

Introduction to Homotopy Theory

(Notes based on lectures by Sergiy Maksymenko)

Abstract

Homotopy theory studies spaces up to a homotopy, which is a continuous deformation of one continuous function to another. This document is a work in progress done during a course audit. These notes are taken purposefully in English to strengthen intuition and simplify lookup of concepts in related literature.

Contents

1 Point-set Topology	1
2 Homotopy	2
2.1 Homotopy and equivalence	2
2.2 Contractible Spaces	2
2.2.1 Trees	3
3 Quotient Spaces and Maps	3
3.1 Quotients and Groups	3
3.2 Cones of Topological Spaces	3
4 Retractions and Deformation Retractions	3
5 Classes of Homotopy Maps	3
5.1 Mappings of S^1 to Itself	3
6 Resources	3

1 Point-set Topology

Definition 1.1 (TOPOLOGICAL SPACE): A topological space is a pair $\langle X, \tau \rangle$, where X is a set and τ , a topology on X , is a collection of subsets ($\tau \subseteq \mathcal{P}(X)$) called open sets, such that:

- $\emptyset \in \tau$.
- $X \in \tau$.
- τ is closed under arbitrary finite intersections.
- τ is closed under arbitrary unions. **Maybe find some cute notation for this.**

Definition 1.2 (TRIVIAL TOPOLOGY): A topological space is called trivial, when the topology on X consists only of \emptyset and X .

Definition 1.3 (DISCRETE TOPOLOGY): A topological space is called discrete, when $\tau = \mathcal{P}(X)$.

Definition 1.4 (CONTINUOUS MAP): Let $\langle X, \tau \rangle$ and $\langle Y, \sigma \rangle$ be topological spaces. A map $f : X \rightarrow Y$ is **continuous** if $\forall s \in \sigma, f^{-1}(s) \in \tau$. In plain English, a map is continuous when a preimage of an open set in Y is an open set in X .

$C(X, Y)$ denotes a set of all continuous maps between X and Y .

Base of topology and methods of inducing topologies on sets were discussed during Lecture 2.

2 Homotopy

Definition 2.1 (HOMOTOPY): Two continuous maps $f, g : X \rightarrow Y$ are homotopic if there is a map called homotopy $H : X \times [0, 1] \rightarrow Y$ that *continuously deforms* f to g , denoted $f \simeq g$ or $f \stackrel{H}{\simeq} g$. In general:

$$\begin{aligned} X \times [0, 1] &\stackrel{H}{\rightarrow} Y \\ H(x, 0) &= f(x) \\ H(x, 1) &= g(x) \\ H(x, t) &= tf(x) + (1 - t)g(x) \end{aligned}$$

Example 2.1: $1 \stackrel{x^t}{\simeq} x$ when viewed as x^0 and x^1 .

$$\begin{aligned} X \times [0, 1] &\stackrel{H}{\rightarrow} Y \\ H(x, 0) &= x^0 \\ H(x, 1) &= x^1 \\ H(x, t) &= x^t \end{aligned}$$

Another possible homotopy between the same functions is $t \cdot x + (1 - t) \cdot 1$, which suggests that there may be many more of them.

Plots?

Example 2.2: $\{\cdot\} \times [0, 1] \rightarrow \mathbb{C}$ with $H(x, t) = e^{2\pi it}$

2.1 Homotopy and equivalence

Theorem 2.2: Homotopy equivalence between spaces is an equivalence relation.

Proof. ...

□

Theorem 2.3: A homotopy between continuous maps is an equivalence relation.

Proof. ...

□

2.2 Contractible Spaces

Definition 2.4 (CONTRACTIBILITY): A space is contractible if it is homotopically equivalent to a point (a constant map).

Definition 2.5 (PATH-CONNECTEDNESS):

Theorem 2.6: Any convex set is contractible.

Definition 2.7 (STAR-CONVEX SET):

Theorem 2.8: Any star-convex set is contractible.

2.2.1 Trees

3 Quotient Spaces and Maps

See lectures 2 and 3.

Brouwer's Fixed Point Theorem was mentioned around here. Bring up the context?.

3.1 Quotients and Groups

3.2 Cones of Topological Spaces

4 Retractions and Deformation Retractions

5 Classes of Homotopy Maps

See lectures 4 and 5.

5.1 Mappings of S^1 to Itself

6 Resources

Course page: <https://sites.google.com/site/kafedramatematikikau/products-services/homotopy-theory>