GENERICITY THEOREMS IN TOPOLOGICAL DYNAMICS.

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1. Introduction.

Some recent theorems in differentiable dynamical systems are of a C^0 nature, referring to C^0 Ω-explosions and C^0 density for example, see [11, 12, 14, 15]. As far as we know however, no one has explained what these theorems imply about the generic homeomorphism of a compact manifold M or the generic C^0 vector field on M. We record here the result of several conversations on this matter.

First the C^0 topology makes $\mathrm{Homeo}(M)$ a Baire space. The usual C^0 metric

$$d(f,g) = \sup_{x \in M} d(f(x), g(x))$$

gives the same topology on $\mathrm{Homeo}(M)$ as does the metric

$$d_H(f,g) = \max(d(f,g), d(f^{-1},g^{-1})) .$$

Under d_H, $\mathrm{Homeo}(M)$ is complete and hence, as a topological space, it has the Baire property: every countable intersection of open dense sets is dense.

A set G is generic (relative to a Baire space $B \supset G$) if G contains a countable intersection of open dense sets. A generic property is one enjoyed by a generic set of elements of B.

Theorem 1. The following properties of $g \in \mathrm{Homeo}(M)$ are generic

(a) g has no C^0 Ω-explosion,

(b) g has no C^0 Ω-implosion,

(c) g is a continuity point of the map $\Omega : \mathrm{Homeo}(M) \to K(M)$ where $K(M)$ is the space of compact subsets of M under the Hausdorff topology, of g,

(f) g has no periodic sinks or sources,

(g) g has infinitely many periodic points of some finite period,

(h) g does not have a fine filtration.

These terms are defined in §2. In §5 theorem 1 is partially generalized for C^0 vector fields.

It is conjectured in [10, 13] that, for every diffeomorphism f of M, the topological entropy $h(f)$ is related to the action of f on homology, $f_* : H_*(M;R) \to H_*(M;R)$, as follows.

Entropy conjecture. $\log|\lambda| \leq h(f)$ for all the eigenvalues λ of f_*. Here we suggest that this is frequently true for homeomorphisms.

"Theorem 2". If $\dim M \neq 4$ then the Entropy Conjecture holds for an open and dense set of Homeo(M). If $\dim M = 4$ then the same is true of any stable component of Homeo(M), i.e. any component containing a somewhere smooth homeomorphism.

We will sketch an argument for proving this "theorem" in the case that $\dim M \neq 4$. It would be very interesting to give a full proof of it.

Remark 1. Recently Anthony Manning has verified the Entropy Conjecture for all homeomorphisms of M^m, $m \leq 3$.

Remark 2. If $\dim M \neq 4$ then by [4] every component of Homeo(M) is stable.

2. *Ω-explosions, filtrations, entropy, etc.*

A point $x \in M$ is called *wandering* for $f \in$ Homeo(M) if there is a neighbourhood U of x in M such that $f^n(U) \cap U = \emptyset$ for all $n \neq 0$. The complement of the wandering points is called the

f *has no C^0 Ω-explosions* if given $\varepsilon > 0$ there is a neighbourhood of f, $\mathcal{U} \subset \mathrm{Homeo}(M)$, such that any $g \in \mathcal{U}$ has $\Omega(g) \subset N_\varepsilon(\Omega(f))$ where $N_\varepsilon(\Omega(f))$ is the ε-neighbourhood of $\Omega(f)$ in M.

f *has no C^0 Ω-implosions* if given $\varepsilon > 0$ there is a neighbourhood of f, $\mathcal{U} \subset \mathrm{Homeo}(M)$, such that any $g \in \mathcal{U}$ has $\Omega(f) \subset N_\varepsilon(\Omega(g))$.

A *filtration* $\mathcal{M}$ for $f \in \mathrm{Homeo}(M)$ is a sequence $\emptyset \subset M_0 \subset \ldots \subset M_k = M$ of compact C^∞ submanifolds M_i with boundary such that

(a) $\dim M_i = \dim M$,

(b) $f(M_i) \subset \mathrm{Int}\, M_i$.

Given a filtration $\mathcal{M}$ for f, $K_\alpha(\mathcal{M}) = \bigcap_{n\in Z} f^n(M_\alpha - M_{\alpha-1})$ is the maximal f-invariant set contained in $M_\alpha - M_{\alpha-1}$. $K(\mathcal{M})$ is defined as $\bigcup_{\alpha=0}^{k} K_\alpha(\mathcal{M})$. If $\mathcal{M}$ is a filtration for f then $\Omega_\alpha \subset K_\alpha(\mathcal{M})$, where $\Omega_\alpha = \Omega \cap (M_\alpha - M_{\alpha-1})$. For any filtration $\mathcal{M}$, $\Omega \subset K(\mathcal{M})$. If $\Omega = K(\mathcal{M})$, $\mathcal{M}$ is called a *fine filtration.* If $\mathcal{N}$ is a filtration for f, defined by $\emptyset \subset N_0 \subset N_1 \subset \ldots \subset N_j = M$ then $\mathcal{N}$ *refines* $\mathcal{M}$ if for any index ℓ there is an index β such that $N_\ell - N_{\ell-1} \subset M_\beta - M_{\beta-1}$. A sequence of filtrations $\mathcal{M}^i$ for f is called *fine* if $\mathcal{M}^{i+1}$ refines $\mathcal{M}^i$ and $\bigcap K(\mathcal{M}^i) = \Omega$.

We now present the concept of entropy à la Bowen [1]. Let (X,d) be a metric space and $T : X \to X$ continuous. A set $E \subset X$ is (n,ε) separated if for any $x,y \in E$ with $x \neq y$ there is a j, $0 \leq j \leq n$, such that the distance $d(T^j x, T^j y) > \varepsilon$. Let $S_n(\varepsilon)$ denote the largest cardinality of the (n,ε) separated sets in X and let

$$S_\varepsilon(T) = \limsup\, (1/n)\log S_n(\varepsilon).$$

The *topological entropy* $h(T)$ of T is then defined by $h(T) = \lim_{\varepsilon\to 0} S_\varepsilon(T)$.

... ...ences of filtrations arose in [12] where of the diffeomorphisms with a fine sequence of filtrations in $\mathrm{Diff}^r(M)$ was posed there. The trouble in proving a theorem of this nature for $r \geq 1$ is a conflict of C^0 closing lemma techniques with the C^r topology. On the other hand our result on the Entropy Conjecture was motivated by [2, 10, 13].

3. Proof of theorem 1.

In [15] Floris Takens proves (a): generically $g \in \mathrm{Homeo}(M)$ has no C^0 Ω-explosion. Also (a) and (b) imply (c); and (a) is equivalent to (d) [12]. This leaves (b), (e), (f), (g) and (h). Behind their proofs lies the idea of a permanent periodic orbit $\mathcal{O}$ – a periodic g-orbit such that any g' near g in $\mathrm{Homeo}(M)$ has a periodic orbit $\mathcal{O}'$ near $\mathcal{O}$. For example if $\mathcal{O} = \{p, gp, \ldots, g^k p\}$ is a periodic sink (topological attractor) then by the Brouwer Fixed Point Theorem it is permanent.

(3.1) Lemma. If $\mathcal{O}$ is a periodic f-orbit and U is a neighbourhood of $p \in \mathcal{O}$ in M then there exists a homeomorphism $c : M \to M$ such that $c|M-U$ = identity and $\mathcal{O}$ is a periodic sink for $c \circ g$.

Proof. c is a very sharp contraction toward p. To construct a suitable c it is only necessary to dominate any local repulsiveness of f^k at p, k being the period of $\mathcal{O}$. Since there are no derivative restrictions on c this can easily be done.

To prove (b), (e) we imitate the proof of the General Density Theorem in [9] replacing hyperbolic periodic points by permanent ones. Consider $\mathrm{perm}(f) = \{p \in M ; p$ is a permanent periodic point of $f\}$. By construction, the map $f \mapsto \overline{\mathrm{perm}(f)}$, $\mathrm{Homeo}(M) \to K(M)$ = compact subsets of M is lower semi-continuous. Let G be the residual set of its continuity points. We claim

$$(1) \quad \Omega(g) = \overline{\mathrm{perm}(g)}, \; g \in G .$$

produces g' near g in Homeo(M) having a periodic point x' near x in M. Lemma 3.1 produces g'' near g' having x' as a permanent periodic point. Hence

$$x \in \limsup_{f \to g} \overline{\mathrm{perm}(f)}$$

contradicting the continuity of $\overline{\mathrm{perm}}$ at g. This proves (1). Clearly (1) implies (e) and (b). It remains to verify (f), (g) and (h).

Because we are working in the C^0 topology isolated periodic points are the exception not the rule. This is in contrast to the C^r topology, $r \geq 1$.

<u>(3.2) Lemma.</u> If $\mathcal{O}$ is a periodic f-orbit and U is a neighbourhood of $\mathcal{O}$ in M then there exists $g \in$ Homeo(M) such that $g = f$ off U and g has two distinct permanent periodic orbits $\mathcal{O}'$, $\mathcal{O}''$ in U. If $\dim M \geq 2$ or $\mathcal{O}$ has even period then $\mathcal{O}'$ and $\mathcal{O}''$ have the same period as $\mathcal{O}$. Otherwise they have period $\leq$ twice that of $\mathcal{O}$.

<u>Proof.</u> Let k = period of $\mathcal{O}$ and let q be a point of M near but not equal to some $p \in \mathcal{O}$. Let h be a homeomorphism which equals the identity off U, sends p to p and sends $f^k q$ to q. (If $\dim M = 1$ and k is odd then f^k can reverse orientation at p. In that case we can only make h send p to p and $f^{2k} q$ to q.) The composition $h \cdot f$ is near f and has distinct periodic orbits through p and q. By (3.1) these orbits can be made permanent for some g near $h \cdot f$, completing the proof of (3.2).

As above, let G be the set of continuity points of $f \mapsto \overline{\mathrm{perm}(f)}$. Put $G_{\varepsilon,k} = \{g \in G$; the ε-neighbourhood of each periodic g-orbit of period $\leq k$ contains two distinct permanent g-orbits of period $\leq k\}$.

Suppose $\dim M \geq 2$. By (3.2) $G_{\varepsilon,k}$ is dense; clearly it is orbits of period $\leq k$. This means that $Per_k(g)$ = periodic points of period $\leq k$ is a perfect set. Whenever it is non-empty it is uncountable. This proves (f) and (g) at once. Note that $Per_k(g) \neq \emptyset$ for some k since $\overline{\text{perm}(g)} = \Omega(g) \neq \emptyset$, M being compact.

Suppose $M = S^1$ and $f : S^1 \to S^1$ reverses orientation. Then f has exactly two fixed points but has no other periodic points of odd prime period. Thus $G_{\varepsilon,k}$ is dense in $\text{Homeo}(S^1)$ iff $k \geq 2$. Again, it is clear that $G_{\varepsilon,k}$ is open and this implies that $Per_k(g)$ is perfect for all $k \geq 2$ and all $g \in G_* = \bigcap_{\varepsilon>0,k\geq 2} G_{\varepsilon,k}$. As above, this gives (f) and (g) for $M = S^1$.

Suppose M is several copies of S^1. The same reasoning shows that generically $Per_k(g)$ is perfect for all large k, completing the proof of (f), (g) in all cases.

Finally let us show that condition (h) is generic. From (g) and (3.1) above, we get for each $n \in Z_+$ an open and dense set $A_n \subset \text{Homeo}(M)$ such that if $g \in A_n$ then $g(\overline{U}_i) \subset \text{Int}\, U_i$ for n disjoint open sets $U_i \subset M$. Thus for the generic $g \in \text{Homeo}(M)$ there are infinitely many such disjoint open sets U_i. This implies (h) and the proof of Theorem 1 is complete.

Remark 1. Here is a more precise version of (f), (g).

If $\dim M \geq 2$ then generically $Per_k(g)$ is either empty or is a Cantor set.

If $\dim M = 1$ then generically $Per_k(g)$ is either empty or is a Cantor set or $k \leq$ the number of components of M and $Per_k(g)$ is finite.

To complete the proof of this remark it suffices to permanently destroy large M-open sets in $Per_k(g)$. This is not hard.

4. <u>A sketch of a proof of "Theorem 2".</u>

We produce a dense but first category set of well-behaved homeomorphisms in the case that $\dim M \neq 4$. We proceed as in [13]. Given $f \in \mathrm{Homeo}(M)$ we pick a small triangulation of M . Since $\dim M \neq 4$ we may perturb f on coordinate charts to produce g near f which transversally preserves a small handle decomposition and is smooth on a neighbourhood of Ω, [4]. That is,

a) if M^k is the union of the handles up to index k, then $g(M^k) \subset \mathrm{Int}\, M^k$

b) the image of each core disk h_i^k is transverse to each transverse $(n-k)$-disk th_j^{n-k} .

In this case the non-wandering set Ω can be described, as in [13], by the intersection matrices $\#(g(h_i^k) \cap th_j^{n-k})$. By construction Ω is zero-dimensional and g exhibits a multiple horseshoe or Morse-Smale behaviour at Ω .

Let $\mathcal{U}$ be the C^0 dense set of such homeomorphisms. For $g \in \mathcal{U}$, $\log s(g_*) \leq h(g)$ [2, 13]. On the other hand, each $g \in \mathcal{U}$ is C^0 lower semi-stable [6]. This means that, for any small perturbation g' of g, there is a continuous surjection $\sigma : \Omega(g') \to \Omega(g)$ such that $\sigma g'(x) = g\sigma(x)$ for any $x \in \Omega(g')$. It follows that the entropy of g' is at least as big as that of g. Therefore the relation $\log s(g'_*) \leq h(g')$ is also true for g' near g . This yields an open and dense set $\mathcal{V} \subset \mathrm{Homeo}(M)$ as required.

The above sketch should also work for the stable components of Homeo(M) since "stable" means (essentially) "locally smoothable", and the transversality theory in the preceeding proof should be adaptable to this assumption.

fields on M. A remarkable but easily proved result of Orlicz [8] (see also Choquet's book [3]) says that the generic $X \in \mathcal{X}^0(M)$ generates a continuous flow. It then makes sense to ask whether Theorem 1 remains true for such an X-flow ϕ. (It does - see Theorem 1' below.) One might also ask about the Entropy Conjecture for flows (Theorem 2) but unfortunately its natural generalization is trivial: the time t map of any flow, ϕ_t, induces the identity on $H_*(M)$ because $\phi_t \simeq 1$. On the other hand there might be an interesting Flow Entropy Conjecture if ϕ_t were forced to act on some sort of "transverse homology groups".

Returning to Theorem 1, we shall restate only the part having to do with filtrations. A <u>global Lyapunov function</u> for the continuous flow ϕ is a real valued continuous function on M which strictly decreases on ϕ-trajectories off Ω and is constant along trajectories of Ω. (Ω is the non-wandering set of ϕ.)

<u>Theorem 1'.</u> Generically $X \in \mathcal{X}^0(M)$ generates a flow having a C^∞ global Lyapunov function.

<u>Proof</u>. Takens' proof of (a) extends to flows. Also (a) continues to be equivalent to (d): a fine sequence of filtrations [7]. Such a fine sequence produces a continuous global Lyapunov function. This can be made C^∞ by the smoothing theory of Wilson [16].

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