Problem Set #5
OCEA/EART 172/272: Geophysical Fluid Dynamics
Due: Tuesday, May 26, 2020

1. Exponential Growth: Consider three sin-waves, $\psi_j(x, y, t) = \phi_j(y) sin(k_j x - \omega_j t)$, with $j = 1$, 2, or 3. Let $\phi_j(y, t) = A_j e^{k_j c_{ij} t}$. Let $A_j = 0.001$ so that at $t = 0$, each of the waves start with the same tiny amplitude. The wavenumber k and the phase speed c are related by the dispersion relation, so in this example, we have k and the imaginary part of the phase speed, c_i, occuring in pairs. Consider 3 waves of wavelength $\lambda_j = \frac{2\pi}{k_j} = j * 1000m$. Let the imaginary part of the three waves be $c_{i1} = 100m\, day^{-1}$, $c_{i2} = 300m\, day^{-1}$, $c_{i3} = 350m\, day^{-1}$. Which wave reaches amplitude 1 first, and how long does it take? At that time, what are the amplitudes of the other 2 waves? How long does it take for the fastest growing wave to reach amplitude 10? Describe in words what the superposition of these 3 waves would roughly look like once the one with fastest growth rate reached a sizeable amplitude, like 10. (Note that the $\omega_j t$ in the sin function provides a phase offset here and is not important to the problem. Consider the sum of the waves at an arbitrary time in the future.) You can do this problem numerically quite easily, but the key is to consider the relative amplitudes of the individual waves and how they contribute to the sum.

2. Cushman-Roisin (2nd Edition) Analytical Problem 10-2. Consider both stability criteria in answering this problem.

3. Cushman-Roisin (2nd Edition) Analytical Problem 10-4. Note, that this problem will be most easily done numerically, using something like matlab, but you can argue the answer using graphs as well or instead. Also, treat β as constant in this problem (i.e., not a function of y). Also, note that the domain for this is $-\infty < y < +\infty$.

4. Consider your answer to Cushman-Roisin (2nd Edition) Analytical Problem 10-4.

   (a) Discuss if increases or decreases to U, L, and β can stabilize or destabilize the jet according to the first criterion for instability (that $\beta - \frac{d^2\overline{u}}{dy^2}$ change sign somewhere in the domain).

   (b) Now consider two cases where the jet has parameters (U, L, and β) that are just slightly on the (i) stable or (ii) unstable side of the stability threshold, an consider the impact of making this feature a westward jet (by letting U<0) instead of eastward jet (with $U > 0$). Is the stable eastward jet (i) stable or possibly unstable if moving westward, or is the eastward unstable jet (ii) stable or still possibly unstable when moving westward. Stated differently, do you need a larger or smaller value of β to ensure jet stability for a westward jet than for an eastward jet. Draw a conclusion that westward jets are more or less stable than eastward jets from this exercise.

5. Extra Credit: In the text and in class, we discussed two stabilty criteria. The first involves a change in sign of the pv gradient. In the original (non-GFD) context, this is sometimes referred to as Rayleigh's inflection point criterion: for instability to be possible, the background flow must include an inflection point within the domain. The second criterion, the GFD version of Fjortoff's Theorem, can be expressed that a necessary condition for instability is that

$$(\overline{u} - \overline{u}_0)(\beta - \frac{d^2\overline{u}}{dy^2}) > 0$$

   somewhere in the domain for all values of $\overline{u}_0$. In practice, we set $\overline{u}_0 = \overline{u}(y_c)$ where y_c is the point where the first criterion is satisfied. Consider

$$\overline{u}(y) = U sinh(y/L),$$

   with $-L < y < L$. Is this jet stable or unstable according to the first criterion? How about the second criterion? For this structure, show that this second criterion can be satisfied automatically by the first criterion using any $\overline{u}_0$ EXCEPT the value $\overline{u}(y_c)$.