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Stably Ergodic Dynamical Systems and Partial Hyperbolicity

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In this paper we show that a little hyperbolicity goes a long way toward guaranteeing stable ergodicity, and in fact may be necessary for it. Our main theorem may be interpreted as saying that the same phenomenon producing chaotic behavior (i.e., some hyperbolicity) also leads to robust statistical behavior. Examples to which our theory applies include translations on certain homogeneous spaces and the time-one map of the geodesic flow for a manifold of constant negative curvature. © 1997 Academic Press

1. INTRODUCTION

Boltzman's ergodic hypothesis underlies statistical mechanics and much of physical thinking. Yet, in 1954, Kolmogorov announced that there are no ergodic Hamiltonian systems in a neighborhood of completely integrable ones. In contrast, in 1962, Anosov found the first open sets of ergodic systems, i.e., stably ergodic diffeomorphisms and flows. Anosov systems are totally hyperbolic while completely integrable systems have no hyperbolic behavior at all.

In this paper we study the mixed situation in which a diffeomorphism f is only partially hyperbolic. Under Tf, the tangent bundle splits into three invariant subbundles, an unstable, a center, and a stable subbundle,

$$TM = E^u \oplus E^c \oplus E^s.$$

For example f can be the time one map of an Anosov flow and E^c is the direction tangent to the flow orbits.

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Our main themes are, first, that a little hyperbolicity goes a long way in guaranteeing stably ergodic behavior (which is more prevalent than one might have imagined) and, second, that in fact the former may be necessary for the latter. In both cases we make use of an accessibility concept from control theory applied to the hyperbolic part of the derivative. As far as we know, the accessibility property of hyperbolic systems was first used in the dynamical systems world by Brin and Pesin (1974).

Here are our four main results.

THEOREM A. *Suppose that the C^2, volume preserving diffeomorphism $f\colon M \to M$ is partially hyperbolic and dynamically coherent. If f has the essential accessibility property and its invariant bundles are sufficiently Hölder, then f is ergodic.*

THEOREM B. *In addition to the hypotheses of Theorem* A *suppose that f has the (complete) accessibility property, the invariant bundles of f are C^1, and the spectrum of Tf is sufficiently bunched. Then f is stably ergodic; i.e., f is ergodic and so is every volume preserving diffeomorphism of M that C^2 approximates it.*

THEOREM C. *The time-one map of the geodesic flow on the unit tangent bundle M of a compact Riemannian manifold of constant negative curvature is stably ergodic. (It is ergodic as a diffeomorphism, not merely as a flow, and so are all C^2 small, volume preserving perturbations of it.)*

THEOREM D. *Let Γ be a uniform discrete subgroup of* $\mathrm{SL}(n, \mathbb{R})$. *For $A \in \mathrm{SL}(n, \mathbb{R})$, let $L_A\colon M \to M$ where $M = \mathrm{SL}(n, \mathbb{R})/\Gamma$ and L_A is left translation by A. Then the following four conditions are equivalent:*

(a) *A has an eigenvalue with modulus different from* 1.

(b) *L_A is partially hyperbolic, dynamically coherent, and its hyperbolic invariant foliations have the accessibility property.*

(c) *The Lie algebra generated by the hyperbolic subspaces of* $\mathrm{Ad}(L_A)$ *is the whole Lie algebra* $\mathrm{SL}(n, \mathbb{R})$.

(d) *L_A is stably ergodic among left translations of* $\mathrm{SL}(n, \mathbb{R})$; *i.e., every left translation near L_A is ergodic.*

Theorems A and B are fully explained in Sections 2 and 3. Theorems A, B, and C are proved at the end of Section 4, while Theorem D is proved in Section 5. Together with Matt Grayson we proved a special case of Theorems A, B, and C in 1994. There we perturb the time-one map of the geodesic flow for a surface of constant negative curvature. This makes the manifold M three dimensional.

Theorem A may be interpreted as saying that even for systems that are not totally hyperbolic, the same phenomenon that produces chaotic behavior, i.e., some hyperbolicity, leads to robust statistics in the form of ergodicity.

In our proof of Theorem A we require a high degree of Hölder continuity: the hyperbolic holonomy maps are θ-Hölder and $1 - \theta$ is quite small. We do not know whether this requirement is really necessary, and we conjecture that it is not. The hyperbolic holonomy maps always have *some* positive degree of Hölder continuity, and perhaps this is enough for stable ergodicity. Also we feel it is quite likely that the accessibility hypotheses in Theorems A and B are generic, so we make the following conjecture.

Conjecture 1. Stable ergodicity is an open and dense property among C^2 volume preserving, partially hyperbolic diffeomorphisms. (Openness is clear.)

In particular this conjecture would imply that the generic C^2 volume preserving perturbation of an ergodic automorphism of the n-torus is ergodic. See Grayson *et al.* (1994) for some discussion of this and for an example of an automorphism of the 4-torus in which the conjecture is an open question. Another case in which the conjecture is an open question occurs for a product $A \times \text{id}$: $M \times N \to M \times N$, where A is a C^2 Anosov diffeomorphism. Is the generic, C^2 small, volume preserving perturbation of $A \times \text{id}$ ergodic? See Bonatti and Diaz (1994) for a striking result in this line if topological transitivity replaces ergodicity.

It remains an open question whether the following fifth condition is equivalent to the four in Theorem D.

(e) L_A is stably ergodic among C^2 volume preserving diffeomorphisms of $\mathrm{SL}(n, \mathbb{R})/\Gamma$.

A second way to conjecturally extend Theorem D involves using groups other than $\mathrm{SL}(n, \mathbb{R})$. Let G be a connected Lie group and let Γ be a uniform discrete subgroup of G. For $g \in G$ let L_g: $G/\Gamma \to G/\Gamma$ be left translation by g. Consider the conditions (a)–(e) above, where G replaces $\mathrm{SL}(n, \mathbb{R})$ and L_g replaces L_A.

Conjecture 2. If L_g is stably ergodic among left translations then L_g is partially hyperbolic and hence dynamically coherent. (In the context of Theorem D, this is included in the implication (d) $\Rightarrow$ (b).)

Conjecture 3. Assume that G is semi-simple and has no compact factor. Then the following are equivalent.

(b′) The hyperbolic foliations of L_g have the accessibility property.

(c′) The hyperbolic subspaces of $\mathrm{Ad}(L_g)$ generate the whole Lie algebra of G.

(d′) L_g is stably ergodic among left translations.

(e′) L_g is stably ergodic among C^2 volume preserving diffeomorphisms of G/Γ.

Since early versions of this paper were written, a good deal of progress has been made on these algebraic conjectures. Brezin and Shub (1995) prove

Conjecture 2 under the additional hypothesis that Γ is admissible in the sense of Brezin and Moore. Further they prove equivalence of (b′), (c′), (d′) in Conjecture 3.

We rely throughout this paper on the stable manifold theory that appears in our articles with Hirsch (1977), Grayson (1994), and Wilkinson (1996). We refer to these papers as HPS, GPS, and PSW respectively.

2. PARTIALLY HYPERBOLIC DYNAMICS

A diffeomorphism $f\colon\ M \to M$ of a compact, connected, boundaryless manifold M is *partially hyperbolic* if $Tf\colon\ TM \to TM$ leaves invariant a continuous splitting $TM = E^u \oplus E^c \oplus E^s$, where $E^u \neq 0 \neq E^s$ and, with respect to some fixed Riemann structure on TM, Tf expands E^u, Tf contracts E^s, and for all $p \in M$,

$$\sup \|T_p^s f\| < \inf \mathfrak{m}(T_p^c f) \qquad \text{and} \qquad \sup \|T_p^c f\| < \inf \mathfrak{m}(T_p^u f). \tag{1}$$

$T^u f$, $T^c f$, $T^s f$ are the restrictions of Tf to E^u, E^c, E^s. If the *center bundle* $E^c = 0$ then f is *totally hyperbolic*, or *Anosov*. The notation $\mathfrak{m}(T)$ refers to the *conorm* (or *minimum norm*) of a linear transformation T,

$$\mathfrak{m}(T) = \inf\{|Tv| : |v| = 1\}.$$

When T is invertible, $\mathfrak{m}(T) = \|T^{-1}\|^{-1}$. (1) means that Tf contracts the *stable bundle* E^s more sharply than it contracts the center bundle E^c and it expands the *unstable bundle* E^u more sharply than it expands E^c. According to HPS, if $f'C^1$ approximates f then f' is also partially hyperbolic.

Standing Assumption. The diffeomorphism f is C^2 and partially hyperbolic.[1]

In HPS it is shown that there are unique f-invariant foliations, $\mathcal{W}^u$ and $\mathcal{W}^s$, tangent to E^u and E^s, and their leaves are dynamically characterized as follows. Points p, q belong to the same $\mathcal{W}^s$-leaf if and only if for some (or any) constant μ, $\|T^s f\| < \mu < \mathfrak{m}(T^c f)$, $d(f^n p,\ f^n q)/\mu^n \to 0$ as $n \to \infty$. Similarly, points p, q belong to the same $\mathcal{W}^u$-leaf if and only if for some (or any) constant λ, $\|T^c f\| < \lambda < \mathfrak{m}(T^u f)$, $d(f^n p,\ f^n q)/\lambda^n \to 0$ as $n \to -\infty$. The leaves of

[1]As defined in (1), partial hyperbolicity is an absolute concept. Most of what we prove, however, remains valid when f is relatively partially hyperbolic, i.e., when (1) is replaced by the assumptions that for all $p \in M$, $\|T_p^s f\| < \mathfrak{m}(T_p^c f)$ and $\|T_p^c f\| < \mathfrak{m}(T_p^u f)$.

$\mathcal{W}^u$ and $\mathcal{W}^s$ are C^2 and are the *unstable manifolds* and the *stable manifolds*, respectively.

A continuous path ϕ: $[0, 1] \to M$ is *piecewise* C^1 if there is a partition $0 = t_0 < \cdots < t_n = 1$ such that the restriction of ϕ to each interval $[t_{i-1}, t_i]$ is a C^1 embedding, $1 \le i \le n$. If E, F are subbundles of TM then ϕ is an (E, F)-*path*, or is *subordinate* to (E, F) provided that

$$\phi'(t) \in E \cup F$$

whenever $\phi'(t)$ exists. The pair of subbundles (E, F) has the (complete) *accessibility property* if every pair of points in M can be joined by a piecewise C^1 path that is subordinate to (E, F), while it has the *essential accessibility property* if this is true for *almost* every pair of points in M. Accessibility is discussed further in the next section.

The subbundles E^u and E^s are *uniquely integrable* in the following sense: if ϕ: $[0, 1] \to M$ is a C^1 path everywhere tangent to E^s or everywhere tangent to E^u then ϕ lies in a single $\mathcal{W}^s$-leaf or in a single $\mathcal{W}^u$-leaf. For if ϕ is everywhere tangent to E^s then for $a, b \in [0, 1]$,

$$d(f^n(\phi(a)), f^n(\phi(b))) \le \int_a^b |Tf^n(\phi'(t))|\, dt \le ||T^s f^n||\text{length}(\phi),$$

which tends to zero so rapidly as $n \to \infty$ that $\phi(a)$ and $\phi(b)$ must lie in a common $\mathcal{W}^s$-leaf. A similar analysis holds for E^u when $n \to -\infty$. Thus, the paths subordinate to (E^u, E^s) stay locally in $\mathcal{W}^u$-leaves and $\mathcal{W}^s$-leaves.

Although $\mathcal{W}^u$ and $\mathcal{W}^s$ are uniquely integrable and have C^2 leaves they are not in general C^1 foliations. This leads us to say that f is *dynamically coherent* if E^{cu}, E^c, E^{cs} do integrate to f-invariant foliations $\mathcal{W}^{cu}$, $\mathcal{W}^c$, $\mathcal{W}^{cs}$, and

$$\mathcal{W}^u \text{ and } \mathcal{W}^c \text{ subfoliate } \mathcal{W}^{cu}, \qquad \text{while } \mathcal{W}^c \text{ and } \mathcal{W}^s \text{ subfoliate } \mathcal{W}^{cs}. \quad (2)$$

(One foliation *subfoliates* a second if each leaf of the second is a union of leaves of the first.) The phrase "dynamically coherent" indicates that the unstable, center unstable, center, center stable, and stable orbit classes fit together nicely. See Fig. 1.

Together with Moe Hirsch, in HPS we investigated normally hyperbolic invariant foliations and laminations. It is just a matter of unraveling the definitions to show that if a partially hyperbolic diffeomorphism leaves invariant

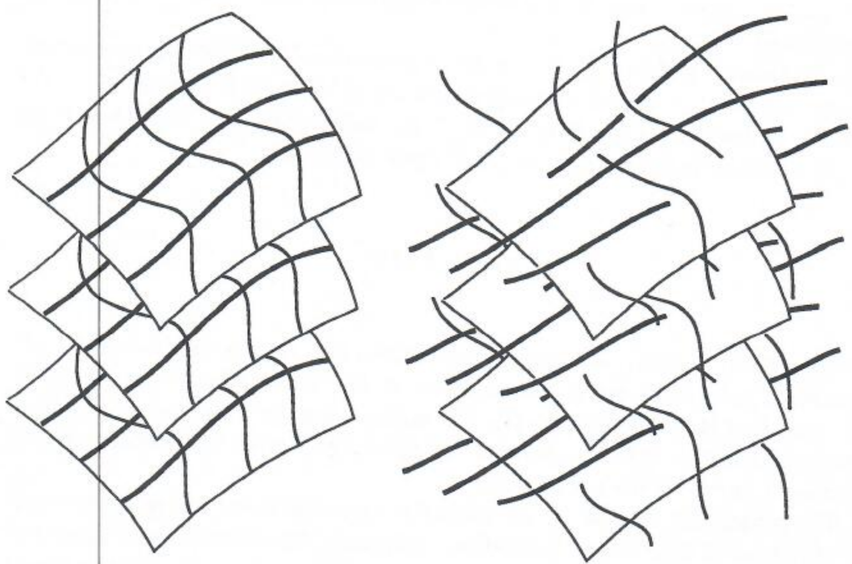

FIG. 1. (a) Coherent foliations. (b) Non-coherent foliations.

a foliation $\mathcal{W}^c$ tangent to E^c then it is 1-normally hyperbolic at $\mathcal{W}^c$, and conversely, if a diffeomorphism is 1-normally hyperbolic at an invariant foliation then it is partially hyperbolic.

It remains to explain the concepts of sufficiently Hölder and spectral bunching. The former condition is a weakening of the assumption in Theorem B that the invariant bundles of f are of class C^1. Although the partially hyperbolic diffeomorphisms having this C^1 bundle property form a non-open set, [2] we can find open conditions that guarantee sufficiently Hölder bundles. Open conditions are needed to prove stable ergodicity. Set

$$\theta_m = \frac{\sqrt{100m^2+1}-1}{10m},$$

where $m = \dim M$. The partially hyperbolic, dynamically coherent diffeomorphism f has *sufficiently Hölder* invariant bundles if

(a) The Hölder exponents of the three bundles E^u, E^c, E^s are greater than θ_m.

[2]This is true except in the special case of dimension two, where partial hyperbolicity implies hyperbolicity and hyperbolicity implies C^1 bundles.

(b) The foliation $\mathcal{W}^u$ is locally uniformly C^1 when restricted to each center unstable leaf W^{cu}, and the foliation $\mathcal{W}^s$ is locally uniformly C^1 when restricted to each center stable leaf W^{cs}.

Not until Eq. (9) in Section 4 do we make explicit use of (a). It is easy to see that (b) implies

(c) The $\mathcal{W}^u$ holonomy maps between center leaves in a common center unstable manifold W^{cu} are locally uniformly Lipschitz, and the same is true of the $\mathcal{W}^s$ holonomy maps between center leaves in a common center stable manifold W^{cs}.

By the *spectrum* of Tf we mean the spectrum of the operator $\sigma \mapsto Tf \circ \sigma \circ f^{-1}$ defined on the space of bounded sections of TM. Spectral bunching conditions are used to prove that f has sufficiently Hölder hyperbolic holonomy. They say that the spectra of $T^u f$, $T^c f$, and $T^s f$ lie in thin, well separated annuli. See Fig. 2.

More precisely, assume that

The spectrum of $T^s f$ lies in the annulus with radii a, b.
The spectrum of $T^c f$ lies in the annulus with radii c, d.
The spectrum of $T^u f$ lies in the annulus with radii e, g.

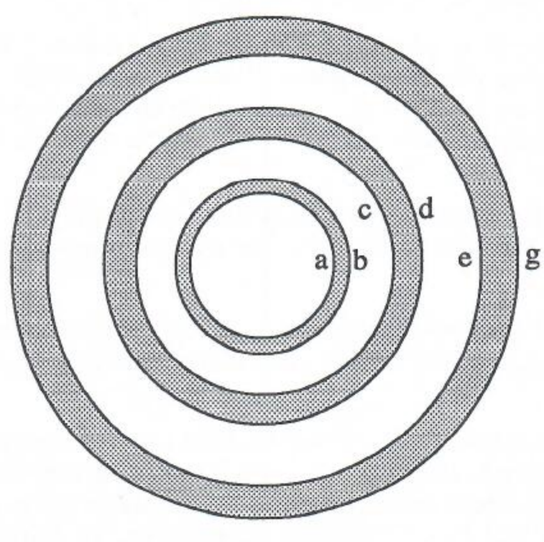

FIG. 2. Bunched spectral annuli.

As we showed in HPS, these conditions are satisfied if and only if there is a Riemann structure on TM *adapted* to Tf in the sense that

$$\begin{aligned} a < |T^s f(v)| < b \qquad &\text{for all } v \in E^s\backslash 0 \\ c < |T^c f(v)| < d \qquad &\text{for all } v \in E^c\backslash 0 \\ e < |T^u f(v)| < g \qquad &\text{for all } v \in E^u\backslash 0. \end{aligned}$$

f has θ-*bunched* spectrum if $0 < \theta < 1$ and

$$\begin{aligned} &\text{(d)} \quad \frac{d}{c} < e \qquad \text{and} \qquad \frac{d}{c} < \frac{1}{b}. \\ &\text{(e)} \quad a^{-\theta} < \frac{e}{d} \qquad \text{and} \qquad g^{\theta} < \frac{e}{d}. \\ &\text{(f)} \quad a^{-\theta} < \frac{c}{b} \qquad \text{and} \qquad g^{\theta} < \frac{c}{b}. \end{aligned}$$

(d) states that the width of the center annulus is thin in comparison to how close to 1 the weakest expansion and weakest contraction are. (e) states that the separation between the center annulus and the unstable annulus, e/d, is large enough to dominate the norm of Tf^{-1}, raised to the power θ. (f) states that the separation between the center annulus and the stable annulus, c/b, is large enough to dominate the norm of Tf, raised to the power θ.

THEOREM 2.1. *If the partially hyperbolic diffeomorphism f has θ-bunched spectrum then so do all diffeomorphism f' that C^1 approximate f. Moreover, E^u, E^c, and E^s are θ-Hölder. If f is dynamically coherent then the restriction of E^u to each center unstable leaf W^{cu} is C^1, and the restriction of E^s to each center stable leaf W^{cs} is C^1.*

Proof. The first assertion is proved in HPS and Shub (1987); the second is proved in HPS, Shub (1987), and PSW; and the third is proved in PSW.

Q.E.D.

COROLLARY. *If the partially hyperbolic diffeomorphism f is dynamically coherent and has θ_m-bunched spectrum then its invariant bundles are sufficiently Hölder.*

Proof. The bundles E^u, E^c, E^s are θ_m-Hölder, and the restriction of E^u to a center unstable leaf is C^1, while the same is true of the restriction of E^s to a center stable leaf. These center unstable and center stable leaves are C^2. A C^1 integrable plane field on a C^2 manifold integrates to a C^1 foliation. Thus the restrictions of $\mathcal{W}^u$ and $\mathcal{W}^s$ to center unstable and center stable leaves give C^1

foliations of those leaves. C^1 foliations have C^1 holonomy maps. Thus, inside the center unstable leaves, the unstable holonomy is C^1 and inside the center stable leaves the stable holonomy is C^1. Q.E.D.

The next result gives a sufficient condition for dynamical coherence. It is a condition that we easily verify in all our examples.

THEOREM 2.2. *If f leaves invariant a foliation $\mathcal{W}^c$ tangent to E^c and $\mathcal{W}^c$ is of class C^1 then f is dynamically coherent.*

Proof. The assumption that $\mathcal{W}^c$ is of class C^1 implies two things:

(a) $\mathcal{W}^c$ is plaque expansive.[3]

(b) $\mathcal{W}^c$ is uniquely integrable.

(i) is Theorem 7.2 of HPS. It is easy to check (ii). Let ϕ be a C^1 curve, everywhere tangent to $E^c = T\mathcal{W}^c$. Express ϕ in a C^1 local chart in which the plaques of $\mathcal{W}^c$ are contained in slices $x_0 \times \mathbb{R}^c \times y_0$. In the chart ϕ remains C^1, and $\phi' \in 0 \times \mathbb{R}^c \times 0$, so ϕ stays in a slice and does not travel from leaf to leaf: $\mathcal{W}^c$ is uniquely integrable.

In Sections 6 and 7 of HPS it is shown that through the leaves of a 1-normally hyperbolic foliation there pass unique, f-invariant C^1 leaf immersed submanifolds W^{cu} and W^{cs}, everywhere tangent to E^{cu} and E^{cs}, respectively. Each W^{cu} is foliated by strong unstable manifolds and each W^{cs} is foliated by strong stable manifolds. Existence of these families of center unstable and center stable leaf immersed submanifolds is true regardless of whether E^{cu} and E^{cs} integrate to foliations. In fact, it is a fundamental, open question whether the W^{cu} leaves and W^{cs} leaves always do fit together to form foliations. We will show that unique integrability of $\mathcal{W}^c$ implies they do.

For $p \in M$, let $W^c(p)$ be the $\mathcal{W}^c$-leaf through p; let $W^{cu}(p)$ and $W^{cs}(p)$ be the center unstable and center stable leaves through $W^c(p)$. Let $q \in W^{cu}(p)$ be given. Through q there pass two manifolds of dimension c—the center manifold $W^c(q)$ and the transverse intersection $W^{cu}(p) \cap W^{cs}(q)$. Both are everywhere tangent to E^c, and so by unique integrability they are equal. Thus $W^c(q) \subset W^{cu}(p)$ and $\mathcal{W}^c$ foliates $W^{cu}(p)$. Since $\mathcal{W}^u$ and $\mathcal{W}^c$ foliate each W^{cu}, the leaves W^{cu} do fit together to form a foliation. For if

[3]Plaque expansiveness is a natural, technical condition of an f-invariant foliation $\mathcal{F}$. The concept is developed in HPS and generalizes orbit expansiveness for hyperbolic dynamics. As in that case, it is used to understand how a perturbation of f affects $\mathcal{F}$. Here is the definition. Let d be a fixed metric on M. A *δ-pseudo-orbit* of f is a sequence $(x_n)_{n\in\mathbb{Z}}$ such that $d(f(x_n), x_{n+1}) < \delta$ for all $n \in \mathbb{Z}$. If x_{n+1} and $f(x_n)$ always lie in the same local $\mathcal{F}$-leaf (or *plaque*) then the δ-pseudo-orbit *respects* $\mathcal{F}$. An f-invariant foliation $\mathcal{F}$ is *plaque expansive* if there exists a $\delta > 0$ such that if (x_n) and (y_n) are δ-pseudo-orbits that respect $\mathcal{F}$ and if $d(x_n, y_n) < \delta$ for all $n \in \mathbb{Z}$ then x_n, y_n belong always to the same plaque of $\mathcal{F}$. Intuitively this means that separated plaques X and Y of $\mathcal{F}$ eventually diverge to a distance $\geq \delta$ apart under f-iteration, even when small errors along $\mathcal{F}$ are permitted.

$q \in W^{cu}(p) \cap W^{cu}(p')$ then $W^c(q)$ is contained in both $W^{cu}(p)$ and $W^{cu}(p')$, and so are all the strong unstable manifolds through points of $W^c(q)$. That is, $W^{cu}(p) = W^{cu}(q) = W^{cu}(p')$, and the leaves W^{cu} form an f-invariant foliation $\mathcal{W}^{cu}$. Similarly, the center stable leaves form an f-invariant foliation $\mathcal{W}^{cs}$, and f is dynamically coherent. See Fig. 3. Q.E.D.

The next result concerns the permanence of dynamical coherence under perturbation.

THEOREM 2.3. *If f is dynamically coherent and its center foliation is plaque expansive then the same is true of each diffeomorphism f' and C^1 approximates f; f' is dynamically coherent and its center foliation is plaque expansive.*

COROLLARY. *If the partially hyperbolic diffeomorphism f leaves invariant a C^1 center foliation then f and all diffeomorphisms f' that C^1 approximate f are dynamically coherent.*

Proof. According to Theorem 2.2, f is dynamically coherent. Its center foliation is C^1 and hence plaque expansive. According to Theorem 2.3, each f' that C^1 approximates f is also dynamically coherent. Q.E.D.

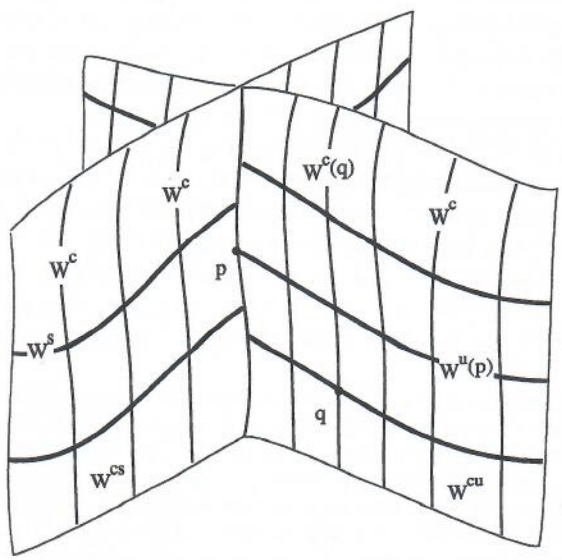

FIG. 3. Dynamical coherence of the invariant foliations of f.

Proof of Theorem 2.3 *under the additional hypothesis:*

not only is the foliation $\mathcal{W}^c$ plaque expansive
but also the foliations $\mathcal{W}^{cu}$ and $\mathcal{W}^{cs}$ are plaque expansive.

The relevant result is Theorem 7.1 of HPS, which describes how perturbations affect an f-invariant, normally hyperbolic, plaque expansive foliation $\mathcal{F}$. Plaque expansivity implies a kind of foliation–structural stability. If f' C^1 approximates f then Tf' leaves invariant a unique bundle that C^0 approximates $T\mathcal{F}$, and the bundle integrates to a unique f'-invariant foliation $\mathcal{F}'$. Applying this to the three normally hyperbolic, plaque expansive foliations $\mathcal{W}^{cu}$, $\mathcal{W}^c$, and $\mathcal{W}^{cs}$, we get f'invariant foliations $\mathcal{W}^{cu}(f')$, $\mathcal{W}^c(f')$, and $\mathcal{W}^{cs}(f')$. The leaf intersection of $\mathcal{W}^{cu}(f')$ and $\mathcal{W}^{cs}(f')$ gives an f'-invariant foliation tangent to the center bundle of f' By uniqueness it is $\mathcal{W}^c(f')$. Thus, $\mathcal{W}^c(f')$ subfoliates $\mathcal{W}^{cu}(f')$ and $\mathcal{W}^{cs}(f')$. As was explained in the proof of Theorem 2.2, the union of the strong unstable leaves through points of a leaf of a normally hyperbolic foliation is a C^1 leaf immersed submanifold, and the family of these is invariant. Thus, for each center leaf $L' = W^c(p, f')$ we form the C^1 leaf immersed submanifold

$$W(p, f') = \bigcup_{q \in L'} W^u(q, f').$$

The family $\{W(p, f')\}$ is f'-invariant and tangent to the center unstable bundle of f' along the center leaves. By uniqueness, $W(p, f') = W^{cu}(p, f')$ and $\mathcal{W}^u(f')$ is seen to subfoliate $\mathcal{W}^{cu}(f')$. Similarly, $\mathcal{W}^s(f')$ subfoliates $\mathcal{W}^{cs}(f')$, completing the proof that f' is dynamically coherent. Q.E.D.

Proof of Theorem 2.3 *in general.* Assume that f' C^1 approximates f. Since $\mathcal{W}^c$ is plaque expansive, Theorem 7.1 of HPS implies that Tf' leaves invariant a unique bundle that C^0 approximates E^c, and the bundle integrates to a unique f'-invariant foliation $\mathcal{W}^c(f')$. Furthermore, there is a canonically defined leaf conjugacy h from $\mathcal{W}^c(f)$ to $\mathcal{W}^c(f')$. The leaf conjugacy is a homeomorphism $h_c \colon M \to M$ that sends leaves of $\mathcal{W}^c(f)$ to leaves of $\mathcal{W}^c(f')$, C^0 approximates the identity map id: $M \to M$, and commutes with the leaf dynamics,[4]

$$h_c(L) = L',$$

[4]When $\mathcal{F}$ is the orbit foliation of an Anosov diffeomorphism or flow, the existence of h_c is Anosov's structural stability theorem.

where $L = W^c(p, f)$ and $L' = W^c(h_c(p), f')$). We claim that dynamical coherence of f implies that the leaf conjugacy h_c carries the foliations $\mathcal{W}^{cu}$ and $\mathcal{W}^{cs}$ to corresponding f'-invariant foliations. This will imply that f' is dynamically coherent. To verify the claim we recall how the leaf conjugacy is constructed.

By compactness of M we can define a uniform tubular neighborhood $U = U_F$ of each center unstable leaf $F = W^{cu}(p)$, and a uniform subtubular neighborhood $V = V_L$ of each $L = W^c(q) \subset W^{cu}(p)$. See Fig. 4.

Since the leaves of $\mathcal{W}^{cu}$ and $\mathcal{W}^c$ are injectively immersed, but not in general embedded, we form U and V in the tangent bundle TM. This prevents self-intersection. Tacitly, we lift f and f' to TM using the smooth exponential of the Riemann structure. Let U^* be the disjoint union of these tubular neighborhoods U,

$$U^* = \sqcup U_F,$$

taking one U for each center unstable leaf F. The diffeomorphism f acts naturally on U^*, $f\colon U^* \to U^*$. It sends the tubular neighborhood of $F = W^{cu}(p)$ to the tubular neighborhood of $fF = W^{cu}(f(p)) = f(W^{cu}(p))$,

$$f\colon U_F \to U_{fF},$$

and it contracts U toward W^{cu}. For $\mathcal{W}^{cu}$ is f-invariant and normally attracting. The diffeomorphism f' also acts naturally on U^*, $f'\colon U^* \to U^*$,

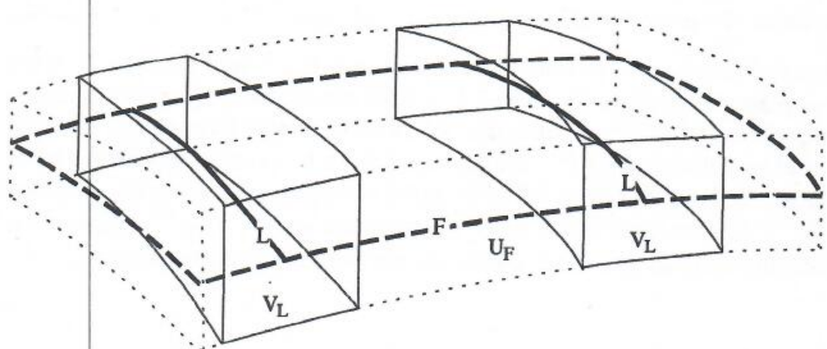

FIG. 4. The tubular neighborhoods U_F of $F = W^{cu}(p)$ and V_L of $L = W^c(p)$.

$$f' \colon U_F \to U_{fF},$$

and it contracts U^*. According to Theorems 6.1, 6.7, and 6.8 of HPS, there is a unique overflowing f'-invariant family of cu-dimensional manifolds

$$F' = W^{cu}(f', F)$$

with $W^{cu}(f', F) \subset U_F$, that C^1 approximate F. In M they are merely known to be leaf immersed submanifolds. We will show that they form a foliation.

The same analysis applies to the disjoint union

$$V^* = \sqcup V_L$$

of the tubular neighborhoods V of the center leaves L of f. By dynamical coherence of f, $\mathcal{W}^c$ subfoliates $\mathcal{W}^{cu}$; i.e., the center leaves L foliate the center unstable leaves F. There is a unique overflowing f'-invariant family of cu-dimensional manifolds H^{cu} in V^* that C^1 approximate the restricted leaves $F \cap V$. By uniqueness

$$H^{cu} = F' \cap V.$$

On the other hand, since the leaf conjugacy h_c C^0 approximates the identity, it does not move the leaf $L = W^c(f, p)$ far:

$$L' = h_c(L) \subset V_L.$$

The union of the local unstable manifolds of f' as q varies in L',

$$W^u(L') = \bigcup_{q \in L'} W^u(q, \varepsilon, f')$$

gives a second overflowing f'-invariant family of cu-dimensional manifolds that C^1 approximate $F \cap V$. By uniqueness

$$H^{cu} = F' \cap V = W^u(L').$$

This shows that F' is subfoliated by $\mathcal{W}^u(f')$ and by $\mathcal{W}^c(f')$, and that h_c carries F to F'. Since h_c is a homeomorphism and $\mathcal{W}^{cu}$ is a foliation, the leaf immersed submanifolds F' are injectively immersed and form a foliation. It is the center unstable foliation of f', and we just observed that it is subfoliated by $\mathcal{W}^u(f')$ and $\mathcal{W}^c(f')$. The same reasoning applies to the center stable manifold foliation and we see that f' is dynamically coherent. Q.E.D.

3. ACCESSIBILITY

Let E, F be continuous subbundles of M. In Section 2 we defined the following concepts:

Accessibility of (E, F). Every pair of points in M can be joined by a piecewise C^1 path ϕ such that $\phi'(t) \in E \cup F$ whenever $\phi'(t)$ exists.

Essential accessibility of (E, F). Almost every pair of points can be joined by such a path.

Only connected manifolds can have these accessibility properties. We are assuming throughout that M is connected. If f is a partially hyperbolic diffeomorphism and (E^u, E^s) has the accessibility or essential accessibility property then we say also that f has these properties.

Accessibility is a concept in control theory. The approach we follow here was developed by Sussman, Lobry, and others; see Lobry (1973), Sussman (1976), and also Gromov (1995).

Let E be a subbundle of TM. If E is C^r then we write $V^r(E)$ for the set of C^r vector fields X subordinate to E,

$$X(p) \in E_p \qquad \text{for all } p \in M.$$

E is *uniquely integrable* (in the control theory sense) if the uniquely integrable fields are dense in $V^0(E)$. Unique integrability does not require that E integrate to a foliation. Bundles of class C^r, $r \geq 1$, are of course uniquely integrable. Also uniquely integrable are the bundles E^u, E^s of a partially hyperbolic diffeomorphism. For any vector field Y in E^u can be approximated by a vector field X in E^u that is smooth on the unstable leaves, and, as we observed in

Section 2, the trajectories of X can not migrate from one unstable leaf to another. The same is true for E^s.

THEOREM 3.1. *If E, F are C^r subbundles of TM, $r \geq 1$, and A_p is the set of points accessible from p by paths subordinate to $E \cup F$ then A_p is a C^r submanifold of M.*

Proof. We follow Lobry (1973). Choose C^r vector fields $X_1, \ldots, X_k$ and $X_{k+1}, \ldots, X_n$ that span E and F. They need not be linearly independent. Let $\phi_{i,t}(x)$ denote the X_i flow. For each t, $\phi_{i,t}$ is a C^r diffeomorphism of M to itself, and the map

$$\begin{aligned} \Phi\colon \mathbb{R}^n \times M &\to M \\ (t_1, \ldots, t_n, p) &\mapsto \phi_{1,t_1} \circ \cdots \circ \phi_{n,t_n}(p) \end{aligned}$$

is C^r. When $(t_1, \ldots, t_n)$ is fixed, Φ defines a diffeomorphism

$$\begin{aligned} \psi\colon\ M &\to M \\ p &\mapsto \phi_{n,t_n} \circ \cdots \circ \phi_{1,t_1}(p) \end{aligned}$$

of M to itself. Let $\mathcal{G}$ be the set of all such diffeomorphisms ψ, for all choices of spanning vector fields X_i, all n-tuples $(t_1, \ldots, t_n)$, and all $n \in \mathbb{N}$. Clearly, $\mathcal{G}$ is a group and $\psi \in \mathcal{G}$ leaves A_p invariant. If $q \in A_p$ then there is a diffeomorphism $\psi = \phi_{n,t_n} \circ \cdots \circ \phi_{1,t_1}$ sending p to q. Thus A_p is topologically homogeneous under ambient diffeomorphisms of M.

The *t-rank* of Φ at $(t_1, \ldots, t_n, p)$ is the rank of the map

$$\phi\colon (t_1, \ldots, t_n) \mapsto \phi_{n,t_n} \circ \cdots \circ \phi_{1,t_1}(p).$$

Let l be the maximal t-rank of Φ, maximized over all choices of spanning vector fields X_i and over all n-tuples $(t_1, \ldots, t_n)$. Let this maximal rank be attained at $(s_1, \ldots, s_n)$. According to the rank theorem, there is a neighborhood U of $(s_1, \ldots, s_n)$ such that $\phi(U)$ is an l-dimensional C^r submanifold of M. Maximality of l implies that if X is subordinate to $E \cup F$ then the restriction of X to $\phi(U)$ is tangent to $\phi(U)$. It follows that $\phi(U)$ is an injectively immersed l-dimensional C^r submanifold of M. Homogeneity implies that the same is true of A_p. Q.E.D.

THEOREM 3.2. *Suppose that E, F are C^1 and (E, F) has the accessibility property. Then there exist vector fields $X_1, \ldots, X_n$ that span E and F, and, for* *each pair of points $p, q \in M$ there is a point $s = (s_1, \ldots, s_n) \in \mathbb{R}^n$ such that $\Phi(s, p) = q$ and the t-rank of Φ at (s, p) is $m = \dim M$.*

Proof. The t-rank can of course be no larger than m. The proof just given shows that for each $p \in M$ there exist vector fields $X_1, \ldots, X_{n^*}$ that span E and F, and there exists a point $s^* = (s_1, \ldots, s_{n^*}) \in \mathbb{R}^{n^*}$ such that for some q^*,

$$\Phi(s^*, p) = q^*$$

and the t-rank of Φ at (s^*, p) is m. Since (E, F) has the accessibility property, we can increase the set of vector fields that span E and F to $X_1, \ldots, X_{n^*}, X_{n^*+1}, \ldots, X_{n^{**}}$ such that for some $(s_{n^*+1}, s_{n^{**}})$,

$$\phi_{n^{**}, s_{n^{**}}} \circ \cdots \circ \phi_{n^*+1, s_{n^*+1}}(q^*) = q.$$

Set $s^{**} = (s_1, \ldots, s_{n^*}, s_{n^*+1}, \ldots, s_{n^{**}})$. Then $\Phi(s^{**}, p) = q$ and the t-rank of Φ at (s^{**}, p) is m. Rank is lower semicontinuous. Hence, for all points (p', q') in some neighborhood W of (p, q) in $M \times M$, the same is true: for some s' near s^{**}, $q' = \Phi(s', p')$ and Φ has t-rank m at (s', p'). Since $M \times M$ is compact, it is covered by finitely many of these neighborhoods $W_1, \ldots, W_N$ and the set of all the vector fields for all the neighborhoods W_j spans E and F, and has the property called for in the theorem. Q.E.D.

If E, F are smooth subbundles of TM we write $\mathcal{L}(E, F)$ for the smallest vector space of smooth vector fields that contains $V^\infty(E) \cup V^\infty(F)$ and is closed under Lie bracket. In other words, we start with all smooth vector fields in E and F, form Lie brackets of them, Lie brackets of the resulting fields, etc., until we stabilize at $\mathcal{L}(E, F)$. The evaluation of $\mathcal{L}(E, F)$ at the point p is the set

$$\mathcal{L}_p(E, F) = \{X(p) : X \in \mathcal{L}(E, F)\}.$$

THEOREM 3.3 (Chow's Theorem). *Let (E, F) be a pair of smooth subbundles of TM such that for each $p \in M$, $\mathcal{L}_p(E, F) = T_pM$. Then (E, F) has the accessibility property.*

Proof. According to Theorem 3.1, A_p is a smooth submanifold of M. Its tangent bundle includes E and F restricted to A_p. The Lie bracket of vector fields tangent to a submanifold is also tangent to the submanifold. Hence

$$\mathcal{L}_p(E, F) \subset T_p(A_p).$$

It is clear that the sets A_p partition M into disjoint subsets since accessibility is an equivalence relation.

Since $\mathcal{L}_p(E, F) = T_pM$, every A_p is an m-dimensional submanifold of M, i.e., an open subset of M. Fix $p_0 \in M$. If $A_{p_0} \neq M$ then there is a point $p \in \partial(A_{p_0})$. Openness of A_p implies that A_p intersects A_{p_0}, so $A_p = A_{p_0}$, contrary to the supposition that p is a boundary point of A_{p_0}. Hence all points of M are accessible from p_0, $A_{p_0} = M$. Q.E.D.

Let $\mathcal{P}^r$ denote the space of all pairs of C^r, uniquely integrable subbundles of TM; equip $\mathcal{P}^r$ with the C^r topology. If $r \geq 1$, integrability implies unique integrability.

THEOREM 3.4. *Suppose that $(E, F) \in \mathcal{P}^1$ has the accessibility property. Then every (E', F') near (E, F) in $\mathcal{P}^0$ also has the accessibility property. (See Grasse, 1984.)*

Proof. Let $X_1, \ldots, X_n$ be as in Theorem 3.2. Fix $p \in M$. For each $q \in M$, Theorem 3.2 provides a point $s = (s_1, \ldots, s_n) \in \mathbb{R}^n$ such that the smooth map

$$\phi\colon (t_1, \ldots, t_n) \mapsto \phi_{1,t_1} \circ \cdots \circ \phi_{n,t_n}(p)$$

sends s to q and has rank m at s. Therefore there exists in $\mathbb{R}^n$ an m-dimensional disc $D = D^m$ through s such that the restriction of ϕ to D is a C^1 diffeomorphism g of D onto a neighborhood U of $q = \phi(s)$, $g\colon D \to U$. See Fig. 5.

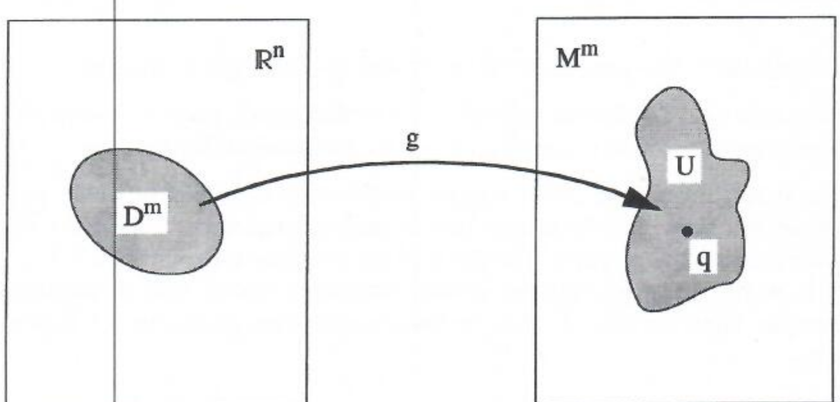

FIG. 5. g sends D^m diffeomorphically onto U.

Let (E', F') approximate (E, F) in $\mathcal{P}^0$ and choose continuous, uniquely integrable vector fields $X_1', \ldots, X_n'$ that span E' and F', and that approximate $X_1, \ldots, X_n$. Denote by $\phi_{i,t}'$ the X_i' flow. Unique integrability implies that the map

$$\begin{aligned} &\phi' \colon \mathbb{R}^n \times M \to M \\ &(t_1, \ldots, t_n, p) \mapsto \phi_{1,t_1}' \circ \cdots \circ \phi_{n,t_n}'(p) \end{aligned}$$

C^0 approximates Φ. Hence, the restriction of ϕ' to D defines a continuous map g' that C^0 approximates g,

$$g' \colon D \to M.$$

The topological index of $g|_{\partial D}$ with respect to q is non-zero and the same is true for the approximation g'. It follows that there exist a neighborhood $\mathcal{U}_q$ of (E, F), in $\mathcal{P}^0$ and a neighborhood U_q of q in M such that if $(E', F') \in \mathcal{U}_q$ then $g'(D) \supset U_q$. Covering M by finitely many of these neighborhoods $U_{q_1}, \ldots, U_{q_N}$, and taking $\mathcal{U} = \mathcal{U}_{q_1} \cap \cdots \cap \mathcal{U}_{q_N}$, we see that if $(E', F') \in \mathcal{U}$ then all points of M are accessible from p by paths subordinate to (E', F'). Hence $A_p' = M$ and (E', F') has the accessibility property. Q.E.D.

We do not use the next theorem and corollary in what follows, but we state them anyway since they serve as motivation for Conjecture 4.

THEOREM 3.5. *The generic $(E, F) \in \mathcal{P}^r$ has the accessibility property, $r \geq 1$.*

COROLLARY. *The generic $(E, F) \in \mathcal{P}^0$ has the accessibility property.*

Conjecture 4. The bundle pair (E^u, E^s) of the generic partially hyperbolic diffeomorphism, volume preserving or not, has the accessibility property.

The proof of Theorem 3.5 is a slight modification of results of Lobry and Sussman that show that the generic pair of smooth tangent vector fields on M has the accessibility property. The proof of the corollary uses Theorem 3.4.

It is worth remarking that the generic continuous vector field is uniquely integrable. Thus, the space $\mathcal{P}^0$ has the Baire property and genericity in it makes sense.

4. JULIENNE GEOMETRY

In GPS we presented two short false proofs of stable ergodicity, and of course we also presented a long correct proof. Here we show how to rescue the second false proof, which was based on the false principle that a bi-Hölder homeomorphism sends each density point of a measurable set to a density point of its image. Below, we show that this density point preservation principle becomes true if we restrict it to measurable sets that are "$(u,\, s)$-saturated" and to bi-Hölder homeomorphisms that are stable or unstable holonomy maps. Besides establishing stable ergodicity in higher dimensions, this will give a shorter, somewhat new proof of the main result in GPS, stable ergodicity of the time one map φ_1 of the geodesic flow for a surface of constant negative curvature. By the way, the first false proof was based on the unproved (and we believe generally incorrect) assumption that the center foliation of a perturbation of φ_1 is absolutely continuous.

We begin with a general estimate responsible for part of the julienne nesting lemma in GPS, and one we use repeatedly. The variables $x,\, y$ are vectors, and F is a matrix.

LEMMA 1. *Suppose that $y(x)$ solves the differential equation $dy/dx = F(x,\, y)$, where F is defined on the set $\{(x,\, y) : |x| \leq h,\, |y| \leq h\}$ and $y(x)$ is defined on the set $\{x : |x| \leq h\}$. If F satisfies the Hölder-like condition $F(x,\, y)| \leq K h^{\theta}$ then $|y(x) - y(0)| \leq K|x|h^{\theta}$.*

Proof. $y(x) - y(0)$ is the average of its derivative with respect to x, the average being taken over the segment $[0,\, x]$. Thus $y(x) - y(0) = \int_0^1 F(sx,\, y(sx))\, ds(x)$ and the assertion is clear. Q.E.D.

We will use this lemma to analyze the holonomy of the invariant foliations $\mathcal{W}^{cu}$, $\mathcal{W}^{u}$, $\mathcal{W}^{c}$, $\mathcal{W}^{s}$, $\mathcal{W}^{cs}$ of a partially hyperbolic diffeomorphism. These foliations, and all the foliations $\mathcal{W}$ we consider in this paper, have at least the following regularity: $\mathcal{W}$ has C^1 leaves $W(p)$ and the map $p \mapsto T_p(W(p))$ is a continuous section of the Grassmann bundle. That is, the tangent field $T\mathcal{W} = \{T_p(W(p)) : p \in M\}$ is a continuous subbundle of TM.

Let $\mathcal{W}$ be such a foliation. At $p \in M$, split T_pM as $X \oplus Y$ where $X = T_p\mathcal{W} = T_pW(p)$. Relative to a fixed, smooth Riemann structure on TM, choose linear orthonormal coordinate frames $x_1,\, \ldots,\, x_k$ in X and $y_1,\, \ldots,\, y_n$ in Y, $k + n = m$. The holonomy of $\mathcal{W}$ near p defines a local function

$$\omega_x \colon Y \to Y \qquad \text{for } x \in X,$$

according to $\exp_p((x,\, \omega_x(y))) = W(\exp_p((0,\, y))) \cap \exp_p(x \times Y)$. See Fig. 6.

Equivalently, $\mathcal{W}$ lifts to a foliation $\overline{\mathcal{W}}$ of a neighborhood of p in T_pM and the leaf of $\overline{\mathcal{W}}$ through (0, y) is the graph of the function $x \mapsto \omega_x(y)$. Since $T\mathcal{W}$ is continuous, $\partial\omega/\partial x$ exists and is continuous. That is, $\omega_x(y)$ solves the continuous partial differential equation

$$\frac{d\omega_x(y)}{dx} = F(x, \omega_x(y)),$$

where $F(x, y)$: $X \to Y$ is the linear transformation whose graph is the plane tangent to $\overline{\mathcal{W}}$ at (x, y), graph $F(x, y) = \{(\xi, F(x, y)(\xi)) : \xi \in X\} = T_{(x, y)}\overline{\mathcal{W}}$. At the origin of T_pM, $\partial\omega/\partial x = F(0, 0) = 0$ since $T_p\mathcal{W} = X$. By continuity of F, $||F|| \leq 1$ on a small neighborhood of (0, 0), and hence

$$\begin{aligned}&\text{if } |x| \leq h, |y| \leq h, \text{ and } h \text{ is small}\\ &\text{then } \omega_x(y) \text{ is defined and } |\omega_x(y)| \leq 2h.\end{aligned}$$

LEMMA 2. *If $T\mathcal{W}$ is θ-Hölder, $0 < \theta < 1$, then for all small h and all x, y with $|x|$, $|y| \leq h$, $\omega_x(y)$ is well defined and for a constant K independent of x, y, h,*

$$|\omega_x(y) - y| \leq K|x|h^\theta \leq Kh^{1+\theta}. \tag{1}$$

Proof. Since graph $F(x, y) = T_{(x, y)}\overline{\mathcal{W}}$ is θ-Hölder, so is F, and since $F(0,0) = 0$, this implies

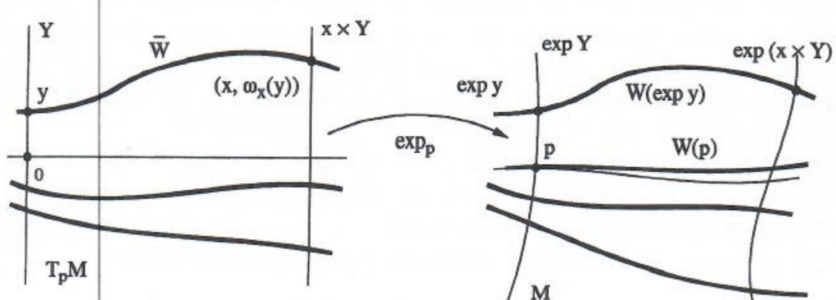

FIG. 6. The $\mathcal{W}$-holonomy defines ω_x.

$$|F(x, y)| \leq Kh^{\theta}$$

when $|x|, |y| \leq 2h$. Thus, if $|x| \leq h$ then the holonomy map $y \mapsto \omega_x(y)$ is a well-defined embedding $Y(h) \to Y(2h)$, and (1) follows from Lemma 1.

Q.E.D.

(1) is a weak form of Hölderness of the holonomy maps. As shown in Wilkinson (1995), the strong form of Hölderness of holonomy maps, $|\omega_x(y) - \omega_x(y')| \leq K|y - y'|^{\theta}$, is not a consequence of Hölderness of the tangent field to the foliation.

Next we discuss some box packing geometry. Let $r = (r_1, \ldots, r_n)$ be an n-vector with positive components. The rectangular *box* with center $v \in \mathbb{R}^n$ and *multi-radius* r is

$$R(v, r) = \{y \in \mathbb{R}^n : \text{for each } i,\ |y_i - v_i| \leq r_i\}.$$

For $\tau > 0$, the τ-*dilation* of this box is $\tau R(v, r) = R(v, \tau r)$. See Fig. 7.

If each component r_i of r satisfies $a \leq r_i \leq b$ then we write $a \leq r \leq b$, and refer to b/a as the *eccentricity* of the box. If all the components of r equal h then R is an n-cube, $Q(v, h)$. Using orthonormal coordinate frames, we identify $\mathbb{R}^m$ with T_pM. For $n \leq m$ we include $\mathbb{R}^n$ in $\mathbb{R}^m$ and think of boxes and cubes in T_pM.

LEMMA 3. *Fix θ, $1/2 < \theta < 1$ and $K > 0$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that if $h < \delta$ and the continuous map ω: $Q(0, h) \to \mathbb{R}^n$ satisfies*

$$|\omega(y) - y| \leq Kh^{1+\theta} \qquad \text{for all } y \in Q(0, h) \tag{2}$$

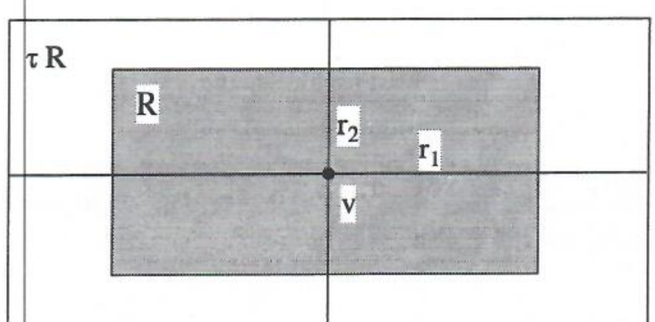

FIG. 7. A box and its τ-dilation.

then each box $R = R(v, r) \subset Q(0, h)$ with $h^{3/2} \le r \le h$ has an ε-squeeze property under ω,

$$(1-\varepsilon)R \subset \omega(R) \subset (1+\varepsilon)R. \tag{3}$$

Proof. Choose θ_0, $1/2 < \theta_0 < \theta$ and define

$$\varepsilon = h^{\theta_0 - 1/2}. \tag{4}$$

Note that $\varepsilon \to 0$ as $h \to 0$. Also, when h is small, $Kh^{1+\theta} \le h^{1+\theta_0}$. Thus (4) implies

$$Kh^{1+\theta} \le h^{1+\theta_0} = \varepsilon h^{3/2} \tag{5}$$

for small h. Now suppose that $y \in R$. Then by (2), (5)

$$\begin{aligned} |\omega_i y - v_i| &= |\omega_i y - y_i + y_i - v_i| \le h^{1+\theta_0} + |y_i - v_i| \\ &\le \varepsilon h^{3/2} + r_i \le \varepsilon r_i + r_i = (1+\varepsilon) r_i, \end{aligned}$$

which implies that $\omega y \in (1+\varepsilon)R$. Thus, $\omega(R) \subset (1+\varepsilon)R$.

The proof that $\omega(R)$ contains $(1-\varepsilon)R$ is of a different character. Under the continuous, linear homotopy

$$\omega_t(y) = ty + (1-t)\omega y,$$

the point $\omega_t(y)$ stays in the cube $Q(y, h^{1+\theta_0})$, which implies that $\omega_t(\partial R)$ is always disjoint from the interior of $(1-\varepsilon)R$. For if $y^* \in \partial R$ then for some i, $|y_i^* - v_i| = r_i$, and if $y \in Q(y^*, h^{1+\theta_0})$ then

$$\begin{aligned} |y_i - v_i| &\ge |y_i^* - v_i| - |y_i - y_i^*| \ge r_i - h^{1+\theta_0} \\ &= r_i - \varepsilon h^{3/2} \ge r_i - \varepsilon r_i = (1-\varepsilon) r_i. \end{aligned}$$

Thus, $y \notin \text{int}((1-\varepsilon)R)$, and thus $\omega_t(\partial R)$ does stay disjoint from the interior of $(1-\varepsilon)R$, $0 \le t \le 1$. Therefore, the index of any point y' interior to $(1-\varepsilon)R$ relative to the restriction of ω_t to ∂R is independent of t. Under the identity map the index is 1, so the index of y' relative to the restriction of ω to ∂R is 1. Therefore $y' \in \omega(R)$; i.e., $\omega(R)$ contains the interior of $(1-\varepsilon)R$. Since $\omega(R)$ is closed and ω is continuous, it also contains $(1-\varepsilon)R$. Q.E.D.

Combining Lemmas 2 and 3, we see that the local holonomy map of a foliation with Hölder tangent field has nice packing properties. Ideally, a holonomy map would be trivial. It would send a box R to its horizontal translate R'. Actually, R is sent to a non-linear box packed between slight dilations of R', provided that R is small, not too small, has good Hölder proportions, and that the horizontal distance between the transversals is appropriately small. ("Good Hölder proportions" means that the width and height satisfy $h^{3/2} \le w \le h$, while "appropriately small" can be considerably larger than the size of the box R.) Formalizing this we say that a family of local embeddings $\omega_x \colon Y \to Y$ indexed by $x \in X$ is *translation-like* if they satisfy the following *box packing condition.*

$$\begin{aligned}&\text{Given } \varepsilon > 0 \text{ there is a } \delta > 0 \text{ such that}\\ &\text{if } R = R(v,r) \subset Y(h) \text{ with } h^{3/2} \le r \le h < \delta \text{ and } |x| \le h \text{ then}\\ &\quad (1-\varepsilon)R \subset \omega_x(R) \subset (1+\varepsilon)R. \end{aligned} \tag{6}$$

See Fig. 8. Summarizing what we have shown, we state

THEOREM 4.1. *The family of local holonomy maps of a foliation with θ-Hölder tangent field, $1/2 < \theta < 1$, is translation-like.*

Next, we turn to the measure theoretic properties of saturated and essentially saturated sets. We recall some of the definitions. Let $f \colon M \to M$ be a partially hyperbolic diffeomorphism with invariant splitting

$$TM = E^u \otimes E^s \otimes E^c,$$

where u, s, c also denote the fiber dimension of the bundles, $u + s + c = m = \dim M$. We always assume that $u, s, c \ge 1$. By E^{cu}, E^{cs}, E^{us} we denote the sums $E^u \oplus E^c$, $E^s \oplus E^c$, $E^c \oplus E^s$. The invariant foliations tangent to E^u, E^s, E^c, E^{cu}, E^{cs} are denoted $\mathcal{W}^u$, $\mathcal{W}^s$, $\mathcal{W}^c$, $\mathcal{W}^{cu}$, $\mathcal{W}^{cs}$. In general there is no foliation tangent to E^{us}. The foliations $\mathcal{W}^u$, etc., have smooth leaves and the plane field tangent to the leaves is continuous. They are foliations. The

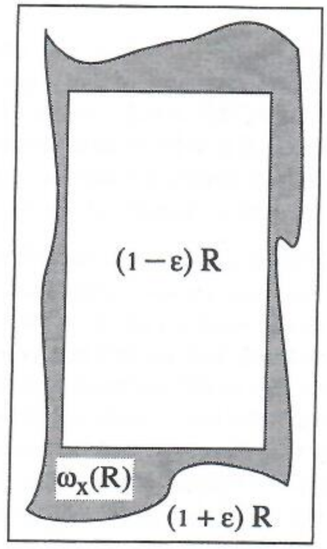

FIG. 8. $\omega_x(R)$ nests between slight dilations of R.

proof of stable ergodicity of f relies on the measure theoretic and Hölder properties of its invariant foliations.

A key property possessed by $\mathcal{W}^u$, $\mathcal{W}^s$ is *absolute continuity*: each local holonomy map sends the sets of measure zero to sets of measure zero. (The measure in question is the natural Riemann measure on the transversal.) Moreover, the Radon–Nikodym derivative of the holonomy map is positive and continuous. See Pugh and Shub (1972). As mentioned above, it is not known whether $\mathcal{W}^c$, $\mathcal{W}^{cu}$, $\mathcal{W}^{cs}$ are absolutely continuous.

Each leaf W of a foliation $\mathcal{W}$ carries a natural measure, its *leaf measure*. If W has dimension k, its leaf measure is a smooth k-dimensional volume form on W. A set A is *completely $\mathcal{W}$-saturated* if it consists of whole leaves: if $p \in A$ when $W(p) \subset A$. An *almost whole leaf* is a set $L(p) \subset W(p)$ such that $W(p)\backslash L(p)$ has leaf measure zero. A measurable set A is *essentially $\mathcal{W}$-saturated* if it almost consists of almost whole leaves of $\mathcal{W}$. More precisely, there is another measurable set A_0 such that the symmetric difference $A\Delta A_0$ is an m-dimensional zero set, and A_0 consists of almost whole leaves of $\mathcal{W}$. If A is essentially $\mathcal{W}^u$-saturated and essentially $\mathcal{W}^s$-saturated then we say it is *essentially (u, s)-saturated.* If A is completely $\mathcal{W}^u$ saturated and completely $\mathcal{W}^s$ saturated, it is *completely (u, s)-saturated.* The following was shown in GPS, pp. 297–298.

THEOREM 4.2. *If the measure preserving, partially hyperbolic diffeomorphism $f\colon M \to M$ is not ergodic then there exists a measurable set $A \subset M$ such that*

(a) *A is invariant under f.*
(b) *$0 <$ measure $A <$ measure M.*
(c) *A is essentially (u, s)-saturated.*

(a) and (b) merely express the non-ergodicity of f, while the standard argument of Hopf shows that (a) and (b) imply (c).

Recall that Lebesgue's Density Theorem states that almost every point of a measurable set A is a density point of A. We will express this in terms of cubes in the tangent bundle as follows. In T_pM we choose orthonormal coordinates $x_1, \ldots, x_u$ in E^u, $y_1, \ldots, y_s$ in E^s, $z_1, \ldots, z_c$ in E^c, and refer to points $v \in T_pM$ as $v = (x, y, z)$. Relative to the maximum coordinate norm $|\,|$ we have cubes $Q(v, r)$ as above. The exponential image of the cube $Q = Q(0, h)$ is a neighborhood $\exp Q$ of p having uniformly bounded eccentricity, and $\exp Q \downarrow p$ as $h \to 0$. That the point p is a density point of A means that the *concentration* [5] of A in $\exp Q$,

$$m(A : \exp Q) = \frac{m(A \cap \exp Q)}{m(\exp Q)},$$

tends to 1 as $\exp Q \downarrow p$. As in GPS we abuse notation and refer to all measures and concentrations of measures as m. Since $\exp_p$ is a local diffeomorphism and its derivative at p is the identity transformation $T_0(T_pM) \to T_pM$, the concentration of $\exp^{-1} A$ in $Q(0, h)$ approximates the concentration of A in $\exp Q$,

$$m(A : \exp Q) - m(\exp^{-1} A : Q) \rightrightarrows 0 \qquad \text{as } h \to 0.$$

The symbol $\rightrightarrows$ denotes uniform convergence. In particular, if p is a density point of A then $m(\exp^{-1} A : Q)$ also tends to 1 as $h \to 0$. (In M we use the smooth Riemann measure, while in T_pM we use the linear Lebesgue measure of the (x, y, z)-coordinates. They are equivalent under $\exp_p$.)

We call $Z = E_p^{us}$, the plane $z = 0$. Let $\overline{\mathcal{W}}^u$ be the foliation $\mathcal{W}^u$ lifted by $\exp_p^{-1}$ to T_pM. Its local holonomy is expressed by maps $\omega^u_{xx'}\colon E_p^{cs} \circlearrowleft$ where $x, x' \in E_p^u$. See Fig. 9. Thus, if (x, y, z) and (x', y', z') lie on a common

[5] The concentration of A in A in $\exp Q$ is the same as the conditional measure of A, conditioned on $\exp Q$. In GPS we referred to it as the density of A in $\exp Q$.

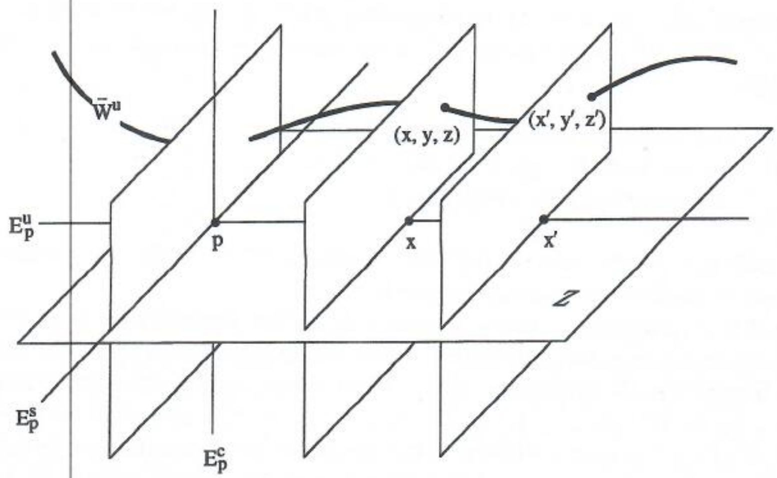

FIG. 9. The local holonomy ω^u of $\overline{\mathcal{W}}^u$.

$\overline{\mathcal{W}}^u$-leaf then $\omega^u_{xx'}(y, z) = (y', z')$. We will consider a box

$$R = R(v, w, h) = Q^{us}(v, w) \times Q^c(0, h),$$

where $Q^{us}(v, w)$ is the cube in Z centered at v having radius w and $Q^c(0, h)$ is the cube centered at the origin in E^c_p having radius h. We say that R has *center* v, *width* w, and *height* h.[6]

THEOREM 4.3. *Assume that E^u, E^s, E^c, are θ-Hölder, $1/2 < \theta < 1$. If $A \subset M$ is essentially (u, s)-saturated and the box $R = R(v, w, h)$ is contained in the cube $Q = Q(0, h)$, with $h^{3/2} \leq w \leq h$, then the concentration of A in* $\exp R$ *approximates its concentration in* $\exp Q$. *That is, as $h \to 0$,*

$$m(A : \exp R) - m(A : \exp Q) \rightrightarrows 0. \tag{7}$$

If p is density point of A then for every such box R, the concentration of $\exp^{-1} A$ *in R tends to 1 as $h \to 0$, and conversely, if for one such box R, the concentration of* $\exp^{-1} A$ *in R tends to 1 as $h \to 0$, then p is a density point of A.*

[6]Note that these quantities w, h are numbers, not vectors, and that our usage of the words "width" and "height" differs from the common meanings by a factor of 2. Also, in the previous multi-radius notation we could write $R = R(v, r)$ where $r = (w, \ldots, w, h, \ldots, h)$ has $(u + s)$ repetitions of w and c repetitions of h.

Proof. The neighborhood $\exp R$ may become arbitrarily eccentric as $h \to 0$, so the boxes R are unsatisfactory for ordinary density point analysis. They are just the sort of highly eccentric neighborhoods usually excluded in Lebesgue's Density Theorem. Although eccentric when judged at linear scale, the boxes R do have bounded *Hölder eccentricity* in the sense that $h^{3/2} \leq w \leq h$. According to Theorem 1 the local holonomy map $\omega^u_{xx'}$ satisfies

$$(1-\varepsilon)S \subset \omega^u_{xx'}(S) \subset (1+\varepsilon)S, \tag{8}$$

where $\varepsilon \to 0$ as $h \to 0$, and $S = S(y)$ is the center stable box

$$S = Q^s(y, w) \times Q^c(0, h),$$

provided that $|x|$, $|x'|$, $|y|$, $|z| \leq h$ and h is small. The center stable measure of the *rim* of S, $(1+\varepsilon)S \backslash (1-\varepsilon)S$, is much less than the measure of S. See Fig. 10.

The *unstable saturate*, $\mathrm{Sat}^u(A)$, is the union of the unstable leaves W^u that are essentially contained in A. That is, $W^u \backslash A$ has leaf measure zero. We then set

$$A^u = \exp^{-1}(\mathrm{Sat}^u(A)).$$

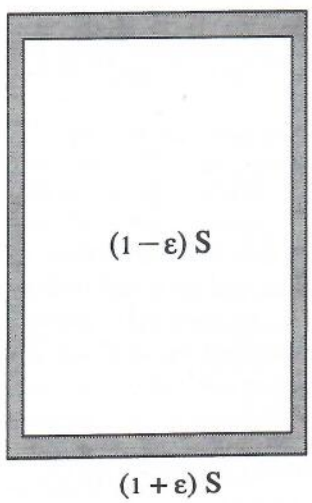

FIG. 10. The rim of S is much thinner than S.

A^u consists of whole $\overline{\mathcal{W}}$-leaves and differs from $\exp^{-1}(A)$ by a zero set. The *x-slice* of A^u is

$$A^u_x = \{(y, z) \in E^{cs}_p : (x, y, z) \in A^u\}.$$

By construction A^u is invariant under the $\overline{\mathcal{W}}^u$ holonomy, $\omega^u_{xx'}(A^u_x) = A^u_{x'}$. Let us compare the concentration of A^u_x and $A^u_{x'}$ in the center stable box S. We know that the Radon–Nikodym derivative of the $\mathcal{W}^u$ holonomy exists and is continuous. See Pugh and Shub (1972). The local holonomy from a transversal to itself is the identity map. Thus, as $h \to 0$, the Radon–Nikodym derivative of $\omega^u_{xx'}$ exists and converges uniformly to 1. According to (8),

$$m(A^u_{x'} \cap (1-\varepsilon)S) \le m(\omega^u_{xx'}(A^u_x \cap S)) \le \sup\, RN m(A^u_x \cap S),$$

where $\sup\, RN < 1 + \delta$, and $\delta \to 0$ as $h \to 0$. Hence

$$\begin{aligned} m(A^u_{x'} : S) = \frac{m(A^u_{x'} \cap S)}{m(S)} &\le \frac{m(A^u_{x'} \cap (1-\varepsilon)S) + m(S \setminus (1-\varepsilon)S)}{m(S)} \\ &\le (1+\delta) m(A^u_x : S) + \delta. \end{aligned}$$

Similarly, $(1 - \delta) m(A^u_x : S) - \delta \le m(A^u_{x'} : S)$. This is valid for all x, $x' \in Q^u(0, h)$. Therefore, for each fixed S, the function

$$a_S \colon x \mapsto m(A^u_x : S)$$

is approximately constant.

Since the linear foliation of T_pM by center stable planes parallel to E^{cs}_p is smooth we can apply Fubini's theorem to find the concentration of A^u in a box $R_0 = R(x_0, y, w, h)$. Since $m(R_0)$ is the product $m(Q^u(x_0, w)) m(Q^s(y, w)) m(Q^c(0, h)) = 2^m w^{u+s} h^c$, the concentration is

$$\begin{aligned} m(A^u : R_0) &= \frac{1}{m(R_0)} \int_{|x - x_0| \le w} m(A^u_x)\, dx \\ &= \frac{1}{m(Q^u(x_0, w))} \int_{|x - x_0| \le w} m(A^u_x : S)\, dx, \end{aligned}$$

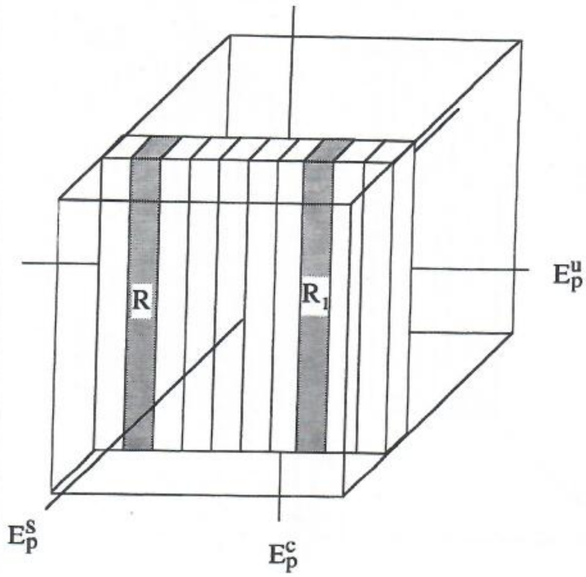

FIG. 11. An unstable slab of boxes.

which is the average over $Q^u(x_0, w)$ of the approximately constant function a_S. Thus, if R_1 is a second box with the same width and height, and whose center has the same y-coordinate $R_1 = R(x_1, y, w, h)$, then

$$m(A^u : R_0) \doteq m(A^u : R_1).$$

This shows that A^u has approximately the same concentration in all boxes of width w and height h in any given *unstable slab*, $Q^u(0, h) \times Q^s(y, w) \times Q^c(0, h)$. See Fig. 11. Since A^u differs from $\exp_p^{-1}(A)$ by a zero set, $\exp^{-1} A$ also has approximately the same concentration in all the boxes of width w and height h in a given unstable slab. The same analysis applies to the stable saturate A^s, and we see that $\exp^{-1} A$ has approximately the same concentration in any two boxes of width w and height h that lie in a common stable slab. A stable slab meets an unstable slab in a box of width w and height h. See Fig. 12. Therefore, $\exp^{-1} A$ has approximately the same concentration in all boxes $R \subset Q$ with width w and height h. It remains to show that these common box concentrations approximate the cube concentration.

Case 1. $h^{3/2} \leq w \leq h^{4/3}$. Fix a box $R^* \subset Q$ of width w and height h. Cover Q by finitely many translates of R^*, say $\{R_j\}$, that meet one another

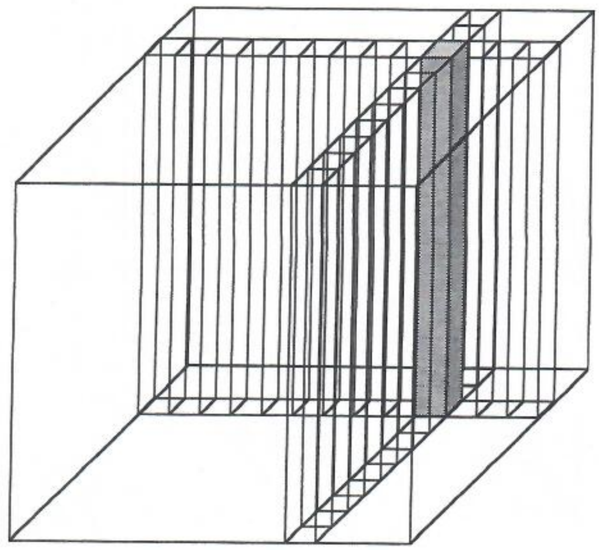

FIG. 12. The intersection of an unstable and stable slab is a box.

only along common faces. Discard any R_j that misses Q and call $U = \bigcup R_j$. Then

$$Q \subset U \subset Q(0, h+w)$$

and $m(U) = \sum m(R_j)$. See Fig. 13 and recall that $Q = Q(0,\, h)$.

Since $w \leq h^{4/3} \ll h$,

$$\frac{m(U \setminus Q)}{m(Q)} \leq \frac{m(Q(0, h+w) \setminus Q)}{m(Q)} \rightrightarrows 0$$

as $h \to 0$. The concentration of $\exp^{-1} A$ in all the boxes R_j is approximately the same as it is in the box R^*, and so $m(\exp^{-1} A : R^*) \doteq m(\exp^{-1} A : U)$. Thus the concentration of $\exp^{-1} A$ in U approximates its concentration in Q, and, applying exp, we conclude that (7) holds for R^*,

$$m(A : \exp R^*) - m(A : \exp Q) \rightrightarrows 0$$

as $h \to 0$.

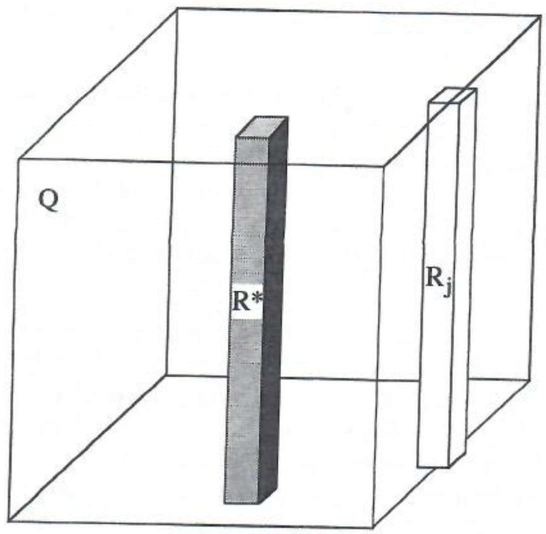

FIG. 13. The cube Q is contained in a union U of boxes R_j.

Case 2. $h^{4/3} \leq w \leq h$. Fix a box $R \subset Q$ of width w and height h. Cover R, rather than Q, with finitely many boxes, say $\{R_j\}$, that meet only along common faces and have width exactly $h^{3/2}$ and height h. These boxes are much thinner than R. Let U be the union of the R_j that meet R. Since $h^{4/3} \leq w$, $h^{3/2}/w \to 0$ as $h \to 0$ and

$$\frac{m(U \setminus R)}{m(R)} \to 0.$$

In Case 1 we showed that the concentration of $\exp^{-1} A$ in all the boxes R_j is nearly the same and is nearly the same as its concentration in Q. Hence,

$$m(\exp^{-1} A : Q) \doteq m(\exp^{-1} A : R_j) \doteq m(\exp^{-1} A : U) \doteq m(\exp^{-1} A : R),$$

which verifies (7) for R, $m(A : \exp Q) - m(A : \exp R) \rightrightarrows 0$ as $h \to 0$. Finally, p is a density point of A if and only if $m(A : \exp Q) \to 1$ as $h \to 0$. By (7) this is equivalent to one, hence all, box concentrations $m(\exp^{-1} A : R)$ tending to 1.

Q.E.D.

Addendum. As $h \to 0$, not only does the box concentration $m(\exp^{-1} A : R)$ uniformly approximate the cube concentration $m(\exp^{-1} A : Q)$, but also the slice concentrations $m(A^u : R_x^{cs})$ and $m(A^s : R_y^{cu})$ uniformly approximate the cube concentration $m(\exp^{-1} A : Q)$.

Proof. The *center stable slice* R_x^{cs} of R at x is the intersection of R with the plane $x \times E_p^{cs}$, and clearly it equals $x \times S$, where S is the center stable box in the proof of the theorem. Thus, $m(A^u : R_x^{cs}) = m(A_x^u : S)$. It was shown that $m(A_x^u : S)$ is approximately constant, and its average value gives the concentration of $\exp^{-1} A$ in R. The latter approximates $m(\exp^{-1} A : Q)$, and therefore so does $m(A^u \colon R_x^{cs})$. The situation with A^s is symmetric.

Q.E.D.

The addendum to Theorem 4.3 describes how an essentially (u, s)-saturated set behaves under the Hölder germ of local holonomy. (All the geometric objects have arbitrarily small size, and their proportions are Hölder controlled, hence the phrase "Hölder germ.") The next result, our main goal in this section, is a bridge between germ behavior and global behavior.

Let f be a partially hyperbolic diffeomorphism of M, and let $E^u \oplus E^c \oplus E^s$ be its invariant splitting. Henceforth, we work under the standing assumption that E^u, E^c, E^s are sufficiently θ-Hölder, i.e.,

$$\theta > \theta_m = \frac{\sqrt{100m^2+1}-1}{10m}. \tag{9}$$

Note that θ_m solves the equation

$$1 - m\left(\frac{1}{\theta_m} - \theta_m\right) = \frac{4}{5}.$$

If $0 \le n \le m$ and $\theta_m < \theta < 1$ then, since $1/\theta - \theta$ decreases to 0 as θ increases to 1,

$$1 - n\left(\frac{1}{\theta} - \theta\right) > 1 - m\left(\frac{1}{\theta} - \theta\right) > 1 - m\left(\frac{1}{\theta_m} - \theta_m\right) = \frac{4}{5}.$$

In particular, (9) implies that $\theta > (\sqrt{101}-1)/10 > 9/10$and for n = the fiber dimension of the unstable or stable bundle,

$$1 + n\theta - n/\theta > 4/5. \tag{10}$$

THEOREM 4.4. *The set of density points of an essentially (u, s)-saturated set is completely (u, s)-saturated. That is, if p_0 is a density point of an essentially*

(u, s)-saturated set A and if $p_1 \in W^u(p_0)$ or $p_1 \in W^s(p_0)$ then p_1 is also a density point of A.

The following covering lemma is the key. It is based on the standard observation that the number of squares in a fine grid which meet the boundary of a unit square is much less than the number needed to cover its interior. The *boundary ratio* at scale ν of a compact set $\Omega \subset \mathbb{R}^n$ is

$$\mathrm{BR}(\Omega, \nu) = \frac{N_\partial}{N},$$

where N_∂ is the number of cubes in the ν-grid in $\mathbb{R}^n$ that meet $\partial\Omega$, and N is the number of cubes in the ν-grid in $\mathbb{R}^n$ that meet Ω.

We will show that the boundary ratio can be made small for embedded cubes that are bi-Hölder or nearly bi-Hölder at appropriate scales. A homeomorphism $\omega\colon Q_0 \to \Omega \subset \mathbb{R}^n$ satisfies the *θ-Hölder cube packing property* (at scale 5/4 with Hölder constant H) provided that $Q_0 = Q(0, w)$ is an n-cube, and if $Q = Q(\nu, \mu) \subset Q_0$ with $w^{5/4} \leq \mu \leq w$ then

$$Q(\omega v, \mu^{1/\theta}/H) \subset \omega(Q) \subset Q(\omega\nu, H\mu^\theta). \tag{11}$$

Note that if ω is a θ-bi-Hölder embedding then it satisfies this packing property for *all* small μ, not merely for μ in some fixed range such as $w^{5/4} \leq \mu \leq w$, and conversely, if ω satisfies this packing property for all small μ then it is θ-bi-Hölder. Thus it is reasonable to say that if ω satisfies the θ-Hölder packing condition then it is *nearly* θ-bi-Hölder. Also note that (10) differs considerably from the box packing condition (6). The latter concerns boxes, not cubes, and it is quite stringent—the rim, which is the outer box minus the inner box, is much thinner than the inner box. In contrast, the rim in (11) can be much thicker then the inner cube. In (6) the rim is controlled at a linear scale while in (11) it is merely controlled at a Hölder scale.

LEMMA 4. *Assume that $\theta < 1$ is large enough that* (10) *holds,* $1 + n\theta - n/\theta > 4/5$. *If $\omega\colon Q_0 \to \Omega \subset \mathbb{R}^n$ satisfies the θ-Hölder cube packing condition* (11) *and w is small then the boundary ratio* $\mathrm{BR}(\Omega, \nu = w^{5/4})$ *is small. See Fig.* 14.

Proof. Precisely, given $\varepsilon > 0$ we assert there is a $\delta = \delta(\varepsilon, n, \theta) > 0$ such that if $w < \delta$ then $\mathrm{BR}(\Omega, \nu = w^{5/4}) < \varepsilon$. We assume $w < 1$, so $\nu < w$. Consider the covering of $Q_0 = Q(0, w)$ by the 2ν-grid. Its cubes have width ν, there are N_1 of them that meet Q_0, and N_2 of them that meet ∂Q_0. Note that this grid covering is done on the domain of ω not on its range Ω. Clearly,

$$
\begin{aligned}
N_1(2\nu)^n &\geq \text{volume of } Q_0 = (2w)^n \\
N_2(2\nu)^n &\leq \text{volume of the } (4\nu)\text{-neighborhood of } \partial Q_0 \\
&\leq K_1(2w)^{n-1}(8\nu),
\end{aligned}
$$

where K_1 is a geometric constant. Thus,

$$N_1 \geq \left(\frac{w}{\nu}\right)^n \qquad \text{and} \qquad N_2 \leq K_2\left(\frac{w}{\nu}\right)^{n-1}. \tag{12}$$

If Q is a 2ν-grid cube that touches Q_0 but is not contained in it, translate Q to a cube Q' so that $Q' \subset Q_0$, and Q' still contains $Q \cap Q_0$. Let $\mathcal{Q}$ be the resulting covering of Q_0. See Fig. 15.

All the cubes of $\mathcal{Q}$ are inside Q_0. Thus, if $Q \in \mathcal{Q}$ and Q touches ∂Q_0 then by (11), ωQ is contained inside a cube of width $H\nu^\theta$. A cube of width $H\nu^\theta$ meets at most $K_3(H\nu^\theta/\nu)^n$ cubes in the ν-grid, where K_3 is a geometric constant. Thus

$$N_\partial \leq K_4\left(\frac{\nu^\theta}{\nu}\right)^n N_2.$$

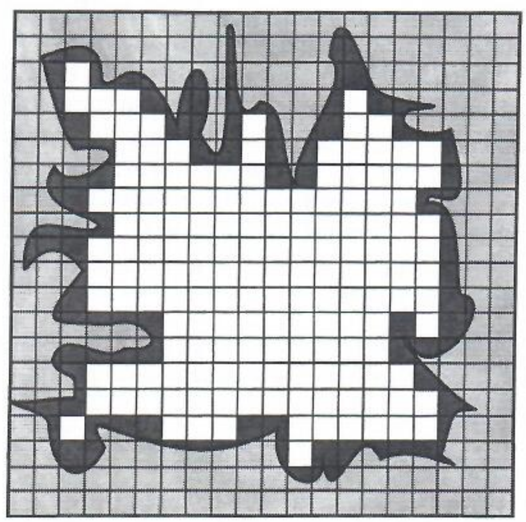

FIG. 14. Despite pathological behavior of ω, the boundary ratio is small.

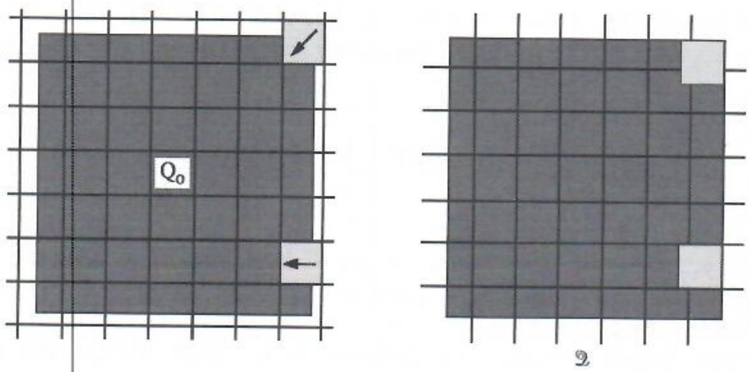

FIG. 15. The covering of Q_0 by ν-grid cubes and two cubes in the translated covering $\mathcal{Q}$.

There are $N_1 - N_2$ cubes $Q \in \mathcal{Q}$ strictly interior to Q_0, and by (11), the ω-image of each contains a cube of width $\nu^{1/\theta}/H$. These cubes of width $\nu^{1/\theta}/H$ are disjoint and hence

$$(N_1 - N_2)(2\nu^{1/\theta}/H)^n \leq \text{volume of } \Omega \leq N\nu^n.$$

Making use of (12) we get

$$\begin{aligned}\mathrm{BR}(\Omega, \nu) &= \frac{N_\partial}{N} \leq \frac{K_5 \nu^{n\theta - n}}{\nu^{n/\theta - n}} \frac{N_2}{N_1} \frac{N_1}{N_1 - N_2} \\ &\leq K_6 \nu^{n\theta - n/\theta} \nu w^{-1} \frac{N_1}{N_1 - N_2} = K_6 w^{(5/4)(1 + n\theta - n/\theta) - 1} \frac{N_1}{N_1 - N_2}.\end{aligned}$$

By (10) the exponent of w is positive. Also, the fraction $N_1/(N_1 - N_2)$ tends to 1 as $w \to 0$ since the cubes of width ν become much smaller than w = width Q_0. It follows that if $w < \delta$ then BR $< \varepsilon$. Q.E.D.

As in GPS, a key concept in the analysis of concentrations is that of a "julienne," a tall, thin, non-linear figure (it resembles a slivered vegetable in a fancy restaurant, or less elegantly a shoestring style french fry) that is fibered by local center manifolds. In this paper the definition of julienne will be slightly different. We take the figures called juliennes in GPS and intersect them with a center stable plane. Such intersections will be the juliennes of this paper. Accordingly, we define a local foliation $\mathcal{L}$ of E_p^{cs} by $\mathcal{L} = E_p^{cs} \cap \overline{\mathcal{W}}^{cu}$. Because $\overline{\mathcal{W}}^{cu}$ meets E_p^{cs} transversally, the leaves of $\mathcal{L}$ have dimension c, and the tangent field $T\mathcal{L}$ is θ-Hölder, since that of $\mathcal{W}^{cu}$ is. See Fig. 16. At the origin, $T\mathcal{L}$ is E^c_p. Given a set $S \subset E^s_p$, we define the (center stable) *julienne* over S as

$$J = J(S, h) = \bigcup_{y \in S} L(y, h),$$

where $L(y, h)$ is the intersection of the $\mathcal{L}$-leaf through y and the set $E^s_p \times Q^c(0, h)$. Thus J has height h and is fibered by leaves of $\mathcal{L}$. Usually S is a stable cube Q^s. See Fig. 17.

The next lemma compares the julienne $J = J(Q^s, h)$ and the box $R = Q^s \times Q^c(0, h)$ whose common base is the stable cube $Q^s = Q^s(\nu, w)$.

LEMMA 5 (Julienne Nesting; see p. 318 of GPS). *Given $\varepsilon > 0$ there is $\delta > 0$ such that if $Q^s \subset Q^s(0, h)$ and $h^{3/2} \le w \le h < \delta$ then*

$$(1 - \varepsilon)R \subset J \subset (1 + \varepsilon)R. \tag{13}$$

See Fig. 18.

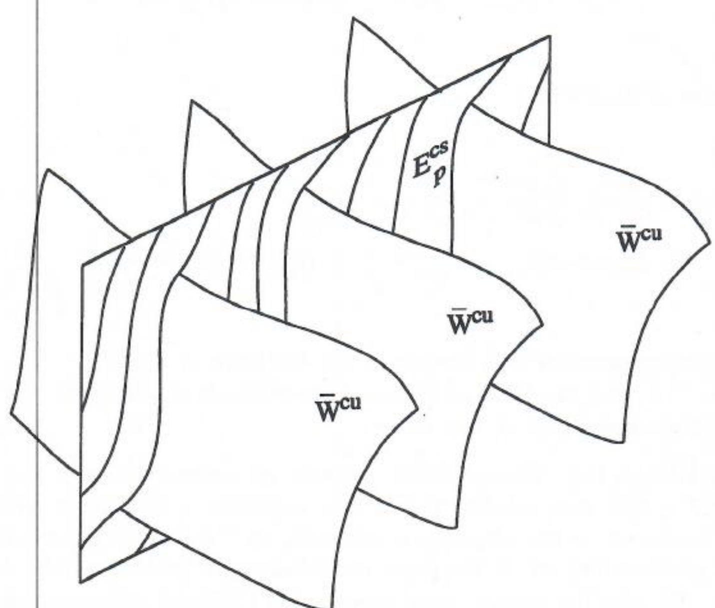

FIG. 16. The foliation $\mathcal{L} = E^{cs}_p \cap \overline{W}^{cu}$.

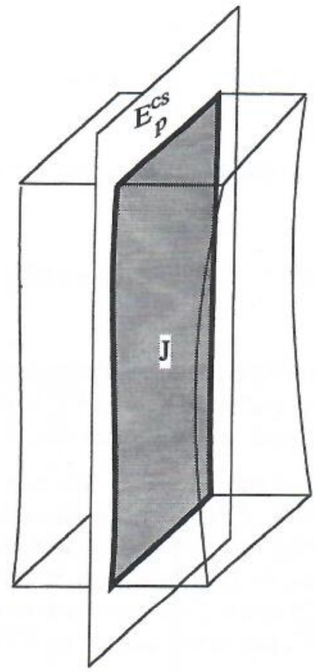

FIG. 17. The julienne J is a slice of what we called a julienne in GPS.

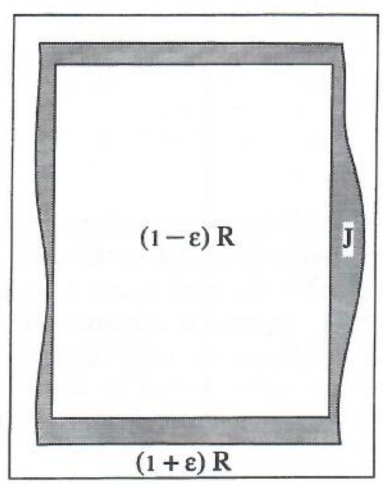

FIG. 18. J nests between slight dilations of R.

Proof. The foliation $\mathcal{L}$ has local holonomy maps

$$\lambda_z : E^s_p(h) \times 0 \to E^s_p \times z.$$

According to Theorem 4.1, the family of holonomy maps $\{\lambda_z\}$ is translation-like. If h is small then

$$(1-\varepsilon)Q^s \subset \lambda_z(Q^s) \subset (1+\varepsilon)Q^s. \tag{14}$$

The map λ_z is the effect of sliding along the $\mathcal{L}$-leaves, and J is fibered by these leaves, so (14) means that the z-slice of J, $\lambda_z(Q^s)$, is packed between the z-slice of $(1-\varepsilon)R$ and of $(1+\varepsilon)R$, which verifies (13). Q.E.D.

So far the analysis has been entirely local. We now turn to holonomy at unit distance. Suppose that $p_1 \in W^u(p_0, 1)$, the unit unstable manifold of p_0. Rescaling the Riemann structure, we may assume that all the leaves of $\mathcal{W}^u$ of size ≤ 2 lie in foliation boxes, so their holonomy is unique and purely local. The exponential images of $E^{cs}_{p_0}$ and $E^{cs}_{p_1}$ are transverse to $\mathcal{W}^u$ at p_0 and p_1. Call them τ_0 and τ_1 and lift the $\mathcal{W}^u$-holonomy map h^u to TM by commutativity of the diagram.

$$\begin{array}{ccc} E^{cs}_{p_0}(\rho) & \xrightarrow{\omega^u} & E^{cs}_{p_1} \\ \exp_{p_0} \downarrow & & \downarrow \exp_{p_1} \\ \tau_0 & \xrightarrow{h^u} & \tau_1 \end{array}$$

According to PSW, ω^u is a θ-bi-Hölder homeomorphism onto a neighborhood of the origin in $E^{cs}_{p_1}$. Near their origins, both center stable planes are foliated by $\mathcal{L}$, their intersection with $\overline{\mathcal{W}}^{cu}$. We write L_0 and L_1 to distinguish $\mathcal{L}$-leaves in $E^{cs}_{p_0}$ from $\mathcal{L}$-leaves in $E^{cs}_{p_1}$. Dynamical coherence implies that the strong unstable foliation and center foliation are subordinate to the center unstable foliation. Each center unstable leaf is fibered by unstable leaves and center leaves. Thus, ω^u sends the $\mathcal{L}$-foliation of $E^{cs}_{p_0}$ to the $\mathcal{L}$-foliation of $E^{cs}_{p_1}$,

$$\omega^u : L_0(\nu) \to L_1(\omega^u \nu).$$

In particular, this lets us express the center unstable holonomy $\omega^{cu}\colon E^s_{p_0}(\rho) \to E^s_{p_1}$ as a composition, $\omega^{cu} = \lambda \circ \omega^u$ where λ is projection along the foliation $\mathcal{L}$ in $E^{cs}_{p_1}$. Thus, ω^u sends $L(y)$ to $L(y_1)$ where $y_1 = \omega^{cu}(y)$. Note that y_1 need not belong to $W^u(y)$, so $\omega^u(y)$ need not equal y_1. See Fig. 19.

LEMMA 6. *$\omega^u\colon L_0(y) \to L_1(y_1)$ is a C^1 diffeomorphism to its image and it has locally uniformly bounded C^1 size.*

Proof. This is Theorem B in PSW. Q.E.D.

We fix a constant $\Lambda \geq 1$ that dominates the Lipschitz behavior of all these unit holonomy maps ω^u, restricted to the $\mathcal{L}$-leaves. Under ω^u, points of a common $\mathcal{L}$-leaf can neither spread apart by a factor more than Λ nor contract together by a factor less than $1/\Lambda$.

Proof of Theorem 4.4. Let A be an essentially (u, s)-saturated set and let $\mathrm{DP}(A)$ be its set of density points. We must show that if $p_0 \in \mathrm{DP}(A)$ and $p_1 \in W^u(p_0)$ or $p_1 \in W^s(p_0)$ then $p_1 \in \mathrm{DP}(A)$. Since all hypotheses and

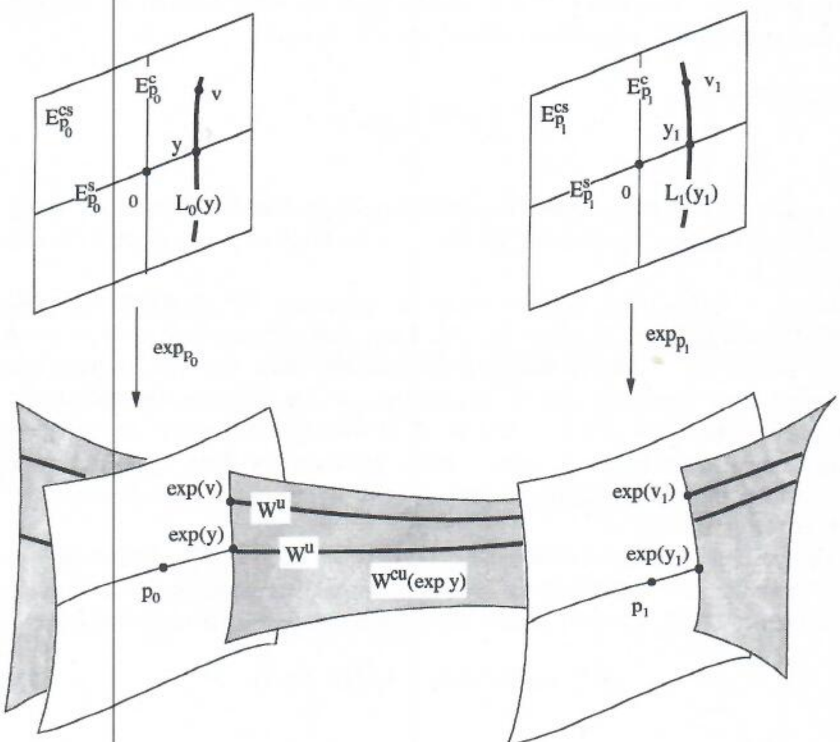

FIG. 19. The unit unstable and center-unstable holonomy, $\omega^u(\nu) = \nu_1$, $\omega^{cu}(y) = y_1$.

lemmas are symmetric in the stable and unstable modes, we may assume that $p_1 \in W^u(p_0)$. It is also no loss of generality to assume that p_1 lies in the unit unstable manifold of p_0. For we can form a finite chain of points from p_0 to p_1, each in the unit unstable manifold of its predecessor, and argue inductively that each point in the chain lies in DP(A).

As discussed in Lemma 6, the holonomy along $\mathcal{W}^u$ is expressed as a θ-bi-Hölder embedding ω^u: $E^{cs}_{p_0}(\rho) \to E^{cs}_{p_1}$. By Theorem 4.3 and its addendum, to show that p_1 is a density point of A it suffices to show that A^u has high concentration in some small, well-shaped center stable box in $E^{cs}_{p_1}$. Recall that A^u is the unstable saturate of A lifted to TM. It is invariant under ω^u.

Specifically, we set $w = h^{6/5}$, $\nu = w^{5/4}$, and $h_1 = h/2\Lambda$. We claim that $m(A^u\colon S_1) \to 1$ as $h_1 \to 0$, where S_1 is the center stable box

$$S_1 = Q^s_1(0, \nu) \times Q^c_1(0, h_1) = R^{cs}_1(0, \nu, h_1) \subset Q^{cs}_1(0, h_1) \subset E^{cs}_{p_1}.$$

We continue to write the subscript 0 or 1 to distinguish objects in $T_{p_0}M$ from those in $T_{p_1}M$. Note that $h^{3/2} = \nu$ and h_1 is on the same order as h. The box S_1 has good Hölder proportions since, when h_1 is small,

$$h_1^{3/2} \leq h^{3/2} = \nu \leq h_1.$$

According to Theorem 4.3 and its addendum, high concentration of A^u in S_1 then implies high concentration of $\exp^{-1} A$ in $Q_1(0, h_1)$, so p_1 is a density point of A.

Let $J_0 = J_0(Q_0, h) = J_0(0, w, h)$ be the julienne in $E^{cs}_{p_0}$, based on the stable cube $Q_0 = Q^s_0(0, w)$ of width w. We know that A^u has high concentration in J_0 as $h \to 0$. Since the Radon–Nikodym derivative of ω^u is bounded and bounded away from zero, and A^u is invariant, A^u has high concentration in the *image julienne* $J_1 = \omega^u(J_0)$. The proof of Theorem 4.4 would be complete (and trivial) if the image julienne were a julienne, but there is no reason to expect this. Instead, using the 2ν-grid, we will *julienne the image julienne*: we will sliver J_1 thinner.

The base of J_1 is the embedded s-cube $\Omega = \omega^{cu}(Q_0) \subset E^s_{p_1}$. Eventually we will show that ω^{cu} satisfies the θ-Hölder cube packing condition, and so small stable cubes in $E^s_{p_1}$ cover Ω nicely. First we check that J_1 is reasonably small,

$$J_1 \subset J_1(\Omega, 2\Lambda h) \subset Q^{cs}_1(0, 2\Lambda h). \tag{15}$$

If $v \in J_0$ then $v \in L_0(y, h)$ for some $y \in Q_0$. Since ω^u is θ-Hölder,

$$|\omega^u(y)| \leq K|y|^\theta \leq Kw^\theta.$$

Now $Kw^\theta \ll h$, since (9) implies that $6\theta/5 > 54/50 > 1$, and thus

$$w^\theta = h^{6\theta/5} \ll h.$$

This means we can apply Lemma 2 to $\omega^u(y)$ and conclude that if $\omega^u(y) \in L_1(y_1)$ with $y_1 = \omega^{cu}(y)$ then for small h,

$$|y_1| \leq Kw^\theta + K'(Kw^\theta)(Kw^\theta)^\theta = Kw^\theta + K''w^{\theta+\theta^2} < 2Kw^\theta. \qquad (16)$$

See Fig. 20. Thus (16) implies

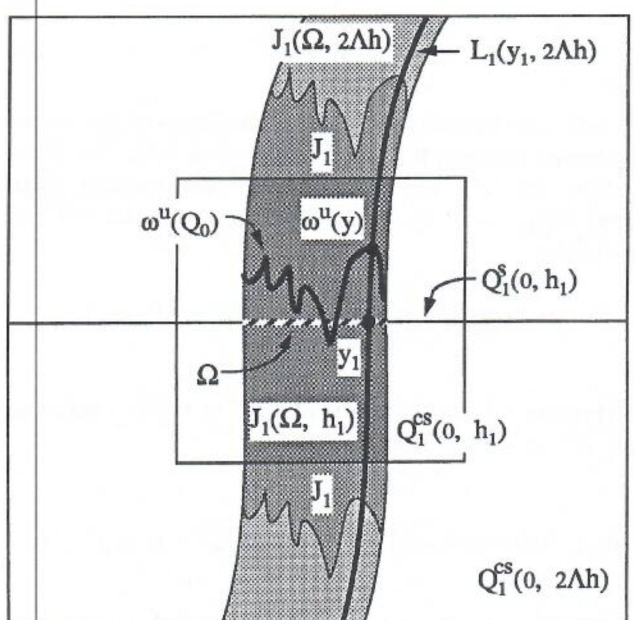

FIG. 20. Julienne inclusions.

$$\Omega \subset Q_1^s(0, 2Kw^\theta) \subset Q_1^s(0, h_1) \subset Q_1^s(0, 2\Lambda h), \tag{17}$$

and therefore the second inclusion of (15) follows from Lemma 2. Since ω^u dilates the $\mathcal{L}$-leaves by at most the factor Λ and $|\omega^u(y) - y_1| \le 2Kw^\theta \ll h$ for small h, we see that

$$\omega^u(L_0(y, h)) \subset L_1(\omega^u(y), \Lambda h) \subset L_1(y_1, 2\Lambda h). \tag{18}$$

Because J_1 is the union of the leaves $L_0(y,\ h)$ for $y \in Q_0$, (17) completes the proof of (15). The same reasoning implies that

$$L_1(y_1, h_1) \subset \omega^u(L_0(y, h)), \tag{19}$$

where $h_1 = h/2\Lambda$. For ω^u shrinks $L_0(y)$ by a factor no less than $1/\Lambda$, and the centerpoint $\omega^u(y)$ of $\omega^u(L_0(y,\ h))$ satisfies $|\omega^u(y) - y_1| \le 2Kw^\theta \ll h$. Letting y vary in Q_0, (19) becomes

$$J_1(\Omega, h_1) \subset J_1. \tag{20}$$

Formulas (15) and (20) express the "relative smallness of the ragged top and bottom" of the image julienne, the crucial estimate in GPS. See Fig. 21.

Next, we show that ω^{cu} has the θ-Hölder cube packing property (11). Consider a cube $Q_0^s(y,\ \mu) \subset Q_0 = Q_0^s(0,\ w)$ in $E^s_{p_0}$, with $w^{5/4} \le \mu \le w$. Since ω^u is θ-Hölder,

$$\omega^u(Q_0^s(y, \mu)) \subset Q_1^{cs}(\omega^u(y, 0), K\mu^\theta).$$

According to Theorem 4.1, the projection along $\mathcal{L}$ in $E^{cs}_{p_1}$ is translation-like, and so for w small, $\omega^{cu} = \lambda \circ \omega^u$ implies

$$\omega^{cu}(Q_0^s(y, \mu)) \subset \lambda(Q_1^{cs}(\omega^u(y, 0), K\mu^\theta)) \subset Q_1^s(\omega^{cu}(y), K\mu^\theta + (Kw^\theta)w^\theta).$$

For the distance we slide along $\mathcal{L}$ is at most Kw^θ, and $T\mathcal{L}$ is θ-Hölder. As we observed above, (9) implies that $2\theta > 9/5 > 5/4$, and so when w is small,

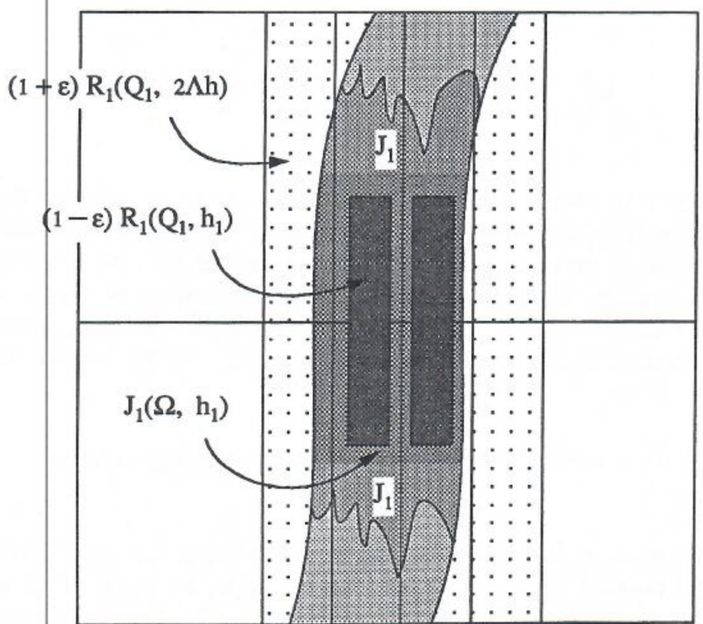

FIG. 21. The image julienne J_1, its ragged top, and the image of the stable cube Q_0.

$w^{2\theta} \ll w^{5/4} \leq \mu$, which implies that the last set is contained in $Q_1^s(\omega^{cu}(y), 2K\mu^\theta)$. That is,

$$\omega^{cu}(Q_0^s(y,\mu)) \subset Q_1^s(\omega^{cu}(y), 2K\mu^\theta),$$

which is half of the θ-Hölder cube packing property. The other half is proved the same way, $(\omega^u)^{-1}$ is θ-Hölder and projection along $\mathcal{L}$ in $E^{cs}_{p_0}$ is also translation-like. The Hölder constant in (11) is $H = 2K$.

Now we can apply Lemma 4 to ω^{cu}. As $h_1 \to 0$, most of the cubes Q_1 in the 2ν-grid on $E^s_{p_1}$ that meet $\Omega = \omega^{cu}(Q_0)$ are interior to it. These cubes Q_1 have width $\nu = w^{5/4} \ll w$. Set

$$\mathcal{Q} = \{Q_1 : Q_1 \text{ meets } \Omega\},$$
$$\mathcal{Q}^* = \{Q_1 : Q_1 \text{ is interior to } \Omega\}.$$

There are N cubes in $\mathcal{Q}$ and $N^* = N - N_\partial$ cubes in $\mathcal{Q}^*$ As $h_1 \to 0$, Lemma 6 states that $N^*/N \to 1$. From (15) and (20), we infer that for small h,

$$\bigcup_{Q_1 \in \mathcal{Q}^*} J_1(Q_1, h_1) \subset J_1 \subset \bigcup_{Q_1 \in \mathcal{Q}} J_1(Q_1, 2\Lambda h). \tag{21}$$

According to Lemma 5, a julienne nests between two boxes, a large one and a small one, the large one being just a slight dilation of the small one. Under the map ω^u, a julienne becomes geometrically much messier, but it has a remnant of the nesting property: an image julienne nests between a union of N large boxes and a union of $N^* = N - N_\partial$ small boxes, the large ones being uniformly bounded dilations of the small ones. See Fig. 22. For by Lemma 5, (21) becomes

$$\bigcup_{Q_1 \in \mathcal{Q}^*} (1-\varepsilon)R_1(Q_1,\, h_1) \subset J_1 \subset \bigcup_{Q_1 \in \mathcal{Q}} (1+\varepsilon)R_1(Q_1,\, 2\Lambda h), \tag{22}$$

where the notation $R_1(Q_1,\, h_1)$ stands for the center stable box $Q_1 \times Q^c(0,\, h_1)$. Thus, J_1 is covered by N big boxes $(1+\varepsilon)R_1(Q_1,\, 2\Lambda h)$, $Q_1 \in \mathcal{Q}$, and it contains N^* small boxes $(1-\varepsilon)R_1(Q_1,\, h_1)$, $Q_1 \in \mathcal{Q}^*$. The small boxes are disjoint. Each big box has volume $(2(1+\varepsilon)\nu)^s(2(1+\varepsilon)2\Lambda h)^c$ and each small one has volume $(2(1-\varepsilon)\nu)^s(2(1-\varepsilon)h/2\Lambda)^c$. The measure ratio of the small

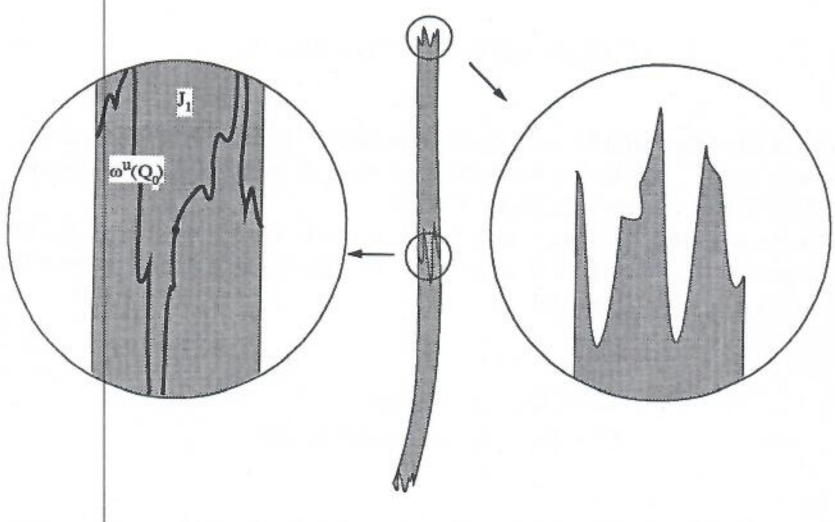

FIG. 22. The image julienne nests between small boxes and big ones.

boxes to the big ones is bounded away from 0 as $h_1 \to 0$. (Its lim inf is $(1/2\Lambda)^{2c}$.) Thus, a major portion of J_1 is filled with the union U of these small boxes $(1-\varepsilon)R_1(Q_1, h_1)$.

The set A^u is highly concentrated in J_0, and as remarked above, it is also highly concentrated in $J_1 = \omega^u(J_0)$. Since U occupies a major portion of J_1, this forces A^u to also be highly concentrated in U,

$$m(A^u\colon \omega^u(J_0)) \doteq 1 \Rightarrow m(A^u\colon U) \doteq 1.$$

It follows that A^u is highly concentrated in at least one of the small boxes $(1-\varepsilon)R_1(Q_1, h_1)$. Since $\varepsilon \to 0$ as $h_1 \to 0$, A^u becomes highly concentrated in the undilated box $R_1 = R_1(Q_1, h_1)$. This box R_1 has the same proportion as the box S_1—it has width ν and height h_1—and these proportions were already shown to be good. By (17), R_1 is contained in the center stable cube $Q_1^{cs}(0, h_1)$. According to the addendum to Theorem 4.3, high concentration of A^u in a single center stable box $R_1 \subset Q_1^{cs}(0, h_1)$ of the right proportion implies high concentration in all other well-shaped center stable boxes (such as S_1), and this implies that $\exp^{-1} A$ has high concentration in the corresponding m-dimensional cube. Therefore p_1 is a density point of A. Q.E.D.

Remark. The actual shapes of the julienne J_0 and its image $J_1 = \omega^u(J_0)$ under $\mathcal{W}^u$-holonomy can be quite messy. The boundary of J_0 consists of two parts: its vertical boundary is the union of the leaves $L_0(y, h)$ in $\mathcal{L}$ that pass through points $y \in \partial Q_0$ while its horizontal boundary is the union of the leaf boundaries $\partial L_0(y, h)$ as y varies in Q_0. Thus J_0 has a square base, a flat horizontal boundary, and a gnarly, wrinkled, striated vertical boundary. Its image is worse, J_1 has a base $\omega^{cu}(Q_0)$ that is homeomorphic (but probably not bi-Hölder homeomorphic) to a cube, an equally awful vertical boundary foliated by the leaves of $\mathcal{L}$ through $\partial\Omega$, and a ragged (no longer flat) horizontal boundary. See Figs. 23 and 24.

In GPS the unstable, center, and stable dimensions were all equal to 1, so the center stable plane had dimension 2. The ragged part of ∂J_1, its horizontal or "top and bottom" part, was treated in the same way we do here: by construction, it is much smaller than the height of J_1. In GPS the base of J_1 was a segment, for the only homeomorph of a 1-cube in a line is a 1-cube. Thus, the base of J_1 presented no pathology. Similarly the vertical boundary of J_1 consisted of two $\mathcal{L}$-leaves, and clearly, individual leaves of $\mathcal{L}$ give no difficulties. The novel part of the proof presented above is Lemma 4—despite the messy vertical boundary of J_1, a major portion of its interior consists of linear boxes.

COROLLARY 1 = THEOREM A. *The diffeomorphism $f\colon M \to M$ of the compact manifold M is ergodic if it satisfies the following hypotheses:*

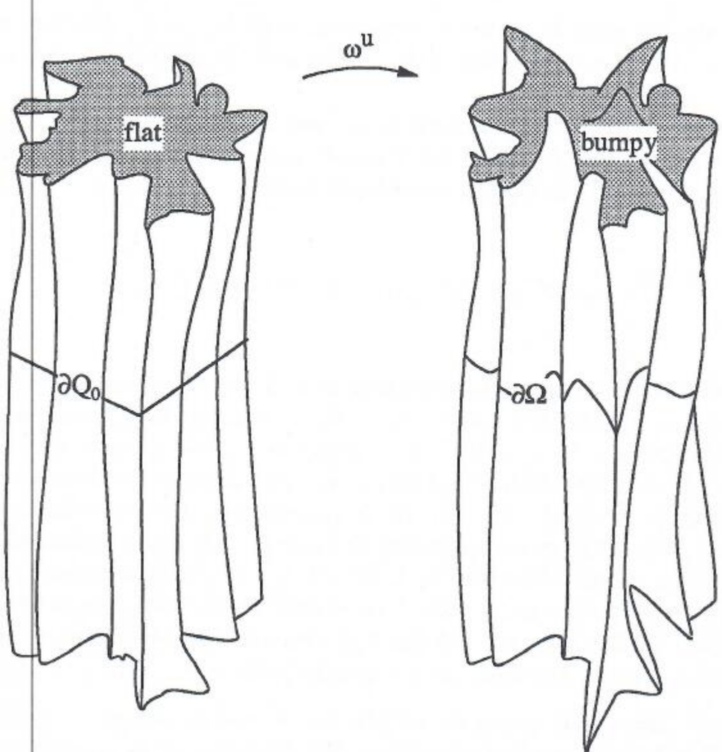

FIG. 23. A julienne and its image when $s = 2$ and $c = 1$.

(i) *f is C^2 and preserves volume.*

(ii) *f is partially hyperbolic with splitting $E^u \oplus E^c \oplus E^s$.*

(iii) *E^u, E^c, E^s are sufficiently θ-Hölder.*

(iv) *E^u, E^c, E^s integrate to invariant, dynamically coherent foliations $\mathcal{W}^u$, $\mathcal{W}^c$, $\mathcal{W}^s$.*

(v) *(E^u, E^s) has the essential accessibility property.*

Proof. By Theorem 4.2, if f is not ergodic then there exists a measurable set $A \subset M$ with intermediate measure that is essentially (u, s)-saturated. Essential accessibility of (E^u, E^s) means that almost every pair of points in $M \times M$ is joined by a finite (E^u, E^s)-path. The set DP(A) $\times$ DP($M \backslash A$) has positive measure in $M \times M$. (Except for a zero set it is $A \times (M \backslash A)$.) Hence there is a (E^u, E^s)-path from some point of DP(A) to some point of DP($M \backslash A$), in obvious contradiction to Theorem 4.4. Q.E.D.

COROLLARY 2 = THEOREM B. *Assume that the diffeomorphism $f: M \to M$ of the compact manifold M satisfies*

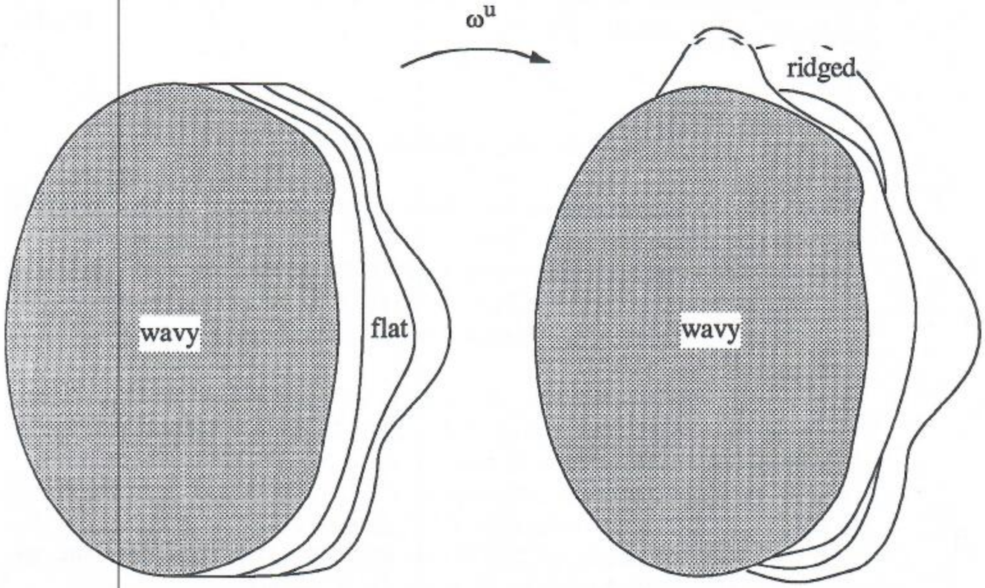

FIG. 24. A julienne and its image when $s = 1$ and $c = 2$.

(i) *f is C^2 and preserves volume.*
(ii) *f is partially hyperbolic with splitting $E^u \oplus E^c \oplus E^s$.*
(iii′) *E^u, E^c, E^s are C^1 and the spectral bunching conditions of Section 2 are valid.*
(iv) *E^u, E^c, E^s integrate to invariant, dynamically coherent foliations $\mathcal{W}^u$, $\mathcal{W}^c$, $\mathcal{W}^s$.*
(v′) *(E^u, E^s) has the (complete) accessibility property.*

Then f is stably ergodic.

Proof. Let f' be a volume preserving diffeomorphism that C^2 approximates f. We claim that f' is ergodic. It suffices to check conditions (i)–(v) in the preceding corollary for f'. (i) is true by assumption. (ii) is true by HPS. (iii) is true by the corollary to Theorem 2.1. (iv) is true by Theorem 2.3. (v) is true by Theorem 3.4. Q.E.D.

COROLLARY 3 = THEOREM C. *The time one map of the geodesic flow on a manifold of constant negative curvature is stably ergodic.*

Proof. It suffices to check conditions (i)–(v′) in the preceding corollary. (i) is true because every geodesic flow is smooth and preserves volume. (ii) is true by the Lobachevsky–Hadamard Theorem. (iii′) is true because M has *constant* negative curvature. (iv) is true because partially hyperbolic *flows* always have dynamically coherent foliations. (v′) is true because $E^u \oplus E^s$ is a contact bundle. See Katok and Kononenko (1996). Q.E.D.

5. ALGEBRAIC STABLE ERGODICITY

In this section we recall some basic facts about translations on homogeneous spaces of Lie groups and prove Theorem D.

Let G be a Lie group, with identity e and right invariant Riemannian metric $<,>$ defined on the tangent bundle to G. Let $\Gamma \subset G$ be a discrete subgroup so G/Γ is a differentiable manifold of the same dimension as G which inherits a Riemannian metric from G.

Given $h \in G$ we denote by L_h, R_h, and C_h the maps defined by $L_h(g) = hg$, $R_h(g) = gh$ and $C_h(g) = hgh^{-1}$ for $g \in G$, i.e., left translation, right translation, and conjugation by h, respectively.

Let $E \subset T_eG$ be a vector subspace of T_eG. Then define $\hat{E} \subset TG$ to be the right invariant vector subbundle of TG defined by $\hat{E}_g = DR_g(e)(E)$. $\hat{E}_g$ defines a vector subbundle of $T(G/\Gamma)$ as well since it is right invariant; we continue to denote this bundle by $\hat{E}$, using $\hat{E}_\Gamma$ if confusion is possible.

PROPOSITION 5.1. *$E \subset T_eG$ is an invariant subspace for $DC_h(e)\colon T_eG \to T_eG$ iff $\hat{E} \subset T(G/\Gamma)$ is an invariant subbundle for $DL_h\colon T(G/\Gamma) \to T(G/\Gamma)$. Moreover, contracting subspaces of E correspond to contracting subbundles $\hat{E}$, expanding correspond to expanding, and in fact any direct sum decompositions filtered by exponential rates of contraction or expansions correspond.*

Proof.

$$\begin{aligned} DL_h(g)\hat{E}_g &= DL_h(g)DR_g(e)E \\ &= DR_g(h)DL_h(e)E \end{aligned}$$

since $R_gL_h = L_hR_g$.

$$\hat{E}_{hg} = DR_{hg}(e)E = DR_g(h)DR_h(e)E.$$

Thus $DL_h(g)\hat{E}_g = \hat{E}_{hg}$ iff

$$\begin{aligned} DL_h(e)E &= DR_h(e)E, \\ DR_h^{-1}(h)DL_h(e)E &= E, \end{aligned}$$

or

$$DC_h(e)E = E.$$

That contracting subspaces correspond to contracting subbundles, etc., follows from the same computation using the right invariance of the Riemannian structure

$$\begin{aligned} DR_{hg}^{-1}(hg)DL_h(g)DR_g(e) &= DR_h^{-1}(h)DR_g^{-1}(hg)DL_h(g)DR_g(e) \\ &= DR_h^{-1}(h)DR_g^{-1}(hg)DR_g(h)DL_h(e) \\ &= DR_h^{-1}(h)DL_h(e) = DC_h(e). \qquad \text{Q.E.D.} \end{aligned}$$

Henceforth we denote $DC_h(e)$ by $\mathrm{Ad}(h)$. If we identify T_eG with the Lie Algebra $\mathfrak{g}$ of right invariant vector fields on G, then it is a standard fact that $\mathrm{Ad}(h)$ is an automorphism of the algebra $\mathfrak{g}$. Since $\mathrm{Ad}(h)$ is linear the main content of this assertion is the following standard proposition.

PROPOSITION 5.2. *For $X, Y \in \mathfrak{g}$,*

$$[\mathrm{Ad}(h)X, \mathrm{Ad}(h)Y] = \mathrm{Ad}(h)[X, Y].$$

Here [,] is the Lie bracket of the vector fields.

Recall that a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is an ideal if $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$. Let e_h^s and e_h^u be the contracting and expanding linear subspaces of $\mathrm{Ad}(h)$ on $\mathfrak{g}$ and e_h^c the central subspace.

PROPOSITION 5.3. *Let $\mathfrak{l}_h \subset \mathfrak{g}$ be the Lie subalgebra of $\mathfrak{g}$ generated by e_h^s and e_h^u. Then $\mathfrak{l}_h$ is an ideal in $\mathfrak{g}$.*

Proof. First note that if

$$\mathrm{Ad}(h)(X) = \lambda X \qquad \text{with } |\lambda| = 1$$

and

$$\mathrm{Ad}(h)(Y) = \mu Y \qquad \text{with } |\mu| < 1$$

then

$$\mathrm{Ad}(h)[X, Y] = \mu\lambda[X, Y] \qquad \text{and} \qquad |\mu\lambda| < 1.$$

It follows that $[e_h^c, e_h^s] \subset e_h^s$ and $[e_h^c, e_h^u] \subset e_h^u$. Now from the Jacobi identity and induction it follows that $[e^c_h, \mathfrak{l}_h] \subset \mathfrak{l}_h$. Since $[e^u_h, \mathfrak{l}_h] \subset \mathfrak{l}_h$ and $[e^s_h, \mathfrak{l}_h] \subset \mathfrak{l}_h$ by definition $\mathfrak{l}_h$ is an ideal. Q.E.D.

COROLLARY 5.1. *If G is a simple group and $e^s_h \oplus e^u_h \neq 0$ then the Lie Algebra generated by e^s_h and e^u_h is all of $\mathfrak{g}$.*

Proof. Since G is simple $\mathfrak{g}$ has no non-trivial ideals. Q.E.D.

PROPOSITION 5.4. *If G is simple and for $h \in G$, e^s_h and $e^u_h \neq 0$, then L_h is a partially hyperbolic dynamically coherent diffeomorphism.*

Proof. It follows from Proposition 5.2 as in Proposition 5.3 that e^s_h, e^c_h, e^u_h, $e^s \oplus e^c$, and $e^u \oplus e^c$ are all subalgebras of $\mathfrak{g}$ and hence tangent to smooth foliations of G. Proposition 5.1 gives the rate conditions. Q.E.D.

We say $g \in G$ has finite order if there is a non-zero integer n such that $g^n = e$.

PROPOSITION 5.5. *If G is compact, then the elements of finite order are dense in G.*

Proof. Let $g \in G$. Then $\overline{\{g^n\}_{n\in\mathbb{Z}}}$ is a compact abelian group, hence a torus product a finite abelian group. As the elements of finite order in the torus are dense we are done. Q.E.D.

COROLLARY 5.2. *If G is a simple group then $e^s_h \oplus e^u_h \neq 0$ iff L_h is partially hyperbolic, dynamically coherent and E^s, E^u have the accessibility property.*

Proof. By Proposition 5.4, $e^s \oplus e^u \neq 0$ iff L_h is partially hyperbolic and dynamically coherent, moreover E^s, E^u are spanned by the right invariant vector fields. Now Chow's Theorem 3.3 and Corollary 5.1 finish the proof. Q.E.D.

We proceed to the proof of Theorem D. First we need two propositions.

PROPOSITION 5.6. *Let $A \in \mathrm{SL}(n, R)$ have eigenvalues $\lambda_1, \dots, \lambda_n$ written with multiplicity. Then the eigenvalues of $\mathrm{Ad}(A)$ are $\lambda_i \lambda_j^{-1}$ for $j \neq i$ and $(n-1)$ ones.*

Proof. If A is diagonal then E_{ij} the matrix with 1 in the ijth place and zero elsewhere is an eigenvector with eigenvalue $\lambda_i \lambda_j^{-1}$. $E_{11} - E_{kk}$ for $k > 1$ is an eigenvector with eigenvalue 1. If A is semi-simple, it is conjugate to a diagonal matrix D and $\mathrm{Ad}(A)$ is conjugate to $\mathrm{Ad}(D)$. So the proposition holds for all semi-simple matrices A. As the semi-simples are open and dense in $\mathrm{SL}(n, R)$ it holds for all of $\mathrm{SL}(n, R)$ by continuity. Q.E.D.

PROPOSITION 5.7. *Let $M \in \mathrm{SL}(n, \mathbb{R})$ have all its eigenvalues on the unit circle. Then there are orthogonal matrices $O_j \in \mathrm{SL}(n, \mathbb{R})$ and $N_j \in \mathrm{SL}(n, \mathbb{R})$ such that $N_j O_j N_j^{-1}$ converges to M as $j \to \infty$.*

Before proving Proposition 5.7 we prove Theorem D.

Proof of Theorem D. (a) is equivalent to (b) and (c) by Propositions 5.1 and 5.6 and Corollaries 5.1 and 5.2 since $\mathrm{SL}(n, \mathbb{R})$ is simple. (b) implies that L_A is ergodic by Theorem 1. As matrices with eigenvalues off the unit circle are open in $\mathrm{SL}(n, \mathbb{R})$, (a) and (b) imply (d). To prove that (d) implies (a), we proceed by contradiction. Suppose that all the eigenvalues of A are of unit modulus and that U is a neighborhood of A. Then there is a conjugate of an orthogonal matrix in U by and hence a finite order matrix B in U. But L_B is not ergodic for finite order B and hence L_A is not stably ergodic. Q.E.D.

Remark. The hypothesis that Γ is uniform discrete may be weakened to Γ discrete and $\mathrm{SL}(n, R)/\Gamma$ of finite volume. Then (a) through (d) remain equivalent. By a theorem of Moore (1966), L_A is ergodic iff $\overline{\{A^m\}_{m\in\mathbb{Z}}}$ is not compact. The rest of the proof is the same. We don't know if stable ergodicity among C^2 volume preserving diffeomorphisms remains true.

We now turn to the proof of Proposition 5.7. First we prove some lemmas. Let $R = R_{c,s} = \left(\begin{smallmatrix} c & s \\ -s & c\end{smallmatrix}\right)$. Let $R_{k,c,s} = R_k$ be the $m \times m$ matrix

$$\begin{pmatrix} I_k & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & I_{m-k-2} \end{pmatrix},$$

where $m - 2 \geq k \geq 0$. Let $\mathcal{R}_{c,s} = \mathcal{R} = R_{m-2}R_{m-3}\cdots R_1R_0$.

LEMMA. *$\mathcal{R}$ has the following form. Each entry is a monomial $\pm s^j c^l$ where $j + l \leq m - 1$. Above the first super diagonal all entries are* 0. *On the first super diagonal all entries equal s. The diagonal is c then $(m-2)$ c^2's then c. The jth subdiagonal is divisible by s^j.*

Proof. The proof is a simple induction on m. Q.E.D.

Remark. Henceforth we assume that $c^2 + s^2 = 1$ so that R_k and hence $\mathcal{R}$ are in the special orthogonal group.

LEMMA. *Let $s_j \to 0$, $c_j \to 1$ as $j \to \infty$ and $a \neq 0$. Let*

$$D_j = \begin{pmatrix} as_j & & & \\ & a^2s_j^2 & O & \\ O & & \ddots & \\ & & & a^m s_j^m \end{pmatrix},$$

i.e., D_j is the $m \times m$ diagonal matrix with ith entry $a^i s_j^i$. Then

$$D_j \mathcal{R}_{c_j,s_j} D_j^{-1} \to \begin{pmatrix} 1 & a^{-1} & & & \\ & 1 & a^{-1} & & O \\ & & 1 & \ddots & \\ O & & & \ddots & a^{-1} \\ & & & & 1 \end{pmatrix}.$$

Proof. The (i, k)th entry of $\mathcal{R}_{c_j,s_j}$ is multiplied by $a^{k-i}s^{k-i}$, so the first superdiagonal of $D_j R_{c_j,s_j} D_j^{-1}$ is a^{-1} while the kth subdiagonal is multiplied by $(as_j)^k$. Thus as $s_j \to 0$ each subdiagonal entry tends to zero.

Let $\hat{\mathcal{R}}_{c,s} = \hat{R}$ be obtained from $\mathcal{R}_{c,s}$ by replacing the (i, k)th entry $\mathcal{R}_{i,k}$ by

$$\begin{pmatrix} \mathcal{R}_{i,k} & O \\ O & \mathcal{R}_{i,k} \end{pmatrix}$$

so $\hat{\mathcal{R}}$ is a $2m \times 2m$ matrix. Otherwise said, $\hat{\mathcal{R}}\mathcal{R} \otimes I$ where $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ or $\hat{\mathcal{R}}$ is the matrix obtained by considering $\mathcal{R}$ as a complex $m \times m$ matrix and then expressing $\mathcal{R}$ as a real $2m \times 2m$ matrix. This last interpretation of $\hat{\mathcal{R}}$ makes the following lemma clear using complex arithmetic instead of real arithmetic in the lemmas above.

LEMMA. *Let $s_j \to 0$, $c_j \to 1$ and $A = \left(\begin{smallmatrix} c_0 & s_0 \\ -s_0 & c_0 \end{smallmatrix}\right)$, $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Let*

$$\hat{D}_j = \begin{pmatrix} S_j A & & & \\ & S_j^2 A^2 & & O \\ O & & \ddots & \\ & & & S_j^m A^m \end{pmatrix}.$$

Then

$$\hat{D}_j \hat{\mathcal{R}}_{c_j,s_j} \hat{D}_j^{-1} \to \begin{pmatrix} I & A^{-1} & & & \\ & I & A^{-1} & & O \\ & & & \ddots & \\ O & & & \ddots & A^{-1} \\ & & & & I \end{pmatrix}.$$

Note. $\hat{\mathcal{R}}_{c_j,s_j}$ is an orthogonal matrix.

LEMMA. *Let*

$$M = \begin{pmatrix} A & I & & O \\ & \ddots & \ddots & \\ O & & A & I \end{pmatrix},$$

where A is either ± 1 or a 2×2 matrix which is a rotation and I the 1×1 or 2×2 identity, respectively. Then there exist orthogonal matrices O_j and invertible matrices N_j of determinant one such that $N_j O_j N_j^{-1} \to M$.

Proof. $M = SU = US$, where

$$S = \begin{pmatrix} A & & \\ & \ddots & \\ & & A \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} I & A^{-1} & & O \\ & & \ddots & \\ & & \ddots & A^{-1} \\ O & & & I \end{pmatrix}.$$

Note that D_j and $\hat{D}_j$ of the lemmas commute with A. Let $D_j \mathcal{R}_j D_j^{-1} \to U$ or $\hat{D}_j \hat{\mathcal{R}}_j \hat{D}_j^{-1} \to U$ as the case may be. Then $D_j S\mathcal{R}_j D_j^{-1} \to SD_j \mathcal{R}_j D_j^{-1} = SU = M$ and $\hat{D}_j S\hat{\mathcal{R}}_j \hat{D}_j^{-1} \to S\hat{D}_j \hat{\mathcal{R}}_j \hat{D}_j^{-1} = SU = M$, respectively. Note that $S\mathcal{R}_j$ and $S\hat{\mathcal{R}}_j$ are orthogonal so we are done. Q.E.D.

Proof of Proposition 5.7. It suffices to prove that for some $A \in \mathrm{SL}(n, R)$ that there exist N_j, O_j as above with $N_j O_j N_j^{-1} \to M'$ and $M' = AMA^{-1}$. So we may assume that

$$M = \begin{pmatrix} M_1 & & O \\ & \ddots & \\ O & & M_l \end{pmatrix},$$

where each M_i is a Jordan block,

$$M_i = \begin{pmatrix} A_i & I & & O \\ & & \ddots & \\ & & \ddots & I \\ O & & & A_i \end{pmatrix}$$

and A_i is either ± 1 or a two by two rotation matrix and I is the one by one or two by two identity. The last lemma now finishes the proof. Q.E.D.

Proposition 5.7 can be rephrased to say that if $A \in \mathrm{SL}(n, R)$ and $\mathrm{Ad}(A)$ has all eigenvalues on the unit circle then A is in the closure of the union of the compact subgroups of $\mathrm{SL}(n, R)$.

We turn our attention to general Lie groups. We make a conjecture in this context which amounts to characterizing stable ergodicity as partial hyperbolicity and essential accessibility.

If A is an automorphism of G such that $A(\Gamma) = \Gamma$ and $g \in G$ then we call the induced diffeomorphism LgA: $G/\Gamma \to G/\Gamma$ affine. Given an affine diffeomorphism $\mathrm{Ad}(g) \cdot DA(e)$ is an automorphism of the Lie algebra $\mathfrak{g}$ of G. The contracting and expanding subspaces of this automorphism generate an ideal $\mathfrak{h}$ in $\mathfrak{g}$. Let $H \subset G$ be the connected normal subgroup of G whose Lie algebra is $\mathfrak{h}$. Let G be a connected Lie group and Γ a uniform discrete subgroup of G.

Conjecture 5. $Lg \cdot A$ is stably ergodic among affine diffeomorphisms of G/Γ iff $L_g \cdot A$ is partially hyperbolic and

$$\overline{H\Gamma} = G.$$

Thus in the Lie group setting stable ergodicity would coincide with partial hyperbolicity and essential accessibility. We don't know how C^2 perturbations affect the ergodicity in the case $\overline{H\Gamma} = G$ but $H \neq G$. This is the situation of the ergodic automorphism of tori for example. They may be stably ergodic and we have conjectured that at least the generic C^2 perturbation is. Conjecture 5 is proven in one direction in Brezin and Shub (1996) with the added hypothesis that the pair G, B is *admissible* in a certain technical sense. It is true for tori and nilmanifolds (see Parry (1970)) and for lattices in semi-simple Lie groups Brezin and Shub (1996).

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