

# Introduction to Homotopy Theory

(Notes based on lectures by Sergiy Maksymenko)

## Abstract

Homotopy theory studies spaces up to a homotopy, which is a continuous deformation of one continuous function to another. This document is a work in progress done during a course audit. These notes are taken purposefully in English to strengthen intuition and simplify lookup of concepts in related literature.

Warning: this document may be edited live during audit so watch out for incorrect statements!

## Contents

<b>1</b>	<b>Set-theoretic Definitions</b>	<b>1</b>
<b>2</b>	<b>Group-theoretic Definitions</b>	<b>2</b>
<b>3</b>	<b>Geometry Definitions</b>	<b>2</b>
<b>4</b>	<b>Point-set Topology</b>	<b>3</b>
4.1	Topology Restrictions . . . . .	4
4.2	Quotient Topology . . . . .	4
<b>5</b>	<b>Homotopies Between Continuous Maps</b>	<b>5</b>
5.1	Contractible Spaces . . . . .	5
<b>6</b>	<b>Quotient Spaces and Maps</b>	<b>6</b>
6.1	Quotients and Groups . . . . .	6
6.2	Cones of Topological Spaces . . . . .	6
<b>7</b>	<b>Retractions, Deformations and Deformation Retractions</b>	<b>6</b>
<b>8</b>	<b>Classes of Homotopy Maps</b>	<b>6</b>
8.1	Mappings of $S^1$ to Itself . . . . .	6
<b>9</b>	<b>Homotopy Types</b>	<b>6</b>
<b>10</b>	<b>Homeomorphisms</b>	<b>7</b>
<b>11</b>	<b>Glueing and Complexes</b>	<b>8</b>
<b>12</b>	<b>Homotopy Groups</b>	<b>9</b>
12.1	Homotopy Groups of Spheres . . . . .	10
12.2	Computing the Fundamental Group . . . . .	10
12.2.1	Computing Fundamental Groups of Surfaces . . . . .	10
<b>13</b>	<b>Resources</b>	<b>10</b>

## 1 Set-theoretic Definitions

**Definition 1.1 (BINARY RELATION):** A binary relation  $R$  on a set  $X$  is a set of ordered pairs of elements of  $X$ .

**Definition 1.2 (EQUIVALENCE RELATION):** An equivalence relation  $\sim$  is a binary relation that is reflexive, symmetric and transitive.

**Definition 1.3 (EQUIVALENCE CLASS OF AN ELEMENT):** Given a set  $X$  and an equivalence relation  $\sim$ , an equivalence class of  $a \in X$ , denoted  $[a]$  is a set  $\{x \in X \mid x \sim a\}$

**Definition 1.4 (QUOTIENT SET):** A quotient set  $X/\sim$  (also said “ $X$  modulo  $\sim$ ” or “ $X$  up to  $\sim$ ”) is a set of all **equivalence classes** of  $X$  with respect to  $\sim$ .

$$X/\sim = \{[x] : x \in X\}$$

**Definition 1.5 (QUOTIENT MAP, PROJECTION):** A surjective mapping that sends a point in  $X$  to its equivalence class, containing it.

$$\pi : X \rightarrow X/\sim$$

**Definition 1.6 (QUOTIENT BY SUBSET MEMBERSHIP):** Given a set  $X$  and subset  $A \subset X$ ,  $X/A$  is a notation that assumes an implicit equivalence relation:

$$\forall \alpha, \beta \in X, \alpha \sim \beta = \begin{cases} \top, & \alpha \in A \wedge \beta \in A \\ \perp, & \text{otherwise} \end{cases}$$

For example:

$$\begin{aligned} X &= \{a, b, c, d, e\} \\ A &= \{a, e\} \subset X \\ X/A &= \{b, c, d, [a]\} \\ [a] &= \{a, e\} \end{aligned}$$

One can say *identification of all points in  $A$  with each other*.

## 2 Group-theoretic Definitions

**Definition 2.1 (PRESENTATION):**

$$\langle S \mid R \rangle$$

Where  $S$  is a generator and  $R$  is a relation.

**Definition 2.2 (COMMUTATOR):** That one for groups: <https://en.wikipedia.org/wiki/Commutator>

**Definition 2.3 (KERNEL (GROUP)):** Analog of a null space of a linear map (turns out it’s also often called a kernel in linear algebra)

**Example 2.4:**

$$\begin{aligned} \mathbb{Z}_5 &= 0, 1, 2, 3, 4 \pmod{5} \\ \{e, a, a^2, a^3, a^4\} &= \langle a \mid a^5 = e \rangle \end{aligned}$$

**Definition 2.5 (FREE GROUP):** Syntax for groups!  $F_2$  if the free group has two generators.

**Definition 2.6 (FREE GROUP BY UNIVERSAL PROPERTY):** See [https://en.wikipedia.org/wiki/Free\\_group#Universal\\_property](https://en.wikipedia.org/wiki/Free_group#Universal_property).

**Example 2.7:**  $F_1 = \mathbb{Z}$  (free cyclic group)

**Theorem 2.8:** Let group  $G$  be generated  $g_1 \dots g_k$ , so every  $h \in G$  is a product of some of its generators.

$$F_k \xrightarrow{\phi} G$$

$$G \cong F_k / \ker \phi$$

$\ker \phi$  is a list of things on the right.

**Example 2.9:** Dihedral group  $\mathcal{D}_5$  — symmetries of the pentagon.

$r$  rotates the pentagon,  $s$  flips it along the vertical axis.

$$\mathcal{D}_5 = \langle r, s \mid r^5 = 1, s^2 = 1, rs = sr^{-1} \rangle$$

See also: [https://en.wikipedia.org/wiki/Dihedral\\_group#Other\\_definitions](https://en.wikipedia.org/wiki/Dihedral_group#Other_definitions)

**Definition 2.10** (FREE PRODUCT WITH AMALGAMATION):

### 3 Geometry Definitions

Most of these constructions are the same **up to** homotopy.

**Definition 3.1** (N-SPHERE, HYPERSPHERE):  $\mathcal{S}^n$ . Generalizes a unit circle ( $\mathcal{S}^1$  — a circle in  $\mathbb{R}^2$ ) and a unit sphere ( $\mathcal{S}^2$  — a sphere in  $\mathbb{R}^3$ ). Given  $\|\cdot\|$  is a norm,

$$\mathcal{S}^n = \{x \mid \|x \in \mathbb{R}^{n+1}\| = 1\}$$

A unit 0-sphere is a pair of endpoints of an interval  $[-1, 1]$  of the real line.

**Definition 3.2** (N-DISK, N-BALL):  $\mathcal{D}^n$ . A disk (a closed ball) is a region contained inside  $\mathcal{S}^n$ .  $\mathcal{D}^1$  is a line segment (in  $\mathbb{R}^1$ ).

$$\mathcal{D}^n = \{x \mid \|x \in \mathbb{R}^n\| \leq 1\}$$

A **ball** usually refers to a disk minus its boundary, an **open ball**, which can be defined using  $<$  instead of  $\leq$ .

$$\mathcal{B}_r^n = \{x \mid \|x \in \mathbb{R}^n\| < r\}$$

**Definition 3.3** (SPHERE MODULO DISK):

$$\mathcal{S}^n = \mathcal{D}^n / \mathcal{S}^{n-1} = \mathcal{D}^n / \partial \mathcal{D}^n$$

For example, identifying the ends (a boundary, which in this case is a zero-dimensional sphere) of a line segment ( $\mathcal{D}^1$ ) as one point produces a circle ( $\mathcal{S}^1$ ) — an object embeddable in  $\mathbb{R}^2$ :

$$\mathcal{S}^1 = \mathcal{D}^1 / \mathcal{S}^0$$

For a 2-sphere, imagine making a dumpling from a disk-shaped piece of dough. It's not exactly a sphere, but topologically we don't care.

$$\mathcal{S}^2 = \mathcal{D}^2 / \mathcal{S}^1$$

**Definition 3.4** (TORUS):  $\mathcal{T}^n$ .

A 2-torus is a product of two circles. You can also build it by glueing two opposite sides of a square. Remember that a square is homeomorphic to a  $\mathcal{D}^2$ . Let  $B$  be a thickened bouquet  $\mathcal{S}^1 \vee \mathcal{S}^1$  (so  $\mathcal{S}^1 \vee \mathcal{S}^1$  is a **Deformation Retraction** of  $B$ ).

$$\mathcal{T}^2 = \mathcal{S}^1 \times \mathcal{S}^1 = \mathcal{D}^2 \bigcup_{f: \partial \mathcal{D}^2 \rightarrow \mathcal{S}^1 \vee \mathcal{S}^1} B$$

**Definition 3.5 (DOUBLE TORUS):** A donut with two holes. Also known as a **genus**-two surface. Can be constructed by folding a hexagon.

## 4 Point-set Topology

**Definition 4.1 (UNIT INTERVAL):** A unit interval (or just an interval)  $I$  is a subset  $[0..1] \subset \mathbb{R}$  of a real line, which is a popular notation in further homotopy buildup.

**Definition 4.2 (A GRAPH OF A FUNCTION):**  $\Gamma(f)$  - a subset of  $\mathbb{R} \times \mathbb{R}$

**Definition 4.3 (TOPOLOGICAL SPACE):** A topological space is a pair  $\langle X, \tau \rangle$ , where  $X$  is a set and  $\tau$ , a topology on  $X$ , is a collection of subsets ( $\tau \subseteq \mathcal{P}(X)$ ) called open sets, such that:

- $\emptyset \in \tau$ .
- $X \in \tau$ .
- $\tau$  is closed under arbitrary finite intersections.
- $\tau$  is closed under arbitrary unions. **Maybe find cute notation for this.**

Posets.

**Definition 4.4 (TRIVIAL TOPOLOGY):** A topological space is called trivial, when the topology on  $X$  consists only of  $\emptyset$  and  $X$ .

**Definition 4.5 (DISCRETE TOPOLOGY):** A topological space is called discrete, when  $\tau = \mathcal{P}(X)$ .

**Definition 4.6 (CONTINUOUS MAP):** Let  $\langle X, \tau \rangle$  and  $\langle Y, \sigma \rangle$  be topological spaces. A map  $f : X \rightarrow Y$  is **continuous** if:

$$\forall s \in \sigma, f^{-1}(s) \in \tau$$

In plain English, a map is continuous when a preimage of an open set in  $Y$  is an open set in  $X$ .

$C(X, Y)$  denotes a set of all continuous maps between  $X$  and  $Y$ .

**Definition 4.7 (HOMEOMORPHISM):** A mapping  $f : X \rightarrow Y$  is a homeomorphism if  $\exists g : Y \rightarrow X$  s.t.  $X \xrightarrow{f} Y \xrightarrow{g} X \xrightarrow{f} Y$ , when  $g \circ f = id_X$  and  $f \circ g = id_Y$ . It is a continuous map that has a continuous inverse.

**Example 4.8:** *How to see that a map  $h : [0..1] \rightarrow [0..1]$  is a homeomorphism?*

- At least it needs to be monotonous.
- Preserves boundaries.

$$H([0..1], 0, 1), ([0..1], 0, 1)_{\text{increasing}} \cup H([0..1], 0, 1), ([0..1], 1, 0)_{\text{decreasing}}$$

Base of topology and methods of inducing topologies on sets were discussed during Lecture 2.

Bonus. Compare topology to a field of sets to a  $\sigma$ -algebra to a Borel  $\sigma$ -algebra. Discussed during Lecture 5.

### 4.1 Topology Restrictions

**Definition 4.9 ( $T_1$  SPACE):**

**Definition 4.10 (HAUSDORFF SPACE):**

## 4.2 Quotient Topology

**Gluing.**

Let  $\langle X, \tau \rangle$  be a topological space and  $\sim$  be an equivalence relation on  $X$ .

**Definition 4.11** (QUOTIENT TOPOLOGICAL SPACE): By analogy of a set and given  $q : \langle X, \tau \rangle \rightarrow \langle X/\sim, \tau_{X/\sim} \rangle$ ,

$$\begin{aligned} X/\sim &= \{[x] : x \in X\} \\ \tau_{X/\sim} &= \{U \subseteq X/\sim \mid q^{-1}(U) \in \tau\} \end{aligned}$$

$\tau_{X/\sim}$  is constructed this way to ensure  $q$  is continuous.

**Theorem 4.12:** Given  $Z$  is a topological space,  $g$  is a surjective map,  $p$  is a quotient map and the following diagram,

$$\begin{array}{ccc} X & \xrightarrow{f = g \circ p} & Z \\ & \searrow p \quad \nearrow g & \\ & X/\sim & \end{array}$$

$g$  is continuous  $\iff f = g \circ p$  is continuous.

**Definition 4.13** (BOUQUET OF SPACES, WEDGE SUM): Glue two spaces by one point.  $A \vee B$ .

## 5 Homotopies Between Continuous Maps

**Definition 5.1** (HOMOTOPY): Two continuous maps  $f, g : X \rightarrow Y$  are homotopic if there is a map called homotopy  $H : X \times [0, 1] \rightarrow Y$  that *continuously deforms*  $f$  to  $g$ , denoted  $f \simeq g$  or  $f \stackrel{H}{\simeq} g$ . In general:

$$\begin{aligned} X \times [0, 1] &\stackrel{H}{\rightarrow} Y \\ H(x, 0) &= f(x) \\ H(x, 1) &= g(x) \\ H(x, t) &= tf(x) + (1 - t)g(x) \end{aligned}$$

**Example 5.2:**  $1 \stackrel{x^t}{\simeq} x$  when viewed as  $x^0$  and  $x^1$ .

$$\begin{aligned} X \times [0, 1] &\stackrel{H}{\rightarrow} Y \\ H(x, 0) &= x^0 \\ H(x, 1) &= x^1 \\ H(x, t) &= x^t \end{aligned}$$

Another possible homotopy between the same functions is  $t \cdot x + (1 - t) \cdot 1$ , which suggests that there may be many more of them.

**Plots?**

**Example 5.3:**  $\{\cdot\} \times [0, 1] \rightarrow \mathbb{C}$  with  $H(x, t) = e^{2\pi it}$

**Theorem 5.4:** A homotopy between continuous maps is an **Equivalence relation**.

*Proof.* ...

□

## 5.1 Contractible Spaces

**Definition 5.5 (CONTRACTIBILITY):** A space is contractible if it is homotopically equivalent to a point (a constant map).

Extra definitions:

- $X \times 0$  is a retraction of  $CX$  (cone over  $X$ )
- homotopic equivalence to a point

**Example 5.6:**  $\mathbb{R}^n$  is contractible to a point.

$$\begin{aligned}\mathbb{R}^n \times [0, 1] &\xrightarrow{H} \mathbb{R}^n \\ H(x, 0) &= 0 \\ H(x, 1) &= x \\ H(x, t) &= tx\end{aligned}$$

**Definition 5.7 (PATH-CONNECTEDNESS):**

**Theorem 5.8:** Any convex set is contractible.

**Lemma 5.9:** Contractibility does not depend on a choice of a point.

**Definition 5.10 (STAR-CONVEX SET):**

**Theorem 5.11:** A star-convex set is contractible to a point.

$$\begin{aligned}A \subset \mathbb{R}^n &\text{ is (star-)convex} \\ a &\in A \\ A \times [0, 1] &\xrightarrow{H} A \\ H(x, 0) &= 0 \\ H(x, 1) &= x \\ H(x, t) &= at + x(1 - t)\end{aligned}$$

**Theorem 5.12:** Assuming we know how to build topologies on trees (as in graph theory trees), every finite tree is a contractible topological space.

## 6 Quotient Spaces and Maps

See lectures 2 and 3. Most of this stuff

### 6.1 Quotients and Groups

### 6.2 Cones of Topological Spaces

Brouwer's Fixed Point Theorem was mentioned around here. Bring up the context?.

## 7 Retractions, Deformations and Deformation Retractions

**Definition 7.1 (RETRACTION):**

Bring up the example with  $\frac{x}{|x|}$  – discussed along with retractions

**Definition 7.2 (DEFORMATION):** A continuous mapping is a deformation into a subspace  $A \subset X$  when

- $H_0 = id_X$

- $\forall t \in [0..1], H_t(A) \subset A$

**Definition 7.3** (DEFORMATION RETRACTION):

## 8 Classes of Homotopy Maps

See lectures 4 and 5.

### 8.1 Mappings of $S^1$ to Itself

$$\pi_1 S^1 = [(S^1, \cdot), (S^1, \cdot)]$$

## 9 Homotopy Types

Lecture 7.

Let  $\langle X, \tau \rangle \langle Y, \sigma \rangle$ .

A Homotopy Type is a synonym to homotopy equivalences between spaces. It is **not** the homotopy type as in Homotopy Type Theory.

**Claim 9.1:** Homeomorphisms and topological spaces form a category.

**Claim 9.2:** Homeomorphisms and topological spaces form a group (for all morphisms with one side fixed).

$\text{Homeo}(X)$  group of all identity maps of  $X$

$g$  is a retraction of  $Y$  to  $X$  on  $g(x)$ ,  $g$  is also left inverse of  $f$

$$g \circ f \simeq id_X$$

**Definition 9.3** (HOMOTOPIC EQUIVALENCE): A mapping  $f : X \rightarrow Y$  is a homotopic equivalence when  $\exists g : Y \rightarrow X$  such that  $g \circ f \simeq id_X$  and  $f \circ g \simeq id_Y$

.

**Theorem 9.4:** Homotopy equivalence between spaces is an **Equivalence relation**.

*Proof.* ...

□

**Example 9.5:**  $f : X \rightarrow \cdot$  is a homotopic equivalence to a point  $\iff X$  is contractible.

$$H_0 = id \simeq H_1 = f$$

**Claim 9.6:** Holes matter!  $\infty \simeq B \simeq o.O$ , look at this graphically, not symbolically.

**Example 9.7:** Glueing of a disk to some space using quotienting. *expanding/reducing?*

**Claim 9.8:** Homotopy Theory cares only about spaces glued from a finite number of  $S^n$  spaces, see Shape Theory.

**Definition 9.9** (SIMPLE HOMOTOPY TYPE): A homotopy type is simple if you use a finite number of expansions and reductions

**Claim 9.10:** This is a cue to working with simplicial complexes.

## ...Interlude ...

$f : X \rightarrow Y$ ,  $f$  is a proper mapping,  $f^{-1}(\text{compact})$  is a compact

**Define compact**

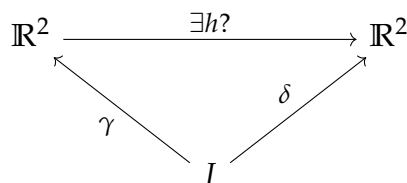
Defining proper homotopy equivalences.

$\mathbb{R}^n \rightarrow \cdot$  is an improper homotopy. Compactness is a bitch.

## 10 Homeomorphisms

A fact about a group of homeomorphisms of an interval: homotopy  $h_t = tf + (1-t)g$  monotonicity  $\iff H^+, H^-$  are convex subset in a vector space  $C(I, I)$  of all continuous functions. *Prove it.*

**Example 10.1:** Given curves  $\gamma$  and  $\delta$  that each enclose some surface. Is there a homeomorphism  $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  s.t.  $h(\gamma) = \delta$ ?



**Example 10.2:** Given  $f(x) = x^2$ . Build a homeomorphism  $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  s.t.  $h(y=0) = \Gamma(f)$ .

$$h(x, y) \mapsto (x, x^2 + y)$$

$$h^{-1}(x, y) \mapsto (x, -x^2 + y)$$

$$h^{-1} \circ h = id_{\mathbb{R}^2}$$

Generalizing,

$$h(x, y) \mapsto (x, f(x) + y)$$

$$h^{-1}(x, y) \mapsto (x, -f(x) + y)$$

$$h^{-1} \circ h = id_{\mathbb{R}^2}$$

**Example 10.3:** Given a graph of a continuous function  $f : [a, b] \rightarrow \mathbb{R}$ ,  $f(a) = f(b) = 0$ ,  $m = \min f(x)$ ,  $M = \max f(x)$ ,  $\epsilon > 0$  that lies within a known rectangle  $K = [a, b] \times [m - \epsilon, M + \epsilon]$ .

Prove,  $\exists h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  ( $h$  is a homeomorphism), s.t.

- $h([a, b] \times 0) = \Gamma(f)$
- $h(z) = z \forall z \in \mathbb{R}^2 \setminus K$ .

This is how we get to locally deform curves on a plane.

**Definition 10.4 (ISOTOPY):** A homotopy  $H : X \times [0, 1] \rightarrow X$  s.t.  $\forall t, H_t$  is a homeomorphism is an isotopy.

**Lemma 10.5 (ALEXANDER'S THEOREM):** Any homeomorphism of a unit interval (or a disk, or any convex set) which is fixed on its boundary  $h/\partial D^2 = id_{\partial D^2}$  is isotopic to  $id_I$  with an isotopy that is also fixed on its boundary.

E.g. there is a homeomorphism that maps a rendering of a letter A inside a disk to rendering of a letter R.

*Proof.* ALEXANDER'S TRICK.

$H : D^2 \times [0, 1] \rightarrow D^2$  is an isotopy.

□

Important result: if you have two continuous deformations of a unit disk, you can have a third continuous deformations between them. It is possible to use differentiability.

See also, Cerf's theorem, [https://en.wikipedia.org/wiki/Cerf\\_theory](https://en.wikipedia.org/wiki/Cerf_theory) and Smale's theorem (don't confuse with Poincare conjecture).



## 11 Glueing and Complexes

We were building up Homotopy Theory definitions to work with CW complexes.

**Definition 11.1 (CW COMPLEX):** Also called a **cell complex**. A space, created by glueing simple spaces together.

Quotienting is the workhorse of glueing. Use quotient topology to set up topologies after glueing two spaces.

**Example 11.2:** Identifying all points from a boundary of a disk (set  $X$ ) and a single point from set  $Y$  as a single equivalence class.

**Definition 11.3 (GLUEING):** Notation:  $X \bigcup_f A$

Big idea: geometric objects have combinatorial structure, there is a lot of machinery that lets you “dumb down” to those structures to simplify working with objects equivalent **up to** a homotopy.

Foreshadowing: [Whitehead theorem](#).

**Example 11.4:** Let’s build a cube. Its vertices are 8 zero-dimensional disks:  $\mathcal{D}^0$ , edges are 1-dimensional disks, faces are 2-dimensional disks.

(NB spaces like  $K^1$  are enough to model graphs in topology)

$$\begin{aligned} K^0 &= \bigsqcup \mathcal{D}_i^0 \\ K^1 &= \left( \bigsqcup \mathcal{D}_i^1 \right) \bigcup_{\phi_i} K^0 \\ K^2 &= \left( \bigsqcup \mathcal{D}_i^2 \right) \bigcup_{\psi_i} K^1 \end{aligned}$$

$K^i$  is an  $i$ -th skeleton. *Interiors* of disks are called **cells**.

**Claim 11.5:** An integer line is a cell complex.

**Definition 11.6 (CW COMPLEX):**

- C is for closure-finite (closure of each cell is contained in a union of finitely many cells)
- W is for weak topology (roughly, every  $K^i$  has a quotient topology)

## 12 Homotopy Groups

Notation abuse: there’s actually a difference between  $\circ$  (map composition) and  $\star$  (group operation, which is denoted as  $\circ$  instead!)

**Definition 12.1 (FUNDAMENTAL GROUP):** [https://en.wikipedia.org/wiki/Homotopy\\_group](https://en.wikipedia.org/wiki/Homotopy_group)

$$\pi_1(X, x_0) = [(\mathcal{S}^1, \cdot), (X, x_0)] = [(I, \partial I), (X, x_0)]$$

where  $[A, B]$  are homotopy classes.

The group multiplication is defined as homotopy composition for **spaces equivalent up to a homotopy**:

$$(\alpha \circ \beta)(t) = \alpha(2t) \text{ for } t \in [0, \frac{1}{2}], \beta(2t - 1) \text{ for } t \in [\frac{1}{2}, 1]$$

Neutral element suggestions: constant mapping doesn’t work.

Let’s use deformations up to a homotopy:

$$[\alpha] \circ [\beta] = [\alpha \circ \beta]$$

$$[\alpha] = [\alpha'] \implies \alpha \simeq \alpha'$$

Now prove it's actually a group

**Theorem 12.2:**

$$[\alpha] \circ [\epsilon] = [\epsilon] \circ [\alpha] = [\alpha]$$

*Proof.* In other words,  $\alpha \circ \epsilon \simeq \alpha$  □

Big idea: we're handling an equivalence class by introducing an extra parameter for a class deformation, and defining operations as parametrized homotopies. Lesson: paying for definitions is hard.

...

**Theorem 12.3:** This has been discussed earlier

$$\pi_1(\mathcal{S}^1) \cong \mathbb{Z}$$

## 12.1 Homotopy Groups of Spheres

The  $i$ -th homotopy group  $\pi_i(\mathcal{S}^n)$  summarizes the different ways in which the  $i$ -dimensional sphere  $\mathcal{S}^i$  can be mapped continuously (in homotopical sense, i.e. maps up to a homotopy) into the  $n$ -dimensional sphere  $\mathcal{S}^n$ .

## 12.2 Computing the Fundamental Group

Problems: given a space, compute its group. Given a group, compute its space.

Two instruments: Cover (a special case of a fibration) and van Kampen's Theorem

**Theorem 12.4:** van Kampen

[https://en.wikipedia.org/wiki/Seifert-van\\_Kampen\\_theorem](https://en.wikipedia.org/wiki/Seifert-van_Kampen_theorem) we were discussing the combinatorial definition

Big idea: express the structure of the fundamental group through the fundamental groups of two open, path-connected subspaces  $A$  and  $B$  that cover  $X$ .

Let  $X$  is a path connected space, and  $A, B$  are its subspaces such that  $X = A \cup B$  and  $A \cap B$  are path connected. ( $\cup$  is generalizable by a product,  $\times$ )

$$\pi_1(A \times B) = \pi_1 A \times \pi_1 B$$

where  $\times$  is an amalgamated free product.

Categorically: the fundamental group is a terminal object in some category.

**Example 12.5:**

$$\pi_1(\mathcal{S}^1 \vee \mathcal{S}^1) = \mathbb{Z} \times \mathbb{Z} = \mathcal{F}_2$$

**Example 12.6:** A 2-disk is path connected.

$$\pi_1(\mathcal{D}^2) = 0$$

**Example 12.7:**

$$\pi_1(\mathcal{S}^2) = 0$$

$A$  and  $B$  are both  $\mathcal{D}^2$ .

### 12.2.1 Computing Fundamental Groups of Surfaces

**Example 12.8:** Fundamental group of  $\mathcal{T}^2$

This is solvable if you look at the torus as a CW complex— note there are multiple ways to compute a complex for a torus. For example, try using a diagonal of a square.

$$\pi_1 \mathcal{T}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle = \mathbb{Z} \oplus \mathbb{Z}$$

## 13 Resources

Course page: <https://sites.google.com/site/kafedramatematikau/products-services/homotopy-theory>