# Introduction to Homotopy Theory

(Notes based on lectures by Sergiy Maksymenko)

#### **Abstract**

Homotopy theory studies spaces up to a homotopy, which is a continuous deformation of one continuous function to another. This documents is a work in progress done during a course audit. These notes are taken purposefully in English to strenghten intuition and simplify lookup of concepts in related literature.

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#### 1 Set-theoretic Definitions

**Definition** 1.1 (BINARY RELATION): A binary relation R on a set X is a set of ordered pairs of elements of X.

**Definition** 1.2 (Equivalence relation  $\sim$  is a binary relation that is reflexive, symmetric and transitive.

**Definition** 1.3 (EQUIVALENCE CLASS OF AN ELEMENT): Given a set X and an equivalence relation  $\sim$ , an equivalence class of  $a \in X$ , denoted [a] is a set  $\{x \in S \mid x \sim a\}$ 

**Definition** 1.4 (QUOTIENT SET): A quotient set  $X/\sim$  (also said "X modulo  $\sim$ ") is a set of all equivalence classes of X with respect to  $\sim$ .

$$X/\sim = \{[x] : x \in X\}$$

**Definition** 1.5 (QUOTIENT MAP): A quotient map is a surjective mapping that sends a point in X to its equivalence class, containing it:  $q: X \to X/\sim$ 

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### 2 Point-set Topology

**Definition** 2.1 (TOPOLOGICAL SPACE): A topological space is a pair  $\langle X, \tau \rangle$ , where X is a set and  $\tau$ , a topology on X, is a collection of subsets ( $\tau \subseteq \mathcal{P}(X)$ ) called open sets, such that:

- $\emptyset \in \tau$ .
- $X \in \tau$ .
- $\tau$  is closed under arbitrary finite intersections.
- $\tau$  is closed under arbitrary unions. Maybe find cute notation for this.

**Definition** 2.2 (TRIVIAL TOPOLOGY): A topological space is called trivial, when the topology on X consists only of  $\emptyset$  and X.

**Definition** 2.3 (DISCRETE TOPOLOGY): A topological space is called discrete, when  $\tau = \mathcal{P}(X)$ .

**Definition** 2.4 (CONTINUOUS MAP): Let  $\langle X, \tau \rangle$  and  $\langle Y, \sigma \rangle$  be topological spaces. A map  $f: X \to Y$  is **continuous** if:

$$\forall s \in \sigma, f^{-1}(s) \in \tau$$

In plain English, a map is continuous when a preimage of an open set in Y is an open set in X. C(X,Y) denotes a set of all continuous maps between X and Y.

Base of topology and methods of inducing topologies on sets were discussed during Lecture 2. Bonus. Compare topology to a field of sets to a  $\sigma$ -algebra to a Borel  $\sigma$ -algebra. Discussed during Lecture 5.

#### 2.1 Topology Restrictions

**Definition** 2.5 ( $T_1$  SPACE):

**Definition 2.6** (HAUSDORFF SPACE):

### 2.2 Quotient Topology

Gluing.

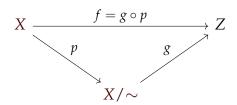
Let  $\langle X, \tau \rangle$  be a topological space and  $\sim$  be an equivalence relation on X.

**Definition 2.7** (QUOTIENT TOPOGICAL SPACE): By analogy of a set and given  $q: \langle X, \tau \rangle \to \langle X/\sim, \tau_{X/\sim} \rangle$ ,

$$X/\sim = \{[x] : x \in X\}$$
  
$$\tau_{X/\sim} = \{U \subseteq X/\sim \mid q^{-1}(U) \in \tau\}$$

 $\tau_{X/\sim}$  is constructed this way to ensure *q* is continuous.

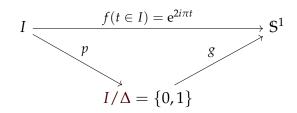
**Theorem** 2.8: Given Z is a topological space, g is a surjective map, p is a quotient map and the following diagram,



*g* is continuous  $\iff f = g \circ p$  is continuous.

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**Example 2.1**: Have  $z \in \mathbb{C}$ ,  $\mathbb{S}^1 = \{|z| = 1\}$ ,  $I = [0..1] \subset \mathbb{R}^1$ .



### 3 Homotopy

**Definition** 3.1 (HOMOTOPY): Two continuous maps  $f,g:X\to Y$  are homotopic if there is a map called homotopy  $H:X\times[0,1]\to Y$  that *continuously deforms* f to g, denoted  $f\simeq g$  or  $f\overset{H}\simeq g$ . In general:

$$X \times [0,1] \xrightarrow{H} Y$$

$$H(x,0) = f(x)$$

$$H(x,1) = g(x)$$

$$H(x,t) = tf(x) + (1-t)g(x)$$

**Example** 3.1:  $1 \stackrel{\chi^t}{\simeq} x$  when viewed as  $x^0$  and  $x^1$ .

$$X \times [0,1] \xrightarrow{H} Y$$

$$H(x,0) = x^{0}$$

$$H(x,1) = x^{1}$$

$$H(x,t) = x^{t}$$

Another possible homotopy between the same functions is  $t \cdot x + (1 - t) \cdot 1$ , which suggsts that there may be many more of them.

Plots?

**Example** 3.2:  $\{\cdot\} \times [0,1] \to \mathbb{C}$  with  $H(x,t) = e^{2\pi i t}$ 

#### 3.1 Homotopy and Equivalence

**Theorem** 3.2: Homotopy equivalence between spaces is an Equivalence relation.

**Theorem** 3.3: A homotopy between continuous maps is an Equivalence relation.

$$Proof.$$
 ...

#### 3.2 Contractible Spaces

**Definition** 3.4 (CONTRACTIBILITY): A space is contractible if it is homotopically equivalent to a point (a constant map).

**Definition 3.5 (PATH-CONNECTEDNESS):** 

**Theorem** 3.6: Any convex set is contractible.

Lemma 3.7: Contractibility does not depend on a choice of a point.

**Definition** 3.8 (STAR-CONVEX SET):

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## 4 Quotient Spaces and Maps

See lectures 2 and 3. Most of this stuff

### 4.1 Quotients and Groups

### 4.2 Cones of Topological Spaces

Brouwer's Fixed Point Theorem was mentioned around here. Bring up the context?.

### 5 Retractions and Deformation Retractions

## 6 Classes of Homotopy Maps

See lectures 4 and 5.

## **6.1** Mappings of $\mathbb{S}^1$ to Itself

#### 7 Resources

Course page: https://sites.google.com/site/kafedramatematikikau/products-services/
homotopy-theory

### References

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