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Stable ergodicity and partial hyperbolicity

Dedicated to the memory of Ricardo Mañé.

1. Introduction

By the results of Anosov in 1967 volume preserving uniformly hyperbolic systems are ergodic and are open. Thus they exhibit robust statistical behavior. Averages are the same for almost all points, not only for the system in question but also for any small perturbation which preserves the same volume. If the perturbation only preserves a close by volume, then the averages of continuous functions are still close by. On the other hand, in 1954, Kolmogorov announced that there are no ergodic Hamiltonian systems in a neighborhood of a completely integrable one. Completely integrable systems have no hyperbolic behavior at all.

In this paper we will review the results of [Grayson, Pugh and Shub, 1994], [Pugh and Shub, 1996], [Pugh, Shub and Wilkinson, 1996], and [Brezin and Shub, 1995] which study the mixed situation in which the system is only partially hyperbolic.

Our themes are:

- 1) A little hyperbolicity goes a long way toward guaranteeing ergodic behavior.
- 2) Stably ergodic systems are considerably more general than one might have feared from Kolmogorov's theorem.
- 3) Some hyperbolicity may be necessary for stable ergodicity.

We consider C^2 diffeomorphisms f of closed manifolds M which preserve a fixed smooth volume on M . We say that f is stably ergodic if there is a neighborhood U of f in the C^2 volume preserving diffeomorphisms of M such that every $g \in U$ is ergodic.

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Each of our main themes is developed in a section below. Finally in section 5, we suggest some generalizations to dissipative systems.

Partial Hyperbolicity and Ergodicity

Main Theorem of this section gives sufficient conditions for a diffeomorphism to be ergodic. We find stably ergodic diffeomorphisms by finding open sets of diffeomorphisms satisfying these conditions. The theorem may be interpreted to say that for systems which are not uniformly hyperbolic, the same phenomenon which produces chaotic behavior i.e. some hyperbolicity may also guarantee ergodicity.

Main Theorem: [Pugh and Shub, 1996] Let $f : M \rightarrow M$ be partially hyperbolic and dynamically coherent. Suppose that the stable and unstable bundles have the accessibility property and that the invariant bundles are sufficiently Hölder. Then f is ergodic.

The accessibility property is a concept from control theory which we apply to the unstable and stable foliations of a partially hyperbolic diffeomorphism. We will soon explain these concepts. Partially hyperbolic diffeomorphisms which are dynamically coherent have some more properties which we will eventually come to. There are three distributions E^s , E^c and E^u , the strong stable, center and strong unstable bundles of the tangent bundle, of C^2 diffeomorphisms are Hölder. That they are sufficiently Hölder is expressed in terms of relationships of the Hölder exponents and is stated in terms of the contraction and expansion constants of the various natural invariant bundles for the derivative. We leave these details to be consulted in [Pugh and Shub, 1996] and [Pugh, Shub and Wilkinson, 1996], but note that foliations of the C^1 tangent bundles are sufficiently Hölder. Partially hyperbolic systems and the accessibility property were to our knowledge first considered in [Brin and Pesin, 1974].

We say that a C^r diffeomorphism $f : M \rightarrow M$ is *partially hyperbolic* iff $r \geq 1$ and there is a continuous Tf -invariant direct sum decomposition

$$TM = E^s \oplus E^c \oplus E^u$$

where E^s and E^u are non-trivial, some Finsler $\|\cdot\|$ on TM and some real constants $0 < b < c < 1 < d < \epsilon < g$ such that

$$a\|v\| < \|Tf(v)\| < b\|v\| \text{ for } v \in E^s - \{0\}$$

$$c\|v\| < \|Tf(v)\| < d\|v\| \text{ for } v \in E^c - \{0\}$$

$$\epsilon\|v\| < \|Tf(v)\| < g\|v\| \text{ for } v \in E^u - \{0\}.$$

Since $Tf : E^c \rightarrow E^c$ may have some contraction and expansion E^s and E^u are strong contracting and strong expanding Tf invariant subbundles. Tangent to E^s and E^u are the strong contracting and strong expanding f invariant foliations which we will denote by W^s and W^u .

Given continuous sub-bundles $F, H \subset TM$ and points $m_0, m_1 \in M$ we say that F is accessible from m_0 iff there is a continuous piecewise C^1 path $\phi[0, 1]$ joining

- c) The Lie Algebra generated by the contracting and expanding directions of Ad_A is the whole Lie Algebra $\text{SL}(n, \mathbb{R})$.
d) L_A is stably ergodic among left translations of $\text{SL}(n, \mathbb{R})$.

ng perturbation of $A \times id : M \times N \rightarrow M \times N$ is stably ergodic for any manifold N . See [Bonatti and Diaz, 1994] for the rather striking topologicality version of this second case of Conjecture 1.

to know a large class of examples of partially hyperbolic diffeomorphisms re stably ergodic. The time one map of the geodesic flow on a compact surface ant negative curvature is the most classically studied partially hyperbolic orphism and it has the accessibility property. In [Grayson, Pugh and Shub, 1996] e proved that it is stably ergodic. Amie Wilkinson [Wilkinson, 1995] removed othesis that the negative curvature be constant. In n-dimensions we have:

em 3. [Pugh and Shub, 1996] The time one map of the geodesic flow on the tangent bundle of a compact n -manifold of constant curvature k , $k < 0$ is stably

- it is ergodic and so are all C^2 small volume preserving perturbations of it.
have also a class of examples which come from the theory of homogeneous of Lie groups. We will assume that our spaces are of the form G/B where G is connected Lie group and B is a closed subgroup which, in addition, is *admissible* in a certain technical sense (see [Brezin and Shub, 1995]) which we will not make precise here. If G is nilpotent, solvable or semi-simple or if B is discrete then the admissibility condition is satisfied.

Let $a \in G$ let L_a denote left translation by a i.e. $L_a(h) = ah$ for all $h \in G$. Then L_a induces a map on G/B which we call L_a as well. Given an automorphism A of $a \in G$ we call $L_a A : G \rightarrow G$ an affine diffeomorphism of G , we also denote it by aA . If $A(B) = B$ then we continue to denote the induced map on G/B by aA and call it an affine diffeomorphism of G/B . We will assume that the measure on G induces a finite measure on G/B which is invariant under left translation and that $A : G/B \rightarrow G/B$ is measure preserving.

an affine diffeomorphism $aA : G \rightarrow G$, aA induces an automorphism of the Lie Algebra \mathfrak{g} of G by $ad(a)DA(\epsilon)$ where ϵ is the identity of G . In particular, $DA(\epsilon)$ is a linear map. Let \mathfrak{g}^s and \mathfrak{g}^u be the generalized eigenspaces of $DA(\epsilon)$ corresponding to the contracting and expanding eigenvalues of $ad(a)DA(\epsilon)$. Let \mathcal{L} be the Lie subalgebra of \mathfrak{g} generated by \mathfrak{g}^s and \mathfrak{g}^u . Then it is not hard to see [Pugh and Shub, 1996] that \mathcal{L} is an ideal in \mathfrak{g} which is $ad(a)DA(\epsilon)$ invariant. As an ideal \mathcal{L} is tangent to the connected normal subgroup which we denote by H and call the *stably* generated subgroup of G .

em 1: Let G/B be a compact manifold and aA be an affine diffeomorphism of G/B . Let r be a positive real. If the eigenvalues of $ad(a)DA(\epsilon)$ are sufficiently close to the three numbers $1, r$ or $\frac{1}{r}$ and $HB = G$, then aA is stably ergodic on G/B .

This theorem has examples for semi-simple groups. We specialize to $\text{SL}(n, \mathbb{R})$.

Let Γ be a uniform discrete subgroup of $\text{SL}(n, \mathbb{R})$.

For $A \in \text{SL}(n, \mathbb{R})$ let $L_A : \text{SL}(n, \mathbb{R})/\Gamma \rightarrow \text{SL}(n, \mathbb{R})/\Gamma$ be given by left translation

em 2: [Pugh and Shub, 1996] The following four conditions are equivalent.

A has an eigenvalue with modulus different from 1.

Conjecture 3: Let dimension $M \geq 2$. For the generic finite dimensional submanifold V contained in $\text{Diff}^r(M)$ and almost every $f \in V$ the equivalence classes of points in the chain recurrent set of f are open in the chain recurrent set.

Conjecture 3 would give a finite spectral decomposition for f where each piece of the decomposition has something akin to the accessibility property.

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