Introduction to Homotopy Theory

(Notes based on lectures by Sergiy Maksymenko)

Abstract

Homotopy theory studies spaces up to a homotopy, which is a continuous deformation of one continuous function to another. This documents is a work in progress done during a course audit. These notes are taken purposefully in English to strenghten intuition and simplify lookup of concepts in related literature.

Warning: this document may be edited live during audit so watch out for incorrect statements!

Contents

1 Set-theoretic Definitions

Definition 1.1 (BINARY RELATION): A binary relation R on a set X is a set of ordered pairs of elements of X.

Definition 1.2 (EQUIVALENCE RELATION): An equivalence relation $\sim$ is a binary relation that is reflexive, symmetric and transitive.

Definition 1.3 (EQUIVALENCE CLASS OF AN ELEMENT): Given a set X and an equivalence relation $\sim$, an equivalence class of $a \in X$, denoted $[a]$ is a set $\{x \in S \mid x \sim a\}$

Definition 1.4 (QUOTIENT SET): A quotient set $X/\sim$ (also said "X *modulo* $\sim$") is a set of all equivalence classes of X with respect to $\sim$.

$$X/\sim = \{[x] : x \in X\}$$

Definition 1.5 (QUOTIENT MAP): A quotient map is a surjective mapping that sends a point in X to its equivalence class, containing it: $q : X \to X/\sim$

Definition 1.6 (QUOTIENT BY SET MEMBERSHIP): **TODO**. X/A

2 Point-set Topology

Definition 2.1 (TOPOLOGICAL SPACE): A topological space is a pair $\langle X, \tau \rangle$, where X is a set and τ, a topology on X, is a collection of subsets ($\tau \subseteq \mathcal{P}(X)$) called open sets, such that:

- $\varnothing \in \tau$.
- $X \in \tau$.
- τ is closed under arbitrary finite intersections.
- τ is closed under arbitrary unions. **Maybe find cute notation for this.**

Posets.

Definition 2.2 (TRIVIAL TOPOLOGY): A topological space is called trivial, when the topology on X consists only of $\varnothing$ and X.

Definition 2.3 (DISCRETE TOPOLOGY): A topological space is called discrete, when $\tau = \mathcal{P}(X)$.

Definition 2.4 (CONTINUOUS MAP): Let $\langle X, \tau \rangle$ and $\langle Y, \sigma \rangle$ be topological spaces. A map $f : X \to Y$ is **continuous** if:

$$\forall s \in \sigma, f^{-1}(s) \in \tau$$

In plain English, a map is continuous when a preimage of an open set in Y is an open set in X.

$C(X, Y)$ denotes a set of all continuous maps between X and Y.

Base of topology and methods of inducing topologies on sets were discussed during Lecture 2.

Bonus. Compare topology to a field of sets to a σ-algebra to a Borel σ-algebra. Discussed during Lecture 5.

2.1 Topology Restrictions

Definition 2.5 (T_1 SPACE):

Definition 2.6 (HAUSDORFF SPACE):

2.2 Quotient Topology

Gluing.

Let $\langle X, \tau \rangle$ be a topological space and $\sim$ be an equivalence relation on X.

Definition 2.7 (QUOTIENT TOPOGICAL SPACE): By analogy of a set and given $q : \langle X, \tau \rangle \to \langle X/{\sim}, \tau_{X/\sim} \rangle$,

$$\begin{aligned} X/{\sim} &= \{[x] : x \in X\} \\ \tau_{X/\sim} &= \{U \subseteq X/{\sim} \mid q^{-1}(U) \in \tau\} \end{aligned}$$

$\tau_{X/\sim}$ is constructed this way to ensure q is continuous.

Theorem 2.8: Given Z is a topological space, g is a surjective map, p is a quotient map and the following diagram,

$$\begin{tikzcd} X \arrow[rr, "f = g \circ p"] \arrow[dr, "p"'] & & Z \\ & X/{\sim} \arrow[ur, "g"'] & \end{tikzcd}$$

g is continuous $\iff$ $f = g \circ p$ is continuous.

Example 2.9: Have $z \in \mathbb{C}$, $\mathbb{S}^1 = \{|z| = 1\}$, $I = [0..1] \subset \mathbb{R}^1$.

$$\begin{tikzcd} I \arrow[rr, "f(t \in I) = e^{2i\pi t}"] \arrow[dr, "p"'] & & \mathbb{S}^1 \\ & I/\Delta = \{0, 1\} \arrow[ur, "g"'] & \end{tikzcd}$$

3 Homotopies Between Continuous Maps

Definition 3.1 (HOMOTOPY): Two continuous maps $f, g : X \to Y$ are homotopic if there is a map called homotopy $H : X \times [0,1] \to Y$ that *continuously deforms* f to g, denoted $f \simeq g$ or $f \overset{H}{\simeq} g$. In general:

$$\begin{aligned} X \times [0,1] &\xrightarrow{H} Y \\ H(x,0) &= f(x) \\ H(x,1) &= g(x) \\ H(x,t) &= tf(x) + (1-t)g(x) \end{aligned}$$

Example 3.2: $1 \overset{x^t}{\simeq} x$ when viewed as x^0 and x^1.

$$\begin{aligned} X \times [0,1] &\xrightarrow{H} Y \\ H(x,0) &= x^0 \\ H(x,1) &= x^1 \\ H(x,t) &= x^t \end{aligned}$$

Another possible homotopy between the same functions is $t \cdot x + (1-t) \cdot 1$, which suggsts that there may be many more of them.

Plots?

Example 3.3: $\{\cdot\} \times [0,1] \to \mathbb{C}$ with $H(x,t) = e^{2\pi i t}$

Theorem 3.4: A homotopy between continuous maps is an **??**.

Proof. … □

3.1 Contractible Spaces

Definition 3.5 (CONTRACTIBILITY): A space is contractible if it is homotopically equivalent to a point (a constant map).

Extra definitions:

- $X \times 0$ **is a retraction of CX (cone over X)**
- **homotopic equivalence to a point**

Example 3.6: $\mathbb{R}^n$ is contractible to a point.

$$\begin{aligned} \mathbb{R}^n \times [0,1] &\xrightarrow{H} \mathbb{R}^n \\ H(x,0) &= 0 \\ H(x,1) &= x \\ H(x,t) &= tx \end{aligned}$$

Definition 3.7 (PATH-CONNECTEDNESS):

Theorem 3.8: Any convex set is contractible.

Lemma 3.9: Contractibility does not depend on a choice of a point.

Definition 3.10 (STAR-CONVEX SET):

Theorem 3.11: A star-convex set is contractible to a point.

$$A \subset \mathbb{R}^n \text{ is (star-)convex}$$
$$a \in A$$
$$A \times [0,1] \xrightarrow{H} A$$
$$H(x,0) = 0$$
$$H(x,1) = x$$
$$H(x,t) = at + x(1-t)$$

Theorem 3.12: Assuming we know how to build topologies on trees (as in graph theory trees), every finite tree is a contractible topological space.

4 Quotient Spaces and Maps

See lectures 2 and 3. Most of this stuff

4.1 Quotients and Groups

4.2 Cones of Topological Spaces

Brouwer's Fixed Point Theorem was mentioned around here. Bring up the context?.

5 Retractions, Deformations and Deformation Retractions

Definition 5.1 (RETRACTION):

Bring up the example with $\frac{x}{|x|} - discussedalongwithretractions$

Definition 5.2 (DEFORMATION): A continuous mapping is a deformation into a subspace $A \subset X$ when

- $H_0 = id_X$
- $\forall t \in [0..1], H_t(A) \subset A$

Definition 5.3 (DEFORMATION RETRACTION):

6 Classes of Homotopy Maps

See lectures 4 and 5.

6.1 Mappings of $\mathbb{S}^1$ to Itself

7 Homotopy Types

Lecture 7.

Let $\langle X, \tau \rangle$ $\langle Y, \sigma \rangle$.

A Homotopy Type is a synonym to homotopy equivalences between spaces. It is **not** the homotopy type as in Homotopy Type Theory.

Claim 7.1: Homeomorphisms and topological spaces form a category.

Claim 7.2: Homeomorphisms and topological spaces form a group (for all morphisms with one side fixed).

$Homeo(X)$ group of all identity maps of X
g is a retraction of Y to X on $g(x)$, g is also left inverse of f
$g \circ f \simeq id_x$

Definition 7.3 (HOMOTOPIC EQUIVALENCE): A mapping $f : X \to Y$ is a homotopic equivalence when $\exists g : Y \to X$ such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$

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Theorem 7.4: Homotopy equivalence between spaces is an **??**.

Proof. ... □

Example 7.5: $f : X \to \cdot$ is a homotopic equivalence to a point $\iff$ X is contractible.
$H_0 = id \simeq H_1 = f$

Claim 7.6: Holes matter! $\infty \simeq$ B $\simeq o\_O$

Example 7.7: Glueing of a disk to some space using quotienting. *expanding/reducing?*

Claim 7.8: Homotopy Theory cares only about spaces glued from a finite number of $\mathcal{S}^n$ spaces, see Shape Theory.

Definition 7.9 (SIMPLE HOMOTOPY TYPE): A homotopy type is simple if you use a finite number of expansions and reductions

Claim 7.10: This is a cue to working with simplicial complexes.

...Interlude...

$f : X \to Y$, f is a proper mapping, f^{-1}(compact) is a compact
Define compact
Defining proper homotopy equivalences.
$\mathbb{R}^n \to \cdot$ is an improper homotopy. Compactness is a bitch.

8 Resources

Course page: https://sites.google.com/site/kafedramatematikikau/products-services/homotopy-theory