Introduction to Homotopy Theory

(Notes based on lectures by Sergiy Maksymenko)

Abstract

Homotopy theory studies spaces up to a homotopy, which is a continuous deformation of one continuous function to another. This documents is a work in progress done during a course audit. These notes are taken purposefully in English to strenghten intuition and simplify lookup of concepts in related literature.

Contents

1 Point-set Topology

Definition 1.1 (TOPOLOGICAL SPACE): A topological space is a pair $\langle X, \tau \rangle$, where X is a set and τ, a topology on X, is a collection of subsets ($\tau \subseteq \mathcal{P}(X)$) called open sets, such that:

- $\varnothing \in \tau$.
- $X \in \tau$.
- τ is closed under arbitrary finite intersections.
- τ is closed under arbitrary unions. **Maybe find cute notation for this.**

Definition 1.2 (TRIVIAL TOPOLOGY): A topological space is called trivial, when the topology on X consists only of $\varnothing$ and X.

Definition 1.3 (DISCRETE TOPOLOGY): A topological space is called discrete, when $\tau = \mathcal{P}(X)$.

Definition 1.4 (CONTINUOUS MAP): Let $\langle X, \tau \rangle$ and $\langle Y, \sigma \rangle$ be topological spaces. A map $f : X \to Y$ is **continuous** if $\forall s \in \sigma, f^{-1}(s) \in \tau$. In plain English, a map is continuous when a preimage of an open set in Y is an open set in X.

$C(X, Y)$ denotes a set of all continuous maps between X and Y.

Base of topology and methods of inducing topologies on sets were discussed during Lecture 2.

Bonus. Compare topology to a field of sets to a σ-algebra to a Borel σ-algebra. Discussed during Lecture 5.

2 Homotopy

Definition 2.1 (Homotopy): Two continuous maps $f, g : X \to Y$ are homotopic if there is a map called homotopy $H : X \times [0,1] \to Y$ that *continuously deforms* f to g, denoted $f \simeq g$ or $f \overset{H}{\simeq} g$. In general:

$$\begin{aligned} X \times [0,1] &\xrightarrow{H} Y \\ H(x,0) &= f(x) \\ H(x,1) &= g(x) \\ H(x,t) &= tf(x) + (1-t)g(x) \end{aligned}$$

Example 2.1: $1 \overset{x^t}{\simeq} x$ when viewed as x^0 and x^1.

$$\begin{aligned} X \times [0,1] &\xrightarrow{H} Y \\ H(x,0) &= x^0 \\ H(x,1) &= x^1 \\ H(x,t) &= x^t \end{aligned}$$

Another possible homotopy between the same functions is $t \cdot x + (1-t) \cdot 1$, which suggsts that there may be many more of them.

Plots?

Example 2.2: $\{\cdot\} \times [0,1] \to \mathbb{C}$ with $H(x,t) = e^{2\pi i t}$

2.1 Homotopy and Equivalence

Definition 2.2 (Equivalence): A binary relation is called an equivalence relation if it's reflexive, symmetric and transitive.

Theorem 2.3: Homotopy equivalence between spaces is an equivalence relation.

Proof. … □

Theorem 2.4: A homotopy between continuous maps is an equivalence relation.

Proof. … □

2.2 Contractible Spaces

Definition 2.5 (Contractibility): A space is contractible if it is homotopically equivalent to a point (a constant map).

Definition 2.6 (Path-connectedness):

Theorem 2.7: Any convex set is contractible.

Lemma 2.8: Contractibility does not depend on a choice of a point.

Definition 2.9 (Star-convex Set):

Theorem 2.10: Any star-convex set is contractible.

3 Quotient Spaces and Maps

See lectures 2 and 3.

Brouwer's Fixed Point Theorem was mentioned around here. Bring up the context?.

3.1 Quotients and Groups

3.2 Cones of Topological Spaces

4 Retractions and Deformation Retractions

5 Classes of Homotopy Maps

See lectures 4 and 5.

5.1 Mappings of $\mathbb{S}^1$ to Itself

6 Resources

Course page: https://sites.google.com/site/kafedramatematikikau/products-services/homotopy-theory