}u$$

$$= 2\log(2)\log(3) - \log(3) + \int_{1/2}^1 \sum_{n=1}^\infty \frac{1}{n(3u)^n}\frac{\log(u)}{u^2}\mathrm{d}u$$

$$\{\text{reverse the order of integration and summation}\}$$

$$= 2\log(2)\log(3) - \log(3) + \sum_{n=1}^{\infty} \frac{1}{n3^n} \int_{1/2}^{1} \frac{\log(u)}{u^{n+2}} \mathrm{d}u$$

$$= 2\log(2)\log(3) - \log(3) - \sum_{n=1}^{\infty} \frac{1}{n3^n} + 3\sum_{n=1}^{\infty} \frac{1}{(n+1)3^{n+1}} + 3\sum_{n=1}^{\infty} \frac{1}{(n+1)^2 3^{n+1}}$$

$$+2(1-\log(2)) \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{2}{3}\right)^n - 3(1-\log(2)) \sum_{n=1}^{\infty} \frac{1}{(n+1)} \left(\frac{2}{3}\right)^{n+1}$$

$$-3\sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \left(\frac{2}{3}\right)^{n+1} = 2-4\log(2)+3\log(2)\log(3)+3\operatorname{Li}_2\left(\frac{1}{3}\right)-3\operatorname{Li}_2\left(\frac{2}{3}\right)$$

$$= 2 - 3\zeta(2) - 4\log(2) + 3\log^2(3) + 6\operatorname{Li}_2\left(\frac{1}{3}\right), \tag{3.207}$$

where for getting the last equality I also used the Dilogarithm function reflection formula[13] (see [44, Chapter 1, p. 5], [100, (5)], [78, Chapter 2, p. 107]),

$$\operatorname{Li}_2(x) + \operatorname{Li}_2(1-x) = \zeta(2) - \log(x)\log(1-x). \tag{3.208}$$

Next, the integral I_2 from (3.206) is a classical one that can be derived in more ways. This time let's choose the Beta function representation in (3.23), and we write

$$\int_{-1}^{1} (1-t)^{x-1} \log(1-t)(1+t)^{y-1} \log(1+t)\mathrm{d}t = \frac{\partial^2}{\partial x \partial y} \left(2^{x+y-1} B(x,y)\right),$$

that leads to $\displaystyle\int_{-1}^{1} \log(1-t)\log(1+t)\mathrm{d}t = \lim_{\substack{x\to 1\\ y\to 1}} \frac{\partial^2}{\partial x \partial y} \left(2^{x+y-1} B(x,y)\right)$, and if we use that $\displaystyle\int_{-1}^{1} \log(1-t)\log(1+t)\mathrm{d}t = 2\int_{0}^{1} \log(1-t)\log(1+t)\mathrm{d}t$, we get

$$I_2 = \int_0^1 \log(1-t)\log(1+t)\mathrm{d}t = \frac{1}{4} \lim_{\substack{x\to 1\\ y\to 1}} \frac{\partial^2}{\partial x \partial y} \left(2^{x+y} B(x,y)\right)$$

[13] It is easy to get the stated formula, and as explained in the first given reference, considering the integration by parts for the integral representation of the Dilogarithm function, $\operatorname{Li}_2(x) = -\displaystyle\int_0^x \frac{\log(1-t)}{t}\mathrm{d}t$, we arrive immediately at the Dilogarithm function reflection formula.

$$= 2 + \log^2(2) - 2\log(2) - \zeta(2), \tag{3.209}$$

where for the last calculations we might prefer to use *Mathematica* (or we can simply do it by hand, which is not hard to do).

Further, passing to the integral I_3 from (3.206), we have

$$I_3 = \int_0^1 \log(2-t)\log(t)\mathrm{d}t \overset{1-t=u}{=} \underbrace{\int_0^1 \log(1-u)\log(1+u)\mathrm{d}u}_{I_2}$$

$$= 2 + \log^2(2) - 2\log(2) - \zeta(2). \tag{3.210}$$

Finally, for the integral I_4 from (3.206), we get

$$I_4 = \int_0^1 \log(t)\log(1-t)\mathrm{d}t = -\int_0^1 \sum_{n=1}^{\infty} \frac{t^n}{n}\log(t)\mathrm{d}t$$

$$\{\text{reverse the order of summation and integration}\}$$

$$= -\sum_{n=1}^{\infty} \frac{1}{n}\int_0^1 t^n \log(t)\mathrm{d}t = \sum_{n=1}^{\infty} \frac{1}{n(n+1)^2} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} - \frac{1}{(n+1)^2}\right)$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) - \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} = 2 - \zeta(2). \tag{3.211}$$

Collecting the values of the integrals I_1, I_2, I_3, and I_4 from (3.207), (3.209), (3.210), and (3.211) in (3.206), we conclude that

$$\int_0^{\infty}\int_0^{\infty} \left(\frac{e^{-x} - e^{-y}}{x-y}\right)\left(\frac{1-e^{-x}}{x}\right)\left(\frac{1-e^{-y}}{y}\right)\mathrm{d}x\mathrm{d}y$$

$$= 2\zeta(2) + 2\log^2(2) - 3\log^2(3) - 6\operatorname{Li}_2\left(\frac{1}{3}\right),$$

and the part i) of the problem is finalized.

For the second part of the problem we'll proceed similarly, and using the result in (3.205), we write

$$\int_0^{\infty}\left(\int_0^{\infty} \left(\frac{e^{-x^2} - e^{-y^2}}{x^2-y^2}\right)\left(\frac{1-e^{-x^2}}{x^2}\right)\left(\frac{1-e^{-y^2}}{y^2}\right)\mathrm{d}x\right)\mathrm{d}y$$

$$= -\int_0^\infty \left(\int_0^\infty \left(\int_0^1 \left(\int_0^1 \left(\int_0^1 e^{-(t+u)x^2-(1-t+v)y^2} \mathrm{d}t\right) \mathrm{d}u\right) \mathrm{d}v\right) \mathrm{d}x\right) \mathrm{d}y$$

$$\{\text{reverse the order of integration}\}$$

$$= -\int_0^1 \left(\int_0^1 \left(\int_0^1 \left(\int_0^\infty \left(\int_0^\infty e^{-(t+u)x^2-(1-t+v)y^2} \mathrm{d}x\right) \mathrm{d}y\right) \mathrm{d}t\right) \mathrm{d}u\right) \mathrm{d}v$$

$$\{\text{make the change of variable } (t+u)x^2 = z \text{ and } (1-t+v)y^2 = w\}$$

$$= -\frac{1}{4}\int_0^1 \left(\int_0^1 \left(\int_0^1 \frac{1}{\sqrt{t+u}\sqrt{1-t+v}}\right.\right.$$

$$\left.\left.\times \left(\int_0^\infty \frac{e^{-w}}{w^{1/2}} \left(\int_0^\infty \frac{e^{-z}}{z^{1/2}} \mathrm{d}z\right) \mathrm{d}w\right) \mathrm{d}t\right) \mathrm{d}u\right) \mathrm{d}v$$

$$\left\{\text{use the fact that } \int_0^\infty x^{-1/2} e^{-x} \mathrm{d}x = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right\}$$

$$= -\frac{\pi}{4}\int_0^1 \left(\int_0^1 \left(\int_0^1 \frac{1}{\sqrt{t+u}\sqrt{1-t+v}} \mathrm{d}t\right) \mathrm{d}u\right) \mathrm{d}v$$

$$\{\text{reverse the order of integration}\}$$

$$= -\frac{\pi}{4}\int_0^1 \left(\int_0^1 \left(\int_0^1 \frac{1}{\sqrt{t+u}\sqrt{1-t+v}} \mathrm{d}u\right) \mathrm{d}v\right) \mathrm{d}t$$

$$= \frac{\pi}{2}\int_0^1 \left(\int_0^1 \frac{\sqrt{t}-\sqrt{1+t}}{\sqrt{1-t+v}} \mathrm{d}v\right) \mathrm{d}t$$

$$= \pi \int_0^1 (\sqrt{1+t}-\sqrt{t})(\sqrt{1-t}-\sqrt{2-t}) \mathrm{d}t$$

$$= \pi \underbrace{\int_0^1 \sqrt{1-t^2} \mathrm{d}t}_{J_1} - \pi \underbrace{\int_0^1 \sqrt{2+t-t^2} \mathrm{d}t}_{J_2} - \pi \underbrace{\int_0^1 \sqrt{t}\sqrt{1-t} \mathrm{d}t}_{J_3}$$

$$+ \pi \underbrace{\int_0^1 \sqrt{2-t}\sqrt{t} \mathrm{d}t}_{J_4}. \tag{3.212}$$

For the integral J_1 in (3.212), it's enough to recall the circle equation, that is $(x-a)^2+(y-b)^2=r^2$ (see [98, (11)]), where if we set $a=b=0$ and $r=1$, we get $x^2+y^2=1$ which is the equation of a circle with the center at $C(0,0)$ and radius $r=1$. Thus, the geometrical interpretation of the integral J_1 is a quarter of the area of a circle with radius $r=1$, that is $\pi/4$. Therefore, we get

$$J_1 = \int_0^1 \sqrt{1-t^2}\mathrm{d}t = \frac{\pi}{4}. \tag{3.213}$$

For the second integral in (3.212), we write

$$J_2 = \int_0^1 \sqrt{2+t-t^2}\mathrm{d}t = \int_0^1 \sqrt{\left(\frac{3}{2}\right)^2 - \left(t-\frac{1}{2}\right)^2}\mathrm{d}t$$

$$\overset{1/2-t=3/2\sin(u)}{=} \frac{9}{4}\int_{-\arcsin(1/3)}^{\arcsin(1/3)} \cos^2(u)\mathrm{d}u = \frac{9}{4}\left(\frac{u}{2}+\frac{1}{4}\sin(2u)\right)\Bigg|_{u=-\arcsin(1/3)}^{u=\arcsin(1/3)}$$

$$= \frac{9}{8}\left(2\arcsin\left(\frac{1}{3}\right) + \sin\left(2\arcsin\left(\frac{1}{3}\right)\right)\right)$$

{make use of the trigonometric identity, $\sin(2\arcsin(x)) = 2x\sqrt{1-x^2}$}

$$= \frac{1}{4}\left(2\sqrt{2} + 9\arcsin\left(\frac{1}{3}\right)\right). \tag{3.214}$$

We may also note we can give a solution by exploiting the geometrical interpretation of the integral. If drawing the graph of the integrand, it's easy to note that the area we look for may be viewed as the sum of the areas of two congruent triangles and the area of a sector, or it can be viewed as well as the sum of a rectangle area and a segment area. For instance, considering the latter version, we have the rectangle of area $\sqrt{2}$ plus the area of segment (see [99, (15)]) (the trigonometric formula is $A_s = r^2(\theta - \sin(\theta))/2)$, where $r = 3/2$ and $\theta = \pi - 2\arcsin(2\sqrt{2}/3)$, and if we use the trigonometric identity $\pi/2 = \arcsin(x) + \arcsin(\sqrt{1-x^2}),\ x > 0$, our angle is $\theta = 2\arcsin(1/3)$, which further leads to the area of segment $9/8(2\arcsin(1/3) - \sin(2\arcsin(1/3)) = 9/4\arcsin(1/3) - \sqrt{2}/2$. Thus, summing the rectangle area and the segment area, we obtain the final result, $1/4(2\sqrt{2} + 9\arcsin(1/3))$.

The calculation of the integral J_3 in (3.212) is straightforward, and we have

$$J_3 = \int_0^1 \sqrt{t(1-t)}\mathrm{d}t \overset{1/2-t=u}{=} \int_{-1/2}^{1/2} \sqrt{\left(\frac{1}{2}\right)^2 - u^2}\mathrm{d}u = \frac{\pi}{8}, \tag{3.215}$$

where I used again the geometrical interpretation of the integral, and the area we look for is half the area of a circle with radius $r = 1/2$.

Finally, for the integral J_4 in (3.212), we have

$$J_4 = \int_0^1 \sqrt{(2-t)t}\,\mathrm{d}t \overset{1-t=u}{=} \int_0^1 \sqrt{1-u^2}\,\mathrm{d}u = J_1 = \frac{\pi}{4}. \tag{3.216}$$

Hence, collecting the values of the integrals J_1, J_2, J_3, and J_4 from (3.213), (3.214), (3.215), and (3.216) in (3.212), we conclude that

$$\int_0^\infty \int_0^\infty \left(\frac{e^{-x^2}-e^{-y^2}}{x^2-y^2}\right)\left(\frac{1-e^{-x^2}}{x^2}\right)\left(\frac{1-e^{-y^2}}{y^2}\right)\mathrm{d}x\mathrm{d}y$$

$$= \frac{3}{8}\pi^2 - \frac{\sqrt{2}}{2}\pi - \frac{9}{4}\pi \arcsin\left(\frac{1}{3}\right),$$

and the part $ii)$ of the problem is finalized.

In the second part of the problem I also made use of the fact that $\int_0^\infty x^{-1/2}e^{-x}\mathrm{d}x = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. How to prove it? One way is based upon the famous Euler's reflection formula, $\Gamma(t)\Gamma(1-t) = \dfrac{\pi}{\sin(\pi t)}$, with $t = 1/2$, and we arrive immediately at $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ (observe that behind the scene it's the Gaussian integral, see [108]).

Once again, starting from simple ideas, which we further develop, we can manage to create/solve beautiful problems. Also, see the details given at the end of Sect. 3.51.

3.38 A Generalized Double Integral Involving a Symmetrical Exponential Integrand and a Limit Related to It

Solution I called this integral a *privileged* one in Sect. 3.35 where we had to use it, since it appeared in *The American Mathematical Monthly*, more exactly, it is part of the limit from the point $ii)$ which I submitted to the journal, the problem **12011** (see [91]). First we want to calculate the integral (it has a very nice value!), and then we'll pass to the limit from the point $ii)$.

For the point $i)$ of the problem, we split the integral which we denote by I_n,

$$I_n = \int_0^\infty \left(\int_0^\infty \frac{x^n-y^n}{e^x-e^y}\mathrm{d}x\right)\mathrm{d}y = \int_0^\infty \left(\int_0^y \frac{x^n-y^n}{e^x-e^y}\mathrm{d}x\right)\mathrm{d}y$$

$$+\int_0^\infty \left(\int_y^\infty \frac{x^n-y^n}{e^x-e^y}\mathrm{d}x\right)\mathrm{d}y,$$

and changing the integration order in the first integral from the right-hand side, and then switching the variables, we arrive at

$$I_n = 2\int_0^\infty \left(\int_y^\infty \frac{x^n - y^n}{e^x - e^y}\mathrm{d}x\right)\mathrm{d}y \overset{x-y=z}{=} 2\int_0^\infty \left(\int_0^\infty \frac{(y+z)^n - y^n}{e^{y+z} - e^y}\mathrm{d}z\right)\mathrm{d}y$$

$$= 2\sum_{k=1}^n \binom{n}{k}\left(\int_0^\infty \frac{z^k}{e^z-1}\mathrm{d}z\right)\left(\int_0^\infty y^{n-k}e^{-y}\mathrm{d}y\right) = 2\sum_{k=1}^n \binom{n}{k} k!(n-k)!\zeta(k+1)$$

$$= 2\cdot n!\sum_{k=1}^n \zeta(k+1), \tag{3.217}$$

where in the calculation I used the well-known integral representation[14] of the product $\zeta(s)\Gamma(s)$, that is $\int_0^\infty \frac{u^{s-1}}{e^u-1}\mathrm{d}u$ (see [78, Chapter 2, p. 96]), and the point $i)$ of the problem is complete.

To calculate the limit from the point $ii)$, we use the result in (3.217) from the point $i)$, and then we obtain that

$$\lim_{n\to\infty}\left(\frac{1}{n!}\int_0^\infty\int_0^\infty \frac{x^n-y^n}{e^x-e^y}\mathrm{d}x\mathrm{d}y - 2n\right) = 2\lim_{n\to\infty}\sum_{i=2}^{n+1}(\zeta(i)-1) = 2\sum_{i=2}^\infty(\zeta(i)-1)$$

$$= 2\sum_{i=2}^\infty\left(\sum_{k=2}^\infty \frac{1}{k^i}\right) = 2\sum_{k=2}^\infty\left(\sum_{i=2}^\infty \frac{1}{k^i}\right) = 2\lim_{n\to\infty}\sum_{k=2}^n \frac{1}{k(k-1)}$$

$$= 2\lim_{n\to\infty}\sum_{k=2}^n\left(\frac{1}{k-1}-\frac{1}{k}\right)$$

$$= 2\lim_{n\to\infty}\left(1-\frac{1}{n}\right) = 2,$$

and the point $ii)$ of the problem is complete.

Alternatively, we might try to attack the integral from the point $i)$ by considering the Cauchy principal value[15] (see [32, 7.1 The Cauchy Principal Value, p. 216], [97]) and then show and use that P.V. $\int_0^\infty \frac{1}{e^x-e^y}\mathrm{d}y = e^{-x}\log(e^x-1)$.

[14] It's straightforward to see, by using the geometric series, that $\int_0^\infty \frac{u^{s-1}}{e^u-1}\mathrm{d}u = \int_0^\infty \frac{u^{s-1}e^{-u}}{1-e^{-u}}\mathrm{d}u = \int_0^\infty u^{s-1}\sum_{k=1}^\infty e^{-ku}\mathrm{d}u = \sum_{k=1}^\infty\int_0^\infty u^{s-1}e^{-ku}\mathrm{d}u = \Gamma(s)\sum_{k=1}^\infty \frac{1}{k^s} = \zeta(s)\Gamma(s)$.

[15] Named after the brilliant French mathematician Augustin-Louis Cauchy (1789–1857), Cauchy principal value is a way of assigning values to improper integrals which in other conditions are divergent.

3.39 A Special Multiple Integral and a Limit of It Involving the Euler–Mascheroni Constant γ, the Euler's Number e, and the Famous π All at Once

Solution We are not away from Sect. 1.37 where we needed the same simple idea to start with as we need here. This time we'll try to evaluate a nice-looking multiple integral and then we'll calculate a limit with the multiple integral which will lead to a fascinating (unexpected) result involving the Euler–Mascheroni constant (γ), the Euler's number (e) and the famous π (if you're interested, an excellent book specialized on the mathematical constants, including the mentioned ones, may be found in [27]).

Using the result in (3.205) in the form $\displaystyle\int_0^1 e^{-ax}e^{(a-1)y}\mathrm{d}a = \frac{e^{-x}-e^{-y}}{y-x}$, where we set $x = x_{n+1}^{2n} + x_{n+2}^{2n} + \ldots + x_{2n}^{2n}$ and $y = x_1^{2n} + x_2^{2n} + \ldots + x_n^{2n}$, and then integrating both sides according to all $2n$ variables, $x_1, x_2, \ldots, x_{2n}$, we get

$$\int_0^\infty \left(\int_0^\infty \left(\cdots \int_0^\infty \frac{e^{-x_{n+1}^{2n}-x_{n+2}^{2n}-\ldots-x_{2n}^{2n}} - e^{-x_1^{2n}-x_2^{2n}-\ldots-x_n^{2n}}}{x_1^{2n}+x_2^{2n}+\ldots+x_n^{2n}-x_{n+1}^{2n}-x_{n+2}^{2n}-\ldots-x_{2n}^{2n}}\mathrm{d}x_1\right)\right.$$
$$\times \mathrm{d}x_2)\cdots \mathrm{d}x_{2n}$$

$$= \int_0^\infty \left(\int_0^\infty \left(\cdots \int_0^1 e^{-a(x_{n+1}^{2n}+x_{n+2}^{2n}+\ldots+x_{2n}^{2n})}e^{(a-1)(x_1^{2n}+x_2^{2n}+\ldots+x_n^{2n})}\mathrm{d}a\right)\mathrm{d}x_1\right)$$
$$\times \mathrm{d}x_2 \cdots \mathrm{d}x_{2n}$$

$$\{\text{reverse the integration order}\}$$

$$= \int_0^1 \left(\int_0^\infty \left(\cdots \int_0^\infty e^{-a(x_{n+1}^{2n}+x_{n+2}^{2n}+\ldots+x_{2n}^{2n})}e^{(a-1)(x_1^{2n}+x_2^{2n}+\ldots+x_n^{2n})}\mathrm{d}x_1\right)\right.$$
$$\times \mathrm{d}x_2)\cdots \mathrm{d}x_{2n}\mathrm{d}a$$

$$= \int_0^1 \left(\int_0^\infty e^{-ax^{2n}}\mathrm{d}x\right)^n \left(\int_0^\infty e^{(a-1)x^{2n}}\mathrm{d}x\right)^n \mathrm{d}a$$

$$= \left(\Gamma\left(1+\frac{1}{2n}\right)\right)^{2n} \int_0^1 a^{-1/2}(1-a)^{-1/2}\mathrm{d}a$$

$$= 2\left(\Gamma\left(1+\frac{1}{2n}\right)\right)^{2n} \arcsin(\sqrt{a})\Big|_{a=0}^{a=1}$$

$$= \pi\left(\Gamma\left(1+\frac{1}{2n}\right)\right)^{2n}, \tag{3.218}$$

where I used that, based upon the change of variable $ax^{2n} = y$, we have

$$\int_0^{\infty} e^{-ax^{2n}} \mathrm{d}x = \frac{a^{-1/(2n)}}{2n} \int_0^{\infty} y^{1/(2n)-1} e^{-y} \mathrm{d}y = \frac{a^{-1/(2n)}}{2n} \Gamma\left(\frac{1}{2n}\right)$$
$$= a^{-1/(2n)} \Gamma\left(1 + \frac{1}{2n}\right),$$

and then, based upon the variable change $(a-1)x^{2n} = -y$, we have that

$$\int_0^{\infty} e^{(a-1)x^{2n}} \mathrm{d}x = \frac{(1-a)^{-1/(2n)}}{2n} \int_0^{\infty} y^{1/(2n)-1} e^{-y} \mathrm{d}y = \frac{(1-a)^{-1/(2n)}}{2n} \Gamma\left(\frac{1}{2n}\right)$$
$$= (1-a)^{-1/(2n)} \Gamma\left(1 + \frac{1}{2n}\right),$$

and the part i) of the problem is finalized.

Using the asymptotic expansion[16] of Gamma function as $x \to 0$ (see [60, Chapter 0, pp. 3–5]), $\Gamma(x) = \dfrac{1}{x} - \gamma + O(x)$, that by the relation $x\Gamma(x) = \Gamma(1+x)$ yields $\Gamma(1+x) = 1 - \gamma x + O(x^2)$, we obtain

$$\lim_{n\to\infty} \left(\Gamma\left(1 + \frac{1}{2n}\right)\right)^{2n} = \lim_{n\to\infty} \left(1 - \frac{\gamma}{2n} + O\left(\frac{1}{n^2}\right)\right)^{2n}$$

$$\left\{\text{use the elementary limit, } \lim_{x\to 0} (1-x)^{1/x} = 1/e\right\}$$

$$= \lim_{n\to\infty} e^{-\gamma + O(1/n)} = e^{-\gamma}. \tag{3.219}$$

[16] In the book *Applied Asymptotic Analysis* by Peter D. Miller, one may find a simple method of deriving an approximation of $\Gamma(x)$, when x is small and positive, by combining the Taylor series with the term by term integration. Since $t^x \approx 1$, for each fixed t, and using that $t^x = \sum_{n=0}^{\infty} \dfrac{(x\log(t))^n}{n!}$, then by multiplying the last equality by e^{-t} and integrating from 0 to ∞, we get $\Gamma(x+1) = 1 + x\displaystyle\int_0^{\infty} \log(t) e^{-t} \mathrm{d}t + \frac{x^2}{2} \int_0^{\infty} \log^2(t) e^{-t} \mathrm{d}t + \cdots = 1 - \gamma x + \left(\frac{\gamma^2}{2} + \frac{\pi^2}{12}\right) x^2 + \cdots$, and from here we obtain that $\Gamma(x) = \dfrac{1}{x} - \gamma + \left(\dfrac{\gamma^2}{2} + \dfrac{\pi^2}{12}\right) x \cdots$, which is in fact the Laurent series for $\Gamma(x)$ about $x = 0$.

Hence, by letting $n \to \infty$ in (3.218), combined with the result in (3.219), we obtain the value of the beautiful limit,

$$\lim_{n\to\infty} \int_0^\infty \int_0^\infty \cdots \int_0^\infty \frac{e^{-x_{n+1}^{2n}-x_{n+2}^{2n}-\ldots-x_{2n}^{2n}} - e^{-x_1^{2n}-x_2^{2n}-\ldots-x_n^{2n}}}{x_1^{2n}+x_2^{2n}+\ldots+x_n^{2n}-x_{n+1}^{2n}-x_{n+2}^{2n}-\ldots-x_{2n}^{2n}}$$
$$\times \mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_{2n} = \frac{\pi}{e^\gamma},$$

and the part ii) of the problem is finalized.

The limit arose based upon the investigations of the multiple integral I created, and it was such a pleasant surprise to discover the neat value of the limit.

3.40 Some Curious Integrals Involving the Hyperbolic Tangent, Also Having Beautiful Connections with the Beta Function

Solution You may find the problem in the present section pretty nice and challenging at the same time, and the thing that may particularly arouse curiosity is the connection with the harmonic numbers we have to prove. Then, it's about the *challenging question* that asks you to calculate the generalization from the third point by using the Beta function, which you may also find nice to try. Based upon the generalization from the third point, one can get various, appealing integrals involving the hyperbolic functions.

The first solution might not be that obvious (which also answers the proposed *challenging question*), and we start with the integral representation of the Beta function in (3.23), where if we replace y by $1-x$ and use that $\mathrm{B}(x, 1-x) = \Gamma(x)\Gamma(1-x) = \dfrac{\pi}{\sin(\pi x)}$, we have

$$\int_{-1}^1 (1-t)^{x-1}(1+t)^{-x}\mathrm{d}t = \frac{\pi}{\sin(\pi x)}. \tag{3.220}$$

Multiplying both sides of (3.220) by $\sin^2(n\pi x)$ and integrating from $x = 0$ to $x = 1$, we get

$$\pi \int_0^1 \frac{\sin^2(n\pi x)}{\sin(\pi x)}\mathrm{d}x = \int_0^1 \left(\int_{-1}^1 (1-t)^{x-1}(1+t)^{-x} \sin^2(n\pi x)\mathrm{d}t \right) \mathrm{d}x$$

$$\{\text{reverse the order of integration}\}$$

$$= \int_{-1}^{1} \frac{1}{1-t} \left(\int_0^1 \left(\frac{1-t}{1+t} \right)^x \sin^2(n\pi x) \mathrm{d}x \right) \mathrm{d}t$$

$$= - \int_{-1}^{1} \frac{1}{2(1-t)\operatorname{arctanh}(t)} \left(\int_0^1 \left(\left(\frac{1-t}{1+t} \right)^x \right)' \sin^2(n\pi x) \mathrm{d}x \right) \mathrm{d}t$$

{integrate the inner integral by parts}

$$= \frac{\pi}{2} \int_{-1}^{1} \frac{n}{(1-t)\operatorname{arctanh}(t)} \left(\int_0^1 \left(\frac{1-t}{1+t} \right)^x \sin(2n\pi x) \mathrm{d}x \right) \mathrm{d}t$$

$$= \frac{n^2\pi^2}{2} \int_{-1}^{1} \frac{t}{(1-t^2)\operatorname{arctanh}(t)((n\pi)^2 + \operatorname{arctanh}^2(t))} \mathrm{d}t, \tag{3.221}$$

where the last equality is obtained by the use of the classical integral,[17] $\int \sin(ax)e^{bx}\mathrm{d}x = \frac{b\sin(ax) - a\cos(ax)}{a^2+b^2}e^{bx} + C$, with the integration limits $x=0$, $x=1$, and $a=2n\pi$, $b=-2\operatorname{arctanh}(t)$.

Now, the integrand of the remaining integral in (3.221) is even, and then we write

$$\frac{n^2\pi^2}{2} \int_{-1}^{1} \frac{t}{(1-t^2)\operatorname{arctanh}(t)((n\pi)^2 + \operatorname{arctanh}^2(t))} \mathrm{d}t$$

$$= n^2\pi^2 \int_0^1 \frac{t}{(1-t^2)\operatorname{arctanh}(t)((n\pi)^2 + \operatorname{arctanh}^2(t))} \mathrm{d}t$$

{make the change of variable $t = \tanh(\pi v)$}

$$= \int_0^\infty \tanh(\pi v) \left(\frac{1}{v} - \frac{v}{n^2+v^2} \right) \mathrm{d}v = \pi \int_0^1 \frac{\sin^2(n\pi x)}{\sin(\pi x)} \mathrm{d}x$$

$$= \frac{\pi}{2} \int_0^1 \frac{1-\cos(2n\pi x)}{\sin(\pi x)} \mathrm{d}x. \tag{3.222}$$

[17] Essentially, we integrate by parts twice, and then we have $\int \sin(ax)e^{bx}\mathrm{d}x = -\frac{1}{a}\int (\cos(ax))'e^{bx}\mathrm{d}x = -\frac{\cos(ax)}{a}e^{bx} + \frac{b}{a}\int \cos(ax)e^{bx}\mathrm{d}x = -\frac{\cos(ax)}{a}e^{bx} + \frac{b}{a^2}\int (\sin(ax))'e^{bx}\mathrm{d}x = -\frac{\cos(ax)}{a}e^{bx} + \frac{b\sin(ax)}{a^2}e^{bx} - \frac{b^2}{a^2}\int \sin(ax)e^{bx}\mathrm{d}x$, whence we obtain immediately that $\int \sin(ax)e^{bx}\mathrm{d}x = \frac{b\sin(ax) - a\cos(ax)}{a^2+b^2}e^{bx} + C$.

Considering the notation $I_n = \int_0^1 \frac{1-\cos(2n\pi x)}{\sin(\pi x)} \mathrm{d}x$ for the last integral in (3.40), we have

$$I_{k+1} - I_k = \int_0^1 \frac{1-\cos((2k+2)\pi x)}{\sin(\pi x)} \mathrm{d}x - \int_0^1 \frac{1-\cos(2k\pi x)}{\sin(\pi x)} \mathrm{d}x$$

$$= \int_0^1 \frac{\cos(2k\pi x)-\cos((2k+2)\pi x)}{\sin(\pi x)} \mathrm{d}x = 2\int_0^1 \sin((2k+1)\pi x)\mathrm{d}x = \frac{4}{\pi}\frac{1}{2k+1},$$

and if we give values to k from $k = 0$ to $k = n-1$ and add up all the resulting relations, we get

$$I_n = \frac{4}{\pi}\sum_{k=0}^{n-1}\frac{1}{2k+1} = \frac{4}{\pi}\sum_{k=1}^{n}\frac{1}{2k-1} = \frac{4}{\pi}\left(H_{2n} - \frac{1}{2}H_n\right). \tag{3.223}$$

Finally, if we plug the result from (3.223) in (3.40), we conclude that

$$\int_0^\infty \tanh(\pi v)\left(\frac{1}{v} - \frac{v}{n^2+v^2}\right)\mathrm{d}v = 2H_{2n} - H_n,$$

and the first solution to the point $iii)$ of the problem is finalized.

For a second solution to the point $iii)$ of the problem, we use the series[18] $\tanh(\pi x) = \frac{8x}{\pi}\sum_{n=1}^\infty \frac{1}{(2n-1)^2+(2x)^2}$ which is derived with the help of the Euler's

---

[18]Recalling the famous Euler's infinite product, $\sin(x) = x\prod_{n=1}^\infty\left(1-\frac{x^2}{n^2\pi^2}\right)$, and then using the identity $\sin(2x) = 2\sin(x)\cos(x)$, we have that $\cos(x) = \prod_{n=1}^\infty\left(\frac{n^2\pi^2-(2x)^2}{n^2\pi^2}\right) \cdot \prod_{n=1}^\infty\left(\frac{n^2\pi^2}{n^2\pi^2-x^2}\right) = \prod_{n=1}^\infty\left(\frac{(2n-1)^2\pi^2-(2x)^2}{(2n-1)^2\pi^2}\right) \cdot \underbrace{\prod_{n=1}^\infty\left(\frac{(2n)^2\pi^2-(2x)^2}{(2n)^2\pi^2}\right)\cdot\prod_{n=1}^\infty\left(\frac{n^2\pi^2}{n^2\pi^2-x^2}\right)}_{1} = \prod_{n=1}^\infty\left(1-\frac{4x^2}{(2n-1)^2\pi^2}\right)$. Upon replacing x by ix in the cosine formula, we get $\cosh(x) = \prod_{n=1}^\infty\left(1+\frac{4x^2}{(2n-1)^2\pi^2}\right)$. At this point, by taking log of both sides of the hyperbolic cosine and differentiating then once with respect to x, we get $\tanh(x) = 8x\sum_{n=1}^\infty\frac{1}{\pi^2(2n-1)^2+(2x)^2}$, where if we replace x by πx, we obtain the desired series, $\tanh(\pi x) = \frac{8x}{\pi}\sum_{n=1}^\infty\frac{1}{(2n-1)^2+(2x)^2}$.

infinite product for the sine, $\sin(x) = x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right)$ (see [23, pp. 251–252]). Then we write that

$$\int_0^{\infty} \tanh(\pi x)\left(\frac{1}{x}-\frac{x}{n^2+x^2}\right)\mathrm{d}x = \frac{8}{\pi}\int_0^{\infty}\sum_{k=1}^{\infty}\frac{x}{(2k-1)^2+(2x)^2}$$
$$\times\left(\frac{1}{x}-\frac{x}{n^2+x^2}\right)\mathrm{d}x$$

{reverse the order of summation and integration}

$$= \frac{8}{\pi}\sum_{k=1}^{\infty}\int_0^{\infty}\frac{x}{(2k-1)^2+(2x)^2}\left(\frac{1}{x}-\frac{x}{n^2+x^2}\right)\mathrm{d}x$$

$$= \frac{8}{\pi}\sum_{k=1}^{\infty}\frac{n^2}{(2k-1)^2-4n^2}\int_0^{\infty}\left(\frac{1}{n^2+x^2}-\frac{4}{(2k-1)^2+4x^2}\right)\mathrm{d}x$$

$$= \frac{8}{\pi}\sum_{k=1}^{\infty}\frac{n^2}{(2k-1)^2-4n^2}\left(\frac{1}{n}\arctan\left(\frac{x}{n}\right)-\frac{2}{2k-1}\arctan\left(\frac{2x}{2k-1}\right)\Bigg|_{x=0}^{x=\infty}\right)$$

$$= 4\sum_{k=1}^{\infty}\frac{n^2}{(2k-1)^2-4n^2}\left(\frac{1}{n}-\frac{2}{2k-1}\right) = 2\sum_{k=1}^{\infty}\left(\frac{1}{2k-1}-\frac{1}{2k-1+2n}\right)$$

$$= 2\sum_{k=1}^{n}\frac{1}{2k-1} = 2\left(H_{2n}-\frac{1}{2}H_n\right) = 2H_{2n}-H_n,$$

and the second solution to the point $iii)$ of the problem is finalized.

The first part of the problem is solved immediately if we make the change of variable $\pi x = y$ in $iii)$ and then plug $n = 1$,

$$\int_0^{\infty} \tanh(x)\left(\frac{1}{x}-\frac{x}{\pi^2+x^2}\right)\mathrm{d}x = 2H_2 - H_1 = 2,$$

and the part $i)$ of the problem is finalized.

Next, to solve the point $ii)$ of the problem, we assume in $iii)$ the extension of n to the reals, $n > 0$, and we write that

$$\int_0^{\infty} \tanh(\pi x)\left(\frac{1}{8x^3+2x}+\frac{1}{64x^3+4x}+\frac{1}{512x^3+8x}+\cdots\right)\mathrm{d}x$$

$$= \int_0^\infty \sum_{n=1}^\infty \frac{\tanh(\pi x)}{8^n x^3 + 2^n x} \mathrm{d}x = \sum_{n=1}^\infty \frac{1}{2^n} \int_0^\infty \tanh(\pi x) \left(\frac{1}{x} - \frac{x}{(2^{-n})^2 + x^2} \right) \mathrm{d}x$$

{make use of the result from the point iii)}

$$= \sum_{n=1}^\infty \left(2^{-n+1} H_{2^{-n+1}} - 2^{-n} H_{2^{-n}} \right) = \lim_{N\to\infty} \sum_{n=1}^N \left(2^{-n+1} H_{2^{-n+1}} - 2^{-n} H_{2^{-n}} \right)$$

$$= 1 - \lim_{N\to\infty} 2^{-N} H_{2^{-N}} = 1,$$

and the part ii) of the problem is finalized.

The result from the point iv) is obtained immediately if we use the identity from the point iii), where we multiply both sides by $1/n^4$ and then consider the summation from $n = 1$ to ∞. So, we write

$$\sum_{n=1}^\infty \frac{1}{n^4} \int_0^\infty \tanh(\pi x) \left(\frac{1}{x} - \frac{x}{n^2 + x^2} \right) \mathrm{d}x = \int_0^\infty \tanh(\pi x) \sum_{n=1}^\infty \left(\frac{1}{x} - \frac{x}{n^2 + x^2} \right) \times \frac{1}{n^4} \mathrm{d}x$$

$$= \int_0^\infty \frac{\tanh(\pi x)}{x^3} \sum_{n=1}^\infty \left(\frac{1}{n^2} - \frac{1}{n^2 + x^2} \right) \mathrm{d}x$$

$$= \int_0^\infty \frac{\tanh(\pi x)}{x^3} \left(\zeta(2) + \frac{1}{2x^2} - \frac{\pi}{2} \frac{\coth(\pi x)}{x} \right) \mathrm{d}x$$

$$= \frac{1}{6} \int_0^\infty \frac{\pi^2 x^2 \tanh(\pi x) + 3\tanh(\pi x) - 3\pi x}{x^5} \mathrm{d}x = 2 \sum_{n=1}^\infty \frac{H_{2n}}{n^4} - \sum_{n=1}^\infty \frac{H_n}{n^4}$$

$$= 32 \sum_{n=1}^\infty \frac{H_{2n}}{(2n)^4} - \sum_{n=1}^\infty \frac{H_n}{n^4} = 16 \left(\sum_{n=1}^\infty \frac{H_n}{n^4} + \sum_{n=1}^\infty (-1)^n \frac{H_n}{n^4} \right) - \sum_{n=1}^\infty \frac{H_n}{n^4}$$

$$= 15 \sum_{n=1}^\infty \frac{H_n}{n^4} - 16 \sum_{n=1}^\infty (-1)^{n-1} \frac{H_n}{n^4}$$

{make use of the results in (3.45), the case $n = 4$, and (4.89)}

$$= \frac{31}{2} \zeta(5) - 7\zeta(2)\zeta(3),$$

whence we obtain that

$$\int_0^\infty \frac{\pi^2 x^2 \tanh(\pi x) + 3\tanh(\pi x) - 3\pi x}{x^5} \mathrm{d}x = 3(31\zeta(5) - 14\zeta(2)\zeta(3)),$$

and the part $iv)$ of the problem is finalized. Note that in the calculations I used that $\coth(\pi x) = \frac{1}{\pi x} + \frac{2x}{\pi} \sum_{k=1}^{\infty} \frac{1}{x^2 + k^2}$, which is obtained by replacing x by $\pi x i$ in the Euler's infinite product for the sine, $\sin(x) = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right)$ (see [23, pp. 251–252]), and then taking log of both sides and differentiating once with respect to x.

For the point $v)$ we assume in $iii)$ the extension of n to the reals, $n > 0$, which is achieved through the use of the Digamma function, and then, differentiating once with respect to n, we get

$$\int_0^\infty \frac{x \tanh(\pi x)}{(n^2 + x^2)^2} \mathrm{d}x = \frac{1}{2n} \left(3\zeta(2) + H_n^{(2)} - 4H_{2n}^{(2)}\right). \tag{3.224}$$

Multiplying both sides of (3.224) by $1/n^2$ and then considering the summation from $n = 1$ to ∞, we have

$$\sum_{n=1}^{\infty} \int_0^\infty \frac{x \tanh(\pi x)}{n^2(n^2 + x^2)^2} \mathrm{d}x = \int_0^\infty \sum_{n=1}^{\infty} \frac{x \tanh(\pi x)}{n^2(n^2 + x^2)^2} \mathrm{d}x$$

$$= \int_0^\infty \tanh(\pi x) \sum_{n=1}^{\infty} \left(\frac{1}{x^3 n^2} - \frac{1}{x(n^2 + x^2)^2} - \frac{1}{x^3(n^2 + x^2)}\right) \mathrm{d}x$$

$$= \int_0^\infty \frac{2\pi^2 x^2 \tanh(\pi x) + 12\tanh(\pi x) - 6\pi^2 x^2 \operatorname{csch}(2\pi x) - 9\pi x}{12x^5} \mathrm{d}x$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^3} \left(3\zeta(2) + H_n^{(2)} - 4H_{2n}^{(2)}\right) = \frac{3}{2}\zeta(2) \sum_{n=1}^{\infty} \frac{1}{n^3} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^3} - 16 \sum_{n=1}^{\infty} \frac{H_{2n}^{(2)}}{(2n)^3}$$

$$= \frac{3}{2}\zeta(2)\zeta(3) + 8\sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n^{(2)}}{n^3} - \frac{15}{2} \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^3}$$

{make use of the results in (4.90) and (6.67)}

$$= 31\zeta(5) - 16\zeta(2)\zeta(3),$$

whence we obtain that

$$\int_0^\infty \frac{2\pi^2x^2\tanh(\pi x)+12\tanh(\pi x)-6\pi^2x^2\operatorname{csch}(2\pi x)-9\pi x}{x^5}\mathrm{d}x$$

$$=12(31\zeta(5)-16\zeta(2)\zeta(3)),$$

and the part $v)$ of the problem is finalized. Note that in the calculations I used the series representation of $\coth(x)$ (see the previous point) together with the fact that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\left(n^2+x^2\right)^2}=\frac{\pi^2x^2\operatorname{csch}^2(\pi x)+\pi x\coth(\pi x)-2}{4x^4}$, which is obtained immediately by differentiation from the series representation of $\coth(x)$.

To add a final note, it is worth mentioning the power of the Euler's infinite product for the sine I used in the present work, which gives an elegant solution to the point $iii)$ of the problem. Also, the solution to the *challenging question* shows once again the huge potential of the Beta function, so useful in many problems.

The integral version proposed at the point $iii)$ of the problem may represent a good researching ground for investigating other versions similar to it and possibly establishing other connections with the realm of the harmonic numbers.

3.41 A Little Integral-Beast from *Inside Interesting Integrals* Together with a Similar Version of It Tamed by Real Methods

Solution The integral from the point $i)$ is one of the integrals that probably will always make me think of the book *Inside Interesting Integrals*, which appears in the *Challenge Problems* section (see [64, Chapter 8, p. 340]), where Paul proposes a beautiful way to attack it by employing the contour integration[19] (for details, see [64, p. 406]).

In this section we'll take Paul's saying, *It is easily done with contour integration, but would (I think) otherwise be pretty darn tough.*, as a challenge and will try to do it differently.

Before passing to the integral from the point $i)$, we prepare the following result,

$$\sum_{n=1}^{\infty}\frac{p^n\sin(nx)}{n!}=\sin(p\sin(x))e^{p\cos(x)}. \tag{3.225}$$

[19] Considering the contour integral $\displaystyle\oint_C \frac{e^{e^{iz}}}{z}\mathrm{d}z$ and choosing the proper contour C as indicated in the given reference (try first to find it alone without using the reference!), we get immediately the desired value.

Proof Based upon the Euler's formula, $e^{ix} = \cos(x) + i\sin(x)$, we write

$$\sum_{n=1}^{\infty}\frac{p^n\sin(nx)}{n!} = \Im\left\{\sum_{n=1}^{\infty}\frac{(pe^{ix})^n}{n!}\right\} = \Im\left\{e^{pe^{ix}}-1\right\} = \Im\left\{e^{p\cos(x)}e^{ip\sin(x)}\right\}$$

$$= \Im\left\{e^{p\cos(x)}(\cos(p\sin(x)) + i\sin(p\sin(x)))\right\} = \sin(p\sin(x))e^{p\cos(x)},$$

and the result in (3.225) is proved.

Upon setting $p = 1$ in (3.225), $\displaystyle\sum_{n=1}^{\infty}\frac{\sin(nx)}{n!} = \sin(\sin(x))e^{\cos(x)}$, and returning to our integral, we have

$$\int_0^{\infty}\frac{\sin(\sin(x))e^{\cos(x)}}{x}\mathrm{d}x = \int_0^{\infty}\frac{1}{x}\sum_{n=1}^{\infty}\frac{\sin(nx)}{n!}\mathrm{d}x = \sum_{n=1}^{\infty}\frac{1}{n!}\int_0^{\infty}\frac{\sin(nx)}{x}\mathrm{d}x$$

$$= \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{1}{n!} = \frac{\pi}{2}(e-1),$$

and the solution to the point *i*) is finalized. The value of the integral is neat, expressed in terms of π and Euler's number (if interested to find more details about these two constants, see [14] and [46]). In the calculations I also used the famous Dirichlet's integral[20](see [30, **3.721.1**, p. 427]), $\displaystyle\int_0^{\infty}\frac{\sin(ax)}{x}\mathrm{d}x = \frac{\pi}{2}\operatorname{sgn}(a), a \in \mathbb{R}$.

Proceeding similarly as in the previous solution, we write that

$$\int_0^{\infty}\frac{\sin(x)\sin(\sin(x))e^{\cos(x)}}{x^2}\mathrm{d}x = \int_0^{\infty}\frac{\sin(x)}{x^2}\sum_{n=1}^{\infty}\frac{\sin(nx)}{n!}\mathrm{d}x$$

$$= \sum_{n=1}^{\infty}\frac{1}{n!}\int_0^{\infty}\frac{\sin(x)\sin(nx)}{x^2}\mathrm{d}x = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n!}\int_0^{\infty}\frac{\cos((n-1)x)-\cos((n+1)x)}{x^2}\mathrm{d}x$$

---

[20] Without loss of generality, it is enough to consider the case $a > 0$, and making the variable change $ax = y$, we write $\displaystyle\int_0^{\infty}\frac{\sin(ax)}{x}\mathrm{d}x = \int_0^{\infty}\frac{\sin(y)}{y}\mathrm{d}y = \int_0^{\infty}\left(\int_0^{\infty}\sin(y)e^{-yt}\mathrm{d}t\right)\mathrm{d}y =$ {reverse the order of integration}$= \displaystyle\int_0^{\infty}\left(\int_0^{\infty}\sin(y)e^{-ty}\mathrm{d}y\right)\mathrm{d}t = \int_0^{\infty}\frac{1}{1+t^2}\mathrm{d}t = \arctan(t)\Big|_{t=0}^{t=\infty} = \frac{\pi}{2}$.

$$= \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n!}\int_0^{\infty}\left(-\frac{1}{x}\right)'(\cos((n-1)x)-\cos((n+1)x))\mathrm{d}x$$

{integrate by parts once}

$$= \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n!}\int_0^{\infty}\frac{(n+1)\sin((n+1)x)-(n-1)\sin((n-1)x)}{x}\mathrm{d}x$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n!}\left((n+1)\int_0^{\infty}\frac{\sin((n+1)x)}{x}\mathrm{d}x-(n-1)\int_0^{\infty}\frac{\sin((n-1)x)}{x}\mathrm{d}x\right)$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{\pi}{2}(n+1)-\frac{\pi}{2}(n-1)\right)=\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{1}{n!}=\frac{\pi}{2}(e-1),$$

and the solution to the point $ii)$ is finalized.

Using the series like the one in (3.225) is a good alternative to the powerful contour integration approach. Moreover, if playing with such series one can get various appealing integrals like the proposed ones. The solution to the integral $i)$ is one of the first solutions I was glad to communicate to Paul immediately after the publishing of his wonderful book, *Inside Interesting Integrals*.

3.42 Ramanujan's Integrals with Beautiful Connections with the Digamma Function and Frullani's Integral

Solution *I ask, "Who's this guy Ramanujan? What did he do?" My father tells me the most incredible story, about the life of Srinivasa Ramanujan. It is the story of an Indian man who overcame incredible odds to become one of the most romantic and influential figures in the history of mathematics. It is the story of a self-taught dropout whose ideas came to him as visions from a goddess. It is the story of a man who had the courage to send his ideas to random mathematicians at the University of Cambridge, and then to accept the invitation of a world-class mathematician who recognized his genius and travel halfway around the world to work with him in England. It is the story of a man who suffered racial prejudice as he strove for accomplishment and recognition. It is a story of a man who would then die tragically at the young age of thirty-two.*—Ken Ono in *My Search For Ramanujan*.

Srinivasa Ramanujan Iyengar (1887–1920) *was an enigmatic Indian mathematician who, without formal education in advanced mathematics, was able to derive thousands of unproved, yet valid, mathematical formulas and identities.*—Ken Ono in *My Search For Ramanujan*.

Ramanujan was an artist. And numbers – and the mathematical language expressing their relationships – were his medium.—Robert Kanigel in *The Man Who Knew Infinity: A Life of the Genius Ramanujan*.

In this section we will experience Ramanujan's creativity and try to know, understand him through mathematics, by two beautiful integrals proposed by him.

To calculate the integral from the point *i*), we rearrange the integrand and prepare it for using the integral representation of Digamma function, $\psi(s+1) = -\gamma + \int_0^1 \frac{x^s-1}{x-1}\mathrm{d}x$ (see [30, **3.265**, p. 333], [61, Chapter 10, p. 127]), and then we write

$$\int_0^{\infty} \frac{(1+ax)^{-p}-(1+bx)^{-q}}{x}\mathrm{d}x$$

$$= \int_0^{\infty} \frac{(1+ax)^{-p}-(1+ax)^{-1}-((1+bx)^{-q}-(1+bx)^{-1})+(1+ax)^{-1}-(1+bx)^{-1}}{x}\mathrm{d}x$$

$$= \int_0^{\infty} \frac{(1+ax)^{-p}-(1+ax)^{-1}}{x}\mathrm{d}x - \int_0^{\infty} \frac{(1+bx)^{-q}-(1+bx)^{-1}}{x}\mathrm{d}x + \int_0^{\infty} \frac{(1+ax)^{-1}-(1+bx)^{-1}}{x}\mathrm{d}x$$

{let the changes of variable $ax = y$ in the first integral, and $bx = y$ in the second one}

$$= \int_0^{\infty} \frac{(1+x)^{-p}-(1+x)^{-1}}{x}\mathrm{d}x - \int_0^{\infty} \frac{(1+x)^{-q}-(1+x)^{-1}}{x}\mathrm{d}x + \int_0^{\infty} \frac{(1+ax)^{-1}-(1+bx)^{-1}}{x}\mathrm{d}x. \tag{3.226}$$

Before calculating the first two integrals, note the third integral is an elementary integral, and we have that $\int \frac{1}{x(1+ax)}\mathrm{d}x = \log(x) - \log(1+ax) + C$, whence

$$\int_0^{\infty} \frac{(1+ax)^{-1}-(1+bx)^{-1}}{x}\mathrm{d}x = \log\left(\frac{1+bx}{1+ax}\right)\Bigg|_{x=0}^{x=\infty} = \log\left(\frac{b}{a}\right). \tag{3.227}$$

On the other hand, this integral can also be viewed as a Frullani's integral (see the conditions and the derivation in [75, pp. 148–151], [64, Chapter 3, pp. 86–87], [20, pp. 432–433] [106]), named after the Italian mathematician Giuliano Frullani (1795–1834),

$$\int_0^{\infty} \frac{f(ax)-f(bx)}{x}\mathrm{d}x = (f(0)-f(\infty))\log\left(\frac{b}{a}\right). \tag{3.228}$$

Setting $f(x) = \dfrac{1}{1+x}$ and applying (3.228), we get the result in (3.227).
Using the result in (3.227) and continuing the work in (3.226), we obtain that

$$\int_0^\infty \frac{(1+ax)^{-p} - (1+bx)^{-q}}{x} \mathrm{d}x$$

$$= \int_0^\infty \frac{(1+x)^{-p} - (1+x)^{-1}}{x} \mathrm{d}x - \int_0^\infty \frac{(1+x)^{-q} - (1+x)^{-1}}{x} \mathrm{d}x + \log\left(\frac{b}{a}\right)$$

$$\left\{\text{make the change of variable } \frac{1}{1+x} = y \text{ in both integrals}\right\}$$

$$= -\int_0^1 \frac{y^{p-1} - 1}{y-1} \mathrm{d}y + \int_0^1 \frac{y^{q-1} - 1}{y-1} \mathrm{d}y + \log\left(\frac{b}{a}\right) = \psi(q) - \psi(p) + \log\left(\frac{b}{a}\right),$$

that finalizes the part i) of the problem.

For a different way of approaching the integral by the Ramanujan's *means*, see [15, Chapter 9, pp. 314–315].

For the Ramanujan's integral from the part ii) of the problem, we make the change of variable $x = e^{-y}$, and then we have

$$\int_0^1 \left(\frac{x^{a-1}}{1-x} - \frac{cx^{b-1}}{1-x^c}\right) \mathrm{d}x = \int_0^\infty \left(\frac{e^{-ay}}{1-e^{-y}} - \frac{ce^{-by}}{1-e^{-cy}}\right) \mathrm{d}y. \tag{3.229}$$

Now we recall the following integral representation of Digamma function (see [61, Chapter 10, p. 125], [79, p. 26]),

$$\psi(x) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}}\right) \mathrm{d}t. \tag{3.230}$$

Letting the variable change $t = cy$ in (3.230), we get

$$\psi(x) = \int_0^\infty \left(\frac{e^{-cy}}{y} - \frac{ce^{-cxy}}{1-e^{-cy}}\right) \mathrm{d}y,$$

and setting $x = b/c$, we have

$$\psi\left(\frac{b}{c}\right) = \int_0^\infty \left(\frac{e^{-cy}}{y} - \frac{ce^{-by}}{1-e^{-cy}}\right) \mathrm{d}y. \tag{3.231}$$

On the other hand, letting $x = a$ in (3.230), we get

$$\psi(a) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-at}}{1-e^{-t}} \right) \mathrm{d}t. \tag{3.232}$$

We return to (3.229), and then we write

$$\int_0^\infty \left(\frac{e^{-ay}}{1-e^{-y}} - \frac{ce^{-by}}{1-e^{-cy}} \right) \mathrm{d}y$$

$$= \int_0^\infty \left(\frac{e^{-cy}}{y} - \frac{ce^{-by}}{1-e^{-cy}} - \left(\frac{e^{-y}}{y} - \frac{e^{-ay}}{1-e^{-y}} \right) + \frac{e^{-y}-e^{-cy}}{y} \right) \mathrm{d}y$$

$$= \int_0^\infty \left(\frac{e^{-cy}}{y} - \frac{ce^{-by}}{1-e^{-cy}} \right) \mathrm{d}y - \int_0^\infty \left(\frac{e^{-y}}{y} - \frac{e^{-ay}}{1-e^{-y}} \right) \mathrm{d}y + \int_0^\infty \frac{e^{-y}-e^{-cy}}{y} \mathrm{d}y$$

{use the specific values of Digamma function from (3.231) and (3.232),}

{and then apply (3.228) for $f(x) = e^{-x}$ to calculate the Frullani's integral}

$$= \psi\left(\frac{b}{c}\right) - \psi(a) + \log(c).$$

Hence, we obtain that

$$\int_0^1 \left(\frac{x^{a-1}}{1-x} - \frac{cx^{b-1}}{1-x^c} \right) \mathrm{d}x = \psi\left(\frac{b}{c}\right) - \psi(a) + \log(c),$$

that finalizes the part ii) of the problem.

For alternative solutions, see [8, p. 154], [53]. One must be very careful with approaching the integral from the second point, it is deceiving and one may be easily led to a result without the log part.

3.43 The Complete Elliptic Integral of the First Kind Ramanujan Is Asked to Calculate in the Movie *The Man Who Knew Infinity* Together with Another Question Originating from His Work

Solution In the movie *The Man Who Knew Infinity* (2015) there is a memorable scene when Ramanujan, at Cambridge, during the class is invited by the cruel Professor Howard to bring his contribution to the calculation of the integral we have at the point i). Amazingly, instead of trying to write the solution, Ramanujan simply writes the final answer (blowing all the coals in the relation with the mentioned professor), a scene that aims to emphasize the genius of Ramanujan.

May the scene presented in the movie be an exaggeration of the Ramanujan's talent? No, I don't think so. If having in mind the evaluation of the Wallis' integral (see [17, p. 113], [30, **3.621.3–3.621.4**, p. 397]) combined with the use of an elementary integral result (presented below), it looks like everything becomes pretty easy to do. Here is how to do it:

First we use that if $1 > a$, we have

$$\int_0^{\pi/2} \frac{\mathrm{d}x}{1 - a\sin^2(x)} = \frac{\pi}{2}\frac{1}{\sqrt{1-a}}. \tag{3.233}$$

To prove the auxiliary result above, we start out by using the fact that $\sin^2(x) + \cos^2(x) = 1$, and then we have

$$\begin{aligned}\int_0^{\pi/2} \frac{\mathrm{d}x}{1 - a\sin^2(x)} &= \int_0^{\pi/2} \frac{\mathrm{d}x}{(1-a)\sin^2(x) + \cos^2(x)}\\ &= \frac{1}{1-a}\int_0^{\pi/2} \frac{(\tan(x))'}{\tan^2(x) + (1/\sqrt{1-a})^2}\mathrm{d}x\\ &= \frac{1}{\sqrt{1-a}} \arctan\left(\sqrt{1-a}\tan(x)\right)\Big|_{x=0}^{x=\pi/2}\\ &= \frac{\pi}{2}\frac{1}{\sqrt{1-a}}.\end{aligned}$$

The indefinite integral form of the integral above may also be found in **2.562.1** from [30].

Thus, based upon the result in (3.233), we obtain that

$$\begin{aligned}K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2(\theta)}} &= \frac{2}{\pi}\int_0^{\pi/2}\left(\int_0^{\pi/2} \frac{1}{1-k^2\sin^2(\theta)\sin^2(\psi)}d\psi\right)d\theta\\ &= \frac{2}{\pi}\int_0^{\pi/2}\left(\int_0^{\pi/2}\sum_{n=0}^{\infty} k^{2n}\sin^{2n}(\theta)\sin^{2n}(\psi)d\psi\right)d\theta\\ &= \frac{2}{\pi}\sum_{n=0}^{\infty} k^{2n}\int_0^{\pi/2}\sin^{2n}(\psi)d\psi\int_0^{\pi/2}\sin^{2n}(\theta)d\theta\end{aligned}$$

$$\left\{\text{make use of the Wallis' integral, } \int_0^{\pi/2}\sin^{2n}(\theta)d\theta = \frac{\pi}{2}\cdot\frac{(2n-1)!!}{(2n)!!}\right\}$$

$$= \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2 k^{2n}$$

$$= \frac{\pi}{2}\left\{1+\left(\frac{1}{2}\right)^2 k^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 k^6+\cdots\right\},$$

which finalizes the part $i)$ of the problem.

For the second part of the problem, we make use of the part $i)$ of the problem where we set $k = i$, and then we obtain

$$1-\left(\frac{1}{2}\right)^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2-\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2+\cdots = \frac{2}{\pi}\int_0^{\pi/2}\frac{d\theta}{\sqrt{1+\sin^2(\theta)}}$$

$$\{\text{make the change of variable } \sin(\theta)=x\}$$

$$= \frac{2}{\pi}\int_0^1 \frac{1}{\sqrt{1-x^4}}\mathrm{d}x \overset{x^4=y}{=} \frac{1}{2\pi}\int_0^1 y^{-3/4}(1-y)^{-1/2}\mathrm{d}y$$

$$\{\text{use the Beta function definition, } \mathrm{B}(x,y)=\int_0^1 u^{x-1}(1-u)^{y-1}\mathrm{d}u\}$$

$$= \frac{1}{2\pi}\,\mathrm{B}\left(\frac{1}{4},\frac{1}{2}\right)$$

$$\left\{\text{use the Beta–Gamma identity, } B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\right\}$$

$$= \frac{1}{2\pi}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)},$$

and if we use the special value, $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$, combined with the Euler's reflection formula, $\Gamma(a)\Gamma(1-a)=\dfrac{\pi}{\sin(\pi a)}$, where we set $a = 1/4$, $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\pi$, we conclude that

$$1-\left(\frac{1}{2}\right)^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2-\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2+\cdots = \frac{\Gamma^2\left(\frac{1}{4}\right)}{\sqrt{(2\pi)^3}},$$

which finalizes the part $ii)$ of the problem.

During the calculations I also used the Beta–Gamma identity, $B(x, y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, and a proof of it may be found in [94].

For an alternative solution to the point $ii)$ of the problem, see [16, Chapter 10, p. 24]. If interested, one may find in *Ramanujan's Notebooks, Part II* by Bruce C. Berndt more such splendid results involving the hypergeometric functions.

3.44 The First Double Integral I Published in *La Gaceta de la RSME*, Together with Another Integral Similar to It

Solution I submitted the double integral from the first point to *La Gaceta de la RSME* and it was published in Vol. 19, No. 3 (2016), the problem **306** (see [41, p. 589]). The thing we don't see immediately (and that, of course, represents a key point of the problem) is the connection with the Inverse tangent integral we already met in Sect. 3.25, where I provided some information about the mentioned function. Then it's about those *strange* integration limits that make us wonder what kind of approach would be useful, fruitful here. *Those integration limits are puzzling! Seriously, do we really have* $\sqrt{3}$–$\sqrt{2}$?

First we want to prove a curious representation of the Inverse tangent integral,

$$\int_a^1 \left(\int_b^1 \frac{1}{(x+y)^2 + (1+xy)^2} \mathrm{d}x \right) \mathrm{d}y = \frac{1}{2} \operatorname{Ti}_2\left(\frac{1-a}{1+a} \cdot \frac{1-b}{1+b} \right), \tag{3.234}$$

where $1 > a, b > 0$.

Proof Make the variable changes $x = \dfrac{1-u}{1+u}$ and $y = \dfrac{1-v}{1+v}$ respectively, and we get

$$\int_a^1 \left(\int_b^1 \frac{1}{(x+y)^2 + (1+xy)^2} \mathrm{d}x \right) \mathrm{d}y = \frac{1}{2} \int_0^{\frac{1-a}{1+a}} \left(\int_0^{\frac{1-b}{1+b}} \frac{1}{1+u^2v^2} \mathrm{d}u \right) \mathrm{d}v$$

$$= \frac{1}{2} \int_0^{\frac{1-a}{1+a}} \frac{\arctan\left(\dfrac{1-b}{1+b} v \right)}{v} \mathrm{d}v \overset{(1-b)/(1+b)v=t}{=} \frac{1}{2} \int_0^{\frac{1-a}{1+a} \cdot \frac{1-b}{1+b}} \frac{\arctan(t)}{t} \mathrm{d}t$$

$$= \frac{1}{2} \operatorname{Ti}_2\left(\frac{1-a}{1+a} \cdot \frac{1-b}{1+b} \right),$$

where I used that $\operatorname{Ti}_2(x) = \displaystyle\int_0^x \frac{\arctan(t)}{t} \mathrm{d}t$, and the proof of the result is complete.

Now, for $a = \sqrt{3} - \sqrt{2}$, we have

$$\frac{1-a}{1+a} = \frac{1-(\sqrt{3}-\sqrt{2})}{1+\sqrt{3}-\sqrt{2}} = \frac{1-(\sqrt{3}-\sqrt{2})}{1+\sqrt{3}-\sqrt{2}} \cdot \frac{1-(\sqrt{3}-\sqrt{2})}{1-(\sqrt{3}-\sqrt{2})}$$

$$= \frac{3+\sqrt{2}-\sqrt{3}-\sqrt{6}}{\sqrt{6}-2} = \frac{3+\sqrt{2}-\sqrt{3}-\sqrt{6}}{\sqrt{6}-2} \cdot \frac{\sqrt{6}+2}{\sqrt{6}+2}$$

$$= \frac{\sqrt{6}-\sqrt{2}}{2} = \frac{\sqrt{3}-1}{\sqrt{2}} = \sqrt{\frac{(\sqrt{3}-1)^2}{2}}$$

$$= \sqrt{2-\sqrt{3}}. \tag{3.235}$$

Therefore, based upon the results in (3.234) and (3.44), we obtain

$$\int_{\sqrt{3}-\sqrt{2}}^{1} \left(\int_{\sqrt{3}-\sqrt{2}}^{1} \frac{1}{(x+y)^2 + (1+xy)^2} \mathrm{d}x \right) \mathrm{d}y = \frac{1}{2} \operatorname{Ti}_2(2-\sqrt{3}), \tag{3.236}$$

where $\operatorname{Ti}_2(2-\sqrt{3})$ is a special value of the Inverse tangent integral (see [44, Chapter 2, p. 45]).

To write $\operatorname{Ti}_2(2-\sqrt{3})$ in known constants, we consider the well-known Ramanujan's formula (see [44, p. 45], [111, (3)]),

$$\sum_{n=1}^{\infty} \frac{\sin(2(2n-1)x)}{(2n-1)^2} = \operatorname{Ti}_2(\tan(x)) - x \log(\tan(x)), \ 0 < x < \frac{\pi}{2}, \tag{3.237}$$

that is proved by differentiation and the use of the Fourier series of $\log(\tan(x))$ (see (3.74)). Then, by setting $x = \dfrac{\pi}{12}$ in (3.237), we get in the left-hand side of the equality,

$$\sum_{n=1}^{\infty} \frac{\sin\left(\dfrac{(2n-1)\,\pi}{6}\right)}{(2n-1)^2} = \frac{1}{2}\left(1 - \frac{1}{3^2} + \frac{1}{5^2} - \cdots\right) + \frac{1}{6}\left(1 - \frac{1}{3^2} + \frac{1}{5^2} - \cdots\right)$$

$$= \frac{2}{3} G, \tag{3.238}$$

where the numbers generated by $\sin\left(\dfrac{(2n-1)\pi}{6}\right)$ are written as $3 \cdot \dfrac{1}{2} - \dfrac{1}{2}$ when getting 1, and $\dfrac{1}{2} - 3 \cdot \dfrac{1}{2}$ when having -1, and then the terms are grouped as seen above.

Using in (3.237) the fact that $\tan\left(\frac{\pi}{12}\right) = 2-\sqrt{3}$ combined with the result from (3.238), we immediately get

$$\operatorname{Ti}_2(2-\sqrt{3}) = \frac{2}{3}G + \frac{\pi}{12}\log(2-\sqrt{3}). \tag{3.239}$$

Hence, by plugging the result from (3.239) in (3.236), we conclude that

$$\int_{\sqrt{3}-\sqrt{2}}^{1}\int_{\sqrt{3}-\sqrt{2}}^{1} \frac{1}{(x+y)^2+(1+xy)^2}\mathrm{d}x\mathrm{d}y = \frac{1}{3}\left(G + \frac{\pi}{8}\log\left(2-\sqrt{3}\right)\right),$$

and the part $i)$ of the problem is finalized.

An alternative solution to the present double integral may be found in *La Gaceta de la RSME*, Vol. 20, No. 3 (2017) (see [42, pp. 566–567]).

For the second point of the problem we need an introduction to the Legendre's chi function[21] of order 2, and then return to Lewin's book in [44, Chapter 1, pp. 18–21]. The Legendre's chi function of order 2 is defined in terms of Dilogarithm function (no wonder why it appears in the referenced book), $\chi_2(x) = \sum_{n=1}^{\infty}\frac{x^{2n-1}}{(2n-1)^2} = \frac{1}{2}(\operatorname{Li}_2(x) - \operatorname{Li}_2(-x))$. Also we may notice the similarity with the Inverse tangent function which has the alternating series version of the Legendre's chi function of order 2.

Now, for the point $ii)$ of the problem, we start out by proving the auxiliary result

$$\int_a^1\left(\int_b^1 \frac{1}{(x+y)(1+xy)}\mathrm{d}x\right)\mathrm{d}y = \int_0^{\frac{1-a}{1+a}}\left(\int_0^{\frac{1-b}{1+b}} \frac{1}{1-x^2y^2}\mathrm{d}x\right)\mathrm{d}y$$
$$= \chi_2\left(\frac{1-a}{1+a}\cdot\frac{1-b}{1+b}\right). \tag{3.240}$$

Proof Letting the variable changes, $x = \dfrac{1-u}{1+u}$ and $y = \dfrac{1-v}{1+v}$, we get

$$\int_a^1\left(\int_b^1 \frac{1}{(x+y)(1+xy)}\mathrm{d}x\right)\mathrm{d}y$$

[21] More generally, we have the notation, $\chi_v(z) = \sum_{n=1}^{\infty}\frac{z^{2n-1}}{(2n-1)^v}$, which represents the Legendre's chi function of order v.

$$= \int_0^{\frac{1-a}{1+a}} \left(\int_0^{\frac{1-b}{1+b}} \frac{1}{1-u^2v^2} \mathrm{d}u \right) \mathrm{d}v = \int_0^{\frac{1-a}{1+a}} \left(\int_0^{\frac{1-b}{1+b}} \sum_{n=1}^{\infty} (uv)^{2n-2} \mathrm{d}u \right) \mathrm{d}v$$

$$= \sum_{n=1}^{\infty} \int_0^{\frac{1-a}{1+a}} u^{2n-2} \mathrm{d}u \int_0^{\frac{1-b}{1+b}} v^{2n-2} \mathrm{d}v = \sum_{n=1}^{\infty} \frac{\left(\frac{1-a}{1+a} \cdot \frac{1-b}{1+b} \right)^{2n-1}}{(2n-1)^2}$$

$$= \chi_2 \left(\frac{1-a}{1+a} \cdot \frac{1-b}{1+b} \right),$$

and the proof of the auxiliary result is finalized.

Then, if we replace a by $\dfrac{1-a}{1+a}$ and b by $\dfrac{1-b}{1+b}$ in (3.240), we obtain

$$\int_{\frac{1-a}{1+a}}^1 \left(\int_{\frac{1-b}{1+b}}^1 \frac{1}{(x+y)(1+xy)} \mathrm{d}x \right) \mathrm{d}y = \int_0^a \left(\int_0^b \frac{1}{1-x^2y^2} \mathrm{d}x \right) \mathrm{d}y = \chi_2(a \cdot b). \tag{3.241}$$

Since we have that

$$\theta = \sqrt{2} + 1 - \sqrt{2\left(\sqrt{2}+1\right)} = 1 + \frac{\left(\sqrt{2} - \sqrt{2(\sqrt{2}+1)}\right)\left(\sqrt{2} + \sqrt{2(\sqrt{2}+1)}\right)}{\sqrt{2} + \sqrt{2(\sqrt{2}+1)}}$$

$$= 1 - \frac{2}{1 + \sqrt{\sqrt{2}+1}} = \frac{\sqrt{\sqrt{2}+1} - 1}{\sqrt{\sqrt{2}+1} + 1} = \frac{\sqrt{\frac{1}{\sqrt{2}-1}} - 1}{\sqrt{\frac{1}{\sqrt{2}-1}} + 1} = \frac{1 - \sqrt{\sqrt{2}-1}}{1 + \sqrt{\sqrt{2}-1}},$$

we see we can choose $a = \sqrt{\sqrt{2}-1}$ and $b = \sqrt{\sqrt{2}-1}$ in (3.241), and then we obtain the desired result,

$$\int_\theta^1 \left(\int_\theta^1 \frac{1}{(x+y)(1+xy)} \mathrm{d}x \right) \mathrm{d}y = \chi_2(\sqrt{2}-1) = \left(\frac{\pi}{4}\right)^2 - \left(\frac{\log(\sqrt{2}+1)}{2} \right)^2.$$

Note that $\chi_2(\sqrt{2}-1)$ is a special value of the Legendre's chi function of order 2, and it can be derived from the well-known identity (see [44, Chapter 1, p. 19], [78, Chapter 2, p. 108]),

$$\chi_2 \left(\frac{1-x}{1+x} \right) + \chi_2(x) = \frac{\pi^2}{8} + \frac{1}{2} \log(x) \log\left(\frac{1+x}{1-x} \right),$$

where we set $x = \sqrt{2} - 1$, and the part ii) of the problem is finalized.

To conclude, we have seen this time that choosing the right variable change made a huge difference for both proposed integrals, and in particular the variable change of the type, $x = \dfrac{1-y}{1+y}$, sometimes is proving to be very useful. For example, remember the problem $A5$, *Evaluate* $\displaystyle\int_0^1 \frac{\log(1+x)}{1+x^2}\mathrm{d}x$, from the 66th Putnam competition, 2005 (see [73]), where such a variable change works perfectly nice and it was also considered in [74] (see (3.92) from the solution to the Sect. 1.18). As a note, the mentioned Putnam integral is an old problem in the mathematical literature, also known as *Serret's integral*, after the French mathematician Joseph Serret (1819–1885), according to [64, p. 54], and an alternative solution you may also find in [62, pp. 121–122], [64, pp. 53–54], and [25, Chapter 4, p. 76].

3.45 An Out-of-Order Integral with an Integrand Expressed in Terms of an Infinite Series and a Generalization of It

Solution *That's a cool, different type of integrand, one involving an infinite series!*, a possible reaction at the sight of the results. The calculations are straightforward if we choose the right *tool*, which in this case is one of the Ramanujan's integrals. More exactly, the generalization presented in this section may be viewed as an application of the first Ramanujan integral from Sect. 1.42.

It's enough to solve the point ii) of the problem since the point i) is the case $\theta = 2$ of the generalization, and making use of the Ramanujan's result in (1.65), we write

$$\int_0^\infty \frac{1}{x}\sum_{n=1}^\infty \left(\left(\theta^n x+1\right)^{-\theta^{n+1}} - \left(\theta^{n+1}x+1\right)^{-\theta^n}\right)\mathrm{d}x$$

$$= \sum_{n=1}^\infty \int_0^\infty \frac{\left(\theta^n x+1\right)^{-\theta^{n+1}} - \left(\theta^{n+1}x+1\right)^{-\theta^n}}{x}\mathrm{d}x$$

$$= \sum_{n=1}^\infty \left(\psi\left(\theta^n\right) - \psi\left(\theta^{n+1}\right) + \log(\theta)\right)$$

$$= \lim_{N\to\infty}\sum_{n=1}^N \left(\psi\left(\theta^n\right) - \psi\left(\theta^{n+1}\right) + \log(\theta)\right)$$

$$= \psi(\theta) + \lim_{N\to\infty}\left(\log(\theta)N - \psi\left(\theta^{N+1}\right)\right),$$

where if we consider the asymptotic behavior of Digamma function (see [1, p. 259], [78, Chapter 1, p. 22]), that is $\psi(x) = \log(x) + O\left(\frac{1}{x}\right)$, we conclude that

$$\int_0^\infty \frac{1}{x} \sum_{n=1}^\infty \left(\left(\theta^n x + 1\right)^{-\theta^{n+1}} - \left(\theta^{n+1} x + 1\right)^{-\theta^n} \right) \mathrm{d}x = \psi(\theta) - \log(\theta),$$

which finalizes the point $ii)$ of the problem.

To get the part $i)$ of the problem, we set $\theta = 2$ in the generalization above and use that $\psi(2) = 1 - \gamma$, based on the series representation of Digamma function, $\psi(x) = -\gamma + \sum_{n=0}^\infty \left(\frac{1}{n+1} - \frac{1}{n+x}\right)$, that further gives

$$\int_0^\infty \frac{1}{x} \sum_{n=1}^\infty \left(\left(2^n x + 1\right)^{-2^{n+1}} - \left(2^{n+1} x + 1\right)^{-2^n} \right) \mathrm{d}x = 1 - \gamma - \log(2),$$

which finalizes the point $i)$ of the problem.

For such problems, one might usually think to reverse the order of summation and integration (assuming its validity) and try to integrate first as in my problem (where we had to make use of a challenging integral by Ramanujan).

3.46 Pretty Charming Ramanujan-Like (Double) Integral Representations of the Riemann Zeta Function and Its Derivative

Solution Soon after obtaining the results meant to be proved in this section, I felt a looking-similarity with at least one of the results by Ramanujan like, for example, $\int_0^1 \left(\frac{x^{c-1}}{1-x} - \frac{bx^{bc-1}}{1-x^b}\right) \sum_{k=0}^\infty x^{ab^k} \mathrm{d}x = \psi\left(\frac{a}{b} + c\right) - \log\left(\frac{a}{b}\right)$ (see [8, p. 154]), and hence that description in the title, *Ramanujan-like (double) integral*, although we speak here about a double integral (and if you're interested, the result I just mentioned can be easily proved by using Ramanujan's integral in (1.66)). In the present section I'll try to approach the part $i)$ of the problem in two different ways, therefore giving two solutions.

For a first solution to the result from the point $i)$, we start with the fact that $\frac{1}{(1-x)(1-xy)} = \sum_{k=1}^\infty x^{k-1} \frac{1-y^k}{1-y}$, and then, making the change of variable $x = 1 - t$, we have

$$\int_0^1\left(\int_0^1 \frac{1}{x(1-(1-x)y)}\sum_{n=1}^{\infty}(xy)^{n^s}\mathrm{d}x\right)\mathrm{d}y = \int_0^1\left(\int_0^1 \sum_{n=1}^{\infty}\frac{((1-t)y)^{n^s}}{(1-t)(1-ty)}\mathrm{d}t\right)\mathrm{d}y$$

$$= \int_0^1\left(\int_0^1 \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} t^{k-1}\frac{1-y^k}{1-y}((1-t)y)^{n^s}\right)\mathrm{d}t\right)\mathrm{d}y$$

$$= \int_0^1\left(\int_0^1 \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\sum_{i=1}^{k} y^{n^s+i-1}t^{k-1}(1-t)^{n^s}\right)\mathrm{d}t\right)\mathrm{d}y$$

$$= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\sum_{i=1}^{k}\int_0^1 y^{n^s+i-1}\left(\int_0^1 t^{k-1}(1-t)^{n^s}\mathrm{d}t\right)\mathrm{d}y\right)$$

$$= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} B(k, n^s+1)\sum_{i=1}^{k}\frac{1}{n^s+i}\right)$$

$$\left\{\text{make use of the identity, } \mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\right\}$$

$$= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{\Gamma(k)\Gamma(n^s+1)}{\Gamma(k+n^s+1)}\sum_{i=1}^{k}\frac{1}{n^s+i}\right) = -\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\lim_{x\to n^s}\frac{d}{dx}\left(\frac{\Gamma(k)\Gamma(x+1)}{\Gamma(k+x+1)}\right)\right)$$

$$= -\sum_{n=1}^{\infty}\left(\lim_{x\to n^s}\frac{d}{dx}\left(\sum_{k=1}^{\infty}\frac{\Gamma(k)\Gamma(x+1)}{\Gamma(k+x+1)}\right)\right) = -\sum_{n=1}^{\infty}\left(\lim_{x\to n^s}\frac{d}{dx}\left(\sum_{k=1}^{\infty}B(k,x+1)\right)\right)$$

$$= -\sum_{n=1}^{\infty}\left(\lim_{x\to n^s}\frac{d}{dx}\left(\sum_{k=1}^{\infty}\int_0^1 y^{k-1}(1-y)^x\mathrm{d}y\right)\right)$$

$$= -\sum_{n=1}^{\infty}\left(\lim_{x\to n^s}\frac{d}{dx}\left(\int_0^1\sum_{k=1}^{\infty} y^{k-1}(1-y)^x\mathrm{d}y\right)\right)$$

$$= -\sum_{n=1}^{\infty}\lim_{x\to n^s}\frac{d}{dx}\left(\int_0^1(1-y)^{x-1}\mathrm{d}y\right)$$

$$\overset{1-y=z}{=} -\sum_{n=1}^{\infty}\lim_{x\to n^s}\frac{d}{dx}\left(\int_0^1 z^{x-1}\mathrm{d}z\right) = -\sum_{n=1}^{\infty}\lim_{x\to n^s}\frac{d}{dx}\left(\frac{1}{x}\right) = \sum_{n=1}^{\infty}\frac{1}{n^{2s}} = \zeta(2s),$$

and the first solution to the point $i)$ of the problem is finalized.

To prove differently the result from the point i), we start with writing that $\dfrac{1}{1-(1-x)y}=\sum_{k=1}^{\infty}((1-x)y)^{k-1}$, and then we have

$$\int_0^1\left(\int_0^1 \frac{1}{x(1-(1-x)y)}\sum_{n=1}^{\infty}(xy)^{n^s}\mathrm{d}x\right)\mathrm{d}y$$

$$=\int_0^1\left(\int_0^1\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}x^{n^s-1}(1-x)^{k-1}y^{n^s+k-1}\right)\mathrm{d}x\right)\mathrm{d}y$$

{reverse the order of summation and integration}

$$=\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\int_0^1\left(\int_0^1 x^{n^s-1}(1-x)^{k-1}y^{n^s+k-1}\mathrm{d}x\right)\mathrm{d}y\right)$$

$$=\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{B(n^s,k)}{k+n^s}\right)=\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{\Gamma(k)\Gamma(n^s)}{\Gamma(k+n^s+1)}\right)$$

$$=\sum_{n=1}^{\infty}\Gamma(n^s)\left(\sum_{k=1}^{\infty}\frac{\Gamma(k)}{\Gamma(k+n^s+1)}\right)$$

$$=\sum_{n=1}^{\infty}\frac{\Gamma(n^s)}{n^s}\left(\sum_{k=1}^{\infty}\left(\frac{\Gamma(k)}{\Gamma(k+n^s)}-\frac{\Gamma(k+1)}{\Gamma(k+n^s+1)}\right)\right)$$

$$=\sum_{n=1}^{\infty}\frac{\Gamma(n^s)}{n^s}\left(\lim_{N\to\infty}\sum_{k=1}^{N}\left(\frac{\Gamma(k)}{\Gamma(k+n^s)}-\frac{\Gamma(k+1)}{\Gamma(k+n^s+1)}\right)\right)$$

$$=\sum_{n=1}^{\infty}\frac{\Gamma(n^s)}{n^s}\left(\frac{1}{\Gamma(n^s+1)}-\lim_{N\to\infty}\frac{\Gamma(N+1)}{\Gamma(N+n^s+1)}\right)=\sum_{n=1}^{\infty}\frac{1}{n^{2s}}=\zeta(2s),$$

where the limit is 0 based on the Stirling's formula, $\Gamma(z+1)\approx\sqrt{2\pi z}\left(\dfrac{z}{e}\right)^z$ (see [1, p. 257], [9, p. 21], [34, p. 86–88], [33, Chapter 2, pp. 165–167], and [78, Chapter 1, p. 8]), and the second solution to the point i) of the problem is finalized.

The second point of the problem is straightforward if differentiating the identity from the point i) with respect to s and then letting $s\to 1$, which gives immediately

$$\int_0^1\int_0^1\frac{\log(xy)}{x(1-(1-x)y)}\sum_{n=1}^{\infty}n\log(n)(xy)^n\mathrm{d}x\mathrm{d}y=\frac{\pi^2}{3}\log\left(\frac{2\pi e^{\gamma}}{A^{12}}\right),$$

where I used that $\zeta'(2) = \dfrac{\pi^2}{6}\log\left(\dfrac{2\pi e^{\gamma}}{A^{12}}\right)$, A is the Glaisher–Kinkelin constant (see [109], [27, pp. 135–136], [78, Chapter 1, p. 25]), and the point $ii)$ of the problem is finalized.

It's pretty enjoyable how things play out during the solutions, the options to consider on the way. The first solution seems to be less obvious if we consider the given form of the geometric series as a starting point.

3.47 The Elementary Calculation of a Fractional Part Integral Naturally Arising in an Exotic Triple Fractional Part Integral

Solution In this section we have arrived in the realm of the fractional part integrals, and some of them are really interesting and tricky as you'll see soon. Particular cases of the proposed generalized integral may be seen as the desired forms to which one might like to reduce some of the problems involving the fractional part integrals, and an immediate example would be the integral in the next section. In the present solution I will try an elementary approach involving the integration by parts that will lead to a useful recurrence relation.

Considering the well-known relation between the fractional part function ([105]) and the integer part of x ([104, 110], [31, pp. 67–70]), $\{x\} = x - \lfloor x \rfloor$, and making the change of variable $x = 1/y$, we write

$$\int_0^1 x^p \left\{\frac{1}{x}\right\}^q \mathrm{d}x = \int_1^\infty y^{-p-2}\{y\}^q \mathrm{d}y$$

$$\{\text{split the integration interval into intervals of the type } [k, k+1)\}$$

$$= \sum_{k=1}^{\infty} \int_k^{k+1} y^{-p-2}\{y\}^q \mathrm{d}y = \sum_{k=1}^{\infty} \int_k^{k+1} y^{-p-2}(y - \lfloor y \rfloor)^q \mathrm{d}y$$

$$\{\text{turn } \lfloor y \rfloor \text{ into } k \text{ since } y \text{ runs on } [k, k+1)\}$$

$$= \sum_{k=1}^{\infty} \underbrace{\int_k^{k+1} y^{-p-2}(y-k)^q \mathrm{d}y}_{I_{p,q,k}}, \tag{3.242}$$

where I denoted the last integral by $I_{p,q,k}$.

Then, integrating by parts, we have

$$I_{p,q,k} = -\int_k^{k+1} \left(\frac{y^{-p-1}}{p+1}\right)' (y-k)^q \, \mathrm{d}y$$

$$= -\frac{y^{-p-1}}{p+1}(y-k)^q \Big|_{y=k}^{y=k+1} + \frac{q}{p+1}\int_k^{k+1} y^{-p-1}(y-k)^{q-1} \, \mathrm{d}y$$

$$= -\frac{1}{(p+1)(k+1)^{p+1}} + \frac{q}{p+1} I_{p-1,q-1,k},$$

whence we obtain, after multiplying both sides by $p+1$, that

$$(p+1)I_{p,q,k} = -\frac{1}{(k+1)^{p+1}} + qI_{p-1,q-1,k}. \tag{3.243}$$

Setting $p = q = n$ in (3.243), we have

$$(n+1)I_{n,n,k} - nI_{n-1,n-1,k} = -\frac{1}{(k+1)^{n+1}}. \tag{3.244}$$

Replacing n by i in (3.244), $(i+1)I_{i,i,k} - iI_{i-1,i-1,k} = -\frac{1}{(k+1)^{i+1}}$, giving values to i from $i = 1$ to n and then summing up all these relations, we get $(n+1)I_{n,n,k} = \frac{1}{k(k+1)} - \sum_{i=1}^{n} \frac{1}{(k+1)^{i+1}}$, or

$$I_{n,n,k} = \frac{1}{k(k+1)(n+1)} - \frac{1}{n+1}\sum_{i=1}^{n} \frac{1}{(k+1)^{i+1}}. \tag{3.245}$$

Thus, by combining the results in (3.242) and (3.245), we have

$$\int_0^1 x^n \left\{\frac{1}{x}\right\}^n \mathrm{d}x = \sum_{k=1}^{\infty} I_{n,n,k} = \sum_{k=1}^{\infty}\left(\frac{1}{k(k+1)(n+1)} - \frac{1}{n+1}\sum_{i=1}^{n}\frac{1}{(k+1)^{i+1}}\right)$$

$$= \frac{1}{n+1}\sum_{k=1}^{\infty}\frac{1}{k(k+1)} - \frac{1}{n+1}\sum_{k=1}^{\infty}\sum_{i=1}^{n}\frac{1}{(k+1)^{i+1}}$$

$$= \frac{1}{n+1} - \frac{1}{n+1}\sum_{i=1}^{n}\sum_{k=1}^{\infty}\frac{1}{(k+1)^{i+1}} = \frac{1}{n+1} - \frac{1}{n+1}\sum_{i=1}^{n}(\zeta(i+1)-1)$$

$$= \frac{1}{n+1} + \sum_{i=1}^{n}\frac{1}{n+1} - \frac{1}{n+1}\sum_{i=1}^{n}\zeta(i+1) = 1 - \frac{1}{n+1}\sum_{i=1}^{n}\zeta(i+1). \tag{3.246}$$

Before treating the case with $p > q > 1$, we calculate another particular case, the initial integral with $q = 1$, and we have, by making the change of variable $x = 1/y$,

$$\int_0^1 x^p \left\{\frac{1}{x}\right\} \mathrm{d}x = \int_1^\infty \frac{\{y\}}{y^{p+2}} \mathrm{d}y$$

{split the integration interval into intervals of the type $[k, k+1)$}

$$= \sum_{k=1}^{\infty} \int_k^{k+1} \frac{\{y\}}{y^{p+2}} \mathrm{d}y = \sum_{k=1}^{\infty} \int_k^{k+1} \frac{y - \lfloor y \rfloor}{y^{p+2}} \mathrm{d}y = \sum_{k=1}^{\infty} \int_k^{k+1} \frac{y-k}{y^{p+2}} \mathrm{d}y$$

$$= \sum_{k=1}^{\infty} \int_k^{k+1} \frac{1}{y^{p+1}} \mathrm{d}y - \sum_{k=1}^{\infty} \int_k^{k+1} \frac{k}{y^{p+2}} \mathrm{d}y$$

$$= \sum_{k=1}^{\infty} \left(-\frac{1}{p y^p} \Bigg|_{y=k}^{y=k+1} \right) - \sum_{k=1}^{\infty} \left(-\frac{k}{(p+1) y^{p+1}} \Bigg|_{y=k}^{y=k+1} \right)$$

$$= \frac{1}{p} \sum_{k=1}^{\infty} \left(\frac{1}{k^p} - \frac{1}{(k+1)^p} \right) - \frac{1}{p+1} \sum_{k=1}^{\infty} \left(\frac{1}{k^p} - \frac{k}{(k+1)^{p+1}} \right)$$

$$= \frac{1}{p(p+1)} \sum_{k=1}^{\infty} \left(\frac{1}{k^p} - \frac{1}{(k+1)^p} \right) - \frac{1}{p+1} \sum_{k=1}^{\infty} \frac{1}{(k+1)^{p+1}}$$

$$= \frac{1}{p(p+1)} - \frac{1}{p+1} (\zeta(p+1) - 1) = \frac{1}{p} - \frac{\zeta(p+1)}{p+1}. \tag{3.247}$$

For the case $p > q > 1$, we use the relation in (3.243) where we multiply both sides by $\dfrac{p!}{q!}$ that leads to

$$\frac{(p+1)!}{q!} I_{p,q,k} = -\frac{p!}{q!(k+1)^{p+1}} + \frac{p!}{(q-1)!} I_{p-1,q-1,k},$$

and exploiting the difference between p and q, we can write the relation above as

$$\begin{aligned} \frac{(p-q+i+2)!}{(i+1)!} I_{p-q+i+1,i+1,k} - \frac{(p-q+i+1)!}{i!} I_{p-q+i,i,k} \\ = -\frac{(p-q+i+1)!}{(i+1)!(k+1)^{p-q+i+2}}. \end{aligned} \tag{3.248}$$

Summing up both sides of (3.47) from $i = 1$ to $i = q - 1$, we obtain

$$\frac{(p+1)!}{q!} I_{p,q,k} = (p-q+2)! I_{p-q+1,1,k} - \sum_{i=1}^{q-1} \frac{(p-q+i+1)!}{(i+1)!(k+1)^{p-q+i+2}}. \tag{3.249}$$

Then, we sum both sides of (3.249), from $k = 1$ to ∞, that yields

$$\frac{(p+1)!}{q!} \sum_{k=1}^{\infty} I_{p,q,k} = (p-q+2)! \sum_{k=1}^{\infty} I_{p-q+1,1,k}$$
$$- \sum_{k=1}^{\infty} \sum_{i=1}^{q-1} \frac{(p-q+i+1)!}{(i+1)!(k+1)^{p-q+i+2}},$$

and based upon (3.242), the previous equality becomes

$$\frac{(p+1)!}{q!} \int_0^1 x^p \left\{\frac{1}{x}\right\}^q \mathrm{d}x$$

$$= (p-q+2)! \int_0^1 x^{p-q+1} \left\{\frac{1}{x}\right\} \mathrm{d}x - \sum_{k=1}^{\infty} \sum_{i=1}^{q-1} \frac{(p-q+i+1)!}{(i+1)!(k+1)^{p-q+i+2}}$$

$$\left\{\text{make use of the result in (3.247) to get the value of } \int_0^1 x^{p-q+1} \left\{\frac{1}{x}\right\} \mathrm{d}x\right\}$$

$$= (p-q+2)! \left(\frac{1}{p-q+1} - \frac{\zeta(p-q+2)}{p-q+2}\right) - \sum_{i=1}^{q-1} \sum_{k=1}^{\infty} \frac{(p-q+i+1)!}{(i+1)!(k+1)^{p-q+i+2}}$$

$$= \frac{(p-q+2)!}{p-q+1} - (p-q+1)!\zeta(p-q+2) - \sum_{i=1}^{q-1} \frac{(p-q+i+1)!}{(i+1)!} (\zeta(p-q+i+2)-1)$$

$$\{\text{make the change of variable } q - i = j \text{ in the remaining sum}\}$$

$$= \frac{(p-q+2)!}{p-q+1} - (p-q+1)!\zeta(p-q+2) - \sum_{j=1}^{q-1} \frac{(p-j+1)!}{(q-j+1)!} (\zeta(p-j+2)-1),$$

whence we obtain that

$$\int_0^1 x^p \left\{\frac{1}{x}\right\}^q \mathrm{d}x = \frac{(p-q+2)!q!}{(p-q+1)(p+1)!} - \frac{(p-q+1)!q!}{(p+1)!} \zeta(p-q+2)$$

$$-\frac{q!}{(p+1)!}\sum_{j=1}^{q-1}\frac{(p-j+1)!}{(q-j+1)!}(\zeta(p-j+2)-1)$$

$$=\frac{(p-q+2)!q!}{(p-q+1)(p+1)!}-\frac{(p-q+1)!q!}{(p+1)!}-\frac{(p-q+1)!q!}{(p+1)!}(\zeta(p-q+2)-1)$$

$$-\frac{q!}{(p+1)!}\sum_{j=1}^{q-1}\frac{(p-j+1)!}{(q-j+1)!}(\zeta(p-j+2)-1)$$

$$\left\{\text{notice that } \frac{(p-q+1)!q!}{(p+1)!}(\zeta(p-q+2)-1) \text{ may be absorbed by the sum}\right\}$$

$$=\frac{q!}{(p+1)!}\left((p-q)!-\sum_{j=1}^{q}\frac{(p-j+1)!}{(q-j+1)!}(\zeta(p-j+2)-1)\right), \tag{3.250}$$

and the case $p > q > 1$ is complete.

Lastly, combining the results from (3.246), (3.247), and (3.250), we obtain, for p, q positive integers, $p \geq q$, that

$$\int_0^1 x^p \left\{\frac{1}{x}\right\}^q \mathrm{d}x$$

$$=\frac{q!}{(p+1)!}\left((p-q)!-\sum_{j=1}^{q}\frac{(p-j+1)!}{(q-j+1)!}(\zeta(p-j+2)-1)\right),$$

and the solution is finalized.

Two particular cases of this generalization are presented by Paul in *Inside Interesting Integrals*, that is ($p = 1, q = 1$) and ($p = 2, q = 1$) (see [64, Chapter 5, pp. 183–184]). For a generalization of the integral (one that expresses the final answer in terms of an infinite series, which is different from the type of my answer above that gives the exact value of the integral), with p, q positive reals, see [29, 2.22, p. 103]. In his solution, the author also arrives at the integral in (3.242) where he makes the change of variable $z = y - k$ and use the Gamma function. The curious reader might try to continue from this point to get another solution.

3.48 The Calculation of a Beautiful Triple Fractional Part Integral with a Cubic Power

Solution I submitted the square power version of the proposed triple integral to *The American Mathematical Monthly*, the problem **11902** (see [87]). The difficulty with this triple integral comes from the very beginning, *How to even start the calculations?* An important objective is to reduce the triple integral to simpler integrals, and I'll show it can be reduced to calculations involving two particular cases of the generalization in the previous section. For getting a solution, we'll simply combine the variable changes with the integration by parts.

Denoting the fractional part integral by $\mathscr{I}$ and making the variable change $x/y = u$, we get

$$\mathscr{I} = \int_0^1 \left(\int_0^1 \left(\int_0^1 \left(\left\{ \frac{x}{y} \right\} \left\{ \frac{y}{z} \right\} \left\{ \frac{z}{x} \right\} \right)^3 \mathrm{d}x \right) \mathrm{d}y \right) \mathrm{d}z$$

$$= \int_0^1 \left(\int_0^1 y \left(\int_0^{1/y} \left(\{u\} \left\{ \frac{y}{z} \right\} \left\{ \frac{z}{uy} \right\} \right)^3 \mathrm{d}u \right) \mathrm{d}y \right) \mathrm{d}z,$$

and after changing the integration order

$$\mathscr{I} = \int_0^1 y \left(\int_0^{1/y} \left(\int_0^1 \left(\{u\} \left\{ \frac{y}{z} \right\} \left\{ \frac{z}{uy} \right\} \right)^3 \mathrm{d}z \right) \mathrm{d}u \right) \mathrm{d}y,$$

we make the change of variable $z/y = t$ that yields

$$\mathscr{I} = \int_0^1 y^2 \left(\int_0^{1/y} \left(\int_0^{1/y} \left(\{u\} \left\{ \frac{1}{t} \right\} \left\{ \frac{t}{u} \right\} \right)^3 \mathrm{d}t \right) \mathrm{d}u \right) \mathrm{d}y. \tag{3.251}$$

Then, recall the Leibniz integral rule, $\frac{d}{dz} \int_{a(z)}^{b(z)} f(x, z) \mathrm{d}x = \int_{a(z)}^{b(z)} \frac{\partial}{\partial z} f(x, z) \mathrm{d}x + f(b(z), z) \frac{d}{dz} b(z) - f(a(z), z) \frac{d}{dz} a(z)$ (see [1, p. 11]), where if applying it for a function of the form, $g(z, x) = \int_0^{b(x)} f(y, z) \mathrm{d}y$, we obtain

$$\frac{d}{dx} \left(\int_0^{a(x)} g(z, x) \mathrm{d}z \right) = \frac{d}{dx} \left(\int_0^{a(x)} \left(\int_0^{b(x)} f(y, z) \mathrm{d}y \right) \mathrm{d}z \right)$$

$$= \frac{d}{dx}(b(x)) \int_0^{a(x)} f(b(x), z) \mathrm{d}z + \frac{d}{dx}(a(x)) \int_0^{b(x)} f(y, a(x)) \mathrm{d}y. \tag{3.252}$$

Using the result from (3.252) in (3.251), where we apply the integration by parts, we have

$$\mathscr{I}=\int_0^1\left(\frac{y^3}{3}\right)'\left(\int_0^{1/y}\left(\int_0^{1/y}\left(\{u\}\left\{\frac{1}{t}\right\}\left\{\frac{t}{u}\right\}\right)^3\mathrm{d}t\right)\mathrm{d}u\right)\mathrm{d}y$$

$$=\frac{y^3}{3}\int_0^{1/y}\left(\int_0^{1/y}\left(\{u\}\left\{\frac{1}{t}\right\}\left\{\frac{t}{u}\right\}\right)^3\mathrm{d}t\right)\mathrm{d}u\Bigg|_{y=0}^{y=1}$$

$$-\frac{1}{3}\int_0^1 y^3\frac{d}{dy}\left(\int_0^{1/y}\left(\int_0^{1/y}\left(\{u\}\left\{\frac{1}{t}\right\}\left\{\frac{t}{u}\right\}\right)^3\mathrm{d}t\right)\mathrm{d}u\right)\mathrm{d}y$$

$$=\frac{1}{3}\int_0^1\left(\int_0^1\left(\{u\}\left\{\frac{1}{t}\right\}\left\{\frac{t}{u}\right\}\right)^3\mathrm{d}t\right)\mathrm{d}u$$

$$-\lim_{y\to 0}\left(\frac{y^3}{3}\int_0^{1/y}\left(\int_0^{1/y}\left(\{u\}\left\{\frac{1}{t}\right\}\left\{\frac{t}{u}\right\}\right)^3\mathrm{d}t\right)\mathrm{d}u\right)$$

$$+\frac{1}{3}\int_0^1\left(\int_0^{1/y}y\left(\{u\}\{y\}\left\{\frac{1}{uy}\right\}\right)^3\mathrm{d}u\right)\mathrm{d}y$$

$$+\frac{1}{3}\int_0^1\left(\int_0^{1/y}y\left(\{ty\}\left\{\frac{1}{y}\right\}\left\{\frac{1}{t}\right\}\right)^3\mathrm{d}t\right)\mathrm{d}y$$

{use that on [0, 1) we have $\{x\}=x$}

$$=\frac{1}{3}\int_0^1\left(\int_0^1 u^3\left(\left\{\frac{1}{t}\right\}\left\{\frac{t}{u}\right\}\right)^3\mathrm{d}t\right)\mathrm{d}u+\frac{1}{3}\int_0^1\left(\int_0^{1/y}y^4\left(\{u\}\left\{\frac{1}{uy}\right\}\right)^3\mathrm{d}u\right)\mathrm{d}y$$

$$+\frac{1}{3}\int_0^1\left(\int_0^{1/y}y\left(\{ty\}\left\{\frac{1}{y}\right\}\left\{\frac{1}{t}\right\}\right)^3\mathrm{d}t\right)\mathrm{d}y, \tag{3.253}$$

where I also used that the limit is 0 by the *Pinching theorem* (see [114]) since

$$0\le\frac{y^3}{3}\int_0^{1/y}\left(\int_0^{1/y}\left(\{u\}\left\{\frac{1}{t}\right\}\left\{\frac{t}{u}\right\}\right)^3\mathrm{d}t\right)\mathrm{d}u\le\frac{y}{3}.$$

Now, in (3.253) we make the changes of variable $uy = w$ in the second integral and $ty = z$ in the third integral, and then we get

$$\int_0^1 \left(\int_0^{1/y} y^4 \left(\{u\}\left\{\frac{1}{uy}\right\}\right)^3 \mathrm{d}u\right) \mathrm{d}y = \int_0^1 \left(\int_0^1 y^3 \left(\left\{\frac{w}{y}\right\}\left\{\frac{1}{w}\right\}\right)^3 \mathrm{d}w\right) \mathrm{d}y, \tag{3.254}$$

and for the third integral we have

$$\int_0^1 \left(\int_0^{1/y} y \left(\{ty\}\left\{\frac{1}{y}\right\}\left\{\frac{1}{t}\right\}\right)^3 \mathrm{d}t\right) \mathrm{d}y = \int_0^1 \left(\int_0^1 \left(\{z\}\left\{\frac{1}{y}\right\}\left\{\frac{y}{z}\right\}\right)^3 \mathrm{d}z\right) \mathrm{d}y$$

{use that on $[0, 1)$ we have $\{z\} = z$}

$$= \int_0^1 \left(\int_0^1 z^3 \left(\left\{\frac{1}{y}\right\}\left\{\frac{y}{z}\right\}\right)^3 \mathrm{d}z\right) \mathrm{d}y. \tag{3.255}$$

Using the results from (3.254) and (3.255) in (3.253), we get

$$\mathscr{I} = \frac{1}{3}\int_0^1 \left(\int_0^1 u^3 \left(\left\{\frac{1}{t}\right\}\left\{\frac{t}{u}\right\}\right)^3 \mathrm{d}t\right) \mathrm{d}u + \frac{1}{3}\int_0^1 \left(\int_0^1 y^3 \left(\left\{\frac{w}{y}\right\}\left\{\frac{1}{w}\right\}\right)^3 \mathrm{d}w\right) \mathrm{d}y$$

$$+\frac{1}{3}\int_0^1 \left(\int_0^1 z^3 \left(\left\{\frac{1}{y}\right\}\left\{\frac{y}{z}\right\}\right)^3 \mathrm{d}z\right) \mathrm{d}y$$

{change the integration order in the first and second integrals}

$$= \frac{1}{3}\int_0^1 \left(\int_0^1 u^3 \left(\left\{\frac{1}{t}\right\}\left\{\frac{t}{u}\right\}\right)^3 \mathrm{d}u\right) \mathrm{d}t + \frac{1}{3}\int_0^1 \left(\int_0^1 y^3 \left(\left\{\frac{1}{w}\right\}\left\{\frac{w}{y}\right\}\right)^3 \mathrm{d}y\right) \mathrm{d}w$$

$$+\frac{1}{3}\int_0^1 \left(\int_0^1 z^3 \left(\left\{\frac{1}{y}\right\}\left\{\frac{y}{z}\right\}\right)^3 \mathrm{d}z\right) \mathrm{d}y$$

{note that all three integrals are equal}

$$= \int_0^1 \left(\int_0^1 u^3 \left(\left\{\frac{1}{t}\right\}\left\{\frac{t}{u}\right\}\right)^3 \mathrm{d}u\right) \mathrm{d}t. \tag{3.256}$$

Next, we split the integral in (3.256), and based upon symmetry we have

$$\mathscr{I} = \int_0^1 \left(\int_0^1 u^3 \left(\left\{\frac{1}{t}\right\}\left\{\frac{t}{u}\right\}\right)^3 \mathrm{d}u\right) \mathrm{d}t$$

$$= \int_0^1 \left(\int_0^t u^3 \left(\left\{ \frac{1}{t} \right\} \left\{ \frac{t}{u} \right\} \right)^3 \mathrm{d}u \right) \mathrm{d}t + \int_0^1 \left(\int_t^1 u^3 \left(\left\{ \frac{1}{t} \right\} \left\{ \frac{t}{u} \right\} \right)^3 \mathrm{d}u \right) \mathrm{d}t. \tag{3.257}$$

If we make the change of variable $u/t = s$ in the first integral from the right-hand side of (3.257), we get

$$\int_0^1 \left(\int_0^t u^3 \left(\left\{ \frac{1}{t} \right\} \left\{ \frac{t}{u} \right\} \right)^3 \mathrm{d}u \right) \mathrm{d}t$$

$$= \int_0^1 \left(\int_0^1 t^4 s^3 \left(\left\{ \frac{1}{t} \right\} \left\{ \frac{1}{s} \right\} \right)^3 \mathrm{d}s \right) \mathrm{d}t = \int_0^1 t^4 \left\{ \frac{1}{t} \right\}^3 \mathrm{d}t \int_0^1 s^3 \left\{ \frac{1}{s} \right\}^3 \mathrm{d}s. \tag{3.258}$$

For the other integral in (3.257), using that $\{t/u\} = t/u$ when u in $(t, 1]$, we have

$$\int_0^1 \left(\int_t^1 u^3 \left(\left\{ \frac{1}{t} \right\} \left\{ \frac{t}{u} \right\} \right)^3 \mathrm{d}u \right) \mathrm{d}t = \int_0^1 \left(\int_t^1 t^3 \left\{ \frac{1}{t} \right\}^3 \mathrm{d}u \right) \mathrm{d}t$$

$$= \int_0^1 (1-t) t^3 \left\{ \frac{1}{t} \right\}^3 \mathrm{d}t = \int_0^1 t^3 \left\{ \frac{1}{t} \right\}^3 \mathrm{d}t - \int_0^1 t^4 \left\{ \frac{1}{t} \right\}^3 \mathrm{d}t. \tag{3.259}$$

Hence, by plugging the results from (3.258) and (3.259) in (3.257), and using the particular cases of the integral from the previous section, $\int_0^1 t^3 \left\{ \frac{1}{t} \right\}^3 \mathrm{d}t = 1 - \frac{1}{4} (\zeta(2) + \zeta(3) + \zeta(4))$ and $\int_0^1 t^4 \left\{ \frac{1}{t} \right\}^3 \mathrm{d}t = \frac{1}{2} - \frac{1}{20} (2\zeta(3) + 3\zeta(4) + 4\zeta(5))$, we conclude that

$$\mathscr{I} = \int_0^1 \int_0^1 \int_0^1 \left(\left\{ \frac{x}{y} \right\} \left\{ \frac{y}{z} \right\} \left\{ \frac{z}{x} \right\} \right)^3 \mathrm{d}x \mathrm{d}y \mathrm{d}z$$

$$= 1 - \frac{3}{8}\zeta(2) - \frac{3}{8}\zeta(3) - \frac{3}{8}\zeta(4) + \frac{21}{320}\zeta(6) + \frac{7}{160}\zeta(8) + \frac{1}{40}\zeta^2(3) + \frac{1}{40}\zeta(2)\zeta(3)$$

$$+ \frac{1}{20}\zeta(2)\zeta(5) + \frac{1}{16}\zeta(3)\zeta(4) + \frac{1}{20}\zeta(3)\zeta(5) + \frac{1}{20}\zeta(4)\zeta(5),$$

and the solution is finalized.

Another way of approaching the problem relies on exploiting the symmetry as you may see in the next section and in [5, 80].

3.49 The Calculation of a Generalized Triple Fractional Part Integral with Positive Integer Powers

Solution The full derivation of the result, using the strategy from the previous section, is given in my article *The evaluation of a special fractional part integral with an integrand raised to positive integer powers* published in *MathProblems journal*, Vol. 6, No. 1 (see [58, pp. 544–550]).

Now, for a different strategy that exploits the symmetry, considering the 6 permutations in the chamber $0 \le x \le y \le z \le 1$, we may assume x is the largest and we split into two cases depending on whether $z \le y$ or $y \le z$, and then we have

$$\mathscr{I}_n = \int_0^1\int_0^1\int_0^1 \left(\left\{\frac{x}{y}\right\}\left\{\frac{y}{z}\right\}\left\{\frac{z}{x}\right\}\right)^n \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= 3\biggl(\int_0^1\left(\int_0^x\left(\int_0^y\left(\left\{\frac{x}{y}\right\}\left\{\frac{y}{z}\right\}\frac{z}{x}\right)^n \mathrm{d}z\right)\mathrm{d}y\right)\mathrm{d}x$$

$$+\int_0^1\left(\int_0^x\left(\int_y^x\left(\left\{\frac{x}{y}\right\}\frac{y}{z}\frac{z}{x}\right)^n \mathrm{d}z\right)\mathrm{d}y\right)\mathrm{d}x\biggr),$$

where I have dropped the fractional part notation on the factors that are smaller than 1. With the change of variable $z = yu$ in the first integral and simply doing the z integral in the second integral gives

$$\mathscr{I}_n = 3\biggl(\int_0^1\left(\int_0^x\left(\int_0^1\left(\left\{\frac{x}{y}\right\}\left\{\frac{1}{u}\right\}\frac{yu}{x}\right)^n y\mathrm{d}u\right)\mathrm{d}y\right)\mathrm{d}x$$

$$+\int_0^1\left(\int_0^x (x-y)\left(\left\{\frac{x}{y}\right\}\frac{y}{x}\right)^n \mathrm{d}y\right)\mathrm{d}x\biggr),$$

and making the change of variable $y = xt$ in both integrals yields

$$\mathscr{I}_n = 3\biggl(\int_0^1 x^2\left(\int_0^1 t^{n+1}\left\{\frac{1}{t}\right\}^n\left(\int_0^1 u^n\left\{\frac{1}{u}\right\}^n \mathrm{d}u\right)\mathrm{d}t\right)\mathrm{d}x$$

$$+\int_0^1 x^2\left(\int_0^1 (1-t)t^n\left\{\frac{1}{t}\right\}^n \mathrm{d}t\right)\mathrm{d}x\biggr) = I_{n+1,n}I_{n,n} + I_{n,n} - I_{n+1,n}$$

$$= 1 - \frac{3}{2(n+1)}\sum_{i=1}^n \zeta(i+1) + \frac{1}{(n+1)^2(n+2)}\left(\sum_{i=1}^n \zeta(i+1)\right)$$

$$\times\left(\sum_{i=1}^n (i+1)\zeta(i+2)\right),$$

where for the penultimate equality I used the integral $I_{p,q} = \int_0^1 x^p \left\{\frac{1}{x}\right\}^q \mathrm{d}x$ in (1.67), and the solution is complete.

Similar solutions to the square power version of this integral I proposed in *The American Mathematical Monthly*, the problem **11902** (see [87]), and that can also be successfully adjusted for the present generalization, may be found in [5], [80].

A generalization of the integral version with $n = 1$ appears as an open problem in [29, 2.58, p. 108].

3.50 A Pair of Cute Fractional Part Integrals Involving the Cotangent Function

Solution In this section we still explore the land of the fascinating fractional part integrals, and now we want to calculate a curious, appealing couple of cotangent integrals. If you went through the previous sections with fractional part integrals and grasped the strategies presented there, it wouldn't be hard to figure out how to initiate a solution. For each integral I'll reduce everything to the calculation of the corresponding series.

I submitted the integral from the point $i)$ to *The American Mathematical Monthly* (writing the statement of the problem with $\tan(x)$ instead of $\cot(x)$), the problem **11924** (see [88]).

Now, for the point $i)$ of the problem it's easy to note that $\int_0^{\pi/2} \frac{\{\cot(x)\}}{\cot(x)}\mathrm{d}x = \int_0^{\pi/2} \frac{\{\tan(x)\}}{\tan(x)}\mathrm{d}x$, which is obtained with the change of variable $x = \pi/2 - y$. Then, using the fact that $\{x\} = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ denotes the integer part of x, we proceed as follows

$$\int_0^{\pi/2} \frac{\{\tan(x)\}}{\tan(x)}\mathrm{d}x = \int_0^{\pi/2} \frac{\tan(x) - \lfloor \tan(x) \rfloor}{\tan(x)}\mathrm{d}x$$

$$= \sum_{k=0}^{\infty} \int_{\arctan(k)}^{\arctan(k+1)} \left(1 - \frac{\lfloor \tan(x) \rfloor}{\tan(x)}\right) \mathrm{d}x$$

$$= \frac{\pi}{4} + \sum_{k=1}^{\infty} \int_{\arctan(k)}^{\arctan(k+1)} \left(1 - \frac{k}{\tan(x)}\right) \mathrm{d}x = \frac{\pi}{4} + \sum_{k=1}^{\infty} (x - k\log(\sin(x)))\Big|_{x=\arctan(k)}^{x=\arctan(k+1)}$$

$$\left\{\text{make use of the fact that } \sin(\arctan(x)) = \frac{x}{\sqrt{1+x^2}}\right\}$$

$$= \frac{\pi}{4} + \sum_{k=1}^{\infty} \left(\arctan(k+1) - \arctan(k) + k \log\left(\frac{k}{\sqrt{1+k^2}} \right) \right.$$
$$\left. - k \log\left(\frac{k+1}{\sqrt{1+(k+1)^2}} \right) \right)$$

$$= \frac{\pi}{4} + \sum_{k=1}^{\infty} \left(\arctan(k+1) - \arctan(k) + k \log\left(\frac{k}{\sqrt{1+k^2}} \right) \right.$$
$$- (k+1) \log\left(\frac{k+1}{\sqrt{1+(k+1)^2}} \right)$$
$$\left. + \log\left(\frac{k+1}{\sqrt{1+(k+1)^2}} \right) \right) = \frac{\pi}{4} + \underbrace{\sum_{k=1}^{\infty} (\arctan(k+1) - \arctan(k))}_{S_1}$$
$$+ \underbrace{\sum_{k=1}^{\infty} \left(k \log\left(\frac{k}{\sqrt{1+k^2}} \right) - (k+1) \log\left(\frac{k+1}{\sqrt{1+(k+1)^2}} \right) \right)}_{S_2}$$
$$+ \underbrace{\sum_{k=1}^{\infty} \log\left(\frac{k+1}{\sqrt{1+(k+1)^2}} \right)}_{S_3}. \tag{3.260}$$

Now, for the series S_1 in (3.260), we write

$$S_1 = \sum_{k=1}^{\infty} (\arctan(k+1) - \arctan(k)) = \lim_{N\to\infty} \sum_{k=1}^{N} (\arctan(k+1) - \arctan(k))$$
$$= \lim_{N\to\infty} \arctan(N+1) - \frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}. \tag{3.261}$$

Further, for the series S_2 in (3.260), we get

$$S_2 = \sum_{k=1}^{\infty} \left(k \log\left(\frac{k}{\sqrt{1+k^2}} \right) - (k+1) \log\left(\frac{k+1}{\sqrt{1+(k+1)^2}} \right) \right)$$

$$= \lim_{N\to\infty} \sum_{k=1}^{N} \left(k \log\left(\frac{k}{\sqrt{1+k^2}}\right) - (k+1)\log\left(\frac{k+1}{\sqrt{1+(k+1)^2}}\right)\right)$$

$$= -\frac{\log(2)}{2} - \lim_{N\to\infty} (N+1)\log\left(\frac{N+1}{\sqrt{1+(N+1)^2}}\right)$$

$$= -\frac{\log(2)}{2} + \frac{1}{2} \lim_{N\to\infty} (N+1)\log\left(1+\frac{1}{(N+1)^2}\right)$$

$$= -\frac{\log(2)}{2} + \underbrace{\lim_{N\to\infty} \frac{1}{2(N+1)}}_{0} \cdot \underbrace{\lim_{N\to\infty} (N+1)^2 \log\left(1+\frac{1}{(N+1)^2}\right)}_{1} = -\frac{\log(2)}{2}, \tag{3.262}$$

where I made use of the elementary limit, $\lim_{x\to 0} \dfrac{\log(1+x)}{x} = 1$.

Lastly, for the series S_3 in (3.260), we have

$$S_3 = \sum_{k=1}^{\infty} \log\left(\frac{k+1}{\sqrt{1+(k+1)^2}}\right) = \frac{1}{2}\sum_{k=1}^{\infty} \log\left(\frac{(k+1)^2}{1+(k+1)^2}\right)$$

$$= -\frac{1}{2}\sum_{k=1}^{\infty} \log\left(\frac{1+(k+1)^2}{(k+1)^2}\right) = -\frac{1}{2}\sum_{k=2}^{\infty} \log\left(1+\frac{1}{k^2}\right) = -\frac{1}{2}\log\left(\frac{\sinh(\pi)}{2\pi}\right), \tag{3.263}$$

and the evaluation of the series is accomplished by using the well-known Euler's infinite product for the sine, $\sin(x) = x\prod_{n=1}^{\infty}\left(1-\dfrac{x^2}{n^2\pi^2}\right)$, (see [23, pp. 251–252]), where by setting $x = \pi i$ gives $\dfrac{\sinh(\pi)}{\pi} = \prod_{n=1}^{\infty}\left(1+\dfrac{1}{n^2}\right)$ or $\dfrac{\sinh(\pi)}{2\pi} = \prod_{n=2}^{\infty}\left(1+\dfrac{1}{n^2}\right)$, and by taking log of both sides of the last equality, we get the precise value of the series in (3.263).

Gathering the results from (3.261), (3.262), and (3.263) and plugging them in (3.260), we obtain the final result,

$$\int_0^{\pi/2} \frac{\{\tan(x)\}}{\tan(x)} \mathrm{d}x = \int_0^{\pi/2} \frac{\{\cot(x)\}}{\cot(x)} \mathrm{d}x = \frac{1}{2}\left(\pi - \log\left(\frac{\sinh(\pi)}{\pi}\right)\right),$$

and the point i) of the problem is finalized.

Alternatively, one can start with the variable change $\cot(x) = u$ and then show that everything reduces to the integral $\Re\left\{\int_0^1 z(\psi(z+i)-\psi(z))\mathrm{d}z\right\}$, which the curious reader might want to calculate.

For slightly different solutions, one may check [81], [6].

I submitted the second problem to *La Gaceta de la RSME* and it was published in Vol. 20, No. 2 (2017), the problem **327** (see [43, p. 327]). Let's start in a similar style as in the previous integral, and then we need to carefully handle with the resulting series, which represent the difficult part of the problem.

For the point $ii)$ of the problem, note that $\int_0^{\pi/2}\{\cot(x)\}\mathrm{d}x = \int_0^{\pi/2}\{\tan(x)\}\mathrm{d}x$ by the change of variable $x = \pi/2 - y$. Now, after using that $\{x\} = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ denotes the integer part of x, our first objective is to turn the integral into a series, and we write

$$\int_0^{\pi/2}\{\tan(x)\}\mathrm{d}x = \int_0^{\pi/2}(\tan(x)-\lfloor\tan(x)\rfloor)\mathrm{d}x$$
$$= \sum_{k=0}^{\infty}\int_{\arctan(k)}^{\arctan(k+1)}(\tan(x)-\lfloor\tan(x)\rfloor)\mathrm{d}x$$
$$= \sum_{k=0}^{\infty}\int_{\arctan(k)}^{\arctan(k+1)}(\tan(x)-k)\mathrm{d}x = \sum_{k=0}^{\infty}(-\log(\cos(x))-kx)\Big|_{x=\arctan(k)}^{x=\arctan(k+1)}$$
$$\left\{\text{make use of the identity } \cos(\arctan(x)) = \frac{1}{\sqrt{1+x^2}}\right\}$$
$$= \sum_{k=0}^{\infty}\left(k\arctan(k)-k\arctan(k+1)-\frac{1}{2}(\log(k^2+1)-\log((k+1)^2+1))\right). \tag{3.264}$$

Rearranging the series in (3.264), we get

$$\int_0^{\pi/2}\{\tan(x)\}\mathrm{d}x = \lim_{N\to\infty}\left(\underbrace{\sum_{k=0}^{N}(k\arctan(k)-(k+1)\arctan(k+1))}_{A_N}\right.$$
$$\left.+\underbrace{\sum_{k=0}^{N}\arctan(k+1)}_{B_N}-\frac{1}{2}\underbrace{\sum_{k=0}^{N}(\log(k^2+1)-\log((k+1)^2+1))}_{C_N}\right). \tag{3.265}$$

For the partial sum A_N in (3.265), we have a simple telescopic sum, and then we write

$$A_N = \sum_{k=0}^{N}(k\arctan(k) - (k+1)\arctan(k+1)) = -(N+1)\arctan(N+1). \tag{3.266}$$

Next, for the partial sum B_N in (3.265), we get

$$B_N = \sum_{k=0}^{N}\arctan(k+1) = \sum_{k=1}^{N+1}\arctan(k) = (N+1)\frac{\pi}{2} - \sum_{k=1}^{N+1}\arctan\left(\frac{1}{k}\right), \tag{3.267}$$

where above I used the identity $\arctan(x) + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2},\ x > 0$.

Then, for the partial sum C_N in (3.265), we deal with another simple telescoping sum, and then we have

$$C_N = \sum_{k=0}^{N}(\log(k^2+1) - \log((k+1)^2+1)) = -\log((N+1)^2+1). \tag{3.268}$$

Returning with the results from (3.266), (3.267), and (3.268) in (3.265), we have

$$\int_0^{\pi/2}\{\tan(x)\}\mathrm{d}x$$

$$= \lim_{N\to\infty}\left((N+1)\frac{\pi}{2} + \frac{1}{2}\log((N+1)^2+1) - (N+1)\arctan(N+1) - \sum_{k=1}^{N+1}\arctan\left(\frac{1}{k}\right)\right)$$

$$\left\{\text{use the identity } \arctan(x) + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2},\ x > 0\right\}$$

$$= \lim_{N\to\infty}\left(\frac{1}{2}\log\left(1 + \frac{1}{(N+1)^2}\right) + (N+1)\arctan\left(\frac{1}{N+1}\right) + \log(N+1) - H_{N+1} - \sum_{k=1}^{N+1}\left(\arctan\left(\frac{1}{k}\right) - \frac{1}{k}\right)\right). \tag{3.269}$$

Letting $N \to \infty$ in (3.269) and using the elementary limit $\lim_{x\to 0}\frac{\arctan(x)}{x} = 1$ and the limit representation of the Euler–Mascheroni constant, $\lim_{n\to\infty}(H_n -$

$\log(n)) = \gamma$ (a dedicated book on the Euler–Mascheroni constant may be found in [34]), we get

$$\int_0^{\pi/2} \{\tan(x)\} \mathrm{d}x = 1 - \gamma - \sum_{k=1}^{\infty} \left(\arctan\left(\frac{1}{k}\right) - \frac{1}{k} \right). \tag{3.270}$$

Since by Taylor series we have that $\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$, the remaining series in (3.270) reduces to

$$\sum_{k=1}^{\infty} \left(\arctan\left(\frac{1}{k}\right) - \frac{1}{k} \right) = \sum_{k=1}^{\infty} \left(\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)k^{2n+1}} \right) - \frac{1}{k} \right)$$

$$= \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)k^{2n+1}} \right) = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{(-1)^n}{(2n+1)k^{2n+1}} \right) = \sum_{n=1}^{\infty} (-1)^n \frac{\zeta(2n+1)}{2n+1}. \tag{3.271}$$

In order to go further with the evaluation of the series in (3.271), we make use of the classical Riemann zeta generating function,[22] $\sum_{k=2}^{\infty} \zeta(k) t^{k-1} = -\psi(1-t) - \gamma$ (see [30, **8.363**, p. 912], [67, Chapter 5, p. 648], and [78, Chapter 3, p. 160]), that if we integrate from $t = 0$ to $t = x$, we get

$$\sum_{k=2}^{\infty} \zeta(k) \frac{x^k}{k} = \log(\Gamma(1-x)) - \gamma x. \tag{3.272}$$

Then we plug $-x$ in (3.272), and we have

$$\sum_{k=2}^{\infty} \zeta(k) \frac{(-x)^k}{k} = \log(\Gamma(1+x)) + \gamma x. \tag{3.273}$$

[22]The zeta series result expressed in terms of Digamma function we need in our calculations is a classical result. Writing $\zeta(k)$ as a series and changing the integration order, we get $\sum_{k=2}^{\infty} \zeta(k) t^{k-1} = \sum_{k=2}^{\infty} \left(\sum_{n=1}^{\infty} \frac{t^{k-1}}{n^k} \right) = \sum_{n=1}^{\infty} \left(\sum_{k=2}^{\infty} \frac{t^{k-1}}{n^k} \right) = \sum_{n=1}^{\infty} \frac{t}{n(n-t)} = -\psi(1-t) - \gamma$, and the last equality comes directly from the series definition of the Digamma function, $\psi(z+1) = -\gamma + \sum_{n=1}^{\infty} \frac{z}{n(n+z)}$, where we set $z = -t$.

Subtracting (3.273) from (3.272) gives

$$\sum_{k=1}^{\infty}\zeta(2k+1)\frac{x^{2k+1}}{2k+1}=\frac{1}{2}\log\left(\frac{\Gamma(1-x)}{\Gamma(1+x)}\right)-\gamma x, \tag{3.274}$$

you may also find in [78, Chapter 3, p. 160].

Further, dividing both sides of (3.274) by x and letting $x=i$, we obtain

$$\sum_{k=1}^{\infty}(-1)^k\frac{\zeta(2k+1)}{2k+1}=\frac{1}{2i}\log\left(\frac{\Gamma(1-i)}{\Gamma(1+i)}\right)-\gamma=-\arg(\Gamma(1+i))-\gamma, \tag{3.275}$$

where in the last equality I used that $\log\left(\dfrac{z}{\bar{z}}\right)=2i\arg(z)$.

Finally, by plugging (3.275) in (3.271) and then (3.271) in (3.270), we conclude that

$$\int_0^{\pi/2}\{\tan(x)\}\mathrm{d}x=\int_0^{\pi/2}\{\cot(x)\}\mathrm{d}x=1+\arg(\Gamma(1+i)),$$

and the point ii) of the problem is finalized.

Alternatively, as at the point i), one can start with the variable change $\cot(x)=u$ and show that the given integral reduces to computing $\Im\left\{\displaystyle\int_0^1 z\psi(z+i)\mathrm{d}z\right\}$, which again the curious reader might want to calculate.

It was a pleasant, unexpected moment to see the connection of my problem with the problem **11592** that appeared in *The American Mathematical Monthly* (see [36]). The strategy of calculating the arctan series in (3.271) follows the same line with the one presented in [29, p. 53]. For another approach you may see the solution in [59].

3.51 Playing with a Resistant Classical Integral Family to the Real Methods That Responds to the Tricks Involving the Use of the Cauchy–Schlömilch Transformation

Solution It's one of the problems I considered to add to the present book from the very beginning, and I was glad to include it (together with the solution) in my first book proposal at Springer at the end of 2015, mainly due to the beautiful way the solution develops when using Cauchy–Schlömilch transformation, which is an approach by real methods. In general, one expects to meet these integrals in the

Complex Analysis books, specifically in their sections dedicated to the calculation of the integrals by contour integration.

To prove the first result (with the integral I), we recollect Cauchy–Schlömilch transformation (see [7, Theorem 2.1., p. 2] and [68, Chapter 2, p. 275]) which says that if f is a continuous function for which the integrals below are convergent, then

$$A = \int_0^\infty f\left(\left(ax - \frac{b}{x}\right)^2\right) \mathrm{d}x = \frac{1}{a}\int_0^\infty f(x^2)\mathrm{d}x, \ \ a, b > 0. \tag{3.276}$$

Proof To prove this result, we let $x = b/(ay)$ in the left-hand side, and we get

$$A = \int_0^\infty f\left(\left(ax - \frac{b}{x}\right)^2\right) \mathrm{d}x = \frac{b}{a}\int_0^\infty f\left(\left(ay - \frac{b}{y}\right)^2\right) \frac{1}{y^2}\mathrm{d}y,$$

or coming back to the notation in x, we have

$$A = \int_0^\infty f\left(\left(ax - \frac{b}{x}\right)^2\right) \mathrm{d}x = \frac{b}{a}\int_0^\infty f\left(\left(ax - \frac{b}{x}\right)^2\right) \frac{1}{x^2}\mathrm{d}x. \tag{3.277}$$

Upon adding up both integrals in (3.277), we have that

$$2A = \frac{1}{a}\int_0^\infty f\left(\left(ax - \frac{b}{x}\right)^2\right)\left(a + \frac{b}{x^2}\right) \mathrm{d}x \overset{ax-b/x=y}{=} \frac{1}{a}\int_{-\infty}^\infty f\left(y^2\right) \mathrm{d}y$$

$$= \frac{1}{a}\int_{-\infty}^\infty f\left(x^2\right) \mathrm{d}x = \frac{2}{a}\int_0^\infty f\left(x^2\right) \mathrm{d}x,$$

whence we arrive at

$$A = \frac{1}{a}\int_0^\infty f\left(x^2\right) \mathrm{d}x,$$

and the proof of the auxiliary result is complete.

To calculate the integral, we first note we may write that $\dfrac{1}{b^2 + x^2} = \displaystyle\int_0^\infty e^{-(b^2+x^2)y}\mathrm{d}y$, and then we have

$$I = \int_0^\infty \frac{\cos(ax)}{b^2 + x^2}\mathrm{d}x = \int_0^\infty \left(\int_0^\infty \cos(ax)e^{-(b^2+x^2)y}\mathrm{d}y\right) \mathrm{d}x$$

{change the integration order assuming the value of the integral preserves}

$$= \int_0^\infty e^{-b^2 y} \left(\int_0^\infty \cos(ax) e^{-yx^2} \mathrm{d}x \right) \mathrm{d}y$$

$$\left\{ \text{make use of the power series, } \cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \right\}$$

$$= \int_0^\infty e^{-b^2 y} \left(\int_0^\infty \sum_{n=0}^{\infty} (-1)^n \frac{(ax)^{2n}}{(2n)!} e^{-yx^2} \mathrm{d}x \right) \mathrm{d}y$$

$$= \int_0^\infty e^{-b^2 y} \sum_{n=0}^{\infty} (-1)^n \frac{a^{2n}}{(2n)!} \left(\int_0^\infty x^{2n} e^{-yx^2} \mathrm{d}x \right) \mathrm{d}y. \tag{3.278}$$

If we denote $P_{2n} = \displaystyle\int_0^\infty x^{2n} e^{-yx^2} \mathrm{d}x$ in (3.278), and then integrate by parts, we have

$$P_{2n} = \frac{1}{2n+1} \int_0^\infty (x^{2n+1})' e^{-yx^2} \mathrm{d}x = \frac{x^{2n+1}}{2n+1} e^{-yx^2} \bigg|_{x=0}^{x=\infty}$$
$$+ \frac{2y}{2n+1} \int_0^\infty x^{2n+2} e^{-yx^2} \mathrm{d}x$$
$$= \frac{2y}{2n+1} \int_0^\infty x^{2n+2} e^{-yx^2} \mathrm{d}x = \frac{2y}{2n+1} P_{2n+2},$$

whence we get that $\dfrac{P_{2n+2}}{P_{2n}} = \dfrac{2n+1}{2y}$, or

$$\frac{P_{2k+2}}{P_{2k}} = \frac{2k+1}{2y}. \tag{3.279}$$

Giving values to k in the recurrence relation in (3.279), from $k = 0$ to $k = n - 1$, and multiplying out all the resulting relations, we get

$$P_{2n} = \frac{1 \cdot 3 \cdots (2n-1)}{(2y)^n} P_0$$

$$\{\text{multiply both numerator and denominator by } 2 \cdot 4 \cdots 2n\}$$

$$= \frac{(2n)!}{4^n n! y^n} P_0$$

$$\left\{\text{use the Gaussian integral, } \int_0^\infty e^{-x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{2}\text{, and make}\right\}$$

$$\left\{\text{the change of variable } x^2 = yz^2 \text{ to get } P_0 = \int_0^\infty e^{-yz^2}\mathrm{d}z = \frac{\sqrt{\pi}}{2\sqrt{y}}\right\}$$

$$= \frac{2\sqrt{\pi}(2n)!}{4^{n+1}n!y^{n+1/2}}. \tag{3.280}$$

Alternatively, one may prove the result in (3.280) by using the differentiation if starting from an integral of the type $\int_0^\infty e^{-yx^2}\mathrm{d}x = \frac{\sqrt{\pi}}{2\sqrt{y}}$.

Now, we plug the result from (3.280) in (3.278), and we obtain that

$$I = \int_0^\infty \frac{\cos(ax)}{b^2+x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{2}\int_0^\infty y^{-1/2}e^{-b^2y}\sum_{n=0}^{\infty}(-1)^n\frac{(a^2/(4y))^n}{n!}\mathrm{d}y$$

$$\left\{\text{use the power series, } e^{-x} = \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!}\right\}$$

$$= \frac{\sqrt{\pi}}{2}\int_0^\infty y^{-1/2}e^{-b^2y-a^2/(4y)}\mathrm{d}y \overset{y=z^2}{=} \sqrt{\pi}\int_0^\infty e^{-b^2z^2-a^2/(4z^2)}\mathrm{d}z$$

$$\{\text{write the power as a perfect square}\}$$

$$= \sqrt{\pi}e^{-ab}\int_0^\infty e^{-(bz-a/(2z))^2}\mathrm{d}z$$

$$\{\text{make use of the Cauchy–Schlömilch transformation in (3.276)}\}$$

$$= \frac{\sqrt{\pi}}{b}e^{-ab}\int_0^\infty e^{-z^2}\mathrm{d}z$$

$$\left\{\text{employ the value of the Gaussian integral, } \int_0^\infty e^{-x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{2}\right\}$$

$$= \frac{\pi}{2b}e^{-ab},$$

and the calculations to the integral I are complete.

The integral I also appears in **3.723.2** from [30], and it is the *source* from which we derive immediately the other three integrals.

If you're interested in a different approach of the integral I, one using the powerful contour integration,[23] you may see [64, Chapter 8, pp. 309–311] and [4, p. 224].

Now, by differentiating I with respect to a, we get

$$J = \int_0^\infty \frac{x\sin(ax)}{b^2 + x^2}\mathrm{d}x = \frac{\pi}{2}e^{-ab}.$$

To get the integral K, we differentiate I with respect to b and obtain

$$K = \int_0^\infty \frac{\cos(ax)}{(b^2 + x^2)^2}\mathrm{d}x = \frac{\pi(1 + ab)}{4b^3}e^{-ab}.$$

Lastly, differentiating I with respect to a and b, we get

$$L = \int_0^\infty \frac{x\sin(ax)}{\left(b^2 + x^2\right)^2}\mathrm{d}x = \frac{\pi a}{4b}e^{-ab},$$

and the solution is finalized.

The Gaussian integral can be calculated by expressing it first in terms of Gamma function with the change of variable $x^2 = y$, and then continuing as presented at the end of Sect. 3.37. Also, a clever approach of the Gaussian integral may be found in Paul's book, *Inside Interesting Integrals* (see [64, pp. 75–77]). However, I think the unbeatable approach is represented by the trick with two Gaussian integrals and the transformation to the polar coordinates (see [115]) as shown in [108].

3.52 Calculating a Somewhat Strange-Looking Quartet of Integrals Involving the Trigonometric Functions

Solution The thing I particularly enjoy about these integrals is how the calculations play out, the creative ways involved, the beautiful picture of mathematical connections revealed when looking over the complete solutions. During the reduction process to simpler calculations, we'll also note connections with alternating harmonic series like $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^3}$, which might not be that easy to calculate

[23] Choosing the contour integral $\displaystyle\oint_C \frac{e^{iaz}}{b^2 + z^2}\mathrm{d}z$ along a semicircle in the upper half-plane (as mentioned in the reference) and integrating in the counterclockwise direction, then showing the line integral along the circular arc vanishes and computing the residue at $z_0 = ib$, we arrive at $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(ax)}{b^2 + x^2}\mathrm{d}x = \frac{\pi}{b}e^{-ab}$.

without a proper approach (but you shouldn't be worried, it is calculated in the sixth chapter).

One of the possible ways to calculate the integral I from the point i) requires the result in (1.11) where if setting $s = 2n$, we get

$$\int_0^\infty \tanh(x)e^{-2nx}\mathrm{d}x = \frac{1}{2}\left(\psi\left(\frac{n+1}{2}\right) - \psi\left(\frac{n}{2}\right) - \frac{1}{n}\right), \tag{3.281}$$

and that we further use to prove the auxiliary result

$$\sum_{n=1}^\infty \left(\psi\left(\frac{n+1}{2}\right) - \psi\left(\frac{n}{2}\right) - \frac{1}{n}\right)\sin(2nx) = -\tan(x)\log(\sin(x)),\ 0 < x < \frac{\pi}{2}. \tag{3.282}$$

Proof Now, if we replace x by y in (3.281), multiply both sides by $\sin(2nx)$ and then consider the sum from $n = 1$ to ∞, we get

$$\sum_{n=1}^\infty \left(\psi\left(\frac{n+1}{2}\right) - \psi\left(\frac{n}{2}\right) - \frac{1}{n}\right)\sin(2nx) = 2\sum_{n=1}^\infty \int_0^\infty \tanh(y)\sin(2nx)e^{-2ny}\mathrm{d}y$$

$$\{\text{reverse the order of integration and summation}\}$$

$$= 2\int_0^\infty \tanh(y)\sum_{n=1}^\infty \sin(2nx)e^{-2ny}\mathrm{d}y = \sin(2x)\int_0^\infty \frac{\tanh(y)}{\cosh(2y)-\cos(2x)}\mathrm{d}y$$

$$= \sin(2x)\int_0^\infty \frac{\sinh(y)}{\cosh(y)(2\cosh^2(y) - 1 - \cos(2x))}\mathrm{d}y$$

$$\left\{\text{make the change of variable } \frac{1}{\cosh(y)} = t\right\}$$

$$= \sin(2x)\int_0^1 \frac{t}{2-(1+\cos(2x))t^2}\mathrm{d}t = \frac{1}{2}\sin(2x)\int_0^1 \frac{t}{1-\cos^2(x)t^2}\mathrm{d}t$$

$$= \frac{1}{2}\sin(2x)\cdot\left(-\frac{\log(1-\cos^2(x)t^2)}{2\cos^2(x)}\Bigg|_{t=0}^{t=1}\right) = -\frac{1}{2}\sin(2x)\cdot\frac{\log(\sin(x))}{\cos^2(x)}$$

$$= -\tan(x)\log(\sin(x)),$$

and the auxiliary proof is complete.

For the calculation of the series, I also used the classical geometric series,[24] $\sum_{n=1}^{\infty} p^n \sin(nx) = \dfrac{p\sin(x)}{1-2p\cos(x)+p^2}$, $|p|<1$, in **1.447.1** from [30], where we get the desired result by replacing x by $2x$ and p by e^{-2y}.

Returning to the integral I, and using the result in (3.282), we have

$$I = \int_0^{\pi/2} x^2 \tan(x)\log(\sin(x))\mathrm{d}x$$

$$= -\int_0^{\pi/2} \sum_{n=1}^{\infty}\left(\psi\left(\frac{n+1}{2}\right)-\psi\left(\frac{n}{2}\right)-\frac{1}{n}\right)x^2\sin(2nx)\mathrm{d}x$$

{reverse the order of summation and integration}

$$= -\sum_{n=1}^{\infty}\left(\psi\left(\frac{n+1}{2}\right)-\psi\left(\frac{n}{2}\right)-\frac{1}{n}\right)\int_0^{\pi/2} x^2\sin(2nx)\mathrm{d}x$$

$$= \frac{1}{4}\sum_{n=1}^{\infty}\left(\psi\left(\frac{n+1}{2}\right)-\psi\left(\frac{n}{2}\right)-\frac{1}{n}\right)\left(\frac{1}{n^3}-(-1)^{n-1}\frac{\pi^2}{2n}+\frac{(-1)^{n-1}}{n^3}\right). \tag{3.283}$$

Now, from the proof of (1.11), we may deduce that

$$\int_0^1 t^{n-1}\frac{1-t}{1+t}\mathrm{d}t = \psi\left(\frac{n+1}{2}\right)-\psi\left(\frac{n}{2}\right)-\frac{1}{n}, \tag{3.284}$$

and then, by plugging the result from (3.284) in (3.283), we have

$$I = \int_0^{\pi/2} x^2 \tan(x)\log(\sin(x))\mathrm{d}x$$

[24]Both series $\sum_{n=1}^{\infty} p^n\sin(nx)$, $|p|<1$ and $\sum_{n=1}^{\infty} p^n\cos(nx)$, $|p|<1$ can be calculated easily if starting with the geometric series, $\sum_{n=1}^{\infty}(pe^{ix})^n = \lim_{N\to\infty}\sum_{n=1}^{N}(pe^{ix})^n = \lim_{N\to\infty} p\dfrac{1-(pe^{ix})^N}{e^{-ix}-p} = \dfrac{p}{e^{-ix}-p} = \dfrac{p(e^{ix}-p)}{(e^{-ix}-p)(e^{ix}-p)} = \dfrac{p(\cos(x)-p)}{1-2p\cos(x)+p^2} + i\dfrac{p\sin(x)}{1-2p\cos(x)+p^2}$. Now, equating the real and imaginary parts, we get immediately that $\sum_{n=1}^{\infty} p^n\sin(nx) = \dfrac{p\sin(x)}{1-2p\cos(x)+p^2}$, $|p|<1$ and $\sum_{n=1}^{\infty} p^n\cos(nx) = \dfrac{p(\cos(x)-p)}{1-2p\cos(x)+p^2}$, $|p|<1$.

$$= \frac{1}{4}\sum_{n=1}^{\infty}\int_0^1 \left(\frac{1}{n^3} - (-1)^{n-1}\frac{\pi^2}{2n} + \frac{(-1)^{n-1}}{n^3}\right) t^{n-1}\frac{1-t}{1+t}\mathrm{d}t$$

{reverse the order of integration and summation}

$$= \frac{1}{4}\int_0^1 \sum_{n=1}^{\infty}\left(\frac{1}{n^3} - (-1)^{n-1}\frac{\pi^2}{2n} + \frac{(-1)^{n-1}}{n^3}\right) t^{n-1}\frac{1-t}{1+t}\mathrm{d}t$$

{calculate the series and split the integral}

$$= \frac{\pi^2}{4}\int_0^1 \frac{\log(1+t)}{1+t}\mathrm{d}t + \frac{1}{4}\int_0^1 \frac{\mathrm{Li}_3(t)}{t}\mathrm{d}t + \frac{1}{2}\int_0^1 \frac{\mathrm{Li}_3(-t)}{1+t}\mathrm{d}t - \frac{\pi^2}{8}\int_0^1 \frac{\log(1+t)}{t}\mathrm{d}t$$

$$-\frac{1}{4}\int_0^1 \frac{\mathrm{Li}_3(-t)}{t}\mathrm{d}t - \frac{1}{2}\int_0^1 \frac{\mathrm{Li}_3(t)}{1+t}\mathrm{d}t = \frac{3}{4}\log^2(2)\zeta(2) - \frac{15}{32}\zeta(4)$$

$$+ \frac{1}{2}\underbrace{\int_0^1 \frac{\mathrm{Li}_3(-t)}{1+t}\mathrm{d}t}_{I_1} - \frac{1}{2}\underbrace{\int_0^1 \frac{\mathrm{Li}_3(t)}{1+t}\mathrm{d}t}_{I_2}. \tag{3.285}$$

For the integral I_1 in (3.285) we might like to integrate by parts, and then we write

$$I_1 = \int_0^1 \frac{\mathrm{Li}_3(-t)}{1+t}\mathrm{d}t = \int_0^1 (\log(1+t))'\,\mathrm{Li}_3(-t)\mathrm{d}t = \underbrace{\log(1+t)\,\mathrm{Li}_3(-t)\Big|_{t=0}^{t=1}}_{-3/4\log(2)\zeta(3)}$$

$$-\int_0^1 \frac{\log(1+t)\,\mathrm{Li}_2(-t)}{t}\mathrm{d}t = -\frac{3}{4}\log(2)\zeta(3) + \frac{1}{2}\left(\mathrm{Li}_2(-t)\right)^2\Big|_{t=0}^{t=1}$$

$$= \frac{5}{16}\zeta(4) - \frac{3}{4}\log(2)\zeta(3). \tag{3.286}$$

Then, for the integral I_2 in (3.285), we write

$$I_2 = \int_0^1 \frac{\mathrm{Li}_3(t)}{1+t}\mathrm{d}t = \int_0^1 \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{(-1)^{k-1}}{n^3}t^{k+n-1}\right)\mathrm{d}t$$

{reverse the order of summation and integration}

$$= \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{(-1)^{k-1}}{n^3}\int_0^1 t^{k+n-1}\mathrm{d}t\right)$$

$$= \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{(-1)^{k-1}}{n^3(n+k)}\right) = \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{(-1)^{k-1}}{kn^3}\right)$$

$$-\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{(-1)^{k-1}}{k^2n^2}\right)+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3}\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+k}\right)\right) = \log(2)\zeta(3)-\frac{5}{4}\zeta(4)$$

$$+\sum_{k=1}^{\infty}(-1)^{k-1}\frac{H_k}{k^3}$$

{the value of the remaining series is given in (4.85)}

$$= \frac{3}{2}\zeta(4)-\frac{3}{4}\log(2)\zeta(3)+\frac{1}{2}\log^2(2)\zeta(2)-\frac{1}{12}\log^4(2)-2\operatorname{Li}_4\left(\frac{1}{2}\right). \tag{3.287}$$

By plugging the results from (3.286) and (3.287) in (3.285), we obtain that

$$I = \int_0^{\pi/2} x^2\tan(x)\log(\sin(x))\mathrm{d}x$$

$$= \frac{1}{24}\log^4(2)+\frac{1}{2}\zeta(2)\log^2(2)-\frac{17}{16}\zeta(4)+\operatorname{Li}_4\left(\frac{1}{2}\right),$$

and the solution to the point $i)$ of the problem is complete.

Passing to the integral J, and making use of the result in (3.282), we have

$$J = \int_0^{\pi/2} x^3\tan(x)\log(\sin(x))\mathrm{d}x$$

$$= -\int_0^{\pi/2}\sum_{n=1}^{\infty}\left(\psi\left(\frac{n+1}{2}\right)-\psi\left(\frac{n}{2}\right)-\frac{1}{n}\right)x^3\sin(2nx)\mathrm{d}x$$

{reverse the order of summation and integration}

$$= -\sum_{n=1}^{\infty}\left(\psi\left(\frac{n+1}{2}\right)-\psi\left(\frac{n}{2}\right)-\frac{1}{n}\right)\int_0^{\pi/2} x^3\sin(2nx)\mathrm{d}x$$

$$= -\sum_{n=1}^{\infty}\left(\psi\left(\frac{n+1}{2}\right)-\psi\left(\frac{n}{2}\right)-\frac{1}{n}\right)\left((-1)^{n-1}\frac{\pi^3}{16n}-(-1)^{n-1}\frac{3\pi}{8n^3}\right)$$

{make use of the result in (3.284)}

$$= -\sum_{n=1}^{\infty} \left((-1)^{n-1} \frac{\pi^3}{16n} - (-1)^{n-1} \frac{3\pi}{8n^3} \right) \int_0^1 t^{n-1} \frac{1-t}{1+t} \mathrm{d}t$$

{reverse the order of integration and summation}

$$= -\int_0^1 \sum_{n=1}^{\infty} \left((-1)^{n-1} \frac{\pi^3}{16n} - (-1)^{n-1} \frac{3\pi}{8n^3} \right) t^{n-1} \frac{1-t}{1+t} \mathrm{d}t$$

{calculate the series and split the integral}

$$= \frac{\pi^3}{8} \int_0^1 \frac{\log(1+t)}{1+t} \mathrm{d}t - \frac{\pi^3}{16} \int_0^1 \frac{\log(1+t)}{t} \mathrm{d}t - \frac{3\pi}{8} \int_0^1 \frac{\mathrm{Li}_3(-t)}{t} \mathrm{d}t + \frac{3\pi}{4} \int_0^1 \frac{\mathrm{Li}_3(-t)}{1+t} \mathrm{d}t$$

{note that the last integral was previously calculated in (3.286)}

$$= \frac{\pi^5}{960} + \frac{\pi^3}{16} \log^2(2) - \frac{9}{16} \pi \log(2) \zeta(3),$$

and the solution to the point ii) of the problem is complete.

Next, to calculate the integral K, we recall the result in (3.282) where if we replace x by $\pi/2 - x$, we have

$$\sum_{n=1}^{\infty} (-1)^{n-1} \left(\psi\left(\frac{n+1}{2}\right) - \psi\left(\frac{n}{2}\right) - \frac{1}{n} \right) \sin(2nx)$$
$$= -\cot(x) \log(\cos(x)), \ 0 < x < \frac{\pi}{2}. \tag{3.288}$$

Now, subtracting (3.52) from (3.282), we get

$$2 \sum_{n=1}^{\infty} \left(\psi\left(n + \frac{1}{2}\right) - \psi(n) - \frac{1}{2n} \right) \sin(4nx)$$
$$= \cot(x) \log(\cos(x)) - \tan(x) \log(\sin(x)), \ 0 < x < \frac{\pi}{2}. \tag{3.289}$$

Considering the result in (3.289), replacing x by t, and then integrating both sides from $t = 0$ to $t = x$, we get

$$2\int_0^x \sum_{n=1}^{\infty}\left(\psi\left(n+\frac{1}{2}\right)-\psi(n)-\frac{1}{2n}\right)\sin(4nt)\mathrm{d}t$$

{reverse the order of summation and integration}

$$=2\sum_{n=1}^{\infty}\left(\psi\left(n+\frac{1}{2}\right)-\psi(n)-\frac{1}{2n}\right)\int_0^x \sin(4nt)\mathrm{d}t$$

$$=\sum_{n=1}^{\infty}\left(\psi\left(n+\frac{1}{2}\right)-\psi(n)-\frac{1}{2n}\right)\frac{\sin^2(2nx)}{n}$$

$$=\int_0^x (\cot(t)\log(\cos(t))-\tan(t)\log(\sin(t)))\mathrm{d}t=\log(\sin(x))\log(\cos(x)).$$

Thus, we have

$$\sum_{n=1}^{\infty}\left(\psi\left(n+\frac{1}{2}\right)-\psi(n)-\frac{1}{2n}\right)\frac{\sin^2(2nx)}{n}$$
$$=\log(\sin(x))\log(\cos(x)),\ 0<x<\frac{\pi}{2}. \tag{3.290}$$

Moreover, if we use in (3.52) that $H_n=\psi(n+1)+\gamma$, combined the fact that

$$\psi\left(n+\frac{1}{2}\right)=2\psi(2n)-\psi(n)-2\log(2)$$

{employ the relation $\psi(n)+\gamma=H_{n-1}$}

$$=2H_{2n-1}-H_{n-1}-\gamma-2\log(2)=2H_{2n}-H_n-\gamma-2\log(2), \tag{3.291}$$

where I used the identity in (3.12), we obtain

$$\sum_{n=1}^{\infty}\left(2H_{2n}-2H_n+\frac{1}{2n}-2\log(2)\right)\frac{\sin^2(2nx)}{n}$$
$$=\log(\sin(x))\log(\cos(x)),\ 0<x<\frac{\pi}{2}. \tag{3.292}$$

Multiplying both sides of (3.52) by x^2 and then integrating from $x=0$ to $x=\pi/2$, we get

$$K=\int_0^{\pi/2} x^2\log(\sin(x))\log(\cos(x))\mathrm{d}x$$

$$= \int_0^{\pi/2} \sum_{n=1}^{\infty} \left(2H_{2n} - 2H_n + \frac{1}{2n} - 2\log(2)\right) \frac{x^2 \sin^2(2nx)}{n} \mathrm{d}x$$

$$\{\text{reverse the order of summation and integration}\}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \left(2H_{2n} - 2H_n + \frac{1}{2n} - 2\log(2)\right) \int_0^{\pi/2} x^2 \sin^2(2nx) \mathrm{d}x$$

$$= \frac{\pi}{16} \sum_{n=1}^{\infty} \frac{1}{n} \left(2H_{2n} - 2H_n + \frac{1}{2n} - 2\log(2)\right) \left(\frac{\pi^2}{3} - \frac{1}{2n^2}\right)$$

$$= \frac{\pi}{16} \left(\sum_{n=1}^{\infty} \frac{H_n}{n^3} - 8 \sum_{n=1}^{\infty} \frac{H_{2n}}{(2n)^3} + \frac{\pi^2}{6} \sum_{n=1}^{\infty} \frac{1}{n^2} + \log(2) \sum_{n=1}^{\infty} \frac{1}{n^3} - \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^4} \right.$$

$$\left. + \frac{2\pi^2}{3} \sum_{n=1}^{\infty} \frac{1}{n} \left(H_{2n} - H_n - \log(2)\right) \right)$$

$$\left\{ \text{make use of the fact that } \sum_{n=1}^{\infty} \frac{H_{2n}}{(2n)^3} = \frac{1}{2} \left(\sum_{n=1}^{\infty} \frac{H_n}{n^3} + \sum_{n=1}^{\infty} (-1)^n \frac{H_n}{n^3} \right) \right\}$$

$$= \frac{\pi}{16} \left(4 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n}{n^3} - 3 \sum_{n=1}^{\infty} \frac{H_n}{n^3} + \frac{\pi^2}{6} \sum_{n=1}^{\infty} \frac{1}{n^2} + \log(2) \sum_{n=1}^{\infty} \frac{1}{n^3} - \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^4} \right.$$

$$\left. + \frac{2\pi^2}{3} \sum_{n=1}^{\infty} \frac{1}{n} \left(H_{2n} - H_n - \log(2)\right) \right)$$

{the first series is given in (4.85), and the second one is the case $n = 3$ in (3.45)}

$$= \frac{19}{2880}\pi^5 + \frac{\pi^3}{48} \log^2(2) - \frac{\pi}{48} \log^4(2) - \frac{3}{8}\pi \log(2)\zeta(3) - \frac{\pi}{2} \operatorname{Li}_4\left(\frac{1}{2}\right)$$

$$+ \frac{\pi^3}{24} \sum_{n=1}^{\infty} \frac{1}{n} \left(H_{2n} - H_n - \log(2)\right). \tag{3.293}$$

For the remaining series that appears in (3.293) (as a note, a very similar version, which is known in the mathematical literature for years, may be found in [38], [79, p. 229]), we can write that

$$\sum_{n=1}^{\infty} \frac{1}{n} (H_{2n} - H_n - \log(2)) = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n} (H_{2n} - H_n - \log(2))$$

$$= \lim_{N\to\infty} \left(2 \sum_{n=1}^{N} \frac{H_{2n}}{2n} - \sum_{n=1}^{N} \frac{H_n}{n} - \log(2) \sum_{n=1}^{N} \frac{1}{n} \right)$$

$$= \lim_{N\to\infty} \left(\sum_{n=1}^{2N} (-1)^n \frac{H_n}{n} + \sum_{n=1}^{2N} \frac{H_n}{n} - \sum_{n=1}^{N} \frac{H_n}{n} - \log(2) H_N \right)$$

{make use of the second equality in (1.5)}

$$= \sum_{n=1}^{\infty} (-1)^n \frac{H_n}{n} + \frac{1}{2} \lim_{N\to\infty} \left(H_{2N}^2 - H_N^2 + H_{2N}^{(2)} - H_N^{(2)} - 2\log(2) H_N \right)$$

$$\left\{ \text{the value of the alternating series is } \sum_{n=1}^{\infty} (-1)^n \frac{H_n}{n} = \frac{1}{2} \left(\log^2(2) - \frac{\pi^2}{6} \right), \right\}$$

{which is straightforward to prove if dividing both sides of (4.5) by x and}

{then considering the integration from $x = -1$ to $x = 0$}

$$= \frac{1}{2} \left(\log^2(2) - \frac{\pi^2}{6} \right) + \frac{1}{2} \lim_{N\to\infty} \Big((H_{2N} - H_N)(H_{2N} + H_N)$$
$$+ H_{2N}^{(2)} - H_N^{(2)} - 2\log(2) H_N \Big)$$

{use the asymptotic expansion, $H_n = \gamma + \log(n) + O(1/n)$, as $n \to \infty$}

$$= \frac{1}{2} \left(\log^2(2) - \frac{\pi^2}{6} \right)$$
$$+ \frac{1}{2} \lim_{N\to\infty} \Big((\log(2) + O(1/N))(2\gamma + 2\log(N) + \log(2) + O(1/N))$$
$$+ H_{2N}^{(2)} - H_N^{(2)} - 2\log(2)(\gamma + \log(N) + O(1/N)) \Big)$$

$$= \log^2(2) - \frac{\pi^2}{12}. \tag{3.294}$$

By plugging the result from (3.294) in (3.293), we get

$$K = \int_0^{\pi/2} x^2 \log(\sin(x)) \log(\cos(x)) \mathrm{d}x$$
$$= \frac{\pi^5}{320} + \frac{\pi^3}{16} \log^2(2) - \frac{\pi}{48} \log^4(2) - \frac{3}{8}\pi \log(2)\zeta(3) - \frac{\pi}{2} \operatorname{Li}_4\left(\frac{1}{2}\right),$$

and the solution to the point $iii)$ of the problem is complete.

An alternative way of calculating the integral K may be found in [48].

Lastly, for calculating the integral L we proceed as before, and multiplying both sides of (3.52) by x^3 and then integrating from $x = 0$ to $x = \pi/2$, we get

$$L = \int_0^{\pi/2} x^3 \log(\sin(x)) \log(\cos(x)) \mathrm{d}x$$
$$= \int_0^{\pi/2} \sum_{n=1}^{\infty} \left(2H_{2n} - 2H_n + \frac{1}{2n} - 2\log(2)\right) \frac{x^3 \sin^2(2nx)}{n} \mathrm{d}x$$

{reverse the order of summation and integration}

$$= \sum_{n=1}^{\infty} \frac{1}{n} \left(2H_{2n} - 2H_n + \frac{1}{2n} - 2\log(2)\right) \int_0^{\pi/2} x^3 \sin^2(2nx) \mathrm{d}x$$
$$= \frac{\pi^2}{128} \sum_{n=1}^{\infty} \frac{1}{n} \left(2H_{2n} - 2H_n + \frac{1}{2n} - 2\log(2)\right) \left(\pi^2 - \frac{3}{n^2}\right)$$
$$= \frac{\pi^4}{256} \sum_{n=1}^{\infty} \frac{1}{n^2} + \frac{3}{64}\pi^2 \log(2) \sum_{n=1}^{\infty} \frac{1}{n^3} - \frac{3}{256}\pi^2 \sum_{n=1}^{\infty} \frac{1}{n^4} + \frac{3}{64}\pi^2 \sum_{n=1}^{\infty} \frac{H_n}{n^3}$$
$$-\frac{3}{8}\pi^2 \sum_{n=1}^{\infty} \frac{H_{2n}}{(2n)^3} + \frac{\pi^4}{64} \sum_{n=1}^{\infty} \frac{1}{n} \left(H_{2n} - H_n - \log(2)\right)$$

{note the fourth series is the case $n = 3$ in (3.45), then for the fifth series use that}

{$\displaystyle\sum_{n=1}^{\infty} \frac{H_{2n}}{(2n)^3} = \frac{1}{2} \sum_{n=1}^{\infty} ((-1)^n + 1) \frac{H_n}{n^3}$, and the sixth series is calculated in (3.294)}

$$= \frac{9}{32} \log(2)\zeta(2)\zeta(3) + \frac{45}{32} \log^2(2)\zeta(4) - \frac{63}{512}\zeta(6) + \frac{3}{16}\pi^2$$
$$\times \left(\sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n}{n^3} - \sum_{n=1}^{\infty} \frac{H_n}{n^3}\right)$$

{the first series is given in (4.85), and the second series is the case $n = 3$ in (3.45)}

$$= \frac{1449}{512}\zeta(6) + \frac{45}{16}\log^2(2)\zeta(4) - \frac{3}{32}\log^4(2)\zeta(2) - \frac{27}{16}\log(2)\zeta(2)\zeta(3)$$
$$-\frac{9}{4}\zeta(2)\operatorname{Li}_4\left(\frac{1}{2}\right),$$

and the solution to the point iv) of the problem is complete.

The strategy presented in the solutions above can be viewed as a powerful way that also allows us to attack similar resistant integrals. The given solutions also answer the proposed *challenging question*.

3.53 Two Beautiful Representations of Catalan's Constant, $G = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \cdots$

Solution As in the title, we may view both results as beautiful representations of the Catalan's constant. I submitted the integral from the first part, in a slightly modified form, to the *MathProblems journal*, Vol. 5, No. 3 (see [57, Problem **131**, p. 442]). If you remember my saying at the end of Sect. 3.17, there I wrote that sometimes finding a simple integral representation of the integrand might be extremely useful. In fact, after the rearrangement of the double integral, it is enough to find a useful integral representation of the numerator of the integrand, and then we'll also see the result in (1.1) from the first section of the current chapter will come into play.

To start with the first point of the problem, we denote the integral by I, and then we write

$$I = \int_0^{\pi/2}\left(\int_0^{\pi/2} \frac{\log\left(\cos\left(\frac{x}{2}\right)\right) - \log\left(\cos\left(\frac{y}{2}\right)\right)}{\cos(x) - \cos(y)}\mathrm{d}x\right)\mathrm{d}y$$

$$\left\{\text{consider the identity } \cos\left(\frac{x}{2}\right) = \sqrt{\frac{1+\cos(x)}{2}}\right\}$$

$$= \int_0^{\pi/2}\left(\int_0^{\pi/2} \frac{\log\left(\sqrt{\frac{1+\cos(x)}{2}}\right) - \log\left(\sqrt{\frac{1+\cos(y)}{2}}\right)}{\cos(x) - \cos(y)}\mathrm{d}x\right)\mathrm{d}y,$$

and making the changes of variable $\cos(x) = u$, $\cos(y) = v$, we get

$$= \frac{1}{2}\int_0^1 \left(\int_0^1 \frac{\log(1+u)-\log(1+v)}{(u-v)\sqrt{1-u^2}\sqrt{1-v^2}}\mathrm{d}u\right)\mathrm{d}v,$$

and since we can write that $\log(1+u)-\log(1+v) = \int_0^1 \left(\frac{u}{1+tu} - \frac{v}{1+tv}\right)\mathrm{d}t$, then the integral becomes

$$I = \frac{1}{2}\int_0^1 \left(\int_0^1 \left(\int_0^1 \frac{1}{(1+tu)(1+tv)\sqrt{1-u^2}\sqrt{1-v^2}}\mathrm{d}t\right)\mathrm{d}u\right)\mathrm{d}v$$

{change the order of integration}

$$= \frac{1}{2}\int_0^1 \left(\int_0^1 \frac{1}{(1+tv)\sqrt{1-v^2}}\left(\int_0^1 \frac{1}{(1+tu)\sqrt{1-u^2}}\,\mathrm{d}u\right)\mathrm{d}v\right)\mathrm{d}t$$

$$= \frac{1}{2}\int_0^1 \frac{\arccos^2(t)}{1-t^2}\mathrm{d}t, \tag{3.295}$$

where to get the last equality I used the result in 1.1.

Further, continuing with the last integral in (3.295), we write

$$I = \frac{1}{2}\int_0^1 \frac{\arccos^2(t)}{1-t^2}\mathrm{d}t \overset{t=\cos(x)}{=} \frac{1}{2}\int_0^{\pi/2} \frac{x^2}{\sin(x)}\mathrm{d}x$$

{apply the integration by parts}

$$= -\int_0^{\pi/2} x\log\left(\tan\left(\frac{x}{2}\right)\right)\mathrm{d}x \overset{x/2=y}{=} -4\int_0^{\pi/4} y\log(\tan(y))\,\mathrm{d}y$$

{recall and use the Fourier series in (3.74)}

$$= 8\int_0^{\pi/4} y\sum_{k=1}^{\infty}\frac{\cos(2(2k-1)y)}{2k-1}\mathrm{d}y = 8\sum_{k=1}^{\infty}\int_0^{\pi/4} y\frac{\cos(2(2k-1)y)}{2k-1}\mathrm{d}y$$

$$= \sum_{k=1}^{\infty}\left(\pi\frac{(-1)^{k-1}}{(2k-1)^2} - \frac{2}{(2k-1)^3}\right) = \pi\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^2} - 2\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3}$$

$$= \pi G - \frac{7}{4}\zeta(3),$$

and the part $i)$ of the problem is finalized.

Passing to the second point of the problem, we may use one[25] of the well-known integral representations of the Catalan's constant, that is $-\int_0^1 \frac{\log(x)}{1+x^2}\mathrm{d}x = G$, and then all reduces to proving that

$$-\int_0^1 \frac{\log(x)}{1+x^2}\mathrm{d}x = \sqrt{2}\sum_{n=1}^{\infty}(-1)^{(n-1)(n-2)/2}\frac{1}{(2n-1)^2} - \int_0^1 \frac{\log(1+\sqrt{1+x^2})}{1+x^2}\mathrm{d}x$$

or

$$\sqrt{2}\sum_{n=1}^{\infty}(-1)^{(n-1)(n-2)/2}\frac{1}{(2n-1)^2} = \int_0^1 \frac{\log\left(\dfrac{1+\sqrt{1+x^2}}{x}\right)}{1+x^2}\mathrm{d}x.$$

Using the integration by parts, we write that

$$\int_0^1 \frac{\log\left(\dfrac{1+\sqrt{1+x^2}}{x}\right)}{1+x^2}\mathrm{d}x = \int_0^1 (\arctan(x))' \log\left(\frac{1+\sqrt{1+x^2}}{x}\right)\mathrm{d}x$$

$$= \arctan(x)\log\left(\frac{1+\sqrt{1+x^2}}{x}\right)\Bigg|_{x=0}^{x=1} + \int_0^1 \frac{\arctan(x)}{x\sqrt{1+x^2}}\mathrm{d}x$$

$$= \frac{\pi}{4}\log(1+\sqrt{2}) + \int_0^1 \frac{\arctan(x)}{x\sqrt{1+x^2}}\mathrm{d}x \overset{x=\tan(y)}{=} \frac{\pi}{4}\log(1+\sqrt{2}) + \int_0^{\pi/4} \frac{y}{\sin(y)}\mathrm{d}y$$

$$= \frac{\pi}{4}\log(1+\sqrt{2}) + \int_0^{\pi/4} \left(\log\left(\tan\left(\frac{y}{2}\right)\right)\right)' y\,\mathrm{d}y$$

$$\{\text{apply the integration by parts}\}$$

$$= \underbrace{\frac{\pi}{4}\log(1+\sqrt{2}) + \left(\log\left(\tan\left(\frac{y}{2}\right)\right) y\Bigg|_{y=0}^{y=\pi/4}\right)}_{0} - \int_0^{\pi/4}\log\left(\tan\left(\frac{y}{2}\right)\right)\mathrm{d}y$$

[25] By simple manipulations it's easy to see that $\int_0^1 \frac{\log(x)}{1+x^2}\mathrm{d}x = \int_0^1 \sum_{n=1}^{\infty}(-1)^{n-1}x^{2n-2}\log(x)\mathrm{d}x = \sum_{n=1}^{\infty}(-1)^{n-1}\int_0^1 x^{2n-2}\log(x)\mathrm{d}x = \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2} = -G$.

$$= -\int_0^{\pi/4} \log\left(\tan\left(\frac{y}{2}\right)\right) \mathrm{d}y \overset{y/2=z}{=} -2\int_0^{\pi/8} \log(\tan(z))\,\mathrm{d}z$$

{recall and use the Fourier series in (3.74)}

$$= 4\int_0^{\pi/8} \sum_{k=1}^{\infty} \frac{\cos(2(2k-1)z)}{2k-1} \mathrm{d}z = 4\sum_{k=1}^{\infty} \int_0^{\pi/8} \frac{\cos(2(2k-1)z)}{2k-1} \mathrm{d}z$$

$$= 2\sum_{k=1}^{\infty} \frac{\sin((2k-1)\pi/4)}{(2k-1)^2}$$

$$= \sqrt{2}\left(1 + \frac{1}{3^2} - \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} - \frac{1}{13^2} - \frac{1}{15^2} + \cdots\right)$$

$$= \sqrt{2}\sum_{n=1}^{\infty} (-1)^{(n-1)(n-2)/2} \frac{1}{(2n-1)^2},$$

and the part $ii)$ of the problem is finalized.

For both results of the problem the calculations have been reduced to the point where I had to make use of the Fourier series of $\log(\tan(x))$, to transform the integrals into useful series.

Often the integrals and series representations of the Catalan's constant are beautiful problems to try and enjoy! More such Catalan's representations may be found in [2, 19, 96].

3.54 Proving Two Equalities with Tough Integrals Involving Logarithms and Polylogarithms

Solution Now we step up toward a problem where we need to prove that two equalities with integrals hold without calculating the integrals (in other words, we have to solve the problem under some restrictions). And the integrals do not look friendly!

Starting with the left-hand side of the equality from the point i), we write

$$3\int_0^{\pi/2} \tan(x)\log^2(\sin(x))\operatorname{Li}_4\left(-\cot^2(x)\right)\mathrm{d}x$$

$$= -\frac{3}{8}\int_0^{\pi/2} \log^2(1-\cos^2(x))\frac{\operatorname{Li}_4\left(\dfrac{\cos^2(x)}{\cos^2(x)-1}\right)}{\cos^2(x)}(\cos^2(x))'\mathrm{d}x$$

$\{$make the change of variable, $\cos^2(x) = y\}$

$$= \frac{3}{8}\int_0^1 \log^2(1-y)\frac{\operatorname{Li}_4\left(\dfrac{y}{y-1}\right)}{y}\mathrm{d}y$$

$\{$make use of the result in (1.12), the case $n = 3\}$

$$= \frac{1}{16}\int_0^1\left(\int_0^1 \frac{\log^3(x)\log^2(1-y)}{1-y+yx}\mathrm{d}x\right)\mathrm{d}y$$

$\{$use the variable changes $x = 1-u$ and $y = 1-v$, and return to the notation in $y, x\}$

$$= \frac{1}{16}\int_0^1\left(\int_0^1 \frac{\log^2(x)\log^3(1-y)}{1-y+yx}\mathrm{d}y\right)\mathrm{d}x$$

$$= \frac{1}{16}\int_0^1 \log^3(1-y)\left(\int_0^1 \frac{\log^2(x)}{1-y+yx}\mathrm{d}x\right)\mathrm{d}y$$

$\{$make use of the result in (1.12), the case $n = 2\}$

$$= -\frac{1}{8}\int_0^1 \log^3(1-y)\frac{\operatorname{Li}_3\left(\dfrac{y}{y-1}\right)}{y}\mathrm{d}y$$

$\{$make the change of variable, $\sin^2(x) = y\}$

$$= 2\int_0^{\pi/2} \cot(x)\log^3\left(\frac{1}{\cos(x)}\right)\operatorname{Li}_3\left(-\tan^2(x)\right)\mathrm{d}x,$$

and the point i) of the problem is finalized.

Acting similarly as before, we write

$$3\int_0^{\pi/2} \tan(x)\log^2(\sin(x))\,\mathrm{Li}_5\left(-\cot^2(x)\right)\mathrm{d}x$$

$$= -\frac{3}{8}\int_0^{\pi/2} \log^2(1-\cos^2(x))\frac{\mathrm{Li}_5\left(\dfrac{\cos^2(x)}{\cos^2(x)-1}\right)}{\cos^2(x)}(\cos^2(x))'\mathrm{d}x$$

$$\{\text{make the change of variable } \cos^2(x) = y\}$$

$$= \frac{3}{8}\int_0^1 \log^2(1-y)\frac{\mathrm{Li}_5\left(\dfrac{y}{y-1}\right)}{y}\mathrm{d}y$$

{make use of the result in (1.12), the case $n = 4$}

$$= -\frac{1}{64}\int_0^1\left(\int_0^1 \frac{\log^4(x)\log^2(1-y)}{1-y+yx}\mathrm{d}x\right)\mathrm{d}y$$

{make the variable changes $x = 1-u$ and $y = 1-v$, and return to the notation in y, x}

$$= -\frac{1}{64}\int_0^1\left(\int_0^1 \frac{\log^2(x)\log^4(1-y)}{1-y+yx}\mathrm{d}y\right)\mathrm{d}x$$

$$= -\frac{1}{64}\int_0^1 \log^4(1-y)\left(\int_0^1 \frac{\log^2(x)}{1-y+yx}\mathrm{d}x\right)\mathrm{d}y$$

{make use of the result in (1.12), the case $n = 2$}

$$= \frac{1}{32}\int_0^1 \log^4(1-y)\frac{\mathrm{Li}_3\left(\dfrac{y}{y-1}\right)}{y}\mathrm{d}y$$

$$\{\text{make the change of variable, } \sin^2(x) = y\}$$

$$= \int_0^{\pi/2} \cot(x)\log^4(\cos(x))\mathrm{Li}_3\left(-\tan^2(x)\right)\mathrm{d}x,$$

and the point ii) of the problem is finalized.

In the next section, we'll prepare to meet similar integrals where we'll want to identify and calculate the harmonic series behind the scene.

3.55 Tough Integrals with Logarithms, Polylogarithms, and Trigonometric and Hyperbolic Functions

Solution If in the previous section we could avoid the calculations of the harmonic series, in this case I think we have to prepare ourselves for encounters with tough series (which also accounts for the *Tough integrals* part from the title of the section). We may also observe some similarities with the previous integrals that make us think to proceed as before for getting the transformation of the integrals into series.

Let's make the change of variable $\sin^2(x) = y$, and then we write that

$$\int_0^{\pi/2} \cot(x)\log(\cos(x))\log^2(\sin(x))\operatorname{Li}_3\left(-\tan^2(x)\right)\mathrm{d}x$$

$$= \frac{1}{16}\int_0^{\pi/2} \log(1-\sin^2(x))\log^2(\sin^2(x))\operatorname{Li}_3\left(-\frac{\sin^2(x)}{1-\sin^2(x)}\right)\frac{(\sin^2(x))'}{\sin^2(x)}\mathrm{d}x$$

$$= \frac{1}{16}\int_0^1 \frac{\log^2(y)\log(1-y)}{y}\operatorname{Li}_3\left(\frac{y}{y-1}\right)\mathrm{d}y$$

$$\left\{\text{make use of the result in (4.11) to express } \frac{\operatorname{Li}_3\left(\frac{y}{y-1}\right)}{y} \text{ as a series}\right\}$$

$$= -\frac{1}{32}\int_0^1 \sum_{n=1}^{\infty} y^{n-1}\frac{H_n^2 + H_n^{(2)}}{n}\log^2(y)\log(1-y)\mathrm{d}y$$

$$\{\text{reverse the order of summation and integration}\}$$

$$= -\frac{1}{32}\sum_{n=1}^{\infty}\frac{H_n^2 + H_n^{(2)}}{n}\int_0^1 y^{n-1}\log^2(y)\log(1-y)\mathrm{d}y$$

$$= -\frac{1}{32}\sum_{n=1}^{\infty}\frac{H_n^2 + H_n^{(2)}}{n}\frac{d^2}{dn^2}\left(\int_0^1 y^{n-1}\log(1-y)\mathrm{d}y\right)$$

$$\{\text{make use of the result in (1.4)}\}$$

$$= -\frac{1}{32}\sum_{n=1}^{\infty}\frac{H_n^2 + H_n^{(2)}}{n}\frac{d^2}{dn^2}\left(-\frac{\psi(n+1)+\gamma}{n}\right)$$

{differentiate and write the result in terms of the generalized harmonic numbers}

$$= -\frac{1}{32}\sum_{n=1}^{\infty}\frac{H_n^2+H_n^{(2)}}{n}\left(2\zeta(3)\frac{1}{n}+2\zeta(2)\frac{1}{n^2}-2\frac{H_n}{n^3}-2\frac{H_n^{(2)}}{n^2}-2\frac{H_n^{(3)}}{n}\right)$$

$$= \frac{1}{16}\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^4}+\frac{1}{16}\sum_{n=1}^{\infty}\frac{H_n^2H_n^{(2)}}{n^3}+\frac{1}{16}\sum_{n=1}^{\infty}\frac{H_n^2H_n^{(3)}}{n^2}+\frac{1}{16}\sum_{n=1}^{\infty}\frac{(H_n^{(2)})^2}{n^3}$$

$$+\frac{1}{16}\sum_{n=1}^{\infty}\frac{H_n^3}{n^4}+\frac{1}{16}\sum_{n=1}^{\infty}\frac{H_n^{(2)}H_n^{(3)}}{n^2}-\frac{1}{16}\zeta(2)\sum_{n=1}^{\infty}\frac{H_n^2}{n^3}-\frac{1}{16}\zeta(2)\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3}$$

$$-\frac{1}{16}\zeta(3)\sum_{n=1}^{\infty}\frac{H_n^2}{n^2}-\frac{1}{16}\zeta(3)\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^2}$$

{the values of the series are given in (4.58), (4.53), (4.59), (4.65), (4.61),}

{(4.63), (4.30), (6.67), (4.29), (4.14), the case $p = 2$, with $n \to \infty$}

$$= \frac{109}{128}\zeta(7)-\frac{23}{32}\zeta(3)\zeta(4)+\frac{1}{16}\zeta(2)\zeta(5),$$

and the solution to the point $i)$ of the problem is complete.

Proceeding similarly for the point $ii)$ of the problem, we employ the change of variable $\sinh^2(x) = y$, and then we have

$$\int_0^{\log(1+\sqrt{2})}\coth(x)\log(\sinh(x))\log\left(2-\cosh^2(x)\right)\operatorname{Li}_2\left(\tanh^2(x)\right)\mathrm{d}x$$

{make use of the identity $\cosh^2(x)-\sinh^2(x)=1$}

$$= \frac{1}{4}\int_0^{\log(1+\sqrt{2})}\log(\sinh^2(x))\log\left(1-\sinh^2(x)\right)$$

$$\times \operatorname{Li}_2\left(\frac{\sinh^2(x)}{1+\sinh^2(x)}\right)\frac{(\sinh^2(x))'}{\sinh^2(x)}\mathrm{d}x$$

$$= \frac{1}{4}\int_0^1\frac{\log(y)\log(1-y)}{y}\operatorname{Li}_2\left(\frac{y}{1+y}\right)\mathrm{d}y$$

{make use of the identity in (4.10)}

$$= \frac{1}{4}\int_0^1 \sum_{n=1}^{\infty}(-1)^{n-1}y^{n-1}\frac{H_n}{n}\log(y)\log(1-y)\,\mathrm{d}y$$

{reverse the order of summation and integration}

$$= \frac{1}{4}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n}\int_0^1 y^{n-1}\log(y)\log(1-y)\,\mathrm{d}y$$

$$= \frac{1}{4}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n}\frac{d}{dn}\left(\int_0^1 y^{n-1}\log(1-y)\mathrm{d}y\right)$$

{make use of the result in (1.4)}

$$= \frac{1}{4}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n}\frac{d}{dn}\left(-\frac{\psi(n+1)+\gamma}{n}\right)$$

{differentiate and write the result in terms of the generalized harmonic numbers}

$$= \frac{1}{4}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n}\left(\frac{H_n}{n^2}+\frac{H_n^{(2)}}{n}-\zeta(2)\frac{1}{n}\right) = \frac{1}{4}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2}{n^3}$$

$$+\frac{1}{4}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n H_n^{(2)}}{n^2} - \frac{1}{4}\zeta(2)\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^2}$$

{make use of the results in (4.93), (4.95), (4.88)}

$$= \frac{73}{128}\zeta(5) - \frac{17}{64}\zeta(2)\zeta(3),$$

and the solution to the point $ii)$ of the problem is complete.

At the opening of Sect. 3.5, where I said a few things about Digamma function, I also mentioned that the extension of the harmonic number H_n, for the non-integer values of n, that is achieved through the Digamma function relation, $H_n = \psi(n+1) + \gamma$, is very useful in many cases, and you may see such examples right above where I used the Digamma function for being able to differentiate.

Then, no concern, every harmonic series involved in the calculations above is fully derived (evaluated) in the sixth chapter.

3.56 A Double Integral Hiding a Beautiful Idea About the Symmetry and (Possibly) an Unexpected Closed-Form

Solution If running *Mathematica*, just out of curiosity, based on the numerical evaluation one might easily arrive to conjecture the value of the double integral is $1/3$, which seems too nice to be true. In fact, this is the right value! *But how to prove it?*

This section is also meant to emphasize the importance of switching from Cartesian coordinates to polar coordinates (see [115]) or vice versa (more exactly, for this problem we will be focused on the latter part, the switch from polar coordinates to Cartesian coordinates). Then, last, but not least, we will have to make use of the *symmetry eye* (supposing the experience gained in the previous sections helped), and wisely exploit the symmetry.

Denoting the integral by I and starting with reversing the order of integration and the change of variable $\tan(y)/\cos(x) = t$, we write

$$I = \int_0^{\pi/4} \left(\int_0^{\pi/4} \arctan(\cos(x)\cot(y)) \sec^3(x) \tan^2(y) \sec^2(y) \mathrm{d}x \right) \mathrm{d}y$$

$$= \int_0^{\pi/4} \left(\int_0^{\pi/4} \arctan(\cos(x)\cot(y)) \sec^3(x) \tan^2(y) \sec^2(y) \mathrm{d}y \right) \mathrm{d}x$$

$$= \int_0^{\pi/4} \left(\int_0^{\sec(x)} \arctan\left(\frac{1}{t}\right) t^2 \mathrm{d}t \right) \mathrm{d}x$$

$$\{\text{switch from polar to Cartesian coordinates}\}$$

$$= \int_0^1 \left(\int_0^x \sqrt{x^2+y^2} \arctan\left(\frac{1}{\sqrt{x^2+y^2}}\right) \mathrm{d}y \right) \mathrm{d}x$$

$$\{\text{exploit the symmetry of the integrand}\}$$

$$= \frac{1}{2} \int_0^1 \left(\int_0^1 \sqrt{x^2+y^2} \arctan\left(\frac{1}{\sqrt{x^2+y^2}}\right) \mathrm{d}x \right) \mathrm{d}y$$

$$= \frac{1}{2} \int_0^1 \left(\int_0^1 \left(\int_0^1 \frac{x^2+y^2}{x^2+y^2+z^2} \mathrm{d}z \right) \mathrm{d}x \right) \mathrm{d}y. \tag{3.296}$$

Now, exploiting the symmetry in (3.296), we write

$$3I = \frac{1}{2}\int_0^1 \left(\int_0^1 \left(\int_0^1 \frac{x^2+y^2}{x^2+y^2+z^2}\mathrm{d}z\right)\mathrm{d}x\right)\mathrm{d}y$$
$$+\frac{1}{2}\int_0^1 \left(\int_0^1 \left(\int_0^1 \frac{y^2+z^2}{x^2+y^2+z^2}\mathrm{d}x\right)\mathrm{d}y\right)\mathrm{d}z$$
$$+\frac{1}{2}\int_0^1 \left(\int_0^1 \left(\int_0^1 \frac{x^2+z^2}{x^2+y^2+z^2}\mathrm{d}y\right)\mathrm{d}x\right)\mathrm{d}z$$

{reverse the integration order in the second and third triple integrals}

$$= \frac{1}{2}\int_0^1 \left(\int_0^1 \left(\int_0^1 \frac{x^2+y^2}{x^2+y^2+z^2}\mathrm{d}z\right)\mathrm{d}x\right)\mathrm{d}y$$
$$+\frac{1}{2}\int_0^1 \left(\int_0^1 \left(\int_0^1 \frac{y^2+z^2}{x^2+y^2+z^2}\mathrm{d}z\right)\mathrm{d}x\right)\mathrm{d}y$$
$$+\frac{1}{2}\int_0^1 \left(\int_0^1 \left(\int_0^1 \frac{x^2+z^2}{x^2+y^2+z^2}\mathrm{d}z\right)\mathrm{d}x\right)\mathrm{d}y$$
$$= \frac{1}{2}\int_0^1 \left(\int_0^1 \left(\int_0^1 \frac{2(x^2+y^2+z^2)}{x^2+y^2+z^2}\mathrm{d}z\right)\mathrm{d}x\right)\mathrm{d}y$$
$$= \int_0^1 \left(\int_0^1 \left(\int_0^1 \mathrm{d}z\right)\mathrm{d}x\right)\mathrm{d}y = 1,$$

whence we conclude that

$$I = \int_0^{\pi/4}\int_0^{\pi/4} \arctan(\cos(x)\cot(y))\sec^3(x)\tan^2(y)\sec^2(y)\mathrm{d}x\mathrm{d}y = \frac{1}{3},$$

and the solution is complete.

To summarize, in this section we saw the power of fruitfully combining the switching from polar coordinates to Cartesian coordinates and the passing from a double integral to a triple integral with a symmetrical integrand, where we could easily exploit the symmetry, a strategy good to keep in our portofolio of strategies for approaching the integrals.

3.57 An Exciting Representation of Catalan's Constant with Trigonometric Functions and Digamma Function

Solution At some point in the writing process of the book I included the problem in Sect. 1.53. However, after pondering more over the beauty of the present result, I considered it is *fair* to assign it a separate section such that all attention to be given to it when the reader arrives at its dedicated section.

If you didn't miss Sect. 1.52, I guess you will easily figure out how to proceed and prove the result.

Now, we already met both series in (3.282) and (3.52) from Sect. 3.52, and then we write that

$$\int_0^{\pi/4} \tan(x) \sum_{n=1}^{\infty} (-1)^{n-1} \left(\psi\left(\frac{n}{2}\right) - \psi\left(\frac{n+1}{2}\right) + \frac{1}{n}\right) \sin(2nx)\mathrm{d}x$$

$$- \int_0^{\pi/4} \cot(x) \sum_{n=1}^{\infty} \left(\psi\left(\frac{n}{2}\right) - \psi\left(\frac{n+1}{2}\right) + \frac{1}{n}\right) \sin(2nx)\mathrm{d}x$$

$$= \int_0^{\pi/4} \log(\cos(x))\mathrm{d}x - \int_0^{\pi/4} \log(\sin(x))\mathrm{d}x = -\int_0^{\pi/4} \log(\tan(x))\mathrm{d}x$$

{recall and use the Fourier series in (3.74)}

$$= 2\int_0^{\pi/4} \sum_{k=1}^{\infty} \frac{\cos(2(2k-1)x)}{2k-1}\mathrm{d}x$$

{reverse the order of summation and integration}

$$= 2\sum_{k=1}^{\infty} \int_0^{\pi/4} \frac{\cos(2(2k-1)x)}{2k-1}\mathrm{d}x = -\sum_{k=1}^{\infty} \frac{\cos(k\pi)}{(2k-1)^2}$$

$$= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{(2k-1)^2} = G,$$

and the solution is complete.

Once we have obtained the functions behind the series, everything nicely reduces to a classical integral representation of the Catalan's constant (which is easy to calculate as seen above). The integral I reduced to the calculations, $-\int_0^{\pi/4} \log(\tan(x))\mathrm{d}x$, is also found in [19, (9), p. 2] and [78, Chapter 1, p. 36].

3.58 Evaluating an Enjoyable Trigonometric Integral Involving the Complete Elliptic Integral of the First Kind at Its Roots

Solution In the present section we find ourselves in the arena with another integral-beast (just remember my *Preface*, if you had a chance to read it, or the first integral from Sect. 1.41, where I used the same words to portray two integrals coming from the Paul's book, *Inside Interesting Integrals*).

Although not one like some of the integrals from the previous sections that eventually got reduced to tough harmonic series, but still a daunting integral, and we see this right from the beginning since we need a (promising) starting point.

A first step is to denote the integral by I and then split it such that we can use a result previously obtained, that is

$$I=\int_0^{\pi/2}\frac{1}{(1+\sin^4(x))\sqrt{1+\sin^2(x)}}\mathrm{d}x=\int_0^{\pi/2}\frac{(1+\sin^4(x))-\sin^4(x)}{(1+\sin^4(x))\sqrt{1+\sin^2(x)}}\mathrm{d}x$$

$$=\underbrace{\int_0^{\pi/2}\frac{1}{\sqrt{1+\sin^2(x)}}\mathrm{d}x}_{I_1}-\underbrace{\int_0^{\pi/2}\frac{\sin^4(x)}{(1+\sin^4(x))\sqrt{1+\sin^2(x)}}\mathrm{d}x}_{I_2},$$

where using that the integral I_1 has been calculated during the solution to the point $ii)$ from Sect. 1.43, we get

$$I=\frac{1}{4\sqrt{2\pi}}\Gamma\left(\frac{1}{4}\right)^2-\underbrace{\int_0^{\pi/2}\frac{\sin^4(x)}{(1+\sin^4(x))\sqrt{1+\sin^2(x)}}\mathrm{d}x}_{I_2}. \tag{3.297}$$

For the remaining integral in (3.297), we write

$$I_2=\int_0^{\pi/2}\frac{\sin^4(x)}{(1+\sin^4(x))\sqrt{1+\sin^2(x)}}\mathrm{d}x$$

$$\{\text{make use of the result in (3.233)}\}$$

$$=\frac{2}{\pi}\int_0^{\pi/2}\left(\int_0^{\pi/2}\frac{\sin^4(x)}{(1+\sin^4(x))(1+\sin^2(x)\sin^2(y))}\mathrm{d}y\right)\mathrm{d}x,$$

and due to the symmetry, it's easy to see all reduces to

$$I_2=\frac{1}{\pi}\left(\int_0^{\pi/2}\left(\int_0^{\pi/2}\frac{\sin^4(x)}{(1+\sin^4(x))(1+\sin^2(x)\sin^2(y))}\mathrm{d}y\right)\mathrm{d}x\right.$$

$$\left.+\int_0^{\pi/2}\left(\int_0^{\pi/2}\frac{\sin^4(y)}{(1+\sin^4(y))(1+\sin^2(x)\sin^2(y))}\mathrm{d}y\right)\mathrm{d}x\right)$$

$$=\frac{1}{\pi}\int_0^{\pi/2}\left(\int_0^{\pi/2}\frac{\sin^4(x)+\sin^4(y)+2\sin^4(x)\sin^4(y)}{(1+\sin^4(x))(1+\sin^4(y))(1+\sin^2(x)\sin^2(y))}\mathrm{d}y\right)\mathrm{d}x$$

$$= \frac{1}{\pi}\int_0^{\pi/2}\left(\int_0^{\pi/2}\frac{(1+\sin^4(x))(1+\sin^4(y))-(1-(\sin(x)\sin(y))^4)}{(1+\sin^4(x))(1+\sin^4(y))(1+\sin^2(x)\sin^2(y))}\mathrm{d}y\right)\mathrm{d}x$$

$$= \frac{1}{\pi}\underbrace{\int_0^{\pi/2}\left(\int_0^{\pi/2}\frac{1}{1+\sin^2(x)\sin^2(y)}\mathrm{d}y\right)\mathrm{d}x}_{I_3}$$

$$-\frac{1}{\pi}\underbrace{\int_0^{\pi/2}\left(\int_0^{\pi/2}\frac{1}{(1+\sin^4(x))(1+\sin^4(y))}\mathrm{d}y\right)\mathrm{d}x}_{I_4}$$

$$+\frac{1}{\pi}\underbrace{\int_0^{\pi/2}\left(\int_0^{\pi/2}\frac{(\sin(x)\sin(y))^2}{(1+\sin^4(x))(1+\sin^4(y))}\mathrm{d}y\right)\mathrm{d}x}_{I_5}. \tag{3.298}$$

By the result in (3.233), the integral I_3 in (3.298) becomes

$$I_3 = \int_0^{\pi/2}\left(\int_0^{\pi/2}\frac{1}{1+\sin^2(x)\sin^2(y)}\mathrm{d}y\right)\mathrm{d}x = \frac{\pi}{2}\underbrace{\int_0^{\pi/2}\frac{1}{\sqrt{1+\sin^2(x)}}\mathrm{d}x}_{I_1}$$

$$= \frac{1}{8}\sqrt{\frac{\pi}{2}}\Gamma\left(\frac{1}{4}\right)^2. \tag{3.299}$$

To get the value of the integral I_4 in (3.298), note that

$$\int_0^{\pi/2}\frac{1}{1+\sin^4(x)}\mathrm{d}x = \frac{1}{2}\int_0^{\pi/2}\frac{1}{1+i\sin^2(x)}\mathrm{d}x + \frac{1}{2}\int_0^{\pi/2}\frac{1}{1-i\sin^2(x)}\mathrm{d}x$$

$$\{\text{make use of the result in (3.233)}\}$$

$$= \frac{\pi}{4}\left(\frac{1}{\sqrt{1+i}}+\frac{1}{\sqrt{1-i}}\right)$$

$$\left\{\text{use the fact that } \frac{1}{\sqrt{1\pm i}} = \frac{1}{\sqrt[4]{2}}\left(\cos\left(\frac{\pi}{8}\right)\mp i\sin\left(\frac{\pi}{8}\right)\right)\right\}$$

$$= \frac{\pi}{2^{5/4}}\cos\left(\frac{\pi}{8}\right) = \frac{\pi}{2^{9/4}}\sqrt{\sqrt{2}+2} = \frac{\pi}{4}\sqrt{\sqrt{2}+1},$$

which immediately leads to

$$I_4 = \int_0^{\pi/2} \left(\int_0^{\pi/2} \frac{1}{(1+\sin^4(x))(1+\sin^4(y))} \mathrm{d}y \right) \mathrm{d}x = \frac{\pi^2}{16}(\sqrt{2}+1). \tag{3.300}$$

Then, for the integral I_5 in (3.298), we write

$$\begin{aligned}
\int_0^{\pi/2} \frac{\sin^2(x)}{1+\sin^4(x)} \mathrm{d}x &= \frac{1}{2} \int_0^{\pi/2} \frac{\sin^2(x)}{1+i\sin^2(x)} \mathrm{d}x + \frac{1}{2} \int_0^{\pi/2} \frac{\sin^2(x)}{1-i\sin^2(x)} \mathrm{d}x \\
&= \frac{1}{2i} \int_0^{\pi/2} \frac{(1+i\sin^2(x))-1}{1+i\sin^2(x)} \mathrm{d}x - \frac{1}{2i} \int_0^{\pi/2} \frac{(1-i\sin^2(x))-1}{1-i\sin^2(x)} \mathrm{d}x \\
&= \frac{i}{2} \int_0^{\pi/2} \frac{1}{1+i\sin^2(x)} \mathrm{d}x - \frac{i}{2} \int_0^{\pi/2} \frac{1}{1-i\sin^2(x)} \mathrm{d}x \\
&\quad \{\text{make use of the result in (3.233)}\} \\
&= \frac{\pi}{4} i \left(\frac{1}{\sqrt{1+i}} - \frac{1}{\sqrt{1-i}} \right) \\
&\quad \left\{\text{use the fact that } \frac{1}{\sqrt{1\pm i}} = \frac{1}{\sqrt[4]{2}} \left(\cos\left(\frac{\pi}{8}\right) \mp i \sin\left(\frac{\pi}{8}\right) \right) \right\} \\
&= \frac{\pi}{2^{5/4}} \sin\left(\frac{\pi}{8}\right) = \frac{\pi}{2^{9/4}} \sqrt{2-\sqrt{2}} = \frac{\pi}{4} \sqrt{\sqrt{2}-1},
\end{aligned}$$

which also immediately leads to

$$I_5 = \int_0^{\pi/2} \left(\int_0^{\pi/2} \frac{(\sin(x)\sin(y))^2}{(1+\sin^4(x))(1+\sin^4(y))} \mathrm{d}y \right) \mathrm{d}x = \frac{\pi^2}{16}(\sqrt{2}-1). \tag{3.301}$$

Next, if we plug the values of the integrals I_3, I_4, and I_5 from (3.299), (3.300), and (3.301) in (3.298), we get

$$I_2 = \int_0^{\pi/2} \frac{\sin^4(x)}{(1+\sin^4(x))\sqrt{1+\sin^2(x)}} \mathrm{d}x = \frac{1}{8} \left(\frac{1}{\sqrt{2\pi}} \Gamma\left(\frac{1}{4}\right)^2 - \pi \right). \tag{3.302}$$

Finally, upon plugging the result from (3.302) in (3.297), we conclude that

$$I = \int_0^{\pi/2} \frac{1}{(1+\sin^4(x))\sqrt{1+\sin^2(x)}} \mathrm{d}x = \frac{1}{8} \left(\pi + \frac{1}{\sqrt{2\pi}} \Gamma\left(\frac{1}{4}\right)^2 \right),$$

and the solution is finalized.

The hard, unnoticeable part of the present solution comes from the very beginning where one is supposed to be aware of the auxiliary integral in (3.233) and then use it in order to turn the integral I_2 into a double integral (and further exploit the symmetry of the integrand).

3.59 Integrating Over an Infinite Product with Factors Containing the Secant and the Hyperbolic Secant with Powers of 2

Solution We prepare now to meet an integral with an integrand expressed in terms of an infinite product. *And what a gorgeous infinite product!*, some of the possible reactions. After I created the infinite product, I couldn't resist the idea to include it under an integral, which gave birth to the present problem.

The strategy to attack the problem involves two divergent series, we let them *clash* and produce the result we need to derive the infinite product.

Let's start the solution with stating an auxiliary result we want to prove and use,

$$\sum_{n=1}^{\infty}\left(\frac{1}{\sin\left(2^{-n}x\right)}-\frac{1}{\sinh\left(2^{-n}x\right)}\right)=\coth\left(\frac{x}{2}\right)-\cot\left(\frac{x}{2}\right). \tag{3.303}$$

Proof Let's note and use the simple identities,[26] $\dfrac{1}{\sin(2^{-n}x)} = \cot(2^{-n-1}x) - \cot(2^{-n}x)$ and $\dfrac{1}{\sinh(2^{-n}x)} = \coth(2^{-n-1}x) - \coth(2^{-n}x)$, and then we write

$$\sum_{n=1}^{N}\left(\frac{1}{\sin(2^{-n}x)}-\frac{1}{\sinh(2^{-n}x)}\right)$$

[26]The right-hand side of the first identity can be written as $\cot(2^{-n-1}x) - \cot(2^{-n}x) = \dfrac{\cos(2^{-n-1}x)}{\sin(2^{-n-1}x)} - \dfrac{\cos(2^{-n}x)}{\sin(2^{-n}x)}$, and if we consider the identity $\sin(2x) = 2\sin(x)\cos(x)$ where we replace x by $2^{-n-1}x$, that is $\sin(2^{-n}) = 2\sin(2^{-n-1}x)\cos(2^{-n-1}x)$, we get $\cot(2^{-n-1}x) - \cot(2^{-n}x) = \dfrac{2\cos^2(2^{-n-1}x) - \cos(2^{-n}x)}{\sin(2^{-n}x)} = \dfrac{1}{\sin(2^{-n}x)}$, and for getting the last equality I used the identity $1+\cos(x) = 2\cos^2\left(\dfrac{x}{2}\right)$. The second identity can be obtained in a similar style, or if we want to make use of the previous identity, then it's enough to replace x by ix that leads immediately to $\dfrac{1}{\sinh(2^{-n}x)} = \coth(2^{-n-1}x) - \coth(2^{-n}x)$.

$$= \sum_{n=1}^{N} \left(\cot(2^{-n-1}x) - \cot(2^{-n}x) - \coth(2^{-n-1}x) + \coth(2^{-n}x) \right)$$

$$= \coth\left(\frac{x}{2}\right) - \cot\left(\frac{x}{2}\right) + \cot(2^{-N-1}x) - \coth(2^{-N-1}x). \tag{3.304}$$

If we let $N \to \infty$ in (3.304) and use that $\cot(x) = 1/x + O(x)$ and $\coth(x) = 1/x + O(x)$, as $x \to 0$, we obtain

$$\sum_{n=1}^{\infty} \left(\frac{1}{\sin(2^{-n}x)} - \frac{1}{\sinh(2^{-n}x)} \right) = \coth\left(\frac{x}{2}\right) - \cot\left(\frac{x}{2}\right),$$

and the proof of the auxiliary result is complete.

One may find in [25, Chapter 7, p. 131] the variant, $\displaystyle\sum_{k=1}^{n} \frac{1}{\sin(2^k x)} = \cot(x) - \cot(2^n x)$, where if setting $x = i$ and letting $n \to \infty$, we also get a solution to the problem **11853** about the calculation of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sinh(2^n)}$ proposed by H. Ohtsuka in *The American Mathematical Monthly* (see [65]).

Considering the result from (3.303), and then integrating from $y = x/2$ to $y = x$, we get

$$\int_{x/2}^{x} \sum_{n=1}^{\infty} \left(\frac{1}{\sin\left(2^{-n}y\right)} - \frac{1}{\sinh\left(2^{-n}y\right)} \right) \mathrm{d}y$$

$$\{\text{reverse the order of summation and integration}\}$$

$$= \sum_{n=1}^{\infty} \int_{x/2}^{x} \left(\frac{1}{\sin\left(2^{-n}y\right)} - \frac{1}{\sinh\left(2^{-n}y\right)} \right) \mathrm{d}y$$

$$= \sum_{n=1}^{\infty} \left(2^n \log(\tan(2^{-n-1}y)) \Big|_{y=x/2}^{y=x} - 2^n \log(\tanh(2^{-n-1}y)) \Big|_{y=x/2}^{y=x} \right)$$

$$= \sum_{n=1}^{\infty} 2^n (\log(1 + \sec(2^{-n-1}x)) - \log(1 + \operatorname{sech}(2^{-n-1}x)))$$

$$= \sum_{n=1}^{\infty} \log\left(\frac{1 + \sec(2^{-n-1}x)}{1 + \operatorname{sech}(2^{-n-1}x)} \right)^{2^n} = \log\left(\prod_{n=1}^{\infty} \left(\frac{1 + \sec(2^{-n-1}x)}{1 + \operatorname{sech}(2^{-n-1}x)} \right)^{2^n} \right)$$

$$= \int_{x/2}^{x} \left(\coth\left(\frac{y}{2}\right) - \cot\left(\frac{y}{2}\right)\right) \mathrm{d}y = 2\left(\log\left(\sinh\left(\frac{y}{2}\right)\right) - \log\left(\sin\left(\frac{y}{2}\right)\right)\right)\Bigg|_{y=x/2}^{y=x}$$
$$= \log\left(\sec^2\left(\frac{x}{4}\right)\cosh^2\left(\frac{x}{4}\right)\right),$$

whence we obtain that $\prod_{n=1}^{\infty}\left(\frac{1+\operatorname{sech}(2^{-n-1}x)}{1+\sec(2^{-n-1}x)}\right)^{2^n} = \frac{\cos^2\left(\frac{x}{4}\right)}{\cosh^2\left(\frac{x}{4}\right)}$, or if we replace x by $2x$, we get

$$\prod_{n=1}^{\infty}\left(\frac{1+\operatorname{sech}(2^{-n}x)}{1+\sec(2^{-n}x)}\right)^{2^n} = \frac{\cos^2\left(\frac{x}{2}\right)}{\cosh^2\left(\frac{x}{2}\right)}. \tag{3.305}$$

Integrating both sides of (3.305) from $x = 0$ to ∞, we obtain

$$\int_0^{\infty}\prod_{n=1}^{\infty}\left(\frac{1+\operatorname{sech}(2^{-n}x)}{1+\sec(2^{-n}x)}\right)^{2^n}\mathrm{d}x$$
$$= \int_0^{\infty}\left(\frac{1+\operatorname{sech}\left(2^{-1}x\right)}{1+\sec\left(2^{-1}x\right)}\right)^{2^1}\cdot\left(\frac{1+\operatorname{sech}\left(2^{-2}x\right)}{1+\sec\left(2^{-2}x\right)}\right)^{2^2}\cdot\left(\frac{1+\operatorname{sech}\left(2^{-3}x\right)}{1+\sec\left(2^{-3}x\right)}\right)^{2^3}\cdots\mathrm{d}x$$
$$= \int_0^{\infty}\frac{\cos^2\left(\frac{x}{2}\right)}{\cosh^2\left(\frac{x}{2}\right)}\mathrm{d}x \overset{x/2=y}{=} 2\int_0^{\infty}\frac{\cos^2(y)}{\cosh^2(y)}\mathrm{d}y = 2\int_0^{\infty}(\tanh(y)-1)'\cos^2(y)\mathrm{d}y$$

{apply the integration by parts}

$$= \underbrace{2(\tanh(y)-1)\cos^2(y)\Bigg|_{y=0}^{y=\infty}}_{2} + 2\int_0^{\infty}(\tanh(y)-1)\sin(2y)\mathrm{d}y$$
$$= 2 - 4\int_0^{\infty}\frac{\sin(2y)}{1+e^{2y}}\mathrm{d}y$$
$$\overset{2y=z}{=} 2 - 2\int_0^{\infty}\frac{\sin(z)}{1+e^z}\mathrm{d}z = 2 - 2\int_0^{\infty}\sin(z)\sum_{n=1}^{\infty}(-1)^{n-1}e^{-nz}\mathrm{d}z$$

{reverse the order of summation and integration}

$$= 2 - 2\sum_{n=1}^{\infty}(-1)^{n-1}\int_0^{\infty}\sin(z)e^{-nz}\mathrm{d}z = 2 - 2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2+1}$$
$$= 1 + \pi\operatorname{csch}(\pi),$$

and in the calculations I used that $\operatorname{csch}(x) = \frac{1}{x} + 2x\sum_{n=1}^{\infty}\frac{(-1)^n}{x^2+\pi^2n^2}$, which is derived by combining the facts that $\operatorname{csch}(x) = \coth(x/2) - \coth(x)$ and $\coth(x) = \frac{1}{x} + 2x\sum_{n=1}^{\infty}\frac{1}{x^2+\pi^2n^2}$, where the latter is obtained by replacing x by xi in the Euler's infinite product for the sine, $\sin(x) = x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right)$, taking log of both sides and then differentiating once with respect to x, and the solution is finalized.

This is an example of a creative way of manipulating two divergent series that finally leads to a beautiful infinite product, which might be an idea of interest also for those who enjoy creating mathematical problems.

3.60 Linking Two Generalized Integrals Involving the Polylogarithm Function to Seductive Series

Solution The key core of the solution is represented by *The Master Theorem of Series*, the relaxed version (the second version) of it, we'll meet in the fourth chapter, a theorem I created for mainly generating helpful identities for the derivation of the harmonic series. So, before proceeding we need to make a short review of the theorem, and then return to solve the proposed problem (see Sect. 4.15).

For both points we need two classical results, that is $\sin\left(\frac{x}{2}\right)\sum_{k=1}^{n}\sin(kx) = \sin\left(\frac{nx}{2}\right)\sin\left(\frac{(n+1)x}{2}\right)$ and $\sin\left(\frac{x}{2}\right)\sum_{k=1}^{n}\cos(kx) = \sin\left(\frac{nx}{2}\right)\cos\left(\frac{(n+1)x}{2}\right)$, which we obtain immediately by considering the real and imaginary parts of the geometric sum, $\sum_{k=1}^{n}e^{ikx}$.

Now, if we consider the relaxed version (the second version) of *The Master Theorem of Series* , where we set $m(k) = \sin(k\theta)$, $\mathscr{M}(k) = \dfrac{\sin\left(\frac{k\theta}{2}\right)\sin\left(\frac{(k+1)\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$, we obtain

$$\frac{1}{\sin\left(\frac{\theta}{2}\right)}\sum_{k=1}^{\infty}\frac{\sin\left(\frac{k\theta}{2}\right)\sin\left(\frac{(k+1)\theta}{2}\right)}{(k+1)(k+n+1)} = \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{\infty}\frac{\sin(k\theta)}{j+k}$$

$$= \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{\infty}\int_0^1 x^{j+k-1}\sin(k\theta)\mathrm{d}x = \frac{1}{n}\int_0^1 \sum_{j=1}^{n} x^{j-1}\sum_{k=1}^{\infty} x^k \sin(k\theta)\mathrm{d}x$$

$$\left\{\text{make use of the result, } \sum_{n=1}^{\infty} p^n \sin(nx) = \frac{p\sin(x)}{1-2p\cos(x)+p^2},\ |p|<1\right\}$$

$$= \frac{1}{n}\int_0^1 \frac{(1-x^n)x\sin(\theta)}{(1-x)(1-2x\cos(\theta)+x^2)}\mathrm{d}x,$$

and using this result, we have for the first point of the problem that

$$\sin(\theta)\sin\left(\frac{\theta}{2}\right)\int_0^1 \frac{x}{(1-x)(1-2x\cos(\theta)+x^2)}(\zeta(m+1)-\operatorname{Li}_{m+1}(x))\mathrm{d}x$$

$$= \sin(\theta)\sin\left(\frac{\theta}{2}\right)\int_0^1 \sum_{n=1}^{\infty}\frac{(1-x^n)x}{n^{m+1}(1-x)(1-2x\cos(\theta)+x^2)}\mathrm{d}x$$

$$\{\text{reverse the order of summation and integration}\}$$

$$= \sin\left(\frac{\theta}{2}\right)\sum_{n=1}^{\infty}\frac{1}{n^{m+1}}\int_0^1 \frac{(1-x^n)x\sin(\theta)}{(1-x)(1-2x\cos(\theta)+x^2)}\mathrm{d}x$$

$$= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{\sin\left(\frac{k\theta}{2}\right)\sin\left(\frac{(k+1)\theta}{2}\right)}{(k+1)(k+n+1)n^m}\right)$$

$$\{\text{reverse the order of summation}\}$$

$$= \sum_{k=1}^{\infty}\frac{\sin\left(\frac{k\theta}{2}\right)\sin\left(\frac{(k+1)\theta}{2}\right)}{k+1}\left(\sum_{n=1}^{\infty}\frac{1}{(k+n+1)n^m}\right)$$

$$\{\text{make use of the result in (6.58)}\}$$

$$= (-1)^{m-1}\sum_{k=1}^{\infty}\frac{H_{k+1}}{(k+1)^{m+1}}\sin\left(\frac{k\theta}{2}\right)\sin\left(\frac{(k+1)\theta}{2}\right)$$

$$+(-1)^{m-1}\sum_{i=2}^{m}(-1)^{i-1}\zeta(i)\sum_{k=1}^{\infty}\frac{\sin\left(\frac{k\theta}{2}\right)\sin\left(\frac{(k+1)\theta}{2}\right)}{(k+1)^{m-i+2}},$$

and the point $i)$ of the problem is finalized.

Next, for the point $ii)$ of the problem, if we consider the relaxed version (the second version) of *The Master Theorem of Series*, where we set $m(k) = \cos(k\theta)$, $\mathscr{M}(k) = \dfrac{\sin\left(\dfrac{k\theta}{2}\right)\cos\left(\dfrac{(k+1)\theta}{2}\right)}{\sin\left(\dfrac{\theta}{2}\right)}$, we obtain

$$\frac{1}{\sin\left(\dfrac{\theta}{2}\right)}\sum_{k=1}^{\infty}\frac{\sin\left(\dfrac{k\theta}{2}\right)\cos\left(\dfrac{(k+1)\theta}{2}\right)}{(k+1)(k+n+1)} = \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{\infty}\frac{\cos(k\theta)}{j+k}$$

$$= \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{\infty}\int_0^1 x^{j+k-1}\cos(k\theta)\mathrm{d}x = \frac{1}{n}\int_0^1\sum_{j=1}^{n}x^{j-1}\sum_{k=1}^{\infty}x^k\cos(k\theta)\mathrm{d}x$$

$$\left\{\text{make use of the result, } \sum_{n=1}^{\infty}p^n\cos(nx) = \frac{p(\cos(x)-p)}{1-2p\cos(x)+p^2},\ |p|<1\right\}$$

$$= \frac{1}{n}\int_0^1\frac{x(1-x^n)(\cos(\theta)-x)}{(1-x)(1-2x\cos(\theta)+x^2)}\mathrm{d}x,$$

and using this result, we have for the second point of the problem that

$$\sin\left(\frac{\theta}{2}\right)\int_0^1\frac{x(\cos(\theta)-x)}{(1-x)(1-2x\cos(\theta)+x^2)}(\zeta(m+1)-\operatorname{Li}_{m+1}(x))\mathrm{d}x$$

$$= \sin\left(\frac{\theta}{2}\right)\int_0^1\sum_{n=1}^{\infty}\frac{x(1-x^n)(\cos(\theta)-x)}{n^{m+1}(1-x)(1-2x\cos(\theta)+x^2)}\mathrm{d}x$$

$$\{\text{reverse the order of summation and integration}\}$$

$$= \sin\left(\frac{\theta}{2}\right)\sum_{n=1}^{\infty}\frac{1}{n^{m+1}}\int_0^1\frac{x(1-x^n)(\cos(\theta)-x)}{(1-x)(1-2x\cos(\theta)+x^2)}\mathrm{d}x$$

$$= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{\sin\left(\dfrac{k\theta}{2}\right)\cos\left(\dfrac{(k+1)\theta}{2}\right)}{(k+1)(k+n+1)n^m}\right)$$

$$\{\text{reverse the order of summation}\}$$

$$= \sum_{k=1}^{\infty} \frac{\sin\left(\frac{k\theta}{2}\right)\cos\left(\frac{(k+1)\theta}{2}\right)}{k+1} \left(\sum_{n=1}^{\infty} \frac{1}{(k+n+1)n^m}\right)$$

{make use of the result in (6.58)}

$$= (-1)^{m-1} \sum_{k=1}^{\infty} \frac{H_{k+1}}{(k+1)^{m+1}} \sin\left(\frac{k\theta}{2}\right)\cos\left(\frac{(k+1)\theta}{2}\right)$$

$$+(-1)^{m-1} \sum_{i=2}^{m} (-1)^{i-1}\zeta(i) \sum_{k=1}^{\infty} \frac{\sin\left(\frac{k\theta}{2}\right)\cos\left(\frac{(k+1)\theta}{2}\right)}{(k+1)^{m-i+2}},$$

and the point $ii)$ of the problem is finalized. Note in the calculations I used that $\sum_{n=1}^{\infty} p^n \sin(nx) = \frac{p\sin(x)}{1-2p\cos(x)+p^2}$, $\sum_{n=1}^{\infty} p^n \cos(nx) = \frac{p(\cos(x)-p)}{1-2p\cos(x)+p^2}$, $|p| < 1$, which are found in **1.447.1** and **1.447.2** from [30]. Both results are proved within Sect. 3.52.

The problem can be seen as an application of *The Master Theorem of Series*, since I created the problem based upon this theorem. It would also be interesting to explore other ways, independent of my theorem, to establish the results.

Having said that, we are at the end of the first major chapter dedicated to the calculation of the integrals, where I hope you had a lot of fun! And the party is not over yet! There is another major chapter that lies in front of us.

For more fun, let's prepare to enter the next chapter dedicated to the calculations of sums and series. Enjoy it!

References

1. Abramowitz, M., Stegun, I.A. (eds.): Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing. Dover Publications, New York (1972)
2. Adamchik, V.S.: Integral and Series Representations for Catalan's Constant. http://www-2.cs.cmu.edu/adamchik/articles/catalan.htm
3. Adamchik, V.S., Kölbig, K.S.: A definite integral of a product of two polylogarithms. SIAM J. Math. Anal. **19**, 926–938 (1988)
4. Agarwal, R.P., Perera, K., Pinelas, S.: An Introduction to Complex Analysis. Springer, New York (2011)
5. Amdeberhan, T., Moll, V.H.: AMM 11902 Solution. https://www.math.temple.edu/~tewodros/solutions/11902.PDF
6. Amdeberhan, T., Moll, V.H.: AMM 11924 Solution. https://www.math.temple.edu/~tewodros/solutions/11924.PDF
7. Amdeberhan, T., Glasser, M.L., Jones, M.C., Moll, V.H., Posey, R., Varela, D.: The Cauchy-Schlömilch transformation. https://arxiv.org/pdf/2004.2445.pdf

8. Andrews, G.E., Berndt, B.C.: Ramanujan's Lost Notebook, Part IV. Springer, New York (2013)
9. Andrews, G.E., Askey, R., Roy, R.: Special Functions. Encyclopedia of Mathematics and its Applications, vol. 71. Cambridge University Press, Cambridge (1999)
10. Artin, E.: The Gamma Function. Holt, Rinehart and Winston, New York (1964)
11. Bailey, D., Borwein, J., Calkin, N., Girgensohn, R., Luke, R., Moll, V.: Experimental Mathematics in Action. A K Peters, Natick (2007)
12. Bak, J., Newman, D.J.: Complex Analysis. Undergraduate Texts in Mathematics, 3rd edn. Springer, New York (2010)
13. Bastien, G.: Elementary methods for evaluating Jordan's sums and analogous Euler's type sums and for setting a sigma sum theorem (2013)
14. Beckmann, P.: A History of π. St. Martin's Press, New York (1971)
15. Berndt, B.: Ramanujan's Notebooks, Part I. Springer, New York (1985)
16. Berndt, B.: Ramanujan's Notebooks, Part II. Springer, New York (1989)
17. Boros, G., Moll, V.H.: Irresistible Integrals, Symbolics, Analysis and Experiments in the Evaluation of Integrals. Cambridge University Press, Cambridge (2004)
18. Borwein, J., Devlin, K.: The Computer as Crucible: An Introduction to Experimental Mathematics. A K Peters, Wellesley (2009)
19. Bradley, D.M.: Representations of Catalan's constant (2001). CiteSeerX 10.1.1.26.1879
20. Bromwich, T.I'A.: An Introduction to the Theory of Infinite Series. Macmillan, London (1908)
21. Chapman, R.: Evaluating $\zeta(2)$. https://empslocal.ex.ac.uk/people/staff/rjchapma/etc/zeta2.pdf
22. Choi, J., Srivastava, H.M.: Explicit evaluation of Euler and related sums. Ramanujan J. **10**, 51–70 (2005)
23. Choudary, A.D.R., Niculescu, C.P.: Real Analysis on Intervals. Springer, New Delhi (2014)
24. De Doelder, P.J.: On some series containing $\psi(x) - \psi(y)$ and $(\psi(x) - \psi(y))^2$ for certain values of x and y. J. Comput. Appl. Math. **37**, 125–141 (1991)
25. Durell, C.V., Robson, A.: Advanced Trigonometry, Dover edition. Dover Publications, New York (2003)
26. Edwards, H.M.: Riemann's Zeta Function. Dover Publications, New York (1974)
27. Finch, S.R.: Mathematical Constants. Cambridge University Press, Cambridge (2003)
28. Freitas, P.: Integrals of polylogarithmic functions, recurrence relations, and associated Euler sums. Math. Comput. **74**(251), 1425–1440 (2005)
29. Furdui, O.: Limits, Series and Fractional Part Integrals. Problems in Mathematical Analysis. Springer, New York (2013)
30. Gradshteyn, I.S., Ryzhik, I.M.: In: Zwillinger, D., Moll, V. (eds.) Table of Integrals, Series, and Products, 8th edn. Academic, New York (2015)
31. Graham, R.L., Knuth, D.E., Patashnik, O.: Concrete Mathematics: A Foundation for Computer Science, 2nd edn. Addison-Wesley, Reading (1994)
32. Hackbusch, W.: Integral Equations: Theory and Numerical Treatment. Birkhäuser, Basel (1995)
33. Hairer, E., Wanner, G.: Analysis by Its History. Springer, New York (1996)
34. Havil, J.: Gamma, Exploring Euler's Constant. Princeton University Press, Princeton (2003)
35. Ivan, M.: Problem 11152, problems and solutions. Am. Math. Mon. **112**, 5, 467–474. https://tandfonline.com/doi/abs/10.1080/00029890.2005.11920217 (2005)
36. Ivan, M.: Problem 11592, problems and solutions. Am. Math. Mon. **118**, 7, 653–660. https://tandfonline.com/doi/abs/10.4169/amer.math.monthly.118.07.653 (2011)
37. Jung, M., Cho, Y.J., Choi, J.: Euler sums evaluatable from integrals. Commun. Korean Math. Soc. **19**, 545–555 (2004)
38. Knuth, D.E.: Evaluation of Porter's constant. Comput. Math. Appl. **2**(2), 137–139 (1976)
39. Kölbig, K.S.: The Polygamma Function $\psi_k(x)$ for $x = 1/4$ and $x = 3/4$. J. Comput. Appl. Math. **75**, 43–46 (1996)
40. Koshy, T.: Catalan Numbers with Applications. Oxford University Press, Oxford (2009)

41. La Gaceta de la RSME (Spain): The Problem 306. http://gaceta.rsme.es/abrir.php?id=1356 (2016)
42. La Gaceta de la RSME (Spain): A solution to the problem 306. http://gaceta.rsme.es/abrir.php?id=1412 (2017)
43. La Gaceta de la RSME (Spain): The Problem 327. http://gaceta.rsme.es/abrir.php?id=1393 (2017)
44. Lewin, L.: Polylogarithms and Associated Functions. North-Holland, New York (1981)
45. Macdonald, I.G.: Symmetric Functions and Hall Polynomials, 2nd edn. Oxford University Press, Oxford (1995)
46. Maor, E.: e: The Story of a Number. Princeton University Press, Princeton (1994)
47. Mathematics Stack Exchange: https://math.stackexchange.com/q/265981
48. Mathematics Stack Exchange: https://math.stackexchange.com/q/331222
49. Mathematics Stack Exchange: https://math.stackexchange.com/q/411410
50. Mathematics Stack Exchange: https://math.stackexchange.com/q/465444
51. Mathematics Stack Exchange: https://math.stackexchange.com/q/795887
52. Mathematics Stack Exchange: https://math.stackexchange.com/q/852464
53. Mathematics Stack Exchange: https://math.stackexchange.com/q/983263
54. Mathematics Stack Exchange: https://math.stackexchange.com/q/1043877
55. Mathematics Stack Exchange: https://math.stackexchange.com/q/1842284
56. Mathematics Stack Exchange: https://math.stackexchange.com/q/1842492
57. MathProblems Journal: Problems and Solutions. Problem 131. Vol. 5, No. 3 (2015). www.mathproblems-ks.org
58. MathProblems Journal: Mathnotes. The evaluation of a special fractional part integral with an integrand raised to positive integer powers. Vol. 6, No. 1 (2016). www.mathproblems-ks.org
59. Mezö, I.: A solution to the problem 327 from La Gaceta de la RSME (Spain). https://sites.google.com/site/istvanmezo81/monthly-problems
60. Miller, P.D.: Applied Asymptotic Analysis. Graduate Studies in Mathematics, vol. 75. American Mathematical Society, Providence (2006)
61. Moll, V.: Special Integrals of Gradshteyn and Ryzhik. The Proofs, vol. I. CRC Press, Taylor and Francis Group/Chapman and Hall, Boca Raton/London (2015)
62. Moll, V.: Special Integrals of Gradshteyn and Ryzhik. The Proofs, vol. II. CRC Press, Taylor and Francis Group/Chapman and Hall, Boca Raton/London (2016)
63. Nahin, P.J.: Dr. Euler's Fabulous Formula: Cures Many Mathematical Ills. Princeton University Press, Princeton (2006)
64. Nahin, P.J.: Inside Interesting Integrals. Springer, New York (2014)
65. Ohtsuka, H.: Problem 11853, problems and solutions. Am. Math. Mon. **122**(7), 700–707. https://tandfonline.com/doi/abs/10.4169/amer.math.monthly.122.7.700 (2015)
66. Piskunov, N.: Differential and Integral Calculus. MIR Publishers, Moscow (1969)
67. Prudnikov, A.P., Brychkov, Yu.A., Marichev, O.I.: Integrals and Series. Vol. 2. Special Functions. Gordon & Breach, New York (1992)
68. Prudnikov, A.P., Brychkov, Yu.A., Marichev, O.I.: Integrals and Series. Vol. 1. Elementary Functions. Gordon & Breach, New York (1998)
69. Putnam Contest: The Problem A4. https://mks.mff.cuni.cz/kalva/putnam/putn77.html (1977)
70. Putnam Contest: The Problem A2. https://mks.mff.cuni.cz/kalva/putnam/putn84.html (1984)
71. Putnam Contest: The Problem A3. https://mks.mff.cuni.cz/kalva/putnam/putn86.html (1986)
72. Putnam Contest: The Problems in 1986. https://kskedlaya.org/putnam-archive/1986.pdf
73. Putnam Contest: The Problem A5. https://kskedlaya.org/putnam-archive/2005.pdf (2005)
74. Putnam Contest: Solutions to the problem A5 in 2005. https://kskedlaya.org/putnam-archive/2005s.pdf
75. Roussos, I.M.: Improper Riemann Integrals. Chapman and Hall/CRC, New York (2013)
76. Silagadze, Z.K.: Problem 11952, problems and solutions. Am. Math. Mon. **124**(1), 83–91. https://tandfonline.com/doi/abs/10.4169/amer.math.monthly.124.1.83 (2017)
77. Sofo, A., Cvijović, D.: Extensions of Euler harmonic sums. Appl. Anal. Discrete Math. **6**, 317–328 (2012)

78. Srivastava, H.M., Choi, J.: Series Associated with the Zeta and Related Functions. Springer (originally published by Kluwer), Dordrecht (2001)
79. Srivastava, H.M., Choi, J.: Zeta and q-Zeta Functions and Associated Series and Integrals. Elsevier Insights, Amsterdam (2012)
80. Tauraso, R.: AMM 11902 Solution. www.mat.uniroma2.it/~tauraso/AMM/AMM11902.pdf
81. Tauraso, R.: AMM 11924 Solution. www.mat.uniroma2.it/~lowtauraso/AMM/AMM11924.pdf
82. Tauraso, R.: AMM 11930 Solution. www.mat.uniroma2.it/~tauraso/AMM/AMM11930.pdf
83. Tauraso, R.: AMM 11952 Solution. www.mat.uniroma2.it/~tauraso/AMM/AMM11952.pdf
84. Tauraso, R.: AMM 11966 Solution. www.mat.uniroma2.it/~tauraso/AMM/AMM11966.pdf
85. Tolstov, G.P.: Fourier Series. Dover Publications, New York (1976)
86. Vălean, C.I.: A new proof for a classical quadratic harmonic series. J. Class. Anal. **8**(2), 155–161 (2016)
87. Vălean, C.I.: Problem 11902, problems and solutions. Am. Math. Mon. **123**(4), 399–406. https://tandfonline.com/doi/abs/10.4169/amer.math.monthly.123.4.399 (2016)
88. Vălean, C.I.: Problem 11924, problems and solutions. Am. Math. Mon. **123**(7), 722–730. https://tandfonline.com/doi/abs/10.4169/amer.math.monthly.123.7.722 (2016)
89. Vălean, C.I.: Problem 11930, problems and solutions. Am. Math. Mon. **123**(8), 831–839. https://tandfonline.com/doi/abs/10.4169/amer.math.monthly.123.08.831 (2016)
90. Vălean, C.I.: Problem 11966, problems and solutions. Am. Math. Mon. **124**(3), 274–281. https://tandfonline.com/doi/abs/10.4169/amer.math.monthly.124.3.274 (2017)
91. Vălean, C.I.: Problem 12011, problems and solutions. Am. Math. Mon. **124**(10), 970–978. https://tandfonline.com/doi/abs/10.4169/amer.math.monthly.124.10.970 (2017)
92. Vălean, C.I.: Problem 12054, problems and solutions. Am. Math. Mon. **125**(6), 562–570. https://tandfonline.com/doi/abs/10.1080/00029890.2018.1460990 (2018)
93. Vălean, C.I., Furdui, O.: Reviving the quadratic series of Au-Yeung. J. Class. Anal. **6**(2), 113–118 (2015)
94. Weisstein, E.W.: Beta Function. http://mathworld.wolfram.com/BetaFunction.html
95. Weisstein, E.W.: Big-O Notation. http://mathworld.wolfram.com/Big-ONotation.html
96. Weisstein, E.W.: Catalan's Constant. http://mathworld.wolfram.com/CatalansConstant.html
97. Weisstein, E.W.: Cauchy Principal Value. http://mathworld.wolfram.com/CauchyPrincipalValue.html
98. Weisstein, E.W.: Circle. http://mathworld.wolfram.com/Circle.html
99. Weisstein, E.W.: Circular Segment. http://mathworld.wolfram.com/CircularSegment.html
100. Weisstein, E.W.: Dilogarithm. http://mathworld.wolfram.com/Dilogarithm.html
101. Weisstein, E.W.: Dirichlet Beta Function. http://mathworld.wolfram.com/DirichletBetaFunction.html
102. Weisstein, E.W.: Dirichlet Eta Function. http://mathworld.wolfram.com/DirichletEtaFunction.html
103. Weisstein, E.W.: Double Factorial. http://mathworld.wolfram.com/DoubleFactorial.html
104. Weisstein, E.W.: Floor Function. http://mathworld.wolfram.com/FloorFunction.html
105. Weisstein, E.W.: Fractional Part. http://mathworld.wolfram.com/FractionalPart.html
106. Weisstein, E.W.: Frullani's Integral. http://mathworld.wolfram.com/FrullanisIntegral.html
107. Weisstein, E.W.: Gamma Function. http://mathworld.wolfram.com/GammaFunction.html
108. Weisstein, E.W.: Gaussian Integral. http://mathworld.wolfram.com/GaussianIntegral.html
109. Weisstein, E.W.: Glaisher-Kinkelin Constant. http://mathworld.wolfram.com/Glaisher-KinkelinConstant.html
110. Weisstein, E.W.: Integer Part. http://mathworld.wolfram.com/IntegerPart.html
111. Weisstein, E.W.: Inverse Tangent Integral. http://mathworld.wolfram.com/InverseTangentIntegral.html
112. Weisstein, E.W.: Legendre Duplication Formula. http://mathworld.wolfram.com/LegendreDuplicationFormula.html
113. Weisstein, E.W.: Lerch Transcendent. http://mathworld.wolfram.com/LerchTranscendent.html

114. Weisstein, E.W.: Pinching Theorem. http://mathworld.wolfram.com/PinchingTheorem.html
115. Weisstein, E.W.: Polar Coordinates. http://mathworld.wolfram.com/PolarCoordinates.html
116. Weisstein, E.W.: Polygamma Function. http://mathworld.wolfram.com/PolygammaFunction.html
117. Weisstein, E.W.: Polylogarithm. http://mathworld.wolfram.com/Polylogarithm.html
118. Weisstein, E.W.: Riemann Zeta Function. http://mathworld.wolfram.com/RiemannZetaFunction.html
119. Weisstein, E.W.: Trilogarithm. http://mathworld.wolfram.com/Trilogarithm.html
120. Weisstein, E.W.: Weierstrass Substitution. http://mathworld.wolfram.com/WeierstrassSubstitution.html
121. Wilf, H.S.: generatingfunctionology, 3rd edn. A K Peters Ltd., Wellesley (2006)
122. Xu, C.: Evaluations of Euler type sums of weight ≤ 5. https://arxiv.org/pdf/1704.03515.pdf (2017)

Chapter 4
Sums and Series

"In their hearts humans plans their course, but the LORD establishes their steps."—Proverbs 16:9

4.1 The First Series Submitted by Ramanujan to the Journal of the Indian Mathematical Society

Prove that

$$1+2\sum_{n=1}^{\infty}\frac{1}{(4n)^3-4n}=\frac{3}{2}\log(2).$$

A (little) challenging question: Calculate the series without using integrals.

4.2 Starting from an Elementary Integral Result and Deriving Two Classical Series in a New Way

Prove that

$$i)\ \sum_{n=1}^{\infty}\frac{(2x)^{2n-1}}{n\binom{2n}{n}}=\frac{\arcsin(x)}{\sqrt{1-x^2}}. \tag{4.1}$$

Then, use $i)$ to prove that

$$ii)\ \frac{1}{2}\sum_{n=1}^{\infty}\frac{(2x)^{2n}}{n^2\binom{2n}{n}}=\arcsin^2(x). \tag{4.2}$$

C. I. Vălean, *(Almost) Impossible Integrals, Sums, and Series*, Problem Books in Mathematics, https://doi.org/10.1007/978-3-030-02462-8_4

4.3 An Extraordinary Series with the Tail of the Riemann Zeta Function Connected to the Inverse Sine Series

Show that

$$\sum_{n=1}^{\infty} \frac{2^{2n}}{n^2 \binom{2n}{n}} \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right)$$

$$= \frac{17}{2}\zeta(4) - 4\log^2(2)\zeta(2) - \frac{1}{3}\log^4(2) - 8\operatorname{Li}_4\left(\frac{1}{2}\right),$$

where ζ denotes the Riemann zeta function and Li_n represents the Polylogarithm function.

4.4 The Evaluation of a Series Involving the Tails of the Series Representations of the Functions $\log\left(\frac{1}{1-x}\right)$ and $\frac{x\arcsin(x)}{\sqrt{1-x^2}}$

Show that

$$\sum_{n=1}^{\infty} \frac{(2n-3)!!}{(2n-2)!!}\left(\frac{1}{2}\log(2) - \sum_{k=1}^{n-1}\frac{1}{k2^{k+1}}\right)\left(\frac{\pi}{4} - \sum_{k=1}^{n-1}\frac{(2k-2)!!}{2^k(2k-1)!!}\right)$$

$$= \sqrt{2}\arctan\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{2}\log\left(\frac{3}{2}\right) - \frac{\pi}{4},$$

where $n!! = \begin{cases} n\cdot(n-2)\cdots 5\cdot 3\cdot 1, & n>0 \text{ odd};\\ n\cdot(n-2)\cdots 6\cdot 4\cdot 2, & n>0 \text{ even};\\ 1, & n=-1,0, \end{cases}$ is the double factorial.

4.5 A Breathtaking Infinite Series Involving the Binomial Coefficient and Expressing a Beautiful Closed-Form

Show that

$$\sum_{n=1}^{\infty} \frac{n}{(2n-1)16^n} \binom{2n}{n}^2 \left(\sum_{k=n}^{\infty} \frac{2^k}{k \binom{2k}{k}} \right) = 1 - \sqrt{2} + \log\left(1+\sqrt{2}\right).$$

4.6 An Eccentric Multiple Series Having the Roots in the Realm of the Botez–Catalan Identity

Let $n \geq 2$ be a natural number. Prove that

$$\sum_{k_1=1}^{\infty} \left(\sum_{k_2=1}^{\infty} \left(\cdots \sum_{k_n=1}^{\infty} (-1)^{\sum_{i=1}^{n} k_i} \left(\log(2) - \sum_{k=1}^{\sum_{i=1}^{n} k_i} \frac{1}{\sum_{i=1}^{n} k_i + k} \right) \right) \cdots \right)$$

$$= (-1)^n \left(\frac{1}{2}\log(2) + \frac{1}{2^{n+1}}\log(2) + \frac{H_n}{2^{n+1}} - \sum_{i=1}^{n} \frac{1}{i 2^{i+1}} - \frac{\pi}{2^{n+2}} \sum_{j=0}^{n-1} \frac{1}{2^j}\binom{2j}{j} \right.$$

$$\left. + \frac{1}{2^{n+1}} \sum_{j=1}^{n-1} \frac{1}{2^j}\binom{2j}{j} \sum_{i=1}^{j} \frac{2^i}{i\binom{2i}{i}} \right),$$

where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ denotes the nth harmonic number.

Examples:

The version with two variables,

$$\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} (-1)^{i+j} \left(\log(2) - \sum_{n=1}^{i+j} \frac{1}{i+j+n} \right) \right) = \frac{1}{8}(5\log(2) - \pi);$$

(continued)

The version with three variables,

$$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^{i+j+k}\left(\log(2)-\sum_{n=1}^{i+j+k}\frac{1}{i+j+k+n}\right)\right)\right)$$

$$=\frac{1}{64}(2+7\pi-36\log(2));$$

The version with four variables,

$$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty}\left(\sum_{l=1}^{\infty}(-1)^{i+j+k+l}\left(\log(2)-\sum_{n=1}^{i+j+k+l}\frac{1}{i+j+k+l+n}\right)\right)\right)\right)$$

$$=\frac{1}{192}(102\log(2)-18\pi-13).$$

4.7 Two Classical Series with Fibonacci Numbers, One Related to the Arctan Function

Calculate

$$S_1=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{F_nF_{n+1}};$$

$$S_2=\sum_{n=-\infty}^{\infty}\arctan\left(\frac{1}{F_{2n-1}}\right),$$

where F_n represents the nth Fibonacci number.

4.8 Two New Infinite Series with Fibonacci Numbers, Related to the Arctan Function

Calculate

$$i)\ \sum_{n=1}^{\infty}\left(\arctan\left(\frac{1}{F_{4n-3}}\right)+\arctan\left(\frac{1}{F_{4n-2}}\right)+\arctan\left(\frac{1}{F_{4n-1}}\right)\right.$$
$$\left.-\arctan\left(\frac{1}{F_{4n}}\right)\right);$$

$$ii)\ \sum_{n=1}^{\infty}(-1)^{n-1}\arctan\left(\frac{1}{F_{2n}}\right),$$

where F_n represents the nth Fibonacci number.

4.9 Useful Series Representations of $\log(1+x)\log(1-x)$ and $\arctan(x)\log(1+x^2)$ from the Notorious *Table of Integrals, Series, and Products* by I.S. Gradshteyn and I.M. Ryzhik

Prove that

$$i)\ -\log(1+x)\log(1-x)$$

$$=\sum_{k=1}^{\infty}\frac{x^{2k}}{k}\sum_{n=1}^{2k-1}\frac{(-1)^{n-1}}{n}=\sum_{k=1}^{\infty}x^{2k}\frac{H_{2k}-H_k}{k}+\frac{1}{2}\sum_{k=1}^{\infty}\frac{x^{2k}}{k^2},\ |x|<1; \tag{4.3}$$

$$ii)\ \sum_{k=1}^{\infty}(-1)^{k-1}x^{2k+1}\frac{H_{2k}}{2k+1}=\frac{1}{2}\arctan(x)\log(1+x^2),\ |x|\le 1, \tag{4.4}$$

where $H_k=\sum_{i=1}^{k}\frac{1}{i}$ denotes the kth harmonic number.

4.10 A Group of Five Useful Generating Functions Related to the Generalized Harmonic Numbers

Show that

$$G_1 = \sum_{n=1}^{\infty} x^n H_n = -\frac{\log(1-x)}{1-x}; \tag{4.5}$$

$$G_2 = \sum_{n=1}^{\infty} x^n H_n^{(m)} = \frac{\operatorname{Li}_m(x)}{1-x}; \tag{4.6}$$

$$G_3 = \sum_{n=1}^{\infty} x^n H_n^2 = \frac{1}{1-x}(\log^2(1-x) + \operatorname{Li}_2(x)); \tag{4.7}$$

$$G_4 = \sum_{n=1}^{\infty} x^n H_n H_n^{(2)} = \frac{1}{1-x}\left(\frac{1}{2}\log(x)\log^2(1-x) + \operatorname{Li}_3(x) + \operatorname{Li}_3(1-x) - \zeta(2)\log(1-x) - \zeta(3)\right); \tag{4.8}$$

$$G_5 = \sum_{n=1}^{\infty} x^n H_n^3 = \frac{1}{1-x}\left(\frac{3}{2}\log(x)\log^2(1-x) - 3\zeta(2)\log(1-x) - \log^3(1-x) + \operatorname{Li}_3(x) + 3\operatorname{Li}_3(1-x) - 3\zeta(3)\right), \tag{4.9}$$

where $|x| < 1$, $H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m}$, $m \geq 1$, is the nth generalized harmonic number of order m, ζ denotes the Riemann zeta function, and Li_n represents the Polylogarithm function.

4.11 Four Members from a Neat Group of Generating Functions Expressed in Terms of Polylogarithm Function

Show that

$$G_1 = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n} H_n = -\frac{\operatorname{Li}_2\left(\frac{x}{x-1}\right)}{x}; \tag{4.10}$$

$$G_2 = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n} (H_n^2 + H_n^{(2)}) = -2\frac{\operatorname{Li}_3\left(\frac{x}{x-1}\right)}{x}; \tag{4.11}$$

$$G_3 = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n} (H_n^3 + 3H_n H_n^{(2)} + 2H_n^{(3)}) = -6\frac{\operatorname{Li}_4\left(\frac{x}{x-1}\right)}{x}; \tag{4.12}$$

$$G_4 = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n} (H_n^4 + 6H_n^2 H_n^{(2)} + 8H_n H_n^{(3)} + 3\left(H_n^{(2)}\right)^2 + 6H_n^{(4)})$$
$$= -24\frac{\operatorname{Li}_5\left(\frac{x}{x-1}\right)}{x}, \tag{4.13}$$

where $|x| \leq 1$, $x \neq 1$, $H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m}$, $m \geq 1$, is the nth generalized harmonic number of order m, ζ denotes the Riemann zeta function, and Li_n represents the Polylogarithm function.

4.12 Two Elementary Harmonic Sums Arising in the Calculation of Harmonic Series

Prove that

$$S_1 = \sum_{k=1}^{n} \frac{H_k^{(p)}}{k^p} = \frac{1}{2}((H_n^{(p)})^2 + H_n^{(2p)}); \tag{4.14}$$

$$S_2 = \sum_{k=1}^{n} \frac{(H_k^{(p)})^2}{k^p} = \frac{1}{3}\left((H_n^{(p)})^3 - H_n^{(3p)}\right) + \sum_{k=1}^{n} \frac{H_k^{(p)}}{k^{2p}}, \tag{4.15}$$

where $H_n^{(p)} = 1 + \frac{1}{2^p} + \cdots + \frac{1}{n^p}$, $p \geq 1$, is the nth generalized harmonic number of order p.

Special cases:

For S_1 with $p = 2$ and $n \to \infty$,

$$\sum_{k=1}^{\infty} \frac{H_k^{(2)}}{k^2} = \frac{7}{4}\zeta(4);$$

For S_1 with $p = 3$ and $n \to \infty$,

$$\sum_{k=1}^{\infty} \frac{H_k^{(3)}}{k^3} = \frac{1}{2}\left(\zeta^2(3) + \zeta(6)\right);$$

For S_2 with $p = 2$ and $n \to \infty$,

$$\sum_{k=1}^{\infty} \frac{(H_k^{(2)})^2}{k^2} = \frac{19}{24}\zeta(6) + \zeta^2(3).$$

4.13 A Strong Generalized Sum, Making a Very Good Cocktail Together with the Identities Generated by *The Master Theorem of Series*

Show that

$$\sum_{k=1}^{n-1}\frac{H_k^{(m)}}{n-k}=\begin{cases} H_n^2-H_n^{(2)}, m=1;\\ \displaystyle\sum_{i=1}^{n}\frac{H_i}{i^{2r-1}}+\sum_{i=1}^{r-1}H_n^{(2r-i)}H_n^{(i)}+\frac{1}{2}(H_n^{(r)})^2-\frac{2r+1}{2}H_n^{(2r)},\ m=2r-1\geq 3;\\ \displaystyle\sum_{i=1}^{n}\frac{H_i}{i^{2r}}+\sum_{i=1}^{r}H_n^{(2r-i+1)}H_n^{(i)}-(r+1)H_n^{(2r+1)}, m=2r\geq 2,\end{cases} \tag{4.16}$$

where $H_n^{(m)}=1+\frac{1}{2^m}+\cdots+\frac{1}{n^m},\ m\geq 1$, is the nth generalized harmonic number of order m.

Examples:

For $m=2$,

$$\sum_{k=1}^{n-1}\frac{H_k^{(2)}}{n-k}=\sum_{k=1}^{n}\frac{H_k}{k^2}+H_nH_n^{(2)}-2H_n^{(3)};$$

For $m=3$,

$$\sum_{k=1}^{n-1}\frac{H_k^{(3)}}{n-k}=\sum_{k=1}^{n}\frac{H_k}{k^3}+H_n^{(3)}H_n+\frac{1}{2}(H_n^{(2)})^2-\frac{5}{2}H_n^{(4)};$$

For $m=4$,

$$\sum_{k=1}^{n-1}\frac{H_k^{(4)}}{n-k}=\sum_{k=1}^{n}\frac{H_k}{k^4}+H_n^{(4)}H_n+H_n^{(3)}H_n^{(2)}-3H_n^{(5)}.$$

A challenging question: Could we derive the generalized sum by elementary manipulations only?

4.14 Four Elementary Sums with Harmonic Numbers, Very Useful in the Calculation of the Harmonic Series of Weight 7

Show that

$$S_1(n) = \sum_{k=1}^{n-1} \frac{H_k^2}{n-k} = H_n^3 - 2H_n H_n^{(2)} + \sum_{k=1}^{n} \frac{H_k}{k^2}; \tag{4.17}$$

$$S_2(n) = \sum_{k=1}^{n-1} \frac{H_k^2}{(n-k)^2} = H_n^2 H_n^{(2)} - 4H_n H_n^{(3)} - (H_n^{(2)})^2 + 2\sum_{k=1}^{n} \frac{H_k}{k^3} + 2H_n \sum_{k=1}^{n} \frac{H_k}{k^2}; \tag{4.18}$$

$$S_3(n) = \sum_{k=1}^{n-1} \frac{H_k^{(2)}}{(n-k)^2} = \left(H_n^{(2)}\right)^2 - 5H_n^{(4)} + 4\sum_{k=1}^{n} \frac{H_k}{k^3}; \tag{4.19}$$

$$S_4(n) = \sum_{k=1}^{n-1} \frac{H_k^3}{n-k} = H_n^4 - 3H_n^2 H_n^{(2)} - \frac{1}{4}\left(H_n^{(2)}\right)^2 - \frac{1}{4} H_n^{(4)} + \sum_{k=1}^{n} \frac{H_k}{k^3}$$

$$- \frac{3}{2} \sum_{k=1}^{n} \frac{H_k^2}{k^2} + 3H_n \sum_{k=1}^{n} \frac{H_k}{k^2}, \tag{4.20}$$

where $H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m},\ m \geq 1$, is the nth generalized harmonic number of order m.

A challenging question: Could we derive the sums by elementary manipulations only?

4.15 The Master Theorem of Series, a New Very Useful Theorem in the Calculation of Many Difficult (Harmonic) Series

(The first version) If k is a positive integer with $\mathscr{M}(k) = m(1) + m(2) + \cdots + m(k)$, and $m(k)$ are real numbers, where $\lim_{k\to\infty} m(k) = 0$, then the following double equality holds

(continued)

$$\sum_{k=1}^{\infty}\frac{\mathscr{M}(k)}{(k+1)(k+n+1)}=m(1)\left(\frac{H_n}{n}-\frac{1}{n+1}\right)+\frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{\infty}\frac{m(k+1)}{j+k+1}$$

$$=\frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{\infty}\frac{m(k)}{j+k},$$

where $H_n = 1+\frac{1}{2}+\cdots+\frac{1}{n}$ is the nth harmonic number.

(The second version, the relaxed version) If k is a positive integer with $\mathscr{M}(k) = m(1)+m(2)+\cdots+m(k)$, and $m(k)$ are real numbers, where $\lim_{k\to\infty}\frac{\mathscr{M}(k)}{k}=0$, then the stated double equality follows.

4.16 The First Application of *The Master Theorem of Series* on the (Generalized) Harmonic Numbers

Show that

$$\sum_{k=1}^{\infty}\frac{H_k^{(m)}}{(k+1)(k+n+1)}=\begin{cases}\dfrac{H_n^2+H_n^{(2)}}{2n}, & m=1;\\ \dfrac{(-1)^{m-1}}{n}\left(\sum_{i=1}^{n}\dfrac{H_i}{i^m}+\sum_{i=2}^{m}(-1)^{i-1}\zeta(i)H_n^{(m-i+1)}\right), & m\geq 2,\end{cases} \tag{4.21}$$

where $H_n^{(m)} = 1+\frac{1}{2^m}+\cdots+\frac{1}{n^m}$, $m\geq 1$, is the nth generalized harmonic number of order m and ζ represents the Riemann zeta function.

4.17 The Second Application of *The Master Theorem of Series* on the Harmonic Numbers

Show that

$$\sum_{k=1}^{\infty}\frac{H_k^2}{(k+1)(k+n+1)}=\frac{H_n^3+3\zeta(2)H_n+3H_nH_n^{(2)}+2H_n^{(3)}}{3n}-\frac{1}{n}\sum_{i=1}^{n}\frac{H_i}{i^2}, \tag{4.22}$$

(continued)

where $H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m}$, $m \geq 1$, is the nth generalized harmonic number of order m and ζ represents the Riemann zeta function.

4.18 The Third Application of *The Master Theorem of Series* on the Harmonic Numbers

Show that

$$\sum_{k=1}^{\infty} \frac{H_k^3}{(k+1)(k+n+1)}$$

$$= \frac{H_n^4 + 6\zeta(2)H_n^2 + 16\zeta(3)H_n + 8H_nH_n^{(3)} + 6H_n^2H_n^{(2)} + 2\zeta(2)H_n^{(2)} + 3H_n^{(4)}}{4n}$$

$$-3\frac{H_n}{n}\sum_{i=1}^{n}\frac{H_i}{i^2} - \frac{2}{n}\sum_{i=1}^{n}\frac{H_i}{i^3} + \frac{3}{2n}\sum_{i=1}^{n}\frac{H_i^2}{i^2}, \tag{4.23}$$

where $H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m}$, $m \geq 1$, is the nth generalized harmonic number of order m and ζ represents the Riemann zeta function.

4.19 The Fourth Application of *The Master Theorem of Series* on the (Generalized) Harmonic Numbers

Show that

$$\sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{(k+1)(k+n+1)}$$

(continued)

$$= 2\zeta(3)\frac{H_n}{n}+\frac{\zeta(2)}{2}\frac{H_n^2}{n}-\frac{\zeta(2)}{2}\frac{H_n^{(2)}}{n}-\frac{H_n^{(4)}}{4n}-\frac{(H_n^{(2)})^2}{4n}-\frac{H_n}{n}\sum_{i=1}^{n}\frac{H_i}{i^2}+\frac{1}{2n}\sum_{i=1}^{n}\frac{H_i^2}{i^2}, \tag{4.24}$$

where $H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m}$, $m \geq 1$, is the nth generalized harmonic number of order m and ζ represents the Riemann zeta function.

4.20 Cool Identities with Ingredients Like the Generalized Harmonic Numbers and the Binomial Coefficient

Show that

$$i)\ \sum_{k=1}^{n-1}\frac{H_k^2 - H_k^{(2)}}{n-k} = H_n^3 - 3H_nH_n^{(2)} + 2H_n^{(3)}; \tag{4.25}$$

$$ii)\ \sum_{k=1}^{\infty}\frac{H_k^2 - H_k^{(2)}}{(k+1)(k+n+1)} = \frac{H_n^3 + 3H_nH_n^{(2)} + 2H_n^{(3)}}{3n}$$

$$= 2\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^4}\binom{n-1}{k-1}; \tag{4.26}$$

$$iii)\ \sum_{k=1}^{\infty}\frac{H_k^3 - 3H_kH_k^{(2)} + 2H_k^{(3)}}{(k+1)(k+n+1)}$$

$$= \frac{H_n^4 + 6H_n^2H_n^{(2)} + 8H_nH_n^{(3)} + 3(H_n^{(2)})^2 + 6H_n^{(4)}}{4n}$$

$$= 6\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^5}\binom{n-1}{k-1}; \tag{4.27}$$

$$iv)\ \sum_{k=1}^{\infty}\frac{H_k^4 - 6H_k^2H_k^{(2)} + 8H_kH_k^{(3)} + 3(H_k^{(2)})^2 - 6H_k^{(4)}}{(k+1)(k+n+1)}$$

$$= \frac{H_n^5 + 10H_n^3H_n^{(2)} + 15H_n(H_n^{(2)})^2 + 20H_n^2H_n^{(3)} + 20H_n^{(2)}H_n^{(3)} + 30H_nH_n^{(4)} + 24H_n^{(5)}}{5n}$$

(continued)

$$= 24 \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^6} \binom{n-1}{k-1}, \tag{4.28}$$

where $H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m}$, $m \geq 1$, is the nth generalized harmonic number of order m.

4.21 Special (and Very Useful) Pairs of Classical Euler Sums Arising in Many Difficult Harmonic Series

Calculate

$$i)\ S_1 = \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^3},\ S_2 = \sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^2};\ ii)\ S_3 = \sum_{n=1}^{\infty} \frac{H_n^{(4)}}{n^2},\ S_4 = \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^4};$$

$$iii)\ S_5 = \sum_{n=1}^{\infty} \frac{H_n^{(5)}}{n^2},\ S_6 = \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^5};\ iv)\ S_7 = \sum_{n=1}^{\infty} \frac{H_n^{(4)}}{n^3},\ S_8 = \sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^4},$$

where $H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m}$, $m \geq 1$, is the nth generalized harmonic number of order m.

4.22 Another Perspective on the Famous Quadratic Series of Au-Yeung Which Leads to an Elementary Solution

Prove that

$$\sum_{n=1}^{\infty} \left(\frac{H_n}{n}\right)^2 = \frac{17}{4}\zeta(4), \tag{4.29}$$

where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ denotes the nth harmonic number and ζ represents the Riemann zeta function.

4.23 Treating a Big Brother Series of the Quadratic Series of Au-Yeung by Elementary Means

Show that

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^3} = \frac{7}{2}\zeta(5) - \zeta(2)\zeta(3), \tag{4.30}$$

where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ denotes the nth harmonic number and ζ represents the Riemann zeta function.

4.24 Calculating Two More Elder Brother Series of the Quadratic Series of Au-Yeung, This Time the Versions with the Powers 4 and 5 in Denominator

Show that

$$i) \sum_{n=1}^{\infty} \frac{H_n^2}{n^4} = \frac{97}{24}\zeta(6) - 2\zeta^2(3); \tag{4.31}$$

$$ii) \sum_{n=1}^{\infty} \frac{H_n^2}{n^5} = 6\zeta(7) - \zeta(2)\zeta(5) - \frac{5}{2}\zeta(3)\zeta(4), \tag{4.32}$$

where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ denotes the nth harmonic number and ζ represents the Riemann zeta function.

4.25 An Advanced Harmonic Series of Weight 5, $\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^2}$, Attacked with a Special Class of Sums

Prove that

$$\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^2} = \zeta(2)\zeta(3) + \zeta(5), \tag{4.33}$$

(continued)

where $H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m}$, $m \geq 1$, is the nth generalized harmonic number of order m and ζ represents the Riemann zeta function.

A (super) challenging question: Is it possible to reduce everything to the calculation of some cases of the classical linear Euler sum, $\displaystyle\sum_{k=1}^{\infty} \frac{H_k}{k^n}$, and the symmetrical pair of series, $\displaystyle\sum_{k=1}^{\infty} \frac{H_k^{(p)}}{k^q} + \sum_{k=1}^{\infty} \frac{H_k^{(q)}}{k^p}$, by elementary series manipulations only?

4.26 An Advanced Harmonic Series of Weight 5, $\sum_{n=1}^{\infty} \frac{H_n^3}{n^2}$, Attacked with a Special Identity

Prove that

$$\sum_{n=1}^{\infty} \frac{H_n^3}{n^2} = \zeta(2)\zeta(3) + 10\zeta(5), \tag{4.34}$$

where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ denotes the nth harmonic number and ζ represents the Riemann zeta function.

4.27 The Evaluation of an Advanced Cubic Harmonic Series of Weight 6, $\sum_{n=1}^{\infty} \left(\frac{H_n}{n}\right)^3$, Treated with Both *The Master Theorem of Series* and Special Logarithmic Integrals of Powers Two and Three

Show that

$$\sum_{n=1}^{\infty} \left(\frac{H_n}{n}\right)^3 = \frac{1}{2}\left(\frac{93}{8}\zeta(6) - 5\zeta^2(3)\right), \tag{4.35}$$

(continued)

where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ denotes the nth harmonic number and ζ represents the Riemann zeta function.

4.28 Another Evaluation of an Advanced Harmonic Series of Weight 6, $\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^3}$, Treated with *The Master Theorem of Series*

Show that

$$\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^3} = \frac{1}{2}\left(5\zeta^2(3) - \frac{101}{24}\zeta(6)\right), \tag{4.36}$$

where $H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m}$, $m \geq 1$, is the nth generalized harmonic number of order m and ζ denotes the Riemann zeta function.

4.29 And Now a Series of Weight 6, $\sum_{n=1}^{\infty} \frac{H_n H_n^{(3)}}{n^2}$, Treated with Both *The Master Theorem of Series* and Special Logarithmic Integrals

Prove that

$$\sum_{n=1}^{\infty} \frac{H_n H_n^{(3)}}{n^2} = \frac{1}{2}\left(\frac{227}{24}\zeta(6) - 3\zeta^2(3)\right), \tag{4.37}$$

where $H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m}$, $m \geq 1$, is the nth generalized harmonic number of order m and ζ denotes the Riemann zeta function.

4.30 An Appealing Exotic Harmonic Series of Weight 6, $\sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^2}$, Derived by Elementary Series Manipulations

Prove that

$$\sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^2} = \frac{41}{12}\zeta(6) + 2\zeta^2(3), \tag{4.38}$$

where $H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m}$, $m \geq 1$, is the nth generalized harmonic number of order m and ζ denotes the Riemann zeta function.

4.31 Another Appealing Exotic Harmonic Series of Weight 6, $\sum_{n=1}^{\infty} \frac{H_n^4}{n^2}$, Derived by Elementary Series Manipulations

Prove that

$$\sum_{n=1}^{\infty} \frac{H_n^4}{n^2} = \frac{979}{24}\zeta(6) + 3\zeta^2(3), \tag{4.39}$$

where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ denotes the nth harmonic number and ζ represents the Riemann zeta function.

4.32 Four Sums with Harmonic Series Involving the Generalized Harmonic Numbers of Order 1, 2, 3, 4, 5, and 6, Originating from *The Master Theorem of Series*

Show, without calculating each series separately, that

$$i)\ \sum_{n=1}^{\infty} \frac{H_n H_n^{(3)}}{n^2} + \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^3} = \frac{21}{8}\zeta(6) + \zeta^2(3); \tag{4.40}$$

(continued)

$$ii) \sum_{n=1}^{\infty} \frac{H_n H_n^{(4)}}{n^2} - \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4} = \frac{1}{2}\left(5\zeta(2)\zeta(5) - \frac{9}{2}\zeta(3)\zeta(4)\right); \quad (4.41)$$

$$iii) \sum_{n=1}^{\infty} \frac{H_n H_n^{(5)}}{n^2} + \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^5} = \frac{1}{2}\left(\frac{17}{36}\zeta(8) + 11\zeta(3)\zeta(5) - 3\zeta(2)\zeta^2(3)\right); \quad (4.42)$$

$$iv) \sum_{n=1}^{\infty} \frac{H_n H_n^{(6)}}{n^2} - \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^6} = 7\zeta(2)\zeta(7) - \frac{19}{3}\zeta(3)\zeta(6) - \frac{15}{4}\zeta(4)\zeta(5) + \zeta^3(3), \quad (4.43)$$

where $H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m}$, $m \geq 1$, is the nth generalized harmonic number of order m and ζ denotes the Riemann zeta function.

4.33 Awesomely Wicked Sums of Series of Weight 7, $\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4}$ and $\sum_{n=1}^{\infty} \frac{H_n H_n^{(3)}}{n^3}$, Originating from a Strong Generalized Sum: The First Part

Prove, without calculating each series separately, that

$$2\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4} + \sum_{n=1}^{\infty} \frac{H_n H_n^{(3)}}{n^3} = 4\zeta(7) + \frac{7}{4}\zeta(3)\zeta(4) - \frac{3}{2}\zeta(2)\zeta(5), \quad (4.44)$$

where $H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m}$, $m \geq 1$, is the nth generalized harmonic number of order m and ζ denotes the Riemann zeta function.

4.34 Awesomely Wicked Sums of Series of Weight 7, $\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4}$ and $\sum_{n=1}^{\infty} \frac{H_n^{(2)} H_n^{(3)}}{n^2}$, Originating from a Strong Generalized Sum: The Second Part

Prove, without calculating each series separately, that

$$\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4} + \sum_{n=1}^{\infty} \frac{H_n^{(2)} H_n^{(3)}}{n^2} = 5\zeta(7) - \frac{1}{2}\zeta(2)\zeta(5) - \frac{3}{4}\zeta(3)\zeta(4), \tag{4.45}$$

where $H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m},\ m \geq 1$, is the nth generalized harmonic number of order m and ζ denotes the Riemann zeta function.

4.35 Awesomely Wicked Sums of Series of Weight 7, $\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4}$, $\sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(3)}}{n^2}$, and $\sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^3}$, Derivation Based upon a New Identity: The Third Part

Prove, without calculating each series separately, that

$$3\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4} + \sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(3)}}{n^2} + \sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^3} = \frac{23}{4}\zeta(3)\zeta(4) - \frac{1}{2}\zeta(2)\zeta(5) + 4\zeta(7), \tag{4.46}$$

where $H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m},\ m \geq 1$, is the nth generalized harmonic number of order m and ζ denotes the Riemann zeta function.

4.36 Deriving More Useful Sums of Harmonic Series of Weight 7

Prove, without calculating each series separately, that

$$i)\ 2\sum_{n=1}^{\infty} \frac{H_n^{(2)} H_n^{(3)}}{n^2} + \sum_{n=1}^{\infty} \frac{(H_n^{(2)})^2}{n^3} = \frac{13}{2}\zeta(3)\zeta(4) - 3\zeta(7); \tag{4.47}$$

(continued)

$$ii)\ 2\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^4}-\sum_{n=1}^{\infty}\frac{(H_n^{(2)})^2}{n^3}=13\zeta(7)-\zeta(2)\zeta(5)-8\zeta(3)\zeta(4); \tag{4.48}$$

$$iii)\ 10\sum_{n=1}^{\infty}\frac{H_n^3}{n^4}-3\sum_{n=1}^{\infty}\frac{H_n^4}{n^3}=75\zeta(7)+5\zeta(2)\zeta(5)-63\zeta(3)\zeta(4), \tag{4.49}$$

where $H_n^{(m)}=1+\frac{1}{2^m}+\cdots+\frac{1}{n^m},\ m\geq 1$, is the nth generalized harmonic number of order m and ζ denotes the Riemann zeta function.

4.37 Preparing the *Weapons* of *The Master Theorem of Series* to Breach the Fortress of the Challenging Harmonic Series of Weight 7: The 1st Episode

Show, without calculating each series separately, that

$$3\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^4}-\sum_{n=1}^{\infty}\frac{H_n^3}{n^4}=4\zeta(2)\zeta(5)+15\zeta(3)\zeta(4)-24\zeta(7), \tag{4.50}$$

where $H_n^{(m)}=1+\frac{1}{2^m}+\cdots+\frac{1}{n^m},\ m\geq 1$, is the nth generalized harmonic number of order m and ζ denotes the Riemann zeta function.

4.38 Preparing the *Weapons* of *The Master Theorem of Series* to Breach the Fortress of the Challenging Harmonic Series of Weight 7: The 2nd Episode

Show, without calculating each series separately, that

$$2\sum_{n=1}^{\infty}\frac{H_n^3}{n^4}-\frac{5}{6}\sum_{n=1}^{\infty}\frac{H_n^4}{n^3}-\sum_{n=1}^{\infty}\frac{H_n^2H_n^{(2)}}{n^3}+\frac{7}{3}\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^4}$$

(continued)

$$= \frac{55}{6}\zeta(7) + \frac{13}{2}\zeta(2)\zeta(5) - \frac{46}{3}\zeta(3)\zeta(4), \tag{4.51}$$

where $H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m},\ m \geq 1$, is the nth generalized harmonic number of order m and ζ denotes the Riemann zeta function.

4.39 Preparing the *Weapons* of *The Master Theorem of Series* to Breach the Fortress of the Challenging Harmonic Series of Weight 7: The 3rd Episode

Show, without calculating each series separately, that

$$3\sum_{n=1}^{\infty} \frac{H_n^3}{n^4} - \frac{5}{4}\sum_{n=1}^{\infty} \frac{H_n^4}{n^3} + \frac{3}{2}\sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^3} + \frac{7}{2}\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4}$$

$$= \frac{15}{4}\zeta(2)\zeta(5) + \frac{11}{2}\zeta(3)\zeta(4) - \frac{29}{4}\zeta(7), \tag{4.52}$$

where $H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m},\ m \geq 1$, is the nth generalized harmonic number of order m and ζ denotes the Riemann zeta function.

4.40 Calculating the Harmonic Series of Weight 7, $\sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^3}$, with the *Weapons* of *The Master Theorem of Series*

Prove that

$$\sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^3} = \frac{19}{2}\zeta(3)\zeta(4) - 2\zeta(2)\zeta(5) - 7\zeta(7), \tag{4.53}$$

where $H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m},\ m \geq 1$, is the nth generalized harmonic number of order m and ζ denotes the Riemann zeta function.

4.41 The Calculation of Two Good-Looking Pairs of Harmonic Series: The Series $\sum_{n=1}^{\infty} \frac{H_n}{n^2} \sum_{k=1}^{n} \frac{H_k}{k^3}$, $\sum_{n=1}^{\infty} \frac{H_n}{n^3} \sum_{k=1}^{n} \frac{H_k}{k^2}$ and $\sum_{n=1}^{\infty} \frac{H_n^2}{n^2} \sum_{k=1}^{n} \frac{H_k}{k^2}$, $\sum_{n=1}^{\infty} \frac{H_n}{n^2} \sum_{k=1}^{n} \frac{H_k^2}{k^2}$

Prove that

$$i) \sum_{n=1}^{\infty} \frac{H_n}{n^2} \left(\frac{H_1}{1^3} + \frac{H_2}{2^3} + \cdots + \frac{H_n}{n^3} \right) = 10\zeta(7) + \frac{9}{2}\zeta(2)\zeta(5) - \frac{23}{2}\zeta(3)\zeta(4); \tag{4.54}$$

$$ii) \sum_{n=1}^{\infty} \frac{H_n}{n^3} \left(\frac{H_1}{1^2} + \frac{H_2}{2^2} + \cdots + \frac{H_n}{n^2} \right) = \frac{23}{2}\zeta(3)\zeta(4) - \frac{11}{2}\zeta(2)\zeta(5) - 4\zeta(7). \tag{4.55}$$

Then, show that

$$iii) \sum_{n=1}^{\infty} \frac{H_n^2}{n^2} \left(\frac{H_1}{1^2} + \frac{H_2}{2^2} + \cdots + \frac{H_n}{n^2} \right) = \frac{45}{16}\zeta(7) - \frac{7}{2}\zeta(2)\zeta(5) + \frac{17}{2}\zeta(3)\zeta(4); \tag{4.56}$$

$$iv) \sum_{n=1}^{\infty} \frac{H_n}{n^2} \left(\frac{H_1^2}{1^2} + \frac{H_2^2}{2^2} + \cdots + \frac{H_n^2}{n^2} \right) = \frac{93}{8}\zeta(7) + \frac{11}{2}\zeta(2)\zeta(5) - \frac{51}{4}\zeta(3)\zeta(4), \tag{4.57}$$

where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ denotes the nth harmonic number and ζ denotes the Riemann zeta function.

4.42 The Calculation of an Essential Harmonic Series of Weight 7: The Series $\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4}$

Prove that

$$\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4} = 2\zeta(2)\zeta(5) + \frac{3}{4}\zeta(3)\zeta(4) - \frac{51}{16}\zeta(7), \tag{4.58}$$

(continued)

where $H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m}$, $m \geq 1$, is the nth generalized harmonic number of order m and ζ denotes the Riemann zeta function.

A (super) challenging question: Derive the result by series manipulations only.

4.43 Plenty of Challenging Harmonic Series of Weight 7 Obtained by Combining the Previous Harmonic Series of Weight 7 with Various Harmonic Series Identities (Derivations by Series Manipulations Only)

Prove that

$$i)\ \sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(3)}}{n^2} = \frac{329}{16}\zeta(7) - \frac{9}{2}\zeta(2)\zeta(5) - 6\zeta(3)\zeta(4); \tag{4.59}$$

$$ii)\ \sum_{n=1}^{\infty} \frac{H_n H_n^{(4)}}{n^2} = \frac{9}{2}\zeta(2)\zeta(5) - \frac{3}{2}\zeta(3)\zeta(4) - \frac{51}{16}\zeta(7); \tag{4.60}$$

$$iii)\ \sum_{n=1}^{\infty} \frac{H_n^3}{n^4} = \frac{231}{16}\zeta(7) + 2\zeta(2)\zeta(5) - \frac{51}{4}\zeta(3)\zeta(4); \tag{4.61}$$

$$iv)\ \sum_{n=1}^{\infty} \frac{H_n^4}{n^3} = \frac{185}{8}\zeta(7) + 5\zeta(2)\zeta(5) - \frac{43}{2}\zeta(3)\zeta(4); \tag{4.62}$$

$$v)\ \sum_{n=1}^{\infty} \frac{H_n^{(2)} H_n^{(3)}}{n^2} = \frac{131}{16}\zeta(7) - \frac{5}{2}\zeta(2)\zeta(5) - \frac{3}{2}\zeta(3)\zeta(4); \tag{4.63}$$

$$vi)\ \sum_{n=1}^{\infty} \frac{H_n H_n^{(3)}}{n^3} = \frac{83}{8}\zeta(7) - \frac{11}{2}\zeta(2)\zeta(5) + \frac{1}{4}\zeta(3)\zeta(4); \tag{4.64}$$

(continued)

$$vii)\ \sum_{n=1}^{\infty} \frac{(H_n^{(2)})^2}{n^3} = 5\zeta(2)\zeta(5) + \frac{19}{2}\zeta(3)\zeta(4) - \frac{155}{8}\zeta(7); \tag{4.65}$$

$$viii)\ \sum_{n=1}^{\infty} \frac{H_n^3 H_n^{(2)}}{n^2} = \frac{83}{16}\zeta(7) - \frac{5}{2}\zeta(2)\zeta(5) + \frac{27}{2}\zeta(3)\zeta(4); \tag{4.66}$$

$$ix)\ \sum_{n=1}^{\infty} \frac{H_n^5}{n^2} = \frac{2051}{16}\zeta(7) + \frac{57}{2}\zeta(2)\zeta(5) + 33\zeta(3)\zeta(4); \tag{4.67}$$

$$x)\ \sum_{n=1}^{\infty} \frac{H_n (H_n^{(2)})^2}{n^2} = 5\zeta(3)\zeta(4) + \frac{13}{2}\zeta(2)\zeta(5) - \frac{217}{16}\zeta(7), \tag{4.68}$$

where $H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m}$, $m \geq 1$, is the nth generalized harmonic number of order m and ζ denotes the Riemann zeta function.

A (super) challenging question: Derive the results by series manipulations only.

4.44 A Member of a Glamorous Series Family Containing the Harmonic Number and the Tail of the Riemann Zeta Function

Prove that

$$\sum_{n=1}^{\infty} \frac{H_n}{n} \left(\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{n^3} \right) = 2\zeta(2)\zeta(3) - \frac{7}{2}\zeta(5), \tag{4.69}$$

where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ denotes the nth harmonic number and ζ denotes the Riemann zeta function.

4.45 More Members of a Glamorous Series Family Containing the Harmonic Number and the Tail of the Riemann Zeta Function

Prove that

$$i)\ \sum_{n=1}^{\infty}\frac{H_n}{n}\left(\zeta(4)-1-\frac{1}{2^4}-\cdots-\frac{1}{n^4}\right)=\frac{5}{48}\zeta(6); \tag{4.70}$$

$$ii)\ \sum_{n=1}^{\infty}\frac{H_n}{n}\left(\zeta(5)-1-\frac{1}{2^5}-\cdots-\frac{1}{n^5}\right)=3\zeta(2)\zeta(5)+\frac{3}{4}\zeta(3)\zeta(4)-6\zeta(7), \tag{4.71}$$

where $H_n=\sum_{k=1}^{n}\frac{1}{k}$ denotes the nth harmonic number and ζ denotes the Riemann zeta function.

4.46 Two Series Generalizations with the Generalized Harmonic Numbers and the Tail of the Riemann Zeta Function

Let $p\geq 2$ be a positive integer. Prove that

$$i)\ \sum_{n=1}^{\infty}\frac{H_n}{n+1}\left(\zeta(p)-1-\frac{1}{2^p}-\cdots-\frac{1}{n^p}\right)=\frac{(-1)^{p-1}}{2(p-1)!}\times\lim_{\substack{x\to 0\\ y\to 0}}\frac{\partial^{p+1}}{\partial x^{p-1}\partial y^2}\mathrm{B}(x,y); \tag{4.72}$$

$$ii)\ \sum_{n=1}^{\infty}\frac{H_n^2-H_n^{(2)}}{n+1}\left(\zeta(p)-1-\frac{1}{2^p}-\cdots-\frac{1}{n^p}\right)=\frac{(-1)^{p}}{3(p-1)!}\times\lim_{\substack{x\to 0\\ y\to 0}}\frac{\partial^{p+2}}{\partial x^{p-1}\partial y^3}\mathrm{B}(x,y), \tag{4.73}$$

(continued)

where $H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m}$, $m \geq 1$, is the nth generalized harmonic number of order m, ζ denotes the Riemann zeta function, and $\mathrm{B}(x, y)$ represents the Beta function defined as $\mathrm{B}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t$.

Examples:

For the generalization at $i)$ with $p = 2$,

$$\sum_{n=1}^{\infty} \frac{H_n}{n+1}\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right) = \frac{5}{4}\zeta(4);$$

For the generalization at $i)$ with $p = 3$,

$$\sum_{n=1}^{\infty} \frac{H_n}{n+1}\left(\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{n^3}\right) = 2(2\zeta(5) - \zeta(2)\zeta(3));$$

For the generalization at $ii)$ with $p = 2$,

$$\sum_{n=1}^{\infty} \frac{H_n^2 - H_n^{(2)}}{n+1}\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right) = 6\zeta(5) - 2\zeta(2)\zeta(3);$$

For the generalization at $ii)$ with $p = 3$,

$$\sum_{n=1}^{\infty} \frac{H_n^2 - H_n^{(2)}}{n+1}\left(\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{n^3}\right) = \frac{35}{8}\zeta(6) - 3\zeta^2(3).$$

4.47 The Art of Mathematics with a Series Involving the Product of the Tails of $\zeta(2)$ and $\zeta(3)$

Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right)\left(\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{n^3}\right)$$
$$= \zeta^2(3) - \frac{61}{48}\zeta(6),$$

(continued)

where ζ denotes the Riemann zeta function.

A (super) challenging question: How about calculating the series without using the particular values of the nonlinear harmonic series of weight 6?

4.48 The Art of Mathematics with Another Splendid Series Involving the Product of the Tails of $\zeta(2)$ and $\zeta(3)$

Prove that

$$\sum_{n=1}^{\infty} \frac{H_n}{n}\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right)\left(\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{n^3}\right)$$

$$= \frac{11}{4}\zeta(3)\zeta(4) - 2\zeta(2)\zeta(5),$$

where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ is the nth harmonic number and ζ denotes the Riemann zeta function.

A (super) challenging question: Could we calculate the given series without using the particular values of the nonlinear harmonic series of weight 7, $\sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(3)}}{n^2}$, $\sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^3}$, $\sum_{n=1}^{\infty} \frac{H_n^{(2)} H_n^{(3)}}{n^2}$, $\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4}$ and $\sum_{n=1}^{\infty} \frac{H_n H_n^{(3)}}{n^3}$?

4.49 Expressing Polylogarithmic Values by Combining the Alternating Harmonic Series and the Non-alternating Harmonic Series with Integer Powers of 2 in Denominator

Prove that

$$i)\ \operatorname{Li}_3\left(\frac{1}{2}\right)$$

$$= \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n2^{n+1}} + \frac{1}{4}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2}{n(n+1)} + \frac{1}{4}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(2)}}{n(n+1)};$$

(continued)

$$ii) \ \operatorname{Li}_4\left(\frac{1}{2}\right)$$

$$= \sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n2^{n+1}} + \frac{1}{12}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^3}{n(n+1)} + \frac{1}{6}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(3)}}{n(n+1)}$$

$$+\frac{1}{4}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n H_n^{(2)}}{n(n+1)};$$

$$iii) \ \operatorname{Li}_5\left(\frac{1}{2}\right)$$

$$= \sum_{n=1}^{\infty} \frac{H_n^{(4)}}{n2^{n+1}} + \frac{1}{48}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^4}{n(n+1)} + \frac{1}{8}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(4)}}{n(n+1)}$$

$$+\frac{1}{8}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2 H_n^{(2)}}{n(n+1)}+\frac{1}{16}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(H_n^{(2)})^2}{n(n+1)}+\frac{1}{6}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n H_n^{(3)}}{n(n+1)},$$

where $H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m}$, $m \geq 1$, denotes the nth generalized harmonic number of order m and Li_n represents the Polylogarithm function.

A challenging question: Is it possible to derive the results without calculating the series?

4.50 Cool Results with Cool Series Involving Summands with the Harmonic Number and the Integer Powers of 2

Show that

$$i) \ \log(2)\sum_{k=1}^{\infty}\frac{H_k}{(k+1)2^{k+1}} + \sum_{k=1}^{\infty}\frac{H_k}{(k+1)^2 2^{k+1}} = \frac{1}{8}\zeta(3) + \frac{1}{3}\log^3(2); \tag{4.74}$$

$$ii) \ \log^2(2)\sum_{k=1}^{\infty}\frac{H_k}{(k+1)2^{k+1}} + \log(2)\sum_{k=1}^{\infty}\frac{H_k}{(k+1)^2 2^k} + \sum_{k=1}^{\infty}\frac{H_k}{(k+1)^3 2^k}$$

(continued)

$$= \frac{1}{4}\left(\zeta(4) + \log^4(2)\right); \tag{4.75}$$

$$iii)\ \log^2(2)\sum_{k=1}^{\infty}\frac{H_k}{(k+1)^2 2^k} + \log(2)\sum_{k=1}^{\infty}\frac{H_k}{(k+1)^3 2^{k-1}} + \sum_{k=1}^{\infty}\frac{H_k}{(k+1)^4 2^{k-1}}$$

$$= \frac{1}{8}\zeta(5) - 2\zeta(2)\zeta(3) - \frac{2}{3}\log^3(2)\zeta(2) + \frac{7}{4}\log^2(2)\zeta(3) - \frac{1}{15}\log^5(2)$$

$$+ 4\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right) + 4\operatorname{Li}_5\left(\frac{1}{2}\right), \tag{4.76}$$

where $H_n = \sum_{k=1}^{n}\frac{1}{k}$ is the nth harmonic number, ζ denotes the Riemann zeta function, and Li_n represents the Polylogarithm function.

4.51 Eight Harmonic Series Involving the Integer Powers of 2 in Denominator

Prove that

$$i)\ \sum_{k=1}^{\infty}\frac{H_k}{(k+1)2^k} = \log^2(2); \tag{4.77}$$

$$ii)\ \sum_{k=1}^{\infty}\frac{H_k}{k2^k} = \frac{1}{2}\zeta(2); \tag{4.78}$$

$$iii)\ \sum_{k=1}^{\infty}\frac{H_k}{(k+1)^2 2^k} = \frac{1}{4}\zeta(3) - \frac{1}{3}\log^3(2); \tag{4.79}$$

$$iv)\ \sum_{k=1}^{\infty}\frac{H_k}{k^2 2^k} = \zeta(3) - \frac{1}{2}\log(2)\zeta(2); \tag{4.80}$$

$$v)\ \sum_{k=1}^{\infty}\frac{H_k}{(k+1)^3 2^k} = \frac{1}{4}\left(\zeta(4) - \log(2)\zeta(3) + \frac{1}{3}\log^4(2)\right); \tag{4.81}$$

(continued)

$$vi)\ \sum_{k=1}^{\infty} \frac{H_k}{k^3 2^k} = \frac{1}{8}\zeta(4) - \frac{1}{8}\log(2)\zeta(3) + \frac{1}{24}\log^4(2) + \operatorname{Li}_4\left(\frac{1}{2}\right); \tag{4.82}$$

$$vii)\ \sum_{k=1}^{\infty} \frac{H_k}{(k+1)^4 2^k}$$

$$= \frac{1}{16}\zeta(5) - \frac{1}{4}\log(2)\zeta(4) + \log^2(2)\zeta(3) - \frac{1}{3}\log^3(2)\zeta(2) - \zeta(2)\zeta(3)$$

$$+ \frac{1}{20}\log^5(2) + 2\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right) + 2\operatorname{Li}_5\left(\frac{1}{2}\right); \tag{4.83}$$

$$viii)\ \sum_{k=1}^{\infty} \frac{H_k}{k^4 2^k}$$

$$= \frac{1}{32}\zeta(5) - \frac{1}{8}\log(2)\zeta(4) + \frac{1}{2}\log^2(2)\zeta(3) - \frac{1}{6}\log^3(2)\zeta(2) - \frac{1}{2}\zeta(2)\zeta(3)$$

$$+ \frac{1}{40}\log^5(2) + \log(2)\operatorname{Li}_4\left(\frac{1}{2}\right) + 2\operatorname{Li}_5\left(\frac{1}{2}\right), \tag{4.84}$$

where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ is the nth harmonic number, ζ denotes the Riemann zeta function, and Li_n represents the Polylogarithm function.

4.52 Let's Calculate Three Classical Alternating Harmonic Series, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^3}$, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(2)}}{n^2}$, and $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2}{n^2}$

Show that

$$i)\ \sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^3}$$

(continued)

$$= \frac{11}{4}\zeta(4) - \frac{7}{4}\log(2)\zeta(3) + \frac{1}{2}\log^2(2)\zeta(2) - \frac{1}{12}\log^4(2) - 2\operatorname{Li}_4\left(\frac{1}{2}\right); \tag{4.85}$$

$$ii) \sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(2)}}{n^2}$$

$$= \frac{1}{6}\log^4(2) - \log^2(2)\zeta(2) + \frac{7}{2}\log(2)\zeta(3) - \frac{51}{16}\zeta(4) + 4\operatorname{Li}_4\left(\frac{1}{2}\right); \tag{4.86}$$

$$iii) \sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2}{n^2}$$

$$= \frac{41}{16}\zeta(4) - \frac{7}{4}\log(2)\zeta(3) + \frac{1}{2}\log^2(2)\zeta(2) - \frac{1}{12}\log^4(2) - 2\operatorname{Li}_4\left(\frac{1}{2}\right), \tag{4.87}$$

where $H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m}$, $m \geq 1$, is the nth generalized harmonic number of order m, Li_n represents the Polylogarithm function, and ζ denotes the Riemann zeta function.

4.53 Then, Let's Calculate Another Pair of Classical Alternating Harmonic Series, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^2}$ and $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^4}$

Show that

$$i) \sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^2} = \frac{5}{8}\zeta(3); \tag{4.88}$$

$$ii) \sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^4} = \frac{59}{32}\zeta(5) - \frac{1}{2}\zeta(2)\zeta(3), \tag{4.89}$$

where $H_n = \sum_{k=1}^{n}\frac{1}{k}$ is the nth harmonic number and ζ denotes the Riemann zeta function.

4.54 A Nice Challenging Trio of Alternating Harmonic Series, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(2)}}{n^3}$, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(3)}}{n^2}$, and $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(4)}}{n}$

Prove that

$$i)\ \sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(2)}}{n^3}=\frac{5}{8}\zeta(2)\zeta(3)-\frac{11}{32}\zeta(5); \tag{4.90}$$

$$ii)\ \sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(3)}}{n^2}=\frac{3}{4}\zeta(2)\zeta(3)-\frac{21}{32}\zeta(5); \tag{4.91}$$

$$iii)\ \sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(4)}}{n}=2\zeta(5)-\frac{3}{8}\zeta(2)\zeta(3)-\frac{7}{8}\log(2)\zeta(4), \tag{4.92}$$

where $H_n^{(m)}=1+\frac{1}{2^m}+\cdots+\frac{1}{n^m}$, $m\geq 1$, is the nth generalized harmonic number of order m and ζ denotes the Riemann zeta function.

4.55 Encountering an Alternating Harmonic Series of Weight 5 with an Eye-Catching Closed-Form, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2}{n^3}$

Show that

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2}{n^3}$$

$$=\frac{2}{15}\log^5(2)-\frac{11}{8}\zeta(2)\zeta(3)-\frac{19}{32}\zeta(5)+\frac{7}{4}\log^2(2)\zeta(3)-\frac{2}{3}\log^3(2)\zeta(2)$$

$$+4\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right)+4\operatorname{Li}_5\left(\frac{1}{2}\right), \tag{4.93}$$

where $H_n=\sum_{k=1}^{n}\frac{1}{k}$ is the nth harmonic number, ζ represents the Riemann zeta function, and Li_n denotes the Polylogarithm function.

4.56 Encountering Another Alternating Harmonic Series of Weight 5 with a Dazzling Closed-Form, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^3}{n^2}$

Show that

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^3}{n^2}$$

$$= \frac{1}{5}\log^5(2) - \log^3(2)\zeta(2) + \frac{21}{8}\log^2(2)\zeta(3) - \frac{27}{16}\zeta(2)\zeta(3) - \frac{9}{4}\zeta(5)$$

$$+ 6\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right) + 6\operatorname{Li}_5\left(\frac{1}{2}\right), \tag{4.94}$$

where $H_n = \sum_{k=1}^{n}\frac{1}{k}$ is the nth harmonic number, ζ represents the Riemann zeta function, and Li_n denotes the Polylogarithm function.

4.57 Yet Another Encounter with a Superb Alternating Harmonic Series of Weight 5, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n H_n^{(2)}}{n^2}$

Show that

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n H_n^{(2)}}{n^2}$$

$$= \frac{23}{8}\zeta(5) - \frac{7}{4}\log^2(2)\zeta(3) + \frac{2}{3}\log^3(2)\zeta(2) + \frac{15}{16}\zeta(2)\zeta(3) - \frac{2}{15}\log^5(2)$$

$$- 4\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right) - 4\operatorname{Li}_5\left(\frac{1}{2}\right), \tag{4.95}$$

where $H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m}$, $m \geq 1$, is the nth generalized harmonic number of order m, ζ represents the Riemann zeta function, and Li_n denotes the Polylogarithm function.

4.58 Fascinating Sums of Two Alternating Harmonic Series Involving the Generalized Harmonic Number

Prove that

$$i)\ \sum_{n=1}^{\infty}(-1)^n \frac{H_n^4}{n} + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(H_n^{(2)})^2}{n}$$

$$= 2\zeta(2)\zeta(3) - 11\zeta(5) + \frac{2}{3}\log^3(2)\zeta(2) - \frac{9}{4}\log^2(2)\zeta(3) + \frac{9}{2}\log(2)\zeta(4) - \frac{\log^5(2)}{10}$$

$$+4\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right) + 8\operatorname{Li}_5\left(\frac{1}{2}\right);$$

$$ii)\ \sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_{2n}}{n^3} + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_{2n}^{(2)}}{n^2} = 2G^2 + \frac{37}{64}\zeta(4),$$

where $H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m}$, $m \geq 1$, is the nth generalized harmonic number of order m, G denotes the Catalan's constant, ζ represents the Riemann zeta function, and Li_n designates the Polylogarithm function.

4.59 An Outstanding Sum of Series Representation of the Particular Value of the Riemann Zeta Function, $\zeta(4)$

Show that

$$\zeta(4)$$

$$= \frac{8}{5}\sum_{n=1}^{\infty}\frac{H_n H_{2n}}{n^2} + \frac{64}{5}\sum_{n=1}^{\infty}\frac{(H_{2n})^2}{(2n+1)^2} + \frac{64}{5}\sum_{n=1}^{\infty}\frac{H_{2n}}{(2n+1)^3}$$

$$-\frac{8}{5}\sum_{n=1}^{\infty}\frac{(H_{2n})^2}{n^2} - \frac{32}{5}\sum_{n=1}^{\infty}\frac{H_n H_{2n}}{(2n+1)^2} - \frac{64}{5}\log(2)\sum_{n=1}^{\infty}\frac{H_{2n}}{(2n+1)^2} - \frac{8}{5}\sum_{n=1}^{\infty}\frac{H_{2n}^{(2)}}{n^2}, \tag{4.96}$$

where $H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m}$, $m \geq 1$, is the nth generalized harmonic number of order m and ζ represents the Riemann zeta function.

4.60 An Excellent Representation of the Particular Value of the Riemann Zeta Function, $\zeta(4)$, with a Triple Series Involving the Factorials and the Generalized Harmonic Numbers

Prove that

$$\zeta(4)$$

$$= \frac{4}{45} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{(i-1)!(j-1)!(k-1)!}{(i+j+k-1)!}$$

$$\times \left((H_{i+j+k-1} - H_{k-1})^2 + H_{i+j+k-1}^{(2)} - H_{k-1}^{(2)} \right),$$

where $H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m}$, $m \geq 1$, denotes the nth generalized harmonic number of order m and ζ represents the Riemann zeta function.

Chapter 5
Hints

5.1 The First Series Submitted by Ramanujan to the Journal of the Indian Mathematical Society

Use that $\sum_{n=1}^{\infty}\left(\frac{1}{4n-3}-\frac{1}{4n-2}+\frac{1}{4n-1}-\frac{1}{4n}\right)=\log(2)$ and $\sum_{n=1}^{\infty}\left(\frac{1}{2n-1}-\frac{1}{2n}\right)=\log(2)$.

5.2 Starting from an Elementary Integral Result and Deriving Two Classical Series in a New Way

For a first solution to the point i) of the problem, make use of the result in (1.1), and for a second solution, wisely employ the Beta function.

5.3 An Extraordinary Series with the Tail of the Riemann Zeta Function Connected to the Inverse Sine Series

Use the Trigamma function integral representation, $\psi^{(1)}(z)=-\int_0^1\frac{t^{z-1}}{1-t}\log(t)\mathrm{d}t$.

C. I. Vălean, *(Almost) Impossible Integrals, Sums, and Series*, Problem Books in Mathematics, https://doi.org/10.1007/978-3-030-02462-8_5

5.4 The Evaluation of a Series Involving the Tails of the Series Representations of the Functions $\log\left(\frac{1}{1-x}\right)$ and $\frac{x\arcsin(x)}{\sqrt{1-x^2}}$

Make use of the results in (3.179) and (3.182).

5.5 A Breathtaking Infinite Series Involving the Binomial Coefficient and Expressing a Beautiful Closed-Form

Use the key identity in (3.173).

5.6 An Eccentric Multiple Series Having the Roots in the Realm of the Botez–Catalan Identity

Exploit the classical Botez–Catalan identity,

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{2n}=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n},\ \forall n\geq 1.$$

5.7 Two Classical Series with Fibonacci Numbers, One Related to the Arctan Function

Recall and use the recurrence relation of the Fibonacci numbers, $F_n = F_{n-1}+F_{n-2}$ where $F_1 = 1$ and $F_2 = 1$, which we combine with Cassini's identity that states that $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$. Then, to calculate S_2, also use the trigonometric identity, $\arctan(a) - \arctan(b) = \arctan\left(\dfrac{a-b}{1+ab}\right)$.

5.8 Two New Infinite Series with Fibonacci Numbers, Related to the Arctan Function

At the point i) split the series into two simpler series. Then, for the series from the point ii) make use and combine Cassini's identity, d'Ocagne's identity, and the recurrence relation of the Fibonacci numbers.

5.9 Useful Series Representations of $\log(1+x)\log(1-x)$ and $\arctan(x)\log(1+x^2)$ from the Notorious *Table of Integrals, Series, and Products* by I.S. Gradshteyn and I.M. Ryzhik

For a first solution to the point $i)$ of the problem, start with combining the Taylor series and the Cauchy product of two series, and for a second solution, start with proving that $\dfrac{\log(1+x)}{1-x} - \dfrac{\log(1-x)}{1+x} = 2\sum_{k=1}^{\infty} x^{2k-1}\varphi(2k-1)$, where $\varphi(k) = \sum_{n=1}^{k} \dfrac{(-1)^{n-1}}{n}$, and then use the idea of telescoping sums. As regards the point $ii)$ of the problem, for a first solution use again the Cauchy product of two series, and for a second solution start with proving that $2\dfrac{x\arctan(x)}{1+x^2} + \dfrac{\log(1+x^2)}{1+x^2} = 2\sum_{k=1}^{\infty}(-1)^{k-1}x^{2k}H_{2k}$, and then, as suggested for the second solution at the previous point, employ the idea of telescoping sums.

5.10 A Group of Five Useful Generating Functions Related to the Generalized Harmonic Numbers

The strategy for all five results is to multiply both sides of each equality by $1-x$ and then use the idea of telescoping sums.

5.11 Four Members from a Neat Group of Generating Functions Expressed in Terms of Polylogarithm Function

For all four results, combine the logarithmic integrals in Sect. 1.3 with the generalized integral in (1.12).

5.12 Two Elementary Harmonic Sums Arising in the Calculation of Harmonic Series

Use the complete homogeneous symmetric polynomial identities.

5.13 A Strong Generalized Sum, Making a Very Good Cocktail Together with the Identities Generated by *The Master Theorem of Series*

Consider $S_{n,m} = \sum_{k=1}^{n-1} \frac{H_k^{(m)}}{n-k} = \sum_{k=1}^{n-1} \frac{H_{n-k}^{(m)}}{k}$ where the last sum is obtained by reversing the order of summing the terms of the initial sum and then exploit the difference $S_{n,m} - S_{n-1,m}$.

5.14 Four Elementary Sums with Harmonic Numbers, Very Useful in the Calculation of the Harmonic Series of Weight 7

For the first sum we can write $S_1(n) = \sum_{k=1}^{n-1} \frac{H_k^2}{n-k} = \sum_{k=1}^{n-1} \frac{H_{n-k}^2}{k}$ and then exploit the difference $S_1(n) - S_1(n-1)$. We may proceed similarly for the other three sums.

5.15 The Master Theorem of Series, a New Very Useful Theorem in the Calculation of Many Difficult (Harmonic) Series

To prove the theorem, we start by making use of the simple result,

$$\frac{1}{n} \sum_{j=1}^{n} \frac{1}{(j+k)(j+k+1)} = \frac{1}{(k+1)(k+n+1)}.$$

5.16 The First Application of *The Master Theorem of Series* on the (Generalized) Harmonic Numbers

A very useful application of *The Master Theorem of Series* is related to the use of the generalized harmonic number by setting $\mathscr{M}(k) = H_k^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{k^m}$ and $m(k) = \frac{1}{k^m}$.

5.17 The Second Application of *The Master Theorem of Series* on the Harmonic Numbers

Another useful and powerful application of *The Master Theorem of Series* is obtained by setting $\mathscr{M}(k) = H_k^2$ and $m(k) = H_k^2 - H_{k-1}^2$. Later in the calculations we make use of the second equality in (1.6).

5.18 The Third Application of *The Master Theorem of Series* on the Harmonic Numbers

Consider *The Master Theorem of Series* where we set $\mathscr{M}(k) = H_k^3$ and $m(k) = H_k^3 - H_{k-1}^3$. During the calculations one also needs to make use of the second equality in (1.7).

5.19 The Fourth Application of *The Master Theorem of Series* on the (Generalized) Harmonic Numbers

As in the previous sections, use *The Master Theorem of Series* where we set $\mathscr{M}(k) = H_k H_k^{(2)}$ and $m(k+1) = H_{k+1}H_{k+1}^{(2)} - H_k H_k^{(2)} = \left(H_k + \dfrac{1}{k+1}\right)\left(H_k^{(2)} + \dfrac{1}{(k+1)^2}\right) - H_k H_k^{(2)} = \dfrac{H_k}{(k+1)^2} + \dfrac{H_k^{(2)}}{k+1} + \dfrac{1}{(k+1)^3}$.

5.20 Cool Identities with Ingredients Like the Generalized Harmonic Numbers and the Binomial Coefficient

Some needed identities are already found in the previous sections.

5.21 Special (and Very Useful) Pairs of Classical Euler Sums Arising in Many Difficult Harmonic Series

Use series with $\zeta(k) - H_n^{(k)}$ instead of $H_n^{(k)}$, which one may combine with Abel's summation, the series version that states that if $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ are two sequences of real numbers with $A_n = \sum_{k=1}^{n} a_k$, then we have that

$$\sum_{k=1}^{\infty} a_k b_k = \lim_{n\to\infty} (A_n b_{n+1}) + \sum_{k=1}^{\infty} A_k (b_k - b_{k+1}). \tag{5.1}$$

5.22 Another Perspective on the Famous Quadratic Series of Au-Yeung Which Leads to an Elementary Solution

Make use of the series identity in (4.21), the case $m = 1$.

5.23 Treating a Big Brother Series of the Quadratic Series of Au-Yeung by Elementary Means

Make use of the result in (4.21), the case $m = 1$.

5.24 Calculating Two More Elder Brother Series of the Quadratic Series of Au-Yeung, This Time the Versions with the Powers 4 and 5 in Denominator

Follow the same strategy as in the previous section and make use of the result in (4.21), the case $m = 1$.

5.25 An Advanced Harmonic Series of Weight 5, $\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^2}$, Attacked with a Special Class of Sums

For a first solution, employ the identity in (4.16), the case $m = 2$. For a second solution, make use of the results in (1.5) and (1.6).

5.26 An Advanced Harmonic Series of Weight 5, $\sum_{n=1}^{\infty} \frac{H_n^3}{n^2}$, Attacked with a Special Identity

For a first solution, make use of the identity in (4.26). For a second solution, make use of the identities in (1.5) and (1.6).

5.27 The Evaluation of an Advanced Cubic Harmonic Series of Weight 6, $\sum_{n=1}^{\infty} \left(\frac{H_n}{n}\right)^3$, Treated with Both *The Master Theorem of Series* and Special Logarithmic Integrals of Powers Two and Three

For a first solution, make use of the identity in (4.26), the first equality, where we multiply both sides by $1/n^2$ and then consider the sum from $n = 1$ to ∞. For a second solution, make use of the results in (1.5) and (1.6) and create a system of relations involving the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{H_n}{n}\right)^3$ and $\displaystyle\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^3}$.

5.28 Another Evaluation of an Advanced Harmonic Series of Weight 6, $\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^3}$, Treated with *The Master Theorem of Series*

For an elementary solution, make use of the identity in (4.21), the case $m = 1$, to establish a relation between the series, $\displaystyle\sum_{n=1}^{\infty} \left(\frac{H_n}{n}\right)^3$ and $\displaystyle\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^3}$, which is combined then with the value of the former series in the previous section. The second approach is common to the one in the previous section.

5.29 And Now a Series of Weight 6, $\sum_{n=1}^{\infty} \frac{H_n H_n^{(3)}}{n^2}$, Treated with Both *The Master Theorem of Series* and Special Logarithmic Integrals

For a first solution, make use of the identity in (4.22). Further, for a second solution involving integrals, make use of the result in (1.4).

5.30 An Appealing Exotic Harmonic Series of Weight 6, $\sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^2}$, Derived by Elementary Series Manipulations

To get a first solution, make use of the identity in (4.24). For a second solution, check the next section.

5.31 Another Appealing Exotic Harmonic Series of Weight 6, $\sum_{n=1}^{\infty} \frac{H_n^4}{n^2}$, Derived by Elementary Series Manipulations

For a first solution, make use of the identities in (4.21), the case $m = 1$, and (4.26), the first equality, to get two relations with the series $\displaystyle\sum_{n=1}^{\infty} \frac{H_n^4}{n^2}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^2}$. For a second solution, combine the calculation of $\displaystyle\sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^2}$ in the previous section with either of the two identities suggested in this section.

5.32 Four Sums with Harmonic Series Involving the Generalized Harmonic Numbers of Order 1, 2, 3, 4, 5, and 6, Originating from *The Master Theorem of Series*

Make use of the result in (4.21).

5.33 Awesomely Wicked Sums of Series of Weight 7, $\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4}$ and $\sum_{n=1}^{\infty} \frac{H_n H_n^{(3)}}{n^3}$, Originating from a Strong Generalized Sum: The First Part

Consider the identity in (4.16), the case $m = 1$.

5.34 Awesomely Wicked Sums of Series of Weight 7, $\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4}$ and $\sum_{n=1}^{\infty} \frac{H_n^{(2)} H_n^{(3)}}{n^2}$, Originating from a Strong Generalized Sum: The Second Part

Make use of the identity in (4.16), the case $m = 2$.

5.35 Awesomely Wicked Sums of Series of Weight 7, $\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4}$, $\sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(3)}}{n^2}$, and $\sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^3}$, Derivation Based upon a New Identity: The Third Part

Use the identity in (4.17).

5.36 Deriving More Useful Sums of Harmonic Series of Weight 7

For the first part of the problem apply Abel's summation, and for the last two points of the section check carefully the other sections in the nearby.

5.37 Preparing the *Weapons* of *The Master Theorem of Series* to Breach the Fortress of the Challenging Harmonic Series of Weight 7: The 1st Episode

Use the first equality of the result in (4.26).

5.38 Preparing the *Weapons* of *The Master Theorem of Series* to Breach the Fortress of the Challenging Harmonic Series of Weight 7: The 2nd Episode

Make use of the first equality of the result in (4.26), multiply both sides by H_n/n^2, and then sum from $n = 1$ to ∞.

5.39 Preparing the *Weapons* of *The Master Theorem of Series* to Breach the Fortress of the Challenging Harmonic Series of Weight 7: The 3rd Episode

Make use of the first equality of the identity in (4.27).

5.40 Calculating the Harmonic Series of Weight 7, $\sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^3}$, with the *Weapons* of *The Master Theorem of Series*

Combine fruitfully the identities from (4.51) and (4.52).

5.41 The Calculation of Two Good-Looking Pairs of Harmonic Series: The Series $\sum_{n=1}^{\infty} \frac{H_n}{n^2} \sum_{k=1}^{n} \frac{H_k}{k^3}$, $\sum_{n=1}^{\infty} \frac{H_n}{n^3} \sum_{k=1}^{n} \frac{H_k}{k^2}$ and $\sum_{n=1}^{\infty} \frac{H_n^2}{n^2} \sum_{k=1}^{n} \frac{H_k}{k^2}$, $\sum_{n=1}^{\infty} \frac{H_n}{n^2} \sum_{k=1}^{n} \frac{H_k^2}{k^2}$

For the first two points of the problem, exploit the identity in (4.21), the case $m = 2$, and for the last two points, use the result in (4.24).

5.42 The Calculation of an Essential Harmonic Series of Weight 7: The Series $\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4}$

For a first solution make use of the identity in (4.19). Then, for a second solution, make use of the identity in (4.18).

5.43 Plenty of Challenging Harmonic Series of Weight 7 Obtained by Combining the Previous Harmonic Series of Weight 7 with Various Harmonic Series Identities (Derivations by Series Manipulations Only)

For $i)$ employ the identity in (4.46), then to get $ii)$ use the identity in (4.41), for $iii)$ consider the identity in (4.50), as regards $iv)$ employ the identity in (4.49), for $v)$ consider the identity in (4.45), to obtain $vi)$ use the identity in (4.44), to get $vii)$ make use of the identity in (4.48), regarding $viii)$ employ the identity in (4.20), for $ix)$ consider the identity (4.28), and finally, to get $x)$, use the identity in (4.27).

5.44 A Member of a Glamorous Series Family Containing the Harmonic Number and the Tail of the Riemann Zeta Function

For a first solution consider Abel's summation as a starting point. Then, for a second solution, use a generalization from one of the next sections.

5.45 More Members of a Glamorous Series Family Containing the Harmonic Number and the Tail of the Riemann Zeta Function

Consider the use of Abel's summation and follow the same way as in the previous section. For a second solution, consider a result in the next section.

5.46 Two Series Generalizations with the Generalized Harmonic Numbers and the Tail of the Riemann Zeta Function

Use that

$$\zeta(p)-1-\frac{1}{2^p}-\cdots-\frac{1}{n^p}=\sum_{k=1}^{\infty}\frac{1}{(n+k)^p}=\frac{(-1)^{p-1}}{(p-1)!}\sum_{k=1}^{\infty}\int_0^1 x^{k+n-1}\log^{p-1}(x)\mathrm{d}x.$$

5.47 The Art of Mathematics with a Series Involving the Product of the Tails of $\zeta(2)$ and $\zeta(3)$

Apply Abel's summation (see (5.1)).

5.48 The Art of Mathematics with Another Splendid Series Involving the Product of the Tails of $\zeta(2)$ and $\zeta(3)$

Make use of Abel's summation applied to the series $\displaystyle\sum_{n=1}^{\infty}\frac{H_n^2(\zeta(2)-H_n^{(2)})}{n^3}$.

5.49 Expressing Polylogarithmic Values by Combining the Alternating Harmonic Series and the Non-alternating Harmonic Series with Integer Powers of 2 in Denominator

Make use of the result in (1.59).

5.50 Cool Results with Cool Series Involving Summands with the Harmonic Number and the Integer Powers of 2

Exploit the identities $\int_0^{1/2} x^k \log(x)\mathrm{d}x = -\frac{\log(2)}{(k+1)2^{k+1}} - \frac{1}{(k+1)^2 2^{k+1}}$ and $\int_0^{1/2} x^k \log^2(x)\mathrm{d}x = \frac{\log^2(2)}{(k+1)2^{k+1}} + \frac{\log(2)}{(k+1)^2 2^k} + \frac{1}{(k+1)^3 2^k}$.

5.51 Eight Harmonic Series Involving the Integer Powers of 2 in Denominator

Make use of the relations in (4.74), (4.75), and (4.76).

5.52 Let's Calculate Three Classical Alternating Harmonic Series, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^3}$, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(2)}}{n^2}$, and $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2}{n^2}$

Reduce the first series to the calculation of the integral $\frac{1}{2}\int_0^1 \frac{\log(1+x)\log^2(x)}{x(1+x)}\mathrm{d}x$. Then, for the second series combine the result in (1.53) with the value of the series from the previous point, and for the third series reduce everything to the calculation of the integral $\int_0^1 \frac{\log(x)}{x(1+x)}(\log^2(1+x) + \mathrm{Li}_2(-x))\mathrm{d}x$.

5.53 Then, Let's Calculate Another Pair of Classical Alternating Harmonic Series, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^2}$ and $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^4}$

Reduce the first series to the calculation of the integral $\displaystyle\int_0^1 \frac{\log^2(1+x)}{x}\mathrm{d}x$, and the second series to the calculation of the integral $\displaystyle\int_0^1 \frac{\log(1+x)\log^3(x)}{x(1+x)}\mathrm{d}x$.

5.54 A Nice Challenging Trio of Alternating Harmonic Series, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(2)}}{n^3}$, $\sum_{n=1}^{\infty}(-)^{n-1}\frac{H_n^{(3)}}{n^2}$, and $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(4)}}{n}$

For the first two points of the section, exploit the result in (1.4) and the Cauchy product of two Polylogarithms to create a system of relations with the given series. For the last point, make use of the Cauchy product of $\operatorname{Li}_4(x)$ and $-\log(1-x)$.

5.55 Encountering an Alternating Harmonic Series of Weight 5 with an Eye-Catching Closed-Form, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2}{n^3}$

Make use of the fact that, $\displaystyle\sum_{n=1}^{\infty} x^n \left(H_n^2 - H_n^{(2)}\right) = \frac{\log^2(1-x)}{1-x}$.

5.56 Encountering Another Alternating Harmonic Series of Weight 5 with a Dazzling Closed-Form, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^3}{n^2}$

Make use of the result in (1.6).

5.57 Yet Another Encounter with a Superb Alternating Harmonic Series of Weight 5, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n H_n^{(2)}}{n^2}$

Exploit the result in (1.5).

5.58 Fascinating Sums of Two Alternating Harmonic Series Involving the Generalized Harmonic Number

For the part $i)$ of the problem, consider the use of the result $\displaystyle\sum_{n=1}^{\infty} x^n\left(H_n^2 - H_n^{(2)}\right) = \frac{\log^2(1-x)}{1-x}$. Then, for the point $ii)$ of the problem reduce the calculations to one of the integrals from the chapter *Integrals* where we can successfully exploit the symmetry.

5.59 An Outstanding Sum of Series Representation of the Particular Value of the Riemann Zeta Function, $\zeta(4)$

Use the first application of *The Master Theorem of Series* in (4.21), the case $m = 1$.

5.60 An Excellent Representation of the Particular Value of the Riemann Zeta Function, $\zeta(4)$, with a Triple Series Involving the Factorials and the Generalized Harmonic Numbers

Prove and use that

$$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty}\frac{\Gamma(i)\Gamma(j)\Gamma(x)}{\Gamma(i+j+x)}\right) = \frac{1}{2}\left(\psi^{(1)}\left(\frac{x}{2}\right) - \psi^{(1)}\left(\frac{1+x}{2}\right)\right).$$

Chapter 6
Solutions

6.1 The First Series Submitted by Ramanujan to the Journal of the Indian Mathematical Society

Solution Let's open the sixth chapter of the book with a series proposed by the famous mathematician Srinivasa Ramanujan in his first problem from a total of 58 problems submitted to the *Journal of the Indian Mathematical Society* between the years 1911 and 1919 (see [8]).

This is another problem that I was very glad to consider together with the solution in my first book proposal at Springer in 2015, since it allows us to finalize it exceptionally beautifully by series manipulations only, with no use of integrals at all. It's one of the easiest problems in this chapter (a high school mathematics background would be enough to calculate it), and to understand and appreciate its beauty I would recommend to consider the proposed *challenging question* that asks you to calculate it without using integrals.

Since we have that $\dfrac{1}{(4n)^3-4n} = \dfrac{1}{2}\left(\dfrac{1}{4n-1}+\dfrac{1}{4n+1}-\dfrac{1}{2n}\right)$ by partial fraction decomposition, we write

$$\sum_{n=1}^{\infty}\frac{1}{(4n)^3-4n} = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{4n-1}+\frac{1}{4n+1}-\frac{1}{2n}\right)$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{4n-1}+\frac{1}{4n+1}-\frac{1}{4n}-\frac{1}{4n}\right)$$

$$\{\text{add and subtract the term } 1/(4n+2)\}$$

C. I. Vălean, *(Almost) Impossible Integrals, Sums, and Series*, Problem Books in Mathematics, https://doi.org/10.1007/978-3-030-02462-8_6

$$= \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{4n-1}-\frac{1}{4n}+\frac{1}{4n+1}-\frac{1}{4n+2}-\frac{1}{4n}+\frac{1}{4n+2}\right)$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{4n-1}-\frac{1}{4n}+\frac{1}{4n+1}-\frac{1}{4n+2}\right)-\frac{1}{4}\sum_{n=1}^{\infty}\left(\frac{1}{2n}-\frac{1}{2n+1}\right). \tag{6.1}$$

Then, noting that $\sum_{n=1}^{\infty}\left(\frac{1}{4n-1}-\frac{1}{4n}+\frac{1}{4n+1}-\frac{1}{4n+2}\right)=\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots=\log(2)-\frac{1}{2}$ and $\sum_{n=1}^{\infty}\left(\frac{1}{2n}-\frac{1}{2n+1}\right)=\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\cdots=1-\log(2)$, where I used the simple fact that $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}=\log(2)$, and plugging these results in (6.1), we get that

$$\sum_{n=1}^{\infty}\frac{1}{(4n)^3-4n}=-\frac{1}{2}+\frac{3}{4}\log(2),$$

which we can write in the expected form,

$$1+2\sum_{n=1}^{\infty}\frac{1}{(4n)^3-4n}=\frac{3}{2}\log(2),$$

and the solution is finalized.

To conclude, all we had to do was to skillfully use two different series representations of $\log(2)$, after employing the partial fraction expansion.

Now that we had to deal with conditionally convergent series $\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots\right)$, it is worth mentioning that an important point that requires a special attention is the *Riemann series theorem* (see [59]) which says that if an infinite series of real numbers is conditionally convergent, by a suitable rearrangements of terms, it can be made to converge to any desired value, or to diverge. In our approach we carefully considered this point and made the rearrangements only in ways that preserve the value of the initial series.

6.2 Starting from an Elementary Integral Result and Deriving Two Classical Series in a New Way

Solution *I have seen these classical results in the book Irresistible Integrals by George Boros and Victor H. Moll!*, the kind of reaction I would expect to receive in this section from the lovers of calculations with integrals that read the book on integrals I just mentioned. That's right, both results appear and are proved in the mentioned book (see [9, Theorem 6.6.2, pp. 119–120] and [9, Exercise 6.6.4, p. 122]).

Now, let's try to get first an alternative solution by exploiting again the power of the integral in Sect. 1.1 from the first chapter, which I successfully used for solving the (famous) Basel problem.

To derive the stated results I'll employ the integral result in (1.1), where upon using the identity $\arcsin(x) + \arccos(x) = \dfrac{\pi}{2}$, we obtain

$$\int_0^1 \frac{1}{(1+yx)\sqrt{1-x^2}}\mathrm{d}x = \frac{\pi}{2\sqrt{1-y^2}} - \frac{\arcsin(y)}{\sqrt{1-y^2}}. \tag{6.2}$$

Returning to the left-hand side of the result in (6.2), we can write

$$\int_0^1 \frac{1}{(1+yx)\sqrt{1-x^2}}\mathrm{d}x = \int_0^1 \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(xy)^{n-1}}{\sqrt{1-x^2}}\mathrm{d}x$$

$$\{\text{reverse the order of summation and integration}\}$$

$$= \sum_{n=1}^{\infty}(-1)^{n-1}y^{n-1}\int_0^1 \frac{x^{n-1}}{\sqrt{1-x^2}}\mathrm{d}x \overset{x=\sin(t)}{=} \sum_{n=1}^{\infty}(-1)^{n-1}y^{n-1}\int_0^{\pi/2} \sin^{n-1}(t)\mathrm{d}t$$

$$\{\text{split the series according to } n \text{ odd and even}\}$$

$$= \sum_{n=1}^{\infty} y^{2n-2}\int_0^{\pi/2}\sin^{2n-2}(t)\mathrm{d}t - \sum_{n=1}^{\infty} y^{2n-1}\int_0^{\pi/2}\sin^{2n-1}(t)\mathrm{d}t$$

$$\left\{\text{use Wallis' integral results, } \int_0^{\pi/2}\sin^{2n}(t)\mathrm{d}t = \frac{\pi}{2^{2n+1}}\binom{2n}{n}\right\}$$

$$\left\{\text{and then } \int_0^{\pi/2}\sin^{2n+1}(t)\mathrm{d}t = \frac{2^{2n}}{2n+1}\frac{1}{\dbinom{2n}{n}}\right\}$$

$$= \frac{\pi}{2}\sum_{n=1}^{\infty}\binom{2n-2}{n-1}\left(\frac{y}{2}\right)^{2n-2} - \sum_{n=1}^{\infty} y^{2n-1}\frac{2^{2n-2}}{2n-1}\frac{1}{\binom{2n-2}{n-1}}$$

$$\{\text{reindex the first series and start from } n=0, \text{ and for the second}\}$$

$$\left\{\text{series employ the simple fact that } \binom{2n-2}{n-1} = \frac{n}{2(2n-1)}\binom{2n}{n}\right\}$$

$$= \frac{\pi}{2}\sum_{n=0}^{\infty}\binom{2n}{n}\left(\frac{y}{2}\right)^{2n} - \sum_{n=1}^{\infty}\frac{(2y)^{2n-1}}{n\binom{2n}{n}}$$

$$\{\text{for the first series recall the generating function of the central binomial}\}$$

$$\left\{\text{coefficients, that is } \sum_{n=0}^{\infty}\binom{2n}{n}y^n = \frac{1}{\sqrt{1-4y}} \text{ where we replace } y \text{ by } y^2/4\right\}$$

$$= \frac{\pi}{2\sqrt{1-y^2}} - \sum_{n=1}^{\infty}\frac{(2y)^{2n-1}}{n\binom{2n}{n}}. \tag{6.3}$$

By combining the results from (6.2) and (6.3), we obtain that

$$\sum_{n=1}^{\infty}\frac{(2x)^{2n-1}}{n\binom{2n}{n}} = \frac{\arcsin(x)}{\sqrt{1-x^2}},$$

and the first solution to the point $i)$ of the problem is finalized (as regards the Wallis' integral results, see [9, p. 113] and [23, **3.621.3–3.621.4**, p. 397]).

For a second solution to the point $i)$ of the problem, we'll want to use the Beta function and the identity connecting the Beta function and Gamma function, $\mathrm{B}(x,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, and then we write

$$\sum_{n=1}^{\infty}\frac{(2x)^{2n-1}}{n\binom{2n}{n}} = \sum_{n=0}^{\infty}\frac{(2x)^{2n+1}}{(n+1)\binom{2n+2}{n+1}} = \sum_{n=0}^{\infty}\frac{2^{2n}x^{2n+1}(n!)^2}{(2n+1)!}$$

$$\left\{\text{use the fact that } \int_0^1 t^n(1-t)^n\mathrm{d}t = \mathrm{B}(n+1,n+1) = \frac{\Gamma(n+1)^2}{\Gamma(2n+2)} = \frac{(n!)^2}{(2n+1)!}\right\}$$

$$= \sum_{n=0}^{\infty} 2^{2n} x^{2n+1} \int_0^1 t^n (1-t)^n \mathrm{d}t = \int_0^1 \sum_{n=0}^{\infty} 2^{2n} x^{2n+1} t^n (1-t)^n \mathrm{d}t$$

$$= \int_0^1 \frac{x}{1 - 4x^2 t(1-t)} \mathrm{d}t$$

$$\overset{(1+u)/2=t}{=} \frac{1}{2} \int_{-1}^1 \frac{x}{1 - x^2 + x^2 u^2} \mathrm{d}u = \int_0^1 \frac{x}{1 - x^2 + x^2 u^2} \mathrm{d}u$$

$$= \frac{1}{\sqrt{1-x^2}} \arctan\left(\frac{x}{\sqrt{1-x^2}}\right)$$

$$= \frac{\arcsin(x)}{\sqrt{1-x^2}},$$

and for the last equality I used the trigonometric identity, $\arcsin(x) = \arctan\left(\frac{x}{\sqrt{1-x^2}}\right)$, and the second solution to the point $i)$ of the problem is finalized.

To get the series from the point $ii)$, we consider the series from the point $i)$ in y which we integrate from $y = 0$ to $y = x$, and then we get

$$\int_0^x \sum_{n=1}^{\infty} \frac{(2y)^{2n-1}}{n\binom{2n}{n}} \mathrm{d}y = \sum_{n=1}^{\infty} \int_0^x \frac{(2y)^{2n-1}}{n\binom{2n}{n}} \mathrm{d}y = \frac{1}{4} \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{n^2\binom{2n}{n}} = \int_0^x \frac{\arcsin(y)}{\sqrt{1-y^2}} \mathrm{d}y$$

$$= \frac{1}{2} \arcsin^2(x),$$

whence we obtain that

$$\frac{1}{2} \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{n^2\binom{2n}{n}} = \arcsin^2(x),$$

and the part $ii)$ of the problem is finalized.

In some problems we might find these series representations very useful, and one such example we'll meet right in the next section where we'll want to make use of the result from the point $ii)$ of the problem. For the series representations of the inverse sine functions with higher powers, you may consult [3, p. 226].

6.3 An Extraordinary Series with the Tail of the Riemann Zeta Function Connected to the Inverse Sine Series

Solution With the picture of the results from the previous section in mind, it's not hard to guess what we might need in the calculation of the present series. Also, the connection with the first integral in Sect. 1.52 will be shown, which will help us to finalize the last step of the solution.

Let's consider writing the tail of the Riemann zeta function in terms of Trigamma function (see [61]) and then express the latter function in terms of the integral representation, $\psi^{(1)}(z) = -\int_0^1 \frac{t^{z-1}}{1-t}\log(t)\mathrm{d}t$ (see [23, **4.251.4**, p. 543]). Therefore, we write that

$$\sum_{n=1}^{\infty} \frac{2^{2n}}{n^2\binom{2n}{n}}\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right) = \sum_{n=1}^{\infty} \frac{2^{2n}}{n^2\binom{2n}{n}}\psi^{(1)}(n+1)$$

$$= -\sum_{n=1}^{\infty} \frac{2^{2n}}{n^2\binom{2n}{n}}\int_0^1 \frac{t^n}{1-t}\log(t)\mathrm{d}t \overset{t=x^2}{=} -\sum_{n=1}^{\infty} \frac{2^{2n+2}}{n^2\binom{2n}{n}}\int_0^1 \frac{x^{2n+1}}{1-x^2}\log(x)\mathrm{d}x$$

{reverse the order of integration and summation}

$$= -2\int_0^1 \frac{\log(x)}{1-x^2}\sum_{n=1}^{\infty} \frac{(2x)^{2n+1}}{n^2\binom{2n}{n}}\mathrm{d}x$$

{employ the inverse sine series in (4.2)}

$$= -8\int_0^1 \frac{x\log(x)\arcsin^2(x)}{1-x^2}\mathrm{d}x \overset{x=\sin(y)}{=} -8\int_0^{\pi/2} y^2\tan(y)\log(\sin(y))\mathrm{d}y$$

{make use of the result in (1.68)}

$$= \frac{17}{2}\zeta(4) - 4\log^2(2)\zeta(2) - \frac{1}{3}\log^4(2) - 8\operatorname{Li}_4\left(\frac{1}{2}\right),$$

and the solution is complete.

Surely, the hard part of the problem is represented by the calculation of the last integral which happily is already evaluated in the third chapter.

6.4 The Evaluation of a Series Involving the Tails of the Series Representations of the Functions $\log\left(\frac{1}{1-x}\right)$ and $\frac{x\arcsin(x)}{\sqrt{1-x^2}}$

Solution Aesthetically speaking, one might easily assert we have in front of our eyes a daunting series, one involving a summand with the product of the tails of two functions, and then all multiplied by a fraction with double factorials.

Despite such a first impression characterized by the adjective *daunting*, the evaluation strategy of the series is pretty simple once we have in hand the *brick* results I employed in Sect. 3.32, and this is what we'll use in the present solution.

Based upon the results given in (3.179) and (3.182), we write that

$$\sum_{n=1}^{\infty}\frac{(2n-3)!!}{(2n-2)!!}\left(\frac{1}{2}\log(2)-\sum_{k=1}^{n-1}\frac{1}{k2^{k+1}}\right)\left(\frac{\pi}{4}-\sum_{k=1}^{n-1}\frac{(2k-2)!!}{2^k(2k-1)!!}\right)$$

$$=\sum_{n=1}^{\infty}\int_0^1\left(\int_0^1\frac{(xy)^{2n-2}y}{((1+x^2)(1+y^2))^n}\mathrm{d}x\right)\mathrm{d}y$$

$$=\int_0^1\left(\int_0^1\sum_{n=1}^{\infty}\frac{(xy)^{2n-2}y}{((1+x^2)(1+y^2))^n}\mathrm{d}y\right)\mathrm{d}x$$

$$=\int_0^1\left(\int_0^1\frac{y}{1+x^2+y^2}\mathrm{d}y\right)\mathrm{d}x=\frac{1}{2}\int_0^1(\log(2+x^2)-\log(1+x^2))\mathrm{d}x$$

$$=\frac{1}{2}\int_0^1 x'(\log(2+x^2)-\log(1+x^2))\mathrm{d}x$$

$$\{\text{apply the integration by parts}\}$$

$$=\underbrace{\frac{1}{2}x(\log(2+x^2)-\log(1+x^2))\Big|_{x=0}^{x=1}}_{1/2\log(3/2)}+2\int_0^1\frac{1}{2+x^2}\mathrm{d}x-\int_0^1\frac{1}{1+x^2}\mathrm{d}x$$

$$=\frac{1}{2}\log\left(\frac{3}{2}\right)+\sqrt{2}\arctan\left(\frac{x}{\sqrt{2}}\right)\Big|_{x=0}^{x=1}-\arctan(x)\Big|_{x=0}^{x=1}$$

$$=\sqrt{2}\arctan\left(\frac{1}{\sqrt{2}}\right)+\frac{1}{2}\log\left(\frac{3}{2}\right)-\frac{\pi}{4},$$

and the solution is finalized.

In this section we saw that sometimes using integral representations of the summand is the key for getting a short, elegant solution. Usually such representations come from the experience gained in the work with lots of integrals and series.

6.5 A Breathtaking Infinite Series Involving the Binomial Coefficient and Expressing a Beautiful Closed-Form

Solution The thing that is not immediately obvious is that one of the results used in the previous section may also be of great help for the current problem, and we see this after bringing it into the form with the central binomial coefficient. It's about the result in (3.182) that can be turned into the useful form already used in (3.173) from the third chapter, Sect. 3.31.

Considering the use of Wallis' integral, $\int_0^{\pi/2} \sin^{2n}(t)\mathrm{d}t = \frac{\pi}{2^{2n+1}}\binom{2n}{n}$, we have

$$\sum_{n=1}^{\infty} \frac{n}{(2n-1)16^n}\binom{2n}{n}^2 \left(\sum_{k=n}^{\infty} \frac{2^k}{k\binom{2k}{k}}\right)$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\int_0^{\pi/2} \frac{n\sin^{2n}(y)}{(2n-1)2^{2n}}\binom{2n}{n}\left(\sum_{k=n}^{\infty} \frac{2^k}{k\binom{2k}{k}}\right)\mathrm{d}y$$

$$\{\text{make use of the result in (3.173)}\}$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\int_0^{\pi/2}\left(\int_0^1 \frac{x^{2n-2}\sin^{2n}(y)}{(1+x^2)^n}\mathrm{d}x\right)\mathrm{d}y$$

$$\{\text{reverse the order of integration and summation}\}$$

$$= \frac{2}{\pi}\int_0^1\left(\int_0^{\pi/2}\sum_{n=1}^{\infty} \frac{x^{2n-2}\sin^{2n}(y)}{(1+x^2)^n}\mathrm{d}y\right)\mathrm{d}x$$

$$= \frac{2}{\pi}\int_0^1\left(\int_0^{\pi/2} \frac{\sin^2(y)}{1+x^2\cos^2(y)}\mathrm{d}y\right)\mathrm{d}x$$

$$= \frac{2}{\pi}\int_0^1\left(\int_0^{\pi/2} \frac{1-\cos^2(y)}{1+x^2\cos^2(y)}\mathrm{d}y\right)\mathrm{d}x$$

$$= \frac{2}{\pi}\int_0^1 \left(\left(1+\frac{1}{x^2}\right)\int_0^{\pi/2}\frac{1}{1+x^2\cos^2(y)}\mathrm{d}y - \frac{1}{x^2}\left(\int_0^{\pi/2}\mathrm{d}y\right)\right)\mathrm{d}x$$

{make use of the integral result in (3.233)}

$$= \int_0^1 \left(\left(1+\frac{1}{x^2}\right)\frac{1}{\sqrt{1+x^2}} - \frac{1}{x^2}\right)\mathrm{d}x = \left(\frac{1}{x} - \frac{\sqrt{1+x^2}}{x} + \operatorname{arcsinh}(x)\right)\Bigg|_{x=0}^{x=1}$$

$$= 1 - \sqrt{2} + \operatorname{arcsinh}(1)$$

$$\{\text{use that } \operatorname{arcsinh}(x) = \log(x+\sqrt{1+x^2})\}$$

$$= 1 - \sqrt{2} + \log(1+\sqrt{2}),$$

and the solution is complete. Note in the calculations some simple, elementary integrals[1] have been used.

As in the previous section, the solution emphasizes again the power of using the proper integral representations for expressing the summand.

6.6 An Eccentric Multiple Series Having the Roots in the Realm of the Botez–Catalan Identity

Solution Let's prepare now for an encounter with an appealing challenging multiple series. *How about the strategies to use here?* Basically, the plan is to first turn the multiple series into an integral, and then try to calculate the integral and finish the solution. Having said that, for a good start we may employ Botez–Catalan identity (you may see it stated below) which one may find very useful in many cases.

For example, a simple classical problem is to use the mentioned identity to prove that $1 - \frac{1}{2} + \frac{1}{3} - \cdots = \log(2)$, and such a solution[2] is fast and elegant.

---

[1] It's easy to observe that $\displaystyle\int \frac{1}{x^2\sqrt{1+x^2}}\mathrm{d}x = \int \frac{\sqrt{1+x^2}}{x^2}\mathrm{d}x - \int \frac{1}{\sqrt{1+x^2}}\mathrm{d}x$, and since we have by the integration by parts that $\displaystyle\int \frac{\sqrt{1+x^2}}{x^2}\mathrm{d}x = -\frac{\sqrt{1+x^2}}{x} + \int \frac{1}{\sqrt{1+x^2}}\mathrm{d}x$, we get immediately that $\displaystyle\int \frac{1}{x^2\sqrt{1+x^2}}\mathrm{d}x = -\frac{\sqrt{1+x^2}}{x} + C$. Alternatively, we could simply use the change of variable $x = \sinh(y)$.

[2] Let's recollect the limit definition of the Euler–Mascheroni constant, $\lim_{n\to\infty}(H_n - \log(n)) = \gamma$, and then, by Botez–Catalan identity, we have $\displaystyle\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k} = \lim_{n\to\infty}\sum_{k=1}^{2n}(-1)^{k-1}\frac{1}{k} =$

Recollecting and using Botez–Catalan identity,

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{2n}=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n},\ \forall n\geq 1,$$

and replacing n by $\displaystyle\sum_{i=1}^{n} k_i$, we get

$$\frac{1}{\left(\displaystyle\sum_{i=1}^{n} k_i\right)+1}+\frac{1}{\left(\displaystyle\sum_{i=1}^{n} k_i\right)+2}+\cdots+\frac{1}{\left(\displaystyle\sum_{i=1}^{n} k_i\right)+\displaystyle\sum_{i=1}^{n} k_i}=\sum_{k=1}^{\sum_{i=1}^{n} k_i}\frac{1}{\left(\displaystyle\sum_{i=1}^{n} k_i\right)+k}$$

$$=1-\frac{1}{2}+\frac{1}{3}-\cdots-\frac{1}{2\displaystyle\sum_{i=1}^{n} k_i}. \tag{6.4}$$

Now, combining the result in (6.4) with the fact that $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}=\log(2)$, we get that

$$\sum_{k_1=1}^{\infty}\left(\sum_{k_2=1}^{\infty}\left(\cdots\sum_{k_n=1}^{\infty}(-1)^{\sum_{i=1}^{n} k_i}\left(\log(2)-\sum_{k=1}^{\sum_{i=1}^{n} k_i}\frac{1}{\left(\displaystyle\sum_{i=1}^{n} k_i\right)+k}\right)\right)\cdots\right)$$

$$=\sum_{k_1=1}^{\infty}\left(\sum_{k_2=1}^{\infty}\left(\cdots\sum_{k_n=1}^{\infty}(-1)^{\sum_{i=1}^{n} k_i}\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2\left(\displaystyle\sum_{i=1}^{n} k_i\right)+k}\right)\right)\cdots\right)$$

$$=\sum_{k_1=1}^{\infty}\left(\sum_{k_2=1}^{\infty}\left(\cdots\sum_{k_n=1}^{\infty}(-1)^{\sum_{i=1}^{n} k_i}\left(\sum_{k=1}^{\infty}(-1)^{k-1}\int_0^1 x^{2(\sum_{i=1}^{n} k_i)+k-1}\mathrm{d}x\right)\right)\cdots\right)$$

$\lim_{n\to\infty}(H_{2n}-H_n)=\lim_{n\to\infty}(H_{2n}-\log(2n)-(H_n-\log(n))+\log(2))=\log(2)$, and the calculations are finalized.

$$= \int_0^1 \sum_{k_1=1}^{\infty} \left(\sum_{k_2=1}^{\infty} \left(\cdots \sum_{k_n=1}^{\infty} (-1)^{\sum_{i=1}^n k_i} \left(\sum_{k=1}^{\infty} (-1)^{k-1} x^{2(\sum_{i=1}^n k_i)+k-1} \right) \right) \cdots \right) \mathrm{d}x$$

$$= \int_0^1 \sum_{k_1=1}^{\infty} \left(\sum_{k_2=1}^{\infty} \left(\cdots \sum_{k_n=1}^{\infty} (-1)^{\sum_{i=1}^n k_i} \frac{x^{2(\sum_{i=1}^n k_i)}}{1+x} \right) \cdots \right) \mathrm{d}x$$

$$= \int_0^1 \frac{1}{1+x} \sum_{k_1=1}^{\infty} (-1)^{k_1} x^{2k_1} \sum_{k_2=1}^{\infty} (-1)^{k_2} x^{2k_2} \cdots \sum_{k_n=1}^{\infty} (-1)^{k_n} x^{2k_n} \mathrm{d}x$$

$$= (-1)^n \int_0^1 \frac{x^{2n}}{(1+x)(1+x^2)^n} \mathrm{d}x$$

{make use of the result in (1.60)}

$$= (-1)^n \left(\frac{1}{2} \log(2) + \frac{1}{2^{n+1}} \log(2) + \frac{H_n}{2^{n+1}} - \sum_{i=1}^{n} \frac{1}{i 2^{i+1}} - \frac{\pi}{2^{n+2}} \sum_{j=0}^{n-1} \frac{1}{2^j} \binom{2j}{j} \right.$$

$$\left. + \frac{1}{2^{n+1}} \sum_{j=1}^{n-1} \frac{1}{2^j} \binom{2j}{j} \sum_{i=1}^{j} \frac{2^i}{i \binom{2i}{i}} \right),$$

and the solution is finalized.

Even after reducing all to the calculation of the integral above, the *challenging* work is still there since the resulting integral requires a careful approach with recurrence relations (as you may see in the third chapter). The case $n = 2$ also appears as a problem in [29, 3.107, p. 158].

6.7 Two Classical Series with Fibonacci Numbers, One Related to the Arctan Function

Solution *A close look at the Fibonacci sequence reveals that it has a fascinating property: every Fibonacci number, except the first two, is the sum of the two immediately preceding Fibonacci numbers.* - Thomas Koshy in *Fibonacci and Lucas Numbers with Applications*.

In this section (and the next one) we are found on the realm of the intriguing Fibonacci numbers. More specifically, we'll approach two classical series with Fibonacci numbers, preparing the ground for the challenge in the next section.

Let's use the recurrence relation $F_n = F_{n-1} + F_{n-2}$ with $F_1 = 1$ and $F_2 = 1$, and note that $\left|\frac{(-1)^{n-1}}{F_n F_{n+1}}\right| = \frac{1}{F_n F_{n+1}} \le \frac{1}{n(n+1)}$ shows the series is absolutely convergent.

Then, if we make use of Cassini's identity,[3] $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$, our series becomes

$$S_1 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+1}} = \sum_{n=1}^{\infty} \frac{F_n F_{n+2} - F_{n+1}^2}{F_n F_{n+1}} = \lim_{N\to\infty} \sum_{n=1}^{N} \left(\frac{F_{n+2}}{F_{n+1}} - \frac{F_{n+1}}{F_n}\right)$$

$$= \lim_{N\to\infty} \left(\frac{F_{N+2}}{F_{N+1}} - \frac{F_2}{F_1}\right) = \phi - 1,$$

that is precisely the desired limit.

The limit, $l = \lim_{N\to\infty} \frac{F_{N+2}}{F_{N+1}}$, which I used above, exists because the series converges as seen at the beginning of the solution.

Thus,

$$\lim_{N\to\infty} \frac{F_{N+2}}{F_{N+1}} = \lim_{N\to\infty} \frac{F_{N+1} + F_N}{F_{N+1}} = \lim_{N\to\infty} \left(1 + \frac{1}{\frac{F_{N+1}}{F_N}}\right) = 1 + \frac{1}{\lim_{N\to\infty} \frac{F_{N+1}}{F_N}},$$

whence the limit is the positive root of $l^2 - l - 1 = 0$, that is $l = \frac{1}{2}\left(1 + \sqrt{5}\right) = \phi$, and the solution to the series S_1 is finalized. The series together with a solution may also be found in [5]. As a fact, one might also like to notice that when replacing the Fibonacci numbers with the Lucas numbers (see [55]) under the limit above, the value of the limit remains the same as stated in [27, p. 36].

To calculate S_2, we first consider the partial sum, that is

$$\sum_{n=-N}^{n=N} \arctan\left(\frac{1}{F_{2n-1}}\right) = \sum_{n=-N}^{N} \arctan\left(\frac{F_{2n-1}}{F_{2n-1}^2}\right). \tag{6.5}$$

Using Cassini's identity, $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$, where we multiply both sides by (-1) and then replace n by $2n-1$, we have

[3] To prove the identity by induction we first see it holds for $n = 2$. Supposing that $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$ is true, then we want to prove that also $F_n F_{n+2} - F_{n+1}^2 = (-1)^{n-1}$ is true. So, using that $F_n = F_{n-1} + F_{n-2}$, we have $F_n F_{n+2} - F_{n+1}^2 = F_n(F_{n+1} + F_n) - F_{n+1}(F_n + F_{n-1}) = F_n F_{n+1} + F_n^2 - F_{n+1}F_n - F_{n+1}F_{n-1} = F_n^2 - F_{n+1}F_{n-1} = (-1)^{n-1}$.

$$F_{2n-1}^2 = 1 + F_{2n}F_{2n-2}. \tag{6.6}$$

On the other hand, from the recurrence relation of the Fibonacci numbers, $F_n = F_{n-1} + F_{n-2}$, where we replace n by $2n$, we get

$$F_{2n-1} = F_{2n} - F_{2n-2}. \tag{6.7}$$

Thus, continuing the work in (6.5) where we use the relations in (6.6) and (6.7), we have

$$\sum_{n=-N}^{N} \arctan\left(\frac{F_{2n-1}}{F_{2n-1}^2}\right) = \sum_{n=-N}^{N} \arctan\left(\frac{F_{2n} - F_{2n-2}}{1 + F_{2n}F_{2n-2}}\right)$$

$$\left\{\text{use the trigonometric identity, } \arctan(a) - \arctan(b) = \arctan\left(\frac{a-b}{1+ab}\right)\right\}$$

$$= \sum_{n=-N}^{N} \left(\arctan\left(F_{2n}\right) - \arctan\left(F_{2n-2}\right)\right)$$

$$\{\text{all but the first and last terms cancel}\}$$

$$= \arctan\left(F_{2N}\right) - \arctan\left(F_{-2N-2}\right). \tag{6.8}$$

Note that $\lim_{N\to\infty} F_{2N} = \infty$, and then based upon Binet's formula, $F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio, which allows us the extension to the negative integers, we get

$$\lim_{N\to\infty} F_{-2N-2} = \frac{\varphi^{-2N-2} - (-\varphi)^{2N+2}}{\sqrt{5}} = -\infty.$$

Hence, letting $N \to \infty$ in (6.8), we conclude that

$$S_2 = \lim_{N\to\infty} \left(\arctan\left(F_{2N}\right) - \arctan\left(F_{-2N-2}\right)\right) = \left(\arctan\left(\infty\right) - \arctan\left(-\infty\right)\right) = \pi,$$

and the solution to the series S_2 is finalized. A very similar series together with a solution that uses the same telescoping idea may be found in [24].

6.8 Two New Infinite Series with Fibonacci Numbers, Related to the Arctan Function

Solution The series with Fibonacci numbers from the first point I submitted to *The American Mathematical Monthly*, the problem **11910** (see [48]). In the solution we'll see the series actually reduces to a sum of two series, a classical one and a new one (which, in fact, was the series I planned to submit initially), where the difficult part is represented by the evaluation of the latter series.

Splitting the initial series into two series, we arrive at

$$\sum_{n=1}^{\infty}\left(\arctan\left(\frac{1}{F_{4n-3}}\right)+\arctan\left(\frac{1}{F_{4n-2}}\right)+\arctan\left(\frac{1}{F_{4n-1}}\right)-\arctan\left(\frac{1}{F_{4n}}\right)\right)$$

$$=\underbrace{\sum_{n=1}^{\infty}\arctan\left(\frac{1}{F_{2n-1}}\right)}_{S_1}+\underbrace{\sum_{n=1}^{\infty}(-1)^{n-1}\arctan\left(\frac{1}{F_{2n}}\right)}_{S_2}=\frac{\pi}{2}+\arctan\left(\frac{1}{\varphi}\right),$$

where I used the facts that $S_1=\lim_{N\to\infty}\sum_{n=1}^{N}\arctan\left(\frac{1}{F_{2n-1}}\right)=\lim_{N\to\infty}\sum_{n=1}^{N}(\arctan(F_{2n})-\arctan(F_{2n-2}))=\lim_{N\to\infty}\arctan(F_{2N})=\frac{\pi}{2}$, based on the point ii) of the previous section, and $S_2=\arctan\left(\frac{1}{\varphi}\right)$, a result from the point ii) of the current section.

For an alternative solution to the proposed series problem, see [44].

To calculate the series from the point ii), we start by placing $(-1)^{n-1}$ inside the argument of arctan, and we get

$$\sum_{n=1}^{\infty}(-1)^{n-1}\arctan\left(\frac{1}{F_{2n}}\right)=\sum_{n=1}^{\infty}\arctan\left(\frac{(-1)^{n-1}}{F_{2n}}\right). \tag{6.9}$$

From Cassini's identity, $F_{n-1}F_{n+1}-F_n^2=(-1)^n$, which we multiply by (-1), we get $(-1)^{n-1}=F_n^2-F_{n-1}F_{n+1}$, that if we combine with $F_{2n}=F_{n+1}^2-F_{n-1}^2$, which is known as d'Ocagne's identity,[4] and then plug these results in (6.9), we obtain

[4]A brief way of proving the d'Ocagne's identity relies on the use of the Binet formula of the Fibonacci numbers, and using $F_n=\frac{\phi^n-(-\phi)^{-n}}{\sqrt{5}}$, we have that $F_{n+1}^2=\left(\frac{\phi^{n+1}-(-\phi)^{-n-1}}{\sqrt{5}}\right)^2=\frac{1}{5}\phi^{2n+2}+\frac{1}{5}(-\phi)^{-2n-2}-\frac{2}{5}(-1)^{n-1}$ and $F_{n-1}^2=$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\arctan\left(\frac{1}{F_{2n}}\right)=\sum_{n=1}^{\infty}\arctan\left(\frac{F_n^2-F_{n-1}F_{n+1}}{F_{n+1}^2-F_{n-1}^2}\right). \tag{6.10}$$

Then, in the recurrence relation of the Fibonacci numbers, $F_n = F_{n-1} + F_{n-2}$, we replace n by $n + 1$, and we get $F_{n+1} = F_n + F_{n-1}$, whence we obtain that $F_n = F_{n+1} - F_{n-1}$. Using this last relation in (6.10), we have

$$\sum_{n=1}^{\infty}(-1)^{n-1}\arctan\left(\frac{1}{F_{2n}}\right)=\sum_{n=1}^{\infty}\arctan\left(\frac{F_n^2-F_{n-1}F_{n+1}}{(F_{n+1}-F_{n-1})(F_{n+1}+F_{n-1})}\right)$$

$$=\sum_{n=1}^{\infty}\arctan\left(\frac{F_n^2-F_{n-1}F_{n+1}}{F_n(F_{n+1}+F_{n-1})}\right)=\sum_{n=1}^{\infty}\arctan\left(\frac{F_n^2-F_{n-1}F_{n+1}}{F_nF_{n+1}+F_nF_{n-1}}\right),$$

and dividing both numerator and denominator by F_nF_{n+1}, we obtain that

$$\sum_{n=1}^{\infty}\arctan\left(\frac{\dfrac{F_n}{F_{n+1}}-\dfrac{F_{n-1}}{F_n}}{1+\dfrac{F_nF_{n-1}}{F_{n+1}F_n}}\right)=\lim_{N\to\infty}\sum_{n=1}^{N}\arctan\left(\frac{\dfrac{F_n}{F_{n+1}}-\dfrac{F_{n-1}}{F_n}}{1+\dfrac{F_nF_{n-1}}{F_{n+1}F_n}}\right)$$

$$=\lim_{N\to\infty}\sum_{n=1}^{N}\left(\arctan\left(\frac{F_n}{F_{n+1}}\right)-\arctan\left(\frac{F_{n-1}}{F_n}\right)\right)=\lim_{N\to\infty}\arctan\left(\frac{F_N}{F_{N+1}}\right)$$

$$=\arctan\left(\frac{1}{\varphi}\right),$$

where in the calculations I used the trigonometric identity, $\arctan(a)-\arctan(b)=\arctan\left(\dfrac{a-b}{1+ab}\right)$, and the limit $\lim_{N\to\infty}\dfrac{F_{N+1}}{F_N}=\varphi$, which is calculated at the point i) from the previous section.

As you may see, the telescoping sums are at the origin of these beautiful series, and to get there one needs to know and wisely use and combine some relations with the Fibonacci numbers (which is in general the way to go for such problems).

$\left(\dfrac{\phi^{n-1}-(-\phi)^{-n+1}}{\sqrt{5}}\right)^2=\dfrac{1}{5}\phi^{2n-2}+\dfrac{1}{5}(-\phi)^{-2n+2}-\dfrac{2}{5}(-1)^{n-1}$, and if considered together, we have $F_{n+1}^2-F_{n-1}^2=\dfrac{1}{5}\phi^{2n}\left(\phi^2-\dfrac{1}{\phi^2}\right)-\dfrac{1}{5}(-\phi)^{-2n}\left(\phi^2-\dfrac{1}{\phi^2}\right)=\dfrac{\phi^{2n}-(-\phi)^{-2n}}{\sqrt{5}}=F_{2n}$, and the identity is proved. Note that $\varphi=\dfrac{1+\sqrt{5}}{2}$ is the golden ratio.

6.9 Useful Series Representations of $\log(1+x)\log(1-x)$ and $\arctan(x)\log(1+x^2)$ from the Notorious *Table of Integrals, Series, and Products* by I.S. Gradshteyn and I.M. Ryzhik

Solution The series representation of $\log(1+x)\log(1-x)$ from the first part of the problem can be pretty useful in the calculations of some integrals that have in the integrand $\log(1+x)\log(1-x)$, and you may find one such example in the first chapter, in Sect. 1.10.

Actually, only the first equality appears in [23], more exactly in **1.516.3**, and the second equality may be found in [9, Exercise 5.2.8, p. 76], which can be derived immediately by using Botez–Catalan identity,

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{2n}=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}=H_{2n}-H_n,\ \forall n\geq 1.$$

For a first solution to the point i) of the problem, we use the Taylor series which we combine with the use of the Cauchy product of two series (see [25, Chapter III, pp. 197–199]), and then we write

$$-\log(1+x)\log(1-x)=\left(\sum_{n=1}^{\infty}\frac{x^n}{n}\right)\left(\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}\right)$$

$$=\sum_{n=1}^{\infty}x^{n+1}\left(\sum_{k=1}^{n}(-1)^{n-k}\frac{1}{k(n-k+1)}\right)$$

$$\{\text{since } \log(1+x)\log(1-x) \text{ is even, the odd powers of } x \text{ in the series must cancel}\}$$

$$=\sum_{n=1}^{\infty}x^{2n}\left(\sum_{k=1}^{2n-1}(-1)^{k-1}\frac{1}{k(2n-k)}\right)$$

$$=\sum_{n=1}^{\infty}\frac{x^{2n}}{n}\left(\frac{1}{2}\sum_{k=1}^{2n-1}\left(\frac{(-1)^{k-1}}{k}+\frac{(-1)^{2n-k-1}}{2n-k}\right)\right)$$

$$=\sum_{n=1}^{\infty}\frac{x^{2n}}{n}\sum_{k=1}^{2n-1}\frac{(-1)^{k-1}}{k}=\sum_{n=1}^{\infty}x^{2n}\frac{H_{2n}-H_n}{n}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{x^{2n}}{n^2},$$

where for getting the last equality I used Botez–Catalan identity, and the first solution to the point i) of the problem is complete.

For a second solution to the point i) of the problem, let's prove first that

$$\frac{\log(1+x)}{1-x} - \frac{\log(1-x)}{1+x} = 2\sum_{k=1}^{\infty} x^{2k-1}\varphi(2k-1), \tag{6.11}$$

where $\varphi(k) = \sum_{n=1}^{k} \frac{(-1)^{n-1}}{n}$.

Multiplying both sides of (6.11) by $1-x^2$, we get for the right-hand side that

$$2\sum_{k=1}^{\infty} x^{2k-1}(1-x^2)\varphi(2k-1) = 2\sum_{k=1}^{\infty} x^{2k-1}\varphi(2k-1) - 2\sum_{k=1}^{\infty} x^{2k+1}\varphi(2k-1)$$

$$= 2\sum_{k=1}^{\infty} x^{2k-1}\varphi(2k-1) - 2\sum_{k=1}^{\infty} x^{2k+1}\left(\varphi(2k+1) - \frac{1}{2k+1} + \frac{1}{2k}\right)$$

$$= 2\sum_{k=1}^{\infty} x^{2k-1}\varphi(2k-1) - 2\sum_{k=1}^{\infty} x^{2k+1}\varphi(2k+1) + 2\sum_{k=1}^{\infty} \frac{x^{2k+1}}{2k+1} - \sum_{k=1}^{\infty} \frac{x^{2k+1}}{k}$$

$$\left\{\text{employ the series, } \operatorname{arctanh}(x) = \sum_{k=1}^{\infty} \frac{x^{2k-1}}{2k-1} \text{ and } \log(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}\right\}$$

$$= \lim_{N\to\infty} \sum_{k=1}^{N} (2x^{2k-1}\varphi(2k-1) - 2x^{2k+1}\varphi(2k+1)) - 2x + 2\operatorname{arctanh}(x) + x\log(1-x^2)$$

$$= \lim_{N\to\infty} (2x\varphi(1) - 2x^{2N+1}\varphi(2N+1)) - 2x + 2\operatorname{arctanh}(x) + x\log(1-x^2)$$

$$= (1+x)\log(1+x) - (1-x)\log(1-x),$$

where I used that $\lim_{N\to\infty} x^{2N+1}\varphi(2N+1) = 0$ because $|x| < 1$ and $\lim_{N\to\infty} \varphi(2N+1) < \infty$. Therefore, the result in (6.11) is true.

Upon integrating back in (6.11), and considering that $\varphi(k) = \sum_{n=1}^{k} \frac{(-1)^{n-1}}{n}$, we get the first equality,

$$-\log(1+x)\log(1-x) = \sum_{k=1}^{\infty} \frac{x^{2k}}{k} \sum_{n=1}^{2k-1} \frac{(-1)^{n-1}}{n}.$$

The second equality follows immediately by employing Botez–Catalan identity mentioned at the beginning of the solution,

$$\sum_{k=1}^{\infty}\frac{x^{2k}}{k}\sum_{n=1}^{2k-1}\frac{(-1)^{n-1}}{n}=\sum_{k=1}^{\infty}x^{2k}\frac{H_{2k}-H_k}{k}+\frac{1}{2}\sum_{k=1}^{\infty}\frac{x^{2k}}{k^2},$$

and the second solution to the point $i)$ of the problem is complete.

To get a first solution to the point $ii)$ of the problem, we proceed as at the previous point and use the Cauchy product of two series (see [33, Chapter III, p.197–199]), and then we have

$$\frac{1}{2}\arctan(x)\log(1+x^2)=\frac{1}{2}\left(\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n}}{n}\right)\left(\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}\right)$$

$$=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n+1}}{2n+1}\sum_{k=1}^{n}\left(\frac{1}{2k}+\frac{1}{2n-2k+1}\right)=\sum_{n=1}^{\infty}(-1)^{n-1}x^{2n+1}\frac{H_{2n}}{2n+1},$$

and the first solution to the point $ii)$ of the problem is complete.

To get a second solution to the point $ii)$ of the problem, we start with showing that

$$2\frac{x\arctan(x)}{1+x^2}+\frac{\log(1+x^2)}{1+x^2}=2\sum_{k=1}^{\infty}(-1)^{k-1}x^{2k}H_{2k},\ |x|<1. \tag{6.12}$$

Multiplying both sides of (6.12) by $1+x^2$, we get for the right-hand side that

$$2\sum_{k=1}^{\infty}(-1)^{k-1}x^{2k}(1+x^2)H_{2k}=2\sum_{k=1}^{\infty}(-1)^{k-1}x^{2k}H_{2k}-2\sum_{k=1}^{\infty}(-1)^{k}x^{2k+2}H_{2k}$$

$$=2\sum_{k=1}^{\infty}(-1)^{k-1}x^{2k}H_{2k}-2\sum_{k=1}^{\infty}(-1)^{k}x^{2k+2}\left(H_{2k+2}-\frac{1}{2k+1}-\frac{1}{2k+2}\right)$$

$$=2\sum_{k=1}^{\infty}(-1)^{k-1}x^{2k}H_{2k}-2\sum_{k=1}^{\infty}(-1)^{k}x^{2k+2}H_{2k+2}+2\sum_{k=1}^{\infty}(-1)^{k}\frac{x^{2k+2}}{2k+1}$$

$$+\sum_{k=1}^{\infty}(-1)^{k}\frac{x^{2k+2}}{k+1}=\lim_{N\to\infty}\sum_{k=1}^{N}\left((-1)^{k-1}2x^{2k}H_{2k}-(-1)^{k}2x^{2k+2}H_{2k+2}\right)$$

$$+2\sum_{k=1}^{\infty}(-1)^{k}\frac{x^{2k+2}}{2k+1}+\sum_{k=1}^{\infty}(-1)^{k}\frac{x^{2k+2}}{k+1}$$

$$\left\{\text{employ the series, } \arctan(x) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^{2k-1}}{2k-1} \text{ and}\right.$$

$$\left.\log(1+x) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}\right\}$$

$$= \lim_{N\to\infty}\left(3x^2 - (-1)^N 2x^{2N+2}H_{2N+2}\right) - 3x^2 + 2x\arctan(x) + \log(1+x^2)$$

$$= 2x\arctan(x) + \log(1+x^2),$$

and therefore the result in (6.12) is true.

Upon integrating both sides of (6.12), we arrive immediately at

$$\sum_{k=1}^{\infty}(-1)^{k-1}x^{2k+1}\frac{H_{2k}}{2k+1} = \frac{1}{2}\arctan(x)\log(1+x^2),$$

and the second solution to the point $ii)$ of the problem is complete.

The result from the point $ii)$ appeared as a proposed problem (which I co-authored) in *La Gaceta de la RSME*, Vol. 19, No. 1 (2016), the problem **290** (see [29, p. 99]). It is also given in [23], in **1.517.3**. Another solution to the result from the point $ii)$ of the problem may be found in [28, pp. 101–103].

The simple strategy involving the telescoping sums showed above is very useful, and you might notice that also in the next section.

6.10 A Group of Five Useful Generating Functions Related to the Generalized Harmonic Numbers

Solution From the title of the section one might easily guess I intend to derive five generating functions related to the generalized harmonic numbers which we may find very useful when working with the harmonic series. For the solutions to the problems in this book we will find the first three generating functions particularly useful. The attack strategy for deriving all five generating functions is based upon the simple idea of wisely using telescoping sums as in the previous section, which I used first in the solution of a beautiful problem I submitted in August, 2016, to the *MathProblems journal*, Vol. 5, No. 4, the problem **140** (see [36]), *Find* $\displaystyle\sum_{n=1}^{\infty}\frac{H_n^3}{(n+1)2^n}$, that won't be difficult to calculate with the right generating function in hand.

For proving the result in (4.5), we multiply both sides by $1-x$ that yields

$$\sum_{n=1}^{\infty} x^n(1-x)H_n = -\log(1-x). \tag{6.13}$$

Now, the left-hand side of (6.13) can be written as follows

$$\sum_{n=1}^{\infty}(x^n H_n - x^{n+1}H_n) = \sum_{n=1}^{\infty} x^n H_n - \sum_{n=1}^{\infty} x^{n+1} H_n$$

$$= \sum_{n=1}^{\infty} x^n H_n - \sum_{n=1}^{\infty} x^{n+1}\left(H_{n+1} - \frac{1}{n+1}\right) = \sum_{n=1}^{\infty} x^n H_n - \sum_{n=1}^{\infty} x^{n+1} H_{n+1} + \sum_{n=1}^{\infty} \frac{x^{n+1}}{n+1}$$

$$= \lim_{N\to\infty}\left(\sum_{n=1}^{N} x^n H_n - \sum_{n=1}^{N} x^{n+1} H_{n+1}\right) - x - \log(1-x)$$

$$= x - \underbrace{\lim_{N\to\infty} x^{N+1} H_{N+1}}_{0} - x - \log(1-x) = -\log(1-x),$$

which ends the solution to the first result.

Further, to prove the result in (4.6), we multiply both sides by $1-x$ that gives

$$\sum_{n=1}^{\infty} x^n(1-x)H_n^{(m)} = \operatorname{Li}_m(x). \tag{6.14}$$

Then, for the left-hand side of (6.14), we write that

$$\sum_{n=1}^{\infty}(x^n H_n^{(m)} - x^{n+1}H_n^{(m)}) = \sum_{n=1}^{\infty} x^n H_n^{(m)} - \sum_{n=1}^{\infty} x^{n+1} H_n^{(m)}$$

$$= \sum_{n=1}^{\infty} x^n H_n^{(m)} - \sum_{n=1}^{\infty} x^{n+1}\left(H_{n+1}^{(m)} - \frac{1}{(n+1)^m}\right)$$

$$= \sum_{n=1}^{\infty} x^n H_n^{(m)} - \sum_{n=1}^{\infty} x^{n+1} H_{n+1}^{(m)} + \sum_{n=1}^{\infty} \frac{x^{n+1}}{(n+1)^m}$$

$$= \lim_{N\to\infty}\left(\sum_{n=1}^{N} x^n H_n^{(m)} - \sum_{n=1}^{N} x^{n+1} H_{n+1}^{(m)}\right) + \operatorname{Li}_m(x) - x$$

$$= x - \underbrace{\lim_{N\to\infty} x^{N+1} H_{N+1}^{(m)}}_{0} + \mathrm{Li}_m(x) - x = \mathrm{Li}_m(x),$$

which ends the solution to the second result.

Further, to prove the result in (4.7), we proceed in the same style as before and multiply both sides by $1-x$ that yields

$$\sum_{n=1}^{\infty} x^n (1-x) H_n^2 = \log^2(1-x) + \mathrm{Li}_2(x). \tag{6.15}$$

Now, the left-hand side of (6.15) can be rearranged as follows

$$\sum_{n=1}^{\infty} x^n (1-x) H_n^2 = \sum_{n=1}^{\infty} (x^n H_n^2 - x^{n+1} H_n^2)$$

$$= \sum_{n=1}^{\infty} \left(x^n H_n^2 - x^{n+1} \left(H_{n+1} - \frac{1}{n+1} \right)^2 \right)$$

$$= \sum_{n=1}^{\infty} \left(x^n H_n^2 - x^{n+1} H_{n+1}^2 + 2x^{n+1} \frac{H_{n+1}}{n+1} - \frac{x^{n+1}}{(n+1)^2} \right)$$

$$= \sum_{n=1}^{\infty} \left(x^n H_n^2 - x^{n+1} H_{n+1}^2 + 2x^{n+1} \frac{H_n}{n+1} + \frac{x^{n+1}}{(n+1)^2} \right)$$

$$= \lim_{N\to\infty} \sum_{n=1}^{N} \left(x^n H_n^2 - x^{n+1} H_{n+1}^2 \right) + 2 \sum_{n=1}^{\infty} x^{n+1} \frac{H_n}{n+1} + \sum_{n=1}^{\infty} \frac{x^{n+1}}{(n+1)^2}$$

$$= x - \underbrace{\lim_{N\to\infty} x^{N+1} H_{N+1}^2}_{0} + 2 \sum_{n=1}^{\infty} \int_0^x t^n H_n \mathrm{d}t + \mathrm{Li}_2(x) - x$$

{reverse the order of integration and summation}

$$= 2 \int_0^x \sum_{n=1}^{\infty} t^n H_n \mathrm{d}t + \mathrm{Li}_2(x)$$

{use the generating function in (4.5)}

$$= -2\int_0^x \frac{\log(1-t)}{1-t}\mathrm{d}t + \mathrm{Li}_2(x) = \log^2(1-x) + \mathrm{Li}_2(x),$$

which ends the solution to the third result.

As you may have guessed, we follow the routine! To prove the result in (4.8), we proceed as in the previous case, and then multiply both sides by $1-x$ that gives

$$\sum_{n=1}^{\infty} x^n(1-x)H_n H_n^{(2)} = (1-x)G(x). \tag{6.16}$$

Considering the left-hand side of (6.16), we have that

$$\sum_{n=1}^{\infty}(x^n - x^{n+1})H_n H_n^{(2)} = \sum_{n=1}^{\infty} x^n H_n H_n^{(2)} - \sum_{n=1}^{\infty} x^{n+1} H_n H_n^{(2)}$$

$$= \sum_{n=1}^{\infty} x^n H_n H_n^{(2)} - \sum_{n=1}^{\infty} x^{n+1}\left(H_{n+1} - \frac{1}{n+1}\right)\left(H_{n+1}^{(2)} - \frac{1}{(n+1)^2}\right)$$

$$= \sum_{n=1}^{\infty} x^n H_n H_n^{(2)} - \sum_{n=1}^{\infty} x^{n+1} H_{n+1} H_{n+1}^{(2)} + \sum_{n=1}^{\infty} x^{n+1}\frac{H_{n+1}}{(n+1)^2}$$

$$+ \sum_{n=1}^{\infty} x^{n+1}\frac{H_{n+1}^{(2)}}{n+1} - \sum_{n=1}^{\infty}\frac{x^{n+1}}{(n+1)^3}$$

$$= \lim_{N\to\infty}\left(\sum_{n=1}^{N} x^n H_n H_n^{(2)} - \sum_{n=1}^{N} x^{n+1} H_{n+1} H_{n+1}^{(2)}\right) + \sum_{n=1}^{\infty} x^{n+1}\frac{H_n + 1/(n+1)}{(n+1)^2}$$

$$+ \sum_{n=1}^{\infty} x^{n+1}\frac{H_n^{(2)} + 1/(n+1)^2}{n+1} - \sum_{n=1}^{\infty}\frac{x^{n+1}}{(n+1)^3}$$

$$= x - \lim_{N\to\infty} x^{N+1} H_{N+1} H_{N+1}^{(2)} + \sum_{n=1}^{\infty} x^{n+1}\frac{H_n}{(n+1)^2} + \sum_{n=1}^{\infty} x^{n+1}\frac{H_n^{(2)}}{n+1} + \sum_{n=1}^{\infty}\frac{x^{n+1}}{(n+1)^3}$$

$$= \sum_{n=1}^{\infty} x^{n+1}\frac{H_n}{(n+1)^2} + \sum_{n=1}^{\infty} x^{n+1}\frac{H_n^{(2)}}{n+1} + \mathrm{Li}_3(x). \tag{6.17}$$

Now, to calculate the first series in (6.17), we recall the result in (4.5) where if we replace x by t and integrate both sides from $t=0$ to $t=x$, we get

$$\int_0^x \sum_{n=1}^{\infty} t^n H_n \mathrm{d}t = \sum_{n=1}^{\infty} \int_0^x t^n H_n \mathrm{d}t = \sum_{n=1}^{\infty} x^{n+1} \frac{H_n}{n+1} = -\int_0^x \frac{\log(1-t)}{1-t} \mathrm{d}t$$
$$= \frac{1}{2} \log^2(1-x),$$

and if we consider

$$\sum_{n=1}^{\infty} t^{n+1} \frac{H_n}{n+1} = \frac{1}{2} \log^2(1-t),$$

and then divide both sides by t and integrate from $t = 0$ to $t = x$, we have

$$\int_0^x \sum_{n=1}^{\infty} t^n \frac{H_n}{n+1} \mathrm{d}t = \sum_{n=1}^{\infty} \int_0^x t^n \frac{H_n}{n+1} \mathrm{d}t = \sum_{n=1}^{\infty} x^{n+1} \frac{H_n}{(n+1)^2}$$
$$= \frac{1}{2} \int_0^x \frac{\log^2(1-t)}{t} \mathrm{d}t$$

{employ the result in (1.8)}

$$= \frac{1}{2} \log(x) \log^2(1-x) + \log(1-x) \operatorname{Li}_2(1-x) - \operatorname{Li}_3(1-x) + \zeta(3). \tag{6.18}$$

Then, for the second series in (6.17), we recall the result in (4.6), the case $m = 2$, where if we replace x by t and integrate both sides from $t = 0$ to $t = x$, we get

$$\int_0^x \sum_{n=1}^{\infty} t^n H_n^{(2)} \mathrm{d}t = \sum_{n=1}^{\infty} \int_0^x t^n H_n^{(2)} \mathrm{d}t = \sum_{n=1}^{\infty} x^{n+1} \frac{H_n^{(2)}}{n+1} = \int_0^x \frac{\operatorname{Li}_2(t)}{1-t} \mathrm{d}t$$

{integrate the last integral by parts}

$$= -\log(1-x) \operatorname{Li}_2(x) - \int_0^x \frac{\log^2(1-t)}{t} \mathrm{d}t$$

{employ the result in (1.8)}

$$= 2 \operatorname{Li}_3(1-x) - \log(1-x) \operatorname{Li}_2(x) - \log(x) \log^2(1-x)$$
$$-2 \log(1-x) \operatorname{Li}_2(1-x) - 2\zeta(3). \tag{6.19}$$

If we collect the results from (6.18) and (6.10) in (6.17), we get

$$\sum_{n=1}^{\infty} x^n (1-x) H_n H_n^{(2)} = \operatorname{Li}_3(x) + \operatorname{Li}_3(1-x) - \log(1-x)\operatorname{Li}_2(x)$$
$$- \log(1-x)\operatorname{Li}_2(1-x) - \frac{1}{2}\log(x)\log^2(1-x) - \zeta(3),$$

and upon using the Dilogarithm function reflection formula (see [30, Chapter 1, p. 5], [53, (5)], and [42, Chapter 2, p. 107]), $\operatorname{Li}_2(x) + \operatorname{Li}_2(1-x) = \zeta(2) - \log(x)\log(1-x)$, we obtain that

$$\sum_{n=1}^{\infty} x^n (1-x) H_n H_n^{(2)} = \frac{1}{2}\log(x)\log^2(1-x) + \operatorname{Li}_3(x) + \operatorname{Li}_3(1-x)$$
$$-\zeta(2)\log(1-x) - \zeta(3),$$

which ends the solution to the fourth result.

The routine, one more time! Lastly, for proving the result in (4.9), we multiply both sides by $1-x$ that yields

$$\sum_{n=1}^{\infty} x^n (1-x) H_n^3 = (1-x) K(x). \tag{6.20}$$

For the left-hand side of (6.20) we proceed as follows

$$\sum_{n=1}^{\infty} x^n (1-x) H_n^3 = \sum_{n=1}^{\infty} (x^n H_n^3 - x^{n+1} H_n^3)$$
$$= \sum_{n=1}^{\infty} \left(x^n H_n^3 - x^{n+1} \left(H_{n+1} - \frac{1}{n+1} \right)^3 \right)$$
$$= \sum_{n=1}^{\infty} \left((x^n H_n^3 - x^{n+1} H_{n+1}^3) + 3x^{n+1} \frac{H_{n+1}^2}{n+1} - 3x^{n+1} \frac{H_{n+1}}{(n+1)^2} + \frac{x^{n+1}}{(n+1)^3} \right)$$
$$= \sum_{n=1}^{\infty} \left((x^n H_n^3 - x^{n+1} H_{n+1}^3) + 3x^{n+1} \frac{(H_n + 1/(n+1))^2}{n+1} - 3x^{n+1} \frac{H_n + 1/(n+1)}{(n+1)^2} \right.$$
$$\left. + \frac{x^{n+1}}{(n+1)^3} \right)$$
$$= \sum_{n=1}^{\infty} (x^n H_n^3 - x^{n+1} H_{n+1}^3) + 3\sum_{n=1}^{\infty} x^{n+1} \frac{H_n^2}{n+1} + 3\sum_{n=1}^{\infty} x^{n+1} \frac{H_n}{(n+1)^2} + \sum_{n=1}^{\infty} \frac{x^{n+1}}{(n+1)^3}$$

$$= \lim_{N\to\infty}\sum_{n=1}^{N}(x^n H_n^3 - x^{n+1}H_{n+1}^3) + 3\sum_{n=1}^{\infty} x^{n+1}\frac{H_n^2}{n+1}$$

$$+3\sum_{n=1}^{\infty} x^{n+1}\frac{H_n}{(n+1)^2} - x + \operatorname{Li}_3(x)$$

$$= x - \underbrace{\lim_{N\to\infty} x^{N+1}H_{N+1}^3}_{0} + 3\sum_{n=1}^{\infty} x^{n+1}\frac{H_n^2}{n+1} + 3\sum_{n=1}^{\infty} x^{n+1}\frac{H_n}{(n+1)^2} - x + \operatorname{Li}_3(x)$$

$$= 3\sum_{n=1}^{\infty} x^{n+1}\frac{H_n^2}{n+1} + 3\sum_{n=1}^{\infty} x^{n+1}\frac{H_n}{(n+1)^2} + \operatorname{Li}_3(x). \tag{6.21}$$

Further, for calculating the first series in (6.21), we write that

$$\sum_{n=1}^{\infty} x^{n+1}\frac{H_n^2}{n+1} = \sum_{n=1}^{\infty}\int_0^x t^n H_n^2 \mathrm{d}t = \int_0^x \sum_{n=1}^{\infty} t^n H_n^2 \mathrm{d}t$$

{make use of the result in (4.7)}

$$= \int_0^x \left(\frac{\log^2(1-t)}{1-t} + \frac{\operatorname{Li}_2(t)}{1-t}\right)\mathrm{d}t = -\frac{1}{3}\log^3(1-x) + \int_0^x \frac{\operatorname{Li}_2(t)}{1-t}\mathrm{d}t$$

$$= -\frac{1}{3}\log^3(1-x) - \int_0^x (\log(1-t))' \operatorname{Li}_2(t)\mathrm{d}t$$

{apply the integration by parts}

$$= -\frac{1}{3}\log^3(1-x) - \log(1-x)\operatorname{Li}_2(x) - \int_0^x \frac{\log^2(1-t)}{t}\mathrm{d}t$$

{employ the result in (1.8)}

$$= -\frac{1}{3}\log^3(1-x) - \log(1-x)\operatorname{Li}_2(x) - \log(x)\log^2(1-x) - 2\log(1-x)\operatorname{Li}_2(1-x)$$

$$+ 2\operatorname{Li}_3(1-x) - 2\zeta(3). \tag{6.22}$$

Since the second series in (6.21) is calculated in (6.18), if we collect the results from (6.22) and (6.18) in (6.21), we get that

$$\sum_{n=1}^{\infty} x^n(1-x)H_n^3$$

$$= \operatorname{Li}_3(x)+3\operatorname{Li}_3(1-x)-3\log(1-x)\operatorname{Li}_2(x)-3\log(1-x)\operatorname{Li}_2(1-x)-\log^3(1-x)$$
$$-\frac{3}{2}\log(x)\log^2(1-x)-3\zeta(3)$$
$$=\frac{3}{2}\log(x)\log^2(1-x)-3\zeta(2)\log(1-x)-\log^3(1-x)+\operatorname{Li}_3(x)+3\operatorname{Li}_3(1-x)-3\zeta(3),$$

where for the last equality I used the Dilogarithm function reflection formula (see [30, Chapter 1, p. 5], [53, (5)], and [42, Chapter 2, p. 107]), $\operatorname{Li}_2(x)+\operatorname{Li}_2(1-x)=\zeta(2)-\log(x)\log(1-x)$, which ends the solution to the fifth result.

The curious reader might also try to use the elementary symmetric polynomials (see [60]), $\sigma_k(x_1,x_2,\ldots,x_n)=\sum_{1\le i_1<i_2<\ldots<i_k\le n} x_{i_1}x_{i_2}\ldots x_{i_k}$, expressed in terms of power sums, $p_k(x_1,x_2,\ldots,x_n)=x_1^k+x_2^k+\ldots+x_n^k$, for proving such results, and then we recall the Newton–Girard formulae (see [56], [31, p. 23]), like $\sigma_1=p_1$, $\sigma_2=(p_1^2-p_2)/2$, $\sigma_3=(p_1^3-3p_1p_2+2p_3)/6$, $\sigma_4=(p_1^4-6p_1^2p_2+8p_1p_3+3p_2^2-6p_4)/24$, and so on, that we may obtain by the recursive relation, $n\sigma_n=\sum_{k=1}^{n}(-1)^{k-1}p_k\sigma_{n-k}$, or the generating function, $\sum_{k=0}^{\infty}t^k\sigma_k=\exp\left(\sum_{k=1}^{\infty}(-1)^{k-1}t^k\frac{p_k}{k}\right)$, which can be derived by using the generating function, $\sum_{k=0}^{\infty}t^k\sigma_k=\prod_{i=1}^{\infty}(1+x_it)$.

We assume $\sum_{n=1}^{\infty}x^n\sigma_{k-1}\left(1,\frac{1}{2},\ldots,\frac{1}{n}\right)=\frac{(-1)^{k-1}}{(k-1)!}\frac{\log^{k-1}(1-x)}{1-x}$ and want to prove by induction that

$$\sum_{n=1}^{\infty}x^n\sigma_k\left(1,\frac{1}{2},\ldots,\frac{1}{n}\right)=\frac{(-1)^k}{k!}\frac{\log^k(1-x)}{1-x},\ |x|<1. \tag{6.23}$$

Note here the well-known result, $\sum_{n=1}^{\infty}x^n\sigma_1\left(1,\frac{1}{2},\ldots,\frac{1}{n}\right)=\sum_{n=1}^{\infty}x^nH_n=-\frac{\log(1-x)}{1-x}$.

Proof If we multiply both sides of (6.23) by $1-x$, we have

$$\sum_{n=1}^{\infty}x^n(1-x)\sigma_k\left(1,\frac{1}{2},\ldots,\frac{1}{n}\right)=\frac{(-1)^k}{k!}\log^k(1-x). \tag{6.24}$$

Then, for the left-hand side of (6.24), we get

$$\sum_{n=1}^{\infty}\left(x^n\sigma_k\left(1,\frac{1}{2},\ldots,\frac{1}{n}\right)-x^{n+1}\sigma_k\left(1,\frac{1}{2},\ldots,\frac{1}{n}\right)\right)$$

$$=\underbrace{\sum_{n=1}^{\infty}\left(x^n\sigma_k\left(1,\frac{1}{2},\ldots,\frac{1}{n-1}\right)-x^{n+1}\sigma_k\left(1,\frac{1}{2},\ldots,\frac{1}{n}\right)\right)}_{0}$$

$$+\sum_{n=1}^{\infty}\frac{x^n}{n}\sigma_{k-1}\left(1,\frac{1}{2},\ldots,\frac{1}{n-1}\right)=\sum_{n=1}^{\infty}\frac{x^n}{n}\sigma_{k-1}\left(1,\frac{1}{2},\ldots,\frac{1}{n-1}\right)$$

$$=\sum_{n=1}^{\infty}\int_0^x y^{n-1}\sigma_{k-1}\left(1,\frac{1}{2},\ldots,\frac{1}{n-1}\right)\mathrm{d}y$$

$$=\int_0^x\sum_{n=1}^{\infty}y^{n-1}\sigma_{k-1}\left(1,\frac{1}{2},\ldots,\frac{1}{n-1}\right)\mathrm{d}y$$

$$=\frac{(-1)^{k-1}}{(k-1)!}\int_0^x\frac{\log^{k-1}(1-y)}{1-y}\mathrm{d}y=\frac{(-1)^k}{k!}\log^k(1-x),$$

which shows the result in (6.24) is true.

More specifically, if we combine now the result in (6.23) with the Newton–Girard formulae, we obtain that $\sum_{n=1}^{\infty}x^nH_n=-\frac{\log(1-x)}{1-x}$, $\sum_{n=1}^{\infty}x^n(H_n^2-H_n^{(2)})=\frac{\log^2(1-x)}{1-x}$, $\sum_{n=1}^{\infty}x^n(H_n^3-3H_nH_n^{(2)}+2H_n^{(3)})=-\frac{\log^3(1-x)}{1-x}$, $\sum_{n=1}^{\infty}x^n(H_n^4-6H_n^2H_n^{(2)}+8H_nH_n^{(3)}+3(H_n^{(2)})^2-6H_n^{(4)})=\frac{\log^4(1-x)}{1-x}$, and so on.

For example, at this point, by combining the second generating function previously given and the generating function G_2 in (4.6), we see we are done with the third point of the problem.

The five main generating functions presented in this section are not new in the mathematical literature. For example, the results in (4.7) and (4.9) may be found in [37] together with a solution, and the result in (4.8) is also stated in [51].

6.11 Four Members from a Neat Group of Generating Functions Expressed in Terms of Polylogarithm Function

Solution Toward the end of the first chapter, in Sects. 1.54 and 1.55, we met challenging integrals where some of the results in this section were very useful, and therefore we want to know how to obtain these generating functions.

Now, for getting a solution we will want to fruitfully combine the logarithmic integrals in Sect. 1.3 with the generalized integral in Sect. 1.6.

To prove the first result, we recall the relation in (1.4), and we have

$$G_1 = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n} H_n = -\sum_{n=1}^{\infty} \int_0^1 (xt)^{n-1} \log(1-t)\mathrm{d}t$$

{reverse the order of integration and summation}

$$= -\int_0^1 \sum_{n=1}^{\infty} (xt)^{n-1} \log(1-t)\mathrm{d}t = -\int_0^1 \frac{\log(1-t)}{1-xt}\mathrm{d}t$$

$$\overset{1-t=z}{=} -\int_0^1 \frac{\log(z)}{1-x+xz}\mathrm{d}z$$

{make use of the result in (1.12), the case $n=1$}

$$= -\frac{\operatorname{Li}_2\left(\dfrac{x}{x-1}\right)}{x},$$

and the first result is proved.

Next, to prove the second result, we recall the relation in (1.5), and we write

$$G_2 = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n} (H_n^2 + H_n^{(2)}) = \sum_{n=1}^{\infty} \int_0^1 (xt)^{n-1} \log^2(1-t)\mathrm{d}t$$

{reverse the order of integration and summation}

$$= \int_0^1 \sum_{n=1}^{\infty} (xt)^{n-1} \log^2(1-t)\mathrm{d}t = \int_0^1 \frac{\log^2(1-t)}{1-xt}\mathrm{d}t \overset{1-t=z}{=} \int_0^1 \frac{\log^2(z)}{1-x+xz}\mathrm{d}z$$

{make use of the result in (1.12), the case $n=2$}

$$= -2\frac{\operatorname{Li}_3\left(\dfrac{x}{x-1}\right)}{x},$$

and the second result is proved.

Further, to prove the third result, we recall the relation in (1.6), and we get

$$G_3 = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n}(H_n^3 + 3H_nH_n^{(2)} + 2H_n^{(3)}) = -\sum_{n=1}^{\infty} \int_0^1 (xt)^{n-1} \log^3(1-t)\mathrm{d}t$$

$$\{\text{reverse the order of integration and summation}\}$$

$$= -\int_0^1 \sum_{n=1}^{\infty}(xt)^{n-1}\log^3(1-t)\mathrm{d}t = -\int_0^1 \frac{\log^3(1-t)}{1-xt}\mathrm{d}t \overset{1-t=z}{=} -\int_0^1 \frac{\log^3(z)}{1-x+xz}\mathrm{d}z$$

$$\{\text{make use of the result in (1.12), the case } n=3\}$$

$$= -6\frac{\operatorname{Li}_4\left(\dfrac{x}{x-1}\right)}{x},$$

and the third result is proved.

Lastly, to prove the fourth result, we recall and employ the relation in (1.7), and we write

$$G_4 = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n}(H_n^4 + 6H_n^2H_n^{(2)} + 8H_nH_n^{(3)} + 3\left(H_n^{(2)}\right)^2 + 6H_n^{(4)})$$
$$= \sum_{n=1}^{\infty} \int_0^1 (xt)^{n-1} \log^4(1-t)\mathrm{d}t$$

$$\{\text{reverse the order of integration and summation}\}$$

$$= \int_0^1 \sum_{n=1}^{\infty}(xt)^{n-1}\log^4(1-t)\mathrm{d}t = \int_0^1 \frac{\log^4(1-t)}{1-xt}\mathrm{d}t \overset{1-t=z}{=} \int_0^1 \frac{\log^4(z)}{1-x+xz}\mathrm{d}z$$

$$\{\text{make use of the result in (1.12), the case } n=4\}$$

$$= -24\frac{\operatorname{Li}_5\left(\dfrac{x}{x-1}\right)}{x},$$

and the fourth result is proved.

These results are good to keep close in our toolbox since in one of the next sections we'll want to use them again. Also, for an alternative approach to the first three points of the problem, see [37].

6.12 Two Elementary Harmonic Sums Arising in the Calculation of Harmonic Series

Solution We are supposed to calculate now two elementary sums with harmonic numbers that perhaps are at least as old as the linear Euler sum, $\sum_{k=1}^{\infty} \frac{H_k}{k^n}$, $n \geq 2$ (see (3.45)), which was derived in the 18th century by Leonhard Euler (1707–1783). We'll find both sums particularly useful during the calculations of the harmonic series, either in the sum form or in the infinite series form.

For the first sum, we consider that $H_k^{(p)} = \sum_{l=1}^{k} \frac{1}{l^p}$, and then we have

$$S_1 = \sum_{k=1}^{n} \frac{H_k^{(p)}}{k^p} = \sum_{k=1}^{n} \sum_{l=1}^{k} \frac{1}{k^p l^p} = \sum_{k=1}^{n} \left(\sum_{l=1}^{n} - \sum_{l=k}^{n} + \sum_{l=k}^{k} \right) \frac{1}{k^p l^p} = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{1}{k^p l^p}$$

$$- \sum_{k=1}^{n} \sum_{l=k}^{n} \frac{1}{k^p l^p} + \sum_{k=1}^{n} \frac{1}{k^{2p}} = (H_n^{(p)})^2 + H_n^{(2p)} - \sum_{k=1}^{n} \sum_{l=1}^{k} \frac{1}{k^p l^p},$$

from which we obtain the desired result

$$S_1 = \sum_{k=1}^{n} \frac{H_k^{(p)}}{k^p} = \frac{1}{2}((H_n^{(p)})^2 + H_n^{(2p)}),$$

and the point i) of the problem is complete. In the calculations I used the simple fact that, by changing the summation order and then swapping the variables, we obtain that $\sum_{k=1}^{n} \sum_{l=k}^{n} \frac{1}{k^p l^p} = \sum_{l=1}^{n} \sum_{k=1}^{l} \frac{1}{k^p l^p} = \sum_{k=1}^{n} \sum_{l=1}^{k} \frac{1}{k^p l^p}$. The result can also be easily established by using Abel's summation (see 2.1).

For the second sum, we consider the following identity

$$\sum_{k=1}^{n} x_k \left(\sum_{l=1}^{k} x_l\right)^2 - \sum_{k=1}^{n} x_k^2 \sum_{l=1}^{k} x_l = \frac{1}{3}\left(\left(\sum_{k=1}^{n} x_k\right)^3 - \sum_{k=1}^{n} x_k^3\right), \tag{6.25}$$

which is a special case of the complete homogeneous symmetric polynomial identities, and we see that immediately if we consider writing $\left(\sum_{l=1}^{k} x_l\right)^2 = 2\sum_{l=1}^{k}\sum_{m=1}^{l} x_l x_m - \sum_{l=1}^{k} x_l^2$ and $\sum_{k=1}^{n}\sum_{l=1}^{k} x_k^2 x_l = \sum_{k=1}^{n}\left(\sum_{l=1}^{n} - \sum_{l=k}^{n} + \sum_{l=k}^{k}\right) x_k^2 x_l = \sum_{k=1}^{n}\sum_{l=1}^{n} x_k^2 x_l + \sum_{k=1}^{n} x_k^3 - \sum_{k=1}^{n}\sum_{l=k}^{n} x_k^2 x_l = \left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} x_k^2\right) + \sum_{k=1}^{n} x_k^3 - \sum_{l=1}^{n} x_l \sum_{k=1}^{l} x_k^2$, and then use that $\sum_{k=1}^{n}\sum_{l=1}^{k}\sum_{m=1}^{l} x_k x_l x_m = \frac{1}{6}\left(\left(\sum_{k=1}^{n} x_k\right)^3 + 3\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} x_k^2\right) + 2\sum_{k=1}^{n} x_k^3\right)$, where I expressed the complete homogeneous symmetric polynomial in terms of power sums, using that $h_3 = (p_1^3 + 3p_1p_2 + 2p_3)/6$ (also you might like to see the details of a similar approach to the last point of Sect. 3.3).

If we set $x_k = 1/k^p$ in (6.25), we obtain the desired result

$$S_2 = \sum_{k=1}^{n} \frac{(H_k^{(p)})^2}{k^p} = \frac{1}{3}\left((H_n^{(p)})^3 - H_n^{(3p)}\right) + \sum_{k=1}^{n} \frac{H_k^{(p)}}{k^{2p}},$$

and the point $ii)$ of the problem is complete.

For an alternative solution we may consider Abel's summation (see 2.1).

6.13 A Strong Generalized Sum, Making a Very Good Cocktail Together with the Identities Generated by *The Master Theorem of Series*

Solution The following generalized sum together with the identities generated by *The Master Theorem of Series* (whose dedicated section we'll meet soon) represents an excellent cocktail that will help us to calculate a wide range of harmonic series, some of them being (rather) tough. Moreover, particular cases of the present generalization may be the keys for getting excellent solutions for pretty resistant harmonic series where one only needs series manipulations, as we'll see in Sect. 6.25.

We denote $S_{n,m} = \sum_{k=1}^{n-1} \frac{H_k^{(m)}}{n-k}$ and use that $\sum_{k=1}^{n-1} \frac{H_k^{(m)}}{n-k} = \sum_{k=1}^{n-1} \frac{H_{n-k}^{(m)}}{k}$ where the last sum is obtained by reversing the order of summing the terms of the initial sum.

Then, if we consider the difference $S_{n,m} - S_{n-1,m}$, we obtain

$$S_{n,m} - S_{n-1,m} = \sum_{k=1}^{n-1} \frac{H_{n-k}^{(m)}}{k} - \sum_{k=1}^{n-2} \frac{H_{n-k-1}^{(m)}}{k} = \sum_{k=1}^{n-1} \frac{H_{n-k}^{(m)} - H_{n-k-1}^{(m)}}{k}$$

$$= \sum_{k=1}^{n-1} \frac{1}{k(n-k)^m} = \sum_{k=1}^{n-1} \frac{1}{k^m(n-k)}, \tag{6.26}$$

where the last sum was obtained by reversing the order of summing the terms in the penultimate sum.

Further, if denoting $s_{n,m} = \sum_{k=1}^{n-1} \frac{1}{k^m(n-k)}$, then we have

$$s_{n,m} = \sum_{k=1}^{n-1} \frac{1}{k^m(n-k)} = \frac{1}{n} \sum_{k=1}^{n-1} \frac{(n-k)+k}{k^m(n-k)} = \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{k^m} + \frac{1}{n} \underbrace{\sum_{k=1}^{n-1} \frac{1}{k^{m-1}(n-k)}}_{s_{n,m-1}}$$

$$= \frac{H_{n-1}^{(m)}}{n} + \frac{1}{n} s_{n,m-1} = \frac{1}{n}\left(H_n^{(m)} - \frac{1}{n^m}\right) + \frac{1}{n} s_{n,m-1}$$

$$= \frac{H_n^{(m)}}{n} - \frac{1}{n^{m+1}} + \frac{1}{n} s_{n,m-1},$$

whence we obtain that $s_{n,m} - \frac{1}{n} s_{n,m-1} = \frac{H_n^{(m)}}{n} - \frac{1}{n^{m+1}}$, or if we replace m by i, $s_{n,i} - \frac{1}{n} s_{n,i-1} = \frac{H_n^{(i)}}{n} - \frac{1}{n^{i+1}}$, multiply both sides of the last relation by n^i, $n^i s_{n,i} - n^{i-1} s_{n,i-1} = n^{i-1} H_n^{(i)} - \frac{1}{n}$, and then give values to i from $i = 1$ to m and sum all the resulting relations, we get

$$\sum_{i=1}^{m} (n^i s_{n,i} - n^{i-1} s_{n,i-1}) = n^m s_{n,m} - s_{n,0} = \sum_{i=1}^{m} \left(n^{i-1} H_n^{(i)} - \frac{1}{n}\right)$$

$$= \sum_{i=1}^{m} n^{i-1} H_n^{(i)} - \sum_{i=1}^{m} \frac{1}{n} = \sum_{i=1}^{m} n^{i-1} H_n^{(i)} - \frac{m}{n},$$

that leads to

$$n^m s_{n,m} = s_{n,0} + \sum_{i=1}^{m} n^{i-1} H_n^{(i)} - \frac{m}{n} = H_{n-1} + \sum_{i=1}^{m} n^{i-1} H_n^{(i)} - \frac{m}{n}$$

$$= H_n + \sum_{i=1}^{m} n^{i-1} H_n^{(i)} - \frac{m+1}{n},$$

or, if dividing the opposite sides by n^m, we get

$$s_{n,m} = \frac{H_n}{n^m} + \sum_{i=1}^{m} \frac{H_n^{(i)}}{n^{m-i+1}} - \frac{m+1}{n^{m+1}}$$

$$\{\text{make the change of variable } m - i + 1 = j \text{ in the sum}\}$$

$$= \frac{H_n}{n^m} + \sum_{j=1}^{m} \frac{H_n^{(m-j+1)}}{n^j} - \frac{m+1}{n^{m+1}}. \tag{6.27}$$

If we plug the result from (6.27) in (6.26), we obtain

$$S_{n,m} - S_{n-1,m} = \frac{H_n}{n^m} - \frac{m+1}{n^{m+1}} + \sum_{j=1}^{m} \frac{H_n^{(m-j+1)}}{n^j}. \tag{6.28}$$

Replacing n by i in (6.28), $S_{i,m} - S_{i-1,m} = \frac{H_i}{i^m} - \frac{m+1}{i^{m+1}} + \sum_{j=1}^{m} \frac{H_i^{(m-j+1)}}{i^j}$, and then summing both sides from $i = 1$ to n, we get

$$\sum_{i=1}^{n} (S_{i,m} - S_{i-1,m}) = S_{n,m} - S_{0,m} = \sum_{i=1}^{n} \frac{H_i}{i^m} - \sum_{i=1}^{n} \frac{m+1}{i^{m+1}} + \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{H_i^{(m-j+1)}}{i^j}$$

$$= \sum_{i=1}^{n} \frac{H_i}{i^m} + \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{H_i^{(m-j+1)}}{i^j} - (m+1) H_n^{(m+1)},$$

and since $S_{0,m} = 0$, we arrive at

$$S_{n,m} = \sum_{i=1}^{n} \frac{H_i}{i^m} + \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{H_i^{(m-j+1)}}{i^j} - (m+1) H_n^{(m+1)}. \tag{6.29}$$

Since the double sum in (6.29) can be split into a sum of sums of the type $\sum_{k=1}^{n}\frac{H_k^{(p)}}{k^q}+\sum_{k=1}^{n}\frac{H_k^{(q)}}{k^p} = H_n^{(p)}H_n^{(q)}+H_n^{(p+q)}$, which is straightforward to prove by Abel's summation (see (2.1)), then, according to m odd and even, we obtain that

$$\sum_{k=1}^{n-1}\frac{H_k^{(m)}}{n-k}=\begin{cases}\displaystyle\sum_{i=1}^{n}\frac{H_i}{i^{2r-1}}+\sum_{i=1}^{r-1}H_n^{(2r-i)}H_n^{(i)}+\frac{1}{2}(H_n^{(r)})^2-\frac{2r+1}{2}H_n^{(2r)}, & m=2r-1\geq 1;\\ \displaystyle\sum_{i=1}^{n}\frac{H_i}{i^{2r}}+\sum_{i=1}^{r}H_n^{(2r-i+1)}H_n^{(i)}-(r+1)H_n^{(2r+1)}, & m=2r\geq 2.\end{cases}$$

It's easy to see that for $m=1$ we have

$$\sum_{k=1}^{n-1}\frac{H_k}{n-k}=\sum_{i=1}^{n}\frac{H_i}{i}+\frac{1}{2}H_n^2-\frac{3}{2}H_n^{(2)}=H_n^2-H_n^{(2)},$$

where I made use of the second equality in (1.5), $\sum_{k=1}^{n}\frac{H_k}{k}=\frac{1}{2}(H_n^2+H_n^{(2)})$.

Hence, we conclude that

$$\sum_{k=1}^{n-1}\frac{H_k^{(m)}}{n-k}=\begin{cases}H_n^2-H_n^{(2)}, & m=1;\\ \displaystyle\sum_{i=1}^{n}\frac{H_i}{i^{2r-1}}+\sum_{i=1}^{r-1}H_n^{(2r-i)}H_n^{(i)}+\frac{1}{2}(H_n^{(r)})^2-\frac{2r+1}{2}H_n^{(2r)}, & m=2r-1\geq 3;\\ \displaystyle\sum_{i=1}^{n}\frac{H_i}{i^{2r}}+\sum_{i=1}^{r}H_n^{(2r-i+1)}H_n^{(i)}-(r+1)H_n^{(2r+1)}, & m=2r\geq 2,\end{cases}$$

and the solution is complete.

To add a final note to the current section, we observe the key strategy was pretty simple and natural, and it was about exploiting the difference of a sum when one of its variables takes different values. In the next section we'll use the same strategy!

The sums of this type have been studied in the mathematical literature. For example, one may find the case $m=1$, in a slightly modified form, in [41, Example 2, p. 846] and [32]. An extension of my generalization, which essentially uses the same idea of getting recurrence relations presented in my original derivation above, may be found in [2, p. 23] (the strategy involving recurrence relations is old in the mathematical literature, and in [41] you may find similar sums evaluated with such a strategy).

6.14 Four Elementary Sums with Harmonic Numbers, Very Useful in the Calculation of the Harmonic Series of Weight 7

Solution Like the previous generalized sum, we'll find the four harmonic sums in this section particularly useful in the calculation of some (advanced) harmonic series. For example, any of the second and third sums is critical in the derivation of the series from Sect. 4.42, depending on the extraction strategy we choose.

Also, as suggested at the end of the previous section, here we'll adopt the same strategy of exploiting the difference of a sum when one of its variables takes different values.

Since $S_1(n) = \sum_{k=1}^{n-1} \frac{H_k^2}{n-k} = \sum_{k=1}^{n-1} \frac{H_{n-k}^2}{k}$, we consider the difference

$$S_1(n) - S_1(n-1) = \sum_{k=1}^{n-1} \frac{H_{n-k}^2}{k} - \sum_{k=1}^{n-2} \frac{H_{n-k-1}^2}{k} = \sum_{k=1}^{n-1} \frac{H_{n-k}^2 - H_{n-k-1}^2}{k}$$

$$= 2\sum_{k=1}^{n-1} \frac{H_{n-k}}{k(n-k)} - \sum_{k=1}^{n-1} \frac{1}{k(n-k)^2}. \tag{6.30}$$

For the first sum in (6.30), we change the order of summing its terms, and we have

$$\sum_{k=1}^{n-1} \frac{H_{n-k}}{k(n-k)} = \sum_{k=1}^{n-1} \frac{H_k}{k(n-k)} = \frac{1}{n}\sum_{k=1}^{n} \frac{H_k}{k} + \frac{1}{n}\sum_{k=1}^{n-1} \frac{H_k}{n-k} - \frac{H_n}{n^2}$$

{use the second equality in (1.5) and the result in (4.16), the case $m = 1$}

$$= \frac{3}{2}\frac{H_n^2}{n} - \frac{1}{2}\frac{H_n^{(2)}}{n} - \frac{H_n}{n^2}. \tag{6.31}$$

Then, for the last sum in (6.30), we have

$$\sum_{k=1}^{n-1} \frac{1}{k(n-k)^2} = \frac{1}{n^2}\sum_{k=1}^{n-1} \frac{1}{k} + \frac{1}{n^2}\sum_{k=1}^{n-1} \frac{1}{n-k} + \frac{1}{n}\sum_{k=1}^{n-1} \frac{1}{(n-k)^2} = 2\frac{H_n}{n^2} + \frac{H_n^{(2)}}{n} - \frac{3}{n^3}. \tag{6.32}$$

If we plug the results from (6.31) and (6.32) in (6.30), we obtain

$$S_1(n) - S_1(n-1) = \frac{3}{n^3} - 4\frac{H_n}{n^2} + 3\frac{H_n^2}{n} - 2\frac{H_n^{(2)}}{n}. \tag{6.33}$$

Replacing n by i in (6.33) and then summing both sides from $i = 1$ to n, where $S_1(0) = 0$ and $S_1(1) = 0$, we get

$$S_1(n) = 3H_n^{(3)} - 4\sum_{i=1}^{n} \frac{H_i}{i^2} + 3\sum_{i=1}^{n} \frac{H_i^2 + H_i^{(2)}}{i} - 5\sum_{i=1}^{n} \frac{H_i^{(2)}}{i}$$

{for the second sum make use of the second equality in (1.6)}

$$= H_n^3 + 3H_nH_n^{(2)} + 5H_n^{(3)} - 4\sum_{i=1}^{n} \frac{H_i}{i^2} - 5\sum_{i=1}^{n} \frac{H_i^{(2)}}{i}. \tag{6.34}$$

It's easy to see the last sum in (6.34) can be rewritten as

$$\sum_{i=1}^{n} \frac{H_i^{(2)}}{i} = H_nH_n^{(2)} + H_n^{(3)} - \sum_{i=1}^{n} \frac{H_i}{i^2}, \tag{6.35}$$

if we use the more general result

$$\sum_{i=1}^{n} \frac{H_i^{(p)}}{i^q} = \sum_{i=1}^{n}\sum_{j=1}^{i} \frac{1}{i^q j^p} = \sum_{j=1}^{n}\sum_{i=j}^{n} \frac{1}{i^q j^p} = \sum_{j=1}^{n} \frac{1}{j^p}\left(\frac{1}{j^q} + H_n^{(q)} - H_j^{(q)}\right)$$

$$= H_n^{(p+q)} + H_n^{(p)}H_n^{(q)} - \sum_{i=1}^{n} \frac{H_i^{(q)}}{i^p}, \tag{6.36}$$

and set $p = 2$ and $q = 1$.

If we plug the result from (6.35) in (6.34), we get

$$S_1(n) = \sum_{k=1}^{n-1} \frac{H_{n-k}^2}{k} = \sum_{k=1}^{n-1} \frac{H_k^2}{n-k} = H_n^3 - 2H_nH_n^{(2)} + \sum_{i=1}^{n} \frac{H_i}{i^2},$$

and the solution to the first sum is complete.

To calculate $S_2(n) = \sum_{k=1}^{n-1} \frac{H_k^2}{(n-k)^2} = \sum_{k=1}^{n-1} \frac{H_{n-k}^2}{k^2}$, we consider the difference

$$S_2(n) - S_2(n-1) = \sum_{k=1}^{n-1} \frac{H_{n-k}^2}{k^2} - \sum_{k=1}^{n-2} \frac{H_{n-k-1}^2}{k^2} = \sum_{k=1}^{n-1} \frac{H_{n-k}^2 - H_{n-k-1}^2}{k^2}$$

$$= 2\sum_{k=1}^{n-1} \frac{H_{n-k}}{k^2(n-k)} - \sum_{k=1}^{n-1} \frac{1}{k^2(n-k)^2}. \tag{6.37}$$

For the first sum in (6.37), we change the order of summing its terms, and we have

$$\sum_{k=1}^{n-1} \frac{H_{n-k}}{k^2(n-k)} = \sum_{k=1}^{n-1} \frac{H_k}{k(n-k)^2} = \frac{1}{n^2}\sum_{k=1}^{n} \frac{H_k}{k} + \frac{1}{n^2}\sum_{k=1}^{n-1} \frac{H_k}{n-k}$$

$$+\frac{1}{n}\sum_{k=1}^{n-1} \frac{H_k}{(n-k)^2} - \frac{H_n}{n^3}$$

$$\{\text{for the first sum make use of the result in (1.5), and for the second}\}$$

$$\{\text{and third sums make use of the cases } m = 1 \text{ and } m = 2 \text{ from (4.16)}\}$$

$$= \frac{3}{2}\frac{H_n^2}{n^2} - \frac{H_n}{n^3} - \frac{1}{2}\frac{H_n^{(2)}}{n^2} + \frac{H_n H_n^{(2)}}{n} - 2\frac{H_n^{(3)}}{n} + \frac{1}{n}\sum_{k=1}^{n} \frac{H_k}{k^2}, \tag{6.38}$$

where in the calculations I used that $\sum_{k=1}^{n-1} \frac{H_k}{(n-k)^2} = \sum_{k=1}^{n-1} \frac{H_{n-k}}{k^2} = \sum_{k=1}^{n-1} \frac{H_k^{(2)}}{n-k}$, and the last equality is obtained by means of Abel's summation.

Then, for the second sum in (6.37), we have

$$\sum_{k=1}^{n-1} \frac{1}{k^2(n-k)^2} = \frac{2}{n^3}\sum_{k=1}^{n-1}\left(\frac{1}{k} + \frac{1}{n-k}\right) + \frac{1}{n^2}\sum_{k=1}^{n-1}\left(\frac{1}{k^2} + \frac{1}{(n-k)^2}\right)$$

$$= 4\frac{H_n}{n^3} + 2\frac{H_n^{(2)}}{n^2} - \frac{6}{n^4}. \tag{6.39}$$

If we plug the results from (6.38) and (6.39) in (6.37), we get

$$S_2(n) - S_2(n-1) = \frac{6}{n^4} + 3\frac{H_n^2}{n^2} + 2\frac{H_n H_n^{(2)}}{n} - 6\frac{H_n}{n^3} - 4\frac{H_n^{(3)}}{n} - 3\frac{H_n^{(2)}}{n^2} + \frac{2}{n}\sum_{k=1}^{n} \frac{H_k}{k^2}. \tag{6.40}$$

Replacing n by i in (6.40) and then summing both sides from $i = 1$ to n, where $S_2(0) = 0$ and $S_2(1) = 0$, we have

$$S_2(n) = 6H_n^{(4)} + 3\sum_{i=1}^{n} \frac{H_i^2}{i^2} + 2\sum_{i=1}^{n} \frac{H_i H_i^{(2)}}{i} - 6\sum_{i=1}^{n} \frac{H_i}{i^3} - 4\sum_{i=1}^{n} \frac{H_i^{(3)}}{i} - 3\sum_{i=1}^{n} \frac{H_i^{(2)}}{i^2}$$

$$+ 2\sum_{i=1}^{n} \frac{1}{i}\sum_{k=1}^{i} \frac{H_k}{k^2}. \tag{6.41}$$

For the second sum in (6.41), we make use of Abel's summation (see (2.1)) where we set $a_i = H_i/i$ and $b_i = H_i^{(2)}$, and using the second equality in (1.5), we are led to

$$\sum_{i=1}^{n} \frac{H_i H_i^{(2)}}{i} = \frac{1}{2}(H_n^2 + H_n^{(2)})H_{n+1}^{(2)} + \frac{1}{2}\sum_{i=1}^{n}(H_i^2 + H_i^{(2)})(H_i^{(2)} - H_{i+1}^{(2)})$$

$$= \frac{1}{2}(H_n^2 + H_n^{(2)})\left(H_n^{(2)} + \frac{1}{(n+1)^2}\right) - \frac{1}{2}\sum_{i=1}^{n}\frac{(H_{i+1} - \frac{1}{i+1})^2 + (H_{i+1}^{(2)} - \frac{1}{(i+1)^2})}{(i+1)^2}$$

$$\{\text{reindex the sum, expand it, and let out the term for } i = n+1\}$$

$$= \frac{1}{2}H_n^2 H_n^{(2)} + \frac{1}{2}(H_n^{(2)})^2 - \frac{1}{2}\sum_{i=1}^{n}\frac{H_i^2}{i^2} + \sum_{i=1}^{n}\frac{H_i}{i^3} - \frac{1}{2}\sum_{i=1}^{n}\frac{H_i^{(2)}}{i^2}$$

$$\{\text{for the last sum make use of the result in (4.14), where we set } p = 2\}$$

$$= \frac{1}{2}H_n^2 H_n^{(2)} + \frac{1}{4}(H_n^{(2)})^2 - \frac{1}{4}H_n^{(4)} + \sum_{i=1}^{n}\frac{H_i}{i^3} - \frac{1}{2}\sum_{i=1}^{n}\frac{H_i^2}{i^2}. \tag{6.42}$$

For the fourth sum in (6.41), based on (6.36) with, say, $p = 3$ and $q = 1$, we have

$$\sum_{i=1}^{n}\frac{H_i^{(3)}}{i} = H_n H_n^{(3)} + H_n^{(4)} - \sum_{i=1}^{n}\frac{H_i}{i^3}. \tag{6.43}$$

Next, for the last sum in (6.41), we apply Abel's summation with $a_i = \dfrac{1}{i}$ and $b_i = \displaystyle\sum_{k=1}^{i}\frac{H_k}{k^2}$, that gives

$$\sum_{i=1}^{n}\frac{1}{i}\sum_{k=1}^{i}\frac{H_k}{k^2} = H_n\sum_{k=1}^{n+1}\frac{H_k}{k^2} - \sum_{i=1}^{n}\frac{(H_{i+1} - 1/(i+1))H_{i+1}}{(i+1)^2}$$

$$\{\text{reindex the second sum and leave out the } (n+1)\text{th term of both sums}\}$$

$$= H_n\sum_{i=1}^{n}\frac{H_i}{i^2} - \sum_{i=1}^{n}\frac{H_i^2}{i^2} + \sum_{i=1}^{n}\frac{H_i}{i^3}. \tag{6.44}$$

Collecting the results from (6.42), (6.43), (4.14), the case $p = 2$, and (6.44) in (6.41), we obtain that

$$S_2(n) = \sum_{k=1}^{n-1} \frac{H_{n-k}^2}{k^2} = \sum_{k=1}^{n-1} \frac{H_k^2}{(n-k)^2}$$

$$= H_n^2 H_n^{(2)} - 4H_n H_n^{(3)} - (H_n^{(2)})^2 + 2\sum_{i=1}^{n} \frac{H_i}{i^3} + 2H_n \sum_{i=1}^{n} \frac{H_i}{i^2},$$

and the solution to the second sum is complete.

Further, for $S_3(n) = \sum_{k=1}^{n-1} \frac{H_k^{(2)}}{(n-k)^2} = \sum_{k=1}^{n-1} \frac{H_{n-k}^{(2)}}{k^2}$, we calculate the difference

$$S_3(n) - S_3(n-1) = \sum_{k=1}^{n-1} \frac{H_{n-k}^{(2)}}{k^2} - \sum_{k=1}^{n-2} \frac{H_{n-k-1}^{(2)}}{k^2} = \sum_{k=1}^{n-1} \frac{1}{k^2(n-k)^2}$$

$$= 4\frac{H_n}{n^3} + 2\frac{H_n^{(2)}}{n^2} - \frac{6}{n^4}, \tag{6.45}$$

where the last sum is calculated in (6.39).

Replacing n by i in (6.14) and then summing both sides from $i = 1$ to n, where $S_3(0) = 0$ and $S_3(1) = 0$, we get

$$S_3(n) = \sum_{k=1}^{n-1} \frac{H_{n-k}^{(2)}}{k^2} = \sum_{k=1}^{n-1} \frac{H_k^{(2)}}{(n-k)^2} = 4\sum_{i=1}^{n} \frac{H_i}{i^3} + 2\sum_{i=1}^{n} \frac{H_i^{(2)}}{i^2} - 6\sum_{i=1}^{n} \frac{1}{i^4}$$

$$= (H_n^{(2)})^2 - 5H_n^{(4)} + 4\sum_{i=1}^{n} \frac{H_i}{i^3},$$

where for the last equality I made use of the result in (4.14), the case $p = 2$, and the solution to the third sum is complete.

Lastly, for $S_4(n) = \sum_{k=1}^{n-1} \frac{H_k^3}{n-k} = \sum_{k=1}^{n-1} \frac{H_{n-k}^3}{k}$, we calculate the difference

$$S_4(n) - S_4(n-1) = \sum_{k=1}^{n-1} \frac{H_{n-k}^3}{k} - \sum_{k=1}^{n-2} \frac{H_{n-k-1}^3}{k} = \sum_{k=1}^{n-1} \frac{H_{n-k}^3 - H_{n-k-1}^3}{k}$$

$$= 3\sum_{k=1}^{n-1} \frac{H_{n-k}^2}{k(n-k)} - 3\sum_{k=1}^{n-1} \frac{H_{n-k}}{k(n-k)^2} + \sum_{k=1}^{n-1} \frac{1}{k(n-k)^3}. \tag{6.46}$$

For the first sum in (6.46), we change the order of summing its terms, and we have

$$\sum_{k=1}^{n-1}\frac{H_{n-k}^2}{k(n-k)}=\sum_{k=1}^{n-1}\frac{H_k^2}{k(n-k)}=\frac{1}{n}\sum_{k=1}^{n-1}\frac{H_k^2+H_k^{(2)}}{k}-\frac{1}{n}\sum_{k=1}^{n-1}\frac{H_k^{(2)}}{k}+\frac{1}{n}\sum_{k=1}^{n-1}\frac{H_k^2}{n-k}$$

$$=\frac{4}{3}\frac{H_n^3}{n}-\frac{H_n^2}{n^2}-2\frac{H_nH_n^{(2)}}{n}-\frac{1}{3}\frac{H_n^{(3)}}{n}+\frac{2}{n}\sum_{k=1}^{n}\frac{H_k}{k^2}, \tag{6.47}$$

where we used that, based on the second equality in (1.6), we have $\displaystyle\sum_{k=1}^{n-1}\frac{H_k^2+H_k^{(2)}}{k}=\frac{1}{3}(H_n^3+3H_nH_n^{(2)}+2H_n^{(3)})-\frac{H_n^2}{n}-\frac{H_n^{(2)}}{n}$, then based on the result in (6.35), we have $\displaystyle\sum_{k=1}^{n-1}\frac{H_k^{(2)}}{k}=H_nH_n^{(2)}+H_n^{(3)}-\frac{H_n^{(2)}}{n}-\sum_{k=1}^{n}\frac{H_k}{k^2}$, and we recognized that $\displaystyle\sum_{k=1}^{n-1}\frac{H_k^2}{n-k}$ is S_1.

Further, for the second sum in (6.46), changing the order of summing its terms, we have

$$\sum_{k=1}^{n-1}\frac{H_{n-k}}{k(n-k)^2}=\sum_{k=1}^{n-1}\frac{H_k}{k^2(n-k)}=\frac{1}{n^2}\sum_{k=1}^{n-1}\frac{H_k}{k}+\frac{1}{n^2}\sum_{k=1}^{n-1}\frac{H_k}{n-k}+\frac{1}{n}\sum_{k=1}^{n-1}\frac{H_k}{k^2}$$

{the first sum is found in (1.5), and note the second sum is (4.16), with $m=1$}

$$=\frac{3}{2}\frac{H_n^2}{n^2}-2\frac{H_n}{n^3}-\frac{1}{2}\frac{H_n^{(2)}}{n^2}+\frac{1}{n}\sum_{k=1}^{n}\frac{H_k}{k^2}. \tag{6.48}$$

Then, considering the last sum in (6.46), we write that

$$\sum_{k=1}^{n-1}\frac{1}{k(n-k)^3}=\frac{1}{n^3}\sum_{k=1}^{n-1}\frac{1}{k}+\frac{1}{n^3}\sum_{k=1}^{n-1}\frac{1}{n-k}+\frac{1}{n^2}\sum_{k=1}^{n-1}\frac{1}{(n-k)^2}+\frac{1}{n}\sum_{k=1}^{n-1}\frac{1}{(n-k)^3}$$

$$=2\frac{H_n}{n^3}+\frac{H_n^{(2)}}{n^2}+\frac{H_n^{(3)}}{n}-\frac{4}{n^4}. \tag{6.49}$$

If we collect the sums from (6.47), (6.48), and (6.49) in (6.46), we get that

$$S_4(n)-S_4(n-1)=4\frac{H_n^3+3H_nH_n^{(2)}+2H_n^{(3)}}{n}-\frac{15}{2}\frac{H_n^2}{n^2}-18\frac{H_nH_n^{(2)}}{n}$$

$$-8\frac{H_n^{(3)}}{n}+8\frac{H_n}{n^3}+\frac{5}{2}\frac{H_n^{(2)}}{n^2}-\frac{4}{n^4}+\frac{3}{n}\sum_{k=1}^{n}\frac{H_k}{k^2}. \tag{6.50}$$

Replacing n by i in (6.50) and then summing both sides from $i = 1$ to n, where $S_4(0) = 0$ and $S_4(1) = 0$, we get

$$S_4(n) = 4\sum_{i=1}^{n} \frac{H_i^3 + 3H_i H_i^{(2)} + 2H_i^{(3)}}{i} - \frac{15}{2}\sum_{i=1}^{n} \frac{H_i^2}{i^2} - 18\sum_{i=1}^{n} \frac{H_i H_i^{(2)}}{i} - 8\sum_{i=1}^{n} \frac{H_i^{(3)}}{i}$$

$$+ 8\sum_{i=1}^{n} \frac{H_i}{i^3} + \frac{5}{2}\sum_{i=1}^{n} \frac{H_i^{(2)}}{i^2} - 4H_n^{(4)} + 3\sum_{i=1}^{n} \frac{1}{i}\sum_{k=1}^{i} \frac{H_k}{k^2}. \tag{6.51}$$

Finally, in (6.51) we use the second equality in (1.7), the results in (6.42), (6.43), to express $\displaystyle\sum_{i=1}^{n} \frac{H_i^{(3)}}{i}$ in terms of $\displaystyle\sum_{i=1}^{n} \frac{H_i}{i^3}$, (4.14), with $p = 2$, (6.44), and we obtain that

$$S_4(n) = \sum_{k=1}^{n-1} \frac{H_k^3}{n-k} = \sum_{k=1}^{n-1} \frac{H_{n-k}^3}{k}$$

$$= H_n^4 - 3H_n^2 H_n^{(2)} - \frac{1}{4}\left(H_n^{(2)}\right)^2 - \frac{1}{4}H_n^{(4)} + \sum_{i=1}^{n} \frac{H_i}{i^3} - \frac{3}{2}\sum_{i=1}^{n} \frac{H_i^2}{i^2} + 3H_n\sum_{i=1}^{n} \frac{H_i}{i^2},$$

and the solution to the fourth sum is complete.

During the calculations we have seen again the usefulness of the harmonic sums in Sect. 1.3. The sum in S_3 is also considered by the generalization in [2, p. 23].

The given solutions also answer the proposed *challenging question*.

6.15 The Master Theorem of Series, a New Very Useful Theorem in the Calculation of Many Difficult (Harmonic) Series

Solution *The Master Theorem of Series* is a theorem I developed for generating identities with the generalized harmonic numbers that we may further exploit for deriving harmonic series or sums of harmonic series of various weights.[5]

---

[5] *What is the weight of a harmonic series?* Considering a harmonic series of the type $\displaystyle\sum_{n=1}^{\infty} \frac{H_n^{(a_1)} H_n^{(a_2)} \cdots H_n^{(a_k)}}{n^q}$, the weight is represented by $w = a_1 + a_2 + \cdots + a_k + q$. Let's also take two examples: $\displaystyle\sum_{n=1}^{\infty} \left(\frac{H_n}{n}\right)^2 = \sum_{n=1}^{\infty} \frac{H_n H_n}{n^2}$, from which we get immediately that the weight is

The theorem was published in the paper *A master theorem of series and an evaluation of a cubic harmonic series* in *Journal of Classical Analysis*, Vol. 10, No. 2, 2017 (see [45]), where I use it to calculate the cubic harmonic series, $\displaystyle\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^3$.

A big challenge for this book was on one hand to add together and calculate harmonic series of weight 4, 5, 6, 7, in a new fashion, where a critical part is played by *The Master Theorem of Series*, and on the other hand, to offer simple, easy to learn techniques based upon real methods to attack these series. Besides approaches for the harmonic series that use integrals, we'll have approaches where one only needs clever series manipulations (where *The Master Theorem of Series* plays a key part)!

For example, if we consult the mathematical literature, we might realize immediately we should *jump* in various papers pretty specialized for collecting the values of the harmonic series we are interested in (and this is if we're lucky enough to have access to those papers), a point that will be understood well from the next sections where I also added references with papers that treat such series.

Is it important to know how to calculate the harmonic series? Definitely! There is a huge spectrum of problems with integrals and series, as in this book, that eventually get reduced to harmonic series of various weights where we need to know how to approach them in order to solve the proposed problems.

To close this little intro, I would say that considering the number of the harmonic series of various weights added in the book, the new style of approaching them by using *The Master Theorem of Series* and the simplicity of most of the solutions (we can finish them by series manipulations only), all together manage to give a distinguishing, unique note to the present book.

To prove the first version of the theorem, we start with considering the partial sum of the series, and using that $\displaystyle\frac{1}{n}\sum_{j=1}^{n}\frac{1}{(j+k)(j+k+1)}=\frac{1}{(k+1)(k+n+1)}$, we have

$$\sum_{k=1}^{N}\frac{\mathscr{M}(k)}{(k+1)(k+n+1)}=\frac{1}{n}\sum_{k=1}^{N}\sum_{j=1}^{n}\frac{\mathscr{M}(k)}{(j+k)(j+k+1)}$$

$$=\frac{1}{n}\sum_{k=1}^{N}\sum_{j=1}^{n}\left(\frac{\mathscr{M}(k)}{j+k}-\frac{\mathscr{M}(k)}{j+k+1}\right)$$

$$=\frac{1}{n}\sum_{k=1}^{N}\sum_{j=1}^{n}\left(\frac{\mathscr{M}(k)}{j+k}-\frac{\mathscr{M}(k+1)-m(k+1)}{j+k+1}\right)$$

$w=1+1+2=4$, since H_n may be viewed as $H_n^{(1)}$, and then the series $\displaystyle\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^2}$, where it's easy to see the weight is $w=1+2+2=5$.

$$= \frac{1}{n}\sum_{k=1}^{N}\sum_{j=1}^{n}\left(\frac{\mathscr{M}(k)}{j+k}-\frac{\mathscr{M}(k+1)}{j+k+1}+\frac{m(k+1)}{j+k+1}\right)$$

$$= \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{N}\left(\frac{\mathscr{M}(k)}{j+k}-\frac{\mathscr{M}(k+1)}{j+k+1}+\frac{m(k+1)}{j+k+1}\right)$$

$$= \frac{1}{n}\sum_{j=1}^{n}\left(\sum_{k=1}^{N}\left(\frac{\mathscr{M}(k)}{j+k}-\frac{\mathscr{M}(k+1)}{j+k+1}\right)+\sum_{k=1}^{N}\frac{m(k+1)}{j+k+1}\right). \tag{6.52}$$

For the first inner sum in (6.52), we have

$$\sum_{k=1}^{N}\left(\frac{\mathscr{M}(k)}{j+k}-\frac{\mathscr{M}(k+1)}{j+k+1}\right)=\frac{\mathscr{M}(1)}{j+1}-\frac{\mathscr{M}(N+1)}{j+N+1}=\frac{m(1)}{j+1}-\frac{\mathscr{M}(N+1)}{j+N+1}. \tag{6.53}$$

Then, by plugging the result from (6.53) in (6.52), we obtain that

$$\sum_{k=1}^{N}\frac{\mathscr{M}(k)}{(k+1)(k+n+1)}=\frac{1}{n}\sum_{j=1}^{n}\left(\frac{m(1)}{j+1}-\frac{\mathscr{M}(N+1)}{j+N+1}+\sum_{k=1}^{N}\frac{m(k+1)}{j+k+1}\right)$$

$$= m(1)\left(\frac{H_n}{n}-\frac{1}{n+1}\right)-\frac{1}{n}\sum_{j=1}^{n}\frac{\mathscr{M}(N+1)}{j+N+1}+\frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{N}\frac{m(k+1)}{j+k+1}. \tag{6.54}$$

Letting now $N\to\infty$ in (6.54), and using the *Stolz–Cesàro theorem* (see [43], [16, pp. 59–61]) to show that $\lim_{N\to\infty}\dfrac{\mathscr{M}(N+1)}{j+N+1}=0$, we obtain that

$$\sum_{k=1}^{\infty}\frac{\mathscr{M}(k)}{(k+1)(k+n+1)}=m(1)\left(\frac{H_n}{n}-\frac{1}{n+1}\right)+\frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{\infty}\frac{m(k+1)}{j+k+1}$$

$$= m(1)\left(\frac{H_n}{n}-\frac{1}{n+1}\right)+\frac{1}{n}\sum_{j=1}^{n}\sum_{k=2}^{\infty}\frac{m(k)}{j+k}$$

$$= m(1)\left(\frac{H_n}{n}-\frac{1}{n+1}\right)+\frac{1}{n}\sum_{j=1}^{n}\left(-\frac{m(1)}{j+1}+\sum_{k=1}^{\infty}\frac{m(k)}{j+k}\right)=\frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{\infty}\frac{m(k)}{j+k},$$

and the solution to the first version is finalized.

Proving the second version is almost identical with the previous proof, except that for the first sum in (6.54), we write that

$$\left|\frac{1}{n}\sum_{j=1}^{n}\frac{\mathscr{M}(N+1)}{j+N+1}\right| \le \frac{1}{n}\sum_{j=1}^{n}\left|\frac{\mathscr{M}(N+1)}{j+N+1}\right| \le \frac{1}{n}\sum_{j=1}^{n}\left|\frac{\mathscr{M}(N+1)}{N+1}\right| = \left|\frac{\mathscr{M}(N+1)}{N+1}\right|,$$

where if we let $N \to \infty$, we conclude that $\lim_{N\to\infty}\frac{1}{n}\sum_{j=1}^{n}\frac{\mathscr{M}(N+1)}{j+N+1} = 0$, and the solution to the second version is finalized.

As seen, I also included a relaxed version of the theorem, one that allows us to approach cases like $m(k) = \sin(k\theta)$ and $m(k) = \cos(k\theta)$ which we met in the last section of the first chapter.

Although the second version of the theorem implies the first version, in practice, and especially in the work with the harmonic series as you'll see in the present book, the readers might find easier to make use of the first version and check $\lim_{k\to\infty} m(k)$ to see if we can apply the theorem. If $\lim_{k\to\infty} m(k)$ doesn't exist (e.g., $m(k) = \sin(k)$), one may try the second version of the theorem and check $\lim_{k\to\infty}\frac{\mathscr{M}(k)}{k}$. Recently, Prof. Necdet Batir, Turkey, proposed in [7] a generalization of my theorem.

6.16 The First Application of *The Master Theorem of Series* on the (Generalized) Harmonic Numbers

Solution This is the first section from a series of four consecutive sections that are dedicated to the applications of *The Master Theorem of Series*. Basically, we'll find particular cases of the present application helpful in many of the next sections, and therefore we want to know how to derive it.

To prove the result, we make use of *The Master Theorem of Series*, the second equality, where we set $\mathscr{M}(k) = H_k^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{k^m}$, $m(k) = \frac{1}{k^m}$, and then we get

$$\sum_{k=1}^{\infty}\frac{H_k^{(m)}}{(k+1)(k+n+1)} = \frac{1}{n}\sum_{j=1}^{n}\underbrace{\sum_{k=1}^{\infty}\frac{1}{k^m(j+k)}}_{S_m}. \tag{6.55}$$

For the series in the right-hand side of (6.55), we note it can be written as

$$S_m = \sum_{k=1}^{\infty}\frac{1}{k^m(j+k)} = \sum_{k=1}^{\infty}\frac{(j+k)-j}{k^{m+1}(j+k)} = \sum_{k=1}^{\infty}\frac{1}{k^{m+1}} - j\sum_{k=1}^{\infty}\frac{1}{k^{m+1}(j+k)}$$

$$= \zeta(m+1) - jS_{m+1},$$

that is

$$S_m = \zeta(m+1) - jS_{m+1}. \tag{6.56}$$

Multiplying both sides of (6.56) by $(-1)^{m-1}j^m$, we have

$$(-1)^{m-1}j^m S_m = (-1)^{m-1}j^m \zeta(m+1) - (-1)^{m-1}j^{m+1}S_{m+1}$$

or

$$(-1)^{m-1}j^m S_m - (-1)^m j^{m+1}S_{m+1} = (-1)^{m-1}j^m \zeta(m+1). \tag{6.57}$$

Replacing m by i in (6.57) and considering the summation from $i = 1$ to $m-1$, we have

$$\sum_{i=1}^{m-1}\left((-1)^{i-1}j^i S_i - (-1)^i j^{i+1}S_{i+1}\right) = jS_1 - (-1)^{m-1}j^m S_m$$

$$= \sum_{i=1}^{m-1}(-1)^{i-1}j^i \zeta(i+1),$$

and since $S_1 = \displaystyle\sum_{k=1}^{\infty}\frac{1}{k(j+k)} = \frac{H_j}{j}$, we get

$$S_m = (-1)^{m-1}\frac{H_j}{j^m} + (-1)^{m-1}\sum_{i=1}^{m-1}(-1)^i j^{i-m}\zeta(i+1),$$

or if reindexing the last sum,

$$S_m = (-1)^{m-1}\frac{H_j}{j^m} + (-1)^{m-1}\sum_{i=2}^{m}(-1)^{i-1}j^{i-m-1}\zeta(i). \tag{6.58}$$

Plugging the result from (6.58) in (6.55), we obtain that

$$\sum_{k=1}^{\infty}\frac{H_k^{(m)}}{(k+1)(k+n+1)} = \frac{1}{n}\sum_{j=1}^{n}\underbrace{\sum_{k=1}^{\infty}\frac{1}{k^m(j+k)}}_{S_m}$$

$$= \frac{1}{n}\sum_{j=1}^{n}\left((-1)^{m-1}\frac{H_j}{j^m} + (-1)^{m-1}\sum_{i=2}^{m}(-1)^{i-1}j^{i-m-1}\zeta(i)\right)$$

$$= \frac{(-1)^{m-1}}{n}\sum_{j=1}^{n}\frac{H_j}{j^m} + \frac{(-1)^{m-1}}{n}\sum_{j=1}^{n}\sum_{i=2}^{m}(-1)^{i-1}j^{i-m-1}\zeta(i)$$

{reverse the summation order in the double sum}

$$= \frac{(-1)^{m-1}}{n}\sum_{j=1}^{n}\frac{H_j}{j^m} + \frac{(-1)^{m-1}}{n}\sum_{i=2}^{m}(-1)^{i-1}\zeta(i)\sum_{j=1}^{n}\frac{1}{j^{m-i+1}}$$

$$= \frac{(-1)^{m-1}}{n}\sum_{j=1}^{n}\frac{H_j}{j^m} + \frac{(-1)^{m-1}}{n}\sum_{i=2}^{m}(-1)^{i-1}\zeta(i)H_n^{(m-i+1)}$$

{replace j by i in the first sum}

$$= \frac{(-1)^{m-1}}{n}\sum_{i=1}^{n}\frac{H_i}{i^m} + \frac{(-1)^{m-1}}{n}\sum_{i=2}^{m}(-1)^{i-1}\zeta(i)H_n^{(m-i+1)}.$$

Since the case $m = 1$ is immediate by using (4.14), with $p = 1$, we conclude that

$$\sum_{k=1}^{\infty}\frac{H_k^{(m)}}{(k+1)(k+n+1)} = \begin{cases} \dfrac{H_n^2 + H_n^{(2)}}{2n}, & m = 1; \\ \dfrac{(-1)^{m-1}}{n}\left(\sum_{i=1}^{n}\dfrac{H_i}{i^m} + \sum_{i=2}^{m}(-1)^{i-1}\zeta(i)H_n^{(m-i+1)}\right), & m \geq 2, \end{cases}$$

and the solution is complete.

The result is known in the mathematical literature, and for a different approach see [39].

6.17 The Second Application of *The Master Theorem of Series* on the Harmonic Numbers

Solution The present result together with one of the particular cases of the generalization from the previous section leads to a key identity for the derivation of some harmonic series, which we will find as a problem in one of the next sections.

In the calculations we'll also need a sum with harmonic numbers that is calculated in Sect. 1.3.

We recall *The Master Theorem of Series*, the first equality, where we set $\mathscr{M}(k) = H_k^2$ and $m(k) = H_k^2 - H_{k-1}^2$, and then we get

$$\sum_{k=1}^{\infty} \frac{H_k^2}{(k+1)(k+n+1)} = (H_1^2 - H_0^2)\left(\frac{H_n}{n} - \frac{1}{n+1}\right) + \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{\infty} \frac{H_{k+1}^2 - H_k^2}{j+k+1}$$

$$= \frac{H_n}{n} - \frac{1}{n+1} + \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{\infty} \frac{2H_k + 1/(k+1)}{(k+1)(j+k+1)}$$

$$= \frac{H_n}{n} - \frac{1}{n+1} + \frac{2}{n}\sum_{j=1}^{n}\sum_{k=1}^{\infty} \frac{H_k}{(k+1)(j+k+1)} + \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{\infty} \frac{1}{(k+1)^2(j+k+1)},$$

and if we make use of the result in (4.21), the case $m = 1$, we get

$$\sum_{k=1}^{\infty} \frac{H_k^2}{(k+1)(k+n+1)}$$

$$= \frac{H_n}{n} - \frac{1}{n+1} + \frac{1}{n}\sum_{j=1}^{n} \frac{H_j^2 + H_j^{(2)}}{j}$$

$$+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{j}\sum_{k=1}^{\infty}\left(\frac{1}{(k+1)^2} - \frac{1}{(k+1)(j+k+1)}\right)$$

$$\left\{\text{use the second equality in (1.6) and the fact that } \sum_{k=1}^{\infty}\frac{1}{k(k+j)} = \frac{H_j}{j}\right\}$$

$$= \frac{H_n}{n} - \frac{1}{n+1} + \frac{H_n^3 + 3H_nH_n^{(2)} + 2H_n^{(3)}}{3n} + \frac{1}{n}\sum_{j=1}^{n}\frac{1}{j}\left(\zeta(2) - 1 + \frac{1}{j+1} - \frac{H_j}{j}\right)$$

$$= \frac{H_n}{n} - \frac{1}{n+1} + \frac{H_n^3 + 3H_nH_n^{(2)} + 2H_n^{(3)}}{3n} + (\zeta(2)-1)\frac{H_n}{n} + \frac{1}{n+1} - \frac{1}{n}\sum_{j=1}^{n}\frac{H_j}{j^2}$$

$$= \frac{H_n^3 + 3\zeta(2)H_n + 3H_nH_n^{(2)} + 2H_n^{(3)}}{3n} - \frac{1}{n}\sum_{j=1}^{n}\frac{H_j}{j^2},$$

and the solution is finalized.

In one of the next sections we'll want to use the present identity for the calculation of a (nasty-looking) series of weight 6. The present result also appears in my article in [45].

6.18 The Third Application of *The Master Theorem of Series* on the Harmonic Numbers

Solution The identity in the present section together with the identity in the next section will play a crucial part in the derivation of another important identity we might like to know and use in the derivation process of the harmonic series of weight 7. As in the previous section, we'll also need a sum with harmonic numbers that is calculated in Sect. 1.3.

Using the same fashion as in the previous solution, we make use of *The Master Theorem of Series*, the first equality, where we set $\mathscr{M}(k) = H_k^3$ and $m(k) = H_k^3 - H_{k-1}^3$, and then we get

$$\sum_{k=1}^{\infty} \frac{H_k^3}{(k+1)(k+n+1)} = (H_1^3 - H_0^3)\left(\frac{H_n}{n} - \frac{1}{n+1}\right) + \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{\infty} \frac{H_{k+1}^3 - H_k^3}{j+k+1}$$

$$= \frac{H_n}{n} - \frac{1}{n+1} + \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{\infty} \frac{H_{k+1}^2 + H_{k+1}H_k + H_k^2}{(k+1)(j+k+1)}$$

$$= \frac{H_n}{n} - \frac{1}{n+1} + \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{\infty} \frac{(H_k + 1/(k+1))^2 + (H_k + 1/(k+1))H_k + H_k^2}{(k+1)(j+k+1)}$$

$$= \frac{H_n}{n} - \frac{1}{n+1} + \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{\infty} \frac{3H_k^2 + 3H_k/(k+1) + 1/(k+1)^2}{(k+1)(j+k+1)}$$

$$= \frac{H_n}{n} - \frac{1}{n+1} + \frac{3}{n}\sum_{j=1}^{n}\sum_{k=1}^{\infty} \frac{H_k^2}{(k+1)(j+k+1)} + \frac{3}{n}\sum_{j=1}^{n}\sum_{k=1}^{\infty} \frac{H_k}{(k+1)^2(j+k+1)}$$

$$+ \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{\infty} \frac{1}{(k+1)^3(j+k+1)}$$

$$= \frac{H_n}{n} - \frac{1}{n+1} + \frac{3}{n}\sum_{j=1}^{n}\sum_{k=1}^{\infty} \frac{H_k^2}{(k+1)(j+k+1)} + \frac{3}{n}\sum_{j=1}^{n}\frac{1}{j}\sum_{k=1}^{\infty} \frac{H_k}{(k+1)^2}$$

$$-\frac{3}{n}\sum_{j=1}^{n}\frac{1}{j}\sum_{k=1}^{\infty}\frac{H_k}{(k+1)(j+k+1)}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{j}\sum_{k=1}^{\infty}\frac{1}{(k+1)^3}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{j^2}\sum_{k=1}^{\infty}\frac{1}{(k+1)^2}$$

$$+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{j^2}\sum_{k=1}^{\infty}\frac{1}{(k+1)(j+k+1)}$$

$$=4\zeta(3)\frac{H_n}{n}-\zeta(2)\frac{H_n^{(2)}}{n}+\frac{1}{n}\sum_{j=1}^{n}\frac{H_j}{j^3}-\frac{3}{n}\sum_{j=1}^{n}\frac{1}{j}\sum_{k=1}^{\infty}\frac{H_k}{(k+1)(j+k+1)}$$

$$+\frac{3}{n}\sum_{j=1}^{n}\sum_{k=1}^{\infty}\frac{H_k^2}{(k+1)(j+k+1)}$$

{make use of the results in (4.21), the case $m=1$, and (4.22)}

$$=4\zeta(3)\frac{H_n}{n}-\zeta(2)\frac{H_n^{(2)}}{n}+\frac{1}{n}\sum_{j=1}^{n}\frac{H_j}{j^3}-\frac{3}{n}\sum_{j=1}^{n}\frac{H_j^2+H_j^{(2)}}{2j^2}$$

$$+\frac{1}{n}\sum_{j=1}^{n}\frac{H_j^3+3H_jH_j^{(2)}+2H_j^{(3)}}{j}+\frac{3\zeta(2)}{n}\sum_{j=1}^{n}\frac{H_j}{j}-\frac{3}{n}\sum_{j=1}^{n}\frac{1}{j}\sum_{i=1}^{j}\frac{H_i}{i^2}$$

{make use of the second equalities in (1.7) and (1.5) (or (4.14), with $p=1$)}

$$=4\zeta(3)\frac{H_n}{n}+\frac{3}{2}\zeta(2)\frac{H_n^2}{n}+\frac{1}{2}\zeta(2)\frac{H_n^{(2)}}{n}+\frac{1}{n}\sum_{j=1}^{n}\frac{H_j}{j^3}-\frac{3}{2n}\sum_{j=1}^{n}\frac{H_j^2}{j^2}-\frac{3}{2n}\sum_{j=1}^{n}\frac{H_j^{(2)}}{j^2}$$

$$+\frac{H_n^4+6H_n^2H_n^{(2)}+8H_nH_n^{(3)}+3\left(H_n^{(2)}\right)^2+6H_n^{(4)}}{4n}-\frac{3}{n}\sum_{j=1}^{n}\frac{1}{j}\sum_{i=1}^{j}\frac{H_i}{i^2}. \tag{6.59}$$

Now, if we change the summation order, the double sum in (6.59) can be written as follows

$$\sum_{j=1}^{n}\sum_{i=1}^{j}\frac{H_i}{i^2j}=\sum_{i=1}^{n}\sum_{j=i}^{n}\frac{H_i}{i^2j}=\sum_{i=1}^{n}\frac{H_i}{i^2}(H_n-H_{i-1})=\sum_{i=1}^{n}\frac{H_i}{i^2}\left(H_n-H_i+\frac{1}{i}\right)$$

$$=H_n\sum_{i=1}^{n}\frac{H_i}{i^2}-\sum_{i=1}^{n}\frac{H_i^2}{i^2}+\sum_{i=1}^{n}\frac{H_i}{i^3}. \tag{6.60}$$

Hence, collecting the results from (6.60) and (4.14), with $p = 2$, in (6.59), we conclude that

$$\sum_{k=1}^{\infty} \frac{H_k^3}{(k+1)(k+n+1)}$$

$$= \frac{H_n^4 + 6\zeta(2)H_n^2 + 16\zeta(3)H_n + 8H_n H_n^{(3)} + 6H_n^2 H_n^{(2)} + 2\zeta(2)H_n^{(2)} + 3H_n^{(4)}}{4n}$$

$$-3\frac{H_n}{n}\sum_{i=1}^{n}\frac{H_i}{i^2} - \frac{2}{n}\sum_{i=1}^{n}\frac{H_i}{i^3} + \frac{3}{2n}\sum_{i=1}^{n}\frac{H_i^2}{i^2},$$

and the solution is finalized.

Additionally, besides its part in the construction of an important identity as mentioned at the beginning of the section and that we'll meet soon, the result above can also be exploited for establishing other relations between series of various weights.

6.19 The Fourth Application of *The Master Theorem of Series* on the (Generalized) Harmonic Numbers

Solution I opened the previous section by also mentioning the result in this section plays a crucial part in the derivation of an important identity (we'll meet right in the next section). As regards the calculations, we'll do them in a similar fashion as before, and we'll see that the hardest part of the problem will be reduced to two particular cases of the generalization from Sect. 4.16.

Recollecting the first equality of *The Master Theorem of Series* where we set $\mathscr{M}(k) = H_k H_k^{(2)}$ and $m(k+1) = H_{k+1}H_{k+1}^{(2)} - H_k H_k^{(2)} = \left(H_k + \frac{1}{k+1}\right)\left(H_k^{(2)} + \frac{1}{(k+1)^2}\right) - H_k H_k^{(2)} = \frac{H_k}{(k+1)^2} + \frac{H_k^{(2)}}{k+1} + \frac{1}{(k+1)^3}$, we obtain

$$\sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{(k+1)(k+n+1)}$$

$$= \frac{H_n}{n} - \frac{1}{n+1} + \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{\infty}\frac{1}{j+k+1}\left(\frac{H_k}{(k+1)^2} + \frac{H_k^{(2)}}{k+1} + \frac{1}{(k+1)^3}\right). \tag{6.61}$$

For the series in the right-hand side of (6.61), we proceed as follows

$$\sum_{k=1}^{\infty} \frac{1}{j+k+1}\left(\frac{H_k}{(k+1)^2}+\frac{H_k^{(2)}}{k+1}+\frac{1}{(k+1)^3}\right)$$

$$=\sum_{k=1}^{\infty} \frac{H_k}{(k+1)^2(j+k+1)}+\sum_{k=1}^{\infty} \frac{H_k^{(2)}}{(k+1)(j+k+1)}+\sum_{k=1}^{\infty} \frac{1}{(k+1)^3(j+k+1)}$$

$$=\frac{1}{j}\sum_{k=1}^{\infty} \frac{H_{k+1}-1/(k+1)}{(k+1)^2}-\frac{1}{j}\sum_{k=1}^{\infty} \frac{H_k}{(k+1)(k+j+1)}+\sum_{k=1}^{\infty} \frac{H_k^{(2)}}{(k+1)(j+k+1)}$$

$$+\frac{1}{j}\sum_{k=1}^{\infty} \frac{1}{(k+1)^3}-\frac{1}{j^2}\sum_{k=1}^{\infty} \frac{1}{(k+1)^2}+\frac{1}{j^2}\sum_{k=1}^{\infty} \frac{1}{(k+1)(k+j+1)}$$

{reindex the first series, expand it and then use the Euler sum in (3.45), the case $n=2$;}

$$\left\{\text{consider } \sum_{k=1}^{\infty} \frac{1}{(k+1)(k+j+1)}=\frac{H_j}{j}-\frac{1}{j+1}, \text{ and use (4.21), with } m=1 \text{ and } m=2\right\}$$

$$=-\frac{1}{j}\sum_{i=1}^{j} \frac{H_i}{i^2}+\zeta(2)\frac{H_j}{j}-\frac{H_j^2}{2j^2}-\frac{H_j^{(2)}}{2j^2}+\frac{2\zeta(3)}{j}-\frac{1}{j+1}-\frac{\zeta(2)}{j^2}+\frac{H_j}{j^3}. \tag{6.62}$$

Returning with the result from (6.62) in (6.61), we get

$$\sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{(k+1)(k+n+1)}=\frac{H_n}{n}-\frac{1}{n+1}$$

$$+\frac{1}{n}\sum_{j=1}^{n}\left(-\frac{1}{j}\sum_{i=1}^{j} \frac{H_i}{i^2}+\zeta(2)\frac{H_j}{j}-\frac{H_j^2}{2j^2}-\frac{H_j^{(2)}}{2j^2}+\frac{2\zeta(3)}{j}-\frac{1}{j+1}-\frac{\zeta(2)}{j^2}+\frac{H_j}{j^3}\right)$$

$$=-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{j}\sum_{i=1}^{j} \frac{H_i}{i^2}+\frac{\zeta(2)}{n}\sum_{j=1}^{n}\frac{H_j}{j}-\frac{1}{2n}\sum_{j=1}^{n}\frac{H_j^2}{j^2}-\frac{1}{2n}\sum_{j=1}^{n}\frac{H_j^{(2)}}{j^2}+\frac{1}{n}\sum_{j=1}^{n}\frac{H_j}{j^3}$$

$$+2\zeta(3)\frac{H_n}{n}-\zeta(2)\frac{H_n^{(2)}}{n}. \tag{6.63}$$

Now, the double sum from (6.63) is calculated in (6.60), which if we combine with the identity in (4.14), the cases $p = 1$ and $p = 2$, we conclude that

$$\sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{(k+1)(k+n+1)}$$

$$= 2\zeta(3)\frac{H_n}{n} + \frac{\zeta(2)}{2}\frac{H_n^2}{n} - \frac{\zeta(2)}{2}\frac{H_n^{(2)}}{n} - \frac{H_n^{(4)}}{4n} - \frac{(H_n^{(2)})^2}{4n} - \frac{H_n}{n}\sum_{i=1}^{n}\frac{H_i}{i^2} + \frac{1}{2n}\sum_{i=1}^{n}\frac{H_i^2}{i^2},$$

and the solution is complete.

Again, as in the previous section, we can also exploit the present result for establishing other relations between series of various weights.

6.20 Cool Identities with Ingredients Like the Generalized Harmonic Numbers and the Binomial Coefficient

Solution Based upon the applications of *The Master Theorem of Series* from the previous sections, and not only on them, we are now able to establish some very useful results. For example, the second identity may help us to calculate the cubic harmonic series, $\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^3$, or to establish other key identities with harmonic series (we'll meet in the next sections) we need in the derivation of the harmonic series. Also, if speaking about the third and fourth results, again, they allow us to get some key identities in the derivation of the harmonic series of weight 7.

In short, we want to know how to derive them since they are of great help in the derivation process of the harmonic series.

The identity from the point $i)$ is straightforward if we combine the identities from (4.17) and (4.16), the case $m = 2$, that gives

$$\sum_{k=1}^{n-1} \frac{H_k^2 - H_k^{(2)}}{n-k} = H_n^3 - 3H_n H_n^{(2)} + 2H_n^{(3)},$$

and the point $i)$ of the problem is finalized.

Alternatively, based on the use of the elementary symmetric polynomials (see [60]), $\sigma_k(x_1, x_2, \ldots, x_n) = \sum_{1\le i_1<i_2<\ldots<i_k\le n} x_{i_1}x_{i_2}\ldots x_{i_k}$, one may notice that the given identity is $\sum_{k=1}^{n-1}\frac{2\sigma_2(1, 1/2, \ldots, 1/k)}{n-k} = 6\sigma_3\left(1, \frac{1}{2}, \ldots, \frac{1}{n}\right)$, which we can

prove using the results in (6.23) with the cases $k = 2, 3$, that is $\sum_{n=1}^{\infty} x^n (H_n^2 - H_n^{(2)}) = \dfrac{\log^2(1-x)}{1-x}$ and $\sum_{n=1}^{\infty} x^n (H_n^3 - 3H_n H_n^{(2)} + 2H_n^{(3)}) = -\dfrac{\log^3(1-x)}{1-x}$, where I expressed the elementary symmetric polynomials $\sigma_k(x_1, x_2, \ldots, x_n)$ in terms of power sums (see also the suggestion for alternative solutions to the end of the solutions to Sect. 4.10). At this point, if we write $-\dfrac{\log^3(1-x)}{1-x} = -\log(1-x) \cdot \dfrac{\log^2(1-x)}{1-x}$ and apply the Cauchy product of two series (see [25, Chapter III, pp. 197–199]), using the generating functions above, and then identifying the coefficients from both sides, we immediately obtain the desired result.

The curious reader might want to use the strategy idea described above and try to easily generalize the result in terms of the elementary symmetric polynomials.

For the point $ii)$ of the problem, we combine the identities from (4.22) and (4.21), the case $m = 2$, that yields

$$\sum_{k=1}^{\infty} \frac{H_k^2 - H_k^{(2)}}{(k+1)(k+n+1)} = \frac{H_n^3 + 3H_n H_n^{(2)} + 2H_n^{(3)}}{3n},$$

which is an identity that also appears in my article in [45] where plays a key part.

Now, to get the second equality, we recall the result in (1.6), and we write that

$$\frac{H_n^3 + 3H_n H_n^{(2)} + 2H_n^{(3)}}{3n} = -\frac{1}{3}\int_0^1 x^{n-1} \log^3(1-x)\mathrm{d}x$$

$$\overset{1-x=y}{=} -\frac{1}{3}\int_0^1 (1-y)^{n-1} \log^3(y)\mathrm{d}y = -\frac{1}{3}\int_0^1 \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k y^k \log^3(y)\mathrm{d}y$$

$$= \frac{1}{3}\sum_{k=0}^{n-1} (-1)^{k-1} \binom{n-1}{k} \int_0^1 y^k \log^3(y)\mathrm{d}y = 2\sum_{k=0}^{n-1} \frac{(-1)^k}{(k+1)^4} \binom{n-1}{k}$$

$$\{\text{reindex the sum and start from } k = 1 \text{ to } k = n\}$$

$$= 2\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^4} \binom{n-1}{k-1},$$

and the point $ii)$ of the problem is finalized.

Then, for the point $iii)$ of the problem, we combine the identities from (4.23), (4.24), and (4.21), the case $m = 3$, that immediately gives

$$\sum_{k=1}^{\infty} \frac{H_k^3 - 3H_k H_k^{(2)} + 2H_k^{(3)}}{(k+1)(k+n+1)} = \frac{H_n^4 + 6H_n^2 H_n^{(2)} + 8H_n H_n^{(3)} + 3(H_n^{(2)})^2 + 6H_n^{(4)}}{4n}.$$

Further, to get the second equality, we recall the result in (1.7), and we write

$$\frac{H_n^4 + 6H_n^2 H_n^{(2)} + 8H_n H_n^{(3)} + 3\left(H_n^{(2)}\right)^2 + 6H_n^{(4)}}{4n} = \frac{1}{4}\int_0^1 x^{n-1}\log^4(1-x)\mathrm{d}x$$

$$\overset{1-x=y}{=} \frac{1}{4}\int_0^1 (1-y)^{n-1}\log^4(y)\mathrm{d}y = \frac{1}{4}\int_0^1 \sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k y^k \log^4(y)\mathrm{d}y$$

$$= \frac{1}{4}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\int_0^1 y^k\log^4(y)\mathrm{d}y = 6\sum_{k=0}^{n-1}\frac{(-1)^k}{(k+1)^5}\binom{n-1}{k}$$

$$\{\text{reindex the sum and start from } k=1 \text{ to } k=n\}$$

$$= 6\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^5}\binom{n-1}{k-1},$$

and the point iii) of the problem is finalized.

The use of the logarithmic integrals from Sect. 1.3 for the second equalities from the second and third points assures a fast, elegant solution.

For the point iv) of the problem, we want to generalize the results obtained with *The Master Theorem of Series*, and we may choose a powerful approach involving the symmetric polynomials which also easily allows us to make generalizations. So, more generally, the results from the points ii), iii), and iv) may be viewed as special cases of the generalization,

$$\sum_{k=1}^{\infty}\frac{\sigma_m(1,1/2,\ldots,1/k)}{(k+1)(k+n+1)} = \frac{1}{n}h_{m+1}\left(1,\frac{1}{2},\ldots,\frac{1}{n}\right) = \sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^{m+2}}\binom{n-1}{k-1}, \tag{6.64}$$

where $\sigma_k(x_1,x_2,\ldots,x_n) = \sum_{1\le i_1<i_2<\ldots<i_k\le n} x_{i_1}x_{i_2}\ldots x_{i_k}$ is the elementary symmetric polynomial and $h_k(x_1,x_2,\ldots,x_n) = \sum_{1\le i_1\le i_2\le\ldots\le i_k\le n} x_{i_1}x_{i_2}\ldots x_{i_k}$ is the complete homogeneous symmetric polynomial.

Using the fact that $\sum_{k=1}^{\infty} x^k\sigma_m\left(1,\frac{1}{2},\ldots,\frac{1}{k}\right) = \frac{(-1)^m}{m!}\frac{\log^m(1-x)}{1-x}$ (see 6.23), we write that

$$\sum_{k=1}^{\infty} \frac{\sigma_m(1, 1/2, \ldots, 1/k)}{(k+1)(k+n+1)} = \frac{1}{n} \sum_{k=1}^{\infty} \sigma_m\left(1, \frac{1}{2}, \ldots, \frac{1}{k}\right)\left(\frac{1}{k+1} - \frac{1}{k+n+1}\right)$$

$$= \frac{1}{n} \sum_{k=1}^{\infty} \sigma_m\left(1, \frac{1}{2}, \ldots, \frac{1}{k}\right) \int_0^1 x^k(1-x^n)\mathrm{d}x = \frac{1}{n} \int_0^1 (1-x^n)$$

$$\times \sum_{k=1}^{\infty} x^k \sigma_m\left(1, \frac{1}{2}, \ldots, \frac{1}{k}\right) \mathrm{d}x$$

$$= \frac{(-1)^m}{n \cdot m!} \int_0^1 (1-x^n) \frac{\log^m(1-x)}{1-x} \mathrm{d}x = \frac{(-1)^{m-1}}{n \cdot (m+1)!} \int_0^1 (1-x^n)(\log^{m+1}(1-x))' \mathrm{d}x$$

$$\{\text{apply the integration by parts}\}$$

$$= \frac{(-1)^{m-1}}{(m+1)!} \int_0^1 x^{n-1} \log^{m+1}(1-x)\mathrm{d}x = \frac{1}{n} h_{m+1}\left(1, \frac{1}{2}, \ldots, \frac{1}{n}\right),$$

where the last equality follows by the generalized logarithmic integral in (3.11), and the first equality in (6.64) is proved.

To prove the second equality in (6.64), we proceed in a similar style, and we write

$$\sum_{k=1}^{\infty} \frac{\sigma_m(1, 1/2, \ldots, 1/k)}{(k+1)(k+n+1)} = \frac{1}{n} \sum_{k=1}^{\infty} \sigma_m\left(1, \frac{1}{2}, \ldots, \frac{1}{k}\right)\left(\frac{1}{k+1} - \frac{1}{k+n+1}\right)$$

$$= \frac{1}{n} \sum_{k=1}^{\infty} \sigma_m\left(1, \frac{1}{2}, \ldots, \frac{1}{k}\right) \int_0^1 x^k(1-x^n)\mathrm{d}x = \frac{1}{n} \int_0^1 (1-x^n)$$

$$\times \sum_{k=1}^{\infty} x^k \sigma_m\left(1, \frac{1}{2}, \ldots, \frac{1}{k}\right) \mathrm{d}x$$

$$= \frac{(-1)^m}{n \cdot m!} \int_0^1 \frac{1-x^n}{1-x} \log^m(1-x)\mathrm{d}x \overset{x=1-y}{=} \frac{(-1)^m}{n \cdot m!} \int_0^1 \frac{1-(1-y)^n}{y} \log^m(y)\mathrm{d}y$$

$$= \frac{(-1)^m}{n \cdot m!} \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \int_0^1 y^{k-1} \log^m(y)\mathrm{d}y$$

$$= \frac{(-1)^m}{m!} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n-1}{k-1} \int_0^1 y^{k-1} \log^m(y)\mathrm{d}y$$

$$\{\text{make use of the result in (1.2)}\}$$

$$= \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^{m+2}} \binom{n-1}{k-1},$$

and the second equality in (6.64) is proved.

Now, based on the result in (6.64), with $m = 4$, and using the relations between the symmetric polynomials and the power sums (see also the ends of the solutions to the Sects. 4.10 and 1.3), we get

$$\sum_{k=1}^{\infty} \frac{H_k^4 - 6H_k^2 H_k^{(2)} + 8H_k H_k^{(3)} + 3(H_k^{(2)})^2 - 6H_k^{(4)}}{(k+1)(k+n+1)}$$

$$= \frac{H_n^5 + 10H_n^3 H_n^{(2)} + 15H_n(H_n^{(2)})^2 + 20H_n^2 H_n^{(3)} + 20H_n^{(2)} H_n^{(3)} + 30H_n H_n^{(4)} + 24H_n^{(5)}}{5n}$$

$$= 24 \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^6} \binom{n-1}{k-1},$$

and the point $iv)$ of the problem is finalized.

6.21 Special (and Very Useful) Pairs of Classical Euler Sums Arising in Many Difficult Harmonic Series

Solution Let's begin with a simple observation: all the Euler sums we want to calculate here may be viewed as particular cases belonging to the generalized series of the type, $\displaystyle\sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q}$. Now, it's known in the mathematical literature the fact that such a generalized series has evaluations in terms of zeta values when $p = 1$ and $q \geq 2$, for $p = q$ and $p + q \geq 4$, and when $p + q \geq 5$ odd and $q \geq 2$. Also, when $p + q$ is even, with $p, q > 1, p \neq q$, we have the *exceptional configurations* $(2, 4), (4, 2)$, as called in the paper *Euler Sums and Contour Integral Representations* by Philippe Flajolet and Bruno Salvy (see [20]).

In this section we'll focus on the series of the type above of weights 5, 6, 7, with $p, q > 1$ and $p \neq q$, and we'll try to calculate them elementarily by using partial fractions, exploiting the symmetry and applying Abel's summation (see (5.1)) when needed.

Since we have, by changing the summation order, that $\displaystyle\sum_{k=1}^{\infty} \frac{H_k^{(p)}}{k^q} = \sum_{k=1}^{\infty} \left(\sum_{n=1}^{k} \frac{1}{n^p k^q} \right) = \sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} \frac{1}{n^p k^q} \right) = \sum_{n=1}^{\infty} \frac{1}{n^p} \left(\frac{1}{n^q} + \zeta(q) - H_n^{(q)} \right) = \sum_{n=1}^{\infty} \frac{1}{n^{p+q}} + \zeta(q) \sum_{n=1}^{\infty} \frac{1}{n^p} - \sum_{n=1}^{\infty} \frac{H_n^{(q)}}{n^p} = \zeta(p+q) + \zeta(p)\zeta(q) - \sum_{k=1}^{\infty} \frac{H_k^{(q)}}{k^p},$
we obtain that

$$\sum_{k=1}^{\infty} \frac{H_k^{(p)}}{k^q} + \sum_{k=1}^{\infty} \frac{H_k^{(q)}}{k^p} = \zeta(p)\zeta(q) + \zeta(p+q), \tag{6.65}$$

which is a result we want to refer to in the course of the solution. The result may also be proved by Abel's summation (see (5.1)).

The first series from the point i) of the problem I needed in my article *A new proof for a classical quadratic harmonic series* published in *Journal of Classical Analysis*, Vol. 8, No. 2, 2016 (see [46]), where I provided with a new way of calculating the series $\displaystyle\sum_{n=1}^{\infty} \frac{H_n^2}{n^3}$.

So, following the line of the solution in the mentioned paper, we start with the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \left(\zeta(2) - H_n^{(2)}\right) = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{1}{n^3(n+k)^2}\right),$$

where swapping the variables in the last double series, we get

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{1}{n^3(n+k)^2}\right) = \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{1}{k^3(n+k)^2}\right) = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{1}{k^3(n+k)^2}\right). \tag{6.66}$$

Summing up both the first and the last series in (6.66), we have

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{1}{n^3(n+k)^2}\right) + \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{1}{k^3(n+k)^2}\right) = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{k^3+n^3}{k^3n^3(n+k)^2}\right)$$

$$= \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{(k+n)^3 - 3kn(k+n)}{k^3n^3(n+k)^2}\right) = \sum_{n=1}^{\infty} \frac{1}{n^3} \sum_{k=1}^{\infty} \frac{1}{k^2} + \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{\infty} \frac{1}{k^3}$$

$$-3\sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{1}{k^2n^2(n+k)}\right) = 2\zeta(2)\zeta(3) - 3\sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{1}{k^2n^2(n+k)}\right),$$

whence we obtain

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{1}{n^3(n+k)^2}\right) = \zeta(2)\zeta(3) - \frac{3}{2}\sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{1}{k^2n^2(n+k)}\right)$$

$$= \zeta(2)\zeta(3) - \frac{3}{2}\sum_{n=1}^{\infty} \frac{1}{n^3} \sum_{k=1}^{\infty} \frac{1}{k^2} + \frac{3}{2}\sum_{n=1}^{\infty} \frac{1}{n^4} \left(\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{n+k}\right)\right)$$

$$\left\{\text{make use of the fact that } \sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{n+k}\right)=H_n\right\}$$

$$=-\frac{1}{2}\zeta(2)\zeta(3)+\frac{3}{2}\sum_{n=1}^{\infty}\frac{H_n}{n^4}=\frac{9}{2}\zeta(5)-2\zeta(2)\zeta(3),$$

where for calculating the last series, I used the Euler sum in (3.45), the case $n=4$, and thus we have

$$\sum_{n=1}^{\infty}\frac{1}{n^3}\left(\zeta(2)-H_n^{(2)}\right)=\frac{9}{2}\zeta(5)-2\zeta(2)\zeta(3),$$

and since $\zeta(2)\sum_{n=1}^{\infty}\frac{1}{n^3}=\zeta(2)\zeta(3)$, we arrive at

$$S_1=\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3}=3\zeta(2)\zeta(3)-\frac{9}{2}\zeta(5), \tag{6.67}$$

and the calculation to the first series from the point $i)$ is finalized.

To calculate the series S_2, we combine the results in (6.65) (with $p=2$ and $q=3$ or $p=3$ and $q=2$) and (6.67), and then we obtain that

$$S_2=\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^2}=\frac{11}{2}\zeta(5)-2\zeta(2)\zeta(3), \tag{6.68}$$

and the calculation to the second series from the point $i)$ is finalized.

Next, for the point $ii)$ of the problem, we start with the series

$$\sum_{n=1}^{\infty}\frac{1}{n^2}(\zeta(4)-H_n^{(4)})=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{1}{n^2(n+k)^4}\right)=\sum_{k=1}^{\infty}\frac{1}{k^4}\sum_{n=1}^{\infty}\frac{1}{n^2}$$
$$-4\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{k^5}\left(\frac{1}{n}-\frac{1}{n+k}\right)\right)+3\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{k^4(n+k)^2}\right)$$
$$+2\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{k^3(k+n)^3}\right)+\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{k^2(n+k)^4}\right)$$

or

$$\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{n^2(n+k)^4}\right)-\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{k^2(n+k)^4}\right)=\frac{7}{4}\zeta(6)$$
$$-4\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{k^5}\left(\frac{1}{n}-\frac{1}{n+k}\right)\right)+2\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{k^3(k+n)^3}\right)$$
$$+3\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{k^4(n+k)^2}\right). \tag{6.69}$$

Based on symmetry, the series in the left-hand side of (6.69) vanish, and then we get

$$3\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{k^4(n+k)^2}\right)=4\sum_{k=1}^{\infty}\frac{H_k}{k^5}-2\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{k^3(k+n)^3}\right)-\frac{7}{4}\zeta(6)$$

{reindex the inner series and start from $n=k+1$}

$$=4\sum_{k=1}^{\infty}\frac{H_k}{k^5}-2\sum_{k=1}^{\infty}\left(\sum_{n=k+1}^{\infty}\frac{1}{k^3n^3}\right)-\frac{7}{4}\zeta(6)$$

$$\left\{\text{since }\sum_{k=1}^{\infty}\left(\sum_{n=1}^{k-1}\frac{1}{k^3n^3}\right)+\sum_{k=1}^{\infty}\frac{1}{k^6}+\sum_{k=1}^{\infty}\left(\sum_{n=k+1}^{\infty}\frac{1}{k^3n^3}\right)=\zeta^2(3),\right.$$
$$\left.\text{and using the fact that}\right\}$$

$$\left\{\sum_{k=1}^{\infty}\left(\sum_{n=1}^{k-1}\frac{1}{k^3n^3}\right)=\sum_{k=1}^{\infty}\left(\sum_{n=k+1}^{\infty}\frac{1}{k^3n^3}\right),\text{ we get }2\sum_{k=1}^{\infty}\left(\sum_{n=k+1}^{\infty}\frac{1}{k^3n^3}\right)\right.$$
$$\left.=\zeta^2(3)-\zeta(6)\right\}$$

$$=4\sum_{k=1}^{\infty}\frac{H_k}{k^5}-\zeta^2(3)-\frac{3}{4}\zeta(6)$$

{use the Euler sum in (3.45), the case $n=5$}

$$=\frac{25}{4}\zeta(6)-3\zeta^2(3),$$

or

$$\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{k^4(n+k)^2}\right)=\frac{25}{12}\zeta(6)-\zeta^2(3). \tag{6.70}$$

Thus, using the result in (6.70), we get

$$\frac{25}{12}\zeta(6)-\zeta^2(3)=\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{k^4(n+k)^2}\right)=\sum_{k=1}^{\infty}\frac{1}{k^4}\left(\zeta(2)-H_k^{(2)}\right)$$

$$=\zeta(2)\sum_{k=1}^{\infty}\frac{1}{k^4}-\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^4}=\frac{7}{4}\zeta(6)-\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^4},$$

whence we get the value of the series S_4,

$$S_4=\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^4}=\zeta^2(3)-\frac{\zeta(6)}{3}, \tag{6.71}$$

and the calculation to the second series from the point ii) is finalized. The series S_4 also appeared in my paper in [45] where I used the same strategy of calculating it.

The series S_3 is extracted by combining the results in (6.65) (with $p=2$ and $q=4$ or $p=4$ and $q=2$) and (6.71),

$$S_3=\sum_{k=1}^{\infty}\frac{H_k^{(4)}}{k^2}=\frac{37}{12}\zeta(6)-\zeta^2(3), \tag{6.72}$$

and the calculation to the first series from the point ii) is finalized.

Further, for the point iii) of the problem, we keep using the same approach, and we start with the series

$$\sum_{n=1}^{\infty}\frac{1}{n^5}\left(\zeta(2)-H_n^{(2)}\right)=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{1}{n^5(n+k)^2}\right)=\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{n^5(n+k)^2}\right)$$

$$=\sum_{k=1}^{\infty}\frac{1}{k^2}\sum_{n=1}^{\infty}\frac{1}{n^5}-2\sum_{k=1}^{\infty}\frac{1}{k^3}\sum_{n=1}^{\infty}\frac{1}{n^4}+3\sum_{k=1}^{\infty}\frac{1}{k^4}\sum_{n=1}^{\infty}\frac{1}{n^3}-4\sum_{k=1}^{\infty}\frac{1}{k^5}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$+5\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{k^6}\left(\frac{1}{n}-\frac{1}{n+k}\right)\right)-\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{k^5(n+k)^2}\right)$$

$$\left\{\text{use the harmonic number representation, } \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+k}\right)=H_k,\right\}$$

$$\{\text{and then make use of the Euler sum in (3.45), the case } n=6\}$$

$$= 20\zeta(7)-4\zeta(3)\zeta(4)-8\zeta(2)\zeta(5)-\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{k^5(n+k)^2}\right). \tag{6.73}$$

Noting in (6.73) that $\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{n^5(n+k)^2}\right)=\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{k^5(n+k)^2}\right)$, we obtain

$$\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{k^5(n+k)^2}\right)=10\zeta(7)-2\zeta(3)\zeta(4)-4\zeta(2)\zeta(5). \tag{6.74}$$

Then, we can write the double series in (6.74) as

$$10\zeta(7)-2\zeta(3)\zeta(4)-4\zeta(2)\zeta(5)$$
$$=\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{k^5(n+k)^2}\right)=\sum_{k=1}^{\infty}\frac{1}{k^5}\left(\zeta(2)-H_k^{(2)}\right)=\zeta(2)\zeta(5)-\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^5},$$

from which we extract the value of the series S_6,

$$S_6=\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^5}=5\zeta(2)\zeta(5)+2\zeta(3)\zeta(4)-10\zeta(7), \tag{6.75}$$

and the calculation to the second series from the point $iii)$ is finalized.

If we consider the results in (6.65) (with $p=2$ and $q=5$ or $p=5$ and $q=2$) and (6.75), we get the value of the series S_5,

$$S_5=\sum_{k=1}^{\infty}\frac{H_k^{(5)}}{k^2}=11\zeta(7)-4\zeta(2)\zeta(5)-2\zeta(3)\zeta(4), \tag{6.76}$$

and the calculation to the first series from the point $iii)$ is finalized.

Then, to pass to the point $iv)$ of the problem, we start with

$$\sum_{n=1}^{\infty}\frac{1}{n^3}\left(\zeta(4)-H_n^{(4)}\right)=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{1}{n^3(n+k)^4}\right)=\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{k=1}^{\infty}\frac{1}{k^4}$$

$$-4\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{k=1}^{\infty}\frac{1}{k^5}+10\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{1}{k^6}\left(\frac{1}{n}-\frac{1}{n+k}\right)\right)-6\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{1}{k^5(n+k)^2}\right)$$

$$-\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{3}{k^4(n+k)^3}\right)-\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{1}{k^3(n+k)^4}\right)$$

{reverse the order of summation in the last four series}

$$=\zeta(3)\zeta(4)-4\zeta(2)\zeta(5)+10\sum_{k=1}^{\infty}\frac{H_k}{k^6}-6\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{k^5(n+k)^2}\right)$$

$$-3\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{k^4(n+k)^3}\right)-\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{k^3(n+k)^4}\right)$$

{use the result in (3.45), the case $n=6$, and the result in (6.74)}

$$=3\zeta(3)\zeta(4)+10\zeta(2)\zeta(5)-20\zeta(7)-3\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{k^4(n+k)^3}\right)$$

$$-\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{k^3(n+k)^4}\right),$$

whence we get that

$$3\zeta(3)\zeta(4)+10\zeta(2)\zeta(5)-20\zeta(7)=2\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{k^3(n+k)^4}\right)$$

$$+3\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{k^4(n+k)^3}\right). \tag{6.77}$$

On the other hand, for the last double series in (6.21), we have

$$\sum_{k=1}^{\infty}\left(\sum_{n=1}^{k-1}\frac{1}{k^4n^3}\right)+\sum_{k=1}^{\infty}\frac{1}{k^7}+\underbrace{\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{k^4(n+k)^3}\right)}_{\sum_{k=1}^{\infty}\left(\sum_{n=k+1}^{\infty}\frac{1}{k^4n^3}\right)}=\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{k^4n^3}\right)=\zeta(3)\zeta(4),$$

which leads to

$$\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{k^4(n+k)^3}\right)=\zeta(3)\zeta(4)-\zeta(7)-\sum_{k=1}^{\infty}\left(\sum_{n=1}^{k-1}\frac{1}{k^4n^3}\right)$$

{change the summation order}

$$=\zeta(3)\zeta(4)-\zeta(7)-\sum_{n=1}^{\infty}\left(\sum_{k=n+1}^{\infty}\frac{1}{k^4n^3}\right)$$

{reindex the inner series}

$$=\zeta(3)\zeta(4)-\zeta(7)-\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{1}{n^3(k+n)^4}\right),$$

or

$$\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{k^4(n+k)^3}\right)+\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{1}{n^3(k+n)^4}\right)=\zeta(3)\zeta(4)-\zeta(7). \qquad (6.78)$$

If we combine the results in (6.21) and (6.78), we get

$$3\zeta(3)\zeta(4)+10\zeta(2)\zeta(5)-20\zeta(7)$$

$$=2\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{k^3(n+k)^4}\right)+3\zeta(3)\zeta(4)-3\zeta(7)-3\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{1}{n^3(k+n)^4}\right)$$

{interchange the variables in the last double series}

$$=3\zeta(3)\zeta(4)-3\zeta(7)-\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{k^3(n+k)^4}\right),$$

or,

$$17\zeta(7)-10\zeta(2)\zeta(5)=\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{k^3(n+k)^4}\right)=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{1}{n^3(n+k)^4}\right)$$

$$=\sum_{n=1}^{\infty}\frac{1}{n^3}\left(\zeta(4)-H_n^{(4)}\right)=\zeta(3)\zeta(4)-\sum_{n=1}^{\infty}\frac{H_n^{(4)}}{n^3},$$

whence we obtain that

$$S_7 = \sum_{n=1}^{\infty} \frac{H_n^{(4)}}{n^3} = 10\zeta(2)\zeta(5) + \zeta(3)\zeta(4) - 17\zeta(7), \tag{6.79}$$

and the calculation to the first series from the point iv) is finalized.

Also, combining the results in (6.21) and (6.78), we have

$$10\zeta(2)\zeta(5) + \zeta(3)\zeta(4) - 18\zeta(7) = \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{1}{k^4(n+k)^3} \right) = \sum_{k=1}^{\infty} \frac{1}{k^4}(\zeta(3) - H_k^{(3)})$$

$$= \zeta(3) \sum_{k=1}^{\infty} \frac{1}{k^4} - \sum_{k=1}^{\infty} \frac{H_k^{(3)}}{k^4} = \zeta(3)\zeta(4) - \sum_{k=1}^{\infty} \frac{H_k^{(3)}}{k^4},$$

whence we get that

$$S_8 = \sum_{k=1}^{\infty} \frac{H_k^{(3)}}{k^4} = 18\zeta(7) - 10\zeta(2)\zeta(5), \tag{6.80}$$

and the calculation to the second series from the point iv) is finalized.

When $p = q$, $p + q \geq 4$, it's easy to note that everything reduces to the well-known result in (4.14), with $n \to \infty$. The generalization of the Euler sum, $\displaystyle\sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q}$, with $p + q$ odd, $q \geq 2$, may be found in [11, 20].

6.22 Another Perspective on the Famous Quadratic Series of Au-Yeung Which Leads to an Elementary Solution

Solution *This identity was surprising and new to us when Enrico Au-Yeung (an undergraduate student in the Faculty of Mathematics in Waterloo) conjectured it on the basis of a computation of 500,000 terms (five digit accuracy!); our first impulse was to perform a higher-order computation to show it to be false.*—David Borwein and Jonathan M. Borwein in *On an intriguing integral and some series related to* $\zeta(4)$ paper. The authors of the previously mentioned paper found an ingenious solution by combining the Fourier series and the *Parseval's theorem* (see [57]) and then reducing all to the evaluation of an integral they calculated by means of contour integration (the details may be found in [10]).

Now, in the following, based upon a simple identity generated with the help of *The Master Theorem of Series*, I will show the calculations are straightforward and all will be finished by simple series manipulations only.

We make use of the series in (4.21), the case $m = 1$, where we multiply both sides by $1/n$ and then consider the summation from $n = 1$ to ∞ that gives

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n^2}{n^2}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^2}=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{H_k}{(k+1)(k+n+1)n}\right)$$

$$\{\text{change the order of summation}\}$$

$$=\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{H_k}{(k+1)(k+n+1)n}\right)$$

$$\left\{\text{use the fact that }\sum_{n=1}^{\infty}\frac{1}{n(n+k+1)}=\frac{H_{k+1}}{k+1}\right\}$$

$$=\sum_{k=1}^{\infty}\frac{H_kH_{k+1}}{(k+1)^2}=\sum_{k=1}^{\infty}\frac{(H_{k+1}-1/(k+1))H_{k+1}}{(k+1)^2}$$

$$\{\text{reindex the series and expand it}\}$$

$$=\sum_{k=1}^{\infty}\frac{H_k^2}{k^2}-\sum_{k=1}^{\infty}\frac{H_k}{k^3}$$

$$\{\text{make use of the Euler sum in (3.45), the case } n=3\}$$

$$=\sum_{k=1}^{\infty}\frac{H_k^2}{k^2}-\frac{5}{4}\zeta(4)=\sum_{n=1}^{\infty}\frac{H_n^2}{n^2}-\frac{5}{4}\zeta(4),$$

from which we get that

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n^2}=\frac{5}{2}\zeta(4)+\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^2}$$

$$\{\text{make use of the identity in (4.14), with } p=2\}$$

$$=3\zeta(4)+\frac{1}{2}\zeta^2(2)=\frac{17}{4}\zeta(4),$$

where I used that $\zeta^2(2)=\frac{5}{2}\zeta(4)$, and the solution is complete.

For a different solution based upon the use of the logarithmic integrals in (1.4) and (1.5), check the article *Reviving the quadratic series of Au-Yeung* in *Journal of Classical Analysis*, Vol. 6, No. 2, 2015 (see [47]). Various approaches of the series are also mentioned in [13, pp. 173–174].

6.23 Treating a Big Brother Series of the Quadratic Series of Au-Yeung by Elementary Means

Solution The present harmonic series represented the *engine* for the creation of another article, *A new proof for a classical quadratic harmonic series* that was published in *Journal of Classical Analysis*, Vol. 8, No. 2, 2016 (see [46]), where I combined the use of the integrals and series in a fruitful way.

Now, as in the previous section, I'll employ the same strategy using a simple identity generated by *The Master Theorem of Series*, and then all will be finished by series manipulations only.

Let's start with the result in (4.21), the case $m = 1$, where we multiply both sides by $1/n^2$ and then consider the summation from $n = 1$ to ∞ that gives

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n^2}{n^3}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3}=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{H_k}{(k+1)(k+n+1)n^2}\right)$$

$$\{\text{change the order of summation}\}$$

$$=\sum_{k=1}^{\infty}\frac{H_k}{k+1}\left(\sum_{n=1}^{\infty}\frac{1}{n^2(k+n+1)}\right)=\sum_{k=1}^{\infty}\frac{H_k}{(k+1)^2}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}-\sum_{n=1}^{\infty}\frac{1}{n(k+n+1)}\right)$$

$$\left\{\text{use that }\sum_{k=1}^{\infty}\frac{1}{k(k+n)}=\frac{H_n}{n}\right\}$$

$$=\sum_{k=1}^{\infty}\frac{H_k}{(k+1)^2}\left(\zeta(2)-\frac{H_{k+1}}{k+1}\right)=\zeta(2)\sum_{k=1}^{\infty}\frac{H_k}{(k+1)^2}-\sum_{k=1}^{\infty}\frac{H_kH_{k+1}}{(k+1)^3}$$

$$=\zeta(2)\sum_{k=1}^{\infty}\frac{H_{k+1}-1/(k+1)}{(k+1)^2}-\sum_{k=1}^{\infty}\frac{(H_{k+1}-1/(k+1))H_{k+1}}{(k+1)^3}$$

$$\{\text{reindex the series and expand them}\}$$

$$=\zeta(2)\sum_{k=1}^{\infty}\frac{H_k}{k^2}-\zeta(2)\sum_{k=1}^{\infty}\frac{1}{k^3}-\sum_{k=1}^{\infty}\frac{H_k^2}{k^3}+\sum_{k=1}^{\infty}\frac{H_k}{k^4}$$

$$\{\text{make use of the Euler sum in (3.45), the cases } n=2 \text{ and } n=4\}$$

$$=3\zeta(5)-\sum_{k=1}^{\infty}\frac{H_n^2}{n^3},$$

whence we obtain that

$$\sum_{k=1}^{\infty} \frac{H_n^2}{n^3} = 2\zeta(5) - \frac{1}{3}\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^3}$$

$$\{\text{make use of the result in (6.67)}\}$$

$$= \frac{7}{2}\zeta(5) - \zeta(2)\zeta(3),$$

and the solution is complete.

For example, the series appeared in the third chapter, in Sect. 3.55, and we'll also want to use it in the evaluation of other harmonic series. It may also be found evaluated in [20, 63].

6.24 Calculating Two More Elder Brother Series of the Quadratic Series of Au-Yeung, This Time the Versions with the Powers 4 and 5 in Denominator

Solution *It's already a routine, right?* If you have read the previous sections, probably you'll figure out immediately the way to go. Again, we make use of the result in (4.21), the case $m = 1$, where if we multiply both sides by $1/n^3$ and then consider the summation from $n = 1$ to ∞, we get

$$\frac{1}{2}\sum_{n=1}^{\infty} \frac{H_n^2}{n^4} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^4} = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} \frac{H_k}{(k+1)(k+n+1)n^3}\right)$$

$$\{\text{change the order of summation}\}$$

$$= \sum_{k=1}^{\infty} \frac{H_k}{k+1}\left(\sum_{n=1}^{\infty} \frac{1}{n^3(k+n+1)}\right) = \sum_{k=1}^{\infty} \frac{H_k}{(k+1)^2}\left(\sum_{n=1}^{\infty} \frac{1}{n^3} - \sum_{n=1}^{\infty} \frac{1}{n^2(k+n+1)}\right)$$

$$= \sum_{k=1}^{\infty} \frac{H_k}{(k+1)^2}\left(\zeta(3) - \frac{1}{k+1}\left(\sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n(k+n+1)}\right)\right)$$

$$\left\{\text{make use of the representation, } \sum_{k=1}^{\infty} \frac{1}{k(k+n)} = \frac{H_n}{n}\right\}$$

$$= \zeta(3)\sum_{k=1}^{\infty} \frac{H_k}{(k+1)^2} - \zeta(2)\sum_{k=1}^{\infty} \frac{H_k}{(k+1)^3} + \sum_{k=1}^{\infty} \frac{H_k H_{k+1}}{(k+1)^4}$$

$$= \zeta(3)\sum_{k=1}^{\infty}\frac{H_{k+1}-1/(k+1)}{(k+1)^2} - \zeta(2)\sum_{k=1}^{\infty}\frac{H_{k+1}-1/(k+1)}{(k+1)^3}$$

$$+\sum_{k=1}^{\infty}\frac{(H_{k+1}-1/(k+1))H_{k+1}}{(k+1)^4}$$

$$\{\text{reindex the series and expand them}\}$$

$$= \zeta(3)\sum_{k=1}^{\infty}\frac{H_k}{k^2} - \zeta(3)\sum_{k=1}^{\infty}\frac{1}{k^3} - \zeta(2)\sum_{k=1}^{\infty}\frac{H_k}{k^3} + \zeta(2)\sum_{k=1}^{\infty}\frac{1}{k^4} + \sum_{k=1}^{\infty}\frac{H_k^2}{k^4} - \sum_{k=1}^{\infty}\frac{H_k}{k^5}$$

$$\{\text{make use of the Euler sum in (3.45), the case } n = 2, 3, 5\}$$

$$= \frac{3}{2}\zeta^2(3) - \frac{35}{16}\zeta(6) + \sum_{n=1}^{\infty}\frac{H_n^2}{n^4},$$

whence we get that

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n^4} = \frac{35}{8}\zeta(6) - 3\zeta^2(3) + \sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4},$$

or if we make use of the result in (6.71),

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n^4} = \frac{97}{24}\zeta(6) - 2\zeta^2(3),$$

and the solution to the point $i)$ of the problem is complete.

The full derivation of the result above is also given in my article, *A master theorem of series and an evaluation of a cubic harmonic series* (see [45]). The value of the series in the form $\displaystyle\sum_{n=1}^{\infty}\frac{H_n^2}{(n+1)^4}$ may be found in [3, 12].

As regards the series from the point $ii)$ of the problem, we make use again of the result in (4.21), the case $m = 1$, where if we multiply both sides by $1/n^4$ and then consider the summation from $n = 1$ to ∞, we get

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n^2}{n^5} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^5} = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{H_k}{(k+1)(k+n+1)n^4}\right)$$

$$\{\text{change the order of summation}\}$$

$$= \sum_{k=1}^{\infty}\frac{H_k}{k+1}\left(\sum_{n=1}^{\infty}\frac{1}{n^4(k+n+1)}\right) = \sum_{k=1}^{\infty}\frac{H_k}{(k+1)^2}\left(\sum_{n=1}^{\infty}\frac{1}{n^4} - \sum_{n=1}^{\infty}\frac{1}{n^3(k+n+1)}\right)$$

$$= \sum_{k=1}^{\infty} \frac{H_k}{(k+1)^2} \left(\zeta(4) - \frac{1}{k+1} \left(\sum_{n=1}^{\infty} \frac{1}{n^3} - \sum_{n=1}^{\infty} \frac{1}{n^2(k+n+1)} \right) \right)$$

$$= \sum_{k=1}^{\infty} \frac{H_k}{(k+1)^2} \left(\zeta(4) - \frac{\zeta(3)}{k+1} + \frac{1}{(k+1)^2} \left(\sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n(k+n+1)} \right) \right)$$

$$\left\{ \text{make use of the representation, } \sum_{k=1}^{\infty} \frac{1}{k(k+n)} = \frac{H_n}{n} \right\}$$

$$= \zeta(4) \sum_{k=1}^{\infty} \frac{H_k}{(k+1)^2} - \zeta(3) \sum_{k=1}^{\infty} \frac{H_k}{(k+1)^3} + \zeta(2) \sum_{k=1}^{\infty} \frac{H_k}{(k+1)^4} - \sum_{k=1}^{\infty} \frac{H_k H_{k+1}}{(k+1)^5}$$

$$= \zeta(4) \sum_{k=1}^{\infty} \frac{H_{k+1} - 1/(k+1)}{(k+1)^2} - \zeta(3) \sum_{k=1}^{\infty} \frac{H_{k+1} - 1/(k+1)}{(k+1)^3}$$

$$+ \zeta(2) \sum_{k=1}^{\infty} \frac{H_{k+1} - 1/(k+1)}{(k+1)^4} - \sum_{k=1}^{\infty} \frac{(H_{k+1} - 1/(k+1))H_{k+1}}{(k+1)^5}$$

$$\{\text{reindex the series and expand them}\}$$

$$= \zeta(4) \sum_{k=1}^{\infty} \frac{H_k}{k^2} - \zeta(4) \sum_{k=1}^{\infty} \frac{1}{k^3} - \zeta(3) \sum_{k=1}^{\infty} \frac{H_k}{k^3} + \zeta(3) \sum_{k=1}^{\infty} \frac{1}{k^4} + \zeta(2) \sum_{k=1}^{\infty} \frac{H_k}{k^4}$$

$$- \zeta(2) \sum_{k=1}^{\infty} \frac{1}{k^5} - \sum_{k=1}^{\infty} \frac{H_k^2}{k^5} + \sum_{k=1}^{\infty} \frac{H_k}{k^6}$$

$$\{\text{make use of the Euler sum in (3.45), the cases } n = 2, 3, 4, 6\}$$

$$= 4\zeta(7) + \zeta(2)\zeta(5) - \frac{11}{4} \zeta(3)\zeta(4) - \sum_{k=1}^{\infty} \frac{H_n^2}{n^5},$$

whence we get that

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^5} = \frac{8}{3} \zeta(7) + \frac{2}{3} \zeta(2)\zeta(5) - \frac{11}{6} \zeta(3)\zeta(4) - \frac{1}{3} \sum_{k=1}^{\infty} \frac{H_n^{(2)}}{n^5},$$

or if we make use of the result in (6.75),

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^5} = 6\zeta(7) - \zeta(2)\zeta(5) - \frac{5}{2}\zeta(3)\zeta(4),$$

and the solution to the point $ii)$ of the problem is complete.

We may also find the series from both points of the problem evaluated in [63], and a generalization of it may be found in [40].

If we look back at the last two sections, where we dealt with similar series, we may notice that sometimes, when using the right tool, we get tremendously simpler solutions (and we talk about pretty challenging series).

6.25 An Advanced Harmonic Series of Weight 5, $\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^2}$, Attacked with a Special Class of Sums

Solution Based on the work in the area of the harmonic series, I'm inclined to think the land of the harmonic series is still far from being fully explored, one filled with mysterious identities waiting for us to discover them. In general, when working with the harmonic series, one of the best strategies is to build up relations between series (a thing I often do in this chapter) by using certain helpful identities, and then try to extract the value of the desired series. The difficulty of this process varies, and a critical factor is, of course, the identity (or the identities) used that could lead sometimes to splendid solutions.

The wonderful thing to happen is that in the following approach I will easily reduce the evaluation of the series to the calculation of some cases of the series of the type $\sum_{k=1}^{\infty} \frac{H_k}{k^n}$ (see (3.45)), and $\sum_{k=1}^{\infty} \frac{H_k^{(p)}}{k^q} + \sum_{k=1}^{\infty} \frac{H_k^{(q)}}{k^p}$.

If we consider the identity in (4.16), the case $m = 2$, where we multiply both sides by $1/n^2$ and then sum from $n = 1$ to ∞, we get

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n-1} \frac{H_k^{(2)}}{n^2(n-k)} \right) = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \frac{H_k}{k^2 n^2} \right) + \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^2} - 2 \sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^2}. \tag{6.81}$$

Now, for the first series in the right-hand side of (6.81) we have, upon changing the summation order, that

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \frac{H_k}{k^2 n^2} \right) = \sum_{k=1}^{\infty} \left(\sum_{n=k}^{\infty} \frac{H_k}{k^2 n^2} \right) = \sum_{k=1}^{\infty} \frac{H_k}{k^2} \left(\frac{1}{k^2} + \frac{1}{(k+1)^2} + \frac{1}{(k+2)^2} + \cdots \right)$$

$$= \sum_{k=1}^{\infty} \frac{H_k}{k^4} + \sum_{k=1}^{\infty} \frac{H_k}{k^2} \left(\frac{1}{(k+1)^2} + \frac{1}{(k+2)^2} + \frac{1}{(k+3)^2} + \cdots \right)$$

$$= \sum_{k=1}^{\infty} \frac{H_k}{k^4} + \sum_{k=1}^{\infty} \frac{H_k}{k^2} \left(\zeta(2) - H_k^{(2)} \right) = \sum_{k=1}^{\infty} \frac{H_k}{k^4} + \zeta(2) \sum_{k=1}^{\infty} \frac{H_k}{k^2} - \sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k^2}$$

$$= \zeta(2)\zeta(3) + 3\zeta(5) - \sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k^2}, \tag{6.82}$$

where in the last equality I made use of the linear Euler sum in (3.45), the cases $n = 2$ and $n = 4$.

Then, if we plug the result from (6.82) in (6.81), we get

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n-1} \frac{H_k^{(2)}}{n^2(n-k)} \right) = \zeta(2)\zeta(3) + 3\zeta(5) - 2 \sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^2}. \tag{6.83}$$

Further, if we consider the left-hand side of the equality in (6.83) and then change the summation order, we obtain

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n-1} \frac{H_k^{(2)}}{n^2(n-k)} \right) = \sum_{k=1}^{\infty} \left(\sum_{n=k+1}^{\infty} \frac{H_k^{(2)}}{n^2(n-k)} \right) = \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{H_k^{(2)}}{n(n+k)^2} \right)$$

$$= \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} H_k^{(2)} \left(\frac{1}{kn(k+n)} - \frac{1}{k(k+n)^2} \right) \right)$$

$$= \sum_{k=1}^{\infty} \frac{H_k^{(2)}}{k} \left(\sum_{n=1}^{\infty} \frac{1}{n(k+n)} \right) - \sum_{k=1}^{\infty} \frac{H_k^{(2)}}{k} \left(\sum_{n=1}^{\infty} \frac{1}{(k+n)^2} \right)$$

$$= \sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k^2} - \sum_{k=1}^{\infty} \frac{H_k^{(2)}}{k} \left(\zeta(2) - H_k^{(2)} \right). \tag{6.84}$$

Applying Abel's summation (see (5.1)) for the last series in (6.84), where we set $a_k = 1/k$ and $b_k = H_k^{(2)}(\zeta(2) - H_k^{(2)})$, we obtain

$$\sum_{k=1}^{\infty} \frac{H_k^{(2)}}{k} \left(\zeta(2) - H_k^{(2)} \right) = \sum_{k=1}^{\infty} H_k \left(H_k^{(2)} \left(\zeta(2) - H_k^{(2)} \right) - H_{k+1}^{(2)} \left(\zeta(2) - H_{k+1}^{(2)} \right) \right)$$

$$= \sum_{k=1}^{\infty} H_k \left(H_k^{(2)} \left(\zeta(2) - H_k^{(2)}\right) - \left(H_k^{(2)} + \frac{1}{(k+1)^2}\right) \left(\zeta(2) - H_k^{(2)} - \frac{1}{(k+1)^2}\right)\right)$$

$$= \sum_{k=1}^{\infty} \left(\frac{H_k}{(k+1)^4} - \zeta(2) \frac{H_k}{(k+1)^2} + 2 \frac{H_k H_k^{(2)}}{(k+1)^2} \right)$$

$$= \sum_{k=1}^{\infty} \left(\frac{H_{k+1} - 1/(k+1)}{(k+1)^4} - \zeta(2) \frac{H_{k+1} - 1/(k+1)}{(k+1)^2} \right.$$

$$\left. +2 \frac{(H_{k+1} - 1/(k+1))(H_{k+1}^{(2)} - 1/(k+1)^2)}{(k+1)^2} \right)$$

{reindex the series and expand it}

$$= \sum_{k=1}^{\infty} \frac{1}{k^5} + \zeta(2) \sum_{k=1}^{\infty} \frac{1}{k^3} - \sum_{k=1}^{\infty} \frac{H_k}{k^4} - \zeta(2) \sum_{k=1}^{\infty} \frac{H_k}{k^2} - 2 \sum_{k=1}^{\infty} \frac{H_k^{(2)}}{k^3} + 2 \sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k^2}$$

$$= 2 \sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k^2} - 2 \sum_{k=1}^{\infty} \frac{H_k^{(2)}}{k^3} - 2\zeta(5), \tag{6.85}$$

where to get the last equality I made use of the classical linear Euler sum in (3.45), the cases $n = 2$ and $n = 4$.

If we plug the result from (6.85) in (6.84), we get

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n-1} \frac{H_k^{(2)}}{n^2(n-k)} \right) = 2\zeta(5) + 2 \sum_{k=1}^{\infty} \frac{H_k^{(2)}}{k^3} - \sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k^2}. \tag{6.86}$$

Then, by combining (6.83) and (6.86), we obtain

$$\zeta(2)\zeta(3) + 3\zeta(5) - 2 \sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^2} = 2\zeta(5) + 2 \sum_{k=1}^{\infty} \frac{H_k^{(2)}}{k^3} - \sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k^2},$$

or

$$\sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k^2} = 2 \left(\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^2} + \sum_{k=1}^{\infty} \frac{H_k^{(2)}}{k^3} \right) - \zeta(2)\zeta(3) - \zeta(5). \tag{6.87}$$

Now, it's a pleasant moment to see in the following we don't need to calculate separately the remaining two series from the right-hand side of (6.87). It's straightforward to show by Abel's summation that $\displaystyle\sum_{k=1}^{\infty} \frac{H_k^{(p)}}{k^q} + \sum_{k=1}^{\infty} \frac{H_k^{(q)}}{k^p} = \zeta(p)\zeta(q) + \zeta(p+q)$, and then we have

$$\sum_{k=1}^{\infty} \frac{H_k^{(2)}}{k^3} + \sum_{k=1}^{\infty} \frac{H_k^{(3)}}{k^2} = \zeta(2)\zeta(3) + \zeta(5). \tag{6.88}$$

Finally, if we plug the result from (6.88) in (6.87), we conclude that

$$\sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k^2} = \zeta(2)\zeta(3) + \zeta(5),$$

and the solution is finalized.

For a second solution involving the integrals, see the second solution in the next section. Also, for a multiple zeta function related approach of the series, check [37]. The present solution also answers the proposed *challenging question*.

6.26 An Advanced Harmonic Series of Weight 5, $\sum_{n=1}^{\infty} \frac{H_n^3}{n^2}$, Attacked with a Special Identity

Solution Before passing to the series of weight 6 from the next sections, we have to face a last challenging series of weight 5. For a first solution by elementary series manipulations, we'll exploit one of the identities generated with the help of *The Master Theorem of Series* which we combine then with the value of the series from the previous section. Then, for a second solution, we'll want to wisely combine identities with logarithmic integrals from Sect. 1.3.

For a first solution by series manipulations only, recall the identity in (4.26), the first equality, where if we multiply both sides by $1/n$ and then consider the sum from $n = 1$ to ∞, we get

$$\frac{1}{3}\sum_{n=1}^{\infty} \frac{H_n^3}{n^2} + \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^2} + \frac{2}{3}\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^2} = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{H_k^2 - H_k^{(2)}}{(k+1)(k+n+1)n} \right)$$

$$\{\text{reverse the order of summation}\}$$

$$= \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{H_k^2 - H_k^{(2)}}{(k+1)(k+n+1)n} \right)$$

$$\left\{ \text{make use of the fact that } \sum_{n=1}^{\infty} \frac{1}{n(n+k+1)} = \frac{H_{k+1}}{k+1} \right\}$$

$$= \sum_{k=1}^{\infty} \frac{(H_k^2 - H_k^{(2)})H_{k+1}}{(k+1)^2} = \sum_{k=1}^{\infty} \frac{((H_{k+1} - 1/(k+1))^2 - (H_{k+1}^{(2)} - 1/(k+1)^2))H_{k+1}}{(k+1)^2}$$

{reindex the series and expand it}

$$= \sum_{k=1}^{\infty} \frac{H_k^3}{k^2} - 2\sum_{k=1}^{\infty} \frac{H_k^2}{k^3} + 2\sum_{k=1}^{\infty} \frac{H_k}{k^4} - \sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k^2}$$

{make use of the linear Euler sum in (3.45), the case $n = 4$}

$$= \sum_{n=1}^{\infty} \frac{H_n^3}{n^2} - 2\sum_{n=1}^{\infty} \frac{H_n^2}{n^3} - \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^2} + 6\zeta(5) - 2\zeta(2)\zeta(3),$$

whence we get that

$$2\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^2} - \frac{2}{3}\sum_{n=1}^{\infty} \frac{H_n^3}{n^2} = 6\zeta(5) - 2\zeta(2)\zeta(3) - 2\sum_{n=1}^{\infty} \frac{H_n^2}{n^3} - \frac{2}{3}\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^2}$$

{make use of the results in (4.30) and (6.68)}

$$= \frac{2}{3}(2\zeta(2)\zeta(3) - 7\zeta(5)),$$

and therefore, we have

$$3\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^2} - \sum_{n=1}^{\infty} \frac{H_n^3}{n^2} = 2\zeta(2)\zeta(3) - 7\zeta(5). \tag{6.89}$$

Finally, if we plug in (6.89) the value of the series $\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^2}$, which is obtained in the previous section, we conclude that

$$\sum_{n=1}^{\infty} \frac{H_n^3}{n^2} = \zeta(2)\zeta(3) + 10\zeta(5),$$

and the first solution is complete.

For a second solution, an approach with integrals, we make use of the identities in (1.5) and (1.6), where if we multiply both sides of the first identity by H_n/n, and then multiply both sides of the second identity by $1/n$, we get

$$\int_0^1 x^{n-1} \frac{H_n}{n} \log^2(1-x)\mathrm{d}x = \frac{H_n^3 + H_n H_n^{(2)}}{n^2} \tag{6.90}$$

and

$$\int_0^1 \frac{x^{n-1}}{n} \log^3(1-x)\mathrm{d}x = -\frac{H_n^3 + 3H_n H_n^{(2)} + 2H_n^{(3)}}{n^2}. \tag{6.91}$$

If we sum up both sides of (6.90) from $n = 1$ to ∞ and change the order of integration and summation, we get

$$\int_0^1 \sum_{n=1}^{\infty} x^{n-1} \frac{H_n}{n} \log^2(1-x)\mathrm{d}x = \sum_{n=1}^{\infty} \frac{H_n^3}{n^2} + \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^2}. \tag{6.92}$$

Now, we consider the generating function of the harmonic numbers in (4.5), where if we divide both sides by t and use that $\log(1-t) = -\sum_{k=1}^{\infty} \frac{t^k}{k}$, we get

$$\sum_{n=1}^{\infty} t^{n-1} H_n = -\frac{\log(1-t)}{t(1-t)} = -\frac{\log(1-t)}{t} - \frac{\log(1-t)}{1-t} = \sum_{k=1}^{\infty} \frac{t^{k-1}}{k} - \frac{\log(1-t)}{1-t}. \tag{6.93}$$

Integrating both sides of (6.93), from $t = 0$ to $t = x$, we have

$$\int_0^x \sum_{n=1}^{\infty} t^{n-1} H_n \mathrm{d}t = \int_0^x \sum_{k=1}^{\infty} \frac{t^{k-1}}{k} \mathrm{d}t - \int_0^x \frac{\log(1-t)}{1-t} \mathrm{d}t,$$

and changing the order of the summation and integration, we get

$$\sum_{n=1}^{\infty} \int_0^x t^{n-1} H_n \mathrm{d}t = \sum_{k=1}^{\infty} \int_0^x \frac{t^{k-1}}{k} \mathrm{d}t + \frac{1}{2} \log^2(1-x)$$

or, after integrating and dividing both sides by x,

$$\sum_{n=1}^{\infty} x^{n-1} \frac{H_n}{n} = \sum_{k=1}^{\infty} \frac{x^{k-1}}{k^2} + \frac{1}{2} \frac{\log^2(1-x)}{x}. \tag{6.94}$$

Plugging the result from (6.94) in the left-hand side of the relation in (6.92), we get

$$\int_0^1 \sum_{n=1}^{\infty} x^{n-1} \frac{H_n}{n} \log^2(1-x)\mathrm{d}x = \int_0^1 \left(\sum_{k=1}^{\infty} \frac{x^{k-1}}{k^2} + \frac{1}{2} \frac{\log^2(1-x)}{x} \right) \times \log^2(1-x)\mathrm{d}x$$

$$= \int_0^1 \sum_{k=1}^{\infty} \frac{x^{k-1}}{k^2} \log^2(1-x)\mathrm{d}x + \frac{1}{2}\int_0^1 \frac{\log^4(1-x)}{x}\mathrm{d}x$$

$$\left\{\text{in the first integral use that } \log^2(1-x) = 2\sum_{n=1}^{\infty} x^{n+1}\frac{H_n}{n+1} \text{(see [30, p. 22],}\right\}$$

{[14, p. 43]), and in the second one make the change of variable, $1 - x = y$}

$$= 2\int_0^1 \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty} \frac{x^{k+n} H_n}{k^2(n+1)}\right)\mathrm{d}x + \frac{1}{2}\int_0^1 \frac{\log^4(x)}{1-x}\mathrm{d}x$$

$$= 2\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\int_0^1 \frac{x^{k+n} H_n}{k^2(n+1)}\mathrm{d}x\right) + \frac{1}{2}\int_0^1 \sum_{n=1}^{\infty} x^{n-1}\log^4(x)\mathrm{d}x$$

$$= 2\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty} \frac{H_n}{k^2(n+1)(k+n+1)}\right) + \frac{1}{2}\sum_{n=1}^{\infty}\int_0^1 x^{n-1}\log^4(x)\mathrm{d}x$$

{reverse the order in the double series}

$$= 2\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} \frac{H_n}{k^2(n+1)(k+n+1)}\right) + 12\sum_{n=1}^{\infty}\frac{1}{n^5}$$

$$= 2\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} \frac{((k+n+1)-k)H_n}{k^2(n+1)^2(k+n+1)}\right) + 12\zeta(5)$$

$$= 2\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} \frac{H_n}{k^2(n+1)^2}\right) - 2\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} \frac{H_n}{k(n+1)^2(k+n+1)}\right) + 12\zeta(5)$$

$$\left\{\text{use that } \sum_{k=1}^{\infty}\frac{1}{k(k+n+1)} = \frac{H_{n+1}}{n+1}\right\}$$

$$= 2\zeta(2)\sum_{n=1}^{\infty}\frac{H_{n+1} - 1/(n+1)}{(n+1)^2} - 2\sum_{n=1}^{\infty}\frac{(H_{n+1} - 1/(n+1))H_{n+1}}{(n+1)^3} + 12\zeta(5)$$

{reindex the series and expand them}

$$= 2\zeta(2)\sum_{n=1}^{\infty}\frac{H_n}{n^2} - 2\sum_{n=1}^{\infty}\frac{H_n^2}{n^3} + 2\sum_{n=1}^{\infty}\frac{H_n}{n^4} - 2\zeta(2)\sum_{n=1}^{\infty}\frac{1}{n^3} + 12\zeta(5)$$

{make use of (4.30) and the linear Euler sums in (3.45), the cases $n = 2, 4$}

$$= 2\zeta(2)\zeta(3) + 11\zeta(5). \tag{6.95}$$

Thus, plugging the result from (6.95) in (6.92), we have that

$$\sum_{n=1}^{\infty} \frac{H_n^3}{n^2} + \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^2} = 2\zeta(2)\zeta(3) + 11\zeta(5). \tag{6.96}$$

Now, we return to the result in (6.91) where we sum both sides from $n = 1$ to ∞ that gives

$$\sum_{n=1}^{\infty} \int_0^1 \frac{x^{n-1}}{n} \log^3(1-x)\mathrm{d}x = -\sum_{n=1}^{\infty} \frac{H_n^3 + 3H_n H_n^{(2)} + 2H_n^{(3)}}{n^2}. \tag{6.97}$$

Changing the integration and summation order in the left-hand side of (6.97), we obtain

$$\sum_{n=1}^{\infty} \int_0^1 \frac{x^{n-1}}{n} \log^3(1-x)\mathrm{d}x = \int_0^1 \sum_{n=1}^{\infty} \frac{x^{n-1}}{n} \log^3(1-x)\mathrm{d}x = -\int_0^1 \frac{\log^4(1-x)}{x}\mathrm{d}x$$

$$= -\sum_{n=1}^{\infty} \frac{H_n^3}{n^2} - 3\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^2} - 2\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^2}$$

{the value of the last series is given in (6.68)}

$$= -\sum_{n=1}^{\infty} \frac{H_n^3}{n^2} - 3\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^2} - 11\zeta(5) + 4\zeta(2)\zeta(3). \tag{6.98}$$

Since we have that

$$-\int_0^1 \frac{\log^4(1-x)}{x}\mathrm{d}x \overset{x=1-y}{=} -\int_0^1 \frac{\log^4(y)}{1-y}\mathrm{d}y = -\int_0^1 \sum_{n=1}^{\infty} y^{n-1} \log^4(y)\mathrm{d}y$$

{change the order of summation and integration}

$$= -\sum_{n=1}^{\infty} \int_0^1 y^{n-1} \log^4(y)\mathrm{d}y = -24\sum_{n=1}^{\infty} \frac{1}{n^5} = -24\zeta(5),$$

by plugging this last result in (6.98), we obtain

$$-\sum_{n=1}^{\infty}\frac{H_n^3}{n^2}-3\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^2}=-13\zeta(5)-4\zeta(2)\zeta(3). \tag{6.99}$$

Now, by combining the relations in (6.96) and (6.99), we conclude that

$$\sum_{n=1}^{\infty}\frac{H_n^3}{n^2}=\zeta(2)\zeta(3)+10\zeta(5);\ \sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^2}=\zeta(2)\zeta(3)+\zeta(5),$$

and the second solution is complete.

To get another relation between the two series, one might exploit the result in (6.23) to get and use $\displaystyle\sum_{n=1}^{\infty}x^n(H_n^3-3H_nH_n^{(2)}+2H_n^{(3)})=-\frac{\log^3(1-x)}{1-x}$.

The series also appears in [63], and a slightly modified version of it, $\displaystyle\sum_{n=1}^{\infty}\frac{H_n^3}{(n+1)^2}$, from which we can get the given series, appears in [20].

6.27 The Evaluation of an Advanced Cubic Harmonic Series of Weight 6, $\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^3$, Treated with Both *The Master Theorem of Series* and Special Logarithmic Integrals of Powers Two and Three

Solution After the exciting experience with the Au-Yeung series, one might naturally think to consider the version with the cubic power instead of the square power. *Is it possible to express the series again in terms of zeta values as I did for the Au-Yeung series?* Yes, it is! An elementary solution by series manipulations, without using integrals at all, may be found in my paper, *A master theorem of series and an evaluation of a cubic harmonic series* published in *Journal of Classical Analysis*, Vol. 10, No. 2, 2017 (see [45]).

For a second solution with integrals, which also wonderfully aims the value of the series $\displaystyle\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^3}$ from the next section, let's start out with the identities in (1.5) and (1.6), where if we multiply both sides of the first identity by H_n/n^2, and then multiply both sides of the second identity by $1/n^2$, we get

$$\int_0^1 x^{n-1}\frac{H_n}{n^2}\log^2(1-x)\mathrm{d}x=\frac{H_n^3+H_nH_n^{(2)}}{n^3} \tag{6.100}$$

and

$$\int_0^1 \frac{x^{n-1}}{n^2}\log^3(1-x)\mathrm{d}x = -\frac{H_n^3 + 3H_n H_n^{(2)} + 2H_n^{(3)}}{n^3}. \tag{6.101}$$

If we sum up both sides of (6.100) from $n = 1$ to ∞ and change the order of integration and summation, we obtain

$$\int_0^1 \sum_{n=1}^{\infty} x^{n-1}\frac{H_n}{n^2}\log^2(1-x)\mathrm{d}x = \sum_{n=1}^{\infty}\frac{H_n^3}{n^3} + \sum_{n=1}^{\infty}\frac{H_n H_n^{(2)}}{n^3}. \tag{6.102}$$

Then, let's start with the result in (6.94) from the previous section where if we integrate from $t = 0$ to $t = x$,

$$\int_0^x \sum_{n=1}^{\infty} t^{n-1}\frac{H_n}{n}\mathrm{d}t = \sum_{n=1}^{\infty}\int_0^x t^{n-1}\frac{H_n}{n}\mathrm{d}t = \sum_{n=1}^{\infty} x^n\frac{H_n}{n^2} = \int_0^x \sum_{k=1}^{\infty}\frac{t^{k-1}}{k^2}\mathrm{d}t$$
$$+\frac{1}{2}\int_0^x \frac{\log^2(1-t)}{t}\mathrm{d}t = \sum_{k=1}^{\infty}\int_0^x \frac{t^{k-1}}{k^2}\mathrm{d}t + \frac{1}{2}\int_0^x \frac{\log^2(1-t)}{t}\mathrm{d}t$$
$$= \sum_{k=1}^{\infty}\frac{x^k}{k^3} + \frac{1}{2}\int_0^x \frac{\log^2(1-t)}{t}\mathrm{d}t,$$

and then divide by x, we get

$$\sum_{n=1}^{\infty} x^{n-1}\frac{H_n}{n^2} = \sum_{k=1}^{\infty}\frac{x^{k-1}}{k^3} + \frac{1}{2x}\int_0^x \frac{\log^2(1-t)}{t}\mathrm{d}t. \tag{6.103}$$

Plugging the result from (6.103) in the left-hand side of the relation in (6.102), we obtain

$$\int_0^1 \sum_{n=1}^{\infty} x^{n-1}\frac{H_n}{n^2}\log^2(1-x)\mathrm{d}x = \int_0^1 \left(\sum_{k=1}^{\infty}\frac{x^{k-1}}{k^3} + \frac{1}{2x}\int_0^x \frac{\log^2(1-t)}{t}\mathrm{d}t\right)\log^2(1-x)\mathrm{d}x$$
$$= \int_0^1 \sum_{k=1}^{\infty}\frac{x^{k-1}}{k^3}\log^2(1-x)\mathrm{d}x + \frac{1}{2}\int_0^1 \frac{\log^2(1-x)}{x}\left(\int_0^x \frac{\log^2(1-t)}{t}\mathrm{d}t\right)\mathrm{d}x. \tag{6.104}$$

Due to the symmetry in the second integral, or by applying the integration by parts with $f' = \dfrac{\log^2(1-x)}{x}$ and $g = \displaystyle\int_0^x \frac{\log^2(1-t)}{t}\mathrm{d}t$, we have

$$\int_0^1 \frac{\log^2(1-x)}{x}\left(\int_0^x \frac{\log^2(1-t)}{t}\mathrm{d}t\right)\mathrm{d}x = \frac{1}{2}\left(\int_0^1 \frac{\log^2(1-x)}{x}\mathrm{d}x\right)^2$$

$$\overset{1-x=y}{=} \frac{1}{2}\left(\int_0^1 \frac{\log^2(x)}{1-x}\mathrm{d}x\right)^2 = \frac{1}{2}\left(\int_0^1 \sum_{k=1}^{\infty} x^{k-1}\log^2(x)\mathrm{d}x\right)^2$$

$$= \frac{1}{2}\left(\sum_{k=1}^{\infty}\int_0^1 x^{k-1}\log^2(x)\mathrm{d}x\right)^2 = \frac{1}{2}\left(2\sum_{k=1}^{\infty}\frac{1}{k^3}\right)^2 = 2\zeta^2(3),$$

and plugging this result in (6.104) yields

$$\int_0^1 \sum_{n=1}^{\infty} x^{n-1}\frac{H_n}{n^2}\log^2(1-x)\mathrm{d}x = \int_0^1 \sum_{k=1}^{\infty}\frac{x^{k-1}}{k^3}\log^2(1-x)\mathrm{d}x + \zeta^2(3)$$

$$\left\{\text{use that } \log^2(1-x) = 2\sum_{n=1}^{\infty} x^{n+1}\frac{H_n}{n+1}\right\}$$

$$= 2\int_0^1 \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{x^{k+n}}{k^3(n+1)}H_n\right)\mathrm{d}x + \zeta^2(3)$$

$$\{\text{change the order of summation and integration}\}$$

$$= 2\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\int_0^1 \frac{x^{k+n}}{k^3(n+1)}H_n\mathrm{d}x\right) + \zeta^2(3)$$

$$= 2\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{H_{n+1}-1/(n+1)}{k^3(n+1)(k+n+1)}\right) + \zeta^2(3) = 2\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{H_n - 1/n}{k^3 n(k+n)}\right) + \zeta^2(3)$$

$$= 2\sum_{n=1}^{\infty}\frac{1}{n^4}\sum_{k=1}^{\infty}\frac{1}{k^2} - 2\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{k=1}^{\infty}\frac{1}{k^3} - 2\sum_{n=1}^{\infty}\frac{1}{n^4}\left(\sum_{k=1}^{\infty}\frac{1}{k(k+n)}\right) - 2\sum_{n=1}^{\infty}\frac{H_n}{n^3}\sum_{k=1}^{\infty}\frac{1}{k^2}$$

$$+2\sum_{n=1}^{\infty}\frac{H_n}{n^2}\sum_{k=1}^{\infty}\frac{1}{k^3} + 2\sum_{n=1}^{\infty}\frac{H_n}{n^3}\left(\sum_{k=1}^{\infty}\frac{1}{k(k+n)}\right) + \zeta^2(3) = \frac{7}{2}\zeta(6) - 2\zeta^2(3) - 2\sum_{n=1}^{\infty}\frac{H_n}{n^5}$$

$$-2\zeta(2)\sum_{n=1}^{\infty}\frac{H_n}{n^3} + 2\zeta(3)\sum_{n=1}^{\infty}\frac{H_n}{n^2} + 2\sum_{n=1}^{\infty}\frac{H_n^2}{n^4} + \zeta^2(3)$$

{consider the values of the series in (4.31) and (3.45), the cases $n = 2, 3, 5$}

$$= \frac{89}{24}\zeta(6). \tag{6.105}$$

If we plug the value from (6.105) in (6.102), we obtain that

$$\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^3 + \sum_{n=1}^{\infty}\frac{H_n H_n^{(2)}}{n^3} = \frac{89}{24}\zeta(6). \tag{6.106}$$

Next, if we sum up both sides of (6.101) from $n = 1$ to ∞ and change the integration and summation order, we get

$$\int_0^1 \sum_{n=1}^{\infty}\frac{x^{n-1}}{n^2}\log^3(1-x)\mathrm{d}x = -\sum_{n=1}^{\infty}\frac{H_n^3}{n^3} - 3\sum_{n=1}^{\infty}\frac{H_n H_n^{(2)}}{n^3} - 2\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^3}. \tag{6.107}$$

The left-hand side of (6.107) can be rewritten as follows

$$\begin{aligned}\int_0^1 \sum_{n=1}^{\infty}\frac{x^{n-1}}{n^2}\log^3(1-x)\mathrm{d}x &= \int_0^1 \frac{\operatorname{Li}_2(x)\log^3(1-x)}{x}\mathrm{d}x \\ &\overset{1-x=y}{=} \int_0^1 \frac{\operatorname{Li}_2(1-y)\log^3(y)}{1-y}\mathrm{d}y\end{aligned}$$

{use the Dilogarithm function reflection formula (see [30, Chapter 1, p. 5],}

{[53, (5)]), [42, Chapter 2, p. 107], $\operatorname{Li}_2(x)+\operatorname{Li}_2(1-x) = \zeta(2)-\log(x)\log(1-x)$}

$$= \zeta(2)\int_0^1 \frac{\log^3(y)}{1-y}\mathrm{d}y - \int_0^1 \frac{\log(1-y)\log^4(y)}{1-y}\mathrm{d}y - \int_0^1 \frac{\log^3(y)\operatorname{Li}_2(y)}{1-y}\mathrm{d}y. \tag{6.108}$$

To calculate the first integral in the right-hand side of (6.108), we write that

$$\begin{aligned}\int_0^1 \frac{\log^3(y)}{1-y}\mathrm{d}y &= \int_0^1 \sum_{k=1}^{\infty} y^{k-1}\log^3(y)\mathrm{d}y = \sum_{k=1}^{\infty}\int_0^1 y^{k-1}\log^3(y)\mathrm{d}y \\ &= -6\sum_{k=1}^{\infty}\frac{1}{k^4} = -6\zeta(4).\end{aligned} \tag{6.109}$$

Next, to calculate the second integral in (6.108), we make use of the generating function of the harmonic numbers in (4.5), and then we have

$$\int_0^1 \frac{\log(1-y)\log^4(y)}{1-y} \mathrm{d}y = -\int_0^1 \sum_{k=1}^{\infty} y^k H_k \log^4(y) \mathrm{d}y$$

{change the order of summation and integration}

$$= -\sum_{k=1}^{\infty} H_k \int_0^1 y^k \log^4(y) \mathrm{d}y = -24 \sum_{k=1}^{\infty} \frac{H_k}{(k+1)^5} = -24 \sum_{k=1}^{\infty} \frac{H_{k+1} - 1/(k+1)}{(k+1)^5}$$

{reindex the series and expand it}

$$= -24 \sum_{k=1}^{\infty} \frac{H_k}{k^5} + 24 \sum_{k=1}^{\infty} \frac{1}{k^6} = 6(2\zeta^2(3) - 3\zeta(6)). \tag{6.110}$$

Lastly, to calculate the third integral in (6.108), we proceed as follows

$$\int_0^1 \frac{\log^3(y)\operatorname{Li}_2(y)}{1-y} \mathrm{d}y = \int_0^1 \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{y^{k+n-1}}{k^2} \log^3(y) \right) \mathrm{d}y$$

{change the order of summation and integration}

$$= \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} \int_0^1 \frac{y^{k+n-1}}{k^2} \log^3(y) \mathrm{d}y \right) = -6 \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{1}{k^2(k+n)^4} \right)$$

$$= -6 \sum_{k=1}^{\infty} \frac{1}{k^2} \left(\zeta(4) - H_k^{(4)} \right) = -6\zeta(4) \sum_{k=1}^{\infty} \frac{1}{k^2} + 6 \sum_{k=1}^{\infty} \frac{H_k^{(4)}}{k^2}$$

{for the second series use the result in (6.72)}

$$= 8\zeta(6) - 6\zeta^2(3). \tag{6.111}$$

Collecting the results from (6.27), (6.110), and (6.111) in (6.108), we get

$$\int_0^1 \sum_{n=1}^{\infty} \frac{x^{n-1}}{n^2} \log^3(1-x) \mathrm{d}x = -\frac{\zeta(6)}{2} - 6\zeta^2(3),$$

which if we related to (6.107), we obtain that

$$\sum_{n=1}^{\infty} \frac{H_n^3}{n^3} + 3 \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^3} + 2 \sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^3} = \frac{\zeta(6)}{2} + 6\zeta^2(3). \tag{6.112}$$

Using that $\displaystyle\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^3} = \frac{1}{2}(\zeta^2(3) + \zeta(6))$, based on the identity in (4.14), the case $p = 3$, with $n \to \infty$, the result in (6.112) becomes

$$\sum_{n=1}^{\infty} \left(\frac{H_n}{n}\right)^3 + 3\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^3} = 5\zeta^2(3) - \frac{\zeta(6)}{2}. \tag{6.113}$$

By combining the results in (6.113) and (6.106) we get a system of relations with the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{H_n}{n}\right)^3$ and $\displaystyle\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^3}$ which gives

$$\sum_{n=1}^{\infty} \left(\frac{H_n}{n}\right)^3 = \frac{1}{2}\left(\frac{93}{8}\zeta(6) - 5\zeta^2(3)\right); \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^3} = \frac{1}{2}\left(5\zeta^2(3) - \frac{101}{24}\zeta(6)\right),$$

and the calculations of the two series are complete.

To get another relation between the two series, one might exploit the result in (6.23) to get and use $\displaystyle\sum_{n=1}^{\infty} x^n (H_n^3 - 3H_n H_n^{(2)} + 2H_n^{(3)}) = -\frac{\log^3(1-x)}{1-x}$.

This second solution aimed to get two birds with one stone by creating a system of two relations with two series like in the second solution from the previous section. The series may also be found calculated in [19]. Also, a slightly modified version of the series, $\displaystyle\sum_{n=1}^{\infty} \frac{H_n^3}{(n+1)^3}$, from which we can extract the given series, appears in [20].

6.28 Another Evaluation of an Advanced Harmonic Series of Weight 6, $\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^3}$, Treated with *The Master Theorem of Series*

Solution In the second solution from the previous section I also approached the present series by a strategy involving integrals. *How about an approach now by series manipulations only?* In this section we'll try to get a solution by series manipulations, and for doing that we might like to employ a particular case of an application of *The Master Theorem of Series*.

By making use of the result in (4.21), the case $m = 1$, where we multiply both sides by H_n/n^2 and then consider the sum from $n = 1$ to ∞, we obtain

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{H_k H_n}{(k+1)(k+n+1)n^2}\right) = \frac{1}{2}\sum_{n=1}^{\infty} \left(\frac{H_n}{n}\right)^3 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^3},$$

or, upon changing the order of summation, we have

$$\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{H_k H_n}{(k+1)(k+n+1)n^2}\right)=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^3+\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n H_n^{(2)}}{n^3}. \tag{6.114}$$

Now, we rearrange the inner series of the double series in the left-hand side of (6.114),

$$\sum_{n=1}^{\infty}\frac{H_n}{n^2(n+k+1)}=\sum_{n=0}^{\infty}\frac{H_n+1/(n+1)}{(n+1)^2(n+k+2)}$$

$$\{\text{leave out the first term of the sum and expand the series}\}$$

$$=\frac{1}{k+2}+\sum_{n=1}^{\infty}\frac{1}{(n+1)^3(n+k+2)}+\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^2(n+k+2)}. \tag{6.115}$$

Then, for the first sum in (6.115), we have

$$\sum_{n=1}^{\infty}\frac{1}{(n+1)^3(n+k+2)}$$

$$=\frac{1}{k+1}\sum_{n=1}^{\infty}\frac{1}{(n+1)^3}-\frac{1}{(k+1)^2}\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}+\frac{1}{(k+1)^2}\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+k+2)}$$

$$\left\{\text{make use of the fact that }\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+k+2)}=\frac{H_{k+2}-1}{k+1}\right\}$$

$$=\frac{\zeta(3)}{k+1}-\frac{\zeta(2)}{(k+1)^2}-\frac{1}{k+2}+\frac{H_{k+1}}{(k+1)^3}. \tag{6.116}$$

For the second series in (6.115), we have

$$\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^2(n+k+2)}=\frac{1}{k+1}\sum_{n=1}^{\infty}\frac{H_n}{n+1}\left(\frac{1}{n+1}-\frac{1}{n+k+2}\right)$$

$$=\frac{1}{k+1}\sum_{n=1}^{\infty}\frac{H_{n+1}-1/(n+1)}{(n+1)^2}-\frac{1}{k+1}\sum_{n=1}^{\infty}\frac{H_n}{(n+1)(n+k+2)}$$

$$\{\text{reindex the first series and expand it}\}$$

$$= \frac{1}{k+1}\sum_{n=1}^{\infty}\frac{H_n}{n^2} - \frac{1}{k+1}\sum_{n=1}^{\infty}\frac{1}{n^3} - \frac{1}{k+1}\sum_{n=1}^{\infty}\frac{H_n}{(n+1)(n+(k+1)+1)}$$

{make use of the Euler sum in (3.45), the case $n = 2$, and then}

{use the result in (4.21), the case $m = 1$, for the last series}

$$= \frac{\zeta(3)}{k+1} - \frac{H_{k+1}^2}{2(k+1)^2} - \frac{H_{k+1}^{(2)}}{2(k+1)^2}. \tag{6.117}$$

Returning with the results from (6.116) and (6.117) in (6.115), we get

$$\sum_{n=1}^{\infty}\frac{H_n}{n^2(n+k+1)} = \frac{2\zeta(3)}{k+1} - \frac{\zeta(2)}{(k+1)^2} + \frac{H_{k+1}}{(k+1)^3} - \frac{H_{k+1}^2}{2(k+1)^2} - \frac{H_{k+1}^{(2)}}{2(k+1)^2}. \tag{6.118}$$

Then, if we plug the result from (6.118) in (6.114), we obtain

$$\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^3 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n H_n^{(2)}}{n^3} = \sum_{k=1}^{\infty}\frac{H_k}{k+1}\left(\sum_{n=1}^{\infty}\frac{H_n}{n^2(k+n+1)}\right)$$

$$= \sum_{k=1}^{\infty}\frac{H_{k+1}-1/(k+1)}{k+1}$$

$$\times\left(\frac{2\zeta(3)}{k+1} - \frac{\zeta(2)}{(k+1)^2} + \frac{H_{k+1}}{(k+1)^3} - \frac{H_{k+1}^2}{2(k+1)^2} - \frac{H_{k+1}^{(2)}}{2(k+1)^2}\right)$$

{reindex the series and expand it}

$$= 2\zeta(3)\sum_{k=1}^{\infty}\frac{H_k}{k^2} - 2\zeta(3)\sum_{k=1}^{\infty}\frac{1}{k^3} - \zeta(2)\sum_{k=1}^{\infty}\frac{H_k}{k^3} + \zeta(2)\sum_{k=1}^{\infty}\frac{1}{k^4} + \frac{3}{2}\sum_{k=1}^{\infty}\frac{H_k^2}{k^4}$$

$$-\sum_{k=1}^{\infty}\frac{H_k}{k^5} - \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{H_k}{k}\right)^3 - \frac{1}{2}\sum_{k=1}^{\infty}\frac{H_k H_k^{(2)}}{k^3} + \frac{1}{2}\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^4}$$

{use the Euler sum in (3.45), the cases $n = 2, 3, 5$}

{and then employ the results in (4.31) and (6.71)}

$$= \frac{89}{24}\zeta(6) - \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^3 - \frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n H_n^{(2)}}{n^3},$$

whence we get

$$\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^3+\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^3}=\frac{89}{24}\zeta(6). \tag{6.119}$$

Finally, by plugging the value of $\displaystyle\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^3$ from (4.35) in (6.119), we obtain that

$$\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^3}=\frac{1}{2}\left(5\zeta^2(3)-\frac{101}{24}\zeta(6)\right),$$

and the solution is complete.

For a second solution based upon the use of integrals, see the second solution in the previous section.

It's worth mentioning the relation established in (4.40) which shows that knowing the value of $\displaystyle\sum_{n=1}^{\infty}\frac{H_nH_n^{(3)}}{n^2}$ leads to the derivation of the present series.

6.29 And Now a Series of Weight 6, $\sum_{n=1}^{\infty}\frac{H_nH_n^{(3)}}{n^2}$, Treated with Both *The Master Theorem of Series* and Special Logarithmic Integrals

Solution At the end of the previous section I gave a reference to a simple identity that relates the series there with the one we want to calculate here. In other words, if we know the value of the series in the previous section we can get the value of the present series by means of the identity in (4.40), which we'll meet in one of the next sections. In the following I'll try to calculate the series both elementarily by series manipulations only, and for doing that I'll use *The Master Theorem of Series*, and on the other hand, I'll also try an approach with integrals.

For a first solution we rely on *The Master Theorem of Series*, more precisely on the result in (4.22) from Sect. 4.17 where if we multiply both sides by H_n/n, and then consider the summation from $n=1$ to ∞, we get

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{H_k^2H_n}{(k+1)n(k+n+1)}\right)$$

$$=\frac{1}{3}\sum_{n=1}^{\infty}\frac{H_n^4}{n^2}+\zeta(2)\sum_{n=1}^{\infty}\frac{H_n^2}{n^2}+\sum_{n=1}^{\infty}\frac{H_n^2H_n^{(2)}}{n^2}+\frac{2}{3}\sum_{n=1}^{\infty}\frac{H_nH_n^{(3)}}{n^2}-\sum_{n=1}^{\infty}\frac{H_n}{n^2}\left(\sum_{i=1}^{n}\frac{H_i}{i^2}\right)$$

{use the values of the series in (4.39), (4.29), (4.38), and (6.121)}

$$= \frac{1615}{72}\zeta(6) + 2\zeta^2(3) + \frac{2}{3}\sum_{n=1}^{\infty} \frac{H_n H_n^{(3)}}{n^2}, \tag{6.120}$$

where I also used that due to the symmetry, since $\sum_{k=1}^{\infty} \frac{H_k}{k^2} = 2\zeta(3)$, we have

$$4\zeta^2(3) = \sum_{k=1}^{\infty} \frac{H_k}{k^2} \sum_{n=1}^{\infty} \frac{H_n}{n^2} = \sum_{k=1}^{\infty} \left(\sum_{n=1}^{k-1} \frac{H_k H_n}{k^2 n^2} \right) + \sum_{k=1}^{\infty} \frac{H_k^2}{k^4} + \sum_{k=1}^{\infty} \left(\sum_{n=k+1}^{\infty} \frac{H_k H_n}{k^2 n^2} \right)$$

$$= 2\sum_{k=1}^{\infty} \left(\sum_{n=1}^{k-1} \frac{H_k H_n}{k^2 n^2} \right) + \sum_{k=1}^{\infty} \frac{H_k^2}{k^4} = 2\sum_{k=1}^{\infty} \left(\sum_{n=1}^{k} \frac{H_k H_n}{k^2 n^2} \right) - \sum_{k=1}^{\infty} \frac{H_k^2}{k^4}$$

{the last series is calculated in (4.31)}

$$= 2\sum_{k=1}^{\infty} \frac{H_k}{k^2} \left(\sum_{n=1}^{k} \frac{H_n}{n^2} \right) - \frac{97}{24}\zeta(6) + 2\zeta^2(3),$$

whence we get that

$$\sum_{k=1}^{\infty} \frac{H_k}{k^2} \left(\sum_{n=1}^{k} \frac{H_n}{n^2} \right) = \frac{97}{48}\zeta(6) + \zeta^2(3). \tag{6.121}$$

Now, for the double series in (6.120), we write

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{H_k^2 H_n}{(k+1)n(k+n+1)} \right) = \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{H_k^2 H_n}{(k+1)n(k+n+1)} \right)$$

{reindex the inner series and start from $n = 0$ to ∞}

$$= \sum_{k=1}^{\infty} \left(\sum_{n=0}^{\infty} \frac{H_k^2 (H_n + 1/(n+1))}{(k+1)(n+1)(k+n+2)} \right)$$

{for $n = 0$ let outside the term of the inner series}

$$= \sum_{k=1}^{\infty} \frac{H_k^2}{(k+1)(k+2)} + \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{H_k^2 (H_n + 1/(n+1))}{(k+1)(n+1)(k+n+2)} \right)$$

$$
= \underbrace{\sum_{k=1}^{\infty} \frac{H_k^2}{(k+1)(k+2)}}_{S_1} + \underbrace{\sum_{k=1}^{\infty} \frac{H_k^2}{k+1} \left(\sum_{n=1}^{\infty} \frac{H_n}{(n+1)(k+n+2)} \right)}_{S_2}
$$

$$
+ \underbrace{\sum_{k=1}^{\infty} \frac{H_k^2}{k+1} \left(\sum_{n=1}^{\infty} \frac{1}{(n+1)^2(k+n+2)} \right)}_{S_3}. \tag{6.122}
$$

We note the series S_1 in (6.122), is the case $n = 1$ in (4.22),

$$
S_1 = \sum_{k=1}^{\infty} \frac{H_k^2}{(k+1)(k+2)} = 1 + \zeta(2). \tag{6.123}
$$

Further, to calculate the series S_2 in (6.122), we make use of the result in (4.21), the case $m = 1$, and then we write

$$
S_2 = \sum_{k=1}^{\infty} \frac{H_k^2}{k+1} \left(\sum_{n=1}^{\infty} \frac{H_n}{(n+1)(n+(k+1)+1)} \right) = \sum_{k=1}^{\infty} \frac{H_k^2}{k+1} \left(\frac{H_{k+1}^2 + H_{k+1}^{(2)}}{2(k+1)} \right)
$$

$$
= \sum_{k=1}^{\infty} \frac{(H_{k+1} - 1/(k+1))^2}{k+1} \left(\frac{H_{k+1}^2 + H_{k+1}^{(2)}}{2(k+1)} \right)
$$

{reindex the series and expand it}

$$
= \frac{1}{2} \sum_{k=1}^{\infty} \frac{H_k^4}{k^2} - \sum_{k=1}^{\infty} \frac{H_k^3}{k^3} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{H_k^2}{k^4} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{H_k^2 H_k^{(2)}}{k^2} - \sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k^3} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{H_k^{(2)}}{k^4}
$$

{use the values of the series from (4.39), (4.35), (4.31), (4.38), (4.36), and (6.71)}

$$
= \frac{81}{4} \zeta(6) + 2\zeta^2(3). \tag{6.124}
$$

Next, for the series S_3 in (6.122), we have

$$
S_3 = \sum_{k=1}^{\infty} \frac{H_k^2}{k+1} \left(\sum_{n=1}^{\infty} \frac{1}{(n+1)^2(k+n+2)} \right)
$$

$$
= \sum_{k=1}^{\infty} \frac{H_k^2}{(k+1)^2} \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} - \sum_{k=1}^{\infty} \frac{H_k^2}{(k+1)^2} \left(\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+k+2)} \right)
$$

$$\left\{\text{use that } \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+k+2)} = \frac{H_{k+1}}{k+1} - \frac{1}{k+2}\right\}$$

$$= (\zeta(2)-1)\sum_{k=1}^{\infty} \frac{(H_{k+1}-1/(k+1))^2}{(k+1)^2}$$

$$-\sum_{k=1}^{\infty} \frac{(H_{k+1}-1/(k+1))^2}{(k+1)^2}\left(\frac{H_{k+1}}{k+1} - \frac{1}{k+2}\right)$$

{reindex the series and expand them}

$$= (\zeta(2)-1)\sum_{k=1}^{\infty} \frac{H_k^2}{k^2} - 2(\zeta(2)-1)\sum_{k=1}^{\infty} \frac{H_k}{k^3} + (\zeta(2)-1)\sum_{k=1}^{\infty} \frac{1}{k^4} - \sum_{k=1}^{\infty} \left(\frac{H_k}{k}\right)^3$$

$$+2\sum_{k=1}^{\infty} \frac{H_k^2}{k^4} - \sum_{k=1}^{\infty} \frac{H_k}{k^5} + \underbrace{\sum_{k=1}^{\infty} \frac{1}{k^4(k+1)}}_{S_4} + \underbrace{\sum_{k=1}^{\infty} \frac{H_k^2}{k^2(k+1)}}_{S_5} - 2\underbrace{\sum_{k=1}^{\infty} \frac{H_k}{k^3(k+1)}}_{S_6}. \tag{6.125}$$

For the series S_4 in (6.125) we make use of the partial fractions, and we write

$$S_4 = \sum_{k=1}^{\infty} \frac{1}{k^4(k+1)} = \sum_{k=1}^{\infty} \left(\frac{1}{k^4} - \frac{1}{k^3} + \frac{1}{k^2} - \frac{1}{k} + \frac{1}{k+1}\right) = \sum_{k=1}^{\infty} \frac{1}{k^4} - \sum_{k=1}^{\infty} \frac{1}{k^3}$$

$$+\sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \zeta(4) - \zeta(3) + \zeta(2) - 1. \tag{6.126}$$

Then, for the series S_5 in (6.125), we proceed as follows

$$S_5 = \sum_{k=1}^{\infty} \frac{H_k^2}{k^2(k+1)} = \sum_{k=1}^{\infty} \left(\frac{H_k^2}{k^2} - \frac{H_k^2}{k} + \frac{(H_{k+1}-1/(k+1))^2}{k+1}\right)$$

$$= \sum_{k=1}^{\infty} \frac{H_k^2}{k^2} - 2\sum_{k=1}^{\infty} \frac{H_{k+1}}{(k+1)^2} + \sum_{k=1}^{\infty} \frac{1}{(k+1)^3} + \sum_{k=1}^{\infty} \left(\frac{H_{k+1}^2}{k+1} - \frac{H_k^2}{k}\right)$$

{reindex the second and third series and start from $k=2$ to ∞}

$$= \sum_{k=1}^{\infty} \frac{H_k^2}{k^2} - 2\sum_{k=2}^{\infty} \frac{H_k}{k^2} + \sum_{k=2}^{\infty} \frac{1}{k^3} - 1$$

{for the first two series use the values in (4.29) and (3.45), the case $n = 2$}

$$= \frac{17}{4}\zeta(4) - 3\zeta(3). \tag{6.127}$$

Now, we calculate the last series from (6.125), and we write

$$S_6 = \sum_{k=1}^{\infty} \frac{H_k}{k^3(k+1)} = \sum_{k=1}^{\infty} \left(\frac{H_k}{k^3} - \frac{H_k}{k^2} + \frac{H_k}{k} - \frac{H_{k+1} - 1/(k+1)}{k+1} \right)$$

$$= \sum_{k=1}^{\infty} \frac{H_k}{k^3} - \sum_{k=1}^{\infty} \frac{H_k}{k^2} + \sum_{k=1}^{\infty} \frac{1}{(k+1)^2} + \sum_{k=1}^{\infty} \left(\frac{H_k}{k} - \frac{H_{k+1}}{k+1} \right)$$

{make use of the classical Euler sum in (3.45), the cases $n = 2, 3$}

$$= \frac{5}{4}\zeta(4) - 2\zeta(3) + \zeta(2). \tag{6.128}$$

At this point, we collect the values of the series S_4, S_5, and S_6 given in (6.126), (6.127), and (6.128), and then plug them in (6.125) that lead to

$$S_3 = \sum_{k=1}^{\infty} \frac{H_k^2}{k+1} \left(\sum_{n=1}^{\infty} \frac{1}{(n+1)^2(k+n+2)} \right)$$

$$= (\zeta(2) - 1) \sum_{k=1}^{\infty} \frac{H_k^2}{k^2} - 2(\zeta(2) - 1) \sum_{k=1}^{\infty} \frac{H_k}{k^3} + (\zeta(2) - 1) \sum_{k=1}^{\infty} \frac{1}{k^4} - \sum_{k=1}^{\infty} \left(\frac{H_k}{k} \right)^3$$

$$+2 \sum_{k=1}^{\infty} \frac{H_k^2}{k^4} - \sum_{k=1}^{\infty} \frac{H_k}{k^5} + \frac{11}{4}\zeta(4) - \zeta(2) - 1$$

{use the Euler sum in (3.45), the cases $n = 3, 5$ together}

{with the values of the series in (4.29), (4.35), and (4.31)}

$$= \frac{16}{3}\zeta(6) - \zeta(2) - \zeta^2(3) - 1. \tag{6.129}$$

Then, by plugging the values of the series S_1, S_2, and S_3 from (6.123), (6.124), and (6.129) in (6.122), we get

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{H_k^2 H_n}{(k+1)n(k+n+1)} \right) = \frac{307}{12}\zeta(6) + \zeta^2(3). \tag{6.130}$$

Hence, by combining (6.120) and (6.130), we conclude that

$$\sum_{n=1}^{\infty} \frac{H_n H_n^{(3)}}{n^2} = \frac{1}{2}\left(\frac{227}{24}\zeta(6) - 3\zeta^2(3)\right),$$

and the first solution is complete.

For a second solution, one involving integrals, let's start out with one of the simpler logarithmic integrals presented in *Integrals* chapter, the identity in (1.4), where if we multiply both sides by $-H_n^{(3)}/n$ and then consider the summation from $n = 1$ to ∞, we get

$$\sum_{n=1}^{\infty} \frac{H_n H_n^{(3)}}{n^2} = -\sum_{n=1}^{\infty} H_n^{(3)} \int_0^1 \left(\int_0^1 (xy)^{n-1} \log(1-x) \mathrm{d}y\right) \mathrm{d}x$$

$$\{\text{change the order of integration and summation}\}$$

$$= -\int_0^1 \left(\int_0^1 \sum_{n=1}^{\infty} (xy)^{n-1} H_n^{(3)} \log(1-x) \mathrm{d}y\right) \mathrm{d}x$$

$$= -\int_0^1 \left(\int_0^1 \frac{\log(1-x)\operatorname{Li}_3(xy)}{xy(1-xy)} \mathrm{d}y\right) \mathrm{d}x$$

$$= -\int_0^1 \left(\int_0^1 \frac{\log(1-x)\operatorname{Li}_3(xy)}{xy} \mathrm{d}y\right) \mathrm{d}x - \int_0^1 \left(\int_0^1 \frac{\log(1-x)\operatorname{Li}_3(xy)}{1-xy} \mathrm{d}y\right) \mathrm{d}x, \tag{6.131}$$

where to get the penultimate equality I used the result in (4.6), the case $m = 3$.

Now, we prepare to calculate the first integral in (6.131), and using that $\int \operatorname{Li}_3(xy)/y \mathrm{d}y = \operatorname{Li}_4(xy) + C$, we get

$$\int_0^1 \left(\int_0^1 \frac{\log(1-x)\operatorname{Li}_3(xy)}{xy} \mathrm{d}y\right) \mathrm{d}x = \int_0^1 \frac{\log(1-x)\operatorname{Li}_4(x)}{x} \mathrm{d}x$$

$$= \int_0^1 \sum_{n=1}^{\infty} \frac{x^{n-1}}{n^4} \log(1-x) \mathrm{d}x$$

$$\{\text{change the order of summation and integration and use the result in (1.4)}\}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^4} \int_0^1 x^{n-1} \log(1-x) \mathrm{d}x = -\sum_{n=1}^{\infty} \frac{H_n}{n^5}$$

$$\{\text{make use of the Euler sum in (3.45), the case } n = 5\}$$

$$= \frac{1}{2}\left(\zeta^2(3) - \frac{7}{2}\zeta(6)\right). \tag{6.132}$$

Further, to calculate the second integral in (6.131), we start with the integration by parts in the inner integral, and then we have

$$\int_0^1 \frac{\log(1-x)\operatorname{Li}_3(xy)}{1-xy}\mathrm{d}y = \int_0^1 \left(-\frac{\log(1-xy)}{x}\right)' \log(1-x)\operatorname{Li}_3(xy)\mathrm{d}y$$

$$= -\frac{\log(1-xy)}{x}\log(1-x)\operatorname{Li}_3(xy)\Bigg|_{y=0}^{y=1} + \frac{\log(1-x)}{x}\int_0^1 \frac{\log(1-xy)\operatorname{Li}_2(xy)}{y}\mathrm{d}y$$

$$= -\frac{\log^2(1-x)\operatorname{Li}_3(x)}{x} + \frac{\log(1-x)}{x}\int_0^1 \frac{\log(1-xy)\operatorname{Li}_2(xy)}{y}\mathrm{d}y$$

$$\left\{\text{note that } \frac{d}{dy}(\operatorname{Li}_2(xy))^2 = -2\frac{\log(1-xy)\operatorname{Li}_2(xy)}{y}\right\}$$

$$= -\frac{\log^2(1-x)\operatorname{Li}_3(x)}{x} - \frac{\log(1-x)(\operatorname{Li}_2(x))^2}{2x},$$

that leads to

$$\int_0^1 \left(\int_0^1 \frac{\log(1-x)\operatorname{Li}_3(xy)}{1-xy}\mathrm{d}y\right)\mathrm{d}x$$

$$= -\int_0^1 \frac{\log^2(1-x)\operatorname{Li}_3(x)}{x}\mathrm{d}x - \frac{1}{2}\int_0^1 \frac{\log(1-x)(\operatorname{Li}_2(x))^2}{x}\mathrm{d}x. \tag{6.133}$$

Now, for the first integral in (6.133), we get

$$\int_0^1 \frac{\log^2(1-x)\operatorname{Li}_3(x)}{x}\mathrm{d}x = \int_0^1 \sum_{n=1}^{\infty} \frac{x^{n-1}}{n^3}\log^2(1-x)\mathrm{d}x$$

$$= \sum_{n=1}^{\infty}\int_0^1 \frac{x^{n-1}}{n^3}\log^2(1-x)\mathrm{d}x$$

$$\{\text{use the identity in (1.5)}\}$$

$$= \sum_{n=1}^{\infty}\frac{H_n^2}{n^4} + \sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4}$$

$$\{\text{make use of the values of the series in (4.31) and (6.71)}\}$$

$$= \frac{89}{24}\zeta(6) - \zeta^2(3). \tag{6.134}$$

Next, the second integral in (6.133) is straightforward, and we get

$$\int_0^1 \frac{\log(1-x)(\operatorname{Li}_2(x))^2}{x} \mathrm{d}x = -\frac{1}{3}(\operatorname{Li}_2(x))^3 \bigg|_{x=0}^{x=1} = -\frac{35}{24}\zeta(6). \tag{6.135}$$

Collecting the results from (6.134) and (6.135) in (6.133), we get

$$\int_0^1 \left(\int_0^1 \frac{\log(1-x)\operatorname{Li}_3(xy)}{1-xy} \mathrm{d}y\right) \mathrm{d}x = \zeta^2(3) - \frac{143}{48}\zeta(6). \tag{6.136}$$

If we plug the results from (6.132) and (6.136) in (6.131), we conclude that

$$\sum_{n=1}^{\infty} \frac{H_n H_n^{(3)}}{n^2} = \frac{1}{2}\left(\frac{227}{24}\zeta(6) - 3\zeta^2(3)\right), \tag{6.137}$$

and the second solution is complete.

As a note before bringing an end to this section, we may notice again during the second solution the extraordinary usefulness of the elementary logarithmic integrals in Sect. 1.3. So good to know!

6.30 An Appealing Exotic Harmonic Series of Weight 6, $\sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^2}$, Derived by Elementary Series Manipulations

Solution A nice observation is that the summand of the given series is in fact the summand of the Au-Yeung series multiplied by $H_n^{(2)}$, which leads to the actual series of weight 6 we want to calculate in this section. As before, the lucky card here is the use of an application of *The Master Theorem of Series* which gives us the possibility of doing the calculations by series manipulations only.

Recalling the identity in (4.24), multiplying both sides by $1/n$ and considering the sum from $n = 1$ to ∞, we obtain, for the left-hand side, that

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{(k+1)(k+n+1)n}\right) = \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty} \frac{H_k H_k^{(2)}}{(k+1)(k+n+1)n}\right)$$

$$= \sum_{k=1}^{\infty} \frac{H_k H_{k+1} H_k^{(2)}}{(k+1)^2} = \sum_{k=1}^{\infty} \frac{(H_{k+1} - 1/(k+1))H_{k+1}(H_{k+1}^{(2)} - 1/(k+1)^2)}{(k+1)^2}$$

$$\{\text{reindex the series and expand it}\}$$

$$= \sum_{k=1}^{\infty} \frac{H_k^2 H_k^{(2)}}{k^2} - \sum_{k=1}^{\infty} \frac{H_k^2}{k^4} - \sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k^3} + \sum_{k=1}^{\infty} \frac{H_k}{k^5}$$

{make use of the results in (4.31), (4.36), and (3.45), the case $n = 5$}

$$= \sum_{k=1}^{\infty} \frac{H_k^2 H_k^{(2)}}{k^2} - \frac{3}{16}\zeta(6) - \zeta^2(3), \tag{6.138}$$

and for the right-hand side, since we have that $\sum_{n=1}^{\infty} \frac{1}{n^2} \left(\sum_{i=1}^{n} \frac{H_i^2}{i^2} \right) = \sum_{i=1}^{\infty} \left(\sum_{n=i}^{\infty} \frac{H_i^2}{i^2 n^2} \right) = \sum_{i=1}^{\infty} \left(\sum_{n=i+1}^{\infty} + \sum_{n=i}^{i} \right) \frac{H_i^2}{i^2 n^2} = \sum_{i=1}^{\infty} \frac{H_i^2}{i^2} \left(\sum_{n=i+1}^{\infty} \frac{1}{n^2} \right) + \sum_{i=1}^{\infty} \frac{H_i^2}{i^4}$, we get

$$\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\sum_{i=1}^{n} \frac{H_i^2}{i^2} \right) - \sum_{n=1}^{\infty} \frac{H_n}{n^2} \left(\sum_{i=1}^{n} \frac{H_i}{i^2} \right) + 2\zeta(3) \sum_{n=1}^{\infty} \frac{H_n}{n^2} + \frac{\zeta(2)}{2} \sum_{n=1}^{\infty} \frac{H_n^2}{n^2}$$
$$- \frac{\zeta(2)}{2} \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2} - \frac{1}{4} \sum_{n=1}^{\infty} \frac{H_n^{(4)}}{n^2} - \frac{1}{4} \sum_{n=1}^{\infty} \frac{(H_n^{(2)})^2}{n^2}$$
$$= \frac{1}{2} \sum_{i=1}^{\infty} \frac{H_i^2}{i^2} \left(\sum_{n=i+1}^{\infty} \frac{1}{n^2} \right) + \frac{1}{2} \sum_{n=1}^{\infty} \frac{H_n^2}{n^4} - \sum_{n=1}^{\infty} \frac{H_n}{n^2} \left(\sum_{i=1}^{n} \frac{H_i}{i^2} \right)$$
$$+ 2\zeta(3) \sum_{n=1}^{\infty} \frac{H_n}{n^2} - \frac{\zeta(2)}{2} \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2} - \frac{1}{4} \sum_{n=1}^{\infty} \frac{H_n^{(4)}}{n^2} - \frac{1}{4} \sum_{n=1}^{\infty} \frac{(H_n^{(2)})^2}{n^2} + \frac{\zeta(2)}{2} \sum_{n=1}^{\infty} \frac{H_n^2}{n^2}$$
$$= -\frac{1}{2} \sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^2} + 2\zeta^2(3) + \frac{79}{16}\zeta(6), \tag{6.139}$$

where for getting the last equality I used the results in (4.31), (6.121), (6.72), (4.29), and (3.45), the case $n = 2$, (4.14), the case $p = 2$, with $n \to \infty$, (4.15), the case $p = 2$, with $n \to \infty$, and the fact that

$$\sum_{i=1}^{\infty} \frac{H_i^2}{i^2} \left(\sum_{n=i+1}^{\infty} \frac{1}{n^2} \right) = \sum_{i=1}^{\infty} \frac{H_i^2}{i^2} \left(\zeta(2) - H_i^{(2)} \right) = \zeta(2) \sum_{i=1}^{\infty} \frac{H_i^2}{i^2} - \sum_{i=1}^{\infty} \frac{H_i^2 H_i^{(2)}}{i^2}$$

{make use of the result in (4.29)}

$$= \frac{119}{16}\zeta(6) - \sum_{i=1}^{\infty} \frac{H_i^2 H_i^{(2)}}{i^2}.$$

Finally, by combining (6.138) and (6.139), we conclude that

$$\sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^2} = \frac{41}{12}\zeta(6) + 2\zeta^2(3),$$

and the solution is finalized.

For a second solution based upon a system of relations with series, check the next section.

6.31 Another Appealing Exotic Harmonic Series of Weight 6, $\sum_{n=1}^{\infty} \frac{H_n^4}{n^2}$, Derived by Elementary Series Manipulations

Solution We have arrived at the last series of weight 6 where a solution by series manipulations is expected as before. For example, an option would be to establish a relation between the current series and the series from the previous section where to further use the value of the latter in order to extract the value of the desired series.

On the other hand, we may establish two relations between the current series and the one from the previous section, and then extract both the value of the desired series, and as a bonus, the one from the previous section.

The main idea is to make a system of two relations with the series $\sum_{n=1}^{\infty} \frac{H_n^4}{n^2}$ and $\sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^2}$. First, we make use of the result in (4.21), the case $m = 1$, where if we multiply both sides by H_n^2/n and then consider the sum from $n = 1$ to ∞, we get

$$\frac{1}{2}\sum_{n=1}^{\infty} \frac{H_n^4}{n^2} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^2} = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{H_k H_n^2}{(k+1)n(k+n+1)} \right)$$

$$\{\text{reverse the order of summation}\}$$

$$= \sum_{k=1}^{\infty} \frac{H_k}{k+1} \sum_{n=1}^{\infty} \frac{H_n^2}{n(k+n+1)}. \tag{6.140}$$

Next, we work on the inner series in (6.140) that we can write as follows

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n(k+n+1)}=\sum_{n=0}^{\infty}\frac{H_{n+1}^2}{(n+1)(k+n+2)}$$

{leave out the term of the series for $n=0$}

$$=\frac{1}{k+2}+\sum_{n=1}^{\infty}\frac{(H_n+1/(n+1))^2}{(n+1)(k+n+2)}$$

$$=\frac{1}{k+2}+\sum_{n=1}^{\infty}\frac{H_n^2}{(n+1)(k+n+2)}+2\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^2(k+n+2)}$$

$$+\sum_{n=1}^{\infty}\frac{1}{(n+1)^3(k+n+2)}. \tag{6.141}$$

The first series in (6.141) we already met in (4.22) from Sect. 4.17, and we have

$$\sum_{n=1}^{\infty}\frac{H_n^2}{(n+1)(k+n+2)}=\frac{H_{k+1}^3+3\zeta(2)H_{k+1}+3H_{k+1}H_{k+1}^{(2)}+2H_{k+1}^{(3)}}{3(k+1)}$$

$$-\frac{1}{k+1}\sum_{i=1}^{k+1}\frac{H_i}{i^2}. \tag{6.142}$$

For the second series in (6.141), we get

$$\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^2(k+n+2)}$$

$$=\frac{1}{k+1}\sum_{n=1}^{\infty}\frac{H_{n+1}}{(n+1)^2}-\frac{1}{k+1}\sum_{n=1}^{\infty}\frac{1}{(n+1)^3}-\frac{1}{k+1}\sum_{n=1}^{\infty}\frac{H_n}{(n+1)(k+n+2)}$$

{use the classical Euler sum in (3.45), the case $n=2$,}

{and then employ the result in (4.21), the case $m=1$}

$$=\frac{\zeta(3)}{k+1}-\frac{H_{k+1}^2+H_{k+1}^{(2)}}{2(k+1)^2}. \tag{6.143}$$

Finally, for the last series in (6.141), we have

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^3(k+n+2)}$$

$$= \frac{1}{k+1} \sum_{n=1}^{\infty} \frac{1}{(n+1)^3} - \frac{1}{(k+1)^2} \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} + \frac{1}{(k+1)^2} \sum_{n=1}^{\infty} \frac{1}{(n+1)(k+n+2)}$$

$$\left\{\text{make use of the fact that } \sum_{n=1}^{\infty} \frac{1}{(n+1)(k+n+2)} = \frac{H_{k+1}}{k+1} - \frac{1}{k+2}\right\}$$

$$= \frac{\zeta(3)}{k+1} - \frac{\zeta(2)}{(k+1)^2} + \frac{H_{k+1}}{(k+1)^3} - \frac{1}{k+2}. \tag{6.144}$$

If we plug the results from (6.142), (6.143), and (6.144) in (6.141), we get

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n(k+n+1)}$$

$$= \frac{H_{k+1}^3}{3(k+1)} - \frac{H_{k+1}^2}{(k+1)^2} + \zeta(2)\frac{H_{k+1}}{k+1} + \frac{H_{k+1}}{(k+1)^3} + \frac{H_{k+1}H_{k+1}^{(2)}}{k+1} - \frac{H_{k+1}^{(2)}}{(k+1)^2}$$

$$+ \frac{2H_{k+1}^{(3)}}{3(k+1)} + \frac{3\zeta(3)}{k+1} - \frac{\zeta(2)}{(k+1)^2} - \frac{1}{k+1} \sum_{i=1}^{k+1} \frac{H_i}{i^2}. \tag{6.145}$$

Then, we plug the result from (6.145) in (6.140), and we get

$$\frac{1}{2} \sum_{n=1}^{\infty} \frac{H_n^4}{n^2} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^2}$$

$$= \sum_{k=1}^{\infty} \frac{H_{k+1} - \frac{1}{k+1}}{k+1} \left(\frac{H_{k+1}^3}{3(k+1)} - \frac{H_{k+1}^2}{(k+1)^2} + \zeta(2)\frac{H_{k+1}}{k+1} + \frac{H_{k+1}}{(k+1)^3} + \frac{H_{k+1}H_{k+1}^{(2)}}{k+1} \right.$$

$$\left. - \frac{H_{k+1}^{(2)}}{(k+1)^2} + \frac{2H_{k+1}^{(3)}}{3(k+1)} + \frac{3\zeta(3)}{k+1} - \frac{\zeta(2)}{(k+1)^2} - \frac{1}{k+1} \sum_{i=1}^{k+1} \frac{H_i}{i^2} \right)$$

$$\{\text{reindex the series, expand it, and replace } k \text{ by } n\}$$

$$= \frac{1}{3} \sum_{n=1}^{\infty} \frac{H_n^4}{n^2} - \frac{4}{3} \sum_{n=1}^{\infty} \frac{H_n^3}{n^3} + 2 \sum_{n=1}^{\infty} \frac{H_n^2}{n^4} + \zeta(2) \sum_{n=1}^{\infty} \frac{H_n^2}{n^2} - 2\zeta(2) \sum_{n=1}^{\infty} \frac{H_n}{n^3} - \sum_{n=1}^{\infty} \frac{H_n}{n^5}$$

$$+\sum_{n=1}^{\infty}\frac{H_n^2H_n^{(2)}}{n^2}-2\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^3}+\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4}+\frac{2}{3}\sum_{n=1}^{\infty}\frac{H_nH_n^{(3)}}{n^2}-\frac{2}{3}\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^3}$$

$$+3\zeta(3)\sum_{n=1}^{\infty}\frac{H_n}{n^2}-3\zeta(3)\sum_{n=1}^{\infty}\frac{1}{n^3}+\zeta(2)\sum_{n=1}^{\infty}\frac{1}{n^4}-\sum_{n=1}^{\infty}\frac{H_n-1/n}{n^2}\left(\sum_{i=1}^{n}\frac{H_i}{i^2}\right)$$

whence we get

$$\frac{1}{6}\sum_{n=1}^{\infty}\frac{H_n^4}{n^2}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n^2H_n^{(2)}}{n^2}$$

$$=2\sum_{n=1}^{\infty}\frac{H_n^2}{n^4}-\frac{4}{3}\sum_{n=1}^{\infty}\frac{H_n^3}{n^3}+\zeta(2)\sum_{n=1}^{\infty}\frac{H_n^2}{n^2}-2\zeta(2)\sum_{n=1}^{\infty}\frac{H_n}{n^3}-\sum_{n=1}^{\infty}\frac{H_n}{n^5}-2\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^3}$$

$$+\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4}+\frac{2}{3}\sum_{n=1}^{\infty}\frac{H_nH_n^{(3)}}{n^2}-\frac{2}{3}\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^3}+3\zeta(3)\sum_{n=1}^{\infty}\frac{H_n}{n^2}-\sum_{n=1}^{\infty}\frac{H_n-1/n}{n^2}\left(\sum_{i=1}^{n}\frac{H_i}{i^2}\right)$$

$$-3\zeta^2(3)+\frac{7}{4}\zeta(6)$$

{make use of the classical Euler sum in (3.45), the cases}

{$n=2, 3, 5$, and then employ the results in (4.31), (4.35), (4.29),}

{(4.36), (6.71), (4.37), and (4.14), the case $p=3$, with $n\to\infty$}

$$=\frac{1453}{144}\zeta(6)-\frac{5}{2}\zeta^2(3)-\sum_{n=1}^{\infty}\frac{H_n-1/n}{n^2}\left(\sum_{i=1}^{n}\frac{H_i}{i^2}\right). \tag{6.146}$$

Now, to calculate the series in (6.146), we proceed as follows

$$\sum_{n=1}^{\infty}\frac{H_n-1/n}{n^2}\left(\sum_{i=1}^{n}\frac{H_i}{i^2}\right)=\sum_{n=1}^{\infty}\frac{H_n}{n^2}\left(\sum_{i=1}^{n}\frac{H_i}{i^2}\right)-\sum_{n=1}^{\infty}\frac{1}{n^3}\left(\sum_{i=1}^{n}\frac{H_i}{i^2}\right)$$

{for the second series change the order of summation}

$$=\sum_{n=1}^{\infty}\frac{H_n}{n^2}\left(\sum_{i=1}^{n}\frac{H_i}{i^2}\right)-\sum_{i=1}^{\infty}\left(\sum_{n=i}^{\infty}\frac{H_i}{i^2n^3}\right)=\sum_{n=1}^{\infty}\frac{H_n}{n^2}\left(\sum_{i=1}^{n}\frac{H_i}{i^2}\right)$$

$$-\sum_{i=1}^{\infty}\frac{H_i}{i^2}\left(\zeta(3)-H_i^{(3)}+\frac{1}{i^3}\right)$$

$$= \sum_{n=1}^{\infty} \frac{H_n}{n^2} \left(\sum_{i=1}^{n} \frac{H_i}{i^2} \right) - \zeta(3) \sum_{i=1}^{\infty} \frac{H_i}{i^2} - \sum_{i=1}^{\infty} \frac{H_i}{i^5} + \sum_{i=1}^{\infty} \frac{H_i H_i^{(3)}}{i^2}$$

$$\{\text{use the linear Euler sum in (3.45), the cases } n = 2, 5, \text{ and}\}$$

$$\{\text{the first and the last series are calculated in (6.121) and (4.37)}\}$$

$$= 5\zeta(6) - 2\zeta^2(3), \tag{6.147}$$

By plugging the result from (6.147) in (6.146), we get

$$\sum_{n=1}^{\infty} \frac{H_n^4}{n^2} - 3 \sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^2} = \frac{733}{24} \zeta(6) - 3\zeta^2(3), \tag{6.148}$$

and the first relation between the two main series has been established.

Then, using the result in (4.26), the first equality, multiplying both sides by H_n/n and considering the sum from $n = 1$ to ∞, we obtain

$$\frac{1}{3} \sum_{n=1}^{\infty} \frac{H_n^4}{n^2} + \sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^2} + \frac{2}{3} \sum_{n=1}^{\infty} \frac{H_n H_n^{(3)}}{n^2} = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{(H_k^2 - H_k^{(2)}) H_n}{(k+1)n(k+n+1)} \right)$$

$$\{\text{reverse the order of summation}\}$$

$$= \sum_{k=1}^{\infty} \frac{H_k^2 - H_k^{(2)}}{k+1} \sum_{n=1}^{\infty} \frac{H_n}{n(k+n+1)}. \tag{6.149}$$

Now we rearrange the inner series in (6.149), and reindexing, we get

$$\sum_{n=1}^{\infty} \frac{H_n}{n(k+n+1)} = \sum_{n=0}^{\infty} \frac{H_{n+1}}{(n+1)(k+n+2)} = \frac{1}{k+2} + \sum_{n=1}^{\infty} \frac{H_n + 1/(n+1)}{(n+1)(k+n+2)}$$

$$= \frac{1}{k+2} + \sum_{n=1}^{\infty} \frac{H_n}{(n+1)(k+n+2)} + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2(k+n+2)}$$

$$\{\text{make use of the result in (4.21), the case } m = 1\}$$

$$= \frac{1}{k+2} + \frac{H_{k+1}^2 + H_{k+1}^{(2)}}{2(k+1)} + \frac{1}{k+1} \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} - \frac{1}{k+1} \sum_{n=1}^{\infty} \frac{1}{(n+1)(k+n+2)}$$

$$\left\{ \text{employ the elementary result, } \sum_{n=1}^{\infty} \frac{1}{(n+1)(k+n+2)} = \frac{H_{k+2}}{k+1} - \frac{1}{k+1} \right\}$$

$$= \frac{1}{k+2} + \frac{H_{k+1}^2 + H_{k+1}^{(2)}}{2(k+1)} + \frac{\zeta(2)-1}{k+1} - \frac{H_{k+2}}{(k+1)^2} + \frac{1}{(k+1)^2}$$

$$= \frac{H_{k+1}^2 + H_{k+1}^{(2)}}{2(k+1)} + \frac{\zeta(2)}{k+1} - \frac{H_{k+1}}{(k+1)^2}. \tag{6.150}$$

Upon plugging the result from (6.150) in (6.149), we obtain

$$\frac{1}{3}\sum_{n=1}^{\infty}\frac{H_n^4}{n^2} + \sum_{n=1}^{\infty}\frac{H_n^2 H_n^{(2)}}{n^2} + \frac{2}{3}\sum_{n=1}^{\infty}\frac{H_n H_n^{(3)}}{n^2}$$

$$= \sum_{k=1}^{\infty}\frac{\left(H_{k+1}-\frac{1}{k+1}\right)^2 - \left(H_{k+1}^{(2)} - \frac{1}{(k+1)^2}\right)}{k+1}\left(\frac{H_{k+1}^2 + H_{k+1}^{(2)}}{2(k+1)} + \frac{\zeta(2)}{k+1} - \frac{H_{k+1}}{(k+1)^2}\right)$$

{reindex the series, expand it, and replace k by n}

$$= \frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n^4}{n^2} - 2\sum_{n=1}^{\infty}\frac{H_n^3}{n^3} + 3\sum_{n=1}^{\infty}\frac{H_n^2}{n^4} + \zeta(2)\sum_{n=1}^{\infty}\frac{H_n^2}{n^2} - 2\zeta(2)\sum_{n=1}^{\infty}\frac{H_n}{n^3} - 2\sum_{n=1}^{\infty}\frac{H_n}{n^5}$$

$$-\zeta(2)\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^2} + \sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{(H_n^{(2)})^2}{n^2} + 2\zeta(2)\sum_{n=1}^{\infty}\frac{1}{n^4},$$

whence we get

$$\frac{1}{6}\sum_{n=1}^{\infty}\frac{H_n^4}{n^2} - \sum_{n=1}^{\infty}\frac{H_n^2 H_n^{(2)}}{n^2}$$

$$= 2\sum_{n=1}^{\infty}\frac{H_n^3}{n^3} - 3\sum_{n=1}^{\infty}\frac{H_n^2}{n^4} - \zeta(2)\sum_{n=1}^{\infty}\frac{H_n^2}{n^2} + 2\zeta(2)\sum_{n=1}^{\infty}\frac{H_n}{n^3} + 2\sum_{n=1}^{\infty}\frac{H_n}{n^5} + \frac{2}{3}\sum_{n=1}^{\infty}\frac{H_n H_n^{(3)}}{n^2}$$

$$+\zeta(2)\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^2} - \sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(H_n^{(2)})^2}{n^2} - 2\zeta(2)\sum_{n=1}^{\infty}\frac{1}{n^4}$$

{make use of the linear Euler sum in (3.45), the cases $n = 3, 5$, then}

{the values of the series in (4.35), (4.31), (4.29), (4.37), (4.14), where we set}

{$p = 2$, with $n \to \infty$, (6.71) and (4.15), where we set $p = 2$, with $n \to \infty$}

$$= \frac{487}{144}\zeta(6) - \frac{3}{2}\zeta^2(3),$$

and thus we have a second relation between the two main series,

$$\frac{1}{6}\sum_{n=1}^{\infty}\frac{H_n^4}{n^2}-\sum_{n=1}^{\infty}\frac{H_n^2H_n^{(2)}}{n^2}=\frac{487}{144}\zeta(6)-\frac{3}{2}\zeta^2(3). \tag{6.151}$$

Combining the results in (6.148) and (6.151), we conclude that

$$\sum_{n=1}^{\infty}\frac{H_n^4}{n^2}=\frac{979}{24}\zeta(6)+3\zeta^2(3);\ \sum_{n=1}^{\infty}\frac{H_n^2H_n^{(2)}}{n^2}=\frac{41}{12}\zeta(6)+2\zeta^2(3),$$

and the solution is complete.

Another solution may be constructed by combining the calculation of $\displaystyle\sum_{n=1}^{\infty}\frac{H_n^2H_n^{(2)}}{n^2}$ in the previous section with either the identity in (6.148) or (6.151).

Also, a slightly modified version of the series $\displaystyle\sum_{n=1}^{\infty}\frac{H_n^4}{(n+1)^2}$, from which we can obtain the given series, appears in [54].

6.32 Four Sums with Harmonic Series Involving the Generalized Harmonic Numbers of Order 1, 2, 3, 4, 5, and 6, Originating from *The Master Theorem of Series*

Solution This section opens the gates to four sums with harmonic series where we'll want to focus on calculating them together rather than trying to calculate each series separately, and this will find pretty useful in some sections. *Why?* Because, as you already saw in the previous sections, the strategy of extracting the values of the harmonic series is based on creating useful relations with harmonic series. On the other hand, in the third chapter, in Sect. 3.36, we saw that such series relations are very useful when trying to calculate some integrals.

For the point i) of the problem, we use the result in (4.21), the case $m=3$, and we have

$$\sum_{k=1}^{\infty}\frac{H_k^{(3)}}{(k+1)(k+n+1)}=\frac{1}{n}\left(\sum_{i=1}^{n}\frac{H_i}{i^3}+\sum_{i=2}^{3}(-1)^{i-1}\zeta(i)H_n^{(3-i+1)}\right)$$

$$=\frac{1}{n}\sum_{i=1}^{n}\frac{H_i}{i^3}-\zeta(2)\frac{H_n^{(2)}}{n}+\zeta(3)\frac{H_n}{n}. \tag{6.152}$$

Upon multiplying the opposite sides of the relation in (6.152) by $1/n$, and then considering the sum from $n = 1$ to ∞, we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \left(\sum_{i=1}^{n} \frac{H_i}{i^3} \right) - \zeta(2) \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2} + \zeta(3) \sum_{n=1}^{\infty} \frac{H_n}{n^2}$$

$$\{\text{for the first series change the summation order}\}$$

$$= \sum_{i=1}^{\infty} \left(\sum_{n=i}^{\infty} \frac{H_i}{i^3 n^2} \right) - \zeta(2) \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2} + \zeta(3) \sum_{n=1}^{\infty} \frac{H_n}{n^2}$$

$$\left\{ \text{use that } \sum_{i=1}^{\infty} \left(\sum_{n=i}^{\infty} \frac{H_i}{i^3 n^2} \right) = \sum_{i=1}^{\infty} \left(\sum_{n=i+1}^{\infty} + \sum_{n=i}^{i} \right) \frac{H_i}{i^3 n^2} \right.$$

$$\left. = \sum_{i=1}^{\infty} \left(\sum_{n=i+1}^{\infty} \frac{H_i}{i^3 n^2} \right) + \sum_{i=1}^{\infty} \frac{H_i}{i^5} \right\}$$

$$\left\{ = \sum_{i=1}^{\infty} \frac{H_i}{i^3} \left(\zeta(2) - H_i^{(2)} \right) + \sum_{i=1}^{\infty} \frac{H_i}{i^5} = \zeta(2) \sum_{i=1}^{\infty} \frac{H_i}{i^3} - \sum_{i=1}^{\infty} \frac{H_i H_i^{(2)}}{i^3} + \sum_{i=1}^{\infty} \frac{H_i}{i^5} \right\}$$

$$= \zeta(2) \sum_{n=1}^{\infty} \frac{H_n}{n^3} - \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^3} + \sum_{n=1}^{\infty} \frac{H_n}{n^5} - \zeta(2) \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2} + \zeta(3) \sum_{n=1}^{\infty} \frac{H_n}{n^2}$$

$$\{\text{make use of the classical Euler sum in (3.45), the cases } n = 2, 3, 5, \text{ and for}\}$$

$$\{\text{the fourth series make use of the result in (4.14), the case } p = 2, \text{ with } n \to \infty\}$$

$$= \frac{7}{8} \zeta(6) + \frac{3}{2} \zeta^2(3) - \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^3} = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{H_k^{(3)}}{(k+1)(k+n+1)n} \right)$$

$$\{\text{reverse the order of summation in the double series}\}$$

$$= \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{H_k^{(3)}}{(k+1)(k+n+1)n} \right)$$

$$\left\{ \text{write that } \sum_{n=1}^{\infty} \frac{1}{n(n+k+1)} = \frac{H_{k+1}}{k+1} \right\}$$

$$= \sum_{k=1}^{\infty} \frac{H_{k+1} H_k^{(3)}}{(k+1)^2} = \sum_{k=1}^{\infty} \frac{H_{k+1} (H_{k+1}^{(3)} - 1/(k+1)^3)}{(k+1)^2}$$

$$\{\text{reindex the series and expand it}\}$$

$$= \sum_{k=1}^{\infty} \frac{H_k H_k^{(3)}}{k^2} - \sum_{k=1}^{\infty} \frac{H_k}{k^5} = \sum_{n=1}^{\infty} \frac{H_n H_n^{(3)}}{n^2} + \frac{1}{2}\zeta^2(3) - \frac{7}{4}\zeta(6),$$

whence we obtain that

$$\sum_{n=1}^{\infty} \frac{H_n H_n^{(3)}}{n^2} + \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^3} = \frac{21}{8}\zeta(6) + \zeta^2(3),$$

and the point $i)$ of the problem is finalized.

Both series have been calculated in the previous sections where I mentioned that knowing the value of either of the two series will lead to the extraction of the other one with the help of this identity.

Next, for the point $ii)$ of the problem, using the result in (4.21), the case $m = 4$, we obtain

$$\sum_{k=1}^{\infty} \frac{H_k^{(4)}}{(k+1)(k+n+1)} = -\frac{1}{n}\left(\sum_{i=1}^{n} \frac{H_i}{i^4} + \sum_{i=2}^{4}(-1)^{i-1}\zeta(i) H_n^{(4-i+1)}\right)$$

$$= -\frac{1}{n}\sum_{i=1}^{n} \frac{H_i}{i^4} + \zeta(2)\frac{H_n^{(3)}}{n} - \zeta(3)\frac{H_n^{(2)}}{n} + \zeta(4)\frac{H_n}{n}. \tag{6.153}$$

Upon multiplying the opposite sides of the relation in (6.153) by $1/n$, and then considering the sum from $n = 1$ to ∞, we obtain

$$-\sum_{n=1}^{\infty} \frac{1}{n^2}\left(\sum_{i=1}^{n} \frac{H_i}{i^4}\right) + \zeta(2)\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^2} - \zeta(3)\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2} + \zeta(4)\sum_{n=1}^{\infty} \frac{H_n}{n^2}$$

$$\{\text{change the summation order in the first series}\}$$

$$= -\sum_{i=1}^{\infty}\left(\sum_{n=i}^{\infty} \frac{H_i}{i^4 n^2}\right) + \zeta(2)\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^2} - \zeta(3)\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2} + \zeta(4)\sum_{n=1}^{\infty} \frac{H_n}{n^2}$$

$$\left\{\text{use that } \sum_{i=1}^{\infty}\left(\sum_{n=i}^{\infty} \frac{H_i}{i^4 n^2}\right) = \sum_{i=1}^{\infty}\left(\sum_{n=i+1}^{\infty} + \sum_{n=i}^{i}\right)\frac{H_i}{i^4 n^2}\right.$$

$$\left. = \sum_{i=1}^{\infty}\left(\sum_{n=i+1}^{\infty} \frac{H_i}{i^4 n^2}\right) + \sum_{i=1}^{\infty} \frac{H_i}{i^6}\right\}$$

$$\left\{= \sum_{i=1}^{\infty} \frac{H_i}{i^4}(\zeta(2) - H_i^{(2)}) + \sum_{i=1}^{\infty} \frac{H_i}{i^6} = \zeta(2)\sum_{i=1}^{\infty} \frac{H_i}{i^4} - \sum_{i=1}^{\infty} \frac{H_i H_i^{(2)}}{i^4} + \sum_{i=1}^{\infty} \frac{H_i}{i^6}\right\}$$

$$= -\zeta(2)\sum_{n=1}^{\infty}\frac{H_n}{n^4}+\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^4}-\sum_{n=1}^{\infty}\frac{H_n}{n^6}+\zeta(2)\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^2}-\zeta(3)\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^2}$$

$$+\zeta(4)\sum_{n=1}^{\infty}\frac{H_n}{n^2}$$

{make use of the classical Euler sum in (3.45), the cases $n = 2, 4, 6,$}

{and also consider the results in (6.68) and (4.14), with $p = 2$}

$$= \sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^4}+\frac{7}{2}\zeta(2)\zeta(5)-\frac{5}{4}\zeta(3)\zeta(4)-4\zeta(7)$$

$$= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{H_k^{(4)}}{(k+1)(k+n+1)n}\right)=\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{H_k^{(4)}}{(k+1)(k+n+1)n}\right)$$

$$= \sum_{k=1}^{\infty}\frac{H_{k+1}H_k^{(4)}}{(k+1)^2}=\sum_{k=1}^{\infty}\frac{H_{k+1}(H_{k+1}^{(4)}-1/(k+1)^4)}{(k+1)^2}$$

{reindex the series and expand it}

$$= \sum_{k=1}^{\infty}\frac{H_kH_k^{(4)}}{k^2}-\sum_{k=1}^{\infty}\frac{H_k}{k^6}=\sum_{n=1}^{\infty}\frac{H_nH_n^{(4)}}{n^2}+\zeta(2)\zeta(5)+\zeta(3)\zeta(4)-4\zeta(7),$$

whence we get that

$$\sum_{n=1}^{\infty}\frac{H_nH_n^{(4)}}{n^2}-\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^4}=\frac{1}{2}\left(5\zeta(2)\zeta(5)-\frac{9}{2}\zeta(3)\zeta(4)\right),$$

and the point $ii)$ of the problem is finalized.

Further, for the point $iii)$ of the problem we make use of the result in (4.21), the case $m = 5$, that gives

$$\sum_{k=1}^{\infty}\frac{H_k^{(5)}}{(k+1)(k+n+1)}=\frac{1}{n}\left(\sum_{i=1}^{n}\frac{H_i}{i^5}+\sum_{i=2}^{5}(-1)^{i-1}\zeta(i)H_n^{(5-i+1)}\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\frac{H_i}{i^5}-\zeta(2)\frac{H_n^{(4)}}{n}+\zeta(3)\frac{H_n^{(3)}}{n}-\zeta(4)\frac{H_n^{(2)}}{n}+\zeta(5)\frac{H_n}{n}. \tag{6.154}$$

Multiplying the opposite sides of the relation in (6.154) by $1/n$ and then considering the summation from $n = 1$ to ∞, we get

$$\sum_{n=1}^{\infty} \frac{1}{n^2}\left(\sum_{i=1}^{n} \frac{H_i}{i^5}\right) - \zeta(2)\sum_{n=1}^{\infty} \frac{H_n^{(4)}}{n^2} + \zeta(3)\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^2} - \zeta(4)\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2} + \zeta(5)\sum_{n=1}^{\infty} \frac{H_n}{n^2}$$

$$\{\text{for the first series change the summation order}\}$$

$$= \sum_{i=1}^{\infty}\left(\sum_{n=i}^{\infty} \frac{H_i}{i^5 n^2}\right) - \zeta(2)\sum_{n=1}^{\infty} \frac{H_n^{(4)}}{n^2} + \zeta(3)\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^2} - \zeta(4)\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2} + \zeta(5)\sum_{n=1}^{\infty} \frac{H_n}{n^2}$$

$$\left\{\text{use that } \sum_{i=1}^{\infty}\left(\sum_{n=i}^{\infty} \frac{H_i}{i^5 n^2}\right) = \sum_{i=1}^{\infty}\left(\sum_{n=i+1}^{\infty} + \sum_{n=i}^{i}\right)\frac{H_i}{i^5 n^2} = \sum_{i=1}^{\infty}\left(\sum_{n=i+1}^{\infty} \frac{H_i}{i^5 n^2}\right)\right.$$

$$\left. + \sum_{i=1}^{\infty} \frac{H_i}{i^7}\right\}$$

$$\left\{= \sum_{i=1}^{\infty} \frac{H_i}{i^5}\left(\zeta(2) - H_i^{(2)}\right) + \sum_{i=1}^{\infty} \frac{H_i}{i^7} = \zeta(2)\sum_{i=1}^{\infty} \frac{H_i}{i^5} - \sum_{i=1}^{\infty} \frac{H_i H_i^{(2)}}{i^5} + \sum_{i=1}^{\infty} \frac{H_i}{i^7}\right\}$$

$$= \zeta(2)\sum_{n=1}^{\infty} \frac{H_n}{n^5} - \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^5} + \sum_{n=1}^{\infty} \frac{H_n}{n^7} - \zeta(2)\sum_{n=1}^{\infty} \frac{H_n^{(4)}}{n^2} + \zeta(3)\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^2}$$

$$-\zeta(4)\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2} + \zeta(5)\sum_{n=1}^{\infty} \frac{H_n}{n^2}$$

$$\{\text{make use of the classical Euler sum in (3.45), the cases } n = 2, 5, 7, \text{ and also}\}$$

$$\{\text{consider the results in (6.72), (6.68), and (4.14), the case } p = 2, \text{ with } n \to \infty\}$$

$$= -\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^5} - \frac{145}{72}\zeta(8) - \frac{3}{2}\zeta(2)\zeta^2(3) + \frac{13}{2}\zeta(3)\zeta(5)$$

$$= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} \frac{H_k^{(5)}}{(k+1)(k+n+1)n}\right) = \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty} \frac{H_k^{(5)}}{(k+1)(k+n+1)n}\right)$$

$$= \sum_{k=1}^{\infty} \frac{H_{k+1} H_k^{(5)}}{(k+1)^2} = \sum_{k=1}^{\infty} \frac{H_{k+1}(H_{k+1}^{(5)} - 1/(k+1)^5)}{(k+1)^2}$$

$$\{\text{reindex the series and expand it}\}$$

$$= \sum_{k=1}^{\infty} \frac{H_k H_k^{(5)}}{k^2} - \sum_{k=1}^{\infty} \frac{H_k}{k^7} = \sum_{n=1}^{\infty} \frac{H_n H_n^{(5)}}{n^2} - \frac{9}{4}\zeta(8) + \zeta(3)\zeta(5),$$

whence we get that

$$\sum_{n=1}^{\infty} \frac{H_n H_n^{(5)}}{n^2} + \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^5} = \frac{1}{2}\left(\frac{17}{36}\zeta(8) + 11\zeta(3)\zeta(5) - 3\zeta(2)\zeta^2(3)\right),$$

and the point iii) of the problem is finalized.

As regards the last point of the problem, we use the result in (4.21), the case $m = 6$, that yields

$$\sum_{k=1}^{\infty} \frac{H_k^{(6)}}{(k+1)(k+n+1)} = -\frac{1}{n}\left(\sum_{i=1}^{n} \frac{H_i}{i^6} + \sum_{i=2}^{6}(-1)^{i-1}\zeta(i) H_n^{(6-i+1)}\right)$$

$$= -\frac{1}{n}\sum_{i=1}^{n} \frac{H_i}{i^6} + \zeta(2)\frac{H_n^{(5)}}{n} - \zeta(3)\frac{H_n^{(4)}}{n} + \zeta(4)\frac{H_n^{(3)}}{n} - \zeta(5)\frac{H_n^{(2)}}{n} + \zeta(6)\frac{H_n}{n}. \tag{6.155}$$

Multiplying the opposite sides of the relation in (6.155) by $1/n$ and then considering the summation from $n = 1$ to ∞, we get

$$-\sum_{n=1}^{\infty} \frac{1}{n^2}\left(\sum_{i=1}^{n} \frac{H_i}{i^6}\right) + \zeta(2)\sum_{n=1}^{\infty} \frac{H_n^{(5)}}{n^2} - \zeta(3)\sum_{n=1}^{\infty} \frac{H_n^{(4)}}{n^2} + \zeta(4)\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^2}$$

$$-\zeta(5)\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2} + \zeta(6)\sum_{n=1}^{\infty} \frac{H_n}{n^2}$$

{change the summation order in the first series}

$$= -\sum_{i=1}^{\infty}\left(\sum_{n=i}^{\infty} \frac{H_i}{i^6 n^2}\right) + \zeta(2)\sum_{n=1}^{\infty} \frac{H_n^{(5)}}{n^2} - \zeta(3)\sum_{n=1}^{\infty} \frac{H_n^{(4)}}{n^2} + \zeta(4)\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^2}$$

$$-\zeta(5)\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2} + \zeta(6)\sum_{n=1}^{\infty} \frac{H_n}{n^2}$$

$$\left\{\text{use that } \sum_{i=1}^{\infty}\left(\sum_{n=i}^{\infty} \frac{H_i}{i^6 n^2}\right) = \sum_{i=1}^{\infty}\left(\sum_{n=i+1}^{\infty} + \sum_{n=i}^{i}\right)\frac{H_i}{i^6 n^2}\right.$$

$$\left. = \sum_{i=1}^{\infty}\left(\sum_{n=i+1}^{\infty} \frac{H_i}{i^6 n^2}\right) + \sum_{i=1}^{\infty} \frac{H_i}{i^8}\right\}$$

$$\left\{= \sum_{i=1}^{\infty} \frac{H_i}{i^6}\left(\zeta(2) - H_i^{(2)}\right) + \sum_{i=1}^{\infty} \frac{H_i}{i^8} = \zeta(2)\sum_{i=1}^{\infty} \frac{H_i}{i^6} - \sum_{i=1}^{\infty} \frac{H_i H_i^{(2)}}{i^6} + \sum_{i=1}^{\infty} \frac{H_i}{i^8}\right\}$$

$$= -\zeta(2)\sum_{n=1}^{\infty}\frac{H_n}{n^6} + \sum_{n=1}^{\infty}\frac{H_n H_n^{(2)}}{n^6} - \sum_{n=1}^{\infty}\frac{H_n}{n^8} + \zeta(2)\sum_{n=1}^{\infty}\frac{H_n^{(5)}}{n^2} - \zeta(3)\sum_{n=1}^{\infty}\frac{H_n^{(4)}}{n^2}$$

$$+\zeta(4)\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^2} - \zeta(5)\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^2} + \zeta(6)\sum_{n=1}^{\infty}\frac{H_n}{n^2}$$

{use the classical Euler sum in (3.45), the cases $n = 2, 6, 8$, and also consider}

{the results in (6.76), (6.72), (6.68), and (4.14), the case $p = 2$, with $n \to \infty$}

$$= 8\zeta(2)\zeta(7) - \frac{16}{3}\zeta(3)\zeta(6) + \zeta^3(3) - \frac{11}{4}\zeta(4)\zeta(5) - 5\zeta(9) + \sum_{n=1}^{\infty}\frac{H_n H_n^{(2)}}{n^6}$$

$$= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{H_k^{(6)}}{(k+1)(k+n+1)n}\right) = \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{H_k^{(6)}}{(k+1)(k+n+1)n}\right)$$

$$= \sum_{k=1}^{\infty}\frac{H_{k+1}H_k^{(6)}}{(k+1)^2} = \sum_{k=1}^{\infty}\frac{(H_{k+1}^{(6)} - 1/(k+1)^6)H_{k+1}}{(k+1)^2}$$

{reindex the series and expand it}

$$= \sum_{k=1}^{\infty}\frac{H_k H_k^{(6)}}{k^2} - \sum_{k=1}^{\infty}\frac{H_k}{k^8}$$

$$= \sum_{n=1}^{\infty}\frac{H_n H_n^{(6)}}{n^2} + \zeta(2)\zeta(7) + \zeta(3)\zeta(6) + \zeta(4)\zeta(5) - 5\zeta(9),$$

whence we obtain that

$$\sum_{n=1}^{\infty}\frac{H_n H_n^{(6)}}{n^2} - \sum_{n=1}^{\infty}\frac{H_n H_n^{(2)}}{n^6} = 7\zeta(2)\zeta(7) - \frac{19}{3}\zeta(3)\zeta(6) - \frac{15}{4}\zeta(4)\zeta(5) + \zeta^3(3),$$

and the point iv) of the problem is finalized.

As a final note, the most difficult series we needed to establish the present results have been the ones evaluated in Sect. 6.21.

6.33 Awesomely Wicked Sums of Series of Weight 7, $\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4}$ and $\sum_{n=1}^{\infty} \frac{H_n H_n^{(3)}}{n^3}$, Originating from a Strong Generalized Sum: The First Part

Solution How did the derivations of the series of weight 4, 5, and 6 seem to you so far in terms of difficulty (assuming you went through those related sections)? One simple observation, as we go toward the series of higher weights, is that the extraction process of the series seems to get more and more complicated. Then keep in mind my *plan* is to derive the series of weight 7 by series manipulations only.

Where should we start from? In a previous section I said that one of the best strategies to extract the harmonic series is to build up (useful) relations with harmonic series. For example, remember that such relations we have in Sect. 4.32. So far so good. Looks like we need more distinct relations in order to be able to extract the series of weight 7.

Let's start by considering the identity in (4.16), the case $m = 1$, where if we multiply both sides by $1/n^5$ and then consider the summation from $n = 1$ to ∞, we get

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n-1} \frac{H_k}{n^5(n-k)} \right) = \sum_{n=1}^{\infty} \frac{H_n^2}{n^5} - \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^5}. \tag{6.156}$$

Using the series values from (4.32) and (6.75) in the right-hand side of (6.156), we obtain

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n-1} \frac{H_k}{n^5(n-k)} \right) = 16\zeta(7) - 6\zeta(2)\zeta(5) - \frac{9}{2}\zeta(3)\zeta(4). \tag{6.157}$$

Further, for left-hand side of (6.157), considering the change of summation order, we have

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n-1} \frac{H_k}{n^5(n-k)} \right) = \sum_{k=1}^{\infty} \left(\sum_{n=k+1}^{\infty} \frac{H_k}{n^5(n-k)} \right)$$

$$\{\text{reindex the inner series and start from } n = 1\}$$

$$= \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{H_k}{n(n+k)^5} \right)$$

$$= \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{H_k}{nk^4(k+n)} \right) - \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{H_k}{k(k+n)^5} \right) - \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{H_k}{k^2(k+n)^4} \right)$$

$$
\begin{aligned}
&-\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{H_k}{k^3(k+n)^3}\right)-\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{H_k}{k^4(k+n)^2}\right)\\
&=\sum_{k=1}^{\infty}\frac{H_k^2}{k^5}-\sum_{k=1}^{\infty}\frac{H_k}{k}\left(\zeta(5)-H_k^{(5)}\right)-\sum_{k=1}^{\infty}\frac{H_k}{k^2}\left(\zeta(4)-H_k^{(4)}\right)\\
&-\sum_{k=1}^{\infty}\frac{H_k}{k^3}\left(\zeta(3)-H_k^{(3)}\right)-\sum_{k=1}^{\infty}\frac{H_k}{k^4}\left(\zeta(2)-H_k^{(2)}\right)\\
&=\sum_{k=1}^{\infty}\frac{H_k^2}{k^5}-\sum_{k=1}^{\infty}\frac{H_k}{k}\left(\zeta(5)-H_k^{(5)}\right)-\zeta(4)\sum_{k=1}^{\infty}\frac{H_k}{k^2}+\sum_{k=1}^{\infty}\frac{H_kH_k^{(4)}}{k^2}-\zeta(3)\sum_{k=1}^{\infty}\frac{H_k}{k^3}\\
&+\sum_{k=1}^{\infty}\frac{H_kH_k^{(3)}}{k^3}-\zeta(2)\sum_{k=1}^{\infty}\frac{H_k}{k^4}+\sum_{n=1}^{\infty}\frac{H_kH_k^{(2)}}{k^4}
\end{aligned}
$$

{make use of the classical Euler sum in (3.45), the cases $n = 2, 3, 4,$}

{the results in (4.32), (4.71), and (4.41), where we use the latter identity}

$$\left\{\text{for expressing } \sum_{k=1}^{\infty}\frac{H_kH_k^{(4)}}{k^2} \text{ in terms of } \sum_{n=1}^{\infty}\frac{H_kH_k^{(2)}}{k^4}\right\}$$

$$=12\zeta(7)-\frac{9}{2}\zeta(2)\zeta(5)-\frac{25}{4}\zeta(3)\zeta(4)+\sum_{k=1}^{\infty}\frac{H_kH_k^{(3)}}{k^3}+2\sum_{n=1}^{\infty}\frac{H_kH_k^{(2)}}{k^4}. \tag{6.158}$$

If we combine (6.157) and (6.158), we obtain that

$$2\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^4}+\sum_{n=1}^{\infty}\frac{H_nH_n^{(3)}}{n^3}=4\zeta(7)+\frac{7}{4}\zeta(3)\zeta(4)-\frac{3}{2}\zeta(2)\zeta(5),$$

and the solution is complete.

You may have noticed that during the calculations we met a series with the tail of the Riemann zeta function of the type, $\sum_{n=1}^{\infty}\frac{H_n}{n}\left(\zeta(k)-H_n^{(k)}\right)$, more exactly the case $k = 5$ we'll meet in Sect. 4.45.

6.34 Awesomely Wicked Sums of Series of Weight 7, $\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4}$ and $\sum_{n=1}^{\infty} \frac{H_n^{(2)} H_n^{(3)}}{n^2}$, Originating from a Strong Generalized Sum: The Second Part

Solution In our pursuit of getting another relation with harmonic series of weight 7, we continue with using another particular case of the generalization used in the previous section. Basically, we proceed similarly as before and do the calculations by series manipulations only.

We return to the generalization given in (4.16), the case $m = 2$, where if we multiply both sides by $1/n^4$ and then consider the summation from $n = 1$ to ∞, we get

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n-1}\frac{H_k^{(2)}}{n^4(n-k)}\right)=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{H_k}{k^2n^4}\right)+\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^4}-2\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^4}. \tag{6.159}$$

Now, changing the summation order for the first series in the right-hand side of (6.159), we get

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{H_k}{k^2n^4}\right)=\sum_{k=1}^{\infty}\left(\sum_{n=k}^{\infty}\frac{H_k}{k^2n^4}\right)=\sum_{k=1}^{\infty}\frac{H_k}{k^2}\left(\frac{1}{k^4}+\frac{1}{(k+1)^4}+\frac{1}{(k+2)^4}+\cdots\right)$$

$$=\sum_{k=1}^{\infty}\frac{H_k}{k^2}\left(\frac{1}{k^4}+\zeta(4)-H_k^{(4)}\right)=\sum_{k=1}^{\infty}\frac{H_k}{k^6}+\zeta(4)\sum_{k=1}^{\infty}\frac{H_k}{k^2}-\sum_{k=1}^{\infty}\frac{H_kH_k^{(4)}}{k^2}$$

{make use of the classical Euler sum in (3.45), the cases $n = 2$ and $n = 6$}

$$=4\zeta(7)+\zeta(3)\zeta(4)-\zeta(2)\zeta(5)-\sum_{k=1}^{\infty}\frac{H_kH_k^{(4)}}{k^2}. \tag{6.160}$$

Then, plugging the results from (6.160) and (6.80) in (6.159) gives

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n-1}\frac{H_k^{(2)}}{n^4(n-k)}\right)=\zeta(3)\zeta(4)+19\zeta(2)\zeta(5)-32\zeta(7)$$

$$+\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^4}-\sum_{n=1}^{\infty}\frac{H_nH_n^{(4)}}{n^2},$$

and if we use the relation in (4.41), we obtain

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n-1}\frac{H_k^{(2)}}{n^4(n-k)}\right)=\frac{13}{4}\zeta(3)\zeta(4)+\frac{33}{2}\zeta(2)\zeta(5)-32\zeta(7). \tag{6.161}$$

For the left-hand side of (6.161), we change the summation order, and then we have

$$\begin{aligned}
\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n-1}\frac{H_k^{(2)}}{n^4(n-k)}\right)&=\sum_{k=1}^{\infty}\left(\sum_{n=k+1}^{\infty}\frac{H_k^{(2)}}{n^4(n-k)}\right)\\
&\{\text{reindex the inner series and start from } n=1 \text{ to } \infty\}\\
&=\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{H_k^{(2)}}{n(n+k)^4}\right)\\
&=\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^3}\left(\sum_{n=1}^{\infty}\frac{1}{n(n+k)}\right)-\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k}\left(\sum_{n=1}^{\infty}\frac{1}{(k+n)^4}\right)\\
&\quad-\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^2}\left(\sum_{n=1}^{\infty}\frac{1}{(k+n)^3}\right)-\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^3}\left(\sum_{n=1}^{\infty}\frac{1}{(k+n)^2}\right)\\
&\left\{\text{for the inner series of the first double series, use that } \sum_{n=1}^{\infty}\frac{1}{n(n+k)}=\frac{H_k}{k}\right\}\\
&=\sum_{k=1}^{\infty}\frac{H_kH_k^{(2)}}{k^4}-\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k}\left(\zeta(4)-H_k^{(4)}\right)-\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^2}\left(\zeta(3)-H_k^{(3)}\right)\\
&\quad-\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^3}\left(\zeta(2)-H_k^{(2)}\right)\\
&=\sum_{k=1}^{\infty}\frac{H_kH_k^{(2)}}{k^4}-\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k}\left(\zeta(4)-H_k^{(4)}\right)-\zeta(3)\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^2}+\sum_{k=1}^{\infty}\frac{H_k^{(2)}H_k^{(3)}}{k^2}\\
&\quad-\zeta(2)\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^3}+\sum_{k=1}^{\infty}\frac{(H_k^{(2)})^2}{k^3}.
\end{aligned}\tag{6.162}$$

Now, for the second series in (6.162), we consider the series $\displaystyle\sum_{k=1}^{\infty}\frac{H_k(\zeta(4)-H_k^{(4)})}{k^2}$ where we apply Abel's summation (see (5.1)) with $a_k=1/k^2$ and $b_k=H_k(\zeta(4)-H_k^{(4)})$ that gives

$$\sum_{k=1}^{\infty} \frac{H_k(\zeta(4) - H_k^{(4)})}{k^2} = \sum_{k=1}^{\infty} H_k^{(2)} \left(H_k(\zeta(4) - H_k^{(4)}) - H_{k+1}(\zeta(4) - H_{k+1}^{(4)}) \right)$$

$$= \sum_{k=1}^{\infty} (H_{k+1}^{(2)} - 1/(k+1)^2) \Big((H_{k+1} - 1/(k+1))(\zeta(4) - H_{k+1}^{(4)} + 1/(k+1)^4)$$

$$-H_{k+1}(\zeta(4) - H_{k+1}^{(4)}) \Big)$$

{reindex the series and carefully expand it}

$$= \sum_{k=1}^{\infty} \frac{1}{k^7} - \sum_{k=1}^{\infty} \frac{H_k}{k^6} - \sum_{k=1}^{\infty} \frac{H_k^{(2)}}{k^5} + \sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k^4} - \sum_{k=1}^{\infty} \frac{H_k^{(4)}}{k^3} + \zeta(4) \sum_{k=1}^{\infty} \frac{1}{k^3}$$

$$- \sum_{k=1}^{\infty} \frac{H_k^{(2)}}{k} \left(\zeta(4) - H_k^{(4)} \right)$$

{make use of the linear Euler sum in (3.45), the case $n = 6$, and}

{the values of the 3rd and 5th series are given in (6.75) and (6.79)}

$$= \sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k^4} - \sum_{k=1}^{\infty} \frac{H_k^{(2)}}{k} \left(\zeta(4) - H_k^{(4)} \right) + 24\zeta(7) - 14\zeta(2)\zeta(5) - \zeta(3)\zeta(4),$$

from which we get

$$\sum_{k=1}^{\infty} \frac{H_k^{(2)}}{k} \left(\zeta(4) - H_k^{(4)} \right)$$

$$= \sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k^4} - \zeta(4) \sum_{k=1}^{\infty} \frac{H_k}{k^2} + \sum_{k=1}^{\infty} \frac{H_k H_k^{(4)}}{k^2} + 24\zeta(7) - 14\zeta(2)\zeta(5) - \zeta(3)\zeta(4)$$

{make use of the classical Euler sum in (3.45), the case $n = 2$, and then}

$$\left\{ \text{express } \sum_{k=1}^{\infty} \frac{H_k H_k^{(4)}}{k^2} \text{ in terms of } \sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k^4} \text{ using the identity in (4.41)} \right\}$$

$$= 2 \sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k^4} + 24\zeta(7) - \frac{23}{2}\zeta(2)\zeta(5) - \frac{21}{4}\zeta(3)\zeta(4). \tag{6.163}$$

Further, for the third series in (6.162), we make use of (4.14) where we set $p = 2$ and then let $n \to \infty$,

$$\sum_{k=1}^{\infty} \frac{H_k^{(2)}}{k^2} = \frac{7}{4}\zeta(4). \tag{6.164}$$

Next, for the sixth series in (6.162), we consider the identity in (4.47),

$$\sum_{k=1}^{\infty} \frac{(H_k^{(2)})^2}{k^3} = \frac{13}{2}\zeta(3)\zeta(4) - 3\zeta(7) - 2\sum_{k=1}^{\infty} \frac{H_k^{(2)} H_k^{(3)}}{k^2}. \tag{6.165}$$

By collecting the series results from (6.163), (6.164), (6.165), and (6.67) in (6.162), we get

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n-1} \frac{H_k^{(2)}}{n^4(n-k)} \right)$$

$$= 16\zeta(2)\zeta(5) + \frac{5}{2}\zeta(3)\zeta(4) - 27\zeta(7) - \sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k^4} - \sum_{k=1}^{\infty} \frac{H_k^{(2)} H_k^{(3)}}{k^2}. \tag{6.166}$$

Finally, by combining (6.161) and (6.166), we conclude that

$$\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4} + \sum_{n=1}^{\infty} \frac{H_n^{(2)} H_n^{(3)}}{n^2} = 5\zeta(7) - \frac{1}{2}\zeta(2)\zeta(5) - \frac{3}{4}\zeta(3)\zeta(4),$$

and the solution is complete.

In the calculations we also needed a result from Sect. 4.36, which is easy to obtain by Abel's summation (as we'll see in its dedicated section).

6.35 Awesomely Wicked Sums of Series of Weight 7, $\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4}$, $\sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(3)}}{n^2}$ and $\sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^3}$, Derivation Based upon a New Identity: The Third Part

Solution An immediate remark from the very beginning of the section is that both here and in the last two sections we have obtained relations where each one contains the series $\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4}$ which seems to play the part of a pivot, central element.

In the following I'll establish a relation between three series of weight 7 which is important in the derivation process of the harmonic series of weight 7, and not only there. For example, the present relation together with the relations obtained in the last two sections will be proving to be the keys for solving an excellent challenging question from one of the next sections (you'll want to discover alone).

Let's start by considering the identity in (4.17) where if we multiply both sides by $1/n^4$, and then consider the sum from $n = 1$ to ∞, we get

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n-1}\frac{H_k^2}{n^4(n-k)}\right)=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{H_k}{n^4k^2}\right)+\sum_{n=1}^{\infty}\frac{H_n^3}{n^4}-2\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^4}. \tag{6.167}$$

For the left-hand side of (6.167), we change the order of summation, and then we have

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n-1}\frac{H_k^2}{n^4(n-k)}\right)=\sum_{k=1}^{\infty}\left(\sum_{n=k+1}^{\infty}\frac{H_k^2}{n^4(n-k)}\right)$$

{reindex the inner series and start from $n = 1$ to ∞}

$$=\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{H_k^2}{n(n+k)^4}\right)=\sum_{k=1}^{\infty}\frac{H_k^2}{k^3}\left(\sum_{n=1}^{\infty}\frac{1}{n(n+k)}\right)-\sum_{k=1}^{\infty}\frac{H_k^2}{k}\left(\sum_{n=1}^{\infty}\frac{1}{(n+k)^4}\right)$$

$$-\sum_{k=1}^{\infty}\frac{H_k^2}{k^2}\left(\sum_{n=1}^{\infty}\frac{1}{(n+k)^3}\right)-\sum_{k=1}^{\infty}\frac{H_k^2}{k^3}\left(\sum_{n=1}^{\infty}\frac{1}{(n+k)^2}\right)$$

$$=\sum_{k=1}^{\infty}\frac{H_k^3}{k^4}-\sum_{k=1}^{\infty}\frac{H_k^2}{k}(\zeta(4)-H_k^{(4)})-\zeta(3)\sum_{k=1}^{\infty}\frac{H_k^2}{k^2}+\sum_{k=1}^{\infty}\frac{H_k^2H_k^{(3)}}{k^2}-\zeta(2)\sum_{k=1}^{\infty}\frac{H_k^2}{k^3}$$

$$+\sum_{k=1}^{\infty}\frac{H_k^2H_k^{(2)}}{k^3}$$

{for the 3rd and 5th series make use of the series results in (4.29) and (4.30)}

$$=\sum_{k=1}^{\infty}\frac{H_k^3}{k^4}-\sum_{k=1}^{\infty}\frac{H_k^2}{k}(\zeta(4)-H_k^{(4)})+\sum_{k=1}^{\infty}\frac{H_k^2H_k^{(3)}}{k^2}+\sum_{k=1}^{\infty}\frac{H_k^2H_k^{(2)}}{k^3}$$

$$-\frac{7}{2}\zeta(2)\zeta(5)-\frac{7}{4}\zeta(3)\zeta(4). \tag{6.168}$$

Now, for the second series in (6.168), we apply Abel's summation (see (5.1)), where we set $a_k = 1/k$ and $b_k = H_k^2(\zeta(4) - H_k^{(4)})$, and then we have

$$\sum_{k=1}^{\infty} \frac{H_k^2}{k}(\zeta(4) - H_k^{(4)}) = \sum_{k=1}^{\infty} H_k \left(H_k^2(\zeta(4) - H_k^{(4)}) - H_{k+1}^2(\zeta(4) - H_{k+1}^{(4)}) \right)$$

$$= \sum_{k=1}^{\infty} (H_{k+1} - 1/(k+1)) \Big((H_{k+1} - 1/(k+1))^2 (\zeta(4) - H_{k+1}^{(4)} + 1/(k+1)^4)$$

$$- H_{k+1}^2(\zeta(4) - H_{k+1}^{(4)}) \Big)$$

$$\{\text{reindex the series and carefully expand it}\}$$

$$= \sum_{k=1}^{\infty} \frac{H_k^3}{k^4} - 2\sum_{k=1}^{\infty} \frac{H_k^2}{k} \left(\zeta(4) - H_k^{(4)}\right) - 3\sum_{k=1}^{\infty} \frac{H_k H_k^{(4)}}{k^2} - 3\sum_{k=1}^{\infty} \frac{H_k^2}{k^5} + \sum_{k=1}^{\infty} \frac{H_k^{(4)}}{k^3}$$

$$+ 3\zeta(4)\sum_{k=1}^{\infty} \frac{H_k}{k^2} + 3\sum_{k=1}^{\infty} \frac{H_k}{k^6} - \zeta(4)\sum_{k=1}^{\infty} \frac{1}{k^3} - \sum_{k=1}^{\infty} \frac{1}{k^7}$$

$$\{\text{make use of the results in (3.45), the cases } n = 2, 6, \text{(4.32), (6.79),}\}$$

$$\left\{ \text{and consider (4.41) to express } \sum_{k=1}^{\infty} \frac{H_k H_k^{(4)}}{k^2} \text{ in terms of } \sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k^4} \right\}$$

$$= \sum_{k=1}^{\infty} \frac{H_k^3}{k^4} - 2\sum_{k=1}^{\infty} \frac{H_k^2}{k} \left(\zeta(4) - H_k^{(4)}\right) - 3\sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k^4} - 24\zeta(7) + \frac{5}{2}\zeta(2)\zeta(5)$$

$$+ \frac{69}{4}\zeta(3)\zeta(4),$$

from which we obtain

$$\sum_{k=1}^{\infty} \frac{H_k^2}{k}(\zeta(4) - H_k^{(4)})$$

$$= \frac{1}{3}\sum_{k=1}^{\infty} \frac{H_k^3}{k^4} - \sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k^4} - 8\zeta(7) + \frac{5}{6}\zeta(2)\zeta(5) + \frac{23}{4}\zeta(3)\zeta(4). \tag{6.169}$$

Next, if we consider the result from (6.169) in (6.168), we get

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n-1} \frac{H_k^2}{n^4(n-k)} \right) = \frac{2}{3}\sum_{k=1}^{\infty} \frac{H_k^3}{k^4} + \sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k^4} + \sum_{k=1}^{\infty} \frac{H_k^2 H_k^{(3)}}{k^2} + \sum_{k=1}^{\infty} \frac{H_k^2 H_k^{(2)}}{k^3}$$

$$- \frac{13}{3}\zeta(2)\zeta(5) - \frac{15}{2}\zeta(3)\zeta(4) + 8\zeta(7). \tag{6.170}$$

Further, for the first series in the right-hand side of (6.167), we change the summation order, and then we write

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{H_k}{n^4k^2}\right)=\sum_{k=1}^{\infty}\left(\sum_{n=k}^{\infty}\frac{H_k}{n^4k^2}\right)=\sum_{k=1}^{\infty}\frac{H_k}{k^2}\left(\frac{1}{k^4}+\frac{1}{(k+1)^4}+\frac{1}{(k+2)^4}+\cdots\right) \tag{6.171}$$

$$=\sum_{k=1}^{\infty}\frac{H_k}{k^2}\left(\frac{1}{k^4}+\zeta(4)-H_k^{(4)}\right)=\sum_{k=1}^{\infty}\frac{H_k}{k^6}+\zeta(4)\sum_{k=1}^{\infty}\frac{H_k}{k^2}-\sum_{k=1}^{\infty}\frac{H_kH_k^{(4)}}{k^2}$$

$\{$make use of the Euler sum in (3.45), the cases $n=2,6\}$

$$=\zeta(3)\zeta(4)-\zeta(2)\zeta(5)+4\zeta(7)-\sum_{k=1}^{\infty}\frac{H_kH_k^{(4)}}{k^2}$$

$\{$employ the result in (4.41)$\}$

$$=\frac{13}{4}\zeta(3)\zeta(4)-\frac{7}{2}\zeta(2)\zeta(5)+4\zeta(7)-\sum_{k=1}^{\infty}\frac{H_kH_k^{(2)}}{k^4}. \tag{6.172}$$

If we plug the result from (6.172) in (6.167), we get

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n-1}\frac{H_k^2}{n^4(n-k)}\right)=\frac{13}{4}\zeta(3)\zeta(4)-\frac{7}{2}\zeta(2)\zeta(5)+4\zeta(7)+\sum_{n=1}^{\infty}\frac{H_n^3}{n^4}-3\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^4}. \tag{6.173}$$

Putting together the results from (6.170) and (6.173), we obtain that

$$4\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^4}-\frac{1}{3}\sum_{n=1}^{\infty}\frac{H_n^3}{n^4}+\sum_{n=1}^{\infty}\frac{H_n^2H_n^{(3)}}{n^2}+\sum_{n=1}^{\infty}\frac{H_n^2H_n^{(2)}}{n^3}$$

$$=\frac{5}{6}\zeta(2)\zeta(5)+\frac{43}{4}\zeta(3)\zeta(4)-4\zeta(7). \tag{6.174}$$

If we plug the result from (4.50) in (6.174), we conclude that

$$3\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^4}+\sum_{n=1}^{\infty}\frac{H_n^2H_n^{(3)}}{n^2}+\sum_{n=1}^{\infty}\frac{H_n^2H_n^{(2)}}{n^3}=\frac{23}{4}\zeta(3)\zeta(4)-\frac{1}{2}\zeta(2)\zeta(5)+4\zeta(7),$$

and the solution is complete.

The campaign of establishing relations with the series of weight 7 will not end here, and it will continue in some of the next sections. We need them in order to extract the series of weight 7.

6.36 Deriving More Useful Sums of Harmonic Series of Weight 7

Solution Continuing the pursuit of getting identities with harmonic series of weight 7, in this section we'll try to derive one of the given results by simply applying Abel's summation, and we'll view the other two as products of the combination of other identities with harmonic series.

For the part $i)$ of the problem we apply Abel's summation (see (5.1)) where we set $a_n = 1/n^2$ and $b_n = H_n^{(2)} H_n^{(3)}$, and we are led to

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)} H_n^{(3)}}{n^2} = \underbrace{\lim_{N\to\infty} H_N^{(2)} H_{N+1}^{(2)} H_{N+1}^{(3)}}_{5/2\zeta(3)\zeta(4)}$$

$$+\sum_{n=1}^{\infty} H_n^{(2)}\left(H_n^{(2)}H_n^{(3)} - \left(H_n^{(2)} + \frac{1}{(n+1)^2}\right)\left(H_n^{(3)} + \frac{1}{(n+1)^3}\right)\right) = \frac{5}{2}\zeta(3)\zeta(4)$$

$$-\sum_{n=1}^{\infty} \frac{H_{n+1}^{(2)} - \frac{1}{(n+1)^2}}{(n+1)^5} - \sum_{n=1}^{\infty} \frac{(H_{n+1}^{(2)} - \frac{1}{(n+1)^2})^2}{(n+1)^3}$$

$$-\sum_{n=1}^{\infty} \frac{(H_{n+1}^{(2)} - \frac{1}{(n+1)^2})(H_{n+1}^{(3)} - \frac{1}{(n+1)^3})}{(n+1)^2}$$

{reindex the series and expand them}

$$= \frac{5}{2}\zeta(3)\zeta(4) - \sum_{n=1}^{\infty} \frac{1}{n^7} + 2\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^5} + \sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^4} - \sum_{n=1}^{\infty} \frac{(H_n^{(2)})^2}{n^3} - \sum_{n=1}^{\infty} \frac{H_n^{(2)} H_n^{(3)}}{n^2}$$

{make use of the values of the series given in (6.75) and (6.80)}

$$= \frac{13}{2}\zeta(3)\zeta(4) - 3\zeta(7) - \sum_{n=1}^{\infty} \frac{(H_n^{(2)})^2}{n^3} - \sum_{n=1}^{\infty} \frac{H_n^{(2)} H_n^{(3)}}{n^2},$$

whence we get

$$2\sum_{n=1}^{\infty} \frac{H_n^{(2)} H_n^{(3)}}{n^2} + \sum_{k=1}^{\infty} \frac{(H_n^{(2)})^2}{n^3} = \frac{13}{2}\zeta(3)\zeta(4) - 3\zeta(7),$$

and the part $i)$ of the problem is complete.

Further, the result from the point $ii)$ is obtained immediately if we combine the identities in (4.45) and (4.47), which yields

$$2\sum_{n=1}^{\infty}\frac{H_n H_n^{(2)}}{n^4}-\sum_{k=1}^{\infty}\frac{(H_n^{(2)})^2}{n^3}=13\zeta(7)-\zeta(2)\zeta(5)-8\zeta(3)\zeta(4),$$

and the part $ii)$ of the problem is complete.

Next, passing to the point $iii)$, we consider the identities in (4.51) and (4.52) where if we multiply the former by $3/2$ and add it to the latter one, we get

$$6\sum_{n=1}^{\infty}\frac{H_n^3}{n^4}-\frac{5}{2}\sum_{n=1}^{\infty}\frac{H_n^4}{n^3}+7\sum_{n=1}^{\infty}\frac{H_n H_n^{(2)}}{n^4}=\frac{27}{2}\zeta(2)\zeta(5)-\frac{35}{2}\zeta(3)\zeta(4)+\frac{13}{2}\zeta(7),$$

that if we also combine with the identity in (4.50), we conclude that

$$10\sum_{n=1}^{\infty}\frac{H_n^3}{n^4}-3\sum_{n=1}^{\infty}\frac{H_n^4}{n^3}=75\zeta(7)+5\zeta(2)\zeta(5)-63\zeta(3)\zeta(4),$$

and the part $iii)$ of the problem is complete.

For getting the last harmonic series identity I needed three of the relations in the next sections. It would also be interesting finding another creative ways of establishing a relation between these two series.

6.37 Preparing the *Weapons* of *The Master Theorem of Series* to Breach the Fortress of the Challenging Harmonic Series of Weight 7: The 1st Episode

Solution Starting with this section and continuing in the next two sections, I'll prepare three results derived with the applications of *The Master Theorem of Series* which are of great help in the derivation of the harmonic series of weight 7.

Let's start out with the first equality of the result in (4.26) where if we multiply both sides by $1/n^3$, and then consider the summation from $n=1$ to ∞, we get

$$\frac{1}{3}\sum_{n=1}^{\infty}\frac{H_n^3}{n^4}+\sum_{n=1}^{\infty}\frac{H_n H_n^{(2)}}{n^4}+\frac{2}{3}\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^4}=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{H_k^2-H_k^{(2)}}{(k+1)(k+n+1)n^3}\right)$$

$$\{\text{reverse the order of summation}\}$$

$$=\sum_{k=1}^{\infty}\frac{H_k^2-H_k^{(2)}}{k+1}\sum_{n=1}^{\infty}\frac{1}{n^3(k+n+1)}. \tag{6.175}$$

Now, it's easy to see that, by using the partial fraction expansion, the inner series in (6.175) can be written as

$$\sum_{n=1}^{\infty} \frac{1}{n^3(k+n+1)} = \frac{1}{k+1}\sum_{n=1}^{\infty}\frac{1}{n^3} - \frac{1}{(k+1)^2}\sum_{n=1}^{\infty}\frac{1}{n^2} + \frac{1}{(k+1)^2}\sum_{n=1}^{\infty}\frac{1}{n(k+n+1)}$$

$$\left\{\text{make use of the fact that } \sum_{n=1}^{\infty}\frac{1}{n(k+n+1)} = \frac{H_{k+1}}{k+1}\right\}$$

$$= \frac{\zeta(3)}{k+1} - \frac{\zeta(2)}{(k+1)^2} + \frac{H_{k+1}}{(k+1)^3}. \tag{6.176}$$

If we plug the result from (6.176) in (6.175), we get

$$\frac{1}{3}\sum_{n=1}^{\infty}\frac{H_n^3}{n^4} + \sum_{n=1}^{\infty}\frac{H_n H_n^{(2)}}{n^4} + \frac{2}{3}\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^4}$$

$$= \sum_{k=1}^{\infty}\frac{(H_{k+1}-1/(k+1))^2 - (H_{k+1}^{(2)} - 1/(k+1)^2)}{k+1}$$

$$\times\left(\frac{\zeta(3)}{k+1} - \frac{\zeta(2)}{(k+1)^2} + \frac{H_{k+1}}{(k+1)^3}\right)$$

$$\{\text{reindex the series and expand it}\}$$

$$= \sum_{k=1}^{\infty}\frac{H_k^3}{k^4} + \zeta(3)\sum_{k=1}^{\infty}\frac{H_k^2}{k^2} - \zeta(2)\sum_{k=1}^{\infty}\frac{H_k^2}{k^3} - 2\sum_{k=1}^{\infty}\frac{H_k^2}{k^5} - 2\zeta(3)\sum_{k=1}^{\infty}\frac{H_k}{k^3} + 2\zeta(2)\sum_{k=1}^{\infty}\frac{H_k}{k^4}$$

$$+2\sum_{k=1}^{\infty}\frac{H_k}{k^6} - \sum_{k=1}^{\infty}\frac{H_k H_k^{(2)}}{k^4} - \zeta(3)\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^2} + \zeta(2)\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^3} + 2\zeta(3)\sum_{k=1}^{\infty}\frac{1}{k^4}$$

$$-2\zeta(2)\sum_{k=1}^{\infty}\frac{1}{k^5}$$

$$\{\text{use the results in (3.45), the cases } n = 3, 4, 6, \text{(4.29),}\}$$

$$\{\text{(4.30), (4.32), (4.14) with } p = 2 \text{ and } n \to \infty, \text{(6.67)}\}$$

$$= \sum_{n=1}^{\infty}\frac{H_n^3}{n^4} - \sum_{n=1}^{\infty}\frac{H_n H_n^{(2)}}{n^4} + 10\zeta(3)\zeta(4) - 4\zeta(2)\zeta(5) - 4\zeta(7),$$

whence we get, also using the result in (6.80), that

$$3\sum_{n=1}^{\infty}\frac{H_n H_n^{(2)}}{n^4}-\sum_{n=1}^{\infty}\frac{H_n^3}{n^4}=4\zeta(2)\zeta(5)+15\zeta(3)\zeta(4)-24\zeta(7), \tag{6.177}$$

and the solution is complete.

The identity used above has proved to be so useful in more situations, and we'll use it again in the next section. Good to know and keep close in our toolbox!

6.38 Preparing the *Weapons* of *The Master Theorem of Series* to Breach the Fortress of the Challenging Harmonic Series of Weight 7: The 2nd Episode

Solution In order to get the present relation, we would like to exploit again the identity used in the previous section, which stems from combining applications of *The Master Theorem of Series*. For proving the result we'll need to use both the values of a bunch of harmonic series and relations amongst the harmonic series of weight 7 previously established.

For the beginning, we make use of the first equality of the identity in (4.26) where if we multiply both sides by H_n/n^2 and then consider the summation over n from $n=1$ to ∞, we get

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{(H_k^2-H_k^{(2)})H_n}{(k+1)(k+n+1)n^2}\right)=\frac{1}{3}\sum_{n=1}^{\infty}\frac{H_n^4}{n^3}+\sum_{n=1}^{\infty}\frac{H_n^2 H_n^{(2)}}{n^3}+\frac{2}{3}\sum_{n=1}^{\infty}\frac{H_n H_n^{(3)}}{n^3}. \tag{6.178}$$

For the left-hand side of (6.178), we change the summation order, and we have

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{(H_k^2-H_k^{(2)})H_n}{(k+1)(k+n+1)n^2}\right)=\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{(H_k^2-H_k^{(2)})H_n}{(k+1)(k+n+1)n^2}\right)$$

$$=\underbrace{\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{(H_k^2-H_k^{(2)})H_n}{(k+1)^2n^2}\right)}_{S_1}-\underbrace{\sum_{k=1}^{\infty}\frac{H_k^2-H_k^{(2)}}{(k+1)^2}\left(\sum_{n=1}^{\infty}\frac{H_n}{n(n+k+1)}\right)}_{S_2}. \tag{6.179}$$

Now, for S_1 in (6.179), considering the classical Euler sum in (3.45), the case $n=2$, we obtain

$$S_1=\sum_{k=1}^{\infty}\frac{H_k^2-H_k^{(2)}}{(k+1)^2}\left(\sum_{n=1}^{\infty}\frac{H_n}{n^2}\right)$$

$$=2\zeta(3)\sum_{k=1}^{\infty}\frac{(H_{k+1}-1/(k+1))^2-H_{k+1}^{(2)}+1/(k+1)^2}{(k+1)^2}$$

$$\{\text{reindex the series and expand it}\}$$

$$= 4\zeta(3)\sum_{k=1}^{\infty}\frac{1}{k^4} - 4\zeta(3)\sum_{k=1}^{\infty}\frac{H_k}{k^3} + 2\zeta(3)\sum_{k=1}^{\infty}\frac{H_k^2}{k^2} - 2\zeta(3)\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^2}$$

$$\{\text{employ the Euler sum in (3.45), the case } n = 3\text{, and then make}\}$$

$$\{\text{use of the results in (4.29) and (4.14), where we set } p = 2 \text{ and let } n \to \infty\}$$

$$= 4\zeta(3)\zeta(4). \tag{6.180}$$

Then, we pass to the series S_2 from (6.179), and we write

$$S_2 = \sum_{k=1}^{\infty}\frac{H_k^2 - H_k^{(2)}}{(k+1)^2}\left(\sum_{n=1}^{\infty}\frac{H_n}{n(n+k+1)}\right)$$

$$\{\text{reindex the inner series and leave out the term for } n = 0\}$$

$$= \sum_{k=1}^{\infty}\frac{H_k^2 - H_k^{(2)}}{(k+1)^2}\left(\frac{1}{k+2} + \sum_{n=1}^{\infty}\frac{H_n + 1/(n+1)}{(n+1)(n+k+2)}\right)$$

$$= \underbrace{\sum_{k=1}^{\infty}\frac{H_k^2 - H_k^{(2)}}{(k+1)^2(k+2)}}_{S_3} + \underbrace{\sum_{k=1}^{\infty}\frac{H_k^2 - H_k^{(2)}}{(k+1)^2}\left(\sum_{n=1}^{\infty}\frac{H_n}{(n+1)(n+k+2)}\right)}_{S_4}$$

$$+ \underbrace{\sum_{k=1}^{\infty}\frac{H_k^2 - H_k^{(2)}}{(k+1)^2}\left(\sum_{n=1}^{\infty}\frac{1}{(n+1)^2(n+k+2)}\right)}_{S_5}. \tag{6.181}$$

For the series S_3 in (6.181), we have

$$S_3 = \sum_{k=1}^{\infty}\frac{H_k^2 - H_k^{(2)}}{(k+1)^2(k+2)} = \sum_{k=1}^{\infty}\frac{H_k^2 - H_k^{(2)}}{(k+1)^2} - \sum_{k=1}^{\infty}\frac{H_k^2 - H_k^{(2)}}{(k+1)(k+2)}$$

$$\{\text{for the second series employ the result in (4.26), the first equality}\}$$

$$= \sum_{k=1}^{\infty}\frac{(H_{k+1} - 1/(k+1))^2 - H_{k+1}^{(2)} + 1/(k+1)^2}{(k+1)^2} - 2$$

$$\{\text{reindex the series and expand it}\}$$

$$= 2\sum_{k=1}^{\infty}\frac{1}{k^4} - 2\sum_{k=1}^{\infty}\frac{H_k}{k^3} + \sum_{k=1}^{\infty}\frac{H_k^2}{k^2} - \sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^2} - 2$$

{use the Euler sum in (3.45), the case $n = 3$, and then employ}

{the results in (4.29) and (4.14), where we set $p = 2$ and let $n \to \infty$}

$$= 2(\zeta(4) - 1). \tag{6.182}$$

Note we could have also used the result in (4.26) to calculate $\displaystyle\sum_{k=1}^{\infty}\frac{H_k^2 - H_k^{(2)}}{(k+1)^2}$.

Returning to the result in (6.181) where we consider S_4, and then use the result in (4.21), the case $m = 1$, we get

$$S_4 = \sum_{k=1}^{\infty}\frac{H_k^2 - H_k^{(2)}}{(k+1)^2}\left(\sum_{n=1}^{\infty}\frac{H_n}{(n+1)(n+k+2)}\right)$$

$$= \frac{1}{2}\sum_{k=1}^{\infty}\frac{(H_k^2 - H_k^{(2)})(H_{k+1}^2 + H_{k+1}^{(2)})}{(k+1)^3}$$

$$= \frac{1}{2}\sum_{k=1}^{\infty}\frac{((H_{k+1} - 1/(k+1))^2 - H_{k+1}^{(2)} + 1/(k+1)^2)(H_{k+1}^2 + H_{k+1}^{(2)})}{(k+1)^3}$$

{reindex the series and expand it}

$$= \frac{1}{2}\sum_{k=1}^{\infty}\frac{H_k^4}{k^3} - \sum_{k=1}^{\infty}\frac{H_k^3}{k^4} - \sum_{k=1}^{\infty}\frac{H_k H_k^{(2)}}{k^4} - \frac{1}{2}\sum_{k=1}^{\infty}\frac{(H_k^{(2)})^2}{k^3} + \sum_{k=1}^{\infty}\frac{H_k^2}{k^5} + \sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^5}$$

$$\left\{\text{use the results in (4.48) to express } \sum_{k=1}^{\infty}\frac{(H_k^{(2)})^2}{k^3} \text{ in terms of } \sum_{k=1}^{\infty}\frac{H_k H_k^{(2)}}{k^4}\right\}$$

{and note the values of the last two series are given in (4.32) and (6.75)}

$$= \frac{1}{2}\sum_{k=1}^{\infty}\frac{H_k^4}{k^3} - \sum_{k=1}^{\infty}\frac{H_k^3}{k^4} - 2\sum_{k=1}^{\infty}\frac{H_k H_k^{(2)}}{k^4} + \frac{5}{2}\zeta(7) + \frac{7}{2}\zeta(2)\zeta(5) - \frac{9}{2}\zeta(3)\zeta(4). \tag{6.183}$$

For the last series in (6.181), we write

$$S_5 = \sum_{k=1}^{\infty}\frac{H_k^2 - H_k^{(2)}}{(k+1)^2}\left(\sum_{n=1}^{\infty}\frac{1}{(n+1)^2(n+k+2)}\right)$$

$$= \sum_{k=1}^{\infty} \frac{H_k^2 - H_k^{(2)}}{(k+1)^2} \left(\frac{1}{k+1} \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} - \frac{1}{k+1} \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+k+2)} \right)$$

$$\left\{ \text{make use of the fact that } \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+k+2)} = \frac{H_{k+1}}{k+1} - \frac{1}{k+2} \right\}$$

$$= \sum_{k=1}^{\infty} \frac{(H_{k+1} - 1/(k+1))^2 - H_{k+1}^{(2)} + 1/(k+1)^2}{(k+1)^2} \left(\frac{\zeta(2)}{k+1} - \frac{H_{k+1}}{(k+1)^2} - \frac{1}{k+2} \right)$$

$$\{\text{reindex the series and expand it}\}$$

$$= \sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k^4} - \sum_{k=1}^{\infty} \frac{H_k^3}{k^4} - \sum_{k=1}^{\infty} \frac{H_k^2}{k^2} + \zeta(2) \sum_{k=1}^{\infty} \frac{H_k^2}{k^3} + 2 \sum_{k=1}^{\infty} \frac{H_k^2}{k^5} + \sum_{k=1}^{\infty} \frac{H_k^{(2)}}{k^2}$$

$$- \zeta(2) \sum_{k=1}^{\infty} \frac{H_k^{(2)}}{k^3} - 2 \sum_{k=1}^{\infty} \frac{H_k}{k^2} + 2 \sum_{k=1}^{\infty} \frac{H_k}{k^3} - 2\zeta(2) \sum_{k=1}^{\infty} \frac{H_k}{k^4}$$

$$-2 \sum_{k=1}^{\infty} \frac{H_k}{k^6} - 2 \sum_{k=1}^{\infty} \frac{1}{k^2} + 2 \sum_{k=1}^{\infty} \frac{1}{k^3}$$

$$-2 \sum_{k=1}^{\infty} \frac{1}{k^4} + 2\zeta(2) \sum_{k=1}^{\infty} \frac{1}{k^5} + 2 \sum_{k=1}^{\infty} \left(\frac{H_k}{k} - \frac{H_k}{k+1} \right) + \sum_{k=1}^{\infty} \left(\frac{H_k^2}{k} - \frac{H_k^2}{k+1} \right)$$

$$- \sum_{k=1}^{\infty} \left(\frac{H_k^{(2)}}{k} - \frac{H_k^{(2)}}{k+1} \right) + 2 \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right)$$

{use the Euler sum in (3.45), the cases $n = 2, 3, 4, 6$ and the results}

{in (4.29), (4.30), (4.32), (4.14) with $p = 2$ and $n \to \infty$, and (6.67)}

$$= \sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k^4} - \sum_{k=1}^{\infty} \frac{H_k^3}{k^4} + 2 \sum_{k=1}^{\infty} \left(\frac{H_k}{k} - \frac{H_{k+1}}{k+1} \right) + \sum_{k=1}^{\infty} \left(\frac{H_k^2}{k} - \frac{H_{k+1}^2}{k+1} \right)$$

$$- \sum_{k=1}^{\infty} \left(\frac{H_k^{(2)}}{k} - \frac{H_{k+1}^{(2)}}{k+1} \right) + 2 \sum_{k=1}^{\infty} \frac{H_{k+1}}{(k+1)^2} + 2 \sum_{k=1}^{\infty} \frac{1}{(k+1)^2} - 2 \sum_{k=1}^{\infty} \frac{1}{(k+1)^3}$$

$$+2 - 2\zeta(2) - 2\zeta(3) - 2\zeta(4) + 4\zeta(7) + 4\zeta(2)\zeta(5) - 8\zeta(3)\zeta(4)$$

$$= \sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k^4} - \sum_{k=1}^{\infty} \frac{H_k^3}{k^4} + 2 - 2\zeta(4) + 4\zeta(7) + 4\zeta(2)\zeta(5) - 8\zeta(3)\zeta(4). \tag{6.184}$$

Now, if we plug the values of the series S_3, S_4, and S_5 from (6.182), (6.183), and (6.184) in (6.181), we obtain

$$S_2 = \sum_{k=1}^{\infty} \frac{H_k^2 - H_k^{(2)}}{(k+1)^2} \left(\sum_{n=1}^{\infty} \frac{H_n}{n(n+k+1)} \right)$$

$$= \frac{1}{2} \sum_{k=1}^{\infty} \frac{H_k^4}{k^3} - 2 \sum_{k=1}^{\infty} \frac{H_k^3}{k^4} - \sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k^4} + \frac{13}{2}\zeta(7) + \frac{15}{2}\zeta(2)\zeta(5) - \frac{25}{2}\zeta(3)\zeta(4). \tag{6.185}$$

Then, by plugging the values of the series S_1 and S_2 from (6.180) and (6.185) in (6.179), we get

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{(H_k^2 - H_k^{(2)})H_n}{(k+1)(k+n+1)n^2} \right)$$

$$= 2 \sum_{k=1}^{\infty} \frac{H_k^3}{k^4} + \sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k^4} - \frac{1}{2} \sum_{k=1}^{\infty} \frac{H_k^4}{k^3} - \frac{15}{2}\zeta(2)\zeta(5) + \frac{33}{2}\zeta(3)\zeta(4) - \frac{13}{2}\zeta(7). \tag{6.186}$$

Finally, if we plug (6.186) in (6.178), we have

$$2 \sum_{n=1}^{\infty} \frac{H_n^3}{n^4} + \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{H_n^4}{n^3} - \frac{15}{2}\zeta(2)\zeta(5) + \frac{33}{2}\zeta(3)\zeta(4) - \frac{13}{2}\zeta(7)$$

$$= \frac{1}{3} \sum_{n=1}^{\infty} \frac{H_n^4}{n^3} + \sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^3} + \frac{2}{3} \sum_{n=1}^{\infty} \frac{H_n H_n^{(3)}}{n^3}$$

$$\left\{ \text{make use of the result in (4.44) to express } \sum_{n=1}^{\infty} \frac{H_n H_n^{(3)}}{n^3} \text{ in terms of } \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4} \right\}$$

$$= \frac{1}{3} \sum_{n=1}^{\infty} \frac{H_n^4}{n^3} + \sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^3} - \frac{4}{3} \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4} + \frac{8}{3}\zeta(7) + \frac{7}{6}\zeta(3)\zeta(4) - \zeta(2)\zeta(5),$$

whence we obtain that

$$2 \sum_{n=1}^{\infty} \frac{H_n^3}{n^4} - \frac{5}{6} \sum_{n=1}^{\infty} \frac{H_n^4}{n^3} - \sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^3} + \frac{7}{3} \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4}$$

$$= \frac{55}{6}\zeta(7) + \frac{13}{2}\zeta(2)\zeta(5) - \frac{46}{3}\zeta(3)\zeta(4),$$

and the solution is complete.

As seen so far, many identities involved in the derivation of the relations with harmonic series of weight 7 have their origins polarized around *The Master Theorem of Series* and the generalized sum in Sect. 4.13.

6.39 Preparing the *Weapons* of *The Master Theorem of Series* to Breach the Fortress of the Challenging Harmonic Series of Weight 7: The 3rd Episode

Solution To go further and get the current relation with harmonic series of weight 7, we might like to rely on another identity that originates from combining applications of *The Master Theorem of Series*. Once we have this job finished, we prepare for the moment of *breaking the ice* and starting to get values of harmonic series of weight 7. Surely, harmonic series of weight 7 we already met like in Sect. 4.21, which are called linear harmonic series, but we also want to get all nonlinear harmonic series of weight 7, which means that the summand of the series involves a product of at least two harmonic numbers.

We make use of the first equality of the identity in (4.27) where if we multiply both sides by $1/n^2$ and then consider the summation from $n=1$ to ∞, we get

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{H_k^3-3H_kH_k^{(2)}+2H_k^{(3)}}{(k+1)(k+n+1)n^2}\right)$$

$$=\frac{1}{4}\sum_{n=1}^{\infty}\frac{H_n^4}{n^3}+\frac{3}{2}\sum_{n=1}^{\infty}\frac{H_n^2H_n^{(2)}}{n^3}+2\sum_{n=1}^{\infty}\frac{H_nH_n^{(3)}}{n^3}+\frac{3}{4}\sum_{n=1}^{\infty}\frac{(H_n^{(2)})^2}{n^3}+\frac{3}{2}\sum_{n=1}^{\infty}\frac{H_n^{(4)}}{n^3}. \tag{6.187}$$

For the left-hand side of the equality in (6.187), we have

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{H_k^3-3H_kH_k^{(2)}+2H_k^{(3)}}{(k+1)(k+n+1)n^2}\right)=\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{H_k^3-3H_kH_k^{(2)}+2H_k^{(3)}}{(k+1)(k+n+1)n^2}\right)$$

$$=\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{H_k^3-3H_kH_k^{(2)}+2H_k^{(3)}}{n^2(k+1)^2}\right)-\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{H_k^3-3H_kH_k^{(2)}+2H_k^{(3)}}{(k+1)^2n(n+k+1)}\right)$$

$$\left\{\text{for the second inner series make use of the fact that } \sum_{n=1}^{\infty}\frac{1}{n(n+k+1)}=\frac{H_{k+1}}{k+1}\right\}$$

$$=\zeta(2)\underbrace{\sum_{k=1}^{\infty}\frac{H_k^3-3H_kH_k^{(2)}+2H_k^{(3)}}{(k+1)^2}}_{S_1}-\underbrace{\sum_{k=1}^{\infty}\frac{(H_k^3-3H_kH_k^{(2)}+2H_k^{(3)})H_{k+1}}{(k+1)^3}}_{S_2}. \tag{6.188}$$

For the first series in (6.188), we have

$$S_1 = \sum_{k=1}^{\infty} \frac{H_k^3 - 3H_k H_k^{(2)} + 2H_k^{(3)}}{(k+1)^2}$$

$$= \sum_{k=1}^{\infty} \frac{(H_{k+1} - \frac{1}{k+1})^3 - 3(H_{k+1} - \frac{1}{k+1})(H_{k+1}^{(2)} - \frac{1}{(k+1)^2}) + 2(H_{k+1}^{(3)} - \frac{1}{(k+1)^3})}{(k+1)^2}$$

$$\{\text{reindex the series and expand it}\}$$

$$= \sum_{k=1}^{\infty} \frac{H_k^3}{k^2} - 3\sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k^2} - 3\sum_{k=1}^{\infty} \frac{H_k^2}{k^3} + 3\sum_{k=1}^{\infty} \frac{H_k^{(2)}}{k^3} + 2\sum_{k=1}^{\infty} \frac{H_k^{(3)}}{k^2} + 6\sum_{k=1}^{\infty} \frac{H_k}{k^4}$$

$$-6\sum_{k=1}^{\infty} \frac{1}{k^5}$$

$$\{\text{make use of the classical Euler sum in (3.45), the case } n = 4,\}$$

$$\{\text{and then the results in (4.34), (4.33), (4.30), (6.67), and (6.68)}\}$$

$$= 6\zeta(5), \tag{6.189}$$

or alternatively,[6] one can calculate the series using the identity in (4.27), the first equality, together with the differentiation. The series in (6.189) is known in the mathematical literature and also appears in [63].

Further, for the second series in (6.188), we write

$$S_2 = \sum_{k=1}^{\infty} \frac{(H_k^3 - 3H_k H_k^{(2)} + 2H_k^{(3)})H_{k+1}}{(k+1)^3}$$

$$= \sum_{k=1}^{\infty} \frac{((H_{k+1} - \frac{1}{k+1})^3 - 3(H_{k+1} - \frac{1}{k+1})(H_{k+1}^{(2)} - \frac{1}{(k+1)^2}) + 2(H_{k+1}^{(3)} - \frac{1}{(k+1)^3}))H_{k+1}}{(k+1)^3}$$

$$\{\text{reindex the series and expand it}\}$$

$$= \sum_{k=1}^{\infty} \frac{H_k^4}{k^3} - 3\sum_{k=1}^{\infty} \frac{H_k^3}{k^4} + 3\sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k^4} + 2\sum_{k=1}^{\infty} \frac{H_k H_k^{(3)}}{k^3} - 3\sum_{k=1}^{\infty} \frac{H_k^2 H_k^{(2)}}{k^3}$$

[6] Multiplying by n both sides of (4.27), the first equality, differentiating with respect to n and then letting $n \to 0$, we get $\sum_{k=1}^{\infty} \frac{H_k^3 - 3H_k H_k^{(2)} + 2H_k^{(3)}}{(k+1)^2} = 6\zeta(5)$, where for differentiation we express the harmonic numbers in terms of Polygamma function.

$$+6\sum_{k=1}^{\infty}\frac{H_k^2}{k^5}-6\sum_{k=1}^{\infty}\frac{H_k}{k^6}$$

{make use of the results in (3.45), the case $n=6$, and (4.32)}

$$=\sum_{k=1}^{\infty}\frac{H_k^4}{k^3}-3\sum_{k=1}^{\infty}\frac{H_k^3}{k^4}+3\sum_{k=1}^{\infty}\frac{H_kH_k^{(2)}}{k^4}+2\sum_{k=1}^{\infty}\frac{H_kH_k^{(3)}}{k^3}-3\sum_{k=1}^{\infty}\frac{H_k^2H_k^{(2)}}{k^3}$$
$$+12\zeta(7)-9\zeta(3)\zeta(4). \tag{6.190}$$

If we plug the series results from (6.189) and (6.190) in (6.188), we get

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{H_k^3-3H_kH_k^{(2)}+2H_k^{(3)}}{(k+1)(k+n+1)n^2}\right)$$
$$=3\sum_{k=1}^{\infty}\frac{H_k^3}{k^4}-\sum_{k=1}^{\infty}\frac{H_k^4}{k^3}-3\sum_{k=1}^{\infty}\frac{H_kH_k^{(2)}}{k^4}-2\sum_{k=1}^{\infty}\frac{H_kH_k^{(3)}}{k^3}+3\sum_{k=1}^{\infty}\frac{H_k^2H_k^{(2)}}{k^3}$$
$$-12\zeta(7)+6\zeta(2)\zeta(5)+9\zeta(3)\zeta(4). \tag{6.191}$$

Now, we combine the results in (6.187) and (6.191) that lead to

$$\frac{5}{4}\sum_{n=1}^{\infty}\frac{H_n^4}{n^3}-3\sum_{n=1}^{\infty}\frac{H_n^3}{n^4}-\frac{3}{2}\sum_{n=1}^{\infty}\frac{H_n^2H_n^{(2)}}{n^3}+4\sum_{n=1}^{\infty}\frac{H_nH_n^{(3)}}{n^3}+\frac{3}{4}\sum_{n=1}^{\infty}\frac{(H_n^{(2)})^2}{n^3}$$
$$+\frac{3}{2}\sum_{n=1}^{\infty}\frac{H_n^{(4)}}{n^3}+3\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^4}=-12\zeta(7)+6\zeta(2)\zeta(5)+9\zeta(3)\zeta(4),$$

and if we make use of the result in (6.79), then the result in (4.44) to express $\sum_{n=1}^{\infty}\frac{H_nH_n^{(3)}}{n^3}$ in terms of $\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^4}$ and afterwards the result in (4.48) to express $\sum_{n=1}^{\infty}\frac{(H_n^{(2)})^2}{n^3}$ as $\sum_{k=1}^{\infty}\frac{H_nH_n^{(2)}}{n^4}$, we finally obtain

$$3\sum_{n=1}^{\infty}\frac{H_n^3}{n^4}-\frac{5}{4}\sum_{n=1}^{\infty}\frac{H_n^4}{n^3}+\frac{3}{2}\sum_{n=1}^{\infty}\frac{H_n^2H_n^{(2)}}{n^3}+\frac{7}{2}\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^4}$$
$$=\frac{15}{4}\zeta(2)\zeta(5)+\frac{11}{2}\zeta(3)\zeta(4)-\frac{29}{4}\zeta(7),$$

and the solution is finalized.

Having shown this last relation, we are ready to enter the next section and obtain the values of the proposed nonlinear harmonic series of weight 7.

6.40 Calculating the Harmonic Series of Weight 7, $\sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^3}$, with the *Weapons* of *The Master Theorem of Series*

Solution Using the relations from the last two sections we get the value of the proposed harmonic series of weight 7 which we'll further use in the extraction process of other harmonic series of weight 7. Everything is straightforward as you'll see!

If we multiply both sides of (4.52) by $-2/3$ and then add (4.51) to the result, we obtain what we need,

$$\sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^3} = \frac{19}{2}\zeta(3)\zeta(4) - 2\zeta(2)\zeta(5) - 7\zeta(7),$$

and the solution is finalized.

Surely, in this section everything went easily and fast since I only used the final *products* of the previous sections (where we had some work to do).

6.41 The Calculation of Two Good-Looking Pairs of Harmonic Series: The Series $\sum_{n=1}^{\infty} \frac{H_n}{n^2} \sum_{k=1}^{n} \frac{H_k}{k^3}$, $\sum_{n=1}^{\infty} \frac{H_n}{n^3} \sum_{k=1}^{n} \frac{H_k}{k^2}$ and $\sum_{n=1}^{\infty} \frac{H_n^2}{n^2} \sum_{k=1}^{n} \frac{H_k}{k^2}$, $\sum_{n=1}^{\infty} \frac{H_n}{n^2} \sum_{k=1}^{n} \frac{H_k^2}{k^2}$

Solution Due to the evaluation of the series from the previous section we are able to extract the values of the first two harmonic series proposed in the current section. If for the first two series we'll only use results obtained in the previous sections, for getting the values of the last two harmonic series things will change as you'll see.

Let's prove first the following key series relation,

$$\sum_{n=1}^{\infty} \frac{H_n}{n^3} \left(\sum_{k=1}^{n} \frac{H_k}{k^2} \right) = \frac{1}{2} \sum_{k=1}^{\infty} \frac{H_k^2 H_k^{(2)}}{k^3} - \frac{1}{2}\zeta(7) - \frac{9}{2}\zeta(2)\zeta(5) + \frac{27}{4}\zeta(3)\zeta(4). \tag{6.192}$$

Proof To prove the result in (6.192), we recall the identity in (4.21), the case $m = 2$, where if we multiply both sides by H_n/n^2 and then consider the summation from $n = 1$ to ∞, we get

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{H_k^{(2)}H_n}{(k+1)(k+n+1)n^2}\right) = \zeta(2)\sum_{n=1}^{\infty}\frac{H_n^2}{n^3} - \sum_{n=1}^{\infty}\frac{H_n}{n^3}\left(\sum_{i=1}^{n}\frac{H_i}{i^2}\right). \tag{6.193}$$

As regards the left-hand side of (6.193), we change the summation order, and we write

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{H_k^{(2)}H_n}{(k+1)(k+n+1)n^2}\right) = \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{H_k^{(2)}H_n}{(k+1)(k+n+1)n^2}\right)$$

$$\{\text{reindex the inner series to start from } n = 0 \text{ to } \infty\}$$

$$\{\text{and let out the term of the inner series for } n = 0\}$$

$$= \sum_{k=1}^{\infty}\frac{H_k^{(2)}}{(k+1)(k+2)} + \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{H_k^{(2)}(H_n + 1/(n+1))}{(k+1)(k+n+2)(n+1)^2}\right)$$

$$= \underbrace{\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{(k+1)(k+2)}}_{S_1} + \underbrace{\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{H_k^{(2)}H_n}{(k+1)(k+n+2)(n+1)^2}\right)}_{S_2}$$

$$+ \underbrace{\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{H_k^{(2)}}{(k+1)(k+n+2)(n+1)^3}\right)}_{S_3}. \tag{6.194}$$

Now, for the series S_1 in (6.194), we consider the identity in (4.21), the case $m = 2$, with $n = 1$,

$$S_1 = \sum_{k=1}^{\infty}\frac{H_k^{(2)}}{(k+1)(k+2)} = \zeta(2) - 1. \tag{6.195}$$

Further, for the series S_2 in (6.194), we have

$$S_2 = \sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k+1}\left(\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^2(n+k+2)}\right)$$

$$= \sum_{k=1}^{\infty} \frac{H_k^{(2)}}{(k+1)^2} \left(\sum_{n=1}^{\infty} \frac{H_n}{(n+1)^2} \right) - \sum_{k=1}^{\infty} \frac{H_k^{(2)}}{(k+1)^2} \left(\sum_{n=1}^{\infty} \frac{H_n}{(n+1)(n+k+2)} \right)$$

$$\left\{ \text{use that } \sum_{n=1}^{\infty} \frac{H_n}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{H_{n+1} - 1/(n+1)}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{H_n}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n^3} = \zeta(3), \right\}$$

{by (3.45), the case $n = 2$, and also employ the result in (4.21), the case $m = 1$}

$$= \zeta(3) \sum_{k=1}^{\infty} \frac{H_{k+1}^{(2)} - 1/(k+1)^2}{(k+1)^2} - \frac{1}{2} \sum_{k=1}^{\infty} \frac{(H_{k+1}^2 + H_{k+1}^{(2)})(H_{k+1}^{(2)} - 1/(k+1)^2)}{(k+1)^3}$$

{reindex the series and expand them}

$$= \zeta(3) \sum_{k=1}^{\infty} \frac{H_k^{(2)}}{k^2} - \zeta(3) \sum_{k=1}^{\infty} \frac{1}{k^4} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{H_k^2}{k^5} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{H_k^{(2)}}{k^5} - \frac{1}{2} \sum_{k=1}^{\infty} \frac{(H_k^{(2)})^2}{k^3}$$

$$- \frac{1}{2} \sum_{k=1}^{\infty} \frac{H_k^2 H_k^{(2)}}{k^3}$$

{make use of the result in (4.14), the case $p = 2$, with $n \to \infty$, (4.32) and (6.75)}

$$= 2\zeta(2)\zeta(5) + \frac{1}{2}\zeta(3)\zeta(4) - 2\zeta(7) - \frac{1}{2} \sum_{k=1}^{\infty} \frac{(H_k^{(2)})^2}{k^3} - \frac{1}{2} \sum_{k=1}^{\infty} \frac{H_k^2 H_k^{(2)}}{k^3}$$

$$\left\{ \text{make use of the result in (4.48) to express } \sum_{n=1}^{\infty} \frac{(H_n^{(2)})^2}{n^3} \text{ as } \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4} \right\}$$

$$= \frac{9}{2}\zeta(7) + \frac{3}{2}\zeta(2)\zeta(5) - \frac{7}{2}\zeta(3)\zeta(4) - \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^3}. \tag{6.196}$$

Then, for the last series in (6.194), S_3, we make use of the result in (4.21) where we set $m = 2$, and then we write

$$S_3 = \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{H_k^{(2)}}{(k+1)(k+n+2)(n+1)^3} \right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{(n+1)^3} \left(\sum_{k=1}^{\infty} \frac{H_k^{(2)}}{(k+1)(k+n+2)} \right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{(n+1)^3} \left(\zeta(2) \frac{H_{n+1}}{n+1} - \frac{1}{n+1} \sum_{i=1}^{n+1} \frac{H_i}{i^2} \right)$$

$$= \zeta(2) \sum_{n=1}^{\infty} \frac{H_{n+1}}{(n+1)^4} - \sum_{n=1}^{\infty} \left(\sum_{i=1}^{n+1} \frac{H_i}{i^2 (n+1)^4} \right)$$

{reindex the series}

$$= 1 - \zeta(2) + \zeta(2) \sum_{n=1}^{\infty} \frac{H_n}{n^4} - \sum_{n=1}^{\infty} \frac{1}{n^4} \left(\sum_{i=1}^{n} \frac{H_i}{i^2} \right)$$

{make use of the Euler sum in (3.45), the case $n = 4$, and at}

{the same time change the summation order in the last series}

$$= 1 - \zeta(2) + 3\zeta(2)\zeta(5) - \frac{5}{2}\zeta(3)\zeta(4) - \sum_{i=1}^{\infty} \frac{H_i}{i^2} \left(\sum_{n=i}^{\infty} \frac{1}{n^4} \right)$$

$$= 1 - \zeta(2) + 3\zeta(2)\zeta(5) - \frac{5}{2}\zeta(3)\zeta(4) - \sum_{i=1}^{\infty} \frac{H_i}{i^2} \left(\frac{1}{i^4} + \frac{1}{(i+1)^4} + \cdots \right)$$

$$= 1 - \zeta(2) + 3\zeta(2)\zeta(5) - \frac{5}{2}\zeta(3)\zeta(4) - \sum_{i=1}^{\infty} \frac{H_i}{i^6} - \sum_{i=1}^{\infty} \frac{H_i}{i^2} \left(\zeta(4) - H_i^{(4)} \right)$$

$$= 1 - \zeta(2) + 3\zeta(2)\zeta(5) - \frac{5}{2}\zeta(3)\zeta(4) - \sum_{i=1}^{\infty} \frac{H_i}{i^6} - \zeta(4) \sum_{i=1}^{\infty} \frac{H_i}{i^2} + \sum_{i=1}^{\infty} \frac{H_i H_i^{(4)}}{i^2}$$

{make use of the Euler sum in (3.45), the cases $n = 2, 6$}

$$= 1 - \zeta(2) + 4\zeta(2)\zeta(5) - \frac{7}{2}\zeta(3)\zeta(4) - 4\zeta(7) + \sum_{i=1}^{\infty} \frac{H_i H_i^{(4)}}{i^2}$$

$$\left\{ \text{make use of the result in (4.41) to express } \sum_{n=1}^{\infty} \frac{H_n H_n^{(4)}}{n^2} \text{ in terms of } \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4} \right\}$$

$$= 1 - \zeta(2) + \frac{13}{2}\zeta(2)\zeta(5) - \frac{23}{4}\zeta(3)\zeta(4) - 4\zeta(7) + \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4}. \tag{6.197}$$

Collecting the results from (6.195), (6.196), and (6.197) in (6.194), we obtain

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{H_k^{(2)}H_n}{(k+1)(k+n+1)n^2}\right)=\frac{1}{2}\zeta(7)+8\zeta(2)\zeta(5)-\frac{37}{4}\zeta(3)\zeta(4)$$
$$-\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n^2H_n^{(2)}}{n^3}. \tag{6.198}$$

By combining the results in (6.193) and (6.41), we get

$$\frac{1}{2}\zeta(7)+8\zeta(2)\zeta(5)-\frac{37}{4}\zeta(3)\zeta(4)-\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n^2H_n^{(2)}}{n^3}$$
$$=\zeta(2)\sum_{n=1}^{\infty}\frac{H_n^2}{n^3}-\sum_{n=1}^{\infty}\frac{H_n}{n^3}\left(\sum_{k=1}^{n}\frac{H_k}{k^2}\right)$$
$$\{\text{make use of the result in (4.30)}\}$$
$$=\frac{7}{2}\zeta(2)\zeta(5)-\frac{5}{2}\zeta(3)\zeta(4)-\sum_{n=1}^{\infty}\frac{H_n}{n^3}\left(\sum_{k=1}^{n}\frac{H_k}{k^2}\right),$$

whence we obtain

$$\sum_{n=1}^{\infty}\frac{H_n}{n^3}\left(\sum_{k=1}^{n}\frac{H_k}{k^2}\right)=\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n^2H_n^{(2)}}{n^3}-\frac{1}{2}\zeta(7)-\frac{9}{2}\zeta(2)\zeta(5)+\frac{27}{4}\zeta(3)\zeta(4),$$

and the proof of the result is complete.

If we combine the results from (6.192) and (4.53), we get immediately that

$$\sum_{n=1}^{\infty}\frac{H_n}{n^3}\left(\frac{H_1}{1^2}+\frac{H_2}{2^2}+\cdots+\frac{H_n}{n^2}\right)=\frac{23}{2}\zeta(3)\zeta(4)-\frac{11}{2}\zeta(2)\zeta(5)-4\zeta(7),$$

and the series from point $ii)$ is calculated.

For the other series, we start with writing that

$$\frac{5}{2}\zeta(3)\zeta(4)=\sum_{k=1}^{\infty}\frac{H_k}{k^2}\left(\sum_{n=1}^{\infty}\frac{H_n}{n^3}\right)=\sum_{k=1}^{\infty}\frac{H_k}{k^2}\left(\left(\sum_{n=1}^{k}+\sum_{n=k}^{\infty}-\sum_{n=k}^{k}\right)\frac{H_n}{n^3}\right)$$
$$=\sum_{k=1}^{\infty}\frac{H_k}{k^2}\left(\sum_{n=1}^{k}\frac{H_n}{n^3}\right)+\sum_{k=1}^{\infty}\frac{H_k}{k^2}\left(\sum_{n=k}^{\infty}\frac{H_n}{n^3}\right)-\sum_{k=1}^{\infty}\frac{H_k^2}{k^5}$$

{reverse the order of summation in the second double series}

$$=\sum_{k=1}^{\infty}\frac{H_k}{k^2}\left(\sum_{n=1}^{k}\frac{H_n}{n^3}\right)+\sum_{n=1}^{\infty}\frac{H_n}{n^3}\left(\sum_{k=1}^{n}\frac{H_k}{k^2}\right)-\sum_{k=1}^{\infty}\frac{H_k^2}{k^5}$$

{make use of the result in (4.32)}

$$=\sum_{k=1}^{\infty}\frac{H_k}{k^2}\left(\sum_{n=1}^{k}\frac{H_n}{n^3}\right)+\sum_{n=1}^{\infty}\frac{H_n}{n^3}\left(\sum_{k=1}^{n}\frac{H_k}{k^2}\right)-6\zeta(7)+\zeta(2)\zeta(5)+\frac{5}{2}\zeta(3)\zeta(4),$$

whence we get

$$\sum_{k=1}^{\infty}\frac{H_k}{k^2}\left(\sum_{n=1}^{k}\frac{H_n}{n^3}\right)+\sum_{n=1}^{\infty}\frac{H_n}{n^3}\left(\sum_{k=1}^{n}\frac{H_k}{k^2}\right)=6\zeta(7)-\zeta(2)\zeta(5). \tag{6.199}$$

If we combine the result in (6.199) with the series from the point $ii)$ calculated above, we obtain that

$$\sum_{n=1}^{\infty}\frac{H_n}{n^2}\left(\frac{H_1}{1^3}+\frac{H_2}{2^3}+\cdots+\frac{H_n}{n^3}\right)=10\zeta(7)+\frac{9}{2}\zeta(2)\zeta(5)-\frac{23}{2}\zeta(3)\zeta(4),$$

and the series from the point $i)$ is calculated.

For the series from the points $iii)$ and $iv)$, we employ the result in (4.24), where multiplying its both sides by H_n/n and then considering the summation from $n=1$ to ∞ give

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{H_kH_k^{(2)}H_n}{(k+1)n(k+n+1)}\right)$$

$$=2\zeta(3)\sum_{n=1}^{\infty}\frac{H_n^2}{n^2}+\frac{\zeta(2)}{2}\sum_{n=1}^{\infty}\frac{H_n^3}{n^2}-\frac{\zeta(2)}{2}\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^2}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{H_nH_n^{(4)}}{n^2}$$

$$-\frac{1}{4}\sum_{n=1}^{\infty}\frac{H_n(H_n^{(2)})^2}{n^2}-\sum_{n=1}^{\infty}\frac{H_n^2}{n^2}\sum_{i=1}^{n}\frac{H_i}{i^2}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n}{n^2}\sum_{i=1}^{n}\frac{H_i^2}{i^2}, \tag{6.200}$$

For the series in the left-hand side of (6.200) we proceed as follows

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{H_kH_k^{(2)}H_n}{(k+1)n(k+n+1)}\right)=\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{H_kH_k^{(2)}H_n}{(k+1)n(k+n+1)}\right)$$

{reindex the inner series to start from $n=0$ to ∞}

{and let out the term of the inner series for $n=0$}

$$= \sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{(k+1)(k+2)} + \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{H_k H_k^{(2)} (H_n + 1/(n+1))}{(k+1)(n+1)(n+k+2)} \right)$$

$$= \underbrace{\sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{(k+1)(k+2)}}_{S_1} + \underbrace{\sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k+1} \left(\sum_{n=1}^{\infty} \frac{H_n}{(n+1)(n+k+2)} \right)}_{S_2}$$

$$+ \underbrace{\sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k+1} \left(\sum_{n=1}^{\infty} \frac{1}{(n+1)^2(n+k+2)} \right)}_{S_3}. \tag{6.201}$$

For the series S_2 in (6.201), we employ the result in (4.21), the case $m = 1$, and we get

$$S_2 = \sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k+1} \left(\sum_{n=1}^{\infty} \frac{H_n}{(n+1)(n+k+2)} \right)$$

$$= \frac{1}{2} \sum_{k=1}^{\infty} \frac{\left(H_{k+1} - \frac{1}{k+1}\right)\left(H_{k+1}^{(2)} - \frac{1}{(k+1)^2}\right)(H_{k+1}^2 + H_{k+1}^{(2)})}{(k+1)^2}$$

{reindex the series and expand it}

$$= \frac{1}{2} \sum_{k=1}^{\infty} \frac{H_k^2}{k^5} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{H_k^{(2)}}{k^5} - \frac{1}{2} \sum_{k=1}^{\infty} \frac{H_k^3}{k^4} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{H_k^3 H_k^{(2)}}{k^2}$$

$$- \frac{1}{2} \sum_{k=1}^{\infty} \frac{H_k^2 H_k^{(2)}}{k^3} - \frac{1}{2} \sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k^4}$$

$$- \frac{1}{2} \sum_{k=1}^{\infty} \frac{(H_k^{(2)})^2}{k^3} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{H_k (H_k^{(2)})^2}{k^2}$$

{use the series values in (4.32), (6.75), (4.61), (4.66), (4.53), (4.58), (4.65), and (4.68)}

$$= \frac{11}{8}\zeta(7) + \frac{1}{2}\zeta(2)\zeta(5) + \frac{11}{2}\zeta(3)\zeta(4). \tag{6.202}$$

Then, for the series S_3 in (6.201), we write

$$S_3 = \sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k+1} \left(\sum_{n=1}^{\infty} \frac{1}{(n+1)^2(n+k+2)} \right) = \sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k+1} \left(\sum_{n=2}^{\infty} \frac{1}{n^2(n+k+1)} \right)$$

$$= \sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k+1} \left(\sum_{n=1}^{\infty} \frac{1}{n^2(n+k+1)} \right) - \sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{(k+1)(k+2)}$$

$$= \zeta(2) \sum_{k=1}^{\infty} \frac{\left(H_{k+1} - \frac{1}{k+1}\right)\left(H_{k+1}^{(2)} - \frac{1}{(k+1)^2}\right)}{(k+1)^2}$$

$$- \sum_{k=1}^{\infty} \frac{\left(H_{k+1} - \frac{1}{k+1}\right)\left(H_{k+1}^{(2)} - \frac{1}{(k+1)^2}\right) H_{k+1}}{(k+1)^3} - S_1$$

{reindex the series and expand them}

$$= \zeta(2) \sum_{k=1}^{\infty} \frac{1}{k^5} - \zeta(2) \sum_{k=1}^{\infty} \frac{H_k}{k^4} - \zeta(2) \sum_{k=1}^{\infty} \frac{H_k^{(2)}}{k^3} + \zeta(2) \sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k^2} - \sum_{k=1}^{\infty} \frac{H_k}{k^6} + \sum_{k=1}^{\infty} \frac{H_k^2}{k^5}$$

$$+ \sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k^4} - \sum_{k=1}^{\infty} \frac{H_k^2 H_k^{(2)}}{k^3} - S_1$$

{use the results in (3.45), the cases $n = 4, 6$, (6.67), (4.33), (4.32), (4.58), and (4.53)}

$$= \frac{93}{16}\zeta(7) + \frac{15}{2}\zeta(2)\zeta(5) - \frac{51}{4}\zeta(3)\zeta(4) - S_1. \tag{6.203}$$

By plugging the results from (6.202) and (6.203) in (6.201), we obtain

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{H_k H_k^{(2)} H_n}{(k+1)n(k+n+1)} \right) = \frac{115}{16}\zeta(7) + 8\zeta(2)\zeta(5) - \frac{29}{4}\zeta(3)\zeta(4). \tag{6.204}$$

Then, for the right-hand side of (6.200), we write

$$2\zeta(3) \sum_{n=1}^{\infty} \frac{H_n^2}{n^2} + \frac{\zeta(2)}{2} \sum_{n=1}^{\infty} \frac{H_n^3}{n^2} - \frac{\zeta(2)}{2} \sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^2} - \frac{1}{4} \sum_{n=1}^{\infty} \frac{H_n H_n^{(4)}}{n^2}$$

$$- \frac{1}{4} \sum_{n=1}^{\infty} \frac{H_n (H_n^{(2)})^2}{n^2} - \sum_{n=1}^{\infty} \frac{H_n^2}{n^2} \sum_{i=1}^{n} \frac{H_i}{i^2} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{H_n}{n^2} \sum_{i=1}^{n} \frac{H_i^2}{i^2}$$

{use the results in (4.29), (4.34), (4.33), (4.60), and (4.68)}

$$= \frac{67}{16}\zeta(7) + \frac{7}{4}\zeta(2)\zeta(5) + \frac{61}{8}\zeta(3)\zeta(4) - \sum_{n=1}^{\infty} \frac{H_n^2}{n^2} \sum_{i=1}^{n} \frac{H_i}{i^2} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{H_n}{n^2} \sum_{i=1}^{n} \frac{H_i^2}{i^2},$$

which if we combine with the result in (6.204), we obtain that

$$\frac{1}{2} \sum_{n=1}^{\infty} \frac{H_n}{n^2} \sum_{i=1}^{n} \frac{H_i^2}{i^2} - \sum_{n=1}^{\infty} \frac{H_n^2}{n^2} \sum_{i=1}^{n} \frac{H_i}{i^2} = 3\zeta(7) + \frac{25}{4}\zeta(2)\zeta(5) - \frac{119}{8}\zeta(3)\zeta(4). \tag{6.205}$$

On the other hand, we may write

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} \sum_{i=1}^{n} \frac{H_i^2}{i^2} = \sum_{i=1}^{\infty} \frac{H_i^2}{i^2} \sum_{n=i}^{\infty} \frac{H_n}{n^2} = \sum_{i=1}^{\infty} \frac{H_i^2}{i^2} \left(2\zeta(3) + \frac{H_i}{i^2} - \sum_{n=1}^{i} \frac{H_n}{n^2} \right)$$

$$= 2\zeta(3) \sum_{i=1}^{\infty} \frac{H_i^2}{i^2} + \sum_{i=1}^{\infty} \frac{H_i^3}{i^4} - \sum_{i=1}^{\infty} \frac{H_i^2}{i^2} \sum_{n=1}^{i} \frac{H_n}{n^2}$$

{make use of the results in (4.29) and (4.61)}

$$= \frac{231}{16}\zeta(7) + 2\zeta(2)\zeta(5) - \frac{17}{4}\zeta(3)\zeta(4) - \sum_{i=1}^{\infty} \frac{H_i^2}{i^2} \sum_{n=1}^{i} \frac{H_n}{n^2},$$

whence we obtain that

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} \sum_{i=1}^{n} \frac{H_i^2}{i^2} + \sum_{n=1}^{\infty} \frac{H_n^2}{n^2} \sum_{i=1}^{n} \frac{H_i}{i^2} = \frac{231}{16}\zeta(7) + 2\zeta(2)\zeta(5) - \frac{17}{4}\zeta(3)\zeta(4). \tag{6.206}$$

Thus, by combining the results from (6.205) and (6.206), we obtain that

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} \sum_{i=1}^{n} \frac{H_i^2}{i^2} = \frac{93}{8}\zeta(7) + \frac{11}{2}\zeta(2)\zeta(5) - \frac{51}{4}\zeta(3)\zeta(4)$$

and

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^2} \sum_{i=1}^{n} \frac{H_i}{i^2} = \frac{45}{16}\zeta(7) - \frac{7}{2}\zeta(2)\zeta(5) + \frac{17}{2}\zeta(3)\zeta(4),$$

and the series from the points $iii)$ and $iv)$ are finalized.

As an observation, if for the first two points of the problem I considered results previously calculated, for the last two points I also used results that are met in the next sections.

Having finalized these series, we prepare to calculate the harmonic series from the next section (where in one of the ways, such series are helpful in the derivation process).

6.42 The Calculation of an Essential Harmonic Series of Weight 7: The Series $\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4}$

Solution The following series result is a critical one that will lead to obtaining the values of more harmonic series of weight 7 presented in the book (recall the previous identities in terms of $\displaystyle\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4}$ and other series of weight 7).

For a first solution, we employ the identity in (4.19), where if we multiply both sides by $1/n^3$, and then consider the sum from $n = 1$ to ∞, we have

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n-1}\frac{H_k^{(2)}}{n^3(n-k)^2}\right) = 4\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{H_k}{n^3k^3}\right) + \sum_{n=1}^{\infty}\frac{\left(H_n^{(2)}\right)^2}{n^3} - 5\sum_{n=1}^{\infty}\frac{H_n^{(4)}}{n^3}. \tag{6.207}$$

For the left-hand side of the result in (6.207), we change the order of summation, and then we have

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n-1}\frac{H_k^{(2)}}{n^3(n-k)^2}\right) = \sum_{k=1}^{\infty}\left(\sum_{n=k+1}^{\infty}\frac{H_k^{(2)}}{n^3(n-k)^2}\right)$$

$$\{\text{reindex the inner series and start from } n = 1 \text{ to } \infty\}$$

$$= \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{H_k^{(2)}}{(n+k)^3n^2}\right)$$

$$= \sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^3}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right) - 3\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^3}\left(\sum_{n=1}^{\infty}\frac{1}{n(n+k)}\right) + \sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^2}\left(\sum_{n=1}^{\infty}\frac{1}{(n+k)^3}\right)$$

$$+2\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^3}\left(\sum_{n=1}^{\infty}\frac{1}{(n+k)^2}\right)$$

$$\left\{\text{use the result in (6.67) and the fact that } \sum_{n=1}^{\infty}\frac{1}{n(n+k)} = \frac{H_k}{k}\right\}$$

$$= \frac{15}{2}\zeta(3)\zeta(4) - \frac{9}{2}\zeta(2)\zeta(5) - 3\sum_{k=1}^{\infty}\frac{H_k H_k^{(2)}}{k^4} + \sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^2}\left(\zeta(3) - H_k^{(3)}\right)$$

$$+2\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^3}\left(\zeta(2) - H_k^{(2)}\right) = \frac{15}{2}\zeta(3)\zeta(4) - \frac{9}{2}\zeta(2)\zeta(5) - 3\sum_{k=1}^{\infty}\frac{H_k H_k^{(2)}}{k^4}$$

$$+\zeta(3)\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^2} - \sum_{k=1}^{\infty}\frac{H_k^{(2)}H_k^{(3)}}{k^2} + 2\zeta(2)\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^3} - 2\sum_{k=1}^{\infty}\frac{(H_k^{(2)})^2}{k^3}$$

{use the results in (4.14), $p = 2$, with $n \to \infty$, and (6.67)}

$$= \frac{97}{4}\zeta(3)\zeta(4) - \frac{27}{2}\zeta(2)\zeta(5) - 3\sum_{n=1}^{\infty}\frac{H_n H_n^{(2)}}{n^4} - \sum_{n=1}^{\infty}\frac{H_n^{(2)}H_n^{(3)}}{n^2} - 2\sum_{n=1}^{\infty}\frac{(H_n^{(2)})^2}{n^3}. \tag{6.208}$$

For the right-hand side of (6.207), using that $\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{H_k}{n^3k^3}\right) = \sum_{k=1}^{\infty}\left(\sum_{n=k}^{\infty}\frac{H_k}{n^3k^3}\right)$ $= \sum_{k=1}^{\infty}\frac{H_k}{k^3}\left(\frac{1}{k^3} + \zeta(3) - H_k^{(3)}\right)$, we write

$$4\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{H_k}{n^3k^3}\right) + \sum_{n=1}^{\infty}\frac{\left(H_n^{(2)}\right)^2}{n^3} - 5\sum_{n=1}^{\infty}\frac{H_n^{(4)}}{n^3}$$

$$= 4\sum_{n=1}^{\infty}\frac{H_n}{n^6} + 4\zeta(3)\sum_{n=1}^{\infty}\frac{H_n}{n^3} - 4\sum_{n=1}^{\infty}\frac{H_nH_n^{(3)}}{n^3} + \sum_{n=1}^{\infty}\frac{\left(H_n^{(2)}\right)^2}{n^3} - 5\sum_{n=1}^{\infty}\frac{H_n^{(4)}}{n^3}$$

{make use of the results in (3.45), the cases $n = 3, 6,$ and (6.79)}

$$= 101\zeta(7) - 54\zeta(2)\zeta(5) - 4\zeta(3)\zeta(4) - 4\sum_{n=1}^{\infty}\frac{H_nH_n^{(3)}}{n^3} + \sum_{n=1}^{\infty}\frac{\left(H_n^{(2)}\right)^2}{n^3}. \tag{6.209}$$

If we combine the results in (6.208) and (6.209) and use the identities in (4.44), (4.45), and (4.48) to express the series $\sum_{n=1}^{\infty}\frac{H_nH_n^{(3)}}{n^3}$, $\sum_{n=1}^{\infty}\frac{H_n^{(2)}H_n^{(3)}}{n^2}$ and $\sum_{n=1}^{\infty}\frac{\left(H_n^{(2)}\right)^2}{n^3}$ in terms of series $\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^4}$, we conclude that

$$\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4} = 2\zeta(2)\zeta(5) + \frac{3}{4}\zeta(3)\zeta(4) - \frac{51}{16}\zeta(7),$$

and the first solution is complete.

For a second solution, the strategy to get the value of the harmonic series relies upon the use of the identity in (4.18), where if we multiply both sides of the relation by $1/n^3$, and then consider the sum from $n = 1$ to ∞, we get

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n-1} \frac{H_k^2}{n^3(n-k)^2} \right)$$
$$= \sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^3} - 4 \sum_{n=1}^{\infty} \frac{H_n H_n^{(3)}}{n^3} - \sum_{n=1}^{\infty} \frac{(H_n^{(2)})^2}{n^3} + 2 \sum_{n=1}^{\infty} \frac{1}{n^3} \left(\sum_{i=1}^{n} \frac{H_i}{i^3} \right)$$
$$+ 2 \sum_{n=1}^{\infty} \frac{H_n}{n^3} \left(\sum_{i=1}^{n} \frac{H_i}{i^2} \right). \tag{6.210}$$

For the left-hand side of (6.210) we change the order of summation, and then we have

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n-1} \frac{H_k^2}{n^3(n-k)^2} \right) = \sum_{k=1}^{\infty} \left(\sum_{n=k+1}^{\infty} \frac{H_k^2}{n^3(n-k)^2} \right)$$

$\{$reindex the inner series and start from $n = 1$ to $\infty\}$

$$= \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{H_k^2}{(n+k)^3 n^2} \right)$$

$$= \sum_{k=1}^{\infty} \frac{H_k^2}{k^3} \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right) - 3 \sum_{k=1}^{\infty} \frac{H_k^2}{k^3} \left(\sum_{n=1}^{\infty} \frac{1}{n(n+k)} \right) + \sum_{k=1}^{\infty} \frac{H_k^2}{k^2} \left(\sum_{n=1}^{\infty} \frac{1}{(n+k)^3} \right)$$
$$+ 2 \sum_{k=1}^{\infty} \frac{H_k^2}{k^3} \left(\sum_{n=1}^{\infty} \frac{1}{(n+k)^2} \right)$$

$$\left\{ \text{use the result in (4.30) and the fact that } \sum_{n=1}^{\infty} \frac{1}{n(n+k)} = \frac{H_k}{k} \right\}$$

$$= \frac{7}{2}\zeta(2)\zeta(5) - \frac{5}{2}\zeta(3)\zeta(4) - 3 \sum_{k=1}^{\infty} \frac{H_k^3}{k^4} + \sum_{k=1}^{\infty} \frac{H_k^2}{k^2} (\zeta(3) - H_k^{(3)})$$

$$+2\sum_{k=1}^{\infty}\frac{H_k^2}{k^3}(\zeta(2)-H_k^{(2)})=\frac{7}{2}\zeta(2)\zeta(5)-\frac{5}{2}\zeta(3)\zeta(4)-3\sum_{k=1}^{\infty}\frac{H_k^3}{k^4}+\zeta(3)\sum_{k=1}^{\infty}\frac{H_k^2}{k^2}$$

$$-\sum_{k=1}^{\infty}\frac{H_k^2H_k^{(3)}}{k^2}+2\zeta(2)\sum_{k=1}^{\infty}\frac{H_k^2}{k^3}-2\sum_{k=1}^{\infty}\frac{H_k^2H_k^{(2)}}{k^3}$$

{the values of the 2nd, 4th, and 5th series are given in (4.29), (4.30), and (4.53)}

$$=14\zeta(7)+\frac{29}{2}\zeta(2)\zeta(5)-\frac{89}{4}\zeta(3)\zeta(4)-3\sum_{k=1}^{\infty}\frac{H_k^3}{k^4}-\sum_{k=1}^{\infty}\frac{H_k^2H_k^{(3)}}{k^2}$$

{consider the result in (4.50) to express $\displaystyle\sum_{k=1}^{\infty}\frac{H_k^3}{k^4}$ using $\displaystyle\sum_{k=1}^{\infty}\frac{H_kH_k^{(2)}}{k^4}$ and then}

{combine the results in (4.46) and (4.53) to express $\displaystyle\sum_{k=1}^{\infty}\frac{H_k^2H_k^{(3)}}{k^2}$ by $\displaystyle\sum_{k=1}^{\infty}\frac{H_kH_k^{(2)}}{k^4}$,}

$$\left\{\text{using that }\sum_{k=1}^{\infty}\frac{H_k^2H_k^{(3)}}{k^2}+3\sum_{k=1}^{\infty}\frac{H_kH_k^{(2)}}{k^4}=11\zeta(7)+\frac{3}{2}\zeta(2)\zeta(5)-\frac{15}{4}\zeta(3)\zeta(4)\right\}$$

$$=\frac{53}{2}\zeta(3)\zeta(4)+25\zeta(2)\zeta(5)-69\zeta(7)-6\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^4}. \tag{6.211}$$

Further, for the right-hand side of (6.210), we have

$$\sum_{n=1}^{\infty}\frac{H_n^2H_n^{(2)}}{n^3}-4\sum_{n=1}^{\infty}\frac{H_nH_n^{(3)}}{n^3}-\sum_{n=1}^{\infty}\frac{(H_n^{(2)})^2}{n^3}+2\sum_{n=1}^{\infty}\frac{1}{n^3}\left(\sum_{i=1}^{n}\frac{H_i}{i^3}\right)$$

$$+2\sum_{n=1}^{\infty}\frac{H_n}{n^3}\left(\sum_{i=1}^{n}\frac{H_i}{i^2}\right)$$

{the values of the first and the last series are given in (4.53) and (4.55),}

{and for the penultimate series change the summation order}

$$=\frac{65}{2}\zeta(3)\zeta(4)-13\zeta(2)\zeta(5)-15\zeta(7)-4\sum_{n=1}^{\infty}\frac{H_nH_n^{(3)}}{n^3}-\sum_{n=1}^{\infty}\frac{(H_n^{(2)})^2}{n^3}$$

$$+2\sum_{i=1}^{\infty}\frac{H_i}{i^3}\left(\sum_{n=i}^{\infty}\frac{1}{n^3}\right)=\frac{65}{2}\zeta(3)\zeta(4)-13\zeta(2)\zeta(5)-15\zeta(7)$$

$$-4\sum_{n=1}^{\infty}\frac{H_nH_n^{(3)}}{n^3}-\sum_{n=1}^{\infty}\frac{(H_n^{(2)})^2}{n^3}+2\sum_{i=1}^{\infty}\frac{H_i}{i^3}\left(\zeta(3)-H_i^{(3)}+\frac{1}{i^3}\right)$$

$$=\frac{65}{2}\zeta(3)\zeta(4)-13\zeta(2)\zeta(5)-15\zeta(7)-6\sum_{n=1}^{\infty}\frac{H_nH_n^{(3)}}{n^3}-\sum_{n=1}^{\infty}\frac{(H_n^{(2)})^2}{n^3}$$

$$+2\zeta(3)\sum_{n=1}^{\infty}\frac{H_n}{n^3}+2\sum_{n=1}^{\infty}\frac{H_n}{n^6}$$

{make use of the classical Euler sum in (3.45), the cases $n=3,6$,}

$$\left\{\text{consider the result in (4.44) to express } \sum_{n=1}^{\infty}\frac{H_nH_n^{(3)}}{n^3} \text{ using } \sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^4},\right\}$$

$$\left\{\text{and use the result in (4.48) to express } \sum_{n=1}^{\infty}\frac{(H_n^{(2)})^2}{n^3} \text{ in terms of } \sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^4}\right\}$$

$$=\frac{29}{2}\zeta(3)\zeta(4)-7\zeta(2)\zeta(5)-18\zeta(7)+10\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^4}. \tag{6.212}$$

Finally, by collecting the results from (6.211) and (6.212) in (6.210), we obtain that

$$\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^4}=2\zeta(2)\zeta(5)+\frac{3}{4}\zeta(3)\zeta(4)-\frac{51}{16}\zeta(7),$$

and the second solution is complete.

Now that the present harmonic series of weight 7 has been calculated, we're ready to jump in the next section and start deriving the remaining harmonic series of weight 7.

The solutions provided also answer the proposed challenging question.

6.43 Plenty of Challenging Harmonic Series of Weight 7 Obtained by Combining the Previous Harmonic Series of Weight 7 with Various Harmonic Series Identities (Derivations by Series Manipulations Only)

Solution Let's prepare for a cascade of derivations of the harmonic series of weight 7! (surely, this is possible now thanks to the work in the previous sections). Essentially, in this section the evaluations of most series will go pretty easily and fast as you'll see below. The solutions provided will also answer the proposed *challenging question*.

For the point i), use the identity in (4.46) which we combine with the values of the series $\sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^3}$ and $\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4}$ given in (4.53) and (4.58) to get

$$\sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(3)}}{n^2} = \frac{329}{16}\zeta(7) - \frac{9}{2}\zeta(2)\zeta(5) - 6\zeta(3)\zeta(4).$$

As regards the point ii), once we know the value of the series $\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4}$, which is given in (4.58), and plug it in the identity in (4.41), we obtain immediately the desired result

$$\sum_{n=1}^{\infty} \frac{H_n H_n^{(4)}}{n^2} = \frac{9}{2}\zeta(2)\zeta(5) - \frac{3}{2}\zeta(3)\zeta(4) - \frac{51}{16}\zeta(7).$$

Further, for the point iii), considering the identity in (4.50) and the value of the series $\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4}$ given in (4.58), we have

$$\sum_{n=1}^{\infty} \frac{H_n^3}{n^4} = \frac{231}{16}\zeta(7) + 2\zeta(2)\zeta(5) - \frac{51}{4}\zeta(3)\zeta(4).$$

Next, for the point iv), since we have just calculated the value of the series $\sum_{n=1}^{\infty} \frac{H_n^3}{n^4}$, we combine it with the identity in (4.49) which immediately gives

$$\sum_{n=1}^{\infty} \frac{H_n^4}{n^3} = \frac{185}{8}\zeta(7) + 5\zeta(2)\zeta(5) - \frac{43}{2}\zeta(3)\zeta(4).$$

We will derive the value of the stated series in v) by combining the value of the series $\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4}$, which is given in (4.58), with the identity in (4.45) that gives

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)} H_n^{(3)}}{n^2} = \frac{131}{16}\zeta(7) - \frac{5}{2}\zeta(2)\zeta(5) - \frac{3}{2}\zeta(3)\zeta(4).$$

The calculation of the series in vi) is achieved by combining the value of the series $\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4}$ given in (4.58) with the identity in (4.44) from which we obtain

$$\sum_{n=1}^{\infty} \frac{H_n H_n^{(3)}}{n^3} = \frac{83}{8}\zeta(7) - \frac{11}{2}\zeta(2)\zeta(5) + \frac{1}{4}\zeta(3)\zeta(4).$$

For the series in vii), let's recall and use the identity in (4.48) that if we combine with the value of the series $\sum_{n=1}^{\infty} \frac{H_n H_n^{(2)}}{n^4}$ given in (4.58), we get immediately the desired value,

$$\sum_{n=1}^{\infty} \frac{(H_n^{(2)})^2}{n^3} = 5\zeta(2)\zeta(5) + \frac{19}{2}\zeta(3)\zeta(4) - \frac{155}{8}\zeta(7).$$

To calculate the series in $viii$), we use the identity in (4.20) where if we multiply both sides by $1/n^3$ and then consider the summation from $n = 1$ to ∞, we get

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n-1} \frac{H_k^3}{n^3(n-k)} \right) = \sum_{n=1}^{\infty} \frac{H_n^4}{n^3} - 3\sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^3} - \frac{1}{4}\sum_{n=1}^{\infty} \frac{\left(H_n^{(2)}\right)^2}{n^3}$$
$$-\frac{1}{4}\sum_{n=1}^{\infty} \frac{H_n^{(4)}}{n^3} + \sum_{n=1}^{\infty} \frac{1}{n^3}\left(\sum_{i=1}^{n} \frac{H_i}{i^3}\right) - \frac{3}{2}\sum_{n=1}^{\infty} \frac{1}{n^3}\left(\sum_{i=1}^{n} \frac{H_i^2}{i^2}\right) + 3\sum_{n=1}^{\infty} \frac{H_n}{n^3}\left(\sum_{k=1}^{n} \frac{H_k}{k^2}\right). \tag{6.213}$$

Considering the left-hand side of (6.213), we write

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n-1} \frac{H_k^3}{n^3(n-k)} \right) = \sum_{k=1}^{\infty} \left(\sum_{n=k+1}^{\infty} \frac{H_k^3}{n^3(n-k)} \right) = \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{H_k^3}{n(n+k)^3} \right)$$
$$= \sum_{k=1}^{\infty} \frac{H_k^3}{k^2}\left(\sum_{n=1}^{\infty} \frac{1}{n(n+k)}\right) - \sum_{k=1}^{\infty} \frac{H_k^3}{k}\left(\sum_{n=1}^{\infty} \frac{1}{(n+k)^3}\right) - \sum_{k=1}^{\infty} \frac{H_k^3}{k^2}\left(\sum_{n=1}^{\infty} \frac{1}{(n+k)^2}\right)$$

$$= \sum_{k=1}^{\infty} \frac{H_k^4}{k^3} - \sum_{k=1}^{\infty} \frac{H_k^3}{k} \left(\zeta(3) - H_k^{(3)}\right) - \sum_{k=1}^{\infty} \frac{H_k^3}{k^2} \left(\zeta(2) - H_k^{(2)}\right)$$

$$= \sum_{k=1}^{\infty} \frac{H_k^4}{k^3} - \sum_{k=1}^{\infty} \frac{H_k^3}{k} \left(\zeta(3) - H_k^{(3)}\right) - \zeta(2) \sum_{k=1}^{\infty} \frac{H_k^3}{k^2} + \sum_{k=1}^{\infty} \frac{H_k^3 H_k^{(2)}}{k^2}. \tag{6.214}$$

For the second series in (6.214), we apply Abel's summation with $a_k = 1/k$ and $b_k = H_k^3(\zeta(3) - H_k^{(3)})$, and then we have

$$\sum_{k=1}^{\infty} \frac{H_k^3}{k} \left(\zeta(3) - H_k^{(3)}\right) = \sum_{k=1}^{\infty} H_k \left(H_k^3 \left(\zeta(3) - H_k^{(3)}\right) - H_{k+1}^3 \left(\zeta(3) - H_{k+1}^{(3)}\right)\right)$$

$$= \sum_{k=1}^{\infty} \left(H_{k+1} - \frac{1}{k+1}\right) \left(\left(H_{k+1} - \frac{1}{k+1}\right)^3 \left(\zeta(3) - H_{k+1}^{(3)} + \frac{1}{(k+1)^3}\right) - H_{k+1}^3 \left(\zeta(3) - H_{k+1}^{(3)}\right)\right)$$

{reindex the series, expand it, and carefully rearrange it}

$$= \sum_{k=1}^{\infty} \frac{1}{k^7} + \zeta(3) \sum_{k=1}^{\infty} \frac{1}{k^4} - 4 \sum_{k=1}^{\infty} \frac{H_k}{k^6} - 4\zeta(3) \sum_{k=1}^{\infty} \frac{H_k}{k^3} - \sum_{k=1}^{\infty} \frac{H_k^{(3)}}{k^4} + 6\zeta(3) \sum_{k=1}^{\infty} \frac{H_k^2}{k^2}$$

$$+6 \sum_{k=1}^{\infty} \frac{H_k^2}{k^5} - 4 \sum_{k=1}^{\infty} \frac{H_k^3}{k^4} + \sum_{k=1}^{\infty} \frac{H_k^4}{k^3} - 6 \sum_{k=1}^{\infty} \frac{H_k^2 H_k^{(3)}}{k^2} + 4 \sum_{k=1}^{\infty} \frac{H_k H_k^{(3)}}{k^3}$$

$$-3 \sum_{k=1}^{\infty} \frac{H_k^3}{k} \left(\zeta(3) - H_k^{(3)}\right)$$

{use the results in (3.45), the cases $n = 3, 6$, (6.80),}

{(4.29), (4.32), (4.61), (4.62), (4.59), and (4.64)}

$$= 10\zeta(2)\zeta(5) + 77\zeta(3)\zeta(4) - \frac{227}{2}\zeta(7) - 3 \sum_{k=1}^{\infty} \frac{H_k^3}{k} \left(\zeta(3) - H_k^{(3)}\right),$$

whence we get that

$$\sum_{k=1}^{\infty} \frac{H_k^3}{k} \left(\zeta(3) - H_k^{(3)}\right) = \frac{1}{4}(10\zeta(2)\zeta(5) + 77\zeta(3)\zeta(4) - \frac{227}{2}\zeta(7)). \tag{6.215}$$

Then, if we plug the result from (6.215) in (6.214), we obtain that

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n-1} \frac{H_k^3}{n^3(n-k)}\right) = \sum_{k=1}^{\infty} \frac{H_k^4}{k^3} - \zeta(2) \sum_{k=1}^{\infty} \frac{H_k^3}{k^2} + \sum_{k=1}^{\infty} \frac{H_k^3 H_k^{(2)}}{k^2}$$

$$-\frac{1}{4}(10\zeta(2)\zeta(5) + 77\zeta(3)\zeta(4) - \frac{227}{2}\zeta(7))$$

{the values of the first two series are given in (4.62) and (4.34)}

$$= \sum_{k=1}^{\infty} \frac{H_k^3 H_k^{(2)}}{k^2} + \frac{103}{2}\zeta(7) - \frac{15}{2}\zeta(2)\zeta(5) - \frac{173}{4}\zeta(3)\zeta(4). \tag{6.216}$$

On the other hand, considering the right-hand side of (6.213), we write that

$$\sum_{n=1}^{\infty} \frac{H_n^4}{n^3} - 3\sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^3} - \frac{1}{4}\sum_{n=1}^{\infty} \frac{\left(H_n^{(2)}\right)^2}{n^3} - \frac{1}{4}\sum_{n=1}^{\infty} \frac{H_n^{(4)}}{n^3} + 3\sum_{n=1}^{\infty} \frac{H_n}{n^3}\left(\sum_{k=1}^{n} \frac{H_k}{k^2}\right)$$

$$+\sum_{n=1}^{\infty} \frac{1}{n^3}\left(\sum_{i=1}^{n} \frac{H_i}{i^3}\right) - \frac{3}{2}\sum_{n=1}^{\infty} \frac{1}{n^3}\left(\sum_{i=1}^{n} \frac{H_i^2}{i^2}\right)$$

{the values of the first five series are given in (4.62), (4.53), (4.65), (6.79), (4.55)}

$$= \frac{1319}{32}\zeta(7) - \frac{37}{4}\zeta(2)\zeta(5) - \frac{145}{8}\zeta(3)\zeta(4) + \sum_{n=1}^{\infty} \frac{1}{n^3}\left(\sum_{i=1}^{n} \frac{H_i}{i^3}\right) - \frac{3}{2}\sum_{n=1}^{\infty} \frac{1}{n^3}\left(\sum_{i=1}^{n} \frac{H_i^2}{i^2}\right)$$

{reverse the summation order in the two remaining series}

$$= \frac{1319}{32}\zeta(7) - \frac{37}{4}\zeta(2)\zeta(5) - \frac{145}{8}\zeta(3)\zeta(4) + \sum_{i=1}^{\infty} \frac{H_i}{i^3}\left(\sum_{n=i}^{\infty} \frac{1}{n^3}\right) - \frac{3}{2}\sum_{i=1}^{\infty} \frac{H_i^2}{i^2}\left(\sum_{n=i}^{\infty} \frac{1}{n^3}\right)$$

$$= \frac{1319}{32}\zeta(7) - \frac{37}{4}\zeta(2)\zeta(5) - \frac{145}{8}\zeta(3)\zeta(4) + \sum_{i=1}^{\infty} \frac{H_i}{i^3}\left(\zeta(3) - H_i^{(3)} + \frac{1}{i^3}\right)$$

$$-\frac{3}{2}\sum_{i=1}^{\infty}\frac{H_i^2}{i^2}\left(\zeta(3)-H_i^{(3)}+\frac{1}{i^3}\right)=\frac{1319}{32}\zeta(7)-\frac{37}{4}\zeta(2)\zeta(5)-\frac{145}{8}\zeta(3)\zeta(4)$$

$$+\zeta(3)\sum_{i=1}^{\infty}\frac{H_i}{i^3}+\sum_{i=1}^{\infty}\frac{H_i}{i^6}-\frac{3}{2}\zeta(3)\sum_{i=1}^{\infty}\frac{H_i^2}{i^2}-\frac{3}{2}\sum_{i=1}^{\infty}\frac{H_i^2}{i^5}-\sum_{i=1}^{\infty}\frac{H_iH_i^{(3)}}{i^3}+\frac{3}{2}\sum_{i=1}^{\infty}\frac{H_i^2H_i^{(3)}}{i^2}$$

{make use of the result in (3.45), the cases $n=3, 6$, (4.29), (4.32), (4.64), and (4.59)}

$$=\frac{907}{16}\zeta(7)-10\zeta(2)\zeta(5)-\frac{119}{4}\zeta(3)\zeta(4). \tag{6.217}$$

Finally, plugging the results from (6.216) and (6.217) in (6.213), we conclude that

$$\sum_{n=1}^{\infty}\frac{H_n^3H_n^{(2)}}{n^2}=\frac{83}{16}\zeta(7)-\frac{5}{2}\zeta(2)\zeta(5)+\frac{27}{2}\zeta(3)\zeta(4).$$

To obtain the value of the series in ix), we consider the identity in (4.28), the first equality, where if we multiply both sides by $1/n$ and then consider the summation from $n=1$ to ∞, we get

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{H_k^4-6H_k^2H_k^{(2)}+8H_kH_k^{(3)}+3(H_k^{(2)})^2-6H_k^{(4)}}{(k+1)(k+n+1)n}\right)$$
$$=\frac{1}{5}\sum_{n=1}^{\infty}\frac{H_n^5}{n^2}+2\sum_{n=1}^{\infty}\frac{H_n^3H_n^{(2)}}{n^2}+3\sum_{n=1}^{\infty}\frac{H_n(H_n^{(2)})^2}{n^2}+4\sum_{n=1}^{\infty}\frac{H_n^2H_n^{(3)}}{n^2}$$
$$+4\sum_{n=1}^{\infty}\frac{H_n^{(2)}H_n^{(3)}}{n^2}+6\sum_{n=1}^{\infty}\frac{H_nH_n^{(4)}}{n^2}+\frac{24}{5}\sum_{n=1}^{\infty}\frac{H_n^{(5)}}{n^2}. \tag{6.218}$$

For the left-hand side of (6.43), we have

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{H_k^4-6H_k^2H_k^{(2)}+8H_kH_k^{(3)}+3(H_k^{(2)})^2-6H_k^{(4)}}{(k+1)(k+n+1)n}\right)$$
$$=\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{H_k^4-6H_k^2H_k^{(2)}+8H_kH_k^{(3)}+3(H_k^{(2)})^2-6H_k^{(4)}}{(k+1)(k+n+1)n}\right)$$
$$=\sum_{k=1}^{\infty}\frac{(H_k^4-6H_k^2H_k^{(2)}+8H_kH_k^{(3)}+3(H_k^{(2)})^2-6H_k^{(4)})H_{k+1}}{(k+1)^2}$$

$$=\sum_{k=1}^{\infty}\frac{((H_{k+1}-1/(k+1))^4-6(H_{k+1}-1/(k+1))^2(H_{k+1}^{(2)}-1/(k+1)^2))H_{k+1}}{(k+1)^2}$$

$$+\sum_{k=1}^{\infty}\frac{(8(H_{k+1}-1/(k+1))(H_{k+1}^{(3)}-1/(k+1)^3)+3(H_{k+1}^{(2)}-1/(k+1)^2)^2)H_{k+1}}{(k+1)^2}$$

$$-6\sum_{k=1}^{\infty}\frac{(H_{k+1}^{(4)}-1/(k+1)^4)H_{k+1}}{(k+1)^2}$$

{reindex the series and expand them}

$$=24\sum_{k=1}^{\infty}\frac{H_k}{k^6}-24\sum_{k=1}^{\infty}\frac{H_k^2}{k^5}+12\sum_{k=1}^{\infty}\frac{H_k^3}{k^4}-4\sum_{k=1}^{\infty}\frac{H_k^4}{k^3}+\sum_{k=1}^{\infty}\frac{H_k^5}{k^2}-12\sum_{k=1}^{\infty}\frac{H_kH_k^{(2)}}{k^4}$$

$$+12\sum_{k=1}^{\infty}\frac{H_k^2H_k^{(2)}}{k^3}-6\sum_{k=1}^{\infty}\frac{H_k^3H_k^{(2)}}{k^2}+3\sum_{k=1}^{\infty}\frac{H_k(H_k^{(2)})^2}{k^2}-8\sum_{k=1}^{\infty}\frac{H_kH_k^{(3)}}{k^3}$$

$$+8\sum_{k=1}^{\infty}\frac{H_k^2H_k^{(3)}}{k^2}-6\sum_{k=1}^{\infty}\frac{H_kH_k^{(4)}}{k^2}$$

{use the values of the series given in (3.45), the case $n=6$,}

{(4.32), (4.61), (4.62), (4.58), (4.53), (4.66), (4.64), (4.59), and (4.60)}

$$=\frac{113}{2}\zeta(7)-48\zeta(2)\zeta(5)-48\zeta(3)\zeta(4)+\sum_{k=1}^{\infty}\frac{H_k^5}{k^2}+3\sum_{k=1}^{\infty}\frac{H_k(H_k^{(2)})^2}{k^2}. \tag{6.219}$$

For the right-hand side of (6.43), we have

$$\frac{1}{5}\sum_{n=1}^{\infty}\frac{H_n^5}{n^2}+2\sum_{n=1}^{\infty}\frac{H_n^3H_n^{(2)}}{n^2}+3\sum_{n=1}^{\infty}\frac{H_n(H_n^{(2)})^2}{n^2}+4\sum_{n=1}^{\infty}\frac{H_n^2H_n^{(3)}}{n^2}+4\sum_{n=1}^{\infty}\frac{H_n^{(2)}H_n^{(3)}}{n^2}$$

$$+6\sum_{n=1}^{\infty}\frac{H_nH_n^{(4)}}{n^2}+\frac{24}{5}\sum_{n=1}^{\infty}\frac{H_n^{(5)}}{n^2}$$

{use the values of the series given in (4.66), (4.59), (4.63), (4.60), and (6.76)}

$$=\frac{3181}{20}\zeta(7)-\frac{126}{5}\zeta(2)\zeta(5)-\frac{108}{5}\zeta(3)\zeta(4)+\frac{1}{5}\sum_{n=1}^{\infty}\frac{H_n^5}{n^2}+3\sum_{n=1}^{\infty}\frac{H_n(H_n^{(2)})^2}{n^2}. \tag{6.220}$$

Lastly, if we plug the results from (6.219) and (6.220) in (6.43), we conclude that

$$\sum_{n=1}^{\infty}\frac{H_n^5}{n^2}=\frac{2051}{16}\zeta(7)+\frac{57}{2}\zeta(2)\zeta(5)+33\zeta(3)\zeta(4).$$

Finally, to get the value of the series in x), we make use of the first equality of the identity in (4.27), where if we multiply both sides by H_n/n and then consider the summation from $n=1$ to ∞, we get

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{(H_k^3-3H_kH_k^{(2)}+2H_k^{(3)})H_n}{(k+1)(k+n+1)n}\right)$$

$$=\frac{1}{4}\sum_{n=1}^{\infty}\frac{H_n^5}{n^2}+\frac{3}{2}\sum_{n=1}^{\infty}\frac{H_n^3H_n^{(2)}}{n^2}+2\sum_{n=1}^{\infty}\frac{H_n^2H_n^{(3)}}{n^2}+\frac{3}{4}\sum_{n=1}^{\infty}\frac{H_n(H_n^{(2)})^2}{n^2}+\frac{3}{2}\sum_{n=1}^{\infty}\frac{H_nH_n^{(4)}}{n^2}. \tag{6.221}$$

For the left-hand side of the equality in (6.221), we write

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{(H_k^3-3H_kH_k^{(2)}+2H_k^{(3)})H_n}{(k+1)(k+n+1)n}\right)=\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{(H_k^3-3H_kH_k^{(2)}+2H_k^{(3)})H_n}{(k+1)(k+n+1)n}\right)$$

$$=\sum_{k=1}^{\infty}\frac{H_k^3-3H_kH_k^{(2)}+2H_k^{(3)}}{k+1}\left(\sum_{n=0}^{\infty}\frac{H_n+1/(n+1)}{(n+1)(n+k+2)}\right)$$

$$=\underbrace{\sum_{k=1}^{\infty}\frac{H_k^3-3H_kH_k^{(2)}+2H_k^{(3)}}{(k+1)(k+2)}}_{S_1}$$

$$+\underbrace{\sum_{k=1}^{\infty}\frac{H_k^3-3H_kH_k^{(2)}+2H_k^{(3)}}{k+1}\left(\sum_{n=1}^{\infty}\frac{H_n}{(n+1)(n+k+2)}\right)}_{S_2}$$

$$+\underbrace{\sum_{k=1}^{\infty}\frac{H_k^3-3H_kH_k^{(2)}+2H_k^{(3)}}{k+1}\left(\sum_{n=1}^{\infty}\frac{1}{(n+1)^2(n+k+2)}\right)}_{S_3}. \tag{6.222}$$

The first series in (6.222) is straightforward in view of the identity in (4.27), if we set $n=1$,

$$S_1 = \sum_{k=1}^{\infty} \frac{H_k^3 - 3H_k H_k^{(2)} + 2H_k^{(3)}}{(k+1)(k+2)} = 6. \tag{6.223}$$

For the second series in (6.222), we write

$$S_2 = \sum_{k=1}^{\infty} \frac{H_k^3 - 3H_k H_k^{(2)} + 2H_k^{(3)}}{k+1} \left(\sum_{n=1}^{\infty} \frac{H_n}{(n+1)(n+k+2)} \right)$$

{make use of the identity in (4.21), the case $m = 1$}

$$= \sum_{k=1}^{\infty} \left(\frac{(H_{k+1} - \frac{1}{k+1})^3 - 3(H_{k+1} - \frac{1}{k+1})(H_{k+1}^{(2)} - \frac{1}{(k+1)^2}) + 2(H_{k+1}^{(3)} - \frac{1}{(k+1)^3})}{k+1} \right.$$

$$\left. \cdot \frac{H_{k+1}^2 + H_{k+1}^{(2)}}{2(k+1)} \right) = -3\sum_{k=1}^{\infty} \frac{H_k^{(2)}}{k^5} - 3\sum_{k=1}^{\infty} \frac{H_k^2}{k^5} + 3\sum_{k=1}^{\infty} \frac{H_k^3}{k^4} - \frac{3}{2}\sum_{k=1}^{\infty} \frac{H_k^4}{k^3} + 3\sum_{k=1}^{\infty} \frac{H_k H_k^{(2)}}{k^4}$$

$$- \sum_{k=1}^{\infty} \frac{H_k^3 H_k^{(2)}}{k^2} + \frac{3}{2}\sum_{k=1}^{\infty} \frac{(H_k^{(2)})^2}{k^3} + \sum_{k=1}^{\infty} \frac{H_k^{(2)} H_k^{(3)}}{k^2} + \sum_{k=1}^{\infty} \frac{H_k^2 H_k^{(3)}}{k^2}$$

$$+ \frac{1}{2}\sum_{k=1}^{\infty} \frac{H_k^5}{k^2} - \frac{3}{2}\sum_{k=1}^{\infty} \frac{H_k (H_k^{(2)})^2}{k^2}$$

{use the values of the series in (6.75), (4.32), and (4.61),}

{(4.62), (4.58), (4.66), (4.65), (4.63), (4.59), and (4.67)}

$$= \frac{2229}{32}\zeta(7) + \frac{39}{4}\zeta(2)\zeta(5) + \frac{15}{2}\zeta(3)\zeta(4) - \frac{3}{2}\sum_{k=1}^{\infty} \frac{H_k (H_k^{(2)})^2}{k^2}. \tag{6.224}$$

Lastly, for the series S_3 in (6.222), we get

$$S_3 = \sum_{k=1}^{\infty} \frac{H_k^3 - 3H_k H_k^{(2)} + 2H_k^{(3)}}{k+1} \left(\sum_{n=1}^{\infty} \frac{1}{(n+1)^2(n+k+2)} \right)$$

$$= \sum_{k=1}^{\infty} \frac{H_k^3 - 3H_k H_k^{(2)} + 2H_k^{(3)}}{k+1} \left(\frac{\zeta(2)}{k+1} - \frac{1}{k+2} - \frac{H_{k+1}}{(k+1)^2} \right)$$

$$= \zeta(2)\sum_{k=1}^{\infty}\frac{H_k^3 - 3H_kH_k^{(2)} + 2H_k^{(3)}}{(k+1)^2} - \sum_{k=1}^{\infty}\frac{H_k^3 - 3H_kH_k^{(2)} + 2H_k^{(3)}}{(k+1)(k+2)}$$

$$-\sum_{k=1}^{\infty}\frac{(H_k^3 - 3H_kH_k^{(2)} + 2H_k^{(3)})H_{k+1}}{(k+1)^3}$$

{use the results in (6.189), (6.223), and (6.190)}

$$= 6\zeta(2)\zeta(5) + 9\zeta(3)\zeta(4) - 12\zeta(7) - 6 - \sum_{k=1}^{\infty}\frac{H_k^4}{k^3} + 3\sum_{k=1}^{\infty}\frac{H_k^3}{k^4} - 3\sum_{k=1}^{\infty}\frac{H_kH_k^{(2)}}{k^4}$$

$$-2\sum_{k=1}^{\infty}\frac{H_kH_k^{(3)}}{k^3} + 3\sum_{k=1}^{\infty}\frac{H_k^2H_k^{(2)}}{k^3}$$

{the values of the series are given in (4.62), (4.61), (4.58), (4.64), and (4.53)}

$$= 6\zeta(2)\zeta(5) + 18\zeta(3)\zeta(4) - 24\zeta(7) - 6. \tag{6.225}$$

If we collect the values of the series S_1, S_2, and S_3 from (6.223), (6.224), and (6.225) in (6.222), we get

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{(H_k^3 - 3H_kH_k^{(2)} + 2H_k^{(3)})H_n}{(k+1)(k+n+1)n}\right) = \frac{63}{4}\zeta(2)\zeta(5) + \frac{51}{2}\zeta(3)\zeta(4) + \frac{1461}{32}\zeta(7)$$

$$-\frac{3}{2}\sum_{n=1}^{\infty}\frac{H_n(H_n^{(2)})^2}{n^2}. \tag{6.226}$$

On the other hand, considering the right-hand side of (6.221), we write

$$\frac{1}{4}\sum_{n=1}^{\infty}\frac{H_n^5}{n^2} + \frac{3}{2}\sum_{n=1}^{\infty}\frac{H_n^3H_n^{(2)}}{n^2} + 2\sum_{n=1}^{\infty}\frac{H_n^2H_n^{(3)}}{n^2} + \frac{3}{4}\sum_{n=1}^{\infty}\frac{H_n(H_n^{(2)})^2}{n^2} + \frac{3}{2}\sum_{n=1}^{\infty}\frac{H_nH_n^{(4)}}{n^2}$$

{use the values of the series given in (4.67), (4.66), (4.59), and (4.60)}

$$= \frac{4875}{64}\zeta(7) + \frac{9}{8}\zeta(2)\zeta(5) + \frac{57}{4}\zeta(3)\zeta(4) + \frac{3}{4}\sum_{n=1}^{\infty}\frac{H_n(H_n^{(2)})^2}{n^2}. \tag{6.227}$$

Collecting the results from (6.226) and (6.227) in (6.221), we conclude that

$$\sum_{n=1}^{\infty} \frac{H_n (H_n^{(2)})^2}{n^2} = 5\zeta(3)\zeta(4) + \frac{13}{2}\zeta(2)\zeta(5) - \frac{217}{16}\zeta(7),$$

and the solutions are complete.

The knowledge on the harmonic series opens the gates to a large panel of problems that involve the use of such series. For example, just remember the triple integral in the first chapter, in Sect. 1.34, the point ii), where we need to know how to handle with the resulting advanced harmonic series (of weight 7).

The interest in the study of the harmonic series has increased in the last years and further investigations and derivations of the harmonic series of weight ≥ 8 have been accomplished as seen in [51] and [52].

6.44 A Member of a Glamorous Series Family Containing the Harmonic Number and the Tail of the Riemann Zeta Function

Solution I submitted the present series to SSMA (School Science and Mathematics Association) and was published in May, 2016 issue, the problem **5406** (see [38]). Splitting the series directly is not advisable since we have a divergence issue because, for example, $\displaystyle\sum_{n=1}^{\infty} \frac{H_n}{n}$ clearly diverges. *Fine! Then, how would we like to act?* Well, we can simply apply Abel's summation.

For an elementary solution by series manipulations only, we start by applying Abel's summation (see (5.1)) with $a_n = \dfrac{H_n}{n}$ and $b_n = \zeta(3) - 1 - \dfrac{1}{2^3} - \cdots - \dfrac{1}{n^3}$, where we also use that $\displaystyle\sum_{k=1}^{n} \frac{H_k}{k} = \frac{1}{2}\left(H_n^2 + H_n^{(2)}\right)$, and then we have

$$\sum_{n=1}^{\infty} \frac{H_n}{n} \left(\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{n^3}\right)$$

$$= \underbrace{\lim_{n\to\infty} \frac{1}{2}\left(H_n^2 + H_n^{(2)}\right)\left(\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{(n+1)^3}\right)}_{0} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{H_n^2 + H_n^{(2)}}{(n+1)^3}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} \frac{H_n^2 + H_n^{(2)}}{(n+1)^3} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{(H_{n+1} - 1/(n+1))^2 + H_{n+1}^{(2)} - 1/(n+1)^2}{(n+1)^3}$$

$$\{\text{reindex the series and expand it}\}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n^2}{n^3} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^3} - \sum_{n=1}^{\infty}\frac{H_n}{n^4}$$

$$\{\text{use the series values given in (4.30), (6.67), and (3.45), the case } n = 4\}$$

$$= 2\zeta(2)\zeta(3) - \frac{7}{2}\zeta(5),$$

and the first solution is finalized.

For a second solution, to calculate the series we first attend a slightly different version of it, where if we use the generalization in (4.72), we get

$$\sum_{n=1}^{\infty}\frac{H_n}{n+1}\left(\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{n^3}\right) = \frac{1}{4}\lim_{\substack{x\to 0\\ y\to 0}}\frac{\partial^4}{\partial x^2\partial y^2}\mathrm{B}(x,y)$$

$$= 2(2\zeta(5) - \zeta(2)\zeta(3)), \tag{6.228}$$

and the calculation of the limit can be done either with *Mathematica* or manually. The double limit with the Beta function can also be approached with Euler sums like that

$$\sum_{n=1}^{\infty}\frac{H_n}{n+1}\left(\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{n^3}\right) = \frac{1}{4}\lim_{\substack{x\to 0\\ y\to 0}}\frac{\partial^4}{\partial x^2\partial y^2}\mathrm{B}(x,y)$$

$$\left\{\text{make use of the Beta function definition, } \mathrm{B}(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\mathrm{d}x\right\}$$

$$= \frac{1}{4}\int_0^1 \frac{\log^2(x)\log^2(1-x)}{x(1-x)}\mathrm{d}x = \frac{1}{4}\int_0^1 \frac{\log^2(x)\log^2(1-x)}{x}\mathrm{d}x$$

$$+\underbrace{\frac{1}{4}\int_0^1 \frac{\log^2(x)\log^2(1-x)}{1-x}\mathrm{d}x}_{\text{let } x = 1-y} = \frac{1}{2}\int_0^1 \frac{\log^2(x)\log^2(1-x)}{x}\mathrm{d}x$$

$$\left\{\text{use that } \sum_{n=1}^{\infty} x^{n+1}\frac{H_n}{n+1} = \frac{1}{2}\log^2(1-x)\right\}$$

$$= \int_0^1 \sum_{n=1}^{\infty} x^n \frac{H_n}{n+1}\log^2(x)\mathrm{d}x = \sum_{n=1}^{\infty}\frac{H_n}{n+1}\int_0^1 x^n\log^2(x)\mathrm{d}x$$

$$= 2\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^4} = 2\sum_{n=1}^{\infty}\frac{H_{n+1} - 1/(n+1)}{(n+1)^4} = 2\sum_{n=1}^{\infty}\frac{H_n - 1/n}{n^4}$$

$$= 2\sum_{n=1}^{\infty}\frac{H_n}{n^4} - 2\sum_{n=1}^{\infty}\frac{1}{n^5}$$

{make use of the Euler sum in (3.45), the case $n = 4$}

$$= 2(2\zeta(5) - \zeta(2)\zeta(3)).$$

On the other hand, the series in (6.228) can be written as

$$\sum_{n=1}^{\infty}\frac{H_n}{n+1}\left(\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{n^3}\right)$$

$$= \sum_{n=1}^{\infty}\frac{H_{n+1} - 1/(n+1)}{n+1}\left(\zeta(3) - H_{n+1}^{(3)} + \frac{1}{(n+1)^3}\right)$$

{reindex the series and carefully expand it}

$$= \sum_{n=1}^{\infty}\frac{H_n}{n}\left(\zeta(3) - H_n^{(3)}\right) + \sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^2} + \sum_{n=1}^{\infty}\frac{H_n}{n^4} - \zeta(3)\sum_{n=1}^{\infty}\frac{1}{n^2} - \sum_{n=1}^{\infty}\frac{1}{n^5}$$

{make use of the results in (6.68) and (3.45), the case $n = 4$}

$$= \sum_{n=1}^{\infty}\frac{H_n}{n}\left(\zeta(3) - H_n^{(3)}\right) + \frac{15}{2}\zeta(5) - 4\zeta(2)\zeta(3),$$

whence we obtain that

$$\sum_{n=1}^{\infty}\frac{H_n}{n}\left(\zeta(3) - H_n^{(3)}\right) = \sum_{n=1}^{\infty}\frac{H_n}{n+1}\left(\zeta(3) - H_n^{(3)}\right) - \frac{15}{2}\zeta(5) + 4\zeta(2)\zeta(3),$$

where if we finally make use of the result in (6.228), we conclude that

$$\sum_{n=1}^{\infty}\frac{H_n}{n}\left(\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{n^3}\right) = 2\zeta(2)\zeta(3) - \frac{7}{2}\zeta(5),$$

and the second solution is finalized.

The same strategies can also be applied for $\displaystyle\sum_{n=1}^{\infty}\frac{H_n}{n}\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right)$ that appeared in [21, p. 149], where the author reduced the series to the calculation of some logarithmic and dilogarithmic integrals (at some point in the past I communicated to the author an elementary solution similar to the first one above).

6.45 More Members of a Glamorous Series Family Containing the Harmonic Number and the Tail of the Riemann Zeta Function

Solution For both series we might like to employ the strategy in the first solution from the previous section, which is fast and only requires Abel's summation and elementary manipulations with series. If in the previous section the resulting harmonic series were of weight 5, now for the point $i)$ of the problem we get harmonic series of weight 6, and for the second point of the problem we obtain harmonic series of weight 7. Passing to the generalized series, if we consider the tail with $\zeta(k)$, $k \geq 2$, then the expected weight of the resulting series is $k+2$.

Let's make again Abel's summation (see (5.1)) our choice here, where if we set $a_n = \dfrac{H_n}{n}$ and $b_n = \zeta(4) - 1 - \dfrac{1}{2^4} - \cdots - \dfrac{1}{n^4}$, we have

$$\sum_{n=1}^{\infty} \frac{H_n}{n}\left(\zeta(4) - 1 - \frac{1}{2^4} - \cdots - \frac{1}{n^4}\right)$$

$$= \underbrace{\lim_{n\to\infty} \frac{1}{2}\left(H_n^2 + H_n^{(2)}\right)\left(\zeta(4) - 1 - \frac{1}{2^4} - \cdots - \frac{1}{(n+1)^4}\right)}_{0} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{H_n^2 + H_n^{(2)}}{(n+1)^4}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} \frac{H_n^2 + H_n^{(2)}}{(n+1)^4} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{(H_{n+1} - 1/(n+1))^2 + H_{n+1}^{(2)} - 1/(n+1)^2}{(n+1)^4}$$

$$\{\text{reindex the series and expand it}\}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} \frac{H_n^2}{n^4} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^4} - \sum_{n=1}^{\infty} \frac{H_n}{n^5}$$

$$\{\text{use the series values given in (4.31), (6.71), and (3.45), the case } n = 5\}$$

$$= \frac{5}{48}\zeta(6),$$

and the solution to the point $i)$ of the problem is complete.

For the point $ii)$ of the problem, using again Abel's summation (see (5.1)), where we set $a_n = \dfrac{H_n}{n}$ and $b_n = \zeta(5) - 1 - \dfrac{1}{2^5} - \cdots - \dfrac{1}{n^5}$, we get

$$\sum_{n=1}^{\infty} \frac{H_n}{n} \left(\zeta(5) - 1 - \frac{1}{2^5} - \cdots - \frac{1}{n^5} \right)$$

$$= \underbrace{\lim_{n\to\infty} \frac{1}{2} \left(H_n^2 + H_n^{(2)} \right) \left(\zeta(5) - 1 - \frac{1}{2^5} - \cdots - \frac{1}{(n+1)^5} \right)}_{0} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{H_n^2 + H_n^{(2)}}{(n+1)^5}$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \frac{H_n^2 + H_n^{(2)}}{(n+1)^5} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(H_{n+1} - 1/(n+1))^2 + H_{n+1}^{(2)} - 1/(n+1)^2}{(n+1)^5}$$

{reindex the series and expand it}

$$= \frac{1}{2} \sum_{n=1}^{\infty} \frac{H_n^2}{n^5} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^5} - \sum_{n=1}^{\infty} \frac{H_n}{n^6}$$

{use the series values given in (4.32), (6.75), and (3.45), the case $n = 6$}

$$= 3\zeta(2)\zeta(5) + \frac{3}{4}\zeta(3)\zeta(4) - 6\zeta(7),$$

and the solution to the point $ii)$ of the problem is complete.

For an alternative solution to both points, make use of the result in (4.72).

6.46 Two Series Generalizations with the Generalized Harmonic Numbers and the Tail of the Riemann Zeta Function

Solution The generalization from the point $i)$ in the present section also gives an alternative way of approaching the series with the tail of the Riemann zeta function like the ones in the last two sections. When we talk about the tail of the Riemann zeta function, we may also recall the series representation of the Polygamma function (see [58]), since the tail of the Riemann zeta function can be expressed in terms of the mentioned function.

For the point $i)$ of the problem, let's note and use that

$$\zeta(p) - 1 - \frac{1}{2^p} - \cdots - \frac{1}{n^p} = \sum_{k=1}^{\infty} \frac{1}{(n+k)^p} = \frac{(-1)^{p-1}}{(p-1)!} \sum_{k=1}^{\infty} \int_0^1 x^{k+n-1} \log^{p-1}(x) \mathrm{d}x, \tag{6.229}$$

where I considered the result in (1.2), and then we get

$$\sum_{n=1}^{\infty}\frac{H_n}{n+1}\left(\zeta(p)-1-\frac{1}{2^p}-\cdots-\frac{1}{n^p}\right)$$

$$=\frac{(-1)^{p-1}}{(p-1)!}\sum_{n=1}^{\infty}\frac{H_n}{n+1}\left(\sum_{k=1}^{\infty}\int_0^1 x^{k+n-1}\log^{p-1}(x)\mathrm{d}x\right)$$

$$\{\text{reverse the order of integration and summation}\}$$

$$=\frac{(-1)^{p-1}}{(p-1)!}\int_0^1\sum_{k=1}^{\infty}x^{k-2}\left(\sum_{n=1}^{\infty}\frac{H_n}{n+1}x^{n+1}\right)\log^{p-1}(x)\mathrm{d}x$$

$$\left\{\text{make use of the fact that }\sum_{n=1}^{\infty}\frac{H_n}{n+1}x^{n+1}=\frac{1}{2}\log^2(1-x)\right\}$$

$$=\frac{(-1)^{p-1}}{2(p-1)!}\int_0^1\frac{\log^{p-1}(x)\log^2(1-x)}{x(1-x)}\mathrm{d}x, \tag{6.230}$$

where we note the integral in (6.230) can be expressed in terms of Beta function which is defined as $\mathrm{B}(x,y)=\displaystyle\int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t$.

Hence, for $p\geq 2$, $p\in\mathbb{N}$, we have that

$$\sum_{n=1}^{\infty}\frac{H_n}{n+1}\left(\zeta(p)-1-\frac{1}{2^p}-\cdots-\frac{1}{n^p}\right)=\frac{(-1)^{p-1}}{2(p-1)!}\lim_{\substack{x\to 0\\ y\to 0}}\frac{\partial^{p+1}}{\partial x^{p-1}\partial y^2}\mathrm{B}(x,y),$$

and the solution to the point $i)$ of the problem is complete.

For the point $ii)$ of the problem we proceed in a similar style. First, if we combine the generating functions in (4.6), the case $m=2$, and (4.7), we have

$$\sum_{n=1}^{\infty}y^n(H_n^2-H_n^{(2)})=\frac{\log^2(1-y)}{1-y}. \tag{6.231}$$

Integrating both sides of (6.231), we get

$$\sum_{n=1}^{\infty}\frac{x^{n+1}}{n+1}(H_n^2-H_n^{(2)})=-\frac{1}{3}\log^3(1-x). \tag{6.232}$$

Now, starting as at the previous series, and using the result in (6.229), we write

$$\sum_{n=1}^{\infty} \frac{H_n^2 - H_n^{(2)}}{n+1} \left(\zeta(p) - 1 - \frac{1}{2^p} - \cdots - \frac{1}{n^p} \right)$$

$$= \frac{(-1)^{p-1}}{(p-1)!} \sum_{n=1}^{\infty} \frac{H_n^2 - H_n^{(2)}}{n+1} \left(\sum_{k=1}^{\infty} \int_0^1 x^{k+n-1} \log^{p-1}(x) \mathrm{d}x \right)$$

{reverse the order of integration and summation}

$$= \frac{(-1)^{p-1}}{(p-1)!} \int_0^1 \sum_{k=1}^{\infty} x^{k-2} \left(\sum_{n=1}^{\infty} x^{n+1} \frac{H_n^2 - H_n^{(2)}}{n+1} \right) \log^{p-1}(x) \mathrm{d}x$$

{make use of the result in (6.232)}

$$= \frac{(-1)^p}{3(p-1)!} \int_0^1 \sum_{k=1}^{\infty} x^{k-2} \log^{p-1}(x) \log^3(1-x) \mathrm{d}x$$

$$= \frac{(-1)^p}{3(p-1)!} \int_0^1 \frac{\log^{p-1}(x) \log^3(1-x)}{x(1-x)} \mathrm{d}x$$

$$= \frac{(-1)^p}{3(p-1)!} \lim_{\substack{x \to 0 \\ y \to 0}} \frac{\partial^{p+2}}{\partial x^{p-1} \partial y^3} \mathrm{B}(x, y),$$

and the solution to the point $ii)$ of the problem is complete.

And now we can prepare ourselves to enter the sections with series involving a product of two tails of the Riemann zeta function!

6.47 The Art of Mathematics with a Series Involving the Product of the Tails of $\zeta(2)$ and $\zeta(3)$

Solution Allow me to say this solution is about the art of mathematics! *Ok. Why?* First we see we have now a series with a product of tails of Riemann zeta function, and all is multiplied by $1/n$. *That doesn't look friendly!* A natural impulse would be to give it a try with Abel's summation as we did in the previous sections for the series with one tail of the Riemann zeta function, and this is a wise choice I'll want to consider. After the application of Abel's summation there will also result two nonlinear harmonic series of weight 6, which happily and amazingly we won't have to compute separately, and this is possible due to an identity generated by *The Master Theorem of Series*!

So, we write that

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)\left(\zeta(3)-1-\frac{1}{2^3}-\cdots-\frac{1}{n^3}\right)$$

$$=\sum_{n=1}^{\infty}\frac{1}{n}(\zeta(2)-H_n^{(2)})(\zeta(3)-H_n^{(3)}),$$

and applying Abel's summation, the series version in (5.1), with $a_n = 1/n$ and $b_n = (\zeta(2)-H_n^{(2)})(\zeta(3)-H_n^{(3)})$, we get that

$$\sum_{n=1}^{\infty}\frac{1}{n}(\zeta(2)-H_n^{(2)})(\zeta(3)-H_n^{(3)}) = \underbrace{\lim_{N\to\infty} H_N(\zeta(2)-H_{N+1}^{(2)})(\zeta(3)-H_{N+1}^{(3)})}_{0}$$

$$+\sum_{n=1}^{\infty} H_n\left((\zeta(2)-H_n^{(2)})(\zeta(3)-H_n^{(3)})-(\zeta(2)-H_{n+1}^{(2)})(\zeta(3)-H_{n+1}^{(3)})\right)$$

$$=\sum_{n=1}^{\infty} H_n\left((\zeta(2)-H_n^{(2)})(\zeta(3)-H_n^{(3)})-(\zeta(2)-H_{n+1}^{(2)})(\zeta(3)-H_{n+1}^{(3)})\right)$$

$$=\sum_{n=1}^{\infty}\left(H_{n+1}-\frac{1}{n+1}\right)\left(\left(\zeta(2)-H_{n+1}^{(2)}+\frac{1}{(n+1)^2}\right)\left(\zeta(3)-H_{n+1}^{(3)}+\frac{1}{(n+1)^3}\right)\right.$$

$$\left.-\left(\zeta(2)-H_{n+1}^{(2)}\right)\left(\zeta(3)-H_{n+1}^{(3)}\right)\right)$$

{reindex the series and expand it}

$$=\zeta(2)\sum_{n=1}^{\infty}\frac{H_n}{n^3}-\zeta(2)\sum_{n=1}^{\infty}\frac{1}{n^4}+\sum_{n=1}^{\infty}\frac{H_n}{n^5}+\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^4}-\sum_{n=1}^{\infty}\frac{1}{n^6}$$

$$+\zeta(3)\sum_{n=1}^{\infty}\frac{H_n}{n^2}-\zeta(3)\sum_{n=1}^{\infty}\frac{1}{n^3}+\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^3}-\left(\sum_{n=1}^{\infty}\frac{H_nH_n^{(3)}}{n^2}+\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^3}\right)$$

{the values of the series are given in (3.45), the cases $n = 2, 3, 5,$}

{(6.71), (4.14), where we set $p = 3$ and let $n\to\infty$, and (4.40)}

$$=\zeta^2(3)-\frac{61}{48}\zeta(6),$$

and the calculation of the series is finalized.

Indeed, the fact that we could avoid so easily the calculation of each nonlinear harmonic series of weight 6 involved (due to *The Master Theorem of Series*) is almost a dream which shows again that in the area of the harmonic series still lie extraordinary ways of calculating them, waiting for us to discover them by mathematical efforts, research.

The solution also answers the proposed challenging question.

The series version without the factor $1/n$,

$$\sum_{n=1}^{\infty}\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)\left(\zeta(3)-1-\frac{1}{2^3}-\cdots-\frac{1}{n^3}\right)=2\zeta(4)-\zeta(2)\zeta(3),$$

and its generalization,

$$\sum_{n=1}^{\infty}\left(\zeta(k)-1-\frac{1}{2^k}-\cdots-\frac{1}{n^k}\right)\left(\zeta(k+1)-1-\frac{1}{2^{k+1}}-\cdots-\frac{1}{n^{k+1}}\right),$$

appeared in the paper *Evaluation of Series Involving the Product of the Tail of* $\zeta(k)$ *and* $\zeta(k+1)$ I co-authored (see [22]). Also, similar series involving the product of the tails of the Riemann zeta function may be found evaluated in [26].

6.48 The Art of Mathematics with Another Splendid Series Involving the Product of the Tails of $\zeta(2)$ and $\zeta(3)$

Solution Once again, allow me to say this solution is about the art of mathematics, and it takes into account the *challenging question*! Compared to the previous series, here we have an additional harmonic number, and that also pushes us in the area of the harmonic series of weight 7.

The proposed *challenging question* gives a special flavor to the problem, since we'll want to calculate the series without making use of the values of some difficult-to-calculate nonlinear harmonic series of weight 7.

To calculate the series we'll employ a slightly different version of the series of weight 7, $\sum_{n=1}^{\infty}\frac{H_n^2 H_n^{(2)}}{n^3}$, which is $\sum_{n=1}^{\infty}\frac{H_n^2(\zeta(2)-H_n^{(2)})}{n^3}$, and in the calculations it won't be necessary to know the value of either of these two series. If we apply Abel's summation in (5.1) for the latter series where we set $a_n = 1/n^3$ and $b_n = H_n^2(\zeta(2)-H_n^{(2)})$, we get

$$\sum_{n=1}^{\infty}\frac{H_n^2(\zeta(2)-H_n^{(2)})}{n^3}=\underbrace{\lim_{N\to\infty} H_N^{(3)} H_{N+1}^2(\zeta(2)-H_{N+1}^{(2)})}_{0}$$

$$+\sum_{n=1}^{\infty} H_n^{(3)}(H_n^2(\zeta(2)-H_n^{(2)})-H_{n+1}^2(\zeta(2)-H_{n+1}^{(2)}))$$

$$=\sum_{n=1}^{\infty}(H_{n+1}^{(3)}-1/(n+1)^3)((H_{n+1}-1/(n+1))^2(\zeta(2)-H_{n+1}^{(2)}+1/(n+1)^2)$$

$$-H_{n+1}^2(\zeta(2)-H_{n+1}^{(2)}))$$

{reindex the series and carefully expand it}

$$=\sum_{n=1}^{\infty}\frac{H_n^2H_n^{(3)}}{n^2}-2\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^4}-2\sum_{n=1}^{\infty}\frac{H_nH_n^{(3)}}{n^3}-\sum_{n=1}^{\infty}\frac{H_n^{(2)}H_n^{(3)}}{n^2}$$

$$-2\sum_{n=1}^{\infty}\frac{H_n}{n}(\zeta(2)-H_n^{(2)})H_n^{(3)}-\sum_{n=1}^{\infty}\frac{1}{n^7}-\zeta(2)\sum_{n=1}^{\infty}\frac{1}{n^5}+2\zeta(2)\sum_{n=1}^{\infty}\frac{H_n}{n^4}+2\sum_{n=1}^{\infty}\frac{H_n}{n^6}$$

$$+\zeta(2)\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^2}+\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^4}+\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^5}-\sum_{n=1}^{\infty}\frac{H_n^2}{n^5}$$

{use the series values in (3.45), the cases $n=4$, 6, (6.68), (6.80), (6.75), and (4.32);}

$$\left\{\text{make use of the result in (4.44) to express } \sum_{n=1}^{\infty}\frac{H_nH_n^{(3)}}{n^3} \text{ as } \sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^4},\right\}$$

$$\left\{\text{and then use the result in (4.45) to express } \sum_{n=1}^{\infty}\frac{H_n^{(2)}H_n^{(3)}}{n^2} \text{ as } \sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^4}\right\}$$

$$=\sum_{n=1}^{\infty}\frac{H_n^2H_n^{(3)}}{n^2}+3\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^4}-2\sum_{n=1}^{\infty}\frac{H_n}{n}(\zeta(2)-H_n^{(2)})H_n^{(3)}$$

$$-4\zeta(7)+8\zeta(2)\zeta(5)-\frac{41}{4}\zeta(3)\zeta(4)$$

$$\left\{\text{employ the result in (4.46) to writer } 3\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^4}+\sum_{n=1}^{\infty}\frac{H_n^2H_n^{(3)}}{n^2} \text{ using } \sum_{n=1}^{\infty}\frac{H_n^2H_n^{(2)}}{n^3}\right\}$$

$$=-\sum_{n=1}^{\infty}\frac{H_n^2H_n^{(2)}}{n^3}-2\sum_{n=1}^{\infty}\frac{H_n}{n}(\zeta(2)-H_n^{(2)})H_n^{(3)}+\frac{15}{2}\zeta(2)\zeta(5)-\frac{9}{2}\zeta(3)\zeta(4),$$

that leads to

$$\sum_{n=1}^{\infty} \frac{H_n^2(\zeta(2) - H_n^{(2)})}{n^3} = \zeta(2) \sum_{n=1}^{\infty} \frac{H_n^2}{n^3} - \sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^3}$$

$$= -\sum_{n=1}^{\infty} \frac{H_n^2 H_n^{(2)}}{n^3} - 2\sum_{n=1}^{\infty} \frac{H_n}{n}(\zeta(2) - H_n^{(2)})H_n^{(3)} + \frac{15}{2}\zeta(2)\zeta(5) - \frac{9}{2}\zeta(3)\zeta(4),$$

whence, also considering the result in (4.30), we obtain

$$\sum_{n=1}^{\infty} \frac{H_n}{n}(\zeta(2) - H_n^{(2)})H_n^{(3)} = 2\zeta(2)\zeta(5) - \zeta(3)\zeta(4). \tag{6.233}$$

Then, in view of the result in (6.233), we write

$$\sum_{n=1}^{\infty} \frac{H_n}{n} \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right) \left(\zeta(3) - 1 - \frac{1}{2^3} - \cdots - \frac{1}{n^3}\right)$$

$$= \sum_{n=1}^{\infty} \frac{H_n}{n} \left(\zeta(2) - H_n^{(2)}\right) \left(\zeta(3) - H_n^{(3)}\right)$$

$$= \zeta(3) \sum_{n=1}^{\infty} \frac{H_n}{n} \left(\zeta(2) - H_n^{(2)}\right) - \sum_{n=1}^{\infty} \frac{H_n}{n} \left(\zeta(2) - H_n^{(2)}\right) H_n^{(3)}$$

{the first series is calculated below and the second series is found in (6.233)}

$$= \frac{11}{4}\zeta(3)\zeta(4) - 2\zeta(2)\zeta(5),$$

where the series $\sum_{n=1}^{\infty} \frac{H_n}{n} \left(\zeta(2) - H_n^{(2)}\right)$ can be calculated entirely by series manipulations if we use Abel's summation with $a_n = \dfrac{H_n}{n}$ and $b_n = \zeta(2) - H_n^{(2)}$ as in the Sect. 6.45. Alternatively, we can also start with the following series version, $\sum_{n=1}^{\infty} \frac{H_n}{n+1} \left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right)$, which is given in (4.72), the case $p = 2$, and then we may write

$$\frac{5}{4}\zeta(4) = \sum_{n=1}^{\infty} \frac{H_n}{n+1} \left(\zeta(2) - H_n^{(2)}\right)$$

$$= \sum_{n=1}^{\infty} \frac{H_{n+1} - 1/(n+1)}{n+1} \left(\zeta(2) - H_{n+1}^{(2)} + \frac{1}{(n+1)^2} \right)$$

{reindex the series and carefully expand it}

$$= \sum_{n=1}^{\infty} \frac{H_n}{n} \left(\zeta(2) - H_n^{(2)} \right) + \sum_{n=1}^{\infty} \frac{H_n}{n^3} - \zeta(2) \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n^4}$$

{use the Euler sum in (3.45), the case $n = 3$ and (4.14), the case $p = 2$}

$$= \sum_{n=1}^{\infty} \frac{H_n}{n} \left(\zeta(2) - H_n^{(2)} \right) - \frac{1}{2} \zeta(4),$$

whence we get the desired value,

$$\sum_{n=1}^{\infty} \frac{H_n}{n} \left(\zeta(2) - H_n^{(2)} \right) = \frac{7}{4} \zeta(4),$$

and the solution is complete.

It's nice to see how beautifully various relations with harmonic series work together when properly combined.

6.49 Expressing Polylogarithmic Values by Combining the Alternating Harmonic Series and the Non-alternating Harmonic Series with Integer Powers of 2 in Denominator

Solution The sums of harmonic series in this section all lead to polylogarithmic values, and to show the equalities hold we might like to combine the result from the point *ii*) in Sect. 1.31 with some generating functions in Sect. 4.11.

For all three results we might like to employ the identity in (1.59). So, if we set $m = 3$ in (1.59), we get

$$\int_0^1 \operatorname{Li}_3 \left(\frac{x}{1+x} \right) \mathrm{d}x = \sum_{n=1}^{\infty} H_{n-1}^{(3)} \left(\log(2) - \sum_{k=1}^{n-1} \frac{1}{k 2^k} \right). \tag{6.234}$$

For the left-hand side of (6.234) we make use of the result in (4.11), and we write

$$\int_0^1 \operatorname{Li}_3\left(\frac{x}{1+x}\right)\mathrm{d}x = \frac{1}{2}\int_0^1 \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}(H_n^2+H_n^{(2)})\mathrm{d}x$$

$$\{\text{reverse the order of summation and integration}\}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2+H_n^{(2)}}{n}\int_0^1 x^n\mathrm{d}x = \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2+H_n^{(2)}}{n(n+1)}. \tag{6.235}$$

For the series in the right-hand side of (6.234), we apply Abel's summation in (5.1), with $a_n = 1$ and $b_n = H_{n-1}^{(3)}\left(\log(2) - \sum_{k=1}^{n-1}\frac{1}{k2^k}\right)$, and then we write

$$\sum_{n=1}^{\infty} H_{n-1}^{(3)}\left(\log(2) - \sum_{k=1}^{n-1}\frac{1}{k2^k}\right) = \underbrace{\lim_{N\to\infty} N H_N^{(3)}\left(\log(2) - \sum_{k=1}^{N}\frac{1}{k2^k}\right)}_{0}$$

$$+\sum_{n=1}^{\infty} n\left(H_{n-1}^{(3)}\left(\log(2)-\sum_{k=1}^{n-1}\frac{1}{k2^k}\right) - \left(H_{n-1}^{(3)}+\frac{1}{n^3}\right)\left(\log(2) - \sum_{k=1}^{n-1}\frac{1}{k2^k}-\frac{1}{n2^n}\right)\right)$$

$$= \sum_{n=1}^{\infty}\frac{H_n^{(3)}}{2^n} - \sum_{n=1}^{\infty}\frac{1}{n^2}\left(\log(2) - \sum_{k=1}^{n-1}\frac{1}{k2^k}\right) = 2\operatorname{Li}_3\left(\frac{1}{2}\right)$$

$$-\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\log(2) - \sum_{k=1}^{n-1}\frac{1}{k2^k}\right), \tag{6.236}$$

where for the last equality I used the result in (4.6), the case $m = 3$.

For the remaining series in (6.236), we apply again Abel's summation in (5.1) with $a_n = 1/n^2$ and $b_n = \log(2) - \sum_{k=1}^{n-1}\frac{1}{k2^k}$ that gives

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\log(2) - \sum_{k=1}^{n-1}\frac{1}{k2^k}\right) = \underbrace{\lim_{N\to\infty} H_N^{(2)}\left(\log(2) - \sum_{k=1}^{N}\frac{1}{k2^k}\right)}_{0} + \sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n2^n}$$

$$= \sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n2^n}. \tag{6.237}$$

If we plug the result from (6.237) in (6.236), we have

$$\sum_{n=1}^{\infty} H_{n-1}^{(3)}\left(\log(2)-\sum_{k=1}^{n-1}\frac{1}{k2^k}\right)=2\operatorname{Li}_3\left(\frac{1}{2}\right)-\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n2^n}. \tag{6.238}$$

Collecting the results from (6.235) and (6.238) in (6.234), we conclude that

$$\operatorname{Li}_3\left(\frac{1}{2}\right)=\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n2^{n+1}}+\frac{1}{4}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2}{n(n+1)}+\frac{1}{4}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(2)}}{n(n+1)},$$

and the part $i)$ of the problem is complete.

Passing to the point $ii)$, we set $m=4$ in (1.59), and we have

$$\int_0^1 \operatorname{Li}_4\left(\frac{x}{1+x}\right)\mathrm{d}x=\sum_{n=1}^{\infty} H_{n-1}^{(4)}\left(\log(2)-\sum_{k=1}^{n-1}\frac{1}{k2^k}\right). \tag{6.239}$$

For the left-hand side of (6.239) we make use of (4.12), and we write

$$\begin{aligned}\int_0^1 \operatorname{Li}_4\left(\frac{x}{1+x}\right)\mathrm{d}x&=\frac{1}{6}\int_0^1\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}(H_n^3+3H_nH_n^{(2)}+2H_n^{(3)})\mathrm{d}x\\
&\qquad\{\text{reverse the order of summation and integration}\}\\
&=\frac{1}{6}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}(H_n^3+3H_nH_n^{(2)}+2H_n^{(3)})\int_0^1 x^n\mathrm{d}x\\
&=\frac{1}{6}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n(n+1)}(H_n^3+3H_nH_n^{(2)}+2H_n^{(3)}).\end{aligned} \tag{6.240}$$

For the series in the right-hand side of (6.239), we apply Abel's summation in (5.1), with $a_n=1$ and $b_n=H_{n-1}^{(4)}\left(\log(2)-\displaystyle\sum_{k=1}^{n-1}\frac{1}{k2^k}\right)$, and then we write

$$\begin{aligned}\sum_{n=1}^{\infty} H_{n-1}^{(4)}\left(\log(2)-\sum_{k=1}^{n-1}\frac{1}{k2^k}\right)&=\underbrace{\lim_{N\to\infty} N H_N^{(4)}\left(\log(2)-\sum_{k=1}^{N}\frac{1}{k2^k}\right)}_{0}\\
&+\sum_{n=1}^{\infty} n\left(H_{n-1}^{(4)}\left(\log(2)-\sum_{k=1}^{n-1}\frac{1}{k2^k}\right)-\left(H_{n-1}^{(4)}+\frac{1}{n^4}\right)\left(\log(2)-\sum_{k=1}^{n-1}\frac{1}{k2^k}-\frac{1}{n2^n}\right)\right)\end{aligned}$$

$$= \sum_{n=1}^{\infty} \frac{H_n^{(4)}}{2^n} - \sum_{n=1}^{\infty} \frac{1}{n^3} \left(\log(2) - \sum_{k=1}^{n-1} \frac{1}{k2^k} \right) =$$

$$2 \operatorname{Li}_4\left(\frac{1}{2}\right) - \sum_{n=1}^{\infty} \frac{1}{n^3} \left(\log(2) - \sum_{k=1}^{n-1} \frac{1}{k2^k} \right), \tag{6.241}$$

where for the last equality I used the result in (4.6), the case $m = 4$.

For the left series in (6.49), we apply Abel's summation in (5.1) with $a_n = 1/n^3$ and $b_n = \log(2) - \sum_{k=1}^{n-1} \frac{1}{k2^k}$ that yields

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \left(\log(2) - \sum_{k=1}^{n-1} \frac{1}{k2^k} \right) = \underbrace{\lim_{N\to\infty} H_N^{(3)} \left(\log(2) - \sum_{k=1}^{N} \frac{1}{k2^k} \right)}_{0} + \sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n2^n}$$

$$= \sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n2^n}. \tag{6.242}$$

If we plug the result from (6.242) in (6.49), we get

$$\sum_{n=1}^{\infty} H_{n-1}^{(4)} \left(\log(2) - \sum_{k=1}^{n-1} \frac{1}{k2^k} \right) = 2 \operatorname{Li}_4\left(\frac{1}{2}\right) - \sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n2^n}. \tag{6.243}$$

Collecting the results from (6.240) and (6.243) in (6.239), we conclude that

$$\operatorname{Li}_4\left(\frac{1}{2}\right)$$

$$= \sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n2^{n+1}} + \frac{1}{12} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n^3}{n(n+1)} + \frac{1}{6} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n^{(3)}}{n(n+1)}$$

$$+ \frac{1}{4} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n H_n^{(2)}}{n(n+1)},$$

and the part *ii*) of the problem is complete.

Lastly, for the point iii), we set $m = 5$ in (1.59), and we have

$$\int_0^1 \operatorname{Li}_5\left(\frac{x}{1+x}\right)\mathrm{d}x = \sum_{n=1}^{\infty} H_{n-1}^{(5)}\left(\log(2) - \sum_{k=1}^{n-1}\frac{1}{k2^k}\right). \tag{6.244}$$

Now, for the left-hand side of (6.244) we make use of the result in (4.11), and then we write

$$\int_0^1 \operatorname{Li}_5\left(\frac{x}{1+x}\right)\mathrm{d}x$$

$$= \frac{1}{24}\int_0^1 \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}(H_n^4 + 6H_n^2H_n^{(2)} + 8H_nH_n^{(3)} + 3\left(H_n^{(2)}\right)^2 + 6H_n^{(4)})\mathrm{d}x$$

{reverse the order of summation and integration}

$$= \frac{1}{24}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}(H_n^4 + 6H_n^2H_n^{(2)} + 8H_nH_n^{(3)} + 3\left(H_n^{(2)}\right)^2 + 6H_n^{(4)})\int_0^1 x^n\mathrm{d}x$$

$$= \frac{1}{24}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n(n+1)}(H_n^4 + 6H_n^2H_n^{(2)} + 8H_nH_n^{(3)} + 3\left(H_n^{(2)}\right)^2 + 6H_n^{(4)}). \tag{6.245}$$

For the right-hand side of (6.244), we apply Abel's summation in (5.1) with $a_n = 1$ and $b_n = H_{n-1}^{(5)}\left(\log(2) - \sum_{k=1}^{n-1}\frac{1}{k2^k}\right)$ that gives

$$\sum_{n=1}^{\infty} H_{n-1}^{(5)}\left(\log(2) - \sum_{k=1}^{n-1}\frac{1}{k2^k}\right) = \lim_{N\to\infty} NH_N^{(5)}\left(\log(2) - \sum_{k=1}^{N}\frac{1}{k2^k}\right)$$

$$+\sum_{n=1}^{\infty} n\left(H_{n-1}^{(5)}\left(\log(2)-\sum_{k=1}^{n-1}\frac{1}{k2^k}\right) - \left(H_{n-1}^{(5)} + \frac{1}{n^5}\right)\left(\log(2) - \sum_{k=1}^{n-1}\frac{1}{k2^k} - \frac{1}{n2^n}\right)\right)$$

$$= \sum_{n=1}^{\infty}\frac{H_n^{(5)}}{2^n} - \sum_{n=1}^{\infty}\frac{1}{n^4}\left(\log(2) - \sum_{k=1}^{n-1}\frac{1}{k2^k}\right) = 2\operatorname{Li}_5\left(\frac{1}{2}\right)$$

$$-\sum_{n=1}^{\infty}\frac{1}{n^4}\left(\log(2) - \sum_{k=1}^{n-1}\frac{1}{k2^k}\right), \tag{6.246}$$

where for the last equality I used the result in (4.6), the case $m = 5$.

Further, for the remaining series in (6.246), we apply Abel's summation in (5.1) with $a_n = 1/n^4$ and $b_n = \log(2) - \sum_{k=1}^{n-1} \frac{1}{k2^k}$, and we get

$$\sum_{n=1}^{\infty} \frac{1}{n^4}\left(\log(2) - \sum_{k=1}^{n-1} \frac{1}{k2^k}\right) = \underbrace{\lim_{N\to\infty} H_N^{(4)}\left(\log(2) - \sum_{k=1}^{N} \frac{1}{k2^k}\right)}_{0} + \sum_{n=1}^{\infty} \frac{H_n^{(4)}}{n2^n}$$

$$= \sum_{n=1}^{\infty} \frac{H_n^{(4)}}{n2^n}. \tag{6.247}$$

If we plug the result from (6.247) in (6.246), we have

$$\sum_{n=1}^{\infty} H_{n-1}^{(5)}\left(\log(2) - \sum_{k=1}^{n-1} \frac{1}{k2^k}\right) = 2\operatorname{Li}_5\left(\frac{1}{2}\right) - \sum_{n=1}^{\infty} \frac{H_n^{(4)}}{n2^n}. \tag{6.248}$$

Finally, collecting the results from (6.245) and (6.248) in (6.244), we conclude that

$$\operatorname{Li}_5\left(\frac{1}{2}\right)$$

$$= \sum_{n=1}^{\infty} \frac{H_n^{(4)}}{n2^{n+1}} + \frac{1}{48}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^4}{n(n+1)} + \frac{1}{8}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(4)}}{n(n+1)}$$

$$+\frac{1}{8}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2H_n^{(2)}}{n(n+1)} + \frac{1}{16}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(H_n^{(2)})^2}{n(n+1)} + \frac{1}{6}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_nH_n^{(3)}}{n(n+1)},$$

and the part $iii)$ of the problem is complete.

If you enjoy such representations with long sums of harmonic series, then there is some good news: if in this section the sums yielded polylogarithmic values like $\operatorname{Li}_3\left(\frac{1}{2}\right)$, $\operatorname{Li}_4\left(\frac{1}{2}\right)$ and $\operatorname{Li}_5\left(\frac{1}{2}\right)$, in one of the next sections we'll meet a long sum with harmonic series (seven series) leading to $\zeta(4)$!

The solutions also answer the proposed challenging question.

6.50 Cool Results with Cool Series Involving Summands with the Harmonic Number and the Integer Powers of 2

Solution Some of the harmonic series with positive integer powers of 2 in denominator might be pretty challenging. For example, one might possibly find the series like $\displaystyle\sum_{n=1}^{\infty}\frac{H_n}{n^3 2^n}$ and $\displaystyle\sum_{n=1}^{\infty}\frac{H_n}{n^4 2^n}$ rather troublesome, and the relations I propose in this section are meant to facilitate the derivation of such series. Also, trying to remain in the perimeter of the real methods is another challenge, especially for the last relation where it is required to calculate a tough integral (which happily I already calculated in the third chapter).

For the first sum of series, I invoke the identity

$$\int_0^{1/2} x^k \log(x)\mathrm{d}x = -\frac{\log(2)}{(k+1)2^{k+1}} - \frac{1}{(k+1)^2 2^{k+1}}, \tag{6.249}$$

which is proved immediately with the integration by parts.

Multiplying both sides of (6.249) by $-H_k$ and then summing up from $k = 1$ to ∞, we get

$$\log(2)\sum_{k=1}^{\infty}\frac{H_k}{(k+1)2^{k+1}} + \sum_{k=1}^{\infty}\frac{H_k}{(k+1)^2 2^{k+1}} = -\sum_{k=1}^{\infty} H_k \int_0^{1/2} x^k \log(x)\mathrm{d}x$$

$$\{\text{reverse the order of integration and summation}\}$$

$$= -\int_0^{1/2} \log(x)\sum_{k=1}^{\infty} x^k H_k \mathrm{d}x$$

$$\{\text{use the generating function in (4.5)}\}$$

$$= \int_0^{1/2} \frac{\log(x)\log(1-x)}{1-x}\mathrm{d}x \overset{1-x=y}{=} \int_{1/2}^{1} \frac{\log(1-y)\log(y)}{y}\mathrm{d}y$$

$$= -\int_{1/2}^{1}\sum_{n=1}^{\infty}\frac{y^{n-1}}{n}\log(y)\mathrm{d}y$$

$$\{\text{reverse the order of summation and integration}\}$$

$$= -\sum_{n=1}^{\infty}\frac{1}{n}\int_{1/2}^{1} y^{n-1}\log(y)\mathrm{d}y = -\sum_{n=1}^{\infty}\frac{1}{n}\left(\int_0^1 - \int_0^{1/2}\right) y^{n-1}\log(y)\mathrm{d}y$$

$$= -\sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 y^{n-1} \log(y) \mathrm{d}y + \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{1/2} y^{n-1} \log(y) \mathrm{d}y$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^3} - \log(2) \sum_{n=1}^{\infty} \frac{1}{n^2 2^n} - \sum_{n=1}^{\infty} \frac{1}{n^3 2^n} = \zeta(3) - \log(2) \operatorname{Li}_2\left(\frac{1}{2}\right) - \operatorname{Li}_3\left(\frac{1}{2}\right)$$

{use the special values in (3.19) and (3.20)}

$$= \frac{1}{8}\zeta(3) + \frac{1}{3}\log^3(2),$$

and the calculation to the point $i)$ is finalized.

I submitted the following sum of series as a problem to *The American Mathematical Monthly*, the problem **11921** (see [49]).

To prove the result with the second sum of series, we employ the simple fact that

$$\int_0^{1/2} x^k \log^2(x) \mathrm{d}x = \frac{\log^2(2)}{(k+1)2^{k+1}} + \frac{\log(2)}{(k+1)^2 2^k} + \frac{1}{(k+1)^3 2^k}, \tag{6.250}$$

which is proved by applying two times the integration by parts or by simply making use of (1.3), with $a = 1/2, n = 2$.

Multiplying both sides of (6.250) by H_k and then summing up from $k = 1$ to ∞, we get

$$\log^2(2) \sum_{k=1}^{\infty} \frac{H_k}{(k+1)2^{k+1}} + \log(2) \sum_{k=1}^{\infty} \frac{H_k}{(k+1)^2 2^k} + \sum_{k=1}^{\infty} \frac{H_k}{(k+1)^3 2^k}$$

$$= \sum_{k=1}^{\infty} H_k \int_0^{1/2} x^k \log^2(x) \mathrm{d}x = \int_0^{1/2} \log^2(x) \sum_{k=1}^{\infty} x^k H_k \mathrm{d}x$$

{use the generating function in (4.5)}

$$= -\int_0^{1/2} \frac{\log(1-x)\log^2(x)}{1-x} \mathrm{d}x$$

{make use of the result in (1.28), the case $p = 1$}

$$= \frac{1}{4}\left(\zeta(4) + \log^4(2)\right),$$

and the calculation to the point $ii)$ is finalized. The last integral also appears in [17, p.128].

For the last sum of series, we recall the result in (1.18), and using the integral there, we write

$$\int_0^{1/2} \frac{\log^2(x)\log^2(1-x)}{x}\mathrm{d}x$$

$$\left\{\text{make use of the fact that } \sum_{k=1}^{\infty} x^k \frac{H_k}{k+1} = \frac{\log^2(1-x)}{2x}\right\}$$

$$= 2\int_0^{1/2} \sum_{k=1}^{\infty} x^k \frac{H_k}{k+1}\log^2(x)\mathrm{d}x$$

$$\{\text{reverse the order of summation and integration}\}$$

$$= 2\sum_{k=1}^{\infty} \frac{H_k}{k+1}\int_0^{1/2} x^k \log^2(x)\mathrm{d}x$$

$$\{\text{make use of the result in (6.250)}\}$$

$$= \log^2(2)\sum_{k=1}^{\infty} \frac{H_k}{(k+1)^2 2^k} + \log(2)\sum_{k=1}^{\infty} \frac{H_k}{(k+1)^3 2^{k-1}} + \sum_{k=1}^{\infty} \frac{H_k}{(k+1)^4 2^{k-1}}$$

$$= \frac{1}{8}\zeta(5) - 2\zeta(2)\zeta(3) - \frac{2}{3}\log^3(2)\zeta(2) + \frac{7}{4}\log^2(2)\zeta(3) - \frac{1}{15}\log^5(2)$$

$$+4\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right) + 4\operatorname{Li}_5\left(\frac{1}{2}\right),$$

and the calculation to the point iii) is finalized.

Equipped with these relations, we are prepared to enter the next section where we'll want the extract the precise values of such series.

6.51 Eight Harmonic Series Involving the Integer Powers of 2 in Denominator

Solution As mentioned at the end of the previous section, once we are equipped with the proper relations, we may extract the values of the series in this section.

Now, to calculate the series from the point i), we use the result in (4.5), $\sum_{k=1}^{\infty} x^k H_k = -\frac{\log(1-x)}{1-x}$, where if we integrate both sides from $x = 0$ to $x = 1/2$, we get

$$\int_0^{1/2} \sum_{k=1}^{\infty} x^k H_k \mathrm{d}x = \sum_{k=1}^{\infty} \int_0^{1/2} x^k H_k \mathrm{d}x = \sum_{k=1}^{\infty} \frac{H_k}{(k+1)2^{k+1}} = -\int_0^{1/2} \frac{\log(1-x)}{1-x} \mathrm{d}x$$

$$= \frac{1}{2} \log^2(2),$$

from which we obtain

$$\sum_{k=1}^{\infty} \frac{H_k}{(k+1)2^k} = \log^2(2),$$

and the point i) of the problem is finalized.

Further, to calculate the series from the point ii), we make use of the series from the point i), and then we write

$$\log^2(2) = \sum_{k=1}^{\infty} \frac{H_k}{(k+1)2^k} = 2 \sum_{k=1}^{\infty} \frac{H_{k+1} - 1/(k+1)}{(k+1)2^{k+1}}$$

{reindex the series and expand it}

$$= 2 \sum_{k=1}^{\infty} \frac{H_k}{k2^k} - 2 \sum_{k=1}^{\infty} \frac{1}{k^2 2^k} = 2 \sum_{k=1}^{\infty} \frac{H_k}{k2^k} - 2 \operatorname{Li}_2\left(\frac{1}{2}\right)$$

{make use of the special value of Dilogarithm function in (3.19)}

$$= 2 \sum_{k=1}^{\infty} \frac{H_k}{k2^k} - \zeta(2) + \log^2(2),$$

whence we obtain

$$\sum_{k=1}^{\infty} \frac{H_k}{k2^k} = \frac{1}{2} \zeta(2),$$

and the point ii) of the problem is finalized.

Next, to get the value of the series from the point $iii)$, we plug the value of the series from the point $i)$ in the relation in (4.74), and then we immediately have

$$\frac{1}{2}\log^3(2)+\frac{1}{2}\sum_{k=1}^{\infty}\frac{H_k}{(k+1)^2 2^k}=\frac{1}{8}\zeta(3)+\frac{1}{3}\log^3(2),$$

whence we obtain

$$\sum_{k=1}^{\infty}\frac{H_k}{(k+1)^2 2^k}=\frac{1}{4}\zeta(3)-\frac{1}{3}\log^3(2),$$

and the point $iii)$ of the problem is finalized.

Jumping to the next series from the point $iv)$, we make use of the series from the point $iii)$, and then we write

$$\frac{1}{4}\zeta(3)-\frac{1}{3}\log^3(2)=\sum_{k=1}^{\infty}\frac{H_k}{(k+1)^2 2^k}=2\sum_{k=1}^{\infty}\frac{H_{k+1}-1/(k+1)}{(k+1)^2 2^{k+1}}$$

$$\{\text{reindex the series and expand it}\}$$

$$=2\sum_{k=1}^{\infty}\frac{H_k}{k^2 2^k}-2\sum_{k=1}^{\infty}\frac{1}{k^3 2^k}=2\sum_{k=1}^{\infty}\frac{H_k}{k^2 2^k}-2\operatorname{Li}_3\left(\frac{1}{2}\right)$$

$$\{\text{use the special value of Trilogarithm function in (3.20)}\}$$

$$=2\sum_{k=1}^{\infty}\frac{H_k}{k^2 2^k}-\frac{7}{4}\zeta(3)+\log(2)\zeta(2)-\frac{1}{3}\log^3(2),$$

whence we get

$$\sum_{k=1}^{\infty}\frac{H_k}{k^2 2^k}=\zeta(3)-\frac{1}{2}\log(2)\zeta(2),$$

and the point $iv)$ of the problem is finalized.

Then, to calculate the series from the point $v)$, we combine the values of the series from the points $i)$, $iii)$ and the identity in (4.75) that lead to

$$\frac{1}{6}\log^4(2)+\frac{1}{4}\log(2)\zeta(3)+\sum_{k=1}^{\infty}\frac{H_k}{(k+1)^3 2^k}=\frac{1}{4}\left(\zeta(4)+\log^4(2)\right),$$

from which we obtain

$$\sum_{k=1}^{\infty} \frac{H_k}{(k+1)^3 2^k} = \frac{1}{4}\left(\zeta(4) - \log(2)\zeta(3) + \frac{1}{3}\log^4(2)\right),$$

and the point $v)$ of the problem is finalized.

Next, for the part $vi)$ of the problem we make use of the result from the point $v)$, and we write

$$\frac{1}{4}\left(\zeta(4) - \log(2)\zeta(3) + \frac{1}{3}\log^4(2)\right) = \sum_{k=1}^{\infty} \frac{H_k}{(k+1)^3 2^k} = 2\sum_{k=1}^{\infty} \frac{H_{k+1} - 1/(k+1)}{(k+1)^3 2^{k+1}}$$

$$\{\text{reindex the series and expand it}\}$$

$$= 2\sum_{k=1}^{\infty} \frac{H_k}{k^3 2^k} - 2\sum_{k=1}^{\infty} \frac{1}{k^4 2^k} = 2\sum_{k=1}^{\infty} \frac{H_k}{k^3 2^k} - 2\operatorname{Li}_4\left(\frac{1}{2}\right),$$

whence we obtain

$$\sum_{k=1}^{\infty} \frac{H_k}{k^3 2^k} = \frac{1}{8}\zeta(4) - \frac{1}{8}\log(2)\zeta(3) + \frac{1}{24}\log^4(2) + \operatorname{Li}_4\left(\frac{1}{2}\right),$$

and the point $vi)$ of the problem is finalized.

Further, the result from the point $vii)$ is straightforward if we combine the relations in (4.74), (4.75), and (4.76) that give

$$\sum_{k=1}^{\infty} \frac{H_k}{(k+1)^4 2^k}$$

$$= \frac{1}{16}\zeta(5) - \frac{1}{4}\log(2)\zeta(4) + \log^2(2)\zeta(3) - \frac{1}{3}\log^3(2)\zeta(2) - \zeta(2)\zeta(3) + \frac{1}{20}\log^5(2)$$
$$+ 2\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right) + 2\operatorname{Li}_5\left(\frac{1}{2}\right),$$

which finalizes the point $vii)$ of the problem.

Finally, for getting the value of the last series, we manipulate the previous series and write

$$\frac{1}{16}\zeta(5) - \frac{1}{4}\log(2)\zeta(4) + \log^2(2)\zeta(3) - \frac{1}{3}\log^3(2)\zeta(2) - \zeta(2)\zeta(3) + \frac{1}{20}\log^5(2)$$

$$+ 2\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right) + 2\operatorname{Li}_5\left(\frac{1}{2}\right) = \sum_{k=1}^{\infty} \frac{H_k}{(k+1)^4 2^k} = 2\sum_{k=1}^{\infty} \frac{H_{k+1} - 1/(k+1)}{(k+1)^4 2^{k+1}}$$

$$\{\text{reindex the series and expand it}\}$$

$$= 2\sum_{k=1}^{\infty}\frac{H_k}{k^4 2^k} - 2\sum_{k=1}^{\infty}\frac{1}{k^5 2^k} = 2\sum_{k=1}^{\infty}\frac{H_k}{k^4 2^k} - 2\operatorname{Li}_5\left(\frac{1}{2}\right),$$

whence we extract the value of the desired series,

$$\sum_{k=1}^{\infty}\frac{H_k}{k^4 2^k}$$

$$= \frac{1}{32}\zeta(5) - \frac{1}{8}\log(2)\zeta(4) + \frac{1}{2}\log^2(2)\zeta(3) - \frac{1}{6}\log^3(2)\zeta(2) - \frac{1}{2}\zeta(2)\zeta(3) + \frac{1}{40}\log^5(2)$$

$$+\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right) + 2\operatorname{Li}_5\left(\frac{1}{2}\right),$$

and the last part of the problem is finalized.

The second, third, and fourth series together with similar versions of them are given in [15]. A different approach of calculating the last series may be found in [34] and [62]. Also, for another strategy of calculating such series, you may see the approach in [35].

6.52 Let's Calculate Three Classical Alternating Harmonic Series, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^3}$, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(2)}}{n^2}$, and $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2}{n^2}$

Solution In this section we step in the area of the alternating harmonic series, and don't be surprised to find they are hard nuts! For example, let's remember together the non-alternating version of the second proposed series we met in more places, that is $\displaystyle\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^2} = \frac{17}{4}\zeta(4)$, which is simply straightforward with using the sum in (4.14), but things change if adding $(-1)^{n-1}$ in the summand, and getting a solution seems to be a more difficult task.

Also, if you recall Sect. 3.52, there we needed the alternating harmonic series from the first point during some of the calculations.

Let's try to reduce the first series to a convenient integral form. Then, we write

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^3}=\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}H_n\int_0^1 x^{n-1}\log^2(x)\mathrm{d}x$$

$$=\frac{1}{2}\int_0^1\log^2(x)\sum_{n=1}^{\infty}(-1)^{n-1}x^{n-1}H_n\mathrm{d}x=\frac{1}{2}\int_0^1\frac{\log(1+x)\log^2(x)}{x(1+x)}\mathrm{d}x$$

$$=\frac{1}{2}\underbrace{\int_0^1\frac{\log(1+x)\log^2(x)}{x}\mathrm{d}x}_{I_1}-\frac{1}{2}\underbrace{\int_0^1\frac{\log(1+x)\log^2(x)}{1+x}\mathrm{d}x}_{I_2}$$

$$=\frac{11}{4}\zeta(4)-\frac{7}{4}\log(2)\zeta(3)+\frac{1}{2}\log^2(2)\zeta(2)-\frac{1}{12}\log^4(2)-2\operatorname{Li}_4\left(\frac{1}{2}\right),$$

where the integrals I_1 and I_2 are calculated below.

So, for the integral I_1 we get

$$I_1=\int_0^1\frac{\log(1+x)\log^2(x)}{x}\mathrm{d}x=\int_0^1\log^2(x)\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{n-1}}{n}\mathrm{d}x$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int_0^1 x^{n-1}\log^2(x)\mathrm{d}x=2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^4}=\frac{7}{4}\zeta(4).$$

Then, for the integral I_2 we have

$$I_2=\int_0^1\frac{\log(1+x)\log^2(x)}{1+x}\mathrm{d}x=\frac{1}{2}\int_0^1(\log^2(1+x))'\log^2(x)\mathrm{d}x$$

{apply the integration by parts}

$$=\underbrace{\frac{1}{2}\log^2(x)\log^2(1+x)\Big|_{x=0}^{x=1}}_{0}-\int_0^1\frac{\log(x)\log^2(1+x)}{x}\mathrm{d}x$$

{make the change of variable $x=y/(1-y)$ and then expand the integral}

$$=\int_0^{1/2}\frac{\log^3(1-y)}{1-y}\mathrm{d}y+\int_0^{1/2}\frac{\log^3(1-y)}{y}\mathrm{d}y-\int_0^{1/2}\frac{\log^2(1-y)\log(y)}{1-y}\mathrm{d}y$$

$$-\int_0^{1/2}\frac{\log^2(1-y)\log(y)}{y}\mathrm{d}y$$

$$\left\{\text{use that } \int_0^{1/2} \frac{\log^2(1-y)\log(y)}{1-y} \mathrm{d}y = -\frac{1}{3}\log^4(2) + \frac{1}{3}\int_0^{1/2} \frac{\log^3(1-y)}{y} \mathrm{d}y\right\}$$

$$= \frac{2}{3}\underbrace{\int_0^{1/2} \frac{\log^3(1-y)}{y} \mathrm{d}y}_{I_3} - \underbrace{\int_0^{1/2} \frac{\log^2(1-y)\log(y)}{y} \mathrm{d}y}_{I_4} + \frac{1}{12}\log^4(2). \tag{6.251}$$

Then, for the integral I_3 in (6.251), we write

$$I_3 = \int_0^{1/2} \frac{\log^3(1-y)}{y} \mathrm{d}y = \left(\int_0^1 - \int_{1/2}^1\right) \frac{\log^3(1-y)}{y} \mathrm{d}y$$

$$= \int_0^1 \frac{\log^3(1-y)}{y} \mathrm{d}y - \int_{1/2}^1 \frac{\log^3(1-y)}{y} \mathrm{d}y \overset{1-y=z}{=} \int_0^1 \frac{\log^3(z)}{1-z} \mathrm{d}z - \int_0^{1/2} \frac{\log^3(z)}{1-z} \mathrm{d}z$$

$$= \int_0^1 \sum_{n=1}^{\infty} z^{n-1} \log^3(z) \mathrm{d}z - \int_0^{1/2} \sum_{n=1}^{\infty} z^{n-1} \log^3(z) \mathrm{d}z$$

$$\{\text{reverse the order of summation and integration}\}$$

$$= \sum_{n=1}^{\infty} \int_0^1 z^{n-1} \log^3(z) \mathrm{d}z - \sum_{n=1}^{\infty} \int_0^{1/2} z^{n-1} \log^3(z) \mathrm{d}z$$

$$\{\text{make use of the result in (1.3)}\}$$

$$= -6\sum_{n=1}^{\infty} \frac{1}{n^4} + 6\sum_{n=1}^{\infty} \frac{1}{n^4 2^n} + 6\log(2)\sum_{n=1}^{\infty} \frac{1}{n^3 2^n} + 3\log^2(2)\sum_{n=1}^{\infty} \frac{1}{n^2 2^n} + \log^3(2)\sum_{n=1}^{\infty} \frac{1}{n 2^n}$$

$$= \log^4(2) - 6\zeta(4) + 6\operatorname{Li}_4\left(\frac{1}{2}\right) + 6\log(2)\operatorname{Li}_3\left(\frac{1}{2}\right) + 3\log^2(2)\operatorname{Li}_2\left(\frac{1}{2}\right)$$

$$\{\text{use the special values in (3.19) and (3.20)}\}$$

$$= \frac{1}{2}\log^4(2) - \frac{3}{2}\log^2(2)\zeta(2) + \frac{21}{4}\log(2)\zeta(3) - 6\zeta(4) + 6\operatorname{Li}_4\left(\frac{1}{2}\right). \tag{6.252}$$

Next, for the integral I_4 in (6.251), we integrate once by parts that gives

$$I_4 = \int_0^{1/2} \frac{\log^2(1-y)\log(y)}{y}\mathrm{d}y = \frac{1}{2}\int_0^{1/2} (\log^2(y))' \log^2(1-y)\mathrm{d}y$$

$$= \frac{1}{2}\log^2(y)\log^2(1-y)\Big|_{y=0}^{y=1/2} + \int_0^{1/2} \frac{\log(1-y)\log^2(y)}{1-y}\mathrm{d}y$$

$$= \frac{1}{2}\log^4(2) + \int_0^{1/2} \frac{\log(1-y)\log^2(y)}{1-y}\mathrm{d}y$$

{the value of the remaining integral is given in (1.28), the case $p = 1$}

$$= \frac{1}{2}\log^4(2) - \frac{1}{4}\left(\zeta(4) + \log^4(2)\right) = \frac{1}{4}\left(\log^4(2) - \zeta(4)\right). \tag{6.253}$$

The last integral also appears in [17, p. 128]. If we plug the values of the integrals from (6.252) and (6.253) in (6.251), we get

$$I_2 = \int_0^1 \frac{\log^2(x)\log(1+x)}{1+x}\mathrm{d}x$$

$$= \frac{1}{6}\log^4(2) - \log^2(2)\zeta(2) + \frac{7}{2}\log(2)\zeta(3) - \frac{15}{4}\zeta(4) + 4\operatorname{Li}_4\left(\frac{1}{2}\right), \tag{6.254}$$

and the calculation to the series from the point $i)$ is finalized.

Further, to solve the second point of the problem, we combine the result in (1.53) with the value of the series from the previous point. Therefore, we write

$$-\frac{3}{16}\zeta(4) = \int_0^1 \frac{\log(x)\operatorname{Li}_2(x)}{1+x}\mathrm{d}x = \int_0^1 \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}(-1)^{k-1}\frac{x^{k+n-1}}{n^2}\right)\log(x)\mathrm{d}x$$

{reverse the order of summation and integration}

$$= \sum_{k=1}^{\infty}(-1)^{k-1}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^1 x^{k+n-1}\log(x)\mathrm{d}x\right) = \sum_{k=1}^{\infty}(-1)^k\left(\sum_{n=1}^{\infty}\frac{1}{n^2(n+k)^2}\right)$$

$$= \sum_{k=1}^{\infty}(-1)^k\left(\sum_{n=1}^{\infty}\left(\frac{1}{k^2n^2} + \frac{1}{k^2(k+n)^2} - \frac{2}{k^2n(k+n))}\right)\right)$$

$$= \sum_{k=1}^{\infty}(-1)^k\frac{1}{k^2}\sum_{n=1}^{\infty}\frac{1}{n^2} + \sum_{k=1}^{\infty}(-1)^k\frac{1}{k^2}(\zeta(2) - H_k^{(2)}) - 2\sum_{k=1}^{\infty}(-1)^k\frac{H_k}{k^3}$$

$$= -\frac{5}{2}\zeta(4) + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{H_k^{(2)}}{k^2} + 2\sum_{k=1}^{\infty}(-1)^{k-1}\frac{H_k}{k^3}$$

{use the value of the series from the point i)}

$$= 3\zeta(4) - \frac{7}{2}\log(2)\zeta(3) + \log^2(2)\zeta(2) - \frac{1}{6}\log^4(2) - 4\operatorname{Li}_4\left(\frac{1}{2}\right)$$

$$+ \sum_{k=1}^{\infty}(-1)^{k-1}\frac{H_k^{(2)}}{k^2},$$

from which we obtain that

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{H_k^{(2)}}{k^2}$$

$$= \frac{1}{6}\log^4(2) - \log^2(2)\zeta(2) + \frac{7}{2}\log(2)\zeta(3) - \frac{51}{16}\zeta(4) + 4\operatorname{Li}_4\left(\frac{1}{2}\right),$$

and the calculation to the series from the point ii) is finalized.

Next, to solve the last part of the problem, we write

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2}{n^2} = \sum_{n=1}^{\infty}(-1)^n H_n^2\int_0^1 x^{n-1}\log(x)\mathrm{d}x$$

$$= \int_0^1 \log(x)\sum_{n=1}^{\infty}(-1)^n x^{n-1}H_n^2\mathrm{d}x$$

{make use of the generating function in (4.7)}

$$= \int_0^1 \frac{\log(x)}{x(1+x)}(\log^2(1+x) + \operatorname{Li}_2(-x))\mathrm{d}x = \int_0^1 \frac{\log(x)\log^2(1+x)}{x}\mathrm{d}x$$

$$+ \int_0^1 \frac{\log(x)\operatorname{Li}_2(-x)}{x}\mathrm{d}x - \int_0^1 \frac{\log(x)\log^2(1+x)}{1+x}\mathrm{d}x - \int_0^1 \frac{\log(x)\operatorname{Li}_2(-x)}{1+x}\mathrm{d}x,$$

and since for the last integral we have

$$\int_0^1 \frac{\log(x)\operatorname{Li}_2(-x)}{1+x}\mathrm{d}x = \underbrace{\log(1+x)\log(x)\operatorname{Li}_2(-x)\Big|_{x=0}^{x=1}}_{0}$$

$$+\int_0^1 \frac{\log(x)\log^2(1+x)}{x}\mathrm{d}x - \int_0^1 \frac{\log(1+x)\operatorname{Li}_2(-x)}{x}\mathrm{d}x$$

$$= \int_0^1 \frac{\log(x)\log^2(1+x)}{x}\mathrm{d}x - \int_0^1 \frac{\log(1+x)\operatorname{Li}_2(-x)}{x}\mathrm{d}x$$

$$= \int_0^1 \frac{\log(x)\log^2(1+x)}{x}\mathrm{d}x + \frac{1}{2}\left(\operatorname{Li}_2(-x)\right)^2\Big|_{x=0}^{x=1} = \frac{5}{16}\zeta(4)$$

$$+\int_0^1 \frac{\log(x)\log^2(1+x)}{x}\mathrm{d}x,$$

then we obtain that

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2}{n^2} = \underbrace{\int_0^1 \frac{\log(x)\operatorname{Li}_2(-x)}{x}\mathrm{d}x}_{J_1} - \underbrace{\int_0^1 \frac{\log(x)\log^2(1+x)}{1+x}\mathrm{d}x}_{J_2} - \frac{5}{16}\zeta(4). \tag{6.255}$$

Now, to calculate the integral J_1 in (6.255), we write

$$J_1 = \int_0^1 \frac{\log(x)\operatorname{Li}_2(-x)}{x}\mathrm{d}x = \int_0^1 \log(x)\sum_{n=1}^{\infty}(-1)^n\frac{x^{n-1}}{n^2}\mathrm{d}x$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\int_0^1 x^{n-1}\log(x)\mathrm{d}x = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^4} = \frac{7}{8}\zeta(4). \tag{6.256}$$

For the integral J_2 in (6.255), we make the change of variable $x = \dfrac{y}{1-y}$ that yields

$$J_2 = \int_0^1 \frac{\log(x)\log^2(1+x)}{1+x}\mathrm{d}x = \int_0^{1/2} \frac{\log^2(1-y)\log(y)}{1-y}\mathrm{d}y$$

$$-\int_0^{1/2} \frac{\log^3(1-y)}{1-y}\mathrm{d}y$$

$$= \frac{1}{4}\log^4(2) - \frac{1}{3}\int_0^{1/2}(\log^3(1-y))'\log(y)\mathrm{d}y$$

$$\{\text{apply the integration by parts}\}$$

$$= -\frac{1}{12}\log^4(2) + \frac{1}{3}\int_0^{1/2} \frac{\log^3(1-y)}{y} \mathrm{d}y$$

$$= \frac{1}{12}\log^4(2) - \frac{1}{2}\log^2(2)\zeta(2) + \frac{7}{4}\log(2)\zeta(3) - 2\zeta(4) + 2\operatorname{Li}_4\left(\frac{1}{2}\right), \tag{6.257}$$

where the last integral is calculated in (6.252).

Collecting the values of the integrals J_1 and J_2 from (6.256) and (6.257) in (6.255), we obtain that

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2}{n^2}$$

$$= \frac{41}{16}\zeta(4) - \frac{7}{4}\log(2)\zeta(3) + \frac{1}{2}\log^2(2)\zeta(2) - \frac{1}{12}\log^4(2) - 2\operatorname{Li}_4\left(\frac{1}{2}\right),$$

and the series from the point $iii)$ is finalized.

It's worth mentioning that as I did for the series from the point $ii)$, one can establish a nice relation between the series from the point $i)$ and the series from the point $iii)$, that is $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^3} - \sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2}{n^2} = \frac{3}{16}\zeta(4)$, and then reduce all to the calculation of a logarithmic integral that we further attack with simple algebraic identities as I acted for some logarithmic integrals at the beginning of the chapter *Integrals*. As I've recently found, the last mentioned harmonic series relation comes from the work by P.J. De Doelder in [17].

6.53 Then, Let's Calculate Another Pair of Classical Alternating Harmonic Series, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^2}$ and $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^4}$

Solution The journey through the realm of the alternating harmonic series continues in this section. In the following I'll try to provide with a solution to the proposed series which relies on the real methods exclusively.

For example, in the first chapter, in Sect. 1.40 we met a nice integral where we also needed the alternating series from the second point in order to finish it.

The solution to the first part of the problem is obtained immediately if we use the integral result in (1.13),

$$\frac{1}{4}\zeta(3)=\int_0^1 \frac{\log^2(1+x)}{x}\mathrm{d}x=2\int_0^1 \sum_{n=1}^{\infty}(-1)^{n-1}x^n\frac{H_n}{n+1}\mathrm{d}x$$

$$=2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n+1}\int_0^1 x^n\mathrm{d}x=2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{(n+1)^2}$$

$$=2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_{n+1}-1/(n+1)}{(n+1)^2}$$

{reindex the series and expand it}

$$=2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^3}-2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^2}=\frac{3}{2}\zeta(3)-2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^2},$$

whence we obtain that

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^2}=\frac{5}{8}\zeta(3),$$

and the calculations to the point $i)$ are complete. Note that in the solution I used the simple fact that $\log^2(1-x)=2\sum_{n=1}^{\infty}x^{n+1}\frac{H_n}{n+1}$.

Next, for the series from the point $ii)$, we write

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^4}=\frac{1}{6}\sum_{n=1}^{\infty}(-1)^n H_n\int_0^1 x^{n-1}\log^3(x)\mathrm{d}x$$

$$=\frac{1}{6}\int_0^1\log^3(x)\sum_{n=1}^{\infty}(-1)^n x^{n-1}H_n\mathrm{d}x=-\frac{1}{6}\int_0^1\frac{\log(1+x)\log^3(x)}{x(1+x)}\mathrm{d}x$$

$$=\frac{1}{6}\underbrace{\int_0^1\frac{\log(1+x)\log^3(x)}{1+x}\mathrm{d}x}_{I_1}-\frac{1}{6}\underbrace{\int_0^1\frac{\log(1+x)\log^3(x)}{x}\mathrm{d}x}_{I_2}. \tag{6.258}$$

To calculate the first integral in (6.258), we start with the integration by parts that gives

$$
\begin{aligned}
I_1 &= \int_0^1 \frac{\log(1+x)\log^3(x)}{1+x}\mathrm{d}x = \frac{1}{2}\int_0^1 (\log^2(1+x))'\log^3(x)\mathrm{d}x \\
&= \underbrace{\frac{1}{2}\log^2(1+x)\log^3(x)\Big|_{x=0}^{x=1}}_{0} - \frac{3}{2}\int_0^1 \frac{\log^2(1+x)\log^2(x)}{x}\mathrm{d}x \\
&= -\frac{3}{2}\int_0^1 \frac{\log^2(1+x)\log^2(x)}{x}\mathrm{d}x \\
&\quad \left\{\text{make the change of variable } x = \frac{y}{1-y}\right\} \\
&= -\frac{3}{2}\int_0^{1/2} \frac{\log^2(1-y)\log^2\left(\frac{y}{1-y}\right)}{(1-y)y}\mathrm{d}y = 3\underbrace{\int_0^{1/2}\frac{\log^3(1-y)\log(y)}{1-y}\mathrm{d}y}_{I_3} \\
&\quad -\frac{3}{2}\underbrace{\int_0^{1/2}\frac{\log^4(1-y)}{y}\mathrm{d}y}_{I_4} - \frac{3}{2}\underbrace{\int_0^{1/2}\frac{\log^2(1-y)\log^2(y)}{y}\mathrm{d}y}_{I_5} \\
&\quad -\frac{3}{2}\underbrace{\int_0^{1/2}\frac{\log^2(1-y)\log^2(y)}{1-y}\mathrm{d}y}_{I_6} \\
&\quad +3\underbrace{\int_0^{1/2}\frac{\log^3(1-y)\log(y)}{y}\mathrm{d}y}_{I_7} - \frac{3}{2}\underbrace{\int_0^{1/2}\frac{\log^4(1-y)}{1-y}\mathrm{d}y}_{1/5\log^5(2)}.
\end{aligned}
\tag{6.259}
$$

Applying the integration by parts for the integral I_3 in (6.259), we have

$$
\begin{aligned}
I_3 &= \int_0^{1/2}\frac{\log^3(1-y)\log(y)}{1-y}\mathrm{d}y = -\frac{1}{4}\int_0^{1/2}(\log^4(1-y))'\log(y)\mathrm{d}y \\
&= \underbrace{-\frac{1}{4}\log^4(1-y)\log(y)\Big|_{y=0}^{y=1/2}}_{1/4\log^5(2)} + \frac{1}{4}\int_0^{1/2}\frac{\log^4(1-y)}{y}\mathrm{d}y = \frac{1}{4}\log^5(2) + \frac{1}{4}I_4.
\end{aligned}
\tag{6.260}
$$

As regards the integral I_4 in (6.259), let's make the change of variable $1-y=z$, and then we obtain

$$I_4=\int_0^{1/2}\frac{\log^4(1-y)}{y}\mathrm{d}y=\int_{1/2}^1\frac{\log^4(z)}{1-z}\mathrm{d}z=\left(\int_0^1-\int_0^{1/2}\right)\frac{\log^4(z)}{1-z}\mathrm{d}z$$

$$=\int_0^1\frac{\log^4(z)}{1-z}\mathrm{d}z-\int_0^{1/2}\frac{\log^4(z)}{1-z}\mathrm{d}z=\int_0^1\sum_{n=1}^{\infty}z^{n-1}\log^4(z)\mathrm{d}z$$

$$-\int_0^{1/2}\sum_{n=1}^{\infty}z^{n-1}\log^4(z)\mathrm{d}z$$

{reverse the order of summation and integration}

$$=\sum_{n=1}^{\infty}\int_0^1 z^{n-1}\log^4(z)\mathrm{d}z-\sum_{n=1}^{\infty}\int_0^{1/2} z^{n-1}\log^4(z)\mathrm{d}z$$

{make use of the results in (1.2) and (1.3)}

$$=24\sum_{n=1}^{\infty}\frac{1}{n^5}-\log^4(2)\sum_{n=1}^{\infty}\frac{1}{n2^n}-4\log^3(2)\sum_{n=1}^{\infty}\frac{1}{n^22^n}-12\log^2(2)\sum_{n=1}^{\infty}\frac{1}{n^32^n}$$

$$-24\log(2)\sum_{n=1}^{\infty}\frac{1}{n^42^n}-24\sum_{n=1}^{\infty}\frac{1}{n^52^n}$$

$$=24\zeta(5)-\log^5(2)-4\log^3(2)\operatorname{Li}_2\left(\frac{1}{2}\right)-12\log^2(2)\operatorname{Li}_3\left(\frac{1}{2}\right)$$

$$-24\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right)-24\operatorname{Li}_5\left(\frac{1}{2}\right)$$

{use the special values in (3.19) and (3.20)}

$$=24\zeta(5)-\frac{21}{2}\log^2(2)\zeta(3)+4\log^3(2)\zeta(2)-\log^5(2)$$

$$-24\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right)-24\operatorname{Li}_5\left(\frac{1}{2}\right). \tag{6.261}$$

For the integral I_6 in (6.259), we make the change of variable $1-y=z$, and we get

$$I_6=\int_0^{1/2}\frac{\log^2(1-y)\log^2(y)}{1-y}\mathrm{d}y=\int_{1/2}^1\frac{\log^2(z)\log^2(1-z)}{z}\mathrm{d}z$$

$$=\left(\int_0^1-\int_0^{1/2}\right)\frac{\log^2(z)\log^2(1-z)}{z}\mathrm{d}z=\int_0^1\frac{\log^2(z)\log^2(1-z)}{z}\mathrm{d}z-I_5$$

$$= 2\int_0^1 \log^2(z) \sum_{n=1}^{\infty} z^n \frac{H_n}{n+1} \mathrm{d}z - I_5 = 2\sum_{n=1}^{\infty} \frac{H_n}{n+1} \int_0^1 z^n \log^2(z)\mathrm{d}z - I_5$$

$$= 4\sum_{n=1}^{\infty} \frac{H_n}{(n+1)^4} - I_5 = 4\sum_{n=1}^{\infty} \frac{H_{n+1} - 1/(n+1)}{(n+1)^4} - I_5$$

{reindex the series and expand it}

$$= 4\sum_{n=1}^{\infty} \frac{H_n}{n^4} - 4\sum_{n=1}^{\infty} \frac{1}{n^5} - I_5$$

{make use of the classical Euler sum in (3.45), the case $n = 4$}

$$= 8\zeta(5) - 4\zeta(2)\zeta(3) - I_5. \tag{6.262}$$

Further, for the integral I_7 in (6.259), we integrate by parts, and we have

$$I_7 = \int_0^{1/2} \frac{\log^3(1-y)\log(y)}{y} \mathrm{d}y = \frac{1}{2}\int_0^{1/2} (\log^2(y))' \log^3(1-y)\mathrm{d}y$$

$$= \underbrace{\frac{1}{2}\log^2(y)\log^3(1-y)\Big|_{y=0}^{y=1/2}}_{-1/2\log^5(2)} + \frac{3}{2} I_6$$

{make use of the result in (6.262)}

$$= 12\zeta(5) - 6\zeta(2)\zeta(3) - \frac{1}{2}\log^5(2) - \frac{3}{2} I_5. \tag{6.263}$$

Collecting the results from (6.260), (6.261), (6.262), and (6.263) in (6.259), and then using the value of the integral I_5 which is given in (1.18), we get

$$I_1 = \int_0^1 \frac{\log(1+x)\log^3(x)}{1+x} \mathrm{d}x = \frac{87}{16}\zeta(5) - 3\zeta(2)\zeta(3). \tag{6.264}$$

Lastly, for the integral I_2 in (6.258), we write that

$$I_2 = \int_0^1 \frac{\log(1+x)\log^3(x)}{x} \mathrm{d}x = \int_0^1 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{n-1}}{n} \log^3(x)\mathrm{d}x$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} \int_0^1 x^{n-1} \log^3(x)\mathrm{d}x = 6\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^5} = -\frac{45}{8}\zeta(5). \tag{6.265}$$

Hence, by plugging the results from (6.264) and (6.265) in (6.258), we conclude that

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^4} = \frac{59}{32}\zeta(5) - \frac{1}{2}\zeta(2)\zeta(3),$$

and the calculations to the point ii) are complete.

In the mathematical literature, a generalization that covers both points is known,

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{H_k}{k^{2n}} = \left(n+\frac{1}{2}\right)\eta(2n+1) - \frac{1}{2}\zeta(2n+1) - \sum_{i=1}^{n-1}\eta(2i)\zeta(2n-2i+1),$$

and a beautiful solution to it may be found in [6] (the strategy by elementary series manipulations, as shown in the paper, is elegant). It would be interesting finding more ways of deriving such series since they seem to be pretty resistant.

As a final note, remember that thanks to the mentioned series generalization I was able to calculate the generalized integral from the first chapter, in Sect. 1.10. So good to know it!

6.54 A Nice Challenging Trio of Alternating Harmonic Series, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(2)}}{n^3}$, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(3)}}{n^2}$, and $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(4)}}{n}$

Solution *Ah, I think I met the non-alternating series versions of the first two proposed alternating series in one of the previous sections!* Yeah, the non-alternating series versions of the first two proposed series have already been included in Sect. 4.21. *Now, how about the strategy of attacking the alternating versions?* Well, I think I might like to create a system of two relations with the series from the points i) and ii), and then extract each series. If you enjoy the calculations of series, prepare for an exciting, enjoyable *round*!

In order to get the values of the desired series, we make use of the result in (1.4), where expressing the harmonic number in terms of Digamma function, since $H_n = \psi(n+1)+\gamma$, differentiating twice with respect to n and then returning to the generalized harmonic numbers, we get

$$\int_0^1 x^{n-1}\log^2(x)\log(1-x)\mathrm{d}x = 2\zeta(3)\frac{1}{n} + 2\zeta(2)\frac{1}{n^2} - 2\frac{H_n}{n^3} - 2\frac{H_n^{(2)}}{n^2} - 2\frac{H_n^{(3)}}{n}. \tag{6.266}$$

Using the identity in (6.266) where we multiply both sides by $(-1)^{n-1}/n$ and consider the sum from $n = 1$ to ∞, we get

$$\sum_{n=1}^{\infty}\int_0^1 (-1)^{n-1}\frac{x^{n-1}}{n}\log^2(x)\log(1-x)\mathrm{d}x = \int_0^1 \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{n-1}}{n}$$

$$\times \log^2(x)\log(1-x)\mathrm{d}x$$

$$= \int_0^1 \frac{\log(1-x)\log^2(x)\log(1+x)}{x}\mathrm{d}x$$

$$= 2\zeta(3)\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2} + 2\zeta(2)\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^3} - 2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^4}$$

$$-2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(2)}}{n^3} - 2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(3)}}{n^2}$$

{make use of the result in (4.89)}

$$= \frac{7}{2}\zeta(2)\zeta(3) - \frac{59}{16}\zeta(5) - 2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(2)}}{n^3} - 2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(3)}}{n^2},$$

whence we get that

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(2)}}{n^3} + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(3)}}{n^2}$$

$$= \frac{7}{4}\zeta(2)\zeta(3) - \frac{59}{32}\zeta(5) - \frac{1}{2}\int_0^1 \frac{\log(1-x)\log^2(x)\log(1+x)}{x}\mathrm{d}x$$

{make use of the generalized integral in (1.19), the case $n = 1$}

$$= \frac{11}{8}\zeta(2)\zeta(3) - \zeta(5), \tag{6.267}$$

and we have obtained a first relation between the first two alternating harmonic series.

To get a second relation between the first two harmonic series, we might like to make use of the Cauchy product of two series (see [25, Chapter III, pp. 197–199]), the version which states that if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are absolutely convergent, then we have

$$\left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right) = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} a_k b_{n-k+1}\right). \tag{6.268}$$

Now, considering $\operatorname{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ and $\operatorname{Li}_3(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^3}$ and applying the Cauchy product formula in (6.268), we get

$$\operatorname{Li}_2(x)\operatorname{Li}_3(x) = \left(\sum_{n=1}^{\infty} \frac{x^n}{n^2}\right)\left(\sum_{n=1}^{\infty} \frac{x^n}{n^3}\right) = \sum_{n=1}^{\infty} x^{n+1}\left(\sum_{k=1}^{n} \frac{1}{k^2(n-k+1)^3}\right)$$

$$= \sum_{n=1}^{\infty} x^{n+1}\Biggl(3\sum_{k=1}^{n} \frac{1}{k(n+1)^4} + 3\sum_{k=1}^{n} \frac{1}{(n-k+1)(n+1)^4} + \sum_{k=1}^{n} \frac{1}{k^2(n+1)^3}$$

$$+2\sum_{k=1}^{n} \frac{1}{(n-k+1)^2(n+1)^3} + \sum_{k=1}^{n} \frac{1}{(n-k+1)^3(n+1)^2}\Biggr)$$

$$= 6\sum_{n=1}^{\infty} x^{n+1} \frac{H_{n+1} - \frac{1}{n+1}}{(n+1)^4} + 3\sum_{n=1}^{\infty} x^{n+1} \frac{H_{n+1}^{(2)} - \frac{1}{(n+1)^2}}{(n+1)^3} + \sum_{n=1}^{\infty} x^{n+1} \frac{H_{n+1}^{(3)} - \frac{1}{(n+1)^3}}{(n+1)^2}$$

$$\{\text{reindex the series and expand them}\}$$

$$= 6\sum_{n=1}^{\infty} x^n \frac{H_n}{n^4} + 3\sum_{n=1}^{\infty} x^n \frac{H_n^{(2)}}{n^3} + \sum_{n=1}^{\infty} x^n \frac{H_n^{(3)}}{n^2} - 10\operatorname{Li}_5(x). \tag{6.269}$$

If we plug $x = -1$ in (6.269), and use the result in (4.89), we obtain a second relation between the first two alternating harmonic series,

$$3\sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n^{(2)}}{n^3} + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n^{(3)}}{n^2} = \frac{21}{8}\zeta(2)\zeta(3) - \frac{27}{16}\zeta(5). \tag{6.270}$$

Finally, by combining the relations in (6.267) and (6.270), we conclude that

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n^{(2)}}{n^3} = \frac{5}{8}\zeta(2)\zeta(3) - \frac{11}{32}\zeta(5)$$

and

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n^{(3)}}{n^2} = \frac{3}{4}\zeta(2)\zeta(3) - \frac{21}{32}\zeta(5),$$

and the solutions to the points $i)$ and $ii)$ are complete.

Returning to the Cauchy product of two series as in (6.268), and using the version in the *Mertens' theorem* (see [18, pp. 82–83]), with both series converging, and at least one converging absolutely, where we consider $\operatorname{Li}_4(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^4}$ and $-\log(1-x)=\sum_{n=1}^{\infty}\frac{x^n}{n}$, we have

$$-\operatorname{Li}_4(x)\log(1-x)=\left(\sum_{n=1}^{\infty}\frac{x^n}{n^4}\right)\left(\sum_{n=1}^{\infty}\frac{x^n}{n}\right)=\sum_{n=1}^{\infty}x^{n+1}\left(\sum_{k=1}^{n}\frac{1}{k^4(n-k+1)}\right)$$

$$=\sum_{n=1}^{\infty}x^{n+1}\left(\sum_{k=1}^{n}\frac{1}{k(n+1)^4}+\sum_{k=1}^{n}\frac{1}{(n-k+1)(n+1)^4}+\sum_{k=1}^{n}\frac{1}{k^2(n+1)^3}\right.$$

$$\left.+\sum_{k=1}^{n}\frac{1}{k^3(n+1)^2}+\sum_{k=1}^{n}\frac{1}{k^4(n+1)}\right)$$

$$=2\sum_{n=1}^{\infty}x^{n+1}\frac{H_{n+1}-\frac{1}{n+1}}{(n+1)^4}+\sum_{n=1}^{\infty}x^{n+1}\frac{H_{n+1}^{(2)}-\frac{1}{(n+1)^2}}{(n+1)^3}+\sum_{n=1}^{\infty}x^{n+1}\frac{H_{n+1}^{(3)}-\frac{1}{(n+1)^3}}{(n+1)^2}$$

$$+\sum_{n=1}^{\infty}x^{n+1}\frac{H_{n+1}^{(4)}-\frac{1}{(n+1)^4}}{n+1}$$

{reindex the series and expand them}

$$=2\sum_{n=1}^{\infty}x^n\frac{H_n}{n^4}+\sum_{n=1}^{\infty}x^n\frac{H_n^{(2)}}{n^3}+\sum_{n=1}^{\infty}x^n\frac{H_n^{(3)}}{n^2}+\sum_{n=1}^{\infty}x^n\frac{H_n^{(4)}}{n}-5\operatorname{Li}_5(x). \tag{6.271}$$

If we plug $x=-1$ in (6.271), combined with the values of the series from the points $i)$, $ii)$, and (4.89), we conclude that

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(4)}}{n}=2\zeta(5)-\frac{3}{8}\zeta(2)\zeta(3)-\frac{7}{8}\log(2)\zeta(4),$$

and the solution to the point $iii)$ is complete.

It's so pleasant to see the power of the real methods when wisely combining some simple results. The first alternating harmonic series together with other alternating series versions of weight 5 may be found in [62] too. Also, alternating harmonic series of various weights (including the weight 5) are given in [4, 6, 20, 54].

6.55 Encountering an Alternating Harmonic Series of Weight 5 with an Eye-Catching Closed-Form, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2}{n^3}$

Solution Having derived in the previous sections a few key alternating harmonic series, we are prepared now to take a new beautiful challenge with an apparently daunting alternating harmonic series (involving the squared harmonic number!).

One of the useful results we might like to employ can be obtained by combining the generating functions in (4.6), the case $m = 2$, and (4.7) that immediately gives

$$\sum_{n=1}^{\infty} x^n \left(H_n^2 - H_n^{(2)}\right) = \frac{\log^2(1-x)}{1-x}. \tag{6.272}$$

Now, in (6.272) we replace x by $-x$, multiply both sides by $\log^2(x)/x$, and then integrate from $x = 0$ to $x = 1$, and we have

$$\int_0^1 \sum_{n=1}^{\infty}(-1)^n x^{n-1} \log^2(x) \left(H_n^2 - H_n^{(2)}\right) \mathrm{d}x = \sum_{n=1}^{\infty}(-1)^n \left(H_n^2 - H_n^{(2)}\right) \times \int_0^1 x^{n-1} \log^2(x) \mathrm{d}x$$

$$= 2\sum_{n=1}^{\infty}(-1)^n \frac{H_n^2 - H_n^{(2)}}{n^3} = 2\sum_{n=1}^{\infty}(-1)^{n-1} \frac{H_n^{(2)}}{n^3} - 2\sum_{n=1}^{\infty}(-1)^{n-1} \frac{H_n^2}{n^3}$$

$$= \int_0^1 \frac{\log^2(x)\log^2(1+x)}{x(1+x)} \mathrm{d}x = \underbrace{\int_0^1 \frac{\log^2(x)\log^2(1+x)}{x} \mathrm{d}x}_{I_1} - \underbrace{\int_0^1 \frac{\log^2(x)\log^2(1+x)}{1+x} \mathrm{d}x}_{I_2}. \tag{6.273}$$

For the integral I_1 in (6.55), we write that

$$I_1 = \int_0^1 \frac{\log^2(x)\log^2(1+x)}{x} \mathrm{d}x = 2\int_0^1 \sum_{n=1}^{\infty}(-1)^{n-1} \frac{x^n}{n+1} H_n \log^2(x) \mathrm{d}x$$

{reverse the order of summation and integration}

$$= 2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n+1}\int_0^1 x^n \log^2(x)\mathrm{d}x = 4\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{(n+1)^4}$$

$$= 4\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_{n+1}-1/(n+1)}{(n+1)^4}$$

{reindex the series and expand it}

$$= 4\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^5} - 4\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^4} = \frac{15}{4}\zeta(5) - 4\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^4}$$

{make use of the result in (4.89)}

$$= 2\zeta(2)\zeta(3) - \frac{29}{8}\zeta(5). \tag{6.274}$$

Further, for the integral I_2 in (6.55), we make the change of variable $x = y/(1-y)$, and then we have

$$I_2 = \int_0^1 \frac{\log^2(x)\log^2(1+x)}{1+x}\mathrm{d}x = \int_0^{1/2}\frac{\log^2(1-y)(\log(y)-\log(1-y))^2}{1-y}\mathrm{d}y$$

$$= \int_0^{1/2}\frac{\log^4(1-y)}{1-y}\mathrm{d}y + \int_0^{1/2}\frac{\log^2(1-y)\log^2(y)}{1-y}\mathrm{d}y$$

$$-2\int_0^{1/2}\frac{\log^3(1-y)\log(y)}{1-y}\mathrm{d}y$$

{for the middle integral combine the results in (6.262) and (1.18)}

$$= \frac{63}{8}\zeta(5) + \frac{2}{3}\log^3(2)\zeta(2) - \frac{7}{4}\log^2(2)\zeta(3) - 2\zeta(2)\zeta(3) + \frac{4}{15}\log^5(2)$$

$$-4\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right) - 4\operatorname{Li}_5\left(\frac{1}{2}\right) - 2\int_0^{1/2}\frac{\log^3(1-y)\log(y)}{1-y}\mathrm{d}y$$

$$\left\{\text{integrating by parts, } \int_0^{1/2}\frac{\log^3(1-y)\log(y)}{1-y}\mathrm{d}y = \frac{1}{4}\log^5(2)\right.$$

$$\left.+\frac{1}{4}\int_0^{1/2}\frac{\log^4(1-y)}{y}\mathrm{d}y\right\}$$

$$= \frac{63}{8}\zeta(5) + \frac{2}{3}\log^3(2)\zeta(2) - \frac{7}{4}\log^2(2)\zeta(3) - 2\zeta(2)\zeta(3) - \frac{7}{30}\log^5(2)$$

$$-4\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right)-4\operatorname{Li}_5\left(\frac{1}{2}\right)-\frac{1}{2}\int_0^{1/2}\frac{\log^4(1-y)}{y}\mathrm{d}y$$

{the value of the last integral is given in (6.261)}

$$=\frac{4}{15}\log^5(2)-\frac{4}{3}\log^3(2)\zeta(2)+\frac{7}{2}\log^2(2)\zeta(3)-\frac{33}{8}\zeta(5)-2\zeta(2)\zeta(3)$$

$$+8\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right)+8\operatorname{Li}_5\left(\frac{1}{2}\right). \tag{6.275}$$

Plugging the results from (6.274) and (6.275) in (6.55), we get that

$$\int_0^1\frac{\log^2(x)\log^2(1+x)}{x(1+x)}\mathrm{d}x$$

$$=\frac{1}{2}\zeta(5)-\frac{7}{2}\log^2(2)\zeta(3)+\frac{4}{3}\log^3(2)\zeta(2)+4\zeta(2)\zeta(3)-\frac{4}{15}\log^5(2)$$

$$-8\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right)-8\operatorname{Li}_5\left(\frac{1}{2}\right). \tag{6.276}$$

Hence, if we consider in (6.55) the value in (6.276) and the value of the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(2)}}{n^3}$ which is given in (4.90), we obtain the desired value

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2}{n^3}$$

$$=\frac{2}{15}\log^5(2)-\frac{11}{8}\zeta(2)\zeta(3)-\frac{19}{32}\zeta(5)+\frac{7}{4}\log^2(2)\zeta(3)-\frac{2}{3}\log^3(2)\zeta(2)$$

$$+4\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right)+4\operatorname{Li}_5\left(\frac{1}{2}\right),$$

and the solution is finalized.

For an alternative way of establishing a relation between $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2}{n^3}$ and $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(2)}}{n^3}$, make use of the identity in (4.16). A slightly different series variant, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2}{(n+1)^3}$, from which we can get the given series may be found in [54]. The value of the alternating harmonic series may also be found in [62].

6.56 Encountering Another Alternating Harmonic Series of Weight 5 with a Dazzling Closed-Form, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^3}{n^2}$

Solution Let's make use of the result in (1.6) where if we multiply the opposite sides by $(-1)^{n-1}/n$ and then consider the summation from $n=1$ to ∞, we get

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^3}{n^2}+3\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_nH_n^{(2)}}{n^2}+2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(3)}}{n^2}$$

{the value of the last series is given in (4.91)}

$$=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^3}{n^2}+3\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_nH_n^{(2)}}{n^2}+\frac{3}{2}\zeta(2)\zeta(3)-\frac{21}{16}\zeta(5)$$

$$=-\int_0^1\frac{\log^3(1-x)\log(1+x)}{x}\mathrm{d}x$$

{use the algebraic identity, $A^3B=\frac{1}{8}((A+B)^4-(A-B)^4-8AB^3)$}

{where we consider replacing A by $\log(1-x)$ and B by $\log(1+x)$}

$$=-\frac{1}{8}\int_0^1\frac{\log^4(1-x^2)}{x}\mathrm{d}x+\frac{1}{8}\int_0^1\frac{\log^4\left(\frac{1-x}{1+x}\right)}{x}\mathrm{d}x$$
$$+\int_0^1\frac{\log(1-x)\log^3(1+x)}{x}\mathrm{d}x$$

{the values of the first two integrals are given in (3.27) and (3.28)}

$$=\frac{69}{16}\zeta(5)+\int_0^1\frac{\log(1-x)\log^3(1+x)}{x}\mathrm{d}x$$

{make use of the identity, $\log^3(1+x)=3\sum_{n=1}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}(H_n^2-H_n^{(2)})$, which is}

{obtained by integrating and combining the identities in (4.6), with $m=2$, and (4.7)}

$$= \frac{69}{16}\zeta(5) + 3\int_0^1 \sum_{n=1}^{\infty}(-1)^n x^n \log(1-x)\frac{H_n^2 - H_n^{(2)}}{n+1}\mathrm{d}x = \frac{69}{16}\zeta(5)$$

$$+3\sum_{n=1}^{\infty}(-1)^n \frac{H_n^2 - H_n^{(2)}}{n+1}\int_0^1 x^n \log(1-x)\mathrm{d}x = \frac{69}{16}\zeta(5) + 3\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2 H_{n+1}}{(n+1)^2}$$

$$-3\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_{n+1}H_n^{(2)}}{(n+1)^2} = \frac{69}{16}\zeta(5) + 3\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(H_{n+1} - 1/(n+1))^2 H_{n+1}}{(n+1)^2}$$

$$-3\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_{n+1}(H_{n+1}^{(2)} - 1/(n+1)^2)}{(n+1)^2}$$

{reindex the series and expand them}

$$= 6\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2}{n^3} - 6\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^4} + 3\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n H_n^{(2)}}{n^2}$$

$$-3\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^3}{n^2} + \frac{69}{16}\zeta(5)$$

{the values of the first two series are given in (4.93) and (4.89)}

$$= \frac{4}{5}\log^5(2) - 4\log^3(2)\zeta(2) + \frac{21}{2}\log^2(2)\zeta(3) - \frac{21}{4}\zeta(2)\zeta(3) - \frac{165}{16}\zeta(5)$$

$$+24\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right) + 24\operatorname{Li}_5\left(\frac{1}{2}\right) + 3\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n H_n^{(2)}}{n^2} - 3\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^3}{n^2},$$

from which we obtain that

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^3}{n^2}$$

$$= \frac{1}{5}\log^5(2) - \log^3(2)\zeta(2) + \frac{21}{8}\log^2(2)\zeta(3) - \frac{27}{16}\zeta(2)\zeta(3) - \frac{9}{4}\zeta(5)$$

$$+6\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right) + 6\operatorname{Li}_5\left(\frac{1}{2}\right),$$

and the solution is finalized.

A slightly different series variant, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^3}{(n+1)^2}$, from which we can get the given series may be found in [54]. The value of the series is also given in [62].

6.57 Yet Another Encounter with a Superb Alternating Harmonic Series of Weight 5, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n H_n^{(2)}}{n^2}$

Solution The technique to employ here reduces to establishing a relation between the series to calculate and the alternating series from the previous section. Recalling and using the identity in (1.5), where we multiply the opposite sides by $(-1)^{n-1}H_n/n$ and then consider the summation from $n=1$ to ∞, we get

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^3}{n^2}+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n H_n^{(2)}}{n^2}=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n}\int_0^1 x^{n-1}\log^2(1-x)\mathrm{d}x$$

$$=\sum_{n=1}^{\infty}(-1)^{n-1}H_n\int_0^1\left(\int_0^1 (xy)^{n-1}\log^2(1-x)\mathrm{d}y\right)\mathrm{d}x$$

{reverse the order of integration and summation}

$$=\int_0^1\left(\int_0^1 \log^2(1-x)\sum_{n=1}^{\infty}(-1)^{n-1}(xy)^{n-1}H_n\mathrm{d}y\right)\mathrm{d}x$$

$$=\int_0^1\left(\int_0^1 \frac{\log^2(1-x)\log(1+xy)}{xy(1+xy)}\mathrm{d}y\right)\mathrm{d}x$$

$$=-\int_0^1\frac{\log^2(1-x)\operatorname{Li}_2(-x)}{x}\mathrm{d}x-\frac{1}{2}\int_0^1\frac{\log^2(1-x)\log^2(1+x)}{x}\mathrm{d}x. \tag{6.277}$$

For the first integral in (6.277), we employ the identity in (1.5) and write

$$\int_0^1\frac{\log^2(1-x)\operatorname{Li}_2(-x)}{x}\mathrm{d}x=\int_0^1\log^2(1-x)\sum_{n=1}^{\infty}(-1)^n\frac{x^{n-1}}{n^2}\mathrm{d}x$$

{reverse the order of summation and integration}

$$=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\int_0^1 x^{n-1}\log^2(1-x)\mathrm{d}x=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left(\frac{H_n^2+H_n^{(2)}}{n}\right)$$

$$=\sum_{n=1}^{\infty}(-1)^n\frac{H_n^2}{n^3}+\sum_{n=1}^{\infty}(-1)^n\frac{H_n^{(2)}}{n^3}$$

{make use of the results in (4.93) and (4.90)}

$$= \frac{15}{16}\zeta(5) - \frac{7}{4}\log^2(2)\zeta(3) + \frac{2}{3}\log^3(2)\zeta(2) + \frac{3}{4}\zeta(2)\zeta(3) - \frac{2}{15}\log^5(2)$$

$$- 4\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right) - 4\operatorname{Li}_5\left(\frac{1}{2}\right). \tag{6.278}$$

Finally, if we plug the results from (6.278), (1.16), and (4.94) in (6.277), we conclude that

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n H_n^{(2)}}{n^2}$$

$$= \frac{23}{8}\zeta(5) - \frac{7}{4}\log^2(2)\zeta(3) + \frac{2}{3}\log^3(2)\zeta(2) + \frac{15}{16}\zeta(2)\zeta(3) - \frac{2}{15}\log^5(2)$$

$$-4\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right) - 4\operatorname{Li}_5\left(\frac{1}{2}\right),$$

and the solution is finalized.

To get another relation between the two series, one might exploit the result in (6.23) to get and use $\sum_{n=1}^{\infty} x^n (H_n^3 - 3H_n H_n^{(2)} + 2H_n^{(3)}) = -\frac{\log^3(1-x)}{1-x}$.

The value of the alternating harmonic series may also be found in [62].

6.58 Fascinating Sums of Two Alternating Harmonic Series Involving the Generalized Harmonic Number

Solution Both sums of series in this section can be beautifully related to integrals from the first chapter. Rather than trying to calculate each series separately, I might like to find a way to evaluate both of them at once, for each point of the problem. *So, how would we like to nicely do it?*

For the part $i)$ of the problem, let's combine first the results in (4.6), the case $m = 2$, and (4.7) to get $\sum_{n=1}^{\infty} x^n \left(H_n^2 - H_n^{(2)}\right) = \frac{\log^2(1-x)}{1-x}$, where if we replace x by $-x$ and then divide both sides by x, we have

$$\sum_{n=1}^{\infty}(-1)^n x^{n-1}\left(H_n^2 - H_n^{(2)}\right) = \frac{\log^2(1+x)}{x(1+x)}. \tag{6.279}$$

Multiplying both sides of (6.279) by $\log^2(1-x)$ and then integrating from $x=0$ to $x=1$, we obtain

$$\int_0^1 \sum_{n=1}^{\infty}(-1)^n \left(H_n^2 - H_n^{(2)}\right) x^{n-1}\log^2(1-x)\mathrm{d}x$$

{reverse the order of summation and integration}

$$=\sum_{n=1}^{\infty}(-1)^n \left(H_n^2 - H_n^{(2)}\right)\int_0^1 x^{n-1}\log^2(1-x)\mathrm{d}x$$

{employ the integral result in (1.5)}

$$=\sum_{n=1}^{\infty}(-1)^n\frac{H_n^4-(H_n^{(2)})^2}{n}=\sum_{n=1}^{\infty}(-1)^n\frac{H_n^4}{n}+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(H_n^{(2)})^2}{n}$$

$$=\int_0^1\frac{\log^2(1-x)\log^2(1+x)}{x(1+x)}\mathrm{d}x$$

$$=\int_0^1\frac{\log^2(1-x)\log^2(1+x)}{x}\mathrm{d}x-\int_0^1\frac{\log^2(1-x)\log^2(1+x)}{1+x}\mathrm{d}x$$

{the values of the integrals are given in (1.16) and (1.17)}

$$=2\zeta(2)\zeta(3)-11\zeta(5)+\frac{2}{3}\log^3(2)\zeta(2)-\frac{9}{4}\log^2(2)\zeta(3)+\frac{9}{2}\log(2)\zeta(4)-\frac{1}{10}\log^5(2)$$

$$+4\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right)+8\operatorname{Li}_5\left(\frac{1}{2}\right),$$

and the part $i)$ of the problem is finalized.

The curious reader might be interested in finding the closed-forms of these series separately, and this can be obtained pretty fast with the previous results in hand. A useful relation can be built by using the identity in (1.7), where if we multiply both sides by $(-1)^{n-1}$ and then consider the summation from $n=1$ to ∞, we are led to

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^4}{n}+6\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2H_n^{(2)}}{n}+8\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_nH_n^{(3)}}{n}+3\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(H_n^{(2)})^2}{n}$$

$$=-6\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(4)}}{n}+\int_0^1\frac{\log^4(1-x)}{1+x}\mathrm{d}x$$

$$\overset{1-x=y}{=} -6\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(4)}}{n}+\frac{1}{2}\int_0^1 \frac{\log^4(y)}{1-y/2}\mathrm{d}y$$

$$=\frac{9}{4}\zeta(2)\zeta(3)+\frac{21}{4}\log(2)\zeta(4)-12\zeta(5)+24\operatorname{Li}_5\left(\frac{1}{2}\right), \tag{6.280}$$

where to get the last equality I used the geometric series and the result in (4.92).

To get another relation, we use the series representation of $\dfrac{\log^4(1-x)}{1-x}$ given at the end of the solutions to the Sect. 4.10, where if we divide both sides by x, then replace x by $-x$ and integrate from $x=0$ to $x=1$, we get

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^4}{n}-6\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2H_n^{(2)}}{n}+8\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_nH_n^{(3)}}{n}$$

$$+3\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(H_n^{(2)})^2}{n}$$

$$=6\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(4)}}{n}-\int_0^1\frac{\log^4(1+x)}{x(1+x)}\mathrm{d}x$$

$$\overset{x/(1+x)=y}{=} 6\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^{(4)}}{n}-\int_0^{1/2}\frac{\log^4(1-y)}{y}\mathrm{d}y$$

$$=24\operatorname{Li}_5\left(\frac{1}{2}\right)+24\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right)-4\log^3(2)\zeta(2)+\frac{21}{2}\log^2(2)\zeta(3)$$

$$-\frac{21}{4}\log(2)\zeta(4)-12\zeta(5)-\frac{9}{4}\zeta(2)\zeta(3)+\log^5(2), \tag{6.281}$$

where for the last equality I made use of the results in (4.92) and (6.261).

To obtain a last useful relation that makes possible the extraction of the remaining alternating harmonic series of weight 5, we need a clever approach, and differentiate the relation in (1.5) with respect to n, then multiply both sides by $(-1)^{n-1}H_n$ and consider the summation from $n=1$ to ∞ that leads to

$$2\zeta(3)\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n}+2\zeta(2)\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2}{n}-\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^3}{n^2}-\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_nH_n^{(2)}}{n^2}$$

$$-2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_nH_n^{(3)}}{n}-2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2H_n^{(2)}}{n}$$

{use the results in (4.5), (4.7), (4.94), and (4.95)}

$$=\log^3(2)\zeta(2)-\frac{15}{8}\log^2(2)\zeta(3)-\frac{5}{2}\log(2)\zeta(4)-\frac{5}{8}\zeta(5)+\frac{13}{4}\zeta(2)\zeta(3)-\frac{1}{15}\log^5(2)$$

$$-2\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right)-2\operatorname{Li}_5\left(\frac{1}{2}\right)-2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_nH_n^{(3)}}{n}-2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2H_n^{(2)}}{n}$$

$$=\int_0^1\frac{\log^2(1-x)\log(x)\log(1+x)}{x}\mathrm{d}x-\int_0^1\frac{\log^2(1-x)\log(x)\log(1+x)}{1+x}\mathrm{d}x. \tag{6.282}$$

Since we have $\log^2(1-x)\log(1+x)=\frac{1}{6}\left(\log^3(1-x^2)-\log^3\left(\frac{1-x}{1+x}\right)\right.$ $\left.-2\log^3(1+x)\right)$, then both integrals above can be reduced to simpler integrals. For example, with the logarithmic identity above, the variable change $(1-x)/(1+x)=y$ for the integral with $\log^3((1-x)/(1+x))$, and some rearrangements of the resulting integrals, we get for the first integral in (6.282) that

$$\int_0^1\frac{\log^2(1-x)\log(x)\log(1+x)}{x}\mathrm{d}x=\frac{1}{3}\int_0^1\frac{x\log^3(x)\log(1-x^2)}{1-x^2}\mathrm{d}x$$
$$+\frac{1}{6}\int_0^1\frac{\log(x)\log^3(1-x^2)}{x}\mathrm{d}x-\frac{1}{3}\int_0^1\frac{\log^3(x)\log(1-x)}{1-x}\mathrm{d}x$$
$$+\frac{1}{3}\int_0^1\frac{\log^3(x)\log(1+x)}{1+x}\mathrm{d}x-\frac{1}{3}\int_0^1\frac{\log(x)\log^3(1+x)}{x}\mathrm{d}x$$

{in the first two integrals make the change of variable $x^2=y$}

{and for the last two integrals make use of the integration by parts}

$$=-\frac{1}{2}\int_0^1\frac{\log^2(x)\log^2(1+x)}{x(1+x)}\mathrm{d}x-\frac{13}{48}\int_0^1\frac{\log^3(x)\log(1-x)}{1-x}\mathrm{d}x$$

{make use of the result in (6.276)}

$$=\frac{2}{15}\log^5(2)-\frac{2}{3}\log^3(2)\zeta(2)+\frac{7}{4}\log^2(2)\zeta(3)-\frac{7}{2}\zeta(5)-\frac{3}{8}\zeta(2)\zeta(3)$$
$$+4\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right)+4\operatorname{Li}_5\left(\frac{1}{2}\right), \tag{6.283}$$

where above I also used that $\int_0^1\frac{\log^3(x)\log(1-x)}{1-x}\mathrm{d}x=-\int_0^1\sum_{n=1}^{\infty}x^nH_n\log^3(x)\mathrm{d}x$ $=-\sum_{n=1}^{\infty}H_n\int_0^1 x^n\log^3(x)\mathrm{d}x=6\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^4}=6\sum_{n=1}^{\infty}\frac{H_{n+1}-1/(n+1)}{(n+1)^4}=$ $6\sum_{n=1}^{\infty}\frac{H_n}{n^4}-6\sum_{n=1}^{\infty}\frac{1}{n^5}=6(2\zeta(5)-\zeta(2)\zeta(3))$, and for the last equality I also used the result in (3.45), the case $n=4$. The integral in (6.283) is also given in [50], and a solution using the same starting strategy that exploits the algebraic identities is given in [33].

For the other integral in (6.282), we use the same algebraic identity as before, together with the variable change $(1-x)/(1+x) = y$ for the integral with $\log^3((1-x)/(1+x))$ and some rearrangements of the resulting integrals, that gives

$$\int_0^1 \frac{\log^2(1-x)\log(x)\log(1+x)}{1+x}\mathrm{d}x = \frac{1}{3}\int_0^1 \frac{\log^3(x)\log(1+x)}{1+x}\mathrm{d}x$$

$$-\frac{1}{3}\int_0^1 \frac{\log(x)\log^3(1+x)}{1+x}\mathrm{d}x - \frac{1}{6}\int_0^1 \frac{(1-x)\log^3(x)\log(1-x^2)}{1-x^2}\mathrm{d}x$$

$$+\frac{1}{6}\int_0^1 \frac{(1-x)\log(x)\log^3(1-x^2)}{1-x^2}\mathrm{d}x$$

$$\{\text{integrate by parts the first two integrals, and in the last}\}$$

$$\{\text{two integrals make the change of variable } x^2 = y\}$$

$$= \frac{1}{12}\int_0^1 \frac{\log^4(1+x)}{x}\mathrm{d}x - \frac{1}{2}\int_0^1 \frac{\log^2(x)\log^2(1+x)}{x}\mathrm{d}x$$

$$-\frac{1}{24}\int_0^1 \frac{\log(1-x)\log^3(x)}{x}\mathrm{d}x$$

$$+\frac{1}{96}\int_0^1 \frac{\log(1-x)\log^3(x)}{1-x}\mathrm{d}x + \frac{1}{24}\int_0^1 \frac{\log^3(1-x)\log(x)}{(1-x)\sqrt{x}}\mathrm{d}x$$

$$-\frac{1}{96}\int_0^1 \frac{\log(1-x)\log^3(x)}{(1-x)\sqrt{x}}\mathrm{d}x$$

$$= \frac{121}{16}\zeta(5) - \frac{15}{8}\log(2)\zeta(4) + \frac{21}{8}\log^2(2)\zeta(3) - \frac{2}{3}\log^3(2)\zeta(2)$$

$$-3\zeta(2)\zeta(3) - \frac{\log^5(2)}{15} - 2\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right) - 2\operatorname{Li}_5\left(\frac{1}{2}\right), \tag{6.284}$$

where in the calculations I used the results in (3.30) and (6.274), and the facts that $\displaystyle\int_0^1 \frac{\log(1-x)\log^3(x)}{x}\mathrm{d}x = -\int_0^1 \sum_{n=1}^{\infty}\frac{x^{n-1}}{n}\log^3(x)\mathrm{d}x = -\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1 x^{n-1}\log^3(x)\mathrm{d}x = 6\sum_{n=1}^{\infty}\frac{1}{n^5} = 6\zeta(5)$, $\displaystyle\int_0^1 \frac{\log^3(1-x)\log(x)}{(1-x)\sqrt{x}}\mathrm{d}x = \lim_{\substack{a\to 1/2\\ b\to 0}}\frac{\partial^4}{\partial a\partial b^3}\mathrm{B}(a,b) = 186\zeta(5)-90\log(2)\zeta(4)+84\log^2(2)\zeta(3)-24\log^3(2)\zeta(2)-78\zeta(2)\zeta(3)$ and then $\displaystyle\int_0^1 \frac{\log(1-x)\log^3(x)}{(1-x)\sqrt{x}}\mathrm{d}x = \lim_{\substack{a\to 1/2\\ b\to 0}}\frac{\partial^4}{\partial a^3\partial b}\mathrm{B}(a,b) = 372\zeta(5)$

$-\,180\log(2)\zeta(4)-126\zeta(2)\zeta(3)$. For the last two integrals, we may also combine the facts that $\sum_{n=1}^{\infty} x^n H_n = -\frac{\log(1-x)}{1-x}$ and $\sum_{n=1}^{\infty} x^n (H_n^3 - 3H_n H_n^{(2)} + 2H_n^{(3)}) = -\frac{\log^3(1-x)}{1-x}$ together with results of *The Master Theorem of Series* like (4.21), the case $m = 1$, and (4.27), and reduce everything to computing limits in terms of Polygamma function. The limits above can be done either manually or with the aid of *Mathematica*.

Now, if we plug the results from (6.283) and (6.284) in (6.282), we get

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2 H_n^{(2)}}{n}+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n H_n^{(3)}}{n}=\frac{1}{2}\log^3(2)\zeta(2)-\frac{1}{2}\log^2(2)\zeta(3)$$
$$-\frac{35}{16}\log(2)\zeta(4)+\frac{167}{32}\zeta(5)+\frac{5}{16}\zeta(2)\zeta(3)-\frac{2}{15}\log^5(2)$$
$$-\,4\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right)-4\operatorname{Li}_5\left(\frac{1}{2}\right). \tag{6.285}$$

At this point, combining the results in (6.280), (6.281), and (6.285) and the main result of the point $i)$ of the problem, we are able to extract the rest of the alternating series of weight 5, which is a beautiful moment! Note that besides the two series we wanted to calculate, we get two more series as a bonus of the strategy I employed!

So, we obtain that

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^4}{n}$$
$$=\frac{3}{10}\log^5(2)-\frac{4}{3}\log^3(2)\zeta(2)+\frac{9}{4}\log^2(2)\zeta(3)+\frac{11}{4}\log(2)\zeta(4)-\frac{83}{16}\zeta(5)$$
$$-\frac{11}{8}\zeta(2)\zeta(3)+4\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right)+8\operatorname{Li}_5\left(\frac{1}{2}\right), \tag{6.286}$$

then,

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(H_n^{(2)})^2}{n}$$

$$= \frac{\log^5(2)}{5} - \frac{2}{3}\log^3(2)\zeta(2) + \frac{29}{4}\log(2)\zeta(4) - \frac{259}{16}\zeta(5) + \frac{5}{8}\zeta(2)\zeta(3)$$

$$+ 8\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right) + 16\operatorname{Li}_5\left(\frac{1}{2}\right), \tag{6.287}$$

next,

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n^2 H_n^{(2)}}{n}$$

$$= \frac{7}{8}\log(2)\zeta(4) - \frac{7}{8}\log^2(2)\zeta(3) + \frac{1}{3}\log^3(2)\zeta(2) + \frac{3}{8}\zeta(2)\zeta(3)$$

$$-\frac{1}{12}\log^5(2) - 2\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right),$$

and finally,

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n H_n^{(3)}}{n}$$

$$= \frac{1}{6}\log^3(2)\zeta(2) + \frac{3}{8}\log^2(2)\zeta(3) - \frac{49}{16}\log(2)\zeta(4) + \frac{167}{32}\zeta(5) - \frac{1}{16}\zeta(2)\zeta(3)$$

$$-\frac{\log^5(2)}{20} - 2\log(2)\operatorname{Li}_4\left(\frac{1}{2}\right) - 4\operatorname{Li}_5\left(\frac{1}{2}\right),$$

and the derivation of the last alternating series of weight 5 is complete.

To solve the second part of the problem, use that $H_{2n} = \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{2n+k}\right)$ and $H_{2n}^{(2)} = \sum_{k=1}^{\infty}\left(\frac{1}{k^2} - \frac{1}{(2n+k)^2}\right)$, and then we write

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_{2n}}{n^3} + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_{2n}^{(2)}}{n^2} = \sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{H_{2n}}{n^3} + \frac{H_{2n}^{(2)}}{n^2}\right)$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^{n-1}\left(\frac{1}{kn^3}-\frac{1}{(2n+k)n^3}+\frac{1}{k^2n^2}-\frac{1}{(2n+k)^2n^2}\right)\right)\\
&= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^{n-1}\left(\frac{2}{k^2n^2}-\frac{4}{k^2(k+2n)^2}\right)\right) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\\
&\quad -4\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^{n-1}\frac{1}{k^2(k+2n)^2}\right)\\
&= \frac{5}{2}\zeta(4)+4\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^{n-1}\frac{1}{k^2}\int_0^1 x^{k+2n-1}\log(x)\mathrm{d}x\right)\\
&= \frac{5}{2}\zeta(4)+4\int_0^1\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}(-1)^{n-1}\frac{1}{k^2}x^{k+2n-1}\right)\log(x)\mathrm{d}x\\
&= \frac{5}{2}\zeta(4)+4\int_0^1\frac{x\log(x)\operatorname{Li}_2(x)}{1+x^2}\mathrm{d}x
\end{aligned}
$$

$$\{\text{make use of the result in (1.56), with } n=1\}$$

$$= 2G^2+\frac{37}{64}\zeta(4),$$

and the part $ii)$ of the problem is finalized.

Alternatively, one may start with the Cauchy product (see [33, Chapter III, pp. 307–571]) of $(\operatorname{Li}_2(x))^2$ that leads immediately to $2\sum_{n=1}^{\infty}x^n\frac{H_n}{n^3}+\sum_{n=1}^{\infty}x^n\frac{H_n^{(2)}}{n^2}=\frac{1}{2}\left((\operatorname{Li}_2(x))^2+6\operatorname{Li}_4(x)\right)$, and then the sum of series becomes $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_{2n}}{n^3}+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_{2n}^{(2)}}{n^2}=-\Re\{2((\operatorname{Li}_2(i))^2+6\operatorname{Li}_4(i))\}$, which leads to the value obtained above.

6.59 An Outstanding Sum of Series Representation of the Particular Value of the Riemann Zeta Function, $\zeta(4)$

Solution For this section I chose a curious sum with harmonic series which is derived with the help of an application of *The Master Theorem of Series*. I won't try to calculate each series separately, but I'll approach all the series at once.

We might like to make use of the first application of *The Master Theorem of Series* in (4.21), the case $m = 1$, where if we replace n by $2n$ and then consider the summation over both sides from $n = 1$ to ∞, we get

$$\frac{1}{4}\sum_{n=1}^{\infty}\frac{(H_{2n})^2}{n^2} + \frac{1}{4}\sum_{n=1}^{\infty}\frac{H_{2n}^{(2)}}{n^2} = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{H_k}{(k+1)(k+2n+1)n}\right)$$

$$\{\text{split the inner series according to } k \text{ odd and even}\}$$

$$= \frac{1}{4}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{H_{2k-1}}{k(k+n)n}\right) + 2\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{H_{2k}}{(2k+1)(2k+2n+1)2n}\right)$$

$$\{\text{reverse the order of summation in both series}\}$$

$$= \frac{1}{4}\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{H_{2k-1}}{k(k+n)n}\right) + 2\sum_{k=1}^{\infty}\frac{H_{2k}}{2k+1}\left(\sum_{n=1}^{\infty}\frac{1}{(2k+2n+1)2n}\right)$$

$$= \frac{1}{4}\sum_{k=1}^{\infty}\frac{H_k H_{2k-1}}{k^2} + 2\sum_{k=1}^{\infty}\frac{H_{2k}}{2k+1}\left(\sum_{n=1}^{\infty}\frac{1}{(2k+2n+1)2n}\right). \tag{6.288}$$

Now, for the inner series in (6.288), we write

$$\sum_{n=1}^{\infty}\frac{1}{(2k+2n+1)2n} = \frac{1}{2k+1}\sum_{n=1}^{\infty}\left(\frac{1}{2n} - \frac{1}{2n+2k+1}\right)$$

$$= \frac{1}{2k+1}\sum_{n=1}^{\infty}\left(\frac{1}{2n} - \frac{1}{2n+1} + \frac{1}{2n+1} - \frac{1}{2n+2k+1}\right)$$

$$= \frac{1}{2k+1}\sum_{n=1}^{\infty}\left(\frac{1}{2n} - \frac{1}{2n+1}\right) + \frac{1}{2k+1}\sum_{n=1}^{\infty}\left(\frac{1}{2n+1} - \frac{1}{2n+1+2k}\right)$$

$$= \frac{1-\log(2)}{2k+1} + \frac{1}{2k+1}\sum_{i=1}^{k}\frac{1}{2i+1}$$

$$= \frac{1-\log(2)}{2k+1} + \frac{1}{2k+1}\left(H_{2k} - \frac{1}{2}H_k - 1 + \frac{1}{2k+1}\right)$$

$$= \frac{1}{(2k+1)^2} + \frac{H_{2k}}{2k+1} - \frac{H_k}{2(2k+1)} - \frac{\log(2)}{2k+1}. \tag{6.289}$$

If we plug the result from (6.289) in (6.288), we get

$$\frac{1}{4}\sum_{n=1}^{\infty}\frac{(H_{2n})^2}{n^2}+\frac{1}{4}\sum_{n=1}^{\infty}\frac{H_{2n}^{(2)}}{n^2}=\frac{1}{4}\sum_{n=1}^{\infty}\frac{H_n(H_{2n}-1/(2n))}{n^2}$$

$$+2\sum_{n=1}^{\infty}\frac{H_{2n}}{2n+1}\left(\frac{1}{(2n+1)^2}+\frac{H_{2n}}{2n+1}-\frac{H_n}{2(2n+1)}-\frac{\log(2)}{2n+1}\right),$$

where using the simple fact that $\displaystyle\sum_{n=1}^{\infty}\frac{H_n}{n^3}=\frac{5}{4}\zeta(4)$, multiplying both sides of the relation above by 32/5 and then rearranging the series, we conclude that

$$\zeta(4)$$

$$=\frac{8}{5}\sum_{n=1}^{\infty}\frac{H_nH_{2n}}{n^2}+\frac{64}{5}\sum_{n=1}^{\infty}\frac{(H_{2n})^2}{(2n+1)^2}+\frac{64}{5}\sum_{n=1}^{\infty}\frac{H_{2n}}{(2n+1)^3}$$

$$-\frac{8}{5}\sum_{n=1}^{\infty}\frac{(H_{2n})^2}{n^2}-\frac{32}{5}\sum_{n=1}^{\infty}\frac{H_nH_{2n}}{(2n+1)^2}-\frac{64}{5}\log(2)\sum_{n=1}^{\infty}\frac{H_{2n}}{(2n+1)^2}-\frac{8}{5}\sum_{n=1}^{\infty}\frac{H_{2n}^{(2)}}{n^2},$$

and the solution is complete.

Surely, an alternative to this approach is to simply try to calculate every single harmonic series, and there would be some work to do!

6.60 An Excellent Representation of the Particular Value of the Riemann Zeta Function, $\zeta(4)$, with a Triple Series Involving the Factorials and the Generalized Harmonic Numbers

Solution Well, here we are ... at the last section of the present chapter and at the same time at the last section of the book, where I prepared a special representation of $\zeta(4)$ with factorials and harmonic numbers!

The magic behind the solution is simply based on writing the Gamma function expression in terms of a product with Beta function,

$$\frac{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_n)}{\Gamma(a_1+a_2+\cdots+a_n)}$$

$$=\frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(a_1+a_2)}\cdot\frac{\Gamma(a_1+a_2)\Gamma(a_3)}{\Gamma(a_1+a_2+a_3)}\cdots\frac{\Gamma(a_1+a_2+\cdots+a_{n-1})\Gamma(a_n)}{\Gamma(a_1+a_2+\cdots+a_n)}$$

$$=\mathrm{B}(a_1,a_2)\cdot\mathrm{B}(a_1+a_2,a_3)\cdots\mathrm{B}(a_1+a_2+\cdots+a_{n-1},a_n), \tag{6.290}$$

which is a simple idea that can be used for obtaining many wonderful results. Recently I found the present result is also given in [42, Chapter 1, p. 11].

Setting $n = 3$, $a_1 = i$, $a_2 = j$, and $a_3 = x$, in (6.290), we get $\dfrac{\Gamma(i)\Gamma(j)\Gamma(x)}{\Gamma(i+j+x)} = \mathrm{B}(i, j) \cdot \mathrm{B}(i+j, x)$, and then we have that

$$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty}\frac{\Gamma(i)\Gamma(j)\Gamma(x)}{\Gamma(i+j+x)}\right) = \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty}\mathrm{B}(i, j)\cdot\mathrm{B}(i+j, x)\right)$$

$$= \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty}\int_0^1 t^{i-1}(1-t)^{j-1}\left(\int_0^1 u^{i+j-1}(1-u)^{x-1}\mathrm{d}u\right)\mathrm{d}t\right)$$

{reverse the order of integration and summation}

$$= \int_0^1 u(1-u)^{x-1}\left(\int_0^1 \sum_{i=1}^{\infty}(ut)^{i-1}\left(\sum_{j=1}^{\infty}(u(1-t))^{j-1}\right)\mathrm{d}t\right)\mathrm{d}u$$

$$= \int_0^1 u(1-u)^{x-1}\left(\int_0^1 \frac{1}{(1-ut)(1-u(1-t))}\mathrm{d}t\right)\mathrm{d}u$$

$$= \int_0^1 \frac{u(1-u)^{x-1}}{2-u}\left(\int_0^1\left(\frac{1}{1-ut}+\frac{1}{1-u(1-t)}\right)\mathrm{d}t\right)\mathrm{d}u$$

$$= 2\int_0^1 \frac{(1-u)^{x-1}\log(1-u)}{u-2}\mathrm{d}u \overset{u=1-v}{=} -2\int_0^1 \frac{v^{x-1}\log(v)}{1+v}\mathrm{d}v$$

{make use of the result in (1.10) which we differentiate once}

$$= \frac{1}{2}\left(\psi^{(1)}\left(\frac{x}{2}\right) - \psi^{(1)}\left(\frac{1+x}{2}\right)\right). \tag{6.291}$$

Differentiating two times with respect to x the opposite sides of (6.291), replacing x by k and then considering the summation from $k = 1$ to ∞, and changing the summation order, we get

$$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{\Gamma(i)\Gamma(j)\Gamma(k)}{\Gamma(i+j+k)}\left((H_{i+j+k-1}-H_{k-1})^2 + H_{i+j+k-1}^{(2)} - H_{k-1}^{(2)}\right)\right)\right)$$

$$= \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{(i-1)!(j-1)!(k-1)!}{(i+j+k-1)!}\right.\right.$$

$$\left.\left.\times\left((H_{i+j+k-1}-H_{k-1})^2 + H_{i+j+k-1}^{(2)} - H_{k-1}^{(2)}\right)\right)\right)$$

$$= \frac{1}{8}\sum_{k=1}^{\infty}\left(\psi^{(3)}\left(\frac{k}{2}\right)-\psi^{(3)}\left(\frac{1+k}{2}\right)\right)=\frac{1}{8}\lim_{N\to\infty}\sum_{k=1}^{N}\left(\psi^{(3)}\left(\frac{k}{2}\right)-\psi^{(3)}\left(\frac{1+k}{2}\right)\right)$$

$$= \frac{1}{8}\lim_{N\to\infty}\left(\psi^{(3)}\left(\frac{1}{2}\right)-\psi^{(3)}\left(\frac{1+N}{2}\right)\right) = \frac{45}{4}\zeta(4),$$

where in the calculations above I used the Polygamma function reflection formula, $(-1)^m\psi^{(m)}(1-x)-\psi^{(m)}(x)=\pi\frac{d^m}{dx^m}\cot(\pi x)$, with $m=3$ and $x=1/2$, to get $\psi^{(3)}(1/2)=\pi^4$, and then the asymptotic expansion, $\psi^{(3)}(x)=2/x^3+O(1/x^4)$, as $x\to\infty$, which is obtained by differentiation from the asymptotic expansion of Digamma function (see [1, p. 259], [42, Chapter 1, p. 22]), and the solution is complete.

Having arrived at the end of the book I remember I closed my Preface with writing *Let's start the journey in the fascinating world of integrals, sums, and series and have much fun!* Now it would be the time to ask you if you enjoyed the journey in the fascinating world of integrals, sums, and series I proposed in this book! Hope you had a pleasant time with reading my book!

Cheers!

Cornel Ioan Vălean

References

1. Abramowitz, M., Stegun, I.A. (eds.): Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing. Dover Publications, New York (1972)
2. Adegoke, K.: On generalized harmonic numbers, Tornheim double series and linear Euler sums (2016)
3. Bailey, D., Borwein, J., Calkin, N., Girgensohn, R., Luke, R., Moll, V.: Experimental Mathematics in Action. A K Peters, Natick (2007)
4. Bailey, D.H., Borwein, J.M., Girgensohn, R.: Experimental evaluation of Euler sums. Exp. Math. **3**(1), 17–30 (1994)
5. Basin, S.L.: Problem B-23. Fibonacci Quart. **2**(1), 78–79 (1964)
6. Bastien, G.: Elementary methods for evaluating Jordan's sums and analogous Euler's type sums and for setting a sigma sum theorem (2013)
7. Batir, N.: Remarks on Vălean's Master Theorem of Series. J. Class. Anal. **13**(1), 79–82 (2018)
8. Berndt, B.C., Choi, Y.-S., Kang, S.-Y.: The problems submitted by Ramanujan to the Journal of the Indian Mathematical Society. Contemp. Math. 00 (1997)
9. Boros, G., Moll, V.H.: Irresistible Integrals, Symbolics, Analysis and Experiments in the Evaluation of Integrals. Cambridge University Press, Cambridge (2004)
10. Borwein, D., Borwein, J.M.: On an intriguing integral and some series related to $\zeta(4)$. Proc. Am. Math. Soc. **123**, 1191–1198 (1995)
11. Borwein, D., Borwein, J.M., Girgensohn, R.: Explicit evaluation of Euler sums. Proc. Edinb. Math. Soc. (2) **38**(2), 277–294 (1995)
12. Borwein, J., Bailey, D.: Mathematics by Experiment: Plausible Reasoning in the 21st Century, 2nd edn. A K Peters Ltd., Wellesley (2008)

13. Borwein, J., Bailey, D., Girgensohn, R.: Experimentation in Mathematics: Computational Paths to Discovery. A K Peters, Natick (2004)
14. Borwein, J., Devlin, K.: The Computer as Crucible: An Introduction to Experimental Mathematics. A K Peters, Wellesley (2009)
15. Choi, J., Srivastava, H.M.: Explicit evaluation of Euler and related sums. Ramanujan J. **10**, 51–70 (2005)
16. Choudary, A.D.R., Niculescu, C.P.: Real Analysis on Intervals. Springer, New Delhi (2014)
17. De Doelder, P.J.: On some series containing $\psi(x) - \psi(y)$ and $(\psi(x) - \psi(y))^2$ for certain values of x and y. J. Comput. Appl. Math. **37**, 125–141 (1991)
18. Duren, P.L.: Invitation to Classical Analysis. American Mathematical Society, Providence (2012)
19. Dutta, R.: Evaluation of a cubic Euler sum. J. Class. Anal. **9**(2), 151–159 (2016)
20. Flajolet, P., Salvy, B.: Euler Sums and Contour Integral Representations. Exp. Math. **7**, 15–35 (1998)
21. Furdui, O.: Limits, Series and Fractional Part Integrals. Problems in Mathematical Analysis. Springer, New York (2013)
22. Furdui, O., Vălean, C.I.: Evaluation of series involving the product of the tail of $\zeta(k)$ and $\zeta(k+1)$. Mediterr. J. Math. **13**, 517–526 (2016)
23. Gradshteyn, I.S., Ryzhik, I.M.: In: Zwillinger, D., Moll, V. (eds.) Table of Integrals, Series, and Products, 8th edn. Academic, New York (2015)
24. Guy, A.R.: Problem B-218. Fibonacci Quart. **10**(3), 335–336 (1972)
25. Hairer, E., Wanner, G.: Analysis by Its History. Springer, New York (1996)
26. Hoffman, M.E.: Sums of Products of Riemann Zeta Tails. Mediterr. J. Math. **13**, 2771–2781 (2016)
27. Huntley, H.E.: The Divine Proportion: A Study in Mathematical Beauty. Dover Publications, New York (1970)
28. La Gaceta de la RSME (Spain): A solution to the problem 290 (2017). http://gaceta.rsme.es/abrir.php?id=1375
29. La Gaceta de la RSME (Spain): The Problem 290 (2016). http://gaceta.rsme.es/abrir.php?id=1307
30. Lewin, L.: Polylogarithms and Associated Functions. North-Holland, New York (1981)
31. Macdonald, I.G.: Symmetric Functions and Hall Polynomials, 2nd edn. Oxford University Press, Oxford (1995)
32. Mathematics Stack Exchange: https://math.stackexchange.com/q/919572
33. Mathematics Stack Exchange: https://math.stackexchange.com/q/932244
34. Mathematics Stack Exchange: https://math.stackexchange.com/q/973408
35. Mathematics Stack Exchange: https://math.stackexchange.com/q/1855171
36. MathProblems Journal: Problems and Solutions. Problem 140. Vol. 5, No. 4 (2015). www.mathproblems-ks.org
37. Mezö, I.: Nonlinear Euler sum. Pac. J. Math. **272**, 201–226 (2014)
38. School Science and Mathematics Association (SSMA): Problem 5406. https://ssma.org/wp-content/uploads/2016/05/May-2016-.pdf (2016)
39. Sofo, A.: Harmonic number sums in higher powers. J. Math. Anal. **2**(2), 15–22 (2011)
40. Sofo, A.: New results containing quadratic harmonic numbers. J. Class. Anal. **9**(2), 117–125 (2016)
41. Spieß, J.: Some identities involving harmonic numbers. Math. Comp. **55**(192), 839–863 (1990)
42. Srivastava, H.M., Choi, J.: Series Associated with the Zeta and Related Functions. Springer (originally published by Kluwer), Dordrecht (2001)
43. Stolz, O.: Vorlesungen über allgemeine Arithmetik: nach den Neueren Ansichten, pp. 173–175. Teubners, Leipzig (1885)
44. Tauraso, R.: AMM 11910 Solution. www.mat.uniroma2.it/~lowtauraso/AMM/AMM11910.pdf.
45. Vălean, C.I.: A master theorem of series and an evaluation of a cubic harmonic series. J. Class. Anal. **10**(2), 97–107 (2017)

46. Vălean, C.I.: A new proof for a classical quadratic harmonic series. J. Class. Anal. **8**(2), 155–161 (2016)
47. Vălean, C.I., Furdui, O.: Reviving the quadratic series of Au-Yeung. J. Class. Anal. **6**(2), 113–118 (2015)
48. Vălean, C.I.: Problem 11910, problems and solutions. Am. Math. Mon. **123**(5), 504–511. https://tandfonline.com/doi/abs/10.4169/amer.math.monthly.123.5.504 (2016)
49. Vălean, C.I.: Problem 11921, problems and solutions. Am. Math. Mon. **123**(6), 613–620. https://tandfonline.com/doi/abs/10.4169/amer.math.monthly.123.6.613 (2016)
50. Vermaseren, J.A.M.: Harmonic Sums, Mellin Transforms and Integrals. https://arxiv.org/pdf/hep-ph/9806280v1.pdf
51. Wang, W., Lyu, Y.: Euler sums and Stirling sums. J. Number Theory **185**, 160–193 (2018)
52. Wang, W., Xu, C.: Euler sums of weights 10 and 11, and some special types (2017). https://www.researchgate.net/publication/318504251
53. Weisstein, E.W.: Dilogarithm. http://mathworld.wolfram.com/Dilogarithm.html
54. Weisstein, E.W.: Euler Sum. http://mathworld.wolfram.com/EulerSum.html
55. Weisstein, E. W.: Lucas Number. http://mathworld.wolfram.com/LucasNumber.html
56. Weisstein, E. W.: Newton-Girard Formulas. http://mathworld.wolfram.com/Newton-GirardFormulas.html
57. Weisstein, E. W.: Parseval's Theorem. http://mathworld.wolfram.com/ParsevalsTheorem.html
58. Weisstein, E. W.: Polygamma Function. http://mathworld.wolfram.com/PolygammaFunction.html
59. Weisstein, E. W.: Riemann Series Theorem. http://mathworld.wolfram.com/RiemannSeriesTheorem.html
60. Weisstein, E. W.: Symmetric Polynomial. http://mathworld.wolfram.com/SymmetricPolynomial.html
61. Weisstein, E. W.: Trigamma Function. http://mathworld.wolfram.com/TrigammaFunction.html
62. Xu, C.: Evaluations of Euler type sums of weight ≤ 5. https://arxiv.org/pdf/1704.03515.pdf (2017)
63. Zheng, D.-Y.: Further summation formulae related to generalized harmonic numbers. J. Math. Anal. Appl. **335**, 692–706 (2007)

Index

C. I. Vălean, *(Almost) Impossible Integrals, Sums, and Series*, Problem Books in Mathematics, https://doi.org/10.1007/978-3-030-02462-8