Introduction to Homotopy Theory

(Notes based on lectures by Sergiy Maksymenko)

Abstract

Homotopy theory studies spaces up to a homotopy, which is a continuous deformation of one continuous function to another. This documents is a work in progress done during a course audit. These notes are taken purposefully in English to strenghten intuition and simplify lookup of concepts in related literature.

Warning: this document may be edited live during audit so watch out for incorrect statements!

Contents

1 Set-theoretic Definitions

Definition 1.1 (BINARY RELATION): A binary relation R on a set X is a set of ordered pairs of elements of X.

Definition 1.2 (EQUIVALENCE RELATION): An equivalence relation \sim is a binary relation that is reflexive, symmetric and transitive.

Definition 1.3 (EQUIVALENCE CLASS OF AN ELEMENT): Given a set X and an equivalence relation \sim , an equivalence class of $a \in X$, denoted [a] is a set $\{x \in S \mid x \sim a\}$

Definition 1.4 (QUOTIENT SET): A quotient set X/\sim (also said "X modulo \sim ") is a set of all equivalence classes of X with respect to \sim .

$$X/\sim = \{[x] : x \in X\}$$

Definition 1.5 (QUOTIENT MAP): A quotient map is a surjective mapping that sends a point in X to its equivalence class, containing it: $q: X \to X/\sim$

Definition 1.6 (QUOTIENT BY SET MEMBERSHIP): TODO. X/A

2 Point-set Topology

Definition 2.1 (TOPOLOGICAL SPACE): A topological space is a pair $\langle X, \tau \rangle$, where X is a set and τ , a topology on X, is a collection of subsets ($\tau \subseteq \mathcal{P}(X)$) called open sets, such that:

- $\emptyset \in \tau$.
- $X \in \tau$.
- ullet au is closed under arbitrary finite intersections.
- τ is closed under arbitrary unions. Maybe find cute notation for this.

Posets.

Definition 2.2 (Trivial Topology): A topological space is called trivial, when the topology on X consists only of \emptyset and X.

Definition 2.3 (DISCRETE TOPOLOGY): A topological space is called discrete, when $\tau = \mathcal{P}(X)$.

2017/03/20 at 13:02:48

Definition 2.4 (Continuous MAP): Let $\langle X, \tau \rangle$ and $\langle Y, \sigma \rangle$ be topological spaces. A map $f: X \to Y$ is **continuous** if:

$$\forall s \in \sigma, f^{-1}(s) \in \tau$$

In plain English, a map is continuous when a preimage of an open set in Y is an open set in X. C(X,Y) denotes a set of all continuous maps between X and Y.

Base of topology and methods of inducing topologies on sets were discussed during Lecture 2. Bonus. Compare topology to a field of sets to a σ -algebra to a Borel σ -algebra. Discussed during Lecture 5.

2.1 Topology Restrictions

Definition 2.5 (T_1 SPACE):

Definition 2.6 (HAUSDORFF SPACE):

2.2 Quotient Topology

Gluing.

Let $\langle X, \tau \rangle$ be a topological space and \sim be an equivalence relation on X.

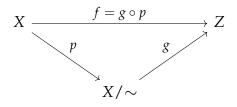
Definition 2.7 (QUOTIENT TOPOGICAL SPACE): By analogy of a set and given $q: \langle X, \tau \rangle \to \langle X/\sim, \tau_{X/\sim} \rangle$,

$$X/\sim = \{[x] : x \in X\}$$

$$\tau_{X/\sim} = \{U \subseteq X/\sim \mid q^{-1}(U) \in \tau\}$$

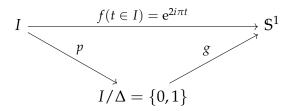
 $\tau_{\rm X/\sim}$ is constructed this way to ensure *q* is continuous.

Theorem 2.8: Given Z is a topological space, g is a surjective map, p is a quotient map and the following diagram,



g is continuous \iff $f = g \circ p$ is continuous.

Example 2.9: Have $z \in \mathbb{C}$, $\mathbb{S}^1 = \{|z| = 1\}$, $I = [0..1] \subset \mathbb{R}^1$.



2017/03/20 at 13:02:48

3 Homotopies Between Continuous Maps

Definition 3.1 (Homotopy): Two continuous maps $f,g:X\to Y$ are homotopic if there is a map called homotopy $H:X\times [0,1]\to Y$ that *continuously deforms* f to g, denoted $f\simeq g$ or $f\overset{H}\simeq g$. In general:

$$X \times [0,1] \xrightarrow{H} Y$$

$$H(x,0) = f(x)$$

$$H(x,1) = g(x)$$

$$H(x,t) = tf(x) + (1-t)g(x)$$

Example 3.2: $1 \stackrel{x^t}{\simeq} x$ when viewed as x^0 and x^1 .

$$X \times [0,1] \xrightarrow{H} Y$$

$$H(x,0) = x^{0}$$

$$H(x,1) = x^{1}$$

$$H(x,t) = x^{t}$$

Another possible homotopy between the same functions is $t \cdot x + (1 - t) \cdot 1$, which suggsts that there may be many more of them.

Plots?

Example 3.3: $\{\cdot\} \times [0,1] \to \mathbb{C}$ with $H(x,t) = e^{2\pi i t}$

Theorem 3.4: A homotopy between continuous maps is an ??.

3.1 Contractible Spaces

Definition 3.5 (CONTRACTIBILITY): A space is contractible if it is homotopically equivalent to a point (a constant map).

Extra definitions:

- $X \times 0$ is a retraction of CX (cone over X)
- homotopic equivalence to a point

Example 3.6: \mathbb{R}^n is contractible to a point.

$$\mathbb{R}^{n} \times [0,1] \xrightarrow{H} \mathbb{R}^{n}$$

$$H(x,0) = 0$$

$$H(x,1) = x$$

$$H(x,t) = tx$$

Definition 3.7 (PATH-CONNECTEDNESS):

Theorem 3.8: Any convex set is contractible.

Lemma 3.9: Contractibility does not depend on a choice of a point.

Definition 3.10 (STAR-CONVEX SET):

Theorem 3.11: A star-convex set is contractible to a point.

$$A \subset \mathbb{R}^n$$
 is (star-)convex
 $a \in A$
 $A \times [0,1] \xrightarrow{H} A$
 $H(x,0) = 0$
 $H(x,1) = x$
 $H(x,t) = at + x(1-t)$

Theorem 3.12: Assuming we know how to build topologies on trees (as in graph theory trees), every finite tree is a contractible topological space.

4 Quotient Spaces and Maps

See lectures 2 and 3. Most of this stuff

4.1 Quotients and Groups

4.2 Cones of Topological Spaces

Brouwer's Fixed Point Theorem was mentioned around here. Bring up the context?.

5 Retractions, Deformations and Deformation Retractions

Definition 5.1 (RETRACTION):

Bring up the example with $\frac{x}{|x|}$ – discussed along with retractions

Definition 5.2 (Deformation): A continuous mapping is a deformation into a subspace $A \subset X$ when

- $H_0 = id_X$
- $\forall t \in [0..1], H_t(A) \subset A$

Definition 5.3 (DEFORMATION RETRACTION):

6 Classes of Homotopy Maps

See lectures 4 and 5.

6.1 Mappings of \mathbb{S}^1 to Itself

7 Homotopy Types

Lecture 7.

Let
$$\langle X, \tau \rangle \langle Y, \sigma \rangle$$
.

A Homotopy Type is a synonym to homotopy equivalences between spaces. It is **not** the homotopy type as in Homotopy Type Theory.

Claim 7.1: Homeomorphisms and topological spaces form a category.

2017/03/20 at 13:02:48 4

Claim 7.2: Homeomorphisms and topological spaces form a group (for all morphisms with one side fixed).

Homeo(X) group of all identity maps of X g is a retraction of Y to X on g(x), g is also left inverse of f $g\circ f\simeq id_x$

Definition 7.3 (Homotopic Equivalence): A mapping $f: X \to Y$ is a homotopic equivalence when $\exists g: Y \to X$ such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$

.

Theorem 7.4: Homotopy equivalence between spaces is an ??.

Proof. ... \Box

Example 7.5: $f: X \to \cdot$ is a homotopic equivalence to a point $\iff X$ is contractible. $H_0 = id \simeq H_1 = f$

Claim 7.6: Holes matter! $\infty \simeq B \simeq o O$

Example 7.7: Glueing of a disk to some space using quotienting. *expanding/reducing?*

Claim 7.8: Homotopy Theory cares only about spaces glued from a finite number of S^n spaces, see Shape Theory.

Definition 7.9 (SIMPLE HOMOTOPY TYPE): A homotopy type is simple if you use a finite number of expansions and reductions

Claim 7.10: This is a cue to working with simplicial complexes.

...Interlude ...

 $f: X \to Y$, f is a proper mapping, $f^{-1}(\text{compact})$ is a compact Define compact

Defining proper homotopy equivalences.

 $\mathbb{R}^n \to \cdot$ is an improper homotopy. Compactness is a bitch.

8 Resources

Course page: https://sites.google.com/site/kafedramatematikikau/products-services/
homotopy-theory

2017/03/20 at 13:02:48 5