

Difference Logic

Satisfiability Checking Seminar

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Outline

- ▶ Main Literature
- ▶ Difference Logic
- ▶ Example Problem: Job Scheduling
- ▶ SAT Checking
- ▶ Constraint Graph And Negative Cycles
- ▶ Conclusion

Main Literature

- ▶ [\[Cotton et al. 2004\]](#) Scott Cotton, Eugene Asarin, Oded Maler and Peter Niebert. **“Some progress in satisfiability checking for difference logic”**. In Formal Techniques, Modelling and Analysis of Timed and Fault-Tolerant Systems, pages 263–276. Springer, 2004.
- ▶ [\[Cormen et al. 2009\]](#) Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest and Clifford Stein. **“Introduction to algorithms”**. MIT press, third edition, 2009.
Note: the chapter 24 **“Single-Source Shortest Paths”** is relevant for the topic.

Difference Logic

- ▶ Difference logic – a special case of linear arithmetic logic,
- ▶ in which constraints have the following form:

$$x - y \prec c$$

x, y – variables, c – constant and $\prec \in \{<, \leq\}$ – comparison operator.

- ▶ $x, y, c \in \mathbb{Z}$ or \mathbb{R} .

Difference Logic

A couple of examples:

$$\phi_1 = (p \vee q) \wedge (p \rightarrow (u - v < 3.3)) \wedge (q \rightarrow (v - w < -5.15))$$

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$$\begin{aligned}\phi_2 = & (u - v < 1) \wedge (v - w < 5) \\ & \wedge (w - x \leq -3) \wedge (x - y < 1) \\ & \wedge (y - z \leq -5) \wedge (y - v \leq 0)\end{aligned}$$

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SAT $u = 0, v = 3, w = 0, x = 3, y = 3, z = 8$

$$\begin{aligned}\phi_3 &= (u - v < 1) \wedge (v - w < 5) \\ &\quad \wedge (w - x \leq -3) \wedge (x - y < -3) \\ &\quad \wedge (y - z \leq -5) \wedge (y - w < 4)\end{aligned}$$

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UNSAT $(w - x \leq -3) \wedge (x - y < -3) \wedge (y - w < 4) \Rightarrow 0 < -2$

Difference Logic

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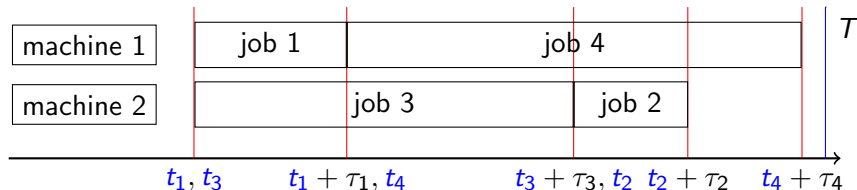
$$\begin{aligned} & (v = -3) \\ \Leftrightarrow & (\neg((v < -3) \vee (v > -3))) \\ \Leftrightarrow & (\neg((v < -3) \vee (-v < 3))) \\ \Leftrightarrow & (\neg((v - 0 < -3) \vee (0 - v < 3))) \end{aligned}$$

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Example Problem: Job Scheduling



- ▶ $p_{mj} = \text{True}$ if job j is scheduled on machine m :
e.g. $p_{11} = p_{14} = p_{23} = p_{22} = \text{True}$
- ▶ job i starts at t_i and lasts τ_i
- ▶ a machine cannot process two or more jobs simultaneously:

$$(p_{mi} \wedge p_{mj}) \rightarrow ((t_i + \tau_i \leq t_j) \vee (t_j + \tau_j \leq t_i)) \Leftrightarrow$$

$$(p_{mi} \wedge p_{mj}) \rightarrow ((t_i - t_j \leq -\tau_i) \vee (t_j - t_i \leq -\tau_j))$$
- ▶ the overall processing time should not exceed T :

$$t_i + \tau_i \leq T \Leftrightarrow t_i - 0 \leq T - \tau_i$$

Example Problem: Job Scheduling

$$\phi = \bigwedge_{j=1}^4 (p_{1j} \vee p_{2j}) \quad \wedge$$

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$$\bigwedge_{j=1}^4 (t_j \geq 0) \quad \wedge \quad \bigwedge_{j=1}^4 (t_j \leq T - \tau_j) \quad \wedge$$

General time constraints

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General time constraints

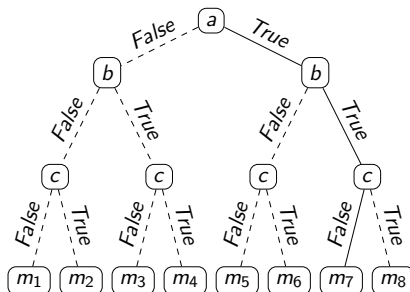
$$\bigwedge_{m=1}^2 \bigwedge_{i=1}^3 \bigwedge_{j=i+1}^4 ((p_{mi} \wedge p_{mj}) \rightarrow ((t_i - t_j \leq -\tau_i) \vee (t_j - t_i \leq -\tau_j)))$$

No time overlap rule

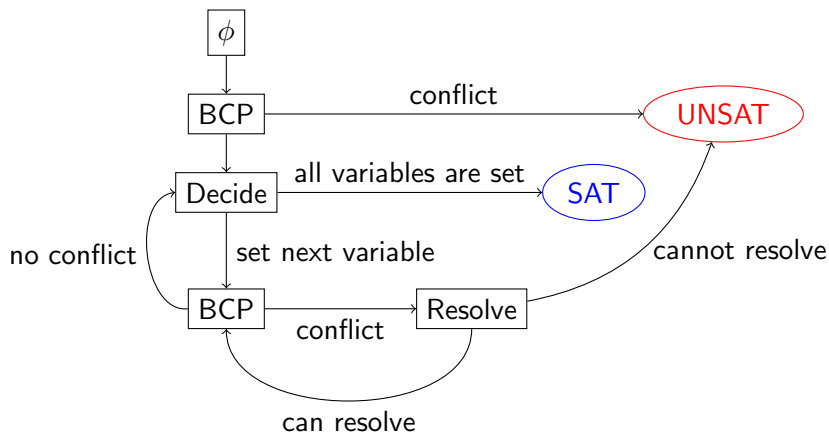
SAT Checking

$$\phi = (a \vee b) \wedge (b \vee c) \wedge (c \vee a) \wedge \dots$$

SAT checking = intelligent search in the model space.
The model space can be represented as a tree.



SAT Checking



SAT Checking

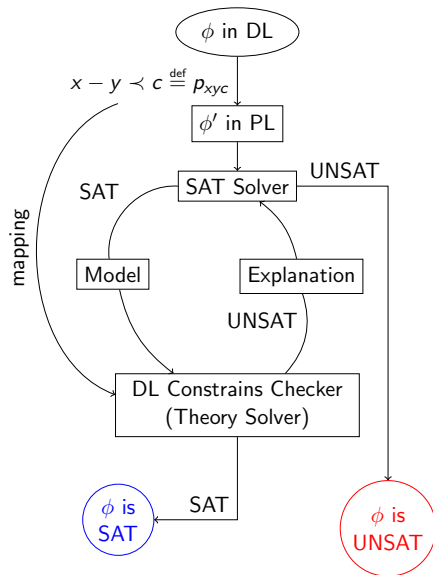


Figure: Lazy approach
Alex Ryndin

SAT Checking

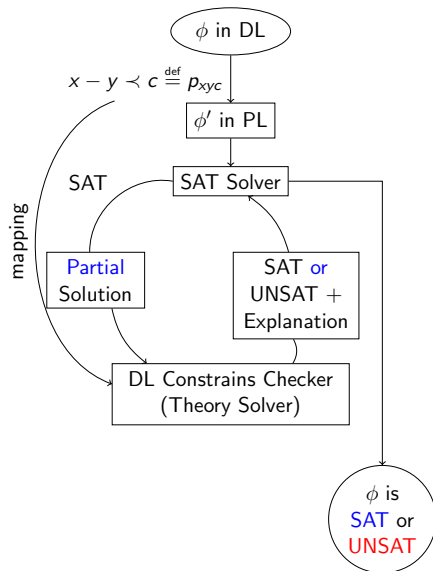
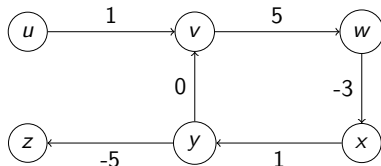


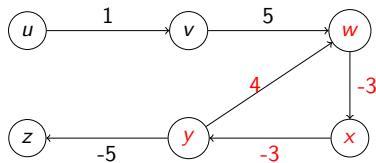
Figure: Incremental approach

Constraint Graph And Negative Cycles



$$\begin{aligned} & (u - v < 1) \\ & \wedge (v - w < 5) \\ & \wedge (w - x \leq -3) \\ & \wedge (x - y < 1) \\ & \wedge (y - z \leq -5) \\ & \wedge (y - v \leq 0) \end{aligned}$$

SAT



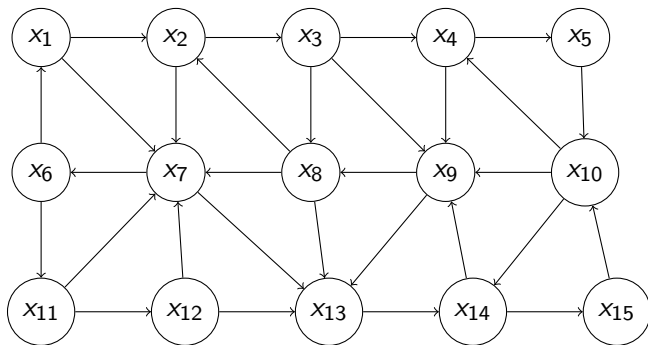
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UNSAT : $0 < -2$

Constraint Graph And Negative Cycles

- ▶ First idea: enumerate all cycles
 - ▶ and check if they are negative
i.e. correspond to conflicts like e.g. $0 < -1$, $0 \leq -5$ etc.
 - ▶ or they have zero weight and an edge with a strict inequality
i.e. correspond to $0 < 0$ conflict.
- ▶ Any problems with this approach?

Constraint Graph And Negative Cycles



- ▶ **Problem:** a graph can have an enormous number of cycles
- ▶ E.g. extreme case: **fully connected directed** graph with n vertices
 - ▶ Number of **simple** cycles = $\sum_{i=2}^n \binom{i}{n} \cdot (i-1)!$
 - ▶ Factorial grows even faster than exponent \Rightarrow the problem becomes intractable.

Constraint Graph And Negative Cycles

- ▶ Use Bellman-Ford algorithm for this task [Cormen et al. 2009]
 - ▶ $O(|V| \cdot |E|)$ time complexity
 - ▶ $s \in V$ – source vertex (can be selected e.g. randomly)
 - ▶ $\Gamma = (V, E, \text{weight})$ – directed graph
 - ▶ $d \in V \mapsto \mathbb{R}$ – distance estimate function
 - ▶ Note: the algorithm runs once for the whole graph, not once for each vertex (i.e. $|V|$ times).

BELLMAN-FORD(Γ, s)

- 1 set $d(x) = \infty \quad \forall x \in V$
- 2 $d(s) = 0$

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- 4 **do for** each edge $(x, y) \in E$
- 5 **do if** $d(x) + \text{weight}(x, y) < d(y)$
- 6 **then** $d(y) = d(x) + \text{weight}(x, y)$

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8  do if  $d(x) + \text{weight}(x, y) < d(y)$ 
9      then return False
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10 return True
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Constraint Graph And Negative Cycles

- ▶ [Cotton et al. 2004] uses the **admissible graph** Γ_d to find a negative or zero weight cycle in the original constraint graph Γ
- ▶ Terminology:
 - ▶ Reduced cost function $r(x, y) = \text{weight}(x, y) + d(x) - d(y)$
 - ▶ Admissible edge: $r(x, y) \leq 0$
 - ▶ Admissible graph Γ_d – a graph consisting of admissible edges
- ▶ Implications:
 - ▶ Γ_d is **dynamic** because it depends on d which **changes during the execution of the algorithm**
 - ▶ If $r(x, y) < 0$ then the edge (x, y) can be "relaxed" i.e. used to improve $d(y)$
 - ▶ Γ_d consists of edges which might potentially be used to improve d
 - ▶ Intuition: if Γ_d has a **cycle** then this cycle might be used to **update d infinitely**

Constraint Graph And Negative Cycles

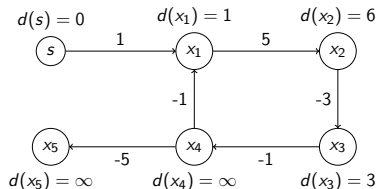


Figure: Γ

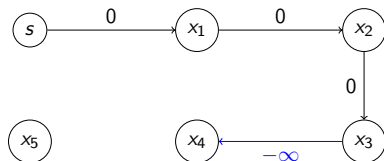


Figure: Γ_d

$$r(x, y) = \text{weight}(x, y) + d(x) - d(y)$$

$$\begin{array}{lll} r(s, x_1) = 0 & r(x_1, x_2) = 0 & r(x_2, x_3) = 0 \\ r(x_3, x_4) = -\infty & r(x_4, x_1) = \infty & r(x_4, x_5) = \emptyset \end{array}$$

Γ_d has no cycles. Let us relax the edge (x_3, x_4)

Constraint Graph And Negative Cycles

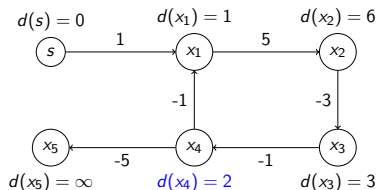


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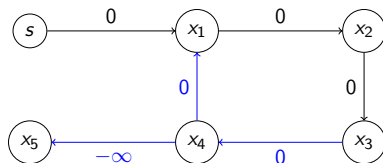


Figure: Γ_d

$$r(x, y) = \text{weight}(x, y) + d(x) - d(y)$$

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Now Γ_d has a cycle: $x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow x_4 \rightarrow x_1$. This cycle in Γ has indeed zero weight.

Constraint Graph And Negative Cycles

Theorem

*Given a constraint graph Γ
and*

*a series of distance estimating
functions $(d_0, d_1, d_2, d_3, \dots)$,
 Γ has a negative or zero cycle
if and only if*

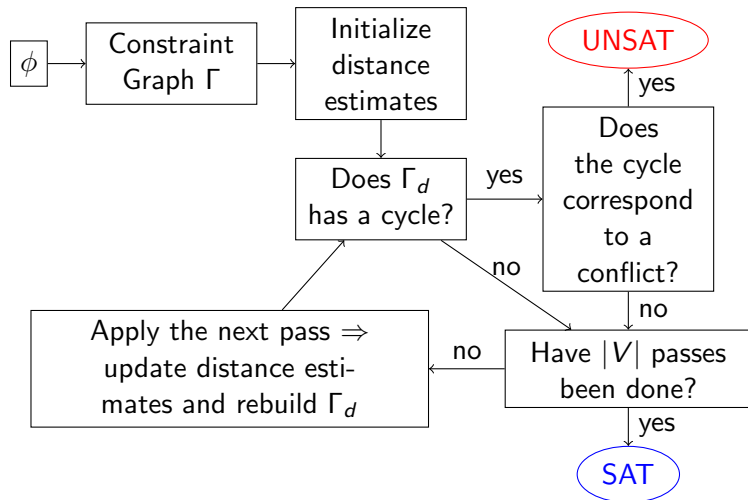
Γ_d has a cycle under some distance estimate d_k .

Proof.

see [Cotton et al. 2004].



Constraint Graph And Negative Cycles



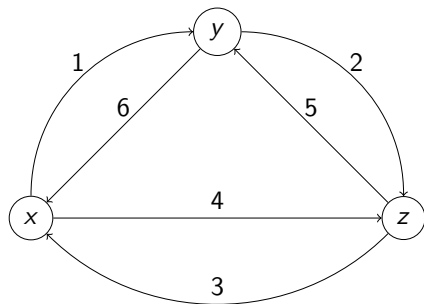
Conclusion

- ▶ Many timing problems (logistics, planning, scheduling, circuits checking) can be expressed in DL. Therefore it is important to have an efficient algorithm for checking SAT of a DL formula.
- ▶ Conjunction of DL constraints can be represented by a constraint graph Γ .
 - ▶ A negative cycle corresponds to a conflict $0 \prec c$ where $c < 0$ and $\prec \in \{<, \leq\}$.
 - ▶ A zero weight cycle with a **strict inequality** edge corresponds to a conflict $0 < 0$.
- ▶ There is no need to enumerate all cycles in Γ . Bellman-Ford algorithm can be used to detect a negative cycle in $O(|V| \cdot |E|)$ operations.
- ▶ A cycle in admissible graph Γ_d corresponds to a negative or zero weight cycle in the corresponding constraint graph Γ .

Thank you

Thank you for your attention

Backup Slide. Number of simple cycles formula explained



- ▶ a fully connected *directed* graph with n vertices.

- ▶ Number of *simple* cycles =

$$\sum_{i=2}^n \binom{i}{n} \cdot (i-1)!$$

- ▶ Example for $i = 3$
(Figure on the left)

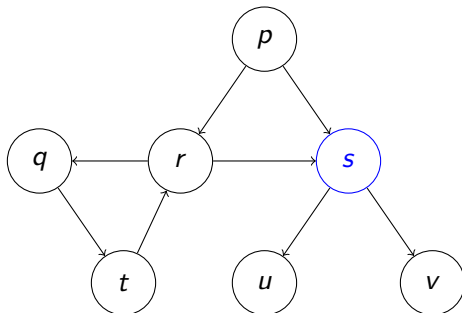
- ▶ There are $i! = 6$ permutations of the vertices which describe 2 cycles:

$(x, y, z), (y, z, x), (z, x, y)$

$(x, z, y), (z, y, x), (y, x, z)$

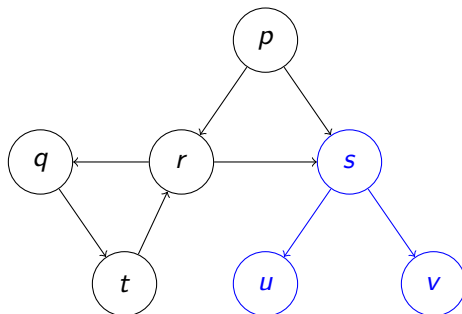
- ▶ Each cycle is described by i permutations which can be produced from each other by **shifting**. Therefore, there are $\frac{i!}{i} = (i-1)!$ cycles.

Backup Slide. Multiple runs of Bellman-Ford



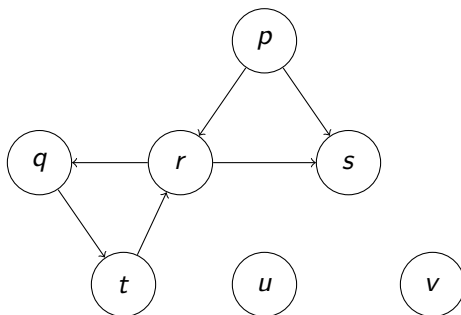
In this example each edge has weight **-1**.
Suppose that **s** is selected as the source vertex.

Backup Slide. Multiple runs of Bellman-Ford



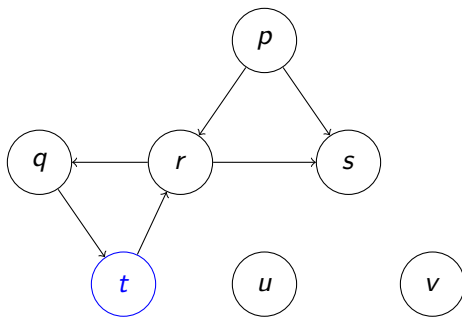
Bellman-Ford processes only a subgraph.
No cycles have been detected.

Backup Slide. Multiple runs of Bellman-Ford



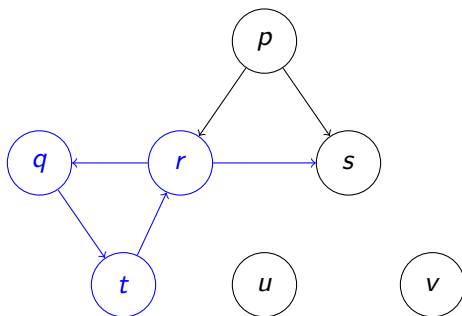
Discard the edges that have been processed
because **we do not need to process them twice.**
Select the next source vertex and run Bellman-Ford again.

Backup Slide. Multiple runs of Bellman-Ford



Suppose, the next source vertex is t .

Backup Slide. Multiple runs of Bellman-Ford



Bellman-Ford finds the negative cycle $t \rightarrow r \rightarrow q \rightarrow t$.

Since the processed edges have been discarded, we do not process the edges $(s; u)$ and $(s; v)$ again, and therefore the complexity for the whole graph stays

$$O(|V| \cdot |E|)$$

i.e. multiple runs of Bellman-Ford do not increase it.