Introduction to Homotopy Theory

(Notes based on lectures by Sergiy Maksymenko)

Abstract

Homotopy theory studies spaces up to a homotopy, which is a continuous deformation of one continuous function to another. This documents is a work in progress done during a course audit. These notes are taken purposefully in English to strenghten intuition and simplify lookup of concepts in related literature.

Warning: this document may be edited live during audit so watch out for incorrect statements!

Contents

1 Set-theoretic Definitions

Definition 1.1 (Binary relation): A binary relation R on a set X is a set of ordered pairs of elements of X.

Definition 1.2 (Equivalence relation): An equivalence relation $\sim$ is a binary relation that is reflexive, symmetric and transitive.

Definition 1.3 (Equivalence class of an element): Given a set X and an equivalence relation $\sim$, an equivalence class of $a \in X$, denoted $[a]$ is a set $\{x \in S \mid x \sim a\}$

Definition 1.4 (Quotient Set): A quotient set $X/\sim$ (also said "X *modulo* $\sim$") is a set of all equivalence classes of X with respect to $\sim$.

$$X/\sim = \{[x] : x \in X\}$$

Definition 1.5 (QUOTIENT MAP): A quotient map is a surjective mapping that sends a point in X to its equivalence class, containing it: $q : X \to X/\sim$

Definition 1.6 (QUOTIENT BY SET MEMBERSHIP): **TODO**. X/A

2 Point-set Topology

Definition 2.1 (UNIT INTERVAL): A unit interval (or just an interval) I is a subset $[0..1] \subset \mathbb{R}$ of a real line, which is a popular notation in further homotopy buildup.

Definition 2.2 (A GRAPH OF A FUNCTION): $\Gamma(f)$ - a subset of $\mathbb{R} \times \mathbb{R}$

Definition 2.3 (TOPOLOGICAL SPACE): A topological space is a pair $\langle X, \tau \rangle$, where X is a set and τ, a topology on X, is a collection of subsets ($\tau \subseteq \mathcal{P}(X)$) called open sets, such that:

- $\varnothing \in \tau$.
- $X \in \tau$.
- τ is closed under arbitrary finite intersections.
- τ is closed under arbitrary unions. **Maybe find cute notation for this.**

Posets.

Definition 2.4 (TRIVIAL TOPOLOGY): A topological space is called trivial, when the topology on X consists only of $\varnothing$ and X.

Definition 2.5 (DISCRETE TOPOLOGY): A topological space is called discrete, when $\tau = \mathcal{P}(X)$.

Definition 2.6 (CONTINUOUS MAP): Let $\langle X, \tau \rangle$ and $\langle Y, \sigma \rangle$ be topological spaces. A map $f : X \to Y$ is **continuous** if:

$$\forall s \in \sigma, f^{-1}(s) \in \tau$$

In plain English, a map is continuous when a preimage of an open set in Y is an open set in X.

$C(X, Y)$ denotes a set of all continuous maps between X and Y.

Definition 2.7 (HOMEOMORPHISM): A mapping $f : X \to Y$ is a homeomorphism if $\exists g : Y \to X$ s.t. $X \xrightarrow{f} Y \xrightarrow{g} X \xrightarrow{f} Y$, when $g \circ f = id_x$ and $f \circ g = id_Y$. It is a continuous map that has a continuous inverse.

Example 2.8: *How to see that a map $h : [0..1] \to [0..1]$ is a homeomorphism?*

- At least it needs to be monotonous.
- Preserves boundaries.

$H(([0..1], 0, 1), ([0..1], 0, 1))_{\text{increasing}} \bigcup H(([0..1], 0, 1), ([0..1], 1, 0))_{\text{decreasing}}$

Base of topology and methods of inducing topologies on sets were discussed during Lecture 2.

Bonus. Compare topology to a field of sets to a σ-algebra to a Borel σ-algebra. Discussed during Lecture 5.

2.1 Topology Restrictions

Definition 2.9 (T_1 SPACE):

Definition 2.10 (HAUSDORFF SPACE):

2.2 Quotient Topology

Gluing.

Let $\langle X, \tau \rangle$ be a topological space and $\sim$ be an equivalence relation on X.

Definition 2.11 (QUOTIENT TOPOGICAL SPACE): By analogy of a set and given $q : \langle X, \tau \rangle \to \langle X/\sim, \tau_{X/\sim} \rangle$,

$$
\begin{aligned}
X/\sim &= \{[x] : x \in X\} \\
\tau_{X/\sim} &= \{U \subseteq X/\sim \mid q^{-1}(U) \in \tau\}
\end{aligned}
$$

$\tau_{X/\sim}$ is constructed this way to ensure q is continuous.

Theorem 2.12: Given Z is a topological space, g is a surjective map, p is a quotient map and the following diagram,

$$
\begin{tikzcd}
X \arrow[rr, "f = g \circ p"] \arrow[dr, "p"'] & & Z \\
& X/\sim \arrow[ur, "g"'] &
\end{tikzcd}
$$

g is continuous $\iff$ $f = g \circ p$ is continuous.

3 Homotopies Between Continuous Maps

Definition 3.1 (HOMOTOPY): Two continuous maps $f, g : X \to Y$ are homotopic if there is a map called homotopy $H : X \times [0, 1] \to Y$ that *continuously deforms* f to g, denoted $f \simeq g$ or $f \overset{H}{\simeq} g$. In general:

$$
\begin{aligned}
X \times [0,1] &\xrightarrow{H} Y \\
H(x, 0) &= f(x) \\
H(x, 1) &= g(x) \\
H(x, t) &= t f(x) + (1 - t) g(x)
\end{aligned}
$$

Example 3.2: $1 \overset{x^t}{\simeq} x$ when viewed as x^0 and x^1.

$$
\begin{aligned}
X \times [0,1] &\xrightarrow{H} Y \\
H(x, 0) &= x^0 \\
H(x, 1) &= x^1 \\
H(x, t) &= x^t
\end{aligned}
$$

Another possible homotopy between the same functions is $t \cdot x + (1 - t) \cdot 1$, which suggsts that there may be many more of them.

Plots?

Example 3.3: $\{\cdot\} \times [0, 1] \to \mathbb{C}$ with $H(x, t) = e^{2\pi i t}$

Theorem 3.4: A homotopy between continuous maps is an Equivalence relation.

Proof. ... □

3.1 Contractible Spaces

Definition 3.5 (CONTRACTIBILITY): A space is contractible if it is homotopically equivalent to a point (a constant map).

Extra definitions:

- $X \times 0$ **is a retraction of CX (cone over X)**
- **homotopic equivalence to a point**

Example 3.6: $\mathbb{R}^n$ is contractible to a point.

$$\mathbb{R}^n \times [0,1] \xrightarrow{H} \mathbb{R}^n$$
$$H(x,0) = 0$$
$$H(x,1) = x$$
$$H(x,t) = tx$$

Definition 3.7 (PATH-CONNECTEDNESS):

Theorem 3.8: Any convex set is contractible.

Lemma 3.9: Contractibility does not depend on a choice of a point.

Definition 3.10 (STAR-CONVEX SET):

Theorem 3.11: A star-convex set is contractible to a point.

$$A \subset \mathbb{R}^n \text{ is (star-)convex}$$
$$a \in A$$
$$A \times [0,1] \xrightarrow{H} A$$
$$H(x,0) = 0$$
$$H(x,1) = x$$
$$H(x,t) = at + x(1-t)$$

Theorem 3.12: Assuming we know how to build topologies on trees (as in graph theory trees), every finite tree is a contractible topological space.

4 Quotient Spaces and Maps

See lectures 2 and 3. Most of this stuff

4.1 Quotients and Groups

4.2 Cones of Topological Spaces

Brouwer's Fixed Point Theorem was mentioned around here. Bring up the context?.

5 Retractions, Deformations and Deformation Retractions

Definition 5.1 (RETRACTION):

Bring up the example with $\frac{x}{|x|} - discussedalongwithretractions$

Definition 5.2 (DEFORMATION): A continuous mapping is a deformation into a subspace $A \subset X$ when

- $H_0 = id_X$
- $\forall t \in [0..1], H_t(A) \subset A$

Definition 5.3 (DEFORMATION RETRACTION):

6 Classes of Homotopy Maps

See lectures 4 and 5.

6.1 Mappings of S^1 to Itself

$$\pi_1\mathcal{S}^1 = [(\mathcal{S}^1, \cdot), (\mathcal{S}^1, \cdot)]$$

7 Homotopy Types

Lecture 7.

Let $\langle X, \tau \rangle$ $\langle Y, \sigma \rangle$.

A Homotopy Type is a synonym to homotopy equivalences between spaces. It is **not** the homotopy type as in Homotopy Type Theory.

Claim 7.1: Homeomorphisms and topological spaces form a category.

Claim 7.2: Homeomorphisms and topological spaces form a group (for all morphisms with one side fixed).

$Homeo(X)$ group of all identity maps of X

g is a retraction of Y to X on $g(x)$, g is also left inverse of f

$g \circ f \simeq id_x$

Definition 7.3 (Homotopic Equivalence): A mapping $f : X \to Y$ is a homotopic equivalence when $\exists g : Y \to X$ such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$

.

Theorem 7.4: Homotopy equivalence between spaces is an Equivalence relation.

Proof. … □

Example 7.5: $f : X \to \cdot$ is a homotopic equivalence to a point $\iff$ X is contractible.

$H_0 = id \simeq H_1 = f$

Claim 7.6: Holes matter! $\infty \simeq$ B $\simeq o\_O$

Example 7.7: Glueing of a disk to some space using quotienting. *expanding/reducing?*

Claim 7.8: Homotopy Theory cares only about spaces glued from a finite number of $\mathcal{S}^n$ spaces, see Shape Theory.

Definition 7.9 (Simple Homotopy Type): A homotopy type is simple if you use a finite number of expansions and reductions

Claim 7.10: This is a cue to working with simplicial complexes.

. . . Interlude . . .

$f : X \to Y$, f is a proper mapping, $f^{-1}(\text{compact})$ is a compact

Define compact

Defining proper homotopy equivalences.

$\mathbb{R}^n \to \cdot$ is an improper homotopy. Compactness is a bitch.

8 Homeomorphisms

A fact about a group of homeomorphisms of an interval: homotopy $h_t = tf + (1-t)g$ monotonicity $\iff H^+, H^-$ are convex subset in a vector space $C(I, I)$ of all continuous functions. *Prove it*.

Example 8.1: Given curves γ and δ that each enclose some surface. Is there a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $h(\gamma) = \delta$?

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[rr, "\exists h?"] & & \mathbb{R}^2 \\
& I \arrow[ul, "\gamma"] \arrow[ur, "\delta"'] &
\end{tikzcd}$$

Example 8.2: Given $f(x) = x^2$. Build a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $h(y = 0) = \Gamma(f)$.

$$h(x, y) \mapsto (x, x^2 + y)$$
$$h^{-1}(x, y) \mapsto (x, -x^2 + y)$$
$$h^{-1} \circ h = id_{\mathbb{R}^2}$$

Generalizing,

$$h(x, y) \mapsto (x, f(x) + y)$$
$$h^{-1}(x, y) \mapsto (x, -f(x) + y)$$
$$h^{-1} \circ h = id_{\mathbb{R}^2}$$

Example 8.3: Given a graph of a continuous function $f : [a, b] \to \mathbb{R}$, $f(a) = f(b) = 0$, $m = \min f(x)$, $M = \max f(x)$, $\epsilon > 0$ that lies within a known rectangle $K = [a, b] \times [m - \epsilon, M + \epsilon]$.

Prove, $\exists h : \mathbb{R}^2 \to \mathbb{R}^2$ (h is a homeomorphism), s.t.

- $h([a, b] \times 0) = \Gamma(f)$
- $h(z) = z \ \forall z \in \mathbb{R}^2 \setminus K$.

This is how we get to locally deform curves on a plane.

Definition 8.4 (Isotopy): A homotopy $H : X \times [0..1] \to X$ s.t. $\forall t$, H_t is a homeomorphism is an isotopy.

Lemma 8.5 (Alexander's Theorem): Any homeomorphism of a unit interval (or a disk, or any convex set) which is fixed on its boundary $h/\partial\mathcal{D}^2 = id_{\mathcal{D}^2}$ is isotopic to id_I with an isotopy that is also fixed on is boundary.

E.g. there is a homeomorphism that maps a rendering of a letter A inside a disk to rendering of a letter R.

Proof. Alexander's Trick.

$H : \mathcal{D}^2 \times [0, 1] \to \mathcal{D}^2$ is an isotopy.

□

Important result: if you have two continuous deformations of a unit disk, you can have a third continuous deformations between them. It is possible to use differentiability.

See also, Cerf's theorem, https://en.wikipedia.org/wiki/Cerf_theory and Smale's theorem (don't confuse with Poincare conjecture).

9 Glueing

We were building up Homotopy Theory definitions to work with CW complexes.

Definition 9.1 (CW COMPLEXES (CELL COMPLEXES)): Spaces, created by glueing simple spaces together.

Quotienting is the workhorse of glueing. Use quotient topology to set up topologies after glueing two spaces.

Example 9.2: Identifying all points from a boundary of a disk (set X) and a single point from set Y as a single equivalence class.

Notation: $X \bigcup_f A$
Big idea: geometric objects have combinatorial structure, there is a lot of machinery that lets you "dumb down" to those structures to simplify working with objects equivalent up to a homotopy.
Foreshadowing: Whitehead theorem.

Example 9.3: Let's build a cube. Its vertices are 8 zero-dimensional disks: $\mathcal{D}^0$, edges are 1-dimensional disks, faces are 2-dimensional disks.
(NB spaces like K^1 are enough to model graphs in topology)

$$
\begin{aligned}
K^0 &= \bigsqcup \mathcal{D}_i^0 \\
K^1 &= \left(\bigsqcup \mathcal{D}_i^1\right) \bigcup_{\phi_i} K^0 \\
K^2 &= \left(\bigsqcup \mathcal{D}_i^2\right) \bigcup_{\psi_i} K^1
\end{aligned}
$$

K^i is an i-th skeleton. *Interiors* of disks are called **cells**.

Claim 9.4: An integer line is a cell complex.

Definition 9.5 (CW COMPLEX):
- C is for closure-finite (closure of each cell is contained in a union of finitely many cells)
- W is for weak topology (roughly, every K^i has a quotient topology)

10 $\pi_n(X, x)$

11 Resources

Course page: https://sites.google.com/site/kafedramatematikikau/products-services/homotopy-theory