DIFFERENTIABILITY AND CONTINUITY OF INVARIANT MANIFOLDS*

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INTRODUCTION

We discuss conditions that guarantee the existence of an ergodic attractor for an abstract, discrete time-dynamical system. We begin with a compact differentiable manifold, B, and a diffeomorphism, $f\colon B \to B$; that is, f is smooth, takes B into B, and has a smooth inverse. There are two cases to consider: (a) f preserves a smooth volume element, μ, in which case $f(B) = B$ and we say that f is volume preserving, and (b) f does not preserve a smooth volume element. Here we abuse language and say that f is dissipative.

Since the topology of the manifold B plays no role in our considerations, there will be no loss of generality in considering B as a ball in some Euclidean space, $B \subset R^m$. In case b, we imagine that $f(B)$ is strictly inside B (FIGURE 1).

We now define an ergodic attractor. Our emphasis is on measure theory, so our definition is different from Smale's in these same proceedings. What we lose conceptually in ignoring open and closed sets we gain in the ability to state a theorem.

DEFINITION. A measurable set, $A \subset B$, and an f-invariant measure, μ, on A will be called an *ergodic attractor* if there is a set of positive Lebesgue measure, $V \subset B$, such that

1. $f^n(x) \to A$ as $n \to \infty$ for all $x \in V$, and
2. for any continuous $\phi\colon B \to R$, $1/n \sum_{k=0}^{n-1} \phi(f^k(x)) \to \int_A \phi \, d\mu$ for Lebesgue almost all x in V as $n \to \infty$.

The methods we use have a long history. We don't want to survey the history, so we simply list some of the main characters: Poincaré, Hadmard, Birkhoff, Peron, Morse, Hedlund, Hopf, Anosov, Sinai, Smale, Ruelle, Bowen, and Pesin. Generally speaking, the techniques involved are hyperbolicity, stable manifold theory, and ergodic theory. We will give heuristic arguments that are far from formal to show how the theory goes. The latest advances in the theory on which this paper is based are due to Pesin, who used Osceledec's multiplicative ergodic theorem to prove an almost everywhere stable manifold theorem in the volume preserving case,[1] and Ruelle, who extended

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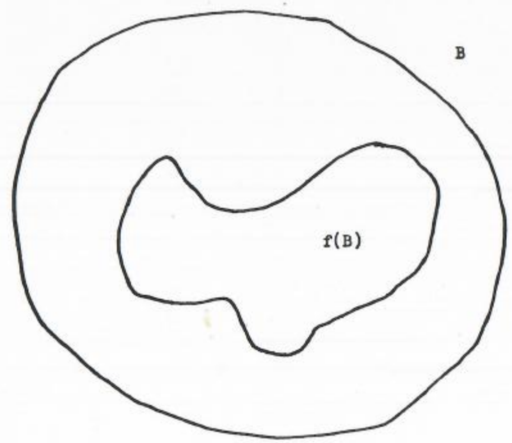

FIGURE 1. The dissipative case.

these results to the dissipative case.[2,3] Our own interest in the subject is a r questions of Ruelle. We begin our heuristic arguments.

A METHOD OF PROVING THAT TIME AVERAGES CONVERGE TO A CONSTANT

Suppose we can divide our space simultaneously into contracting and sets, which we will call leaves (FIGURE 2). Given two points, x, y, on a contr we have $d(f^n(x), f^n(y)) \to 0$ as $n \to \infty$. So, for a continuous ϕ,

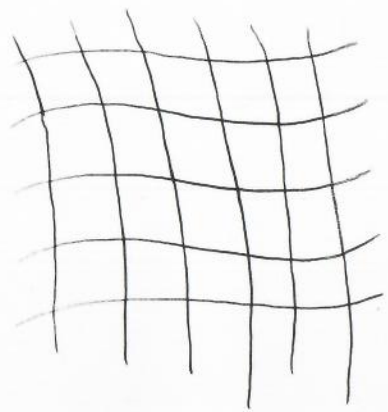

FIGURE 2. Contracting and expanding leaves.

$$\lim \frac{1}{N} \sum_{n=0}^{N-1} \phi(f^n(x)) \text{ exists if and only if}$$

$$\lim \frac{1}{N} \sum_{n=0}^{N-1} \phi(f^n(y)) \text{ exists}$$

and if they exist, then they are equal. Thus, forward time averages are constant on contracting leaves. On expanding leaves, $d(f^{-n}(x), f^{-n}(y)) \to 0$ as $n \to \infty$, so backward time averages are constant on expanding leaves. To prove that we get the same constant for almost all contracting leaves, we would like to invoke Birkhoff's ergodic theorem (so we are assuming an invariant measure) to say backwards time averages equal forward time averages almost everywhere and to push the constant that we get for one contracting leaf along the expanding leaves to conclude that we get the same constant on each contracting leaf. This argument works, provided that we can prove various technical properties about the decompositions of the space into expanding and contracting leaves and about the holonomic map from one contracting leaf to another along the expanding leaves. This last property is the absolute continuity of the holonomy map.

Getting the Structure from Infinitesimal Data

If tangent vectors are exponentially stretched or contracted, then we expect to mirror the linear structure nonlinearly for the diffeomorphism f itself. For simplicity's sake, we will consider the case in which there is no neutral behavior. An orbit, $\{f^n(x)\}$, $-\infty < n < \infty$, will be called uniformly hyperbolic if the space of tangents $T_{f^n(x)}$ splits as a direct sum, $T_{f^n(x)} = E^S_{f^n(x)} \oplus E^u_{f^n(x)}$, of two subspaces such that the derivative Df: $T_{f^n(x)} B \to T_{f^{n+1}(x)} B$ respects this splitting (that is, $Df(E^S_{f^n(x)}) = E^S_{f^{n+1}(x)}$ and $Df(E^u_{f^n(x)}) = (E^u_{f^{n+1}(x)})$ and, moreover, the E^S spaces are uniformly contracted and the E^u subspaces are uniformly expanded (that is, there are constants $C > 0$, $0 \leq \lambda < 1$ such that $\|Df^m_{f^n(x)}(v)\| \leq C\lambda^m\|v\|$ for v in $E^S_{f^n(x)}$ and $\|Df^{-m}_{f^n(x)} v\| \leq C\lambda^m\|v\|$ for v in $E^u_{f^n(x)}$ and these constants C, λ are independent of n and m.)

In Figure 3, the linear subspaces that are invariant by $Df_{f^n(x)}$ are exhibited. Standard stable manifold theory proves the existence of nonlinear disks tangent to these linear subspaces. The disks tangent to the E^u directions are called unstable and those tangent to the E^S directions are called stable. These disks are invariant in the sense that f maps unstable disks onto unstable disks and stable disks into stable disks. The unstable disks are expanded by f and the stable disks are contracted by f. The disks are as smooth as f and are uniform in size.

Anosov systems and Axiom A systems posit uniform hyperbolicity of orbits with the same constants C, λ for all the points in the sets in question. The stable and unstable discs produce the structure of Figure 2.

It is possible to prove the existence of these stable and unstable disks under weaker hypotheses.

Definition. An orbit, $\{f^n(x)\}$, $-\infty < n < \infty$, will be called *subexponentially hyperbolic* if

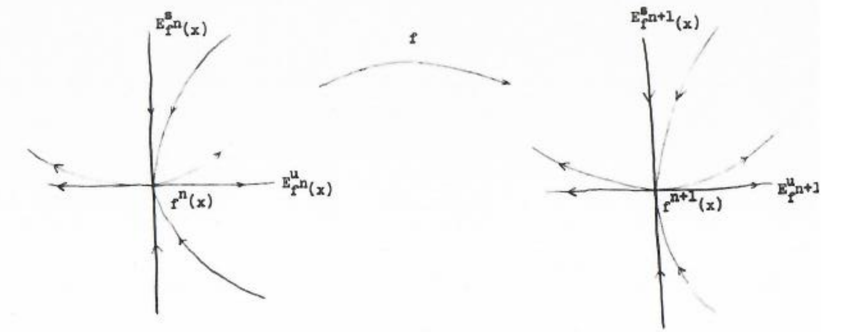

Figure 3. Local invariant manifolds.

a. The space of tangents $T_{f^n(x)}$ splits as a direct sum, $T_{f^n(x)} = E^S_{f^n(x)} \oplus E^u_{f^n(x)}$, suc
the derivative Df: $T_{f^n(x)} B \to T_{f^{n+1}(x)} B$ respects this splitting.

b. There is a constant $0 < \lambda < 1$ such that, given $\epsilon > 0$, there are con
$A_\epsilon(f^n(x))$, $A'_\epsilon(f^n(x))$ that satisfy

1. $A_\epsilon(f^{n+m}(x)) \leq A_\epsilon(f^n(x))e^{\epsilon|m|}$,

   $A'_\epsilon(f^{n+m}(x)) \leq A'_\epsilon(f^n(x))e^{\epsilon|m|}$.

2. For all n, all v in $E^S_{f^n(x)}$, all v in $E^u_{f^n(x)}$, and $m > 0$,

$$\|D_{f^n(x)} f^m(v)\| \leq A_\epsilon(f^n(x))\lambda^m \|v\|,$$

$$\|D_{f^n(x)} f^{-m}(v)\| \leq A'_\epsilon(f^n(x))\lambda^m \|v\|.$$

This definition is the same as that for uniform hyperbolicity except th
constant C is not constant along the orbit. It has been replaced by a seque
constants that may be chosen to grow slower than any exponential—hence, "
ponential." The difference between the uniform and subexponential cases is n
less that, while the angles between the E^u and E^S subspaces in the uniform c
bounded away from 0, as in Figure 3, the angles in the subexponential ca
decay subexponentially, as in Figure 4.

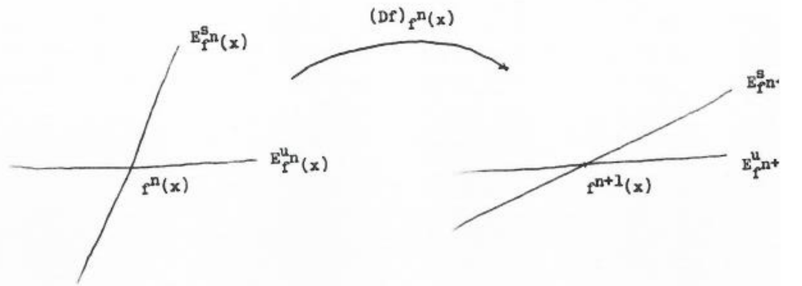

Figure 4. Subexponential decay of angles.

THEOREM. Suppose that f is a diffeomorphism and that its derivative satisfies a Hölder condition.†

Let $\{f^n(x)\}$ $-\infty < n < \infty$ be a subexponentially hyperbolic orbit. Then there are nonlinear disks tangent to the subspaces $E^u_{f^n(x)}$ and $E^S_{f^n(x)}$, which are called unstable and stable, respectively. f expands the unstable disk at $f^n(x)$ onto the unstable disk at $f^{n+1}(x)$ and contracts the stable disk at $f^n(x)$ into the stable disk at $f^{n+1}(x)$. The disks are as smooth as f is.

The disks are no longer necessarily uniform in size, but the size varies at worst subexponentially. This theorem can be found in Pesin and Ruelle.[1-3] All we have to add is that the disks are C^r when f is C^r for integer values of $r > 1$. The case $r = 1$ remains open.

INVARIANT MEASURES

Given a tangent vector v at the point x, we can study its exponential growth rate by considering

$$\overline{\lim}\, \frac{1}{k} \log \|Df^k_x(v)\| \equiv \chi(v),$$

the Lyapunov characteristic exponent of v. Since the derivatives of f are bounded and f is a diffeomorphism, the chain rule shows that $(1/k) \log \|Df^k_x(v)\|$ is a bounded sequence of numbers. $\overline{\text{Lim}}$ is read lim sup and means the supremum of all possible subsequential limits of this sequence. It is easy to see that, for any point x, there are at most m distinct Lyapunov exponents that can occur for tangent vectors at x, where m is the dimension of B. It would be much better if we could replace $\overline{\lim}$ by lim. Oseledec has proven a multiplicative ergodic theorem that allows one to do this for almost every x with respect to an invariant measure. For simplicity's sake, we won't state the general theorem and will content ourselves with a corollary that is sufficient for our purposes. If μ is an f-invariant measure we say that μ has no zero exponents if, for almost all x with respect to μ, $\chi(v) \neq 0$ for all $v \in T_xB$, $v \neq 0$.

COROLLARY OF OSELEDEC'S MULTIPLICATIVE ERGODIC THEOREM. Let μ be an invariant probability measure for f with no zero exponents; then, for almost all x, the orbit through x is subexponentially hyperbolic. Moreover, the spaces E^S_x and E^u_x depend measurably on x.

Now, if f is a diffeomorphism and the derivative satisfies a Hölder condition, we may apply the results of the previous section to obtain expanding and contracting leaves near the support of μ. But now, instead of having the spaghetti-like grid of FIGURE 2, which holds in the uniform Anosov and Axiom A cases, it looks more like the spaghetti has been cooked and put back in the ball (FIGURE 5).

Fixing a sufficiently large C and a λ close enough to 1, we can find sets of large measure such that

$$A_i(x),\ A'(x) < C \text{ and } \lambda_x < \lambda.$$

†Twice continuously differentiable is more than enough to guarantee this.

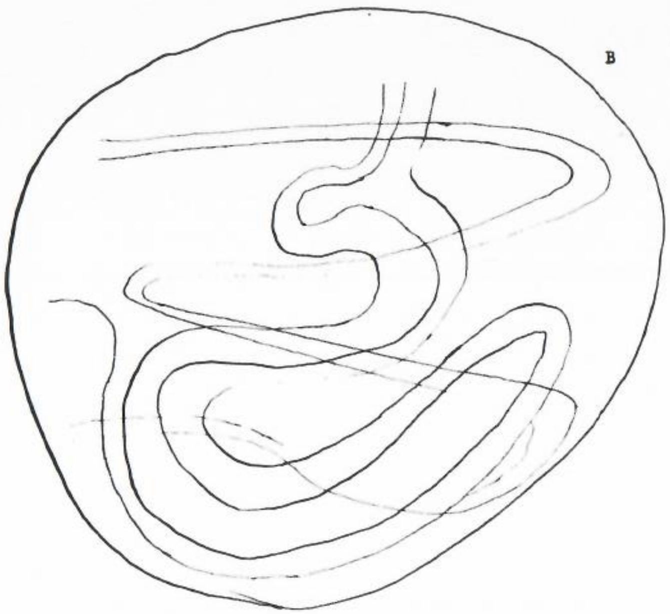

FIGURE 5. Spaghetti-like contracting and expanding leaves.

For these points x, the stable and unstable manifolds will be uniformly lar
locally, we can get a picture on some measurable set that looks like FIGU
The measure of the intersections may, however, be meager in the Lebe
along the contracting or expanding leaves. We get around this problem by
that μ induces absolutely continuous conditional measures on the unstable d

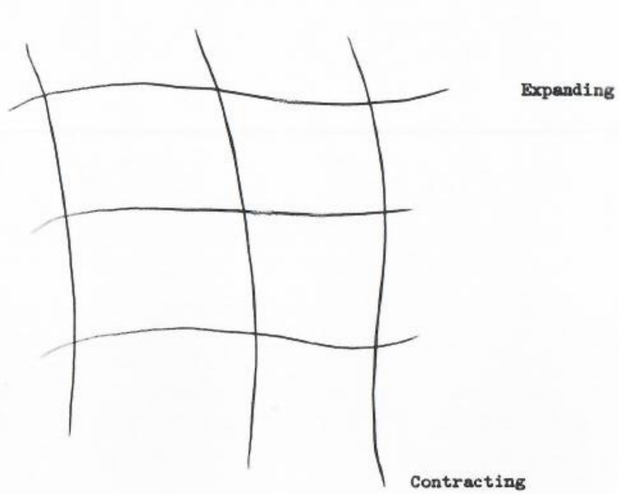

FIGURE 6. A uniform portion of FIGURE 5.

these intersections look like a lace doily in each expanding disk and one can prove that the transition maps along the contracting disks are absolutely continuous. We may fill in the stack of doilies to a set of positive measure in B (FIGURE 7) and prove a theorem by the argument given in the section entitled "A Method of Proving that Time Averages Converge to a Constant."

MAIN THEOREM. Suppose that f is a diffeomorphism and that the derivative satisfies a Hölder condition. If μ is an invariant probability measure for f such that (a) μ has no zero exponents and (b) μ induces absolutely continuous measures on unstable

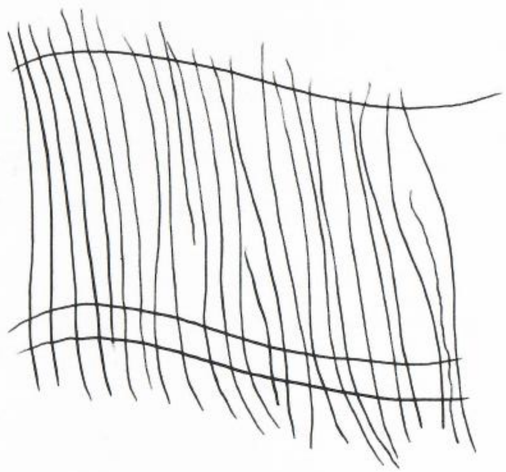

FIGURE 7. A stack of doilies (side view).

disks, then there is an ergodic attractor in the support of μ and the ergodic measure is μ (normalized).

WHEN IS THERE SUCH A MEASURE?

Volume Preserving

The conditional measure is always smooth on the expanding disks, so, if there are no zero exponents, there is always an erogodic attractor for Lebesgue measure. This is the case that was originally dealt with by Pesin.[1]

Dissipative

Here we have a question that seems important and was raised by Ruelle.[2,3] Let $\chi^+(x)$ be the sum of the positive exponents at x counted with multiplicity. If there are no zero exponents and $h_\mu = \int \chi^+ \, d\mu$, then is μ absolutely continuous on unstable disks? This condition is made necessary by the results Katok reported at this conference.

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