NOTES

ON THE DENSITY OF NEUTRAL HYDROGEN IN INTERGALACTIC SPACE

Recent spectroscopic observations by Schmidt (1965) of the quasi-stellar source 3C 9, which is reported by him to have a redshift of 2.01, and for which Lyman-α is in the visible spectrum, make possible the determination of a new very low value for the density of neutral hydrogen in intergalactic space. It is observed that the continuum of the source continues (though perhaps somewhat weakened) to the blue of Ly-α; the line as seen on the plates has some structure but no obvious asymmetry. Consider, however, the fate of photons emitted to the blue of Ly-α. As we move away from the source along the line of sight, the source becomes redshifted to observers locally at rest in the expansion, and for one such observer, the frequency of any such photon coincides with the rest frequency of Ly-α in his frame and can be scattered by neutral hydrogen in his vicinity. The calculation of the size of the effect is very easily performed as follows:

Let us consider a cosmological model with the metric

$$ds^2 = dt^2 - R^2(t)\,(du^2 + \sigma^2(u)d\gamma^2)\,,$$

where

$$\sigma(u) = \begin{cases} \sin u\,, \\ u \\ \sinh u \end{cases}$$

depending on the curvature, and $d\gamma$ is the increment in angle.

The probability of scattering of a photon in a *proper* length interval $dl = R(t)du$ at cosmic time t is clearly

$$dp = n(t)\sigma(\nu_s)dl\,,$$

where $n(t)$ is the number of neutral hydrogen atoms per unit volume at time t (assumed uniform for the present) and $\sigma(\nu)$ is the radiative excitation cross-section for the Ly-α transition. This has the form

$$\sigma(\nu) = \frac{\pi e^2}{m c} f g(\nu - \nu_a)\,,$$

where f is the oscillator strength (here $f = 0.416$), and g the profile function, which is strongly peaked at $\nu = \nu_a$, the Ly-α frequency (2.46×10^{15} sec^{-1}), and has unit integral:

$$\int_{-\infty}^{\infty} g(x)\,dx = 1\,.$$

Let the redshift of the object observed be z_0, and suppose the redshift of the resonance scattering layer as seen from here is z; $z < z_0$, and if the observed frequency is ν,

$$\nu_s = \nu(1 + z)$$

is the frequency seen by an observer stationary with respect to the scattering layer. Thus the total optical depth at ν is

$$p = \int_0^{z_0} dp = \int_0^{z_0} n[t(z)]\,\sigma[\nu(1+z)]\,\frac{dl}{dz}\,dz\,.$$

Now dl/dz is clearly something like cH^{-1}. It can in fact be shown easily using the development angle formalism (Sandage 1961*b*) that

$$\frac{dl}{dz}=R(t)\frac{du}{dz}=\frac{cH_0^{-1}}{(1+z)^2(1+2q_0z)^{1/2}}$$

for the relativistic models with vanishing pressure and cosmological constant, and with deceleration parameter q_0 (see, e.g., Sandage 1961*a* for a discussion of these models). For the steady-state model, $u = z$, and so

$$\frac{dl}{dz}=\frac{cH^{-1}}{1+z}.$$

We then have, for the relativistic models,

$$p=\int_1^{Hz_0}\left\{\frac{n[t(z)]\pi e^2 f}{mH_0\nu(1+z)^2(1+2q_0z)^{1/2}}\right\}g[\nu(1+z)-\nu_a]\nu d(1+z).$$

The function g is strongly peaked at zero; its width depends on the intergalactic temperature, but even at 10^6 ° K its width expressed in velocity units is only $2\times10^{-4}\,c$ (compared to the redshift, which is of the order of 2). Thus we can take the factor in braces out of the integral, evaluated at $(1+z)=\nu_a/\nu$; the integral that is left is unity, and we obtain (for $H_0 = 10^{-10}\ \mathrm{yr}^{-1}$)

$$p=\frac{n_s}{(1+z)(1+2q_0z)^{1/2}}\left(\frac{\pi e^2 f}{m\nu_a H_0}\right)\simeq(5\times10^{10}\ \mathrm{cm}^3)\frac{n_s}{(1+z)(1+2q_0z)^{1/2}}.$$

Here n_s is the number density of neutral hydrogen in the scattering region. The flux will be reduced, of course, by a factor e^{-p}, but it is difficult to say just how large the effect is on the plates of 3C 9. Intensity tracings were made of two plates, and tracings of the neighboring night-sky spectra allowed approximate subtraction of the night-sky contribution. The blueward component and the wing of the line are noticeably depressed (they are enhanced on the plates by a strong, broad night-sky feature at 3640 Å that gives the line an almost symmetric profile before the sky is removed), but the exact amount is difficult to measure; best estimates place the depression at about 40 per cent, which corresponds to an optical depth of about $\frac{1}{2}$. This yields, for $q_0 = \frac{1}{2}$, a number density $n_s = 6\times10^{-11}\ \mathrm{cm}^{-3}$, or a mass density $\rho_s = 1\times10^{-34}\ \mathrm{gm\ cm}^{-3}$—a figure five orders of magnitude below the limit (for the *present* density, which should be 27 times smaller because of the expansion) obtained from 21-cm observations by Field (1962).

For the $q_0 = \frac{1}{2}$ model, the total density at $z = 2$ is $5\times10^{-28}\ \mathrm{gm\ cm}^{-3}$; thus only about one part in 5×10^6 of the total mass at that time could have been in the form of intergalactic neutral hydrogen. For the steady-state model $\rho_s = 2\times10^{-35}$ and the (constant) total density is 4×10^{-29}; the factor here is somewhat less, about 2×10^6.

We are thus led to the conclusion that either the present cosmological ideas about the density are grossly incorrect, and that space is very nearly empty, or that the matter exists in some other form. Oort has shown that only about 1 per cent of the $q_0 = \frac{1}{2}$ density is accounted for by galaxies, and it has been generally assumed that the remainder exists as an intergalactic gas which is presumably mostly or entirely hydrogen. It is possible that this interpretation is still valid but that essentially all of the hydrogen is ionized; this conclusion can be defended if we are allowed to make the intergalactic electron temperature high enough. Field and Henry (1964) have shown that the temperature cannot exceed 2×10^6 ° K for the steady state and about 4×10^5 ° K for the evolving models in the vicinity of $q_0 = \frac{1}{2}$. Higher temperatures than these would result in more free-free (bremsstrahlung) emission from the intergalactic medium than the total flux observed in the X-ray region. One finds (Allen 1963) that the mean recombination time for ionized hydrogen is about

$$t_r = 1.2 \times 10^{11}\, T^{1/2} n^{-1}\, [\ln (1+1.58 \times 10^6/T)]^{-1}\,,$$

where T is the electron temperature and n the total number density. Consider first the $q_0 = \frac{1}{2}$ relativistic case. At $n = 3 \times 10^{-4}$ (the value at $z = 2$, assuming essentially all the density in the form of hydrogen) and $T = 2 \times 10^5$ ° K (corresponding to a present value of 2×10^4 and adiabatic expansion), $t_r = 7 \times 10^{16}$ sec, or 2×10^9 years. At $z = 2$, the age of the universe was 2×10^9 years; we must thus find a mechanism by which the ionization level may be *maintained* at 5×10^6. This requires a process with a mean ionization time per hydrogen atom of $t_{\text{ion}} = 1.6\ 10^{10}$ sec, or 500 years. Electron collisions are inadequate; the mean lifetime for collisional ionization is about (Allen 1963)

$$t_{\text{coll}} = 1.75 \times 10^{10}\, T^{-1/2} n^{-1} (1 + T/1.6 \times 10^6) \exp (1.58 \times 10^5/T)\,,$$

or about 10^4 years. Consider now radiative ionization: we find that we need a mean intensity at the Lyman limit of 1.2×10^{-20} erg cm^{-2} (c/s)$^{-1}$ sec^{-1} ster^{-1}, if the spectrum does not fall off too rapidly to the blue.

The flux can come from three sources; normal galaxies, radiogalaxies, and QSS's, and the intergalactic medium itself. The contribution from the first two sources can be estimated roughly, and almost certainly does not exceed 3×10^{-24} units at $z = 2$, of which about 10 per cent is from quasi-stellar sources (assuming that one can extrapolate the visual radiation into the UV with a spectral index of -0.7, and assuming a present space density of [600 Mpc]$^{-3}$).

The intergalactic medium is more promising. For $q_0 = \frac{1}{2}$, the computations of Field and Henry give, for $T = 2 \times 10^5$ ° K at $z = 2$, an intensity 6×10^{-21} units at the Lyman limit. This is a lower bound, and correction of the Gaunt factor (Karzas and Latter 1961) for this temperature and frequency range gives an estimate of about 1.1×10^{-20} units, which is very nearly equal to the required value; the treatment of the singularity at $t = 0$ is quite uncertain, and the excellent agreement is probably fortuitous. Going to larger q_0 increases the free-free intensity, but this is more than compensated by the increased level of ionization required and the decreased recombination time. If we decrease the temperature slightly, the recombination time will drop but will be exactly compensated by an increase in the free-free intensity down to a present temperature of about 3×10^3 ° K, below which the intensity drops precipitously. We have computed this equilibrium assuming that all the mass is in the gas; we find, however, that the number density of neutrals is independent of the gas density, since the recombination time is proportional to n^{-1} and the ionization time to n^{-2}. Lower densities do, of course, make collisional ionization relatively more important. Because of this independence of total number, the dependence on q_0 enters only through the factor $(1 + 2q_0 z)^{1/2}$ in dl/dz, and differences between reasonable models are smaller by far than the uncertainty in the treatment of the free-free intensity integrals near $t = 0$. It seems therefore inappropriate at this time to discuss differences between relativistic models with small q_0. As we go to temperatures (at $z = 2$) higher than about 5×10^5, the recombination time increases more rapidly than the radiative ionization time; collisions become more important, and the ionization ratio decreases.

For the steady state, the observed ionization ratio is obtained *collisionally* at a temperature of about 7×10^5 ° K; t_{coll} is about 1.5×10^{12} sec, and the radiative ionization time is very much longer, about 10^{13} sec. If the older density of 2×10^{-29} g/cm^3 is taken, the equilibrium temperature drops to about 4×10^5. D. W. Sciama (private communication) has pointed out that recombinations which result in reionization should be deleted from the equilibrium considerations, but it is easily shown that the medium is optically thin in the ionization continuum, so that high-energy photons are redshifted past the limit before they can ionize.

One interesting consequence of an essentially completely ionized medium is that there appears to be an appreciable optical depth in Thompson scattering for very distant objects. Since the cross-section for this process is independent of frequency, its only effect is to attenuate the over-all level of radiation. The optical depth to redshift z is

$$p = \int \sigma n dl ,$$

which for the relativistic models is

$$p = \frac{n_0 \sigma}{H_0} \left[\frac{3q_0 + q_0 z - 1}{3q_0^2} (1 + 2q_0 z)^{1/2} - \frac{3q_0 - 1}{3q_0^2} \right]$$

and for the steady state is

$$p = \frac{n_0 \sigma}{H} \ln (1 + z) .$$

At $z = 2$, these quantities are 0.38, 0.23, and 0.18 for $q_0 = 1$, $q_0 = \frac{1}{2}$, and the steady state, respectively, and are in such a direction as to *decrease* the sensitivity of the m–log z relation to the model. In addition, these figures assume that all the material is in the ionized intergalactic medium, a fact that is certainly not known a priori; thus one does not know quite how much to correct for the effect, and the interpretation of data from very distant sources is no longer a straightforward application of the usual simple cosmological tests that involve only the parameters of the metric tensor plus knowledge of intrinsic source properties.

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James E. Gunn
Bruce A. Peterson

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Mount Wilson and Palomar Observatories
Carnegie Institution of Washington
California Institute of Technology

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SOME CONSEQUENCES OF LARGE REDSHIFTS

The detection of redshifts numerically greater than unity (Schmidt 1965) makes it necessary to compare the effects of selecting different models of the Universe. I believe that the contents of Table 1 below and the deductions which can be drawn from them may be of interest. The table gives luminosity distances, D, and local linear diameters, l, for sources of radiation whose redshifts, z, and angular diameters, a'', are regarded as known. The values of D and l depend on the choice of model of the Universe and vary from model to model even though the redshift of a source and the Hubble constant remain fixed. The general relativity models may be conveniently distinguished from one another by assigning numerical values to two parameters: the acceleration factor, q_0, and the density parameter, σ_0 (McVittie 1965, eqs. [8.321] and [8.402]). The metric of a model universe is

$$ds^2 = dt^2 - \frac{R^2(t)}{c^2}\frac{(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2)}{(1 + kr^2/4)^2}, \tag{1}$$

in the usual notation. The Hubble constant is H and its reciprocal (a time interval) is T_0. Then the cosmical constant, λ, and the space-curvature constant, k, are given by

$$\lambda T_0^2 = -3(q_0 - \sigma_0), \tag{2}$$

$$k\left(\frac{cT_0}{R_0}\right)^2 = -(q_0 + 1 - 3\sigma_0), \tag{3}$$

where R_0 is the value of the scale factor, R, at the instant at which the observations are being made by an observer located at $r = 0$ in the space-time (1). The relations (2) and (3) are valid in models where the pressure is zero.

The luminosity distance, D, of a source of radiation is that distance parameter which has the following property: if P_e is the energy in watts emitted isotropically by the source, as measured *at the source*, then the flux received by the observer is

$$L = P_e/(4\pi D^2)\,. \tag{4}$$

It can be proved that in all models (McVittie 1965, § 8.6 [iii])

$$D = cT_0\mathfrak{D}(z, q_0, \sigma_0)\,, \tag{5}$$

where $\mathfrak{D}$ is a dimensionless function of the redshift z, which also involves q_0 and σ_0. The model (Model 1) in which

$$q_0 = \sigma_0 = 1 \tag{6}$$

happens to have the property that $\mathfrak{D} = z$ for all values of z however large these may be. The luminosity distance of a source of redshift, z, in this model is therefore

$$D_1 = cT_0 z\,. \tag{7}$$

In other models one may write

$$D = D_1 g(z)\,, \tag{8}$$

where

$$g(z) = \frac{\mathfrak{D}(z)}{z}. \tag{9}$$

Hence if D_1 has been calculated from equation (7) for a given value of T_0, the luminosity distance of the same source in some other model is calculable from equation (8) once the function $g(z)$ has been computed from equation (9). The method for finding $\mathfrak{D}(z)$, and consequently $g(z)$, is given elsewhere (McVittie 1965, § 8.6 [iii]).

The results for six specimen models are

Model 2:

A3(vii-*b*).[1]— $q_0 = \frac{19}{8}$, $\sigma_0 = \frac{27}{8}$ $(k = +1, \lambda > 0)$.

$$g_2(z) = \frac{2}{3\sqrt{3}} \frac{1+z}{z} \sin \omega ,$$

$$\omega = \ln \left\{ \frac{2+\sqrt{3}}{2-\sqrt{3}} \frac{[1+3(1+z)]^{1/2} - \sqrt{3}}{[1+3(1+z)]^{1/2} + \sqrt{3}} \right\}.$$

Model 3:

A1 (iv).— $q_0 = \sigma_0 = 2$ $(k = +1, \lambda = 0)$.

$$g_3(z) = \frac{1}{4z}[2z - 1 + (1+4z)^{1/2}].$$

Model 4:

A2 (v).— $q_0 = 1, \sigma_0 = 0$ $(k = -1, \lambda < 0)$.

$$g_4(z) = \frac{1+z}{z}\{[2(1+z)^2 - 1]^{1/2} - (1+z)\}.$$

Model 5:

A1 (iii).— $q_0 = \sigma_0 = 0.5$ (Einstein–de Sitter, $k = 0$, $\lambda = 0$).

$$g_5(z) = \frac{2}{z}[(1+z) - (1+z)^{1/2}].$$

Model 6:

A1 (i).— $q_0 = \sigma_0 = 0$ (Milne's model, $k = -1$, $\lambda = 0$).

$$g_6(z) = 1 + \tfrac{1}{2}z .$$

Model 7:

A2 (vi-*b*).— $q_0 = -1$, $\sigma_0 = 0$ (de Sitter universe, $k = 0$, $\lambda > 0$).

$$g_7(z) = 1 + z .$$

If the local diameter, l, of a source of radiation is small, then the source subtends at the observer an angular diameter in seconds of arc given by (McVittie 1965, § 8.6 [ii] and eq. [8.617])

$$a'' = \frac{1}{206265} \frac{l(1+z)^2}{D}.$$

Hence in Model 1

$$\frac{l_1}{a''} = (206265)\, cT_0 z (1+z)^{-2}, \tag{10}$$

whereas in Model n, the linear diameter corresponding to the same a'' would be l_n where, by equations (7), (8), and (9),

$$\frac{l_n}{a''} = \frac{l_1}{a''} g_n(z). \tag{11}$$

Table 1 contains the values of D_1 and of l_1/a'' calculated for $H = 100$ km/sec/Mpc, which means that $T_0 = 9.778 \times 10^9$ years. The g-factors for the six models are also listed; of these g_6 is also given in Table 4 of Schmidt (1965). The redshifts include those of quasi-stellar sources.

[1] These identifying descriptions refer to sections of the Appendix in McVittie (1965).

The effect on the luminosity distance of a source produced by changing the model universe is exemplified by the case of 3C 147 ($z = 0.545$). It would have a luminosity distance of 1634 Mpc if Model 1 were employed, 1152 Mpc ($= 1634 \times 0.766$) if Model 2 were used, and 2525 Mpc ($= 1634 \times 1.545$) if Model 7 were chosen. Similarly the range for 3C 9 ($z = 2.012$) runs from 3425 Mpc for Model 2 to 18160 Mpc for Model 7. Corresponding variations in the linear diameters occur according as one model or another is employed. It is also worth noting that the luminosity distance cT_0, which is 2.998×10^9 pc, has no particular significance. It is certainly not the radius of the "observable Universe," which is defined by $z = \infty$.

A source of radiation that emits energy at the rate $F(\nu_e)d\nu_e$ watts at the emission frequency ν_e in the band width $d\nu_e$ produces at the observer a flux density $S(\nu)$ W $(c/s)^{-1}$ m^{-2} at frequency ν where[2]

$$S(\nu) = \frac{(1+z)F_n(\nu_e)}{4\pi D_n^2} = \frac{(1+z)F_n(\nu_e)}{4\pi(cT_0)^2 g_n^2 z^2},$$

TABLE 1

LUMINOSITY DISTANCES, LINEAR DIMENSIONS, AND g-FACTORS

Object	z	$D_1 \times 10^{-9}$ pc	$l_1/a'' \times 10^{-3}$ pc	g_2	g_3	g_4	g_5	g_6	g_7
3C 273 . .	0 158	0 474	1 712	0 907	0 939	1 018	1 036	1 079	1 158
3C 48 . .	0 368	1 103	2 858	819	889	1 071	1 078	1 184	1 368
3C 47. .	0 425	1 274	3 041	800	878	1 089	1 088	1 212	1 425
3C 295	0 461	1 382	3 138	789	872	1 100	1 094	1 230	1 461
3C 147	0 545	1 634	3 318	766	859	1 127	1 108	1 272	1 545
3C 254 .	0 734	2 200	3 548	722	835	1 193	1 137	1 367	1 734
. . . .	0 848	2 542	3 608	700	823	1 235	1 152	1 424	1 848
. . . . . .	1 000	2 998	3 633	674	809	1 292	1 172	1 500	2 000
3C 245	1 029	3 084	3 632	670	807	1 302	1 175	1.514	2 029
CTA 102	1 037	3 108	3 632	669	806	1 306	1 176	1 518	2 037
3C 287 .	1 055	3 162	3 630	666	804	1 312	1 178	1.528	2 055
. . .	1 300	3 897	3 571	634	787	1 407	1 205	1 650	2 300
. . . .	1 700	5 096	3 389	593	764	1 565	1 244	1 850	2 700
. . . .	2 000	5 995	3 229	569	.750	1 685	1 268	2 000	3 000
3C 9	2 012	6 031	3 223	568	749	1 689	1 269	2 006	3 012
. .	3 000	8 993	2 725	0 513	0 717	2 090	1 333	2 500	4 000

the calculation being performed for Model n. Hence $F_n(\nu_e)$ in W $(c/s)^{-1}$ is

$$F_n(\nu_e) = \left[4\pi(cT_0)^2 \frac{z^2}{(1+z)} S(\nu)\right] g_n^2(z), \tag{12}$$

and in Model 1

$$F_1(\nu_e) = 4\pi(cT_0)^2 z^2 (1+z)^{-1} S(\nu)\ ; \tag{13}$$

or, if $H = 100$ km/sec/Mpc,

$$F_1(\nu_e) = 1.08 \times 10^{53} z^2 (1+z)^{-1} S(\nu) \tag{14}$$

as given by Schmidt (1965).[3] Thus in any model it follows that

$$F_n(\nu_e) = F_1(\nu_e) g_n^2(z)\ . \tag{15}$$

[2] See McVittie (1965), eq (8 830) with

$$F_n(\nu_e) = 4\pi P[\nu(1+z), t] = 4\pi P(\nu_e, t).$$

[3] In Schmidt's work $F(\nu_{em})$ stands for the particular expression (14) of $F_1(\nu_e)$, and $f(\nu_{obs})$ is written for $S(\nu)$.

Suppose that two sources, J and J', are selected. Then it is clear from equations (12) and (15) that the ratio

$$\frac{F_n'(\nu_e)}{F_n(\nu_e)}=\frac{F_1'}{F_1}\left[\frac{g_n(z')}{g_n(z)}\right]^2 \tag{16}$$

is independent of the particular value assigned to the Hubble constant. Relative intrinsic power outputs can therefore be calculated for each model once the redshifts z, z', and observed flux densities $S(\nu)$, $S'(\nu)$ are known. For example, let J be 3C 273 and J' be 3C 9 and let ν_e be 10^{15} c/s (=emitted wavelength of 3000 Å). Then from Schmidt's (1965) Table 4, $F_1' = 10^{23\ 6}$ and $F_1 = 10^{23\ 8}$ or $F_1'/F_1 = 10^{-0\ 2}$, a number independent of the value of H. Hence in Model 2

$$\frac{F_2'(10^{15})}{F_2(10^{15})}=10^{-0\ 2}\left(\frac{0.56}{0.91}\right)^2=0.24\,,$$

and therefore 3C 9 is calculated to have about one-quarter of the emission, at 10^{15} c/s, of 3C 273. But if Model 7 is employed instead,

$$\frac{F_7'(10^{15})}{F_7(10^{15})}=10^{-0\ 2}\left(\frac{3}{1.16}\right)^2=4.2\,,$$

and so 3C 9 now turns out to have over four times the emission of 3C 273 at 10^{15} c/s. The change of model has therefore raised the emission of 3C 9 relative to that of 3C 273 by

TABLE 2

INTRINSIC EMISSIONS AT 10^{15} C/S

Object	log F_1	log F_2	log F_3	log F_4	log F_5	log F_6	log F_7
3C 273 . . .	23 8	23 7	23 7	23 8	23 8	23 9	23 9
3C 48 . . .	23 0	22 8	22 9	23 1	23 1	23 1	23 3
3C 47. . .	22 5	22 3	22 4	22 6	22 6	22 7	22 8
3C 147 .	23 1	22 9	23 0	23 2	23 2	23 3	23 5
3C 254 .	22 9	22 6	22 7	23 1	23 0	23 2	23 4
3C 245. .	23 5	23 2	23 3	23 7	23 6	23 9	24 1
CTA 102	23 5	23 2	23 3	23 7	23 6	23 9	24 1
3C 287. . .	23 4	23 0	23 2	23 6	23 5	23 8	24 0
3C 9 .	23 6	23 1	23 3	24 1	23 8	24 2	24 6
Mean	23 26	22 98	23 10	23 43	23 37	23 54	23 74
σ . .	0 41	0 39	0 40	0 48	0 41	0 49	0 52

a factor of 18. However, if the significent quantity is taken to be log F_n, the results for the quasi-stellar sources are shown in Table 2. In this table, log F_1 is taken from Schmidt's Table 4 and log F_n (rounded off to one decimal) is calculated from formula (15) and the g-factors of our Table 1. The mean value of log F_n and the standard deviation, σ, from the mean are also shown. Schmidt's conclusion that the intrinsic emissions of the nine sources are "similar" is seen to be true if Models 1, 2, 3, or 5 are employed in the calculations. These models all contain matter of high density and therefore are models in which gravitational effects are pronounced. The similarity of the sources is perhaps less conspicuous in Models 4, 6, and 7, which are models empty of matter and in which, therefore, the effects of motion, but not those of gravitation, are taken into account. These conclusions, of course, follow when the models are regarded as general relativity models.

The steady-state theory employs the metric of Model 7, and the laws of the propagation of radiation in this theory are identical with those of general relativity. Thus luminosity distances, angular diameters, and emission outputs are computed in the same fashion in both theories. From the present point of view, therefore, the predictions of the steady-state theory are identical with those of the de Sitter universe of general relativity. The factors g_7 are the largest of those found in Table 1 and lead to the greatest luminosity distances and the largest linear diameters. Model 7 also makes the intrinsic emissions of the four remotest sources larger than that of 3C 273. There is indeed a hint of an increase of intrinsic emission with redshift, in other words, as the instant of emission occurs earlier and earlier in the history of the Universe. It is perhaps ironical that this effect should be suggested, however inconclusively, in the model of the steady-state theory where the properties of the Universe are assumed to be unchanging in time.

The determination of the general-relativity model universe that best fits the data obtained from a set of sources of radiation such as the quasi-stellar sources, is the inverse of the problem of calculating the quantities $F_n(\nu_e)$. The flux-density formula may now be written

$$\log S(\nu) = \log\left[\frac{F(\nu_e)}{4\pi(cT_0)^2}\right] + \log\left[\frac{1+z}{\mathfrak{D}^2(z, q_0, \sigma_0)}\right], \tag{17}$$

and it is required to find q_0, σ_0, and T_0 $(=1/H)$ from a set of observed pairs of values of S and z. Strictly speaking, this is possible only if $F(\nu_e)$ could be known for each source independently of its S and z. Since this information is not obtainable, one may perhaps argue from the results of Table 2 that there is a good prima facie case for assuming that $\log F$ (10^{15}) is constant for these sources, though its value is unknown. With this hypothesis, it is possible to find q_0 and σ_0 from equation (17) and the observed pairs of values of (S,z). The process would also give T_0 as a multiple of the (constant) intrinsic emission F. It may be that Schmidt has carried out these operations and thus picked out our Models 1 and 6, but his a priori assignment of a value to H (or T_0) renders this interpretation of his work unlikely. In this connection it is unfortunate that Schmidt does not list the numbers he used for S—which he calls $f(\nu_{\text{obs}})$—together with the errors he assigned to them.

A final point which is brought out by the column giving l_1/a'' in Table 1, is the existence of a maximum for l_1 which, it can be proved (McVittie 1965, p. 223), occurs at exactly $z = 1$ in Model 1. This is the counterpart of a theorem that is usually stated the other way around. It is shown that if l is kept fixed and the source is located successively at positions of increasing z, then a minimum of a'' will occur at a finite value of z. For more distant positions, a'' increases again. Among the models in which this effect is found are the high-density Models 1, 2, 3, and 5. Models 4, 6, and 7 are zero-density models and therefore do not show this effect (McVittie 1965, (8.6) [ii]). Thus it appears that objects are now being observed whose redshifts would place them beyond the turning point for angular diameters, in certain models at least. That this effect can be exploited to distinguish observationally between the two classes of models remains to be seen. One serious obstacle lies in the difficulty of measuring accurate angular diameters whether in the optical, or in the radio, domain.

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G. C. McVittie

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University of Illinois Observatory
Urbana, Illinois

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