Difference Logic

Satisfiability Checking Seminar

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Abstract

This report describes the difference logic (DL) and a graph-based approach for solving satisfiability (SAT) problem of DL formulas. There is of course a simplex-based algorithm which solves SAT problem of any linear arithmetic (LA) constraints. However, it is not efficient compared to the algorithm described here because simplex does not utilize the simple structure of DL constraints.

Efficiently solving SAT problem of DL constraints is very important because a lot of timing related problems can be described by this logic e.g. scheduling problems, detecting race conditions in digital circuits etc.

The report is organized as follows. Chapter 1 introduces difference logic. Chapter 2 gives theoretical background on SAT checking. Chapter 3 describes a graph-based approach to solving SAT problem of DL. Chapter 4 draws a conclusion.

1 Introduction

1.1 Difference Logic

DL is a special case of an LA logic. [1] and [2, p.5] define DL as follows:

Definition 1.1 (Difference Logic) *Let $\mathcal{B} = \{b_1, b_2, \dots\}$ be a set of Boolean variables and $\mathcal{X} = \{x_1, x_2, \dots\}$ be a set of numerical variables over a domain $\mathbb{D}$. The domain $\mathbb{D}$ is either the Integers $\mathbb{Z}$ or the Reals $\mathbb{R}$. The difference logic over $\mathcal{B}$ and $\mathcal{X}$ is called $DL(\mathcal{X}, \mathcal{B})$ and given by the following grammar:*

$$\phi \stackrel{\text{def}}{=} b \mid (x - y \prec c) \mid \neg\phi \mid \phi \wedge \phi$$

where $b \in \mathcal{B}$, $x, y \in \mathcal{X}$, $c \in \mathbb{D}$ is a constant and $\prec \in \{<, \leq\}$ is a comparison operator.

The remaining Boolean connectives $\vee, \rightarrow, \leftrightarrow, \dots$ can be defined in the usual ways in terms of conjunction $\wedge$ and negation $\neg$.

Examples of DL formulas are given below:

$$\phi_1 = (p \vee q \vee r) \wedge (p \rightarrow (u - v < 3)) \wedge (q \rightarrow (v - w < -5)) \wedge (r \rightarrow (w - x < 0)) \quad (1)$$

$$\phi_2 = (u - v < 1) \wedge (v - w < 5) \wedge (w - x \leq -3) \wedge (x - y < 1) \wedge (y - z \leq -5) \wedge (y - v \leq 0) \quad (2)$$

$$\phi_3 = (u - v < 1) \wedge (v - w < 5) \wedge (w - x \leq -3) \wedge (x - y < -3) \wedge (y - z \leq -5) \wedge (y - w < 4) \quad (3)$$

2 Preliminaries

2.1 Solving SAT Problem of Propositional Logic

Most of the SAT solvers employ a variation of the Davis-Putnam-Logemann-Loveland (DPLL) algorithm [3, 4] for solving SAT problem of the propositional logic (PL). One such basic SAT checking algorithm is given below:

Algorithm 1 A basic SAT checking algorithm for solving SAT problem of PL. It takes a PL formula to be checked for SAT and returns SAT status of the formula (SAT or UNSAT). It also returns a model i.e. an assignment, which evaluates the formula to $True$, in case if the formula is SAT.

```
CHECK( PL formula φ)
 1  model ← ∅
 2  (inferredAssignments, conflictingClauses) ← PROPAGATE(φ, model)
 3  conflictHasArisen ← conflictingClauses ≠ ∅
 4  if conflictHasArisen
 5     then return (UNSAT, NIL)
 6  model ← model ∪ inferredAssignments
 7  while True
 8  do (nextVariable, value) ← DECIDE(φ, model)
 9     allVariablesHaveAlreadyBeenAssigned ← nextVariable = NIL
10     if allVariablesHaveAlreadyBeenAssigned
11        then return (SAT, model)
12     model ← model ∪ {nextVariable ← value}
13     repeat
14           (inferredAssignments, conflictingClauses) ← PROPAGATE(φ, model)
15           model ← model ∪ inferredAssignments
16           conflictHasArisen ← conflictingClauses ≠ ∅
17           if conflictHasArisen
18              then (newModelWithoutConflict, φ_a) ← RESOLVE(φ, model, conflictingClauses)
19                   conflictHasNotBeenResolved ← newModelWithoutConflict = NIL
20                   if conflictHasNotBeenResolved
21                      then return (UNSAT, NIL)
22                   model ← newModelWithoutConflict
23                   φ ← φ ∧ φ_a
24        until ¬conflictHasArisen
```

The Algorithm 1 is quite generic. It is more of a template. One needs to plug into this template his own implementations of the following functions:

PROPAGATE This function calculates all assignments that follow from a given PL formula and a given model. It also returns a list of conflicting clauses (if there are any). These clauses will be in conflict if one extends the model by the returned assignments.

DECIDE This function applies some heuristic and selects a variable to be set next and a value.

RESOLVE This function resolves a conflict or returns NIL if it cannot be resolved.

In the Algorithm 1 a model is represented by a mapping from PL variables of the input PL formula to Booleans. It is a simplification. Real-world SAT solvers maintain a lot of

additional information such as decision levels, assignment order of variables for every decision level, an implicant for every variable (i.e. the clause from which the variable's value was inferred during the PROPAGATE) etc.

Additionally, a care should be taken when resolving a conflict and computing a new model. The solver should make sure it visits each state associated with a model once only.

2.2 Solving SAT Problem of Difference Logic

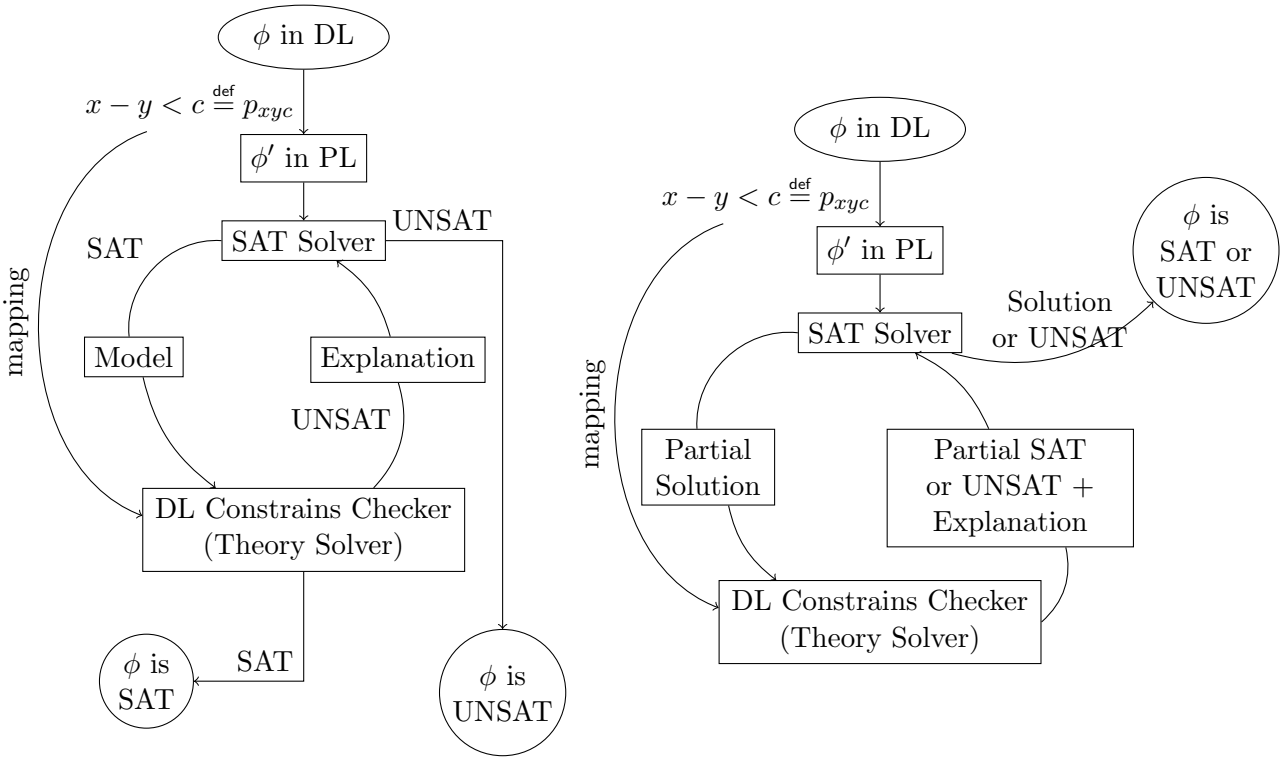

Figure 1: Illustration of the lazy (left) and incremental (right) approaches.

[1] mentions the following main approaches for solving the SAT problem of DL:

- Preprocessing approach. This approach suggests transforming a DL formula into an equivalent PL formula by encoding all intrinsic dependencies between DL constraints in PL. An example of such a dependency could be transitivity:

$$(x - y < a) \wedge (y - z < b) \rightarrow (x - z < a + b) \tag{4}$$

  After the transformation a SAT solver can be used to check SAT of the resulting equivalent PL formula. If the PL formula is SAT then the solution for the original DL formula can be constructed by the reverse transformation.

- Lazy approach (Figure 1 left). This approach suggests substituting each DL constraint $x - y < c$ with a Boolean variable $p_{xyc} \in \mathbb{B}$ thus yielding a PL formula ϕ'. ϕ' represents the "skeleton", the Boolean abstraction over the original DL formula ϕ. Then a SAT solver is used in tandem with a DL constraints checker (the theory solver) to solve the SAT problem. In this approach the SAT solver always computes a complete solution which is then passed to the theory solver.

- Incremental approach (Figure 1 right). This approach is very similar to the lazy one. However, instead of computing a complete solution, the SAT solver invokes the DL constraints checker each time it updates its model. The DL constraints checker should be able to maintain some internal state of the currently received DL constraints and update it incrementally (i.e. add new constraints, delete existing ones). Hence the name of the approach.

3 Topic

3.1 Notation

Throughout this Chapter the following notation is used:

- ϕ is a conjunction of DL constraints. It is being checked for SAT.
- $x - y \prec c$ is a general form of a DL constraint in ϕ where $\prec \in \{<, \leq\}$.
- $\mathbb{D}$ is a domain over which the variables and constants in ϕ are defined (e.g. $\mathbb{R}$).

3.2 Constraint Graph

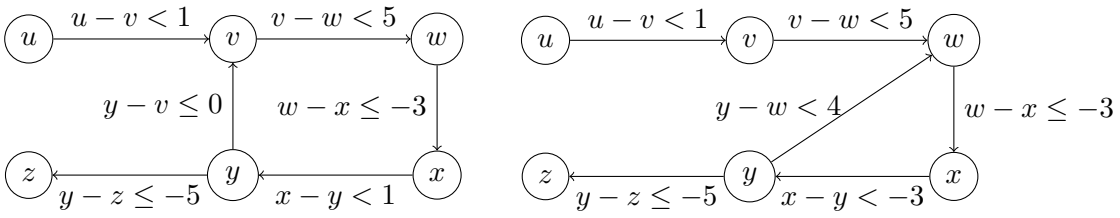

Figure 2: Examples of constraint graphs for Equation 2 (left) and Equation 3 (right).

Constraint graph (Figure 2) is a weighted directed graph which represents ϕ and which is used by a DL constraints checker (Figure 1) to test if ϕ is SAT. In [1] it is defined as follows:

Definition 3.1 (Constraint Graph) *The constraint graph is a graph* $\Gamma = (V, E, weight, op)$ *where:*

- V *is a set of vertices. Each vertex* $x \in V$ *corresponds to one numeric variable occurring in* $x - y \prec c$.
- E *is a set of directed edges. Each edge* $(x, y) \in E$ *corresponds to* $x - y \prec c$.
- $weight(x, y) : E \mapsto \mathbb{D}$ *is a weight function. It maps each edge* $(x, y) \in E$ *to the constant* $c \in \mathbb{D}$ *from the corresponding DL inequality* $x - y \prec c$.
- $op(x, y) : E \mapsto \{<, \leq\}$ *is a function which maps each edge* $(x, y) \in E$ *to the operation* $\prec$ *from the corresponding DL inequality* $x - y \prec c$.

3.3 Negative Cycles in Constraint Graph

There is a direct correspondence between a negative cycle in a constraint graph and SAT of ϕ represented by this graph.

A path in the graph corresponds to a sum of the corresponding constraints. E.g. the path $u \to v \to w \to x$ in the left graph on Figure 2 corresponds to the following sum of the DL inequalities:

$$\begin{array}{r} u - v < \quad 1 \\ v - w < \quad 5 \\ \underline{w - x \leq -3} \\ u - x < \quad 3 \end{array} \tag{5}$$

If at least one strict inequality is present then the resulting inequality will also be strict. This summation along a path can also be expressed with an inferred transitivity constraint (e.g. Equation 4). The transitivity constraint naturally follows from ϕ and therefore must be satisfied in order to satisfy ϕ.

A cycle in the constraint graph corresponds to an inequality $0 \prec c$ which may cause a conflict in the following situations:

- $c < 0$
- $c = 0$ and $\prec$ is $<$ (can be checked with *op* from Definition 3.1)

An example of a conflict can be seen on the right graph on Figure 2. The conflict corresponds to the negative cycle $x \to y \to w \to x$ which corresponds to the following conflicting inequalities:

$$\begin{array}{r} x - y < -3 \\ y - w < \quad 4 \\ \underline{w - x \leq -3} \\ 0 < -2 \end{array} \tag{6}$$

3.4 Bellman-Ford Algorithm for Constraint Graph

[1] uses a Goldberg-Radzik [6] variant of the Bellman-Ford algorithm [5, p.651] to detect negative cycles and thus check ϕ for SAT (Algorithm 2). [6] states that the algorithm has the same worst-case complexity $O(|V| \cdot |E|)$ as Bellman-Ford algorithm but is superior in practice. Terminology and notation used in the algorithm are given below.

Definition 3.2 (Source Vertex) *The source vertex $s \in V$ is a vertex from which the shortest paths to other vertices are computed.*

Definition 3.3 (Shortest Path Weight) *The shortest path weight [5, p.643] $\delta(v)$ from the source vertex to $v \in V$ is defined as the minimal sum of the weights of the edges of a path from the source vertex to v across all such paths (if at least one exist). If there is no path from the source vertex to v then $\delta(v) = \infty$. If there is a negative cycle on a path from the source vertex to v then $\delta(v) = -\infty$.*

Definition 3.4 (Distance Estimating Function [1]) *The distance estimating function $d(v) : V \mapsto \mathbb{D}$ is function which returns an upper bound on the distance from the source vertex to the given vertex $v \in V$.*

Definition 3.5 (Reduced Cost Function [6]) *The reduced cost function $r_d(x, y) : V \mapsto \mathbb{D}$ is defined as follows: $r_d(x, y) = weight(x, y) + d(x) - d(y)$.*

Definition 3.6 (Vertex Status) *The vertex status* $status(x) = \{unreached, labeled, scanned\}$ *is a function on vertices which shows a current state of a vertex* $x \in V$. $status(x) = unreached$ *means* x *has not been explored yet.* $status(x) = labeled$ *means* x *has been explored i.e. the distance estimate for it has been updated at least once and potentially it can be used to improve distance estimates to other vertices.* $status(x) = scanned$ *means* x *has been completely explored and will not be considered further for improving distance estimates.*

Definition 3.7 (Admissible Edge) *Edge* $(x, y) \in E$ *is called admissible if* $r_d(x, y) \leq 0$.

Definition 3.8 (Admissible Graph) *Admissible graph* Γ_d *is a subgraph of* Γ *composed of the admissible edges of* Γ.

In Algorithm 2 distance estimates are iteratively updated. In [5, p.648] this process is called "iterative edge relaxation". It can also be seen as a series of different distance estimate functions $(d_0, d_1, d_2, d_3, \dots)$. Each d_i in this series describes which distance estimates have the vertices at some iteration of the Algorithm 2.

The idea of the Algorithm 2 is to use the dynamically changing graph Γ_d (it changes whenever d changes) to detect negative or zero cycles in the original graph Γ. The following theorem from [1] expresses this idea more formally.

Theorem 3.1 *Given a constraint graph* Γ *and a series of distance estimating functions* $(d_0, d_1, d_2, d_3, \dots)$, Γ *has a negative or zero cycle if and only if* Γ_d *has a cycle under some distance estimate* d_k.

Proof 3.1 $\Rightarrow$ *Case 1.* Γ *has a negative cycle* $N = (x_0, x_1, \dots, x_{n-1}, x_n)$ *with* $x_n = x_0$. *To prove:* Γ_d *also has this cycle. In [5, pp.672-673] it is proven that if a graph has a negative cycle, then the distance estimation process* $(d_0, d_1, d_2, d_3, \dots)$ *will never converge. Therefore, for sufficiently large* k *there always will be some edge from* N *which can further update current* d_k *i.e.* $\exists i, 0 \leq i < n : d_k(x_i) + weight(x_i, w_{i+1}) < d_k(x_{i+1})$.

It can be the case that at some further iteration a path $p = s \to x_{i+1}$ *from the source vertex* s *to* x_{i+1} *will be discovered and this path may give even better improvement than the mentioned one for* d_k *i.e.* $len(p) < d_k(x_i) + weight(x_i, w_{i+1})$ *where* $len(p)$ *is the length of the path* p. *In this situation an edge from the path* p *will be applied to update* d *and therefore edge* (x_i, x_{i+1}) *will be excluded from* Γ_d *because in this case* $r_d(x_i, x_{i+1}) = weight(x_i, x_{i+1}) + d_m(x_i) - d_m(x_{i+1}) = weight(x_i, x_{i+1}) + d_k(x_i) - len(p) > 0$ *where* $m > k$ *is some further iteration in which the path* p *has been used to update distance estimate to the vertex* x_{i+1}, *i.e.* $d_m(x_{i+1}) = len(p)$, *and the distance estimate to the vertex* x_i *has not been updated, i.e.* $d_m(x_i) = d_k(x_i)$. *A simple example is given on Figure 3 on the left where in the very beginning a path* (s, x) *from* s *to* x *will be the shortest known path until the negative edge* (w, x) *is discovered and the path* (s, v, w, x) *will be shorter.*

Since a graph has a finite number of edges, the number of such paths, which connect s *and a vertex in* N, *is also finite. Therefore w.l.o.g. we can assume that the* k *is sufficiently large and all this paths have already been discovered and applied to improve* d. *If these paths are not contained in other negative cycles, they cannot be used in further iterations (see "convergence property" and "path relaxation property" in [5, p.650]).*

Suppose, however, that there is another negative cycle which has a connection with N. *An example is given on Figure 3 on the right where the negative cycle* (u, v, w, z, u) *has the common vertex* w *with another negative cycle* (w, x, y, w). *So this cycle can interfere with* N *and cause some of the edges of* N *to be excluded from* Γ_d *by providing a better update to the vertex* x_{i+1} *than the edge* (x_i, x_{i+1}). *However, in order to provide this update to*

x_{i+1} *algorithm needs to enter this cycle and complete it i.e. follow all its edges and apply updates. Therefore, again, w.l.o.g. we can regard this interfering negative cycle instead of* N*.*

Thus, w.l.o.g. let the regarded cycle N *be a cycle which does not interfere with other cycles and let* k *be sufficiently large such that all paths, which connect* s *with vertices of* N *and which are not contained in any negative cycles, have already been applied to improve* d *and therefore will not be applied in the further iterations. Then the edge* (x_i, x_{i+1})

Case 2. Γ *has no negative cycles but it has a zero weight cycle* $Z = (x_0, x_1, \ldots, x_{n-1}, x_n)$ *with* $x5_n = x_0$*. Then ...*

$\Leftarrow$ *fff*

Some notes to the Algorithm 2:

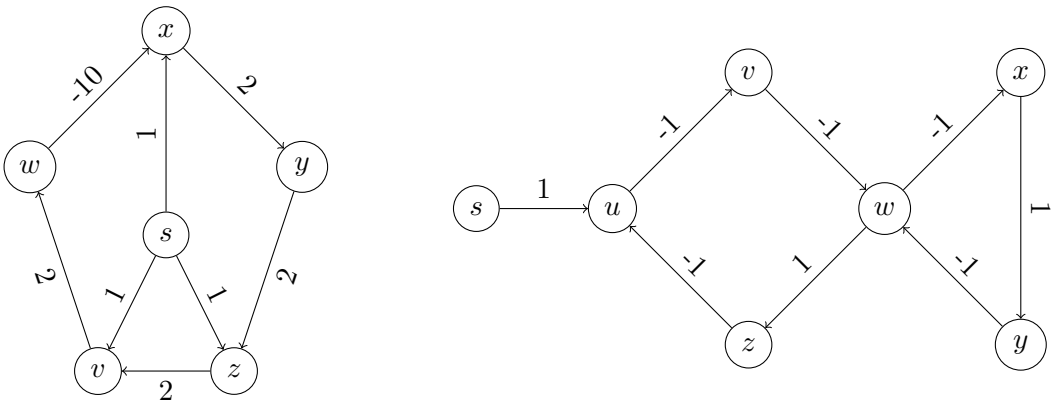

Figure 3: Example of negative cycles in graphs.

- In practice, checking Γ_d for a cycle (line 9) and computing a set of vertices reachable from B (line 15) can be done by only one DFS call. However, it is two logically distinct steps and therefore they are performed separately in the algorithm.
- Unexplored vertices (line 15) are vertices which have status *unreached* i.e. they have not been processed yet at that particular step.
- If B is empty or no unexplored vertices are reachable from it (line 15) then A will be assigned an empty set.

Algorithm 2 An algorithm for checking if ϕ which corresponds to the input constraint graph $\Gamma = (V, E, weight, op)$ is SAT. It returns SAT or UNSAT status and a set of DL constraints corresponding to a conflict (in case of UNSAT). It is based on Bellman-Ford algorithm [5, p.561]. Goldberg-Radzik heuristic [6], which is used here, suggests to scan a graph in a topological order. This algorithm uses depth first search [5, p.603] (DFS) and breadth first search [5, p.594] (BFS) for auxiliary tasks.

GOLDBERG-RADZIK(constraint graph $\Gamma = (V, E, weight, op)$, source vertex $s \in V$)

for each vertex $x \in V$
do $d(x) = \infty$
    $status(x) = unreached$
$d(s) = 0$
$status(s) = labeled$
$A \leftarrow \varnothing$
$B \leftarrow \{s\}$
repeat
        if Γ_d has a cycle C (DFS on Γ_d can be used to check it)
          then $l \leftarrow$ length of C in Γ
                if $l < 0$
                  then return ($UNSAT$, DL constraints corresponding to L)
                if $\exists\ (x, y) \in L$ such that $op(x, y) = <$
                  then return ($UNSAT$, DL constraints corresponding to L)
        for each vertex $x \in B$
        do if x has no outgoing admissible edges
              then $B \leftarrow B \setminus \{x\}$
                    $status(x) = scanned$
        $A \leftarrow$ set of unexplored vertices reachable from B in Γ_d (BFS on Γ_d can be used here)
        $A \leftarrow$ sort A topologically using Γ_d as an input graph (DFS on Γ_d can be used here)
        $B \leftarrow \varnothing$
        for each vertex $x \in A$
        do $status(x) = labeled$
            for each edge $(x, y) \in E$
            do if $d(x) + weight(x, y) < d(y)$
                  then $d(y) \leftarrow d(x) + weight(x, y)$
                        if $status(y) = unreached$
                          then $B \leftarrow B \cup \{y\}$
                        $status(y) \leftarrow labeled$
                        $status(x) \leftarrow scanned$
  until A *is empty*
return $(SAT, \varnothing)$

4 Conclusion

Conclusion on the topic ($\frac{1}{2}$ of a page).

References

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