Introduction to Homotopy Theory
(Notes based on lectures by Sergiy Maksymenko)

Abstract

Homotopy theory studies spaces up to a homotopy, which is a continuous deformation of one continuous function to another. This documents is a work in progress done during a course audit. These notes are taken purposefully in English to strenghten intuition and simplify lookup of concepts in related literature.

Contents

1 Set-theoretic Definitions

Definition 1.1 (BINARY RELATION): A binary relation R on a set X is a set of ordered pairs of elements of X.

Definition 1.2 (EQUIVALENCE RELATION): An equivalence relation $\sim$ is a binary relation that is reflexive, symmetric and transitive.

Definition 1.3 (EQUIVALENCE CLASS OF AN ELEMENT): Given a set X and an equivalence relation $\sim$, an equivalence class of $a \in X$, denoted $[a]$ is a set $\{x \in S \mid x \sim a\}$

Definition 1.4 (QUOTIENT SET): A quotient set $X/\sim$ (also said "X *modulo* $\sim$") is a set of all equivalence classes of X with respect to $\sim$.

$$X/\sim = \{[x] : x \in X\}$$

Definition 1.5 (QUOTIENT MAP): A quotient map is a surjective mapping that sends a point in X to its equivalence class, containing it: $q : X \to X/\sim$

2 Point-set Topology

Definition 2.1 (Topological space): A topological space is a pair $\langle X, \tau \rangle$, where X is a set and τ, a topology on X, is a collection of subsets ($\tau \subseteq \mathcal{P}(X)$) called open sets, such that:

- $\emptyset \in \tau$.
- $X \in \tau$.
- τ is closed under arbitrary finite intersections.
- τ is closed under arbitrary unions. **Maybe find cute notation for this.**

Definition 2.2 (Trivial Topology): A topological space is called trivial, when the topology on X consists only of $\emptyset$ and X.

Definition 2.3 (Discrete Topology): A topological space is called discrete, when $\tau = \mathcal{P}(X)$.

Definition 2.4 (Continuous map): Let $\langle X, \tau \rangle$ and $\langle Y, \sigma \rangle$ be topological spaces. A map $f : X \to Y$ is **continuous** if:

$$\forall s \in \sigma, f^{-1}(s) \in \tau$$

In plain English, a map is continuous when a preimage of an open set in Y is an open set in X.
$C(X, Y)$ denotes a set of all continuous maps between X and Y.

Base of topology and methods of inducing topologies on sets were discussed during Lecture 2.
Bonus. Compare topology to a field of sets to a σ-algebra to a Borel σ-algebra. Discussed during Lecture 5.

2.1 Topology Restrictions

Definition 2.5 (T_1 space):

Definition 2.6 (Hausdorff space):

2.2 Quotient Topology

Gluing.
Let $\langle X, \tau \rangle$ be a topological space and $\sim$ be an equivalence relation on X.
Definition 2.7 (Quotient Topogical Space): By analogy of a set and given $q : \langle X, \tau \rangle \to \langle X/{\sim}, \tau_{X/\sim} \rangle$,

$$X/{\sim} = \{[x] : x \in X\}$$
$$\tau_{X/\sim} = \{U \subseteq X/{\sim} \mid q^{-1}(U) \in \tau\}$$

$\tau_{X/\sim}$ is constructed this way to ensure q is continuous.

Theorem 2.8: Given Z is a topological space, g is a surjective map, p is a quotient map and the following diagram,

$$\begin{tikzcd} X \arrow[rr, "f = g \circ p"] \arrow[dr, "p"'] & & Z \\ & X/{\sim} \arrow[ur, "g"'] & \end{tikzcd}$$

g is continuous $\iff$ $f = g \circ p$ is continuous.

Example 2.1: Have $z \in \mathbb{C}$, $\mathbb{S}^1 = \{|z| = 1\}$, $I = [0..1] \subset \mathbb{R}^1$.

$$\begin{tikzcd}
I \arrow[rr, "f(t \in I) = e^{2i\pi t}"] \arrow[dr, "p"'] & & \mathbb{S}^1 \\
& I/\Delta = \{0,1\} \arrow[ur, "g"'] &
\end{tikzcd}$$

3 Homotopy

Definition 3.1 (HOMOTOPY): Two continuous maps $f, g : X \to Y$ are homotopic if there is a map called homotopy $H : X \times [0,1] \to Y$ that *continuously deforms* f to g, denoted $f \simeq g$ or $f \overset{H}{\simeq} g$. In general:

$$\begin{aligned}
X \times [0,1] &\xrightarrow{H} Y \\
H(x,0) &= f(x) \\
H(x,1) &= g(x) \\
H(x,t) &= tf(x) + (1-t)g(x)
\end{aligned}$$

Example 3.1: $1 \overset{x^t}{\simeq} x$ when viewed as x^0 and x^1.

$$\begin{aligned}
X \times [0,1] &\xrightarrow{H} Y \\
H(x,0) &= x^0 \\
H(x,1) &= x^1 \\
H(x,t) &= x^t
\end{aligned}$$

Another possible homotopy between the same functions is $t \cdot x + (1-t) \cdot 1$, which suggsts that there may be many more of them.

Plots?

Example 3.2: $\{\cdot\} \times [0,1] \to \mathbb{C}$ with $H(x,t) = e^{2\pi i t}$

3.1 Homotopy and Equivalence

Theorem 3.2: Homotopy equivalence between spaces is an Equivalence relation.

Proof. … □

Theorem 3.3: A homotopy between continuous maps is an Equivalence relation.

Proof. … □

3.2 Contractible Spaces

Definition 3.4 (CONTRACTIBILITY): A space is contractible if it is homotopically equivalent to a point (a constant map).

Definition 3.5 (PATH-CONNECTEDNESS):

Theorem 3.6: Any convex set is contractible.

Lemma 3.7: Contractibility does not depend on a choice of a point.

Definition 3.8 (STAR-CONVEX SET):

Theorem 3.9: Any star-convex set is contractible.

4 Quotient Spaces and Maps

See lectures 2 and 3. Most of this stuff

4.1 Quotients and Groups

4.2 Cones of Topological Spaces

Brouwer's Fixed Point Theorem was mentioned around here. Bring up the context?.

5 Retractions and Deformation Retractions

6 Classes of Homotopy Maps

See lectures 4 and 5.

6.1 Mappings of S^1 to Itself

7 Resources

Course page: https://sites.google.com/site/kafedramatematikikau/products-services/homotopy-theory

References