Relative Equilibria and Diagonals

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In the question of whether the relative equilibrium classes, $\overline{R}_e$, are finite, one of the first things to do seems to be to show that the Newtonian potential function $-\sum_{i\neq j} \frac{m_i m_j}{\|x_i - x_j\|}$ restricted to the sphere $\Sigma\, m_i x_i^2 = 1$ and $\Sigma\, m_i x_i = 0$ has no singularities in a neighborhood of the diagonals $x_i = x_j$, $i \neq j$, where the function is not defined. Here we let $x_i \in R^k$, $i = 1,\ldots,n$, and $m_i \in R_+$, $i = 1,\ldots,n$.

We proceed as follows: The derivative of $-\sum_{i\neq j} \frac{m_i m_j}{\|x_i - x_j\|}$ is

$\sum_{i\neq j} \frac{m_i m_j \langle x_i - x_j,\ v_i - v_j \rangle}{\|x_i - x_j\|^3}$. Let $x_1,\ldots,x_{k_1}, x_{k_1+1},\ldots,x_{k_2},\ldots,x_*,\ldots,x_n = X$

be a point on the generalized diagonals where we have grouped the equal terms $x_1 = \ldots = x_{k_1}$, $x_{k_1+1} = \ldots = x_{k_2}$, $x_* = \ldots = x_n$. Let x_j^i denote the i^{th} component of x_j. Without loss of generality we may assume $k_1 \geq 2$ and $x_n^1 \neq x_1^1$. We represent a point in a neighborhood of X by $X+\delta$ where $\delta = (\delta_1,\ldots,\delta_n)$. We will pick a vector $V(\delta) = (v_1(\delta),\ldots,v_n(\delta))$ defined in this neighborhood so that:

(a) $\|V(\delta)\|$ is bounded

(b) $$\sum_{i\neq j} \frac{m_i m_j \langle x_i+\delta_i-(x_j+\delta_j),\ v_i(\delta)-v_j(\delta) \rangle}{\|x_i+\delta_i-(x_j+\delta_j)\|^3} \longrightarrow +\infty$$

as $\delta_i - \delta_j \longrightarrow 0$ for $i,j \leq k_1$ and $i \neq j$.

(c) $V(\delta)$ is in tangent space to the sphere $\Sigma_{i=1}^n\, m_i x_i^2 = 1$ and $\Sigma_{i=1}^n\, m_i x_i = 0$ if $X+\delta$ lies in the sphere.

We do this by solving the following system of linear equations continuously in a neighborhood of X.

(1) $\Sigma_{i=1}^n\, m_i (x_i + \delta_i) v_i = 0$

(2) $\Sigma_{i=1}^n\, m_i v_i = 0$

(3) $v_1 - v_j = \delta_1 - \delta_j$ for $2 \leq j \leq k_1$

(4) $v_{k_1+1} = v_{k_1+2} = \ldots = v_{*-1} = 0$

(5) $v_* = v_{*+1} = \ldots = v_n$

If we have solved these equations then writing down the derivative

$$\Sigma \frac{m_i m_j < x_i - x_j, v_i - v_j >}{\|x_i - x_j\|^3}$$

we see that at $X+\delta$ if it is defined its value is

$$\geq \sum_{\substack{i \neq j \\ i,j \leq k_1}} \frac{m_i m_j < \delta_i - \delta_j, \delta_i - \delta_j >}{\|\delta_i - \delta_j\|^3} + C$$ where C is a constant which may be chosen for the whole neighborhood if $\|\delta\|$ is small enough. Now

$$\sum_{\substack{i \neq j \\ i,j \leq k_1}} \frac{m_i m_j < \delta_i - \delta_j, \delta_i - \delta_j >}{\|\delta_i - \delta_j\|^3} + C = \sum_{\substack{i \neq j \\ i,j \leq k_1}} \frac{m_i m_j}{\|\delta_i - \delta_j\|} + C$$ which obviously tends to $+\infty$ as any $\delta_i - \delta_j \longrightarrow 0$. Which proves (b). Equations (1) and (2) prove c, and continuity insures that $\|V(\delta)\|$ is bounded.

It remains only to solve the equations for the v_i. We consider the family of linear maps

$$\Sigma\, m_i(x_i + \delta_i)v_i : R^{kn} \longrightarrow R^1$$

$$\Sigma\, m_i v_i \qquad : R^{kn} \longrightarrow R^k$$

$$v_1 - v_i \qquad : R^k \longrightarrow R^k \qquad \text{for } 2 \leq i \leq k_1$$

restricted to the linear subspace of R^{kn} defined by $v_{k_1+1} = \dots = v_{*-1} = 0$ and $v_* = \dots = v_n$. So we have a linear map from a $(k_1+1)k$ dimensional vector space to a k_1k+1 dimensional vector space. We will prove that it is surjective at $\delta = 0$. Then we will be able to solve the equations continuously in a neighborhood and we will be done. To prove the surjectivity we restrict to the k_1k+1 dimensional subspace determined by $v_n^{\,j} = 0$ for $j \neq 1$ and we will show that the map is an isomorphism by calculating the determinant of its matrix. Its matrix is:

$$\begin{pmatrix} m_1 x_1 & m_2 x_2 \cdots\cdots m_{k_1} x_{k_1} & \begin{matrix} \tilde{m} x_n^{\,1} \\ \tilde{m} \\ 0 \\ \vdots \\ 0 \end{matrix} \\ m_1 I & m_2 I \cdots\cdots m_{k_1} I & \\ I & \begin{matrix} -I & & 0 \\ & \ddots & \vdots \\ & & 0 \\ & & -I \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \vdots & & \\ I & & \end{pmatrix}$$

where $\tilde{m} = \Sigma_{j=*}^{n} m_j$, I is the $k \times k$ identity matrix, and $\begin{pmatrix} \tilde{m} x_n^{\,1} \\ \tilde{m} \\ 0 \\ \vdots \end{pmatrix}$ is an ordinary column of numbers. Let $m = \Sigma_{j=1}^{k_1} m_j$. Then after identifying $x_1 = \dots = x_{k_1}$ and column reducing we get:

$$\begin{pmatrix} mx_1 & m_2x_2 & \cdots & m_{k_1}x_{k_1} & \tilde{m}x^1 \\ & & & & \tilde{m}^k \\ mI & m_2I & \cdots & m_{k_1}I & 0 \\ & & & \vdots & \vdots \\ & & & & 0 \\ 0 & -I & & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & & & -I & 0 \end{pmatrix}$$

so that the determinant is equal to the determinant of:

$$\begin{pmatrix} mx_1 & \tilde{m}x^1_n \\ & \tilde{m} \\ & 0 \\ mI & \vdots \\ & 0 \end{pmatrix} \text{ which is } m^{k-1}\det \begin{pmatrix} mx^1_1 & \tilde{m}x^1_k \\ m & \tilde{m} \end{pmatrix}$$

which is $m^{k-1}(\tilde{m}mx^1_1 - m\tilde{m}x^1_n) = m^k\,\tilde{m}(x^1_1 - x^1_n) \neq 0$ because by hypothesis $x^1_1 \neq x^1_n$.
Note that the calculation also gives that the neighborhood of the diagonals in which we may assume that there are no singularities varies continuously with the masses.

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