Chapter 2: Coordinate Systems and Transformation

2.1. Introduction

An orthogonal system is one in which the coordinates are mutually perpendicular.

2.2. Cartesian Coordinates (x, y, z)

The ranges of coordinate system

$$-\infty < x < \infty$$
$$-\infty < y < \infty$$
$$-\infty < z < \infty$$

A vector $\mathbf{A}$ in Cartesian (otherwise known as rectangular) coordinates can be written as

$$\left(A_x, A_y, A_z\right) \quad \text{or} \quad A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$$

where $\mathbf{a}_x$, $\mathbf{a}_x$, $\mathbf{a}_x$ are units vectors along the $x-$, $y-$, and $z-$ direction as shown.

2.3. Circular Cylindrical Coordinate (ρ, ϕ, z)

The ranges of coordinate system

$$0 \leq \rho < \infty$$
$$0 \leq \phi < \infty$$
$$-\infty < z < \infty$$

A vector $\mathbf{A}$ in cylindrical coodinates can be written as

$$\left(A_\rho, A_\phi, A_z\right) \quad \text{or} \quad A_\rho\mathbf{a}_\rho + A_\phi\mathbf{a}_\phi + A_z\mathbf{a}_z$$

The magnitude of $\mathbf{A}$ is

$$|\mathbf{A}| = \left(A_\rho^2 + A_\phi^2 + A_z^2\right)$$

The relation of unit vectors are

$$\mathbf{a}_\rho \bullet \mathbf{a}_\rho = \mathbf{a}_\phi \bullet \mathbf{a}_\phi = \mathbf{a}_z \bullet \mathbf{a}_z = 1$$
$$\mathbf{a}_\rho \bullet \mathbf{a}_\phi = \mathbf{a}_\phi \bullet \mathbf{a}_z = \mathbf{a}_z \bullet \mathbf{a}_\rho = 0$$
$$\mathbf{a}_\rho \times \mathbf{a}_\phi = \mathbf{a}_z$$
$$\mathbf{a}_\phi \times \mathbf{a}_z = \mathbf{a}_\rho$$
$$\mathbf{a}_z \times \mathbf{a}_\rho = \mathbf{a}_\phi$$

The relationships between the variables (x, y, z) of the Cartesian coordinate system and those of the cylindrical system and those of the cylindrical system (ρ, ϕ, z)

$$\rho = \sqrt{x^2 + y^2}$$
$$\phi = \arctan\left(\frac{y}{x}\right)$$
$$z = z$$

or

$$x = \rho \cos(\phi)$$
$$y = \rho \sin(\phi)$$
$$z = z$$

The relationship between $(\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_z)$ and $(\mathbf{a}_\rho, \mathbf{a}_\phi, \mathbf{a}_z)$ are obtained geometrically

$$\mathbf{a}_x = \cos(\phi)\mathbf{a}_\rho - \sin(\phi)\mathbf{a}_\phi$$
$$\mathbf{a}_y = \sin(\phi)\mathbf{a}_\rho + \cos(\phi)\mathbf{a}_\phi$$
$$\mathbf{a}_z = \mathbf{a}_z$$

or

$$\mathbf{a}_\rho = \cos(\phi)\mathbf{a}_x + \sin(\phi)\mathbf{a}_y$$
$$\mathbf{a}_\phi = -\sin(\phi)\mathbf{a}_x + \cos(\phi)\mathbf{a}_y$$
$$\mathbf{a}_z = \mathbf{a}_z$$

The relationship between (A_x, A_y, A_z) and (A_ρ, A_ϕ, A_z) are obtained by substitution

$$\mathbf{A} = \left(A_x \cos(\phi) + A_y \sin(\phi)\right)\mathbf{a}_\rho + \left(-A_x \sin(\phi) + A_y \cos(\phi)\right) + A_z \mathbf{a}_z$$

or

$$A_\rho = A_x \cos(\phi) + A_y \sin(\phi)$$
$$A_\phi = -A_x \sin(\phi) + A_y \cos(\phi)$$
$$A_z = A_z$$

In matrix form, transformation of vector **A** form $\left(A_x, A_y, A_z\right) \rightarrow \left(A_\rho, A_\phi, A_z\right)$

$$\begin{bmatrix} A_p \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

By matrix inversion, transformation of vector **A** form $\left(A_\rho, A_\phi, A_z\right) \rightarrow \left(A_x, A_y, A_z\right)$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_p \\ A_\phi \\ A_z \end{bmatrix}$$

Alternatively, using dot product

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \mathbf{a}_x \bullet \mathbf{a}_\rho & \mathbf{a}_x \bullet \mathbf{a}_\phi & \mathbf{a}_x \bullet \mathbf{a}_z \\ \mathbf{a}_y \bullet \mathbf{a}_p & \mathbf{a}_y \bullet \mathbf{a}_\phi & \mathbf{a}_y \bullet \mathbf{a}_z \\ \mathbf{a}_z \bullet \mathbf{a}_p & \mathbf{a}_z \bullet \mathbf{a}_\phi & \mathbf{a}_z \bullet \mathbf{a}_z \end{bmatrix} \begin{bmatrix} A_p \\ A_\phi \\ A_z \end{bmatrix}$$

2.4. Spherical Coordinates (r, θ, ϕ)

The range of variables

$$0 \le r < \infty$$
$$0 \le \theta \le \pi$$
$$0 \le \phi < 2\pi$$

A vector **A** in spherical coordinates may be written as

$$\left(A_r, A_\theta, A_\phi\right) \;\; or \;\; A_r \mathbf{a}_r + A_\theta \mathbf{a}_\theta + A_\phi \mathbf{a}_\phi$$

The magniture of **A** is

$$|\mathbf{A}| = \sqrt{A_r^2 + A_\theta^2 + A_\phi^2}$$

The unit vector relationships

$$\mathbf{a}_r \bullet \mathbf{a}_r = \mathbf{a}_\theta \bullet \mathbf{a}_\theta = \mathbf{a}_\phi \bullet \mathbf{a}_\phi = 1$$
$$\mathbf{a}_r \bullet \mathbf{a}_\theta = \mathbf{a}_\theta \bullet \mathbf{a}_\phi = \mathbf{a}_\phi \bullet \mathbf{a}_r = 0$$
$$\mathbf{a}_r \times \mathbf{a}_\theta = \mathbf{a}_\phi$$
$$\mathbf{a}_\theta \times \mathbf{a}_\phi = \mathbf{a}_r$$
$$\mathbf{a}_\phi \times \mathbf{a}_r = \mathbf{a}_\theta$$

The space variables (x, y, z) in Cartesian coordinates can be related to variables (r, θ, ϕ) of a spherical coordinate system

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$$
$$\phi = \arctan\left(\frac{y}{x}\right)$$

or

$$x = r\sin(\theta)\cos\phi$$
$$y = r\sin(\theta)\sin\phi$$
$$z = r\cos(\theta)$$

The unit vectors $\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_z$ and $\mathbf{a}_r, \mathbf{a}_\theta, \mathbf{a}_\phi$ are related as follows:

$\mathbf{a}_r, \mathbf{a}_\theta, \mathbf{a}_\phi \rightarrow \mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_z$

$$\mathbf{a}_x = \sin(\theta)\cos(\phi)\mathbf{a}_r + \cos(\theta)\cos(\phi)\mathbf{a}_\theta - \sin(\phi)\mathbf{a}_\phi$$
$$\mathbf{a}_y = \sin(\theta)\sin(\phi)\mathbf{a}_r + \cos(\theta)\sin(\phi)\mathbf{a}_\theta + \cos(\phi)\mathbf{a}_\phi$$
$$\mathbf{a}_z = \cos(\theta)\mathbf{a}_r - \sin(\theta)\mathbf{a}_\theta$$

or $\mathbf{a}_x, \mathbf{a}_y, \mathbf{a}_z \rightarrow \mathbf{a}_r, \mathbf{a}_\theta, \mathbf{a}_\phi$

$$\mathbf{a}_r = \sin(\theta)\cos(\phi)\mathbf{a}_x + \sin(\theta)\sin(\phi)\mathbf{a}_y + \cos(\theta)\mathbf{a}_z$$
$$\mathbf{a}_\theta = \cos(\theta)\sin(\phi)\mathbf{a}_x + \cos(\theta)\sin(\phi)\mathbf{a}_y - \cos(\theta)\mathbf{a}_z$$
$$\mathbf{a}_\phi = -\sin(\phi)\mathbf{a}_x + \cos(\theta)\mathbf{a}_y$$

The components of vector $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{A} = (A_r, A_\theta, A_\phi)$ are related by the equation

$$\mathbf{A} = \left(A_x\sin(\theta)\cos(\phi) + A_y\sin(\theta)\sin(\phi) + A_z\cos(\theta)\right)\mathbf{a}_r$$
$$+\left(A_x\cos(\theta)\cos(\phi) + A_y\cos(\theta)\sin(\phi) - A_z\sin(\theta)\right)\mathbf{a}_\theta$$
$$-\left(-A_x\sin(\phi) + A_y\cos(\phi)\right)\mathbf{a}_\phi$$

or

$$A_r = A_x\sin(\theta)\cos(\phi) + A_y\sin(\theta)\sin(\phi) + A_z\cos(\theta)$$
$$A_\theta = A_x\cos(\theta)\cos(\phi) + A_y\cos(\theta)\sin(\phi) - A_z\sin(\theta)$$
$$A_\phi = -A_x\sin(\phi) + A_y\cos(\phi)$$

In matrix form, $(A_x, A_y, A_z) \rightarrow (A_r, A_\theta, A_\phi)$

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin(\theta)\cos(\phi) & \sin(\theta)\sin(\phi) & \cos(\theta) \\ \cos(\theta)\cos(\phi) & \cos(\theta)\sin(\phi) & -\sin(\theta) \\ -\sin(\phi) & \cos(\phi) & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

By matrix inversion, transformation of vector $\mathbf{A}$ form $(A_r, A_\theta, A_\phi) \quad \to (A_x, A_y, A_z)$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin(\theta)\cos(\phi) & \cos(\theta)\cos(\phi) & -\sin(\phi) \\ \sin(\theta)\sin(\phi) & \cos(\theta)\sin(\phi) & \cos(\phi) \\ \cos(\theta) & -\sin(\theta) & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

Alternatively, using dot product

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \mathbf{a}_r \bullet \mathbf{a}_x & \mathbf{a}_r \bullet \mathbf{a}_y & \mathbf{a}_r \bullet \mathbf{a}_z \\ \mathbf{a}_\theta \bullet \mathbf{a}_x & \mathbf{a}_\theta \bullet \mathbf{a}_y & \mathbf{a}_\theta \bullet \mathbf{a}_z \\ \mathbf{a}_\phi \bullet \mathbf{a}_x & \mathbf{a}_\phi \bullet \mathbf{a}_y & \mathbf{a}_\phi \bullet \mathbf{a}_z \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

The distance between two points is necessary in EM theory. The distance d between two points with position vector $\mathbf{r}_1$ and $\mathbf{r}_2$ is generally given by

$$d = |\mathbf{r}_2 - \mathbf{r}_1|$$

or

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \quad \text{(Cartesian)}$$

$$d^2 = \rho_2^2 + \rho_1^2 - 2\rho_1\rho_2\cos(\phi_2 - \phi_1) + (z_2 - z_1)^2 \quad \text{(cylindrical)}$$

$$d^2 = r_2^2 + r_1^2 - 2r_1r_2\cos(\theta_2)\cos(\theta_1) - 2r_1r_2\sin(\theta_2)\sin(\theta_1)\cos(\phi_2 - \phi_1) \quad \text{(spherical)}$$

Example 2.1

Given point $P(-2,6,3)$ and vector $\mathbf{A} = y\mathbf{a}_x + (x+z)\mathbf{a}_y$, express P and $\mathbf{A}$ in cylindrical and spherical coordinates. Evaluate $\mathbf{A}$ at P in the Cartesian, cylindrical, and spherical systems.

Solution:

$\because \quad P(-2,6,3) \quad \therefore \quad x = -2; \quad y = 6;$ and $z = 3$. Using the formula to get (ρ, ϕ, z)

$$\rho = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + 6^2} = 2\sqrt{10} = 6.3246$$

$$\phi = \arctan\left(\frac{6}{-2}\right) = (180° - 71.565°) = 108.43°$$

$$z = 6$$

$\therefore P(-2,6,3)_{Cartesian} = P(6.3246, 108.43°, 3)_{cylindrical}$

$\because \quad \mathbf{A} = y\mathbf{a}_x + (x+z)\mathbf{a}_y = 6\mathbf{a}_x + (-2+3)\mathbf{a}_y = 6\mathbf{a}_x + \mathbf{a}_y, \quad \therefore A_x = 6; \quad A_y = 1$ and $A_z = 0$

Using the matrix conversion $(A_x, A_y, A_z) \to (A_\rho, A_\phi, A_z)$

$$\begin{bmatrix} A_p \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$= \begin{bmatrix} \cos(108.43°) & \sin(108.43°) & 0 \\ -\sin(108.43°) & \cos(108.43°) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} A_p \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} -0.94816 \\ -6.0084 \\ 0 \end{bmatrix}$$

$\therefore \mathbf{A} = -0.94816\mathbf{a}_\rho + -6.0084\mathbf{a}_\phi$ (cylindrical coordinates)

$\because$ $P(-2,6,3)$ $\therefore$ $x=-2;$ $y=6;$ and $z=3$. Using the formula to get (r,θ,ϕ)

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{(-2)^2+6^2+3^2} = 7$$

$$\theta = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right) = \arctan\left(\frac{\sqrt{(-2)^2+6^2}}{3}\right) = 64.624°$$

$$\phi = \arctan\left(\frac{6}{-2}\right) = (180° - 71.565°) = 108.43°$$

$\therefore$ $P(-2,6,3)_{Cartesian} = P(7, 64.624°, 108.43°)_{spherical}$

$\because$ $\mathbf{A} = y\mathbf{a}_x + (x+z)\mathbf{a}_y = 6\mathbf{a}_x + (-2+3)\mathbf{a}_y = 6\mathbf{a}_x + \mathbf{a}_y,$ $\therefore A_x = 6;$ $A_y = 1$ and $A_z = 0$

Using the matrix conversion $(A_x, A_y, A_z) \to (A_r, A_\theta, A_\phi)$

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin(\theta)\cos(\phi) & \sin(\theta)\sin(\phi) & \cos(\theta) \\ \cos(\theta)\cos(\phi) & \cos(\theta)\sin(\phi) & -\sin(\theta) \\ -\sin(\phi) & \cos(\phi) & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$= \begin{bmatrix} \sin(64.624°)\cos(108.43°) & \sin(64.624°)\sin(108.43°) & \cos(64.624°) \\ \cos(64.624°)\cos(108.43°) & \cos(64.624°)\sin(108.43°) & -\sin(64.624°) \\ -\sin(108.43°) & \cos(108.43°) & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} -0.85668 \\ -0.40634 \\ -6.0084 \end{bmatrix}$$

$\therefore \mathbf{A} = -0.85668\mathbf{a}_r - 0.40634\mathbf{a}_\theta - 6.0084\mathbf{a}_\phi$ (spherical coordinates)

$$\therefore \mathbf{A} = 6\mathbf{a}_x + \mathbf{a}_y = -0.94816\mathbf{a}_\rho + -6.0084\mathbf{a}_\phi = -0.85668\mathbf{a}_r - 0.40634\mathbf{a}_\theta - 6.0084\mathbf{a}_\phi$$

Getting $|\mathbf{A}|$ for all coordinate system

$$|\mathbf{A}|_{Cartesian} = \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{6^2 + 1^2 + 0} = 6.0828$$

$$|\mathbf{A}|_{cylindrical} = \sqrt{A_\rho^2 + A_\phi^2 + A_z^2} = \sqrt{(-0.94816)^2 + (-6.0084)^2 + 0} = 6.0828$$

$$|\mathbf{A}|_{spherical} = \sqrt{A_r^2 + A_\theta^2 + A_\phi^2} = \sqrt{(-0.85668)^2 + (-0.40634)^2 + (-6.0084)^2} = 6.0828$$

Therefore, the answers are correct.