Majorization and Triangular Matrix Polynomials with Prescribed Diagonal

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Majorization

Definition

Given natural vectors $\mathbf{x}$ and $\mathbf{y}$ with non-increasing entries, we say that $\mathbf{x}$ majorizes $\mathbf{y}$ and write $\mathbf{x} \succeq \mathbf{y}$ if

$$\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i \quad \text{for } k = 1, 2, \ldots, n$$

with equality when $k = n$.

- Think of $\mathbf{x}$ and $\mathbf{y}$ as permutation equivalence classes.
- Extends to real vectors.
- The set of all vectors majorized by $\mathbf{x}$ is the convex hull of all $\sigma(\mathbf{x})$ for $\sigma \in S_n$.

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Applications of majorization

Some applications taken from Marshall and Olkin.

- Majorization of real vectors and doubly stochastic matrices:

$$\mathbf{x} \succeq \mathbf{y} \text{ iff } \exists D \text{ doubly stochastic s.t. } \mathbf{y} = D\mathbf{x}.$$

- If H is Hermitian, then the vector of eigenvalues majorizes the vector of diagonal entries.
- Flows in graphs and networks.
- Bounds on condition numbers, unitarily invariant norms and symmetric gauge functions.
- Relationships between sampling without replacement and sampling with replacement.

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Classically...

- Majorization appears in the literature as far back as 1903 (Muirhead, *Proc. Edinburgh Math. Soc.*)
- The term "majorization" was coined by Hardy, Littlewood, and Polya in 1934.

Contemporarily...

- The essential reference is the book by Marshall, Olkin, and Arnold from 2011.
- Paper by Arnold in *Statistical Science* surveys the cornucopia of applications, 2007.

In linear algebra...

- Taslaman, Tisseur, Zaballa link majorization to the diagonal of an upper triangular realization of a given Smith form, 2013.

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Robin Hood transfers

Robin Hood transfer

Consider $\mathbf{x} \in \mathbb{N}^n$ *and construct* $\mathbf{x}'$ *by replacing* x_i *and* x_j *with* x_i' *and* x_j' *such that*

$$x_i > x_i' \geq x_j' > x_j \text{ and } x_i' + x_j' = x_i + x_j.$$

- $\mathbf{x} \succeq \mathbf{x}'$
- If $\mathbf{x} \succeq \mathbf{y}$, then there is a finite sequence

$$\mathbf{x} = \mathbf{x}^{(0)} \succeq \mathbf{x}^{(1)} \succeq \mathbf{x}^{(m)} = \mathbf{y}$$

  such that $\mathbf{x}^{(i+1)}$ is obtained from $\mathbf{x}^{(i)}$ by a single Robin Hood transfer.
- Muirhead's theorem.

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Triangular realizations

Consider the following

Inverse problem

Given a Smith form $S(\lambda)$ and a list of polynomials $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_n\}$, does there exist a triangular matrix polynomial $T(\lambda)$ with $S \sim T$ and diagonal entries given by $\mathcal{D}$?

- A set of conditions was given by Marques de Sá in his 1979 Householder Prize winning dissertation.
  - Involves GCDs of products of polynomials.
  - Checking takes roughly $\mathcal{O}(2^n)$ time.
- We present a simpler set of conditions based on majorization.
  - Checking takes roughly $\mathcal{O}(qn^2)$.
  - Implementable construction.

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Background

Definition

Let $\chi(\lambda)$ be an irreducible polynomial. The *factor count* of a polynomial $p(\lambda)$ for χ is the number of times χ appears as a factor of $p(\lambda)$, denoted

$$|p(\lambda)|_\chi.$$

Definition

The *factor counting vector* of a set $\mathcal{P} = \{p_1(\lambda), p_2(\lambda), \ldots, p_n(\lambda)\}$ is the vector

$$|\mathcal{P}|_\chi := (|p_1|_\chi, |p_2|_\chi, \ldots, |p_n|_\chi).$$

- Straightforward to extend to a collection of irreducible polynomials $\mathcal{F} = \{\chi_1, \chi_2, \ldots, \chi_k\}$.

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Main result

Theorem

Let $\mathcal{S} = \{s_1, s_2, \dots, s_n\}$ be the set of invariant polynomials from the Smith form $S(\lambda)$, and let $\mathcal{D} = \{d_1, d_2, \dots, d_n\}$ be a list of polynomials. The desired realization exists if and only if

$$|\mathcal{S}|_\chi \succeq |\mathcal{D}|_\chi \text{ for all irreducible } \chi \mid \prod_{i=1}^{n} s_i.$$

- Forward implication: directly show inequalities hold.
- Reverse implication: several steps.
  - Muirhead's theorem applied to each factor counting vector.
  - Each Robin Hood transfer can be accomplished by a unimodular transformation that preserves triangularity.

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Forward implication

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1. Assume $|\mathcal{S}|_\chi$ and $|\mathcal{D}|_\chi$ are in non-decreasing order.
2. By Smith's theorem
$$\prod_{i=1}^{k} s_i \mid \prod_{i=1}^{k} d_i.$$
3. Since $|p(\lambda)q(\lambda)|_\chi = |p(\lambda)|_\chi + |q(\lambda)|_\chi$ and $r(\lambda) \mid p(\lambda)$ implies $|r(\lambda)|_\chi \leq |p(\lambda)|_\chi$,
$$\sum_{i=1}^{k} |s_i|_\chi \leq \sum_{i=1}^{k} |d_i|_\chi.$$
4. Equality of determinants gives equality when $k = n$.

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Unimodular transfer lemma

Lemma

Consider

$$T(\lambda) = \begin{bmatrix} p(\lambda)\chi(\lambda)^a & r(\lambda) \\ 0 & q(\lambda)\chi(\lambda)^b \end{bmatrix}$$

with $GCD(\chi, pq) = 1$. *Then* $T(\lambda)$ *is equivalent to*

$$T(\lambda)' = \begin{bmatrix} p(\lambda)\chi(\lambda)^\alpha & r(\lambda)' \\ 0 & q(\lambda)\chi(\lambda)^\beta \end{bmatrix}$$

where $a > \alpha \geq \beta > b$ *and* $\alpha + \beta = a + b$.

- Moves factors of χ along the diagonal to accomplish a Robin Hood transfer.
- Applied to adjacent diagonal entries in a larger triangular matrix.

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with $GCD(\chi, pq) = 1$*. Then* $T(\lambda)$ *is equivalent to*

$$T(\lambda)' = \begin{bmatrix} p(\lambda)\chi(\lambda)^\alpha & r(\lambda)' \\ 0 & q(\lambda)\chi(\lambda)^\beta \end{bmatrix}$$

where $a > \alpha \geq \beta > b$ *and* $\alpha + \beta = a + b$*.*

- Moves factors of χ along the diagonal to accomplish a Robin Hood transfer.
- Applied to adjacent diagonal entries in a larger triangular matrix.

Computational implementation

- Most steps can be implemented without the need for numerical computation.
- Unimodular transfer lemma requires the computation of 2×2 unimodular transformations that take a 2×2 upper triangular matrix to Smith form.
- Two such computations for each Robin Hood transfer.

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Questions???

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