Introduction to Homotopy Theory
(Notes based on lectures by Sergiy Maksymenko)

Abstract

Homotopy theory studies spaces up to a homotopy, which is a continuous deformation of one continuous function to another. This documents is a work in progress done during a course audit. These notes are taken purposefully in English to strenghten intuition and simplify lookup of concepts in related literature.

Warning: this document may be edited live during audit so watch out for incorrect statements!

Contents

1 Set-theoretic Definitions

Definition 1.1 (BINARY RELATION): A binary relation R on a set X is a set of ordered pairs of elements of X.

Definition 1.2 (EQUIVALENCE RELATION): An equivalence relation $\sim$ is a binary relation that is reflexive, symmetric and transitive.

Definition 1.3 (EQUIVALENCE CLASS OF AN ELEMENT): Given a set X and an equivalence relation $\sim$, an equivalence class of $a \in X$, denoted $[a]$ is a set $\{x \in S \mid x \sim a\}$

Definition 1.4 (QUOTIENT SET): A quotient set $X/\sim$ (also said "X *modulo* $\sim$" or "X up to $\sim$") is a set of all equivalence classes of X with respect to $\sim$.

$$X/\sim = \{[x] : x \in X\}$$

Definition 1.5 (QUOTIENT MAP, PROJECTION): A surjective mapping that sends a point in X to its equivalence class, containing it.

$$\pi : X \to X/\sim$$

Definition 1.6 (QUOTIENT BY SUBSET MEMBERSHIP): Given a set X and subset $A \subset X$, X/A is a notation that assumes an implicit equivalence relation:

$$\forall \alpha\beta \in X,\ \alpha \sim \beta = \begin{cases} \top, \alpha \in A \wedge \beta \in A \\ \bot, \text{otherwise} \end{cases}$$

For example:

$$\begin{aligned} X &= \{a, b, c, d, e\} \\ A &= \{a, e\} \subset X \\ X/A &= \{b, c, d, [a]\} \\ [a] &= \{a, e\} \end{aligned}$$

One can say *identification of all points in A with each other*.

2 Group-theoretic Definitions

Definition 2.1 (PRESENTATION):

$$\langle S \mid R \rangle$$

Where S is a generator and R is a relation.

Definition 2.2 (COMMUTATOR): That one for groups: `https://en.wikipedia.org/wiki/Commutator`

Definition 2.3 (KERNEL (GROUP)): Analog of a null space of a linear map (turns out it's also often called a kernel in linear algebra)

Example 2.4:

$$\begin{gathered} \mathbb{Z}_5 = 0, 1, 2, 3, 4 \pmod 5 \\ \{e, a, a^2, a^3, a^4\} = \langle a \mid a^5 = e \rangle \end{gathered}$$

Definition 2.5 (FREE GROUP): Syntax for groups! F_2 if the free group has two generators.

Example 2.6: $F_1 = \mathbb{Z}$ (free cyclic group)

Theorem 2.7: Let group G be generated $g_1 \dots g_k$, so every $h \in G$ is a product of some of its generators.

$$F_k \xrightarrow{\phi} G$$

$$G \cong F_k / \ker \phi$$

3 Geometry Definitions

Definition 3.1 (N-SPHERE, HYPERSPHERE): $\mathcal{S}^n$. Generalizes a unit circle ($\mathcal{S}^1$ — a circle in $\mathbb{R}^2$) and a unit sphere ($\mathcal{S}^2$ — a sphere in $\mathbb{R}^3$). Given $\|\cdot\|$ is a norm,

$$\mathcal{S}^n = \{x \mid \|x \in \mathbb{R}^{n+1}\| = 1\}$$

A unit 0-sphere is a pair of endpoints of an interval $[-1, 1]$ of the real line.

Definition 3.2 (N-DISK, N-BALL): $\mathcal{D}^n$. A disk (a closed ball) is a region contained inside $\mathcal{S}^n$. $\mathcal{D}^1$ is a line segment (in $\mathbb{R}^1$).

$$\mathcal{D}^n = \{x \mid \|x \in \mathbb{R}^n\| \leq 1\}$$

A **ball** usually refers to a disk minus its boundary, an **open ball**, which can be defined using $<$ instead of $\leq$.

$$\mathcal{B}^n_r = \{x \mid \|x \in \mathbb{R}^n\| < r\}$$

Definition 3.3 (SPHERE MODULO DISK):

$$\mathcal{S}^n = \mathcal{D}^n \diagup \mathcal{S}^{n-1} = \mathcal{D}^n \diagup \partial\mathcal{D}^n$$

For example, identifying the ends (a boundary, which in this case is a zero-dimensional sphere) of a line segment ($\mathcal{D}^1$) as one point produces a circle ($\mathcal{S}^1$) — an object embeddable in $\mathbb{R}^2$:

$$\mathcal{S}^1 = \mathcal{D}^1 \diagup \mathcal{S}^0$$

For a 2-sphere:

$$\mathcal{S}^2 = \mathcal{D}^2 \diagup \mathcal{S}^1$$

4 Point-set Topology

Definition 4.1 (UNIT INTERVAL): A unit interval (or just an interval) I is a subset $[0..1] \subset \mathbb{R}$ of a real line, which is a popular notation in further homotopy buildup.

Definition 4.2 (A GRAPH OF A FUNCTION): $\Gamma(f)$ - a subset of $\mathbb{R} \times \mathbb{R}$

Definition 4.3 (TOPOLOGICAL SPACE): A topological space is a pair $\langle X, \tau \rangle$, where X is a set and τ, a topology on X, is a collection of subsets ($\tau \subseteq \mathcal{P}(X)$) called open sets, such that:

- $\varnothing \in \tau$.
- $X \in \tau$.
- τ is closed under arbitrary finite intersections.
- τ is closed under arbitrary unions. **Maybe find cute notation for this.**

Posets.

Definition 4.4 (TRIVIAL TOPOLOGY): A topological space is called trivial, when the topology on X consists only of $\varnothing$ and X.

Definition 4.5 (DISCRETE TOPOLOGY): A topological space is called discrete, when $\tau = \mathcal{P}(X)$.

Definition 4.6 (CONTINUOUS MAP): Let $\langle X, \tau \rangle$ and $\langle Y, \sigma \rangle$ be topological spaces. A map $f : X \to Y$ is **continuous** if:

$$\forall s \in \sigma, f^{-1}(s) \in \tau$$

In plain English, a map is continuous when a preimage of an open set in Y is an open set in X.
$C(X, Y)$ denotes a set of all continuous maps between X and Y.

Definition 4.7 (HOMEOMORPHISM): A mapping $f : X \to Y$ is a homeomorphism if $\exists g : Y \to X$ s.t. $X \xrightarrow{f} Y \xrightarrow{g} X \xrightarrow{f} Y$, when $g \circ f = id_x$ and $f \circ g = id_Y$. It is a continuous map that has a continuous inverse.

Example 4.8: *How to see that a map* $h : [0..1] \to [0..1]$ *is a homeomorphism?*

- At least it needs to be monotonous.
- Preserves boundaries.

$H(([0..1], 0, 1), ([0..1], 0, 1))_{\text{increasing}} \bigcup H(([0..1], 0, 1), ([0..1], 1, 0))_{\text{decreasing}}$

Base of topology and methods of inducing topologies on sets were discussed during Lecture 2.

Bonus. Compare topology to a field of sets to a σ-algebra to a Borel σ-algebra. Discussed during Lecture 5.

4.1 Topology Restrictions

Definition 4.9 (T_1 SPACE):

Definition 4.10 (HAUSDORFF SPACE):

4.2 Quotient Topology

Gluing.

Let $\langle X, \tau \rangle$ be a topological space and $\sim$ be an equivalence relation on X.

Definition 4.11 (QUOTIENT TOPOGICAL SPACE): By analogy of a set and given $q : \langle X, \tau \rangle \to \langle X/\sim, \tau_{X/\sim} \rangle$,

$$X/\sim \; = \{[x] : x \in X\}$$
$$\tau_{X/\sim} = \{U \subseteq X/\sim \mid q^{-1}(U) \in \tau\}$$

$\tau_{X/\sim}$ is constructed this way to ensure q is continuous.

Theorem 4.12: Given Z is a topological space, g is a surjective map, p is a quotient map and the following diagram,

$$\begin{tikzcd} X \arrow[rr, "f = g \circ p"] \arrow[dr, "p"'] & & Z \\ & X/\sim \arrow[ur, "g"'] & \end{tikzcd}$$

g is continuous $\iff$ $f = g \circ p$ is continuous.

5 Homotopies Between Continuous Maps

Definition 5.1 (HOMOTOPY): Two continuous maps $f, g : X \to Y$ are homotopic if there is a map called homotopy $H : X \times [0,1] \to Y$ that *continuously deforms* f to g, denoted $f \simeq g$ or $f \overset{H}{\simeq} g$. In general:

$$
\begin{aligned}
X \times [0,1] &\xrightarrow{H} Y \\
H(x,0) &= f(x) \\
H(x,1) &= g(x) \\
H(x,t) &= tf(x) + (1-t)g(x)
\end{aligned}
$$

Example 5.2: $1 \overset{x^t}{\simeq} x$ when viewed as x^0 and x^1.

$$
\begin{aligned}
X \times [0,1] &\xrightarrow{H} Y \\
H(x,0) &= x^0 \\
H(x,1) &= x^1 \\
H(x,t) &= x^t
\end{aligned}
$$

Another possible homotopy between the same functions is $t \cdot x + (1-t) \cdot 1$, which suggsts that there may be many more of them.

Plots?

Example 5.3: $\{\cdot\} \times [0,1] \to \mathbb{C}$ with $H(x,t) = e^{2\pi i t}$

Theorem 5.4: A homotopy between continuous maps is an Equivalence relation.

Proof. ... □

5.1 Contractible Spaces

Definition 5.5 (CONTRACTIBILITY): A space is contractible if it is homotopically equivalent to a point (a constant map).

Extra definitions:

- $X \times 0$ **is a retraction of CX (cone over X)**
- **homotopic equivalence to a point**

Example 5.6: $\mathbb{R}^n$ is contractible to a point.

$$
\begin{aligned}
\mathbb{R}^n \times [0,1] &\xrightarrow{H} \mathbb{R}^n \\
H(x,0) &= 0 \\
H(x,1) &= x \\
H(x,t) &= tx
\end{aligned}
$$

Definition 5.7 (PATH-CONNECTEDNESS):

Theorem 5.8: Any convex set is contractible.

Lemma 5.9: Contractibility does not depend on a choice of a point.

Definition 5.10 (STAR-CONVEX SET):

Theorem 5.11: A star-convex set is contractible to a point.

$$
\begin{aligned}
A \subset \mathbb{R}^n &\text{ is (star-)convex} \\
a &\in A \\
A \times [0,1] &\xrightarrow{H} A \\
H(x,0) &= 0 \\
H(x,1) &= x \\
H(x,t) &= at + x(1-t)
\end{aligned}
$$

Theorem 5.12: Assuming we know how to build topologies on trees (as in graph theory trees), every finite tree is a contractible topological space.

6 Quotient Spaces and Maps

See lectures 2 and 3. Most of this stuff

6.1 Quotients and Groups

6.2 Cones of Topological Spaces

Brouwer's Fixed Point Theorem was mentioned around here. Bring up the context?.

7 Retractions, Deformations and Deformation Retractions

Definition 7.1 (RETRACTION):

Bring up the example with $\frac{x}{|x|} - discussedalongwithretractions$

Definition 7.2 (DEFORMATION): A continuous mapping is a deformation into a subspace $A \subset X$ when

- $H_0 = id_X$
- $\forall t \in [0..1], H_t(A) \subset A$

Definition 7.3 (DEFORMATION RETRACTION):

8 Classes of Homotopy Maps

See lectures 4 and 5.

8.1 Mappings of $\mathbb{S}^1$ to Itself

$$\pi_1 \mathcal{S}^1 = [(\mathcal{S}^1, \cdot), (\mathcal{S}^1, \cdot)]$$

9 Homotopy Types

Lecture 7.

Let $\langle X, \tau \rangle$ $\langle Y, \sigma \rangle$.

A Homotopy Type is a synonym to homotopy equivalences between spaces. It is **not** the homotopy type as in Homotopy Type Theory.

Claim 9.1: Homeomorphisms and topological spaces form a category.

Claim 9.2: Homeomorphisms and topological spaces form a group (for all morphisms with one side fixed).

$Homeo(X)$ group of all identity maps of X
g is a retraction of Y to X on $g(x)$, g is also left inverse of f
$g \circ f \simeq id_x$

Definition 9.3 (HOMOTOPIC EQUIVALENCE): A mapping $f : X \to Y$ is a homotopic equivalence when $\exists g : Y \to X$ such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$

.

Theorem 9.4: Homotopy equivalence between spaces is an Equivalence relation.

Proof. … $\square$

Example 9.5: $f : X \to \cdot$ is a homotopic equivalence to a point $\iff$ X is contractible.
$H_0 = id \simeq H_1 = f$

Claim 9.6: Holes matter! $\infty \simeq \mathrm{B} \simeq o\_O$

Example 9.7: Glueing of a disk to some space using quotienting. *expanding/reducing?*

Claim 9.8: Homotopy Theory cares only about spaces glued from a finite number of $\mathcal{S}^n$ spaces, see Shape Theory.

Definition 9.9 (SIMPLE HOMOTOPY TYPE): A homotopy type is simple if you use a finite number of expansions and reductions

Claim 9.10: This is a cue to working with simplicial complexes.

…Interlude…

$f : X \to Y$, f is a proper mapping, f^{-1}(compact) is a compact
Define compact
Defining proper homotopy equivalences.
$\mathbb{R}^n \to \cdot$ is an improper homotopy. Compactness is a bitch.

10 Homeomorphisms

A fact about a group of homeomorphisms of an interval: homotopy $h_t = tf + (1-t)g$ monotonicity $\iff$ H^+, H^- are convex subset in a vector space $C(I, I)$ of all continuous functions. *Prove it*.

Example 10.1: Given curves γ and δ that each enclose some surface. Is there a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $h(\gamma) = \delta$?

$$\begin{tikzcd} \mathbb{R}^2 \arrow[rr, "\exists h?"] & & \mathbb{R}^2 \\ & I \arrow[ul, "\gamma"] \arrow[ur, "\delta"'] & \end{tikzcd}$$

Example 10.2: Given $f(x) = x^2$. Build a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $h(y = 0) = \Gamma(f)$.

$$h(x, y) \mapsto (x, x^2 + y)$$
$$h^{-1}(x, y) \mapsto (x, -x^2 + y)$$
$$h^{-1} \circ h = id_{\mathbb{R}^2}$$

Generalizing,

$$h(x, y) \mapsto (x, f(x) + y)$$
$$h^{-1}(x, y) \mapsto (x, -f(x) + y)$$
$$h^{-1} \circ h = id_{\mathbb{R}^2}$$

Example 10.3: Given a graph of a continuous function $f : [a, b] \to \mathbb{R}$, $f(a) = f(b) = 0$, $m = \min f(x)$, $M = \max f(x)$, $\epsilon > 0$ that lies within a known rectangle $K = [a, b] \times [m - \epsilon, M + \epsilon]$.

Prove, $\exists h : \mathbb{R}^2 \to \mathbb{R}^2$ (h is a homeomorphism), s.t.

- $h([a, b] \times 0) = \Gamma(f)$
- $h(z) = z \ \forall z \in \mathbb{R}^2 \setminus K$.

This is how we get to locally deform curves on a plane.

Definition 10.4 (ISOTOPY): A homotopy $H : X \times [0..1] \to X$ s.t. $\forall t$, H_t is a homeomorphism is an isotopy.

Lemma 10.5 (ALEXANDER'S THEOREM): Any homeomorphism of a unit interval (or a disk, or any convex set) which is fixed on its boundary $h / \partial\mathcal{D}^2 = id_{\mathcal{D}^2}$ is isotopic to id_I with an isotopy that is also fixed on is boundary.

E.g. there is a homeomorphism that maps a rendering of a letter A inside a disk to rendering of a letter R.

Proof. ALEXANDER'S TRICK.
$H : \mathcal{D}^2 \times [0, 1] \to \mathcal{D}^2$ is an isotopy.

□

Important result: if you have two continuous deformations of a unit disk, you can have a third continuous deformations between them. It is possible to use differentiability.

See also, Cerf's theorem, `https://en.wikipedia.org/wiki/Cerf_theory` and Smale's theorem (don't confuse with Poincare conjecture).

11 Glueing

We were building up Homotopy Theory definitions to work with CW complexes.

Definition 11.1 (CW COMPLEXES (CELL COMPLEXES)): Spaces, created by glueing simple spaces together.

Quotienting is the workhorse of glueing. Use quotient topology to set up topologies after glueing two spaces.

Example 11.2: Identifying all points from a boundary of a disk (set X) and a single point from set Y as a single equivalence class.

Notation: $X \bigcup_f A$

Big idea: geometric objects have combinatorial structure, there is a lot of machinery that lets you "dumb down" to those structures to simplify working with objects equivalent up to a homotopy.

Foreshadowing: Whitehead theorem.

Example 11.3: Let's build a cube. Its vertices are 8 zero-dimensional disks: $\mathcal{D}^0$, edges are 1-dimensional disks, faces are 2-dimensional disks.
(NB spaces like K^1 are enough to model graphs in topology)

$$K^0 = \bigsqcup \mathcal{D}_i^0$$
$$K^1 = \left(\bigsqcup \mathcal{D}_i^1\right) \bigcup_{\phi_i} K^0$$
$$K^2 = \left(\bigsqcup \mathcal{D}_i^2\right) \bigcup_{\psi_i} K^1$$

K^i is an i-th skeleton. *Interiors* of disks are called **cells**.

Claim 11.4: An integer line is a cell complex.

Definition 11.5 (CW COMPLEX):
- C is for closure-finite (closure of each cell is contained in a union of finitely many cells)
- W is for weak topology (roughly, every K^i has a quotient topology)

12 Homotopy Groups

April 3.
Notation abuse: there's actually a difference between $\circ$ (map composition) and $\star$ (group operation, which is denoted as $\circ$ instead!)

Definition 12.1 (FUNDAMENTAL GROUP):

$$\pi_1(X, x_0) = [(\mathcal{S}^1, \cdot), (X, x_0)] = [(I, \partial I), (X, x_0)]$$

where [A, B] are homotopy classes.
The group multiplication is defined as homotopy composition for **spaces equivalent up to a homotopy**:

$$(\alpha \circ \beta)(t) = \alpha(2t) t \in [0, \frac{1}{2}, \beta(2t-1) t \in [\frac{1}{2}, 1]]$$

Neutral element suggestions: constant mapping doesn't work.
Let's use deformations up to a homotopy:

$$[\alpha] \circ [\beta] = [\alpha \circ \beta]$$

$$[\alpha] = [\alpha'] \implies \alpha \simeq \alpha'$$

Now prove it's actually a group

Theorem 12.2:

$$[\alpha] \circ [\epsilon] = [\epsilon] \circ [\alpha] = [\alpha]$$

Proof. In other words, $\alpha \circ \epsilon \simeq \alpha$ □

Big idea: we're handling an equivalence class by introducing an extra parameter for a class deformation, and defining operations as parametrized homotopies. Lesson: paying for definitions is hard.
...

Theorem 12.3: **This has been discussed earlier**

$$\pi_1(\mathcal{S}^1) \cong \mathbb{Z}$$

12.1 Computing the Fundamental Group

Two instruments: Cover (a special case of a fibration) and Van Kampen's Theorem

Theorem 12.4: (Van Kampen)

For any finite group G, build a 2-dimensional CW-complex X, s.t. $\pi_1(X, x) = G$. Gave birth to combinatorial group theory (see also: word problem).

12.2 Homotopy Groups of Spheres

The i-th homotopy group $\pi_i(\mathcal{S}^n)$ summarizes the different ways in which the i-dimensional sphere $\mathcal{S}^i$ can be mapped continuously (in homotopical sense, i.e. maps up to a homotopy) into the n-dimensional sphere $\mathcal{S}^n$.

13 Resources

Course page: `https://sites.google.com/site/kafedramatematikikau/products-services/homotopy-theory`