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Analysis of the Combined Finite Element and Fourier Interpolation

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Summary. We consider Lagrange interpolation involving trigonometric polynomials of degree $\leqq N$ in one space direction, and piecewise polynomials over a finite element decomposition of mesh size $\leqq h$ in the other space directions. We provide error estimates in non-isotropic Sobolev norms, depending additively on the parameters h and N. An application to the convergence analysis of an elliptic problem, with some numerical results, is given.

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Introduction

In many problems arising from physics, engineering, meteorology, the solution is periodic in one direction (for example the time), and submitted to (nonperiodic) boundary conditions in the remaining directions.

For the numerical approximation of these problems, a natural strategy is the coupling of different methods, such as the Fourier method and the Finite Element method. The discrete solution is then a trigonometric polynomial in the variable of periodicity and a piecewise polynomial in the other variables.

In the error analysis of such schemes, the operator $\mathscr{P}$ matching trigonometric interpolation in one direction and piecewise Lagrange interpolation in the others can be successfully employed. The aim of this paper is to provide error estimates for this operator in the Sobolev norms with respect to the discretization parameters h (the finite element mesh size) and N (the degree of trigonometric polynomials).

The discretization parameters intervene in our estimates in a splitted way, namely as a power of h plus a power of N^{-1}. Moreover, we allow the solution to have different regularities in the different directions. The main interest of these results lies in the fact that h and N^{-1} are not requested to vanish at the same rate. Taking into account the regularities of the solution, the parameters h and N can be freely chosen in order to balance their contributions to the

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error. Hence, when the solution is smooth, the balance can be achieved by choosing N^{-1} asymptotically much larger than h. This is in agreement with the computational experience, which suggests to take advantage of the higher accuracy of spectral methods by working with a low number of frequencies [7].

The function spaces proper to obtain our estimates are the non isotropic Sobolev spaces of $H^{r,s}$-type [8]. Some of their properties are given in Sect. 1.

The main results of this paper, concerning error estimates for the operator $\mathscr{P}$ in L^2 and in H^1 norms are Theorems 2.1, 3.1 and 3.2. The corresponding basic interpolation properties for finite elements [5] and for Fourier methods [9, 12, 4] are employed.

These results are useful whenever a finite element method is combined with the *pseudospectral* (collocation) method (i.e. the commonest Fourier method). If the *spectral* (Galerkin) Fourier method is used instead, the L^2-projection operator over the subspace of trigonometric polynomials of order $\leqq N$ replace the trigonometric interpolation operator in the analysis. This situation is also covered by our results, as stated in Remark 3.1.

We present here only the case of one dimensional finite elements. This is done on purpose in order to simplify the notations and make the arguments as clear as possible. However, as pointed out in Remark 3.2, the extension of our proofs and results to higher dimensions is straightforward. As a matter of fact, one could give a proof of part (ii) of Theorem 3.2 simpler than the one we present. However, such a proof (formally similar to the one of point (i)), only holds for one dimensional finite elements, so we use a more general technique which can be applied also for more dimensions.

In Sect. 4 it is shown how these estimates can be applied to the convergence analysis of an elliptic second order model problem. Some numerical results are also given. Another application to the two dimensional Rayleigh convection problem is contained in [3].

1. Preliminaires and Notations

Let $\mathscr{A} \subset \mathbb{R}^n$ be an open set with locally lipschitz boundary. Throughout the paper $H^s(\mathscr{A}) = W^{s,2}(\mathscr{A})$ is the Sobolev space of order $s \geqq 0$ over $\mathscr{A}$ (cf. [1]), equipped with the Sobolev norm

$$\|u\|_{s,\mathscr{A}}^2 = \sum_{|\alpha| \leqq s} \|D^\alpha u\|_{L^2(\mathscr{A})}^2$$

if s is integer, or

$$\|u\|_{s,\mathscr{A}}^2 = \|u\|_{[s],\mathscr{A}}^2 + |u|_{s,\mathscr{A}}^2$$

if $s = [s] + \sigma$, with $[s]$ = integral part of s, $0 < \sigma < 1$ and

$$|u|_{s,\mathscr{A}}^2 = \sum_{|\alpha| = [s]} \int_{\mathscr{A}} \int_{\mathscr{A}} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2\sigma}} \, dx \, dy.$$

1.1. Non Isotropic Spaces: Some Properties

Let X be a Hilbert space, and denote by $L^2(a,b;X)$ $(a<b)$ the space of measurable functions $f(\eta)$ of (a,b) in X for the Lebesgue measure $d\eta$ and such that $\int_a^b \|f(\eta)\|_X^2 d\eta < \infty$. Set $\Omega=(0,1)$ and $Q=\Omega\times(0,2\pi)\subset\mathbb{R}_x\times\mathbb{R}_y$; following Grisvard [8] (see also Lions-Magenes [10]), let us define for real r, $s\geqq 0$

$$H^{r,s}(Q)=L^2(0,2\pi;H^r(\Omega))\cap H^s(0,2\pi;L^2(\Omega)). \tag{1.1}$$

If s is an integer $H^s(0,2\pi;X)=\{v\,|\,D^{(k)}v\in L^2(0,2\pi;X), 0\leqq k\leqq s\}$. For real s the space $H^s(0,2\pi;X)$ is defined by interpolation. $H^{r,s}(Q)$ is a Hilbert space with the norm

$$\|v\|_{r,s,Q}=(\|v\|^2_{L^2_y(H^r_x)}+\|v\|^2_{H^s_y(L^2_x)})^{\frac{1}{2}},$$

where, for convenience of notations, $H^\alpha_y(H^\beta_x)$ is used instead of $H^\alpha(0,2\pi;H^\beta(\Omega))$, $\alpha\geqq 0$, $\beta\geqq 0$. Similarly, $H^\gamma_x(H^\delta_y)$ will denote the space $H^\gamma(\Omega;H^\delta(0,2\pi))$, $\gamma\geqq 0$, $\delta\geqq 0$.

Lemma 1.1. *For any real $\sigma\geqq 0$, $H^{\sigma,\sigma}(Q)=H^\sigma(Q)$. Moreover if $q=\min(r,s)$, then $H^{r,s}(Q)\subseteq H^q(Q)$.*

Proof. The first part of the result is a consequence of [10], Remark 2.2, p. 6. The second part is then trivial. □

The next result has been proved by Grisvard ([8], Lemma 7.2).

Lemma 1.2. *The following continuous imbedding holds*

$$H^{r,s}(Q)\subseteq H^\gamma_x(H^\delta_y), \quad \text{if } \gamma r^{-1}+\delta s^{-1}=1. \quad \square \tag{1.2}$$

Lemma 1.3. *The following continuous inclusion holds*

$$H^{r,s}(Q)\subseteq C^0(\bar{Q}), \quad \text{if } r^{-1}+s^{-1}<2. \tag{1.3}$$

Proof. The one dimensional subset of $\mathbb{R}^2$

$$I(r,s)=\{(\gamma,\delta)\subseteq\mathbb{R}^2\,|\,\tfrac{1}{2}<\gamma\leqq r, \tfrac{1}{2}<\delta\leqq s, \gamma r^{-1}+\delta s^{-1}=1\}$$

is non empty iff $r^{-1}+s^{-1}<2$. Then there exist $\gamma>\frac{1}{2}$, $\delta>\frac{1}{2}$ such that by Lemma 1.2 and the Sobolev imbedding theorem [1] it follows

$$H^{r,s}(Q)\hookrightarrow H^\gamma_x(H^\delta_y)\hookrightarrow C^0(\bar{\Omega};H^\delta(0,2\pi))\hookrightarrow C^0(\bar{\Omega};C^0([0,2\pi]))=C^0(\bar{Q}). \quad \square$$

Finally we denote by $C^\infty_p(Q)$ the subset of $C^\infty(Q)$ of the functions 2π-periodic with all their derivatives, with respect to the y-direction. Then we set

$$H^{r,s}_p(Q)=\text{closure of } C^\infty_p(Q) \text{ under the norm } \|\cdot\|_{r,s,Q}. \tag{1.4}$$

1.2. Interpolation Operators: Basic Properties

The first part of this Section is devoted to recall some properties of Lagrange finite elements interpolation which will be used in the next Sections. To this end, we introduce some standard notations: $\mathcal{T}_h$=subdivision of Ω into subintervals $I_h=(x_i, x_{i+k})(1 \leqq i \leqq M)$ of length smaller than h; Π^k=Lagrange interpolation operator of degree k over any I_h. Moreover $\mathbb{P}^\ell(\mathcal{A})$ denotes the set of polynomials of degree $\leqq \ell$ over the set $\mathcal{A}$, for any integer $\ell \geqq 0$.

The next Lemma is an easy consequence of Theorem 6.1 of [6] (the case $\theta=1$ follows from the Bramble-Hilbert Lemma [2]).

Lemma 1.4. *Suppose $\rho=j+\theta$, where $0<\theta\leqq 1$ and j is a non-negative integer. Let I be an open interval of $\mathbb{R}$; there exists a positive constant (depending on I and ρ) such that for any $v \in H^\rho(I)$*

$$\operatorname*{Inf}_{p\in\mathbb{P}^j(I)} \|v-p\|_{\rho,I} \leqq C\,|v|_{\rho,I}. \quad \square \tag{1.5}$$

The following estimate holds

$$\|u-\Pi^k u\|_{\mu,\Omega} \leqq C h^{\rho-\mu}\,|u|_{\rho,\Omega} \qquad 0 \leqq \mu \leqq 1, \quad \mu \leqq \rho > \tfrac{1}{2}, \ \rho \leqq k+1. \tag{1.6}$$

It is well known for integral ρ (see e.g. [5], Theorem 3.2.1); for real ρ it can be proved by the same technique but using the more general Lemma 1.4 instead of the Bramble-Hilbert Lemma.

Lemma 1.5. *Assume $u \in H_y^m(H_x^\rho)$ for some $\rho > \frac{1}{2}$ and $m \geqq 1$; then*

$$D_y^m \Pi^k u = \Pi^k D_y^m u \in L^2(Q). \tag{1.7}$$

Proof. By the continuity of the mapping Π^k: $H^\rho(\Omega) \to L^2(\Omega)$, and $x\in\Omega \to D_y^m u(x,\cdot) \in L^2(0,2\pi)$, it follows that $\Pi^k D_y^m u \in L^2(Q)$. Then it is enough to verify that the equality (1.7) holds in $\mathcal{D}'(Q)$.

Let $\{x_j, 1 \leqq j \leqq J = kM-1\}$ denote the set of the nodes of the decomposition $\mathcal{T}_h$; we denote by $\{\phi_j(x)\}$ the relative Lagrange basis. Then $\Pi^k u(x,y) = \sum_j u(x_j, y)\phi_j(x)$, and for any $w \in \mathcal{D}(Q)$ we get

$$\langle D_y \Pi^k u, w\rangle = -\int_0^1 dx \int_0^{2\pi} dy\, \Pi^k u\, \overline{D_y w} = \sum_j \int_0^1 dx\, \phi_j(x) \int_0^{2\pi} dx\, D_y u(x_j, y)\,\bar{w} = \langle \Pi^k D_y u, w\rangle.$$

This proves (1.7) for $m=1$. If $m \geqq 2$ we proceed recursively. $\square$

We introduce now the trigonometric interpolation.

Let $C_p^\infty(0,2\pi)$ be the set of restrictions to $(0,2\pi)$ of the infinitely differentiable functions of $\mathbb{R}_y$ which are 2π-periodic. For any $\sigma \geqq 0$, $H_p^\sigma(0,2\pi)$ will denote the completion of $C_p^\infty(0,2\pi)$ under the Sobolev norm $\|\cdot\|_{\sigma,y} \equiv \|\cdot\|_{H^\sigma(0,2\pi)}$, and $C_p^0([0,2\pi])$ the completion under the maximum norm. Setting $\psi_\ell(y) = \frac{1}{\sqrt{2\pi}}\exp(i\ell y)$ for any $\ell \in \mathbb{Z}$, we define $S_N = \operatorname{span}\{\psi_\ell(y), -N \leqq \ell \leqq N\}$, where $N>0$ is a given integer.

First of all we denote by P_0^N the orthogonal projection operator from $L^2(0,2\pi)$ over S_N. We shall give bounds for the error $I-P_0^N$ in the next Lemma 1.7, whose proof relies on the following result.

Lemma 1.6. *The following inequality*

$$\|w\|_{\sigma,y} \leqq C\,|w|_{\sigma,y} \tag{1.8}$$

holds for any $w \in H_p^\sigma(0,2\pi)$ $(\sigma \geqq 0)$ *such that* $\mu(w) \equiv \int_0^{2\pi} w(y)\,dy = 0$.

Proof. Consider the case $0<\sigma\leqq 1$ and assume by contradiction that there exists a sequence $\{w_n\} \subseteq H_p^\sigma(0,2\pi)$ such that

$$\begin{gathered} \mu(w_n)=0, \qquad \|w_n\|_{\sigma,y}=1 \qquad \text{for } n \geqq 0, \\ \lim_{n\to\infty} |w_n|_{\sigma,y}=0. \end{gathered} \tag{1.9}$$

$\{w_n\}$ is bounded in $H^\sigma(0,2\pi)$, hence by compactness a subsequence $\{w_{n'}\}$ converges to a suitable $\bar{w}$ in $L^2(0,2\pi)$ and a.e. in $(0,2\pi)$. Passing to the limit, we get by (1.9) $\mu(\bar{w})=0$ and $\|\bar{w}\|_{0,y}=1$, so that $\bar{w}$ cannot be constant. On the other hand, by Fatou's lemma we have $|\bar{w}|_{\sigma,y}=0$, which implies $\bar{w}$ constant a.e. This proves the result for $0<\sigma\leqq 1$. Assume now by induction that (1.8) holds for any $\sigma \leqq m$, m integer $\geqq 1$, and prove it for $m<\sigma\leqq m+1$. We get first of all

$$\|w\|_{m,y} \leqq C\,|w|_{m,y} \equiv C\,\|w^{(m)}\|_{0,y}.$$

On the other hand, if $s=\sigma-m$, $w^{(m)} \in H_p^s(0,2\pi)$ and $\mu(w^{(m)})=w^{(m-1)}(2\pi)-w^{(m-1)}(0)=0$ by periodicity. Hence

$$\|w^{(m)}\|_{0,y} \leqq C\,|w^{(m)}|_{s,y} \equiv C\,|w|_{\sigma,y}$$

by the previous part, and the proof is complete. □

Lemma 1.7. *Let* $u \in H_p^\sigma(0,2\pi)$, $\sigma \geqq 0$. *Then*

$$\|u-P_0^N u\|_{v,y} \leqq C N^{v-\sigma}\,|u|_{\sigma,y}, \qquad 0 \leqq v \leqq \sigma. \tag{1.10}$$

Proof. Only the case $\sigma>0$ is non-trivial. Setting $w=u-\mu(u)$, we have

$$\|u-P_0^N u\|_{v,y} = \|w-P_0^N w\|_{v,y} \leqq C N^{v-\sigma}\,\|w\|_{\sigma,y}$$

(cfr. [4], Theorem 1.1). By Lemma 1.6, $\|w\|_{\sigma,y} \leqq C|w|_{\sigma,y}=C|u|_{\sigma,y}$, so (1.10) holds. □

We introduce now the points $y_\ell = 2\pi\ell/(2N+1)$ for $\ell=0,\dots,2N$, and we denote by P^N: $C_p^0(0,2\pi) \to S_N$ the trigonometric interpolation operator at the points $\{y_\ell\}$. Moreover we introduce the discrete "inner product"

$$(u,v)_N = 2\pi/(2N+1) \sum_{\ell=0}^{2N} u(y_\ell)\,\overline{v(y_\ell)} \tag{1.11}$$

which coincides with the L^2-inner product over S_N, ([9]), i.e.,

$$(u,v)_N = \int_0^{2\pi} u(y)\,\overline{v(y)}\,dy, \qquad \forall u,v \in S_N. \tag{1.12}$$

Then $P^N u = \sum_{\ell=-N}^{N} (u, \psi_\ell)_N \psi_\ell$ and we can prove the following result in a similar way to Lemma 1.5.

Lemma 1.8. *Assume* $u \in H_x^m(H_y^\rho)$ *for some* $\rho > 1/2$ *and* $m \geqq 1$. *Then*

$$D_x^m P^N u = P^N D_x^m u \in L^2(Q). \quad \square \tag{1.13}$$

Finally we state the approximation result for P^N.

Lemma 1.9. *Let* $u \in H_p^\sigma(0, 2\pi)$, $\sigma > 1/2$. *Then*

$$\|u - P^N u\|_{\nu,y} \leqq C N^{\nu-\sigma} |u|_{\sigma,y}, \qquad 0 \leqq \nu \leqq \sigma. \tag{1.14}$$

Proof. Similar to the previous one, using this time Theorem 2.1 of [4] instead of Theorem 1.1. $\square$

2. Global Interpolation: L^2-Error Estimates

We introduce the cartesian set $Q_{h,N}$ of the interpolation points $P_{j\ell} = (x_j, y_\ell)$, $1 \leqq j \leqq J$, $0 \leqq \ell \leqq 2N$. Let us set $C_p^0(\bar{Q}) = C^0(\bar{\Omega}) \otimes C_p^0([0, 2\pi])$, and $\mathscr{V}_{h,N} = V_h \otimes S_N$, where

$$V_h = \{v \in H^1(\Omega) \,|\, v_{|I_h} \in \mathbb{P}^k(I_h) \,\forall I_h \in \mathscr{T}_h\}.$$

The operator

$$\mathscr{P} = \Pi^k P^N = P^N \Pi^k \colon \; C_p^0(\bar{Q}) \to \mathscr{V}_{h,N} \tag{2.1}$$

is the (unique) interpolation operator at the points $P_{j\ell} \in Q_{h,N}$. In this section we want to provide upper bounds for $u - \mathscr{P}u$ vanishing with h and N^{-1}.

We will constantly use a reference domain, in agreement with a phylosophy well known for Finite Elements, and first used by Pasciak [12] for Fourier methods.

Let $I_h = (x_i, x_{i+k})$ be the current subinterval of $\mathscr{T}_h$, and set $K = I_h \times (0, 2\pi)$; then $\bar{Q} = \bigcup \bar{K}$. Define $\hat{K} = (0,1) \times (0, 2\pi N)$ and consider the dilation $F\colon K \to \hat{K}$, defined by $F((x,y)^T) = (\hat{x}, \hat{y})^T = (h_k^{-1}(x - x_i), Ny)^T$, where $h_K \leqq h$. For any function w defined on K we set $\hat{w} = w \circ F^{-1}$, so that $\hat{w}(\hat{x}, \hat{y}) = w(x,y) \,\forall (x,y) \in K$. Setting $V_K = \mathbb{P}^k(I_h) \otimes S_N$, and denoting with $\hat{S}_N$ the image of S_N under F^{-1}, it follows that $\hat{V} = \mathbb{P}^k(0,1) \otimes \hat{S}_N$ is the image of V_K under F^{-1}. For convenience $|\hat{w}|_{\nu,\hat{y}}$ and $|\hat{w}|_{\mu,\hat{x}}$ will be used instead of $|\hat{w}(\hat{x}, \cdot)|_{\nu,(0,2\pi N)}$ and $|\hat{w}(\cdot, \hat{y})|_{\mu,(0,1)}$, respectively.

It is an easy matter to check that for any $\mu \geqq 0$ and $\sigma \geqq 0$

$$|w|_{\mu,I_h} = h_K^{\frac{1}{2}-\mu} |\hat{w}|_{\mu,\hat{x}}, \tag{2.2}$$

and

$$|v|_{\sigma,y} = N^{\sigma-\frac{1}{2}} |\hat{v}|_{\sigma,\hat{y}}. \tag{2.3}$$

Moreover, the following lemma states the equivalence of the Sobolev norms of any order over $\hat{S}_N$.

Lemma 2.1. *For any $\sigma \geqq 0$, there exists a constant $C > 0$ independent of N such that*

$$\|\hat{v}\|_{\sigma,\hat{y}} \leqq C\,\|\hat{v}\|_{0,\hat{y}}, \qquad \forall \hat{v} \in \hat{S}_N. \tag{2.4}$$

Proof. By (2.3) we get $|\hat{v}|_{\sigma,\hat{y}} = N^{\frac{1}{2}-\sigma}|v|_{\sigma,y}$. On the other hand by the inverse inequality (see, e.g., [4], Proposition 1.1) we have $|v|_{\sigma,y} \leqq CN^{\sigma}|v|_{0,y}$, and again by (2.3) $|v|_{0,y} = N^{-\frac{1}{2}}|\hat{v}|_{0,\hat{y}}$. This proves the inequality $|\hat{v}|_{\sigma,\hat{y}} \leqq C\,\|\hat{v}\|_{0,\hat{y}}$, whence (2.8). □

We define

$$\hat{\Pi}:\ C^0([0,1]) \to \mathbb{P}^k(0,1), \qquad \hat{\Pi}\hat{w}(\hat{x}_j) = \hat{w}(\hat{x}_j), \qquad \hat{x}_j = (j-1)/k, \qquad 1 \leqq j \leqq k+1,$$

$$\hat{P}:\ C_p^0([0,2\pi]) \to \hat{S}_N, \qquad \hat{P}\hat{v}(\hat{y}_\ell) = \hat{v}(\hat{y}_\ell), \qquad \hat{y}_\ell = 2\pi\ell/(2N+1), \qquad 0 \leqq \ell \leqq 2N,$$

$$\hat{\mathscr{P}}:\ C^0([0,1]) \otimes C_p^0([0,2\pi]) \to \hat{V}, \qquad \hat{\mathscr{P}} = \hat{\Pi}\hat{P} = \hat{P}\hat{\Pi}.$$

Then $\hat{\mathscr{P}}\hat{u}(\hat{x}_j,\hat{y}_\ell) = \hat{u}(\hat{x}_j,\hat{y}_\ell)$. Moreover we have

$$\widehat{\Pi^k w} = \hat{\Pi}^k \hat{w} \qquad \forall w \in C^0(\bar{I}_h), \tag{2.5}$$

$$\widehat{P^N v} = \hat{P}\hat{v} \qquad \forall v \in C_p^0([0,2\pi]), \tag{2.6}$$

$$(\hat{\Pi} - I)\hat{w} = 0 \qquad \forall \hat{w} \in \mathbb{P}^k(0,1), \tag{2.7}$$

$$(\hat{P} - I)\hat{v} = 0 \qquad \forall \hat{v} \in \hat{S}_N, \tag{2.8}$$

$$(\hat{\mathscr{P}} - I)\hat{u} = 0 \qquad \forall \hat{u} \in \hat{V}. \tag{2.9}$$

We shall give now an estimate for the error $\|u - \mathscr{P}u\|_{0,Q}$. Suppose $u \in H_p^{r,s}(Q)$, with $r^{-1} + s^{-1} < 2$. For any $u^* \in \hat{V}$ we have by (2.9)

$$\begin{aligned}\|u - \mathscr{P}u\|_{0,Q}^2 &= N^{-1}\sum_K h_K \|\hat{u} - \hat{\mathscr{P}}\hat{u}\|_{0,\hat{K}}^2 = N^{-1}\sum_K h_K \|(I - \hat{\mathscr{P}})(\hat{u} - u^*)\|_{0,\hat{K}}^2 \\ &\leqq N^{-1}\sum_K h_K \|I - \hat{\mathscr{P}}\|_{\mathscr{L}(H_p^{r,s}(K),L^2(\hat{K}))}^2 \cdot \|\hat{u} - u^*\|_{r,s,\hat{K}}^2\end{aligned} \tag{2.10}$$

Lemma 2.2. *For any $r \geqq 0$, $s \geqq 0$ such that $r^{-1} + s^{-1} < 2$, there exists a positive constant C independent of h and N such that*

$$\|I - \hat{\mathscr{P}}\|_{\mathscr{L}(H_p^{r,s}(\hat{K}),L^2(\hat{K}))} \leqq C. \tag{2.11}$$

Proof. It is sufficient to prove that $\|\hat{\mathscr{P}}\|_{\mathscr{L}(H_p^{r,s}(\hat{K}),L^2(\hat{K}))}$ can be bounded independently of h and N.

For any $\hat{u}$ continuous on $\hat{K}$, and periodic in the y-direction, we have

$$\hat{\mathscr{P}}\hat{u} = \sum_{j=1}^{k+1} \hat{P}\hat{u}(\hat{x}_j,\hat{y})\,L_j(\hat{x}), \quad \text{where} \quad L_j(\hat{x}) = \prod_{i \neq j} (\hat{x} - \hat{x}_i)/(\hat{x}_j - \hat{x}_i).$$

Since $|L_j(\hat{x})| = 1\ \forall \hat{x} \in [0,1]$, by (1.12) it follows that

$$\int_{\hat{K}} |\hat{\mathscr{P}}\hat{u}|^2\, d\hat{x}\, d\hat{y} \leqq (k+1) \sum_{j=1}^{k+1} \int_0^{2\pi N} |\hat{P}\hat{u}(\hat{x}_j, \hat{y})|^2\, d\hat{y}$$

$$= (k+1) N \sum_{j=1}^{k+1} \int_0^{2\pi} |P^N u(x_j, y)|^2\, dy = (k+1)\, 2\pi N/(2N+1) \sum_{j=1}^{k+1} \sum_{\ell=0}^{2N} |u(x_j, y_\ell)|^2$$

$$= (k+1)\, 2\pi N/(2N+1) \sum_{j=1}^{k+1} \sum_{\ell=0}^{2N} |\hat{u}(\hat{x}_j, N y_\ell)|^2.$$

Setting $\hat{K}_{j,\ell} = (\hat{x}_j, \hat{x}_{j+1}) \times (N y_\ell, N y_{\ell+1})$, we have meas $(\hat{K}_{j\ell}) = 2\pi/k$, and $\bar{\hat{K}} = \bigcup \bar{\hat{K}}_{j,l}$. Using the continuity of the Sobolev imbedding $H^{r,s}(\hat{K}_{j\ell}) \subseteq C^0(\bar{\hat{K}}_{j\ell})$ (see Lemma 1.3), it follows that

$$\sum_{j=1}^{k+1} \sum_{\ell=1}^{2N} |\hat{u}(\hat{x}_j, N y_\ell)|^2 \leqq C_1 \sum_{j=1}^{k+1} \sum_{\ell=0}^{2N} \|\hat{u}\|^2_{H^{r,s}(\hat{K}_{j\ell})} \leqq C_1 \|\hat{u}\|^2_{H^{r,s}(\hat{K})}$$

with a constant C_1 independent of h and N. The previous inequality concludes the proof. $\square$

Let $\hat{\Pi}_0$ denote the orthogonal projection operator upon $\mathbb{P}_k(0,1)$ in the $L^2(0,1)$-norm. Since for any $\hat{v} \in H^\rho(0,1)$ and $\rho \geqq 0$ one has

$$\|\hat{v} - \hat{\Pi}_0 \hat{v}\|_{\rho,\hat{x}} \leqq \|I - \hat{\Pi}_0\|_{\mathscr{L}(H^\rho(0,1), H^\rho(0,1))} \cdot \inf_{v^* \in \mathbb{P}_k(0,1)} \|\hat{v} - v^*\|_{\rho,\hat{x}}$$

we get by (1.5)

$$\|\hat{v} - \hat{\Pi}_0 \hat{v}\|_{\rho,\hat{x}} \leqq C\, |\hat{v}|_{\rho,\hat{x}}, \qquad \forall \hat{v} \in H^\rho(0,1),\ 0 \leqq \rho \leqq k+1. \tag{2.12}$$

Similarly, $\hat{P}_0$ will denote the orthogonal projection operator upon $\hat{S}_N$ in the $L^2(0, 2\pi N)$-norm. For any $w \in L^2(0, 2\pi)$ we get $\widehat{P_0^N w} = \hat{P}_0 \hat{w}$. Then by (2.3) and (1.10) we obtain

$$\|\hat{w} - \hat{P}_0 \hat{w}\|_{\sigma,\hat{y}} \leqq C\, |\hat{w}|_{\sigma,\hat{y}}, \qquad \forall \hat{w} \in H_p^\sigma(0, 2\pi N),\ 0 \leqq \sigma, \tag{2.13}$$

for a constant C independent of N.

Lemma 2.3. *Assume $\hat{u} \in H_p^{\rho,\sigma}(\hat{K})$, with $\rho \leqq k+1$. There exists a constant C independent of N and $\hat{u}$ such that*

$$\inf_{u^* \in \hat{V}} \|\hat{u} - u^*\|^2_{\rho,\sigma,\hat{K}} \leqq C \left[\int_0^{2\pi N} d\hat{y}\, |\hat{u}|^2_{\rho,\hat{x}} + \int_0^1 d\hat{x}\, |\hat{u}|^2_{\sigma,\hat{y}} \right]. \tag{2.14}$$

Proof. Take $u^* = \hat{\Pi}_0 \hat{P}_0 \hat{u} = \hat{P}_0 \hat{\Pi}_0 \hat{u}$. For any $\hat{w} \in H^\rho(0,1)$, we have

$$\|\hat{\Pi}_0 \hat{w}\|_{\rho,\hat{x}} \leqq C_\rho \|\hat{\Pi}_0 \hat{w}\|_{0,\hat{x}} \leqq C_\rho \|\hat{w}\|_{0,\hat{x}}$$

by the equivalence of norms in $\mathbb{P}_k(0,1)$ and the definition of $\hat{\Pi}_0$. Then

$$\int_0^{2\pi N} d\hat{y}\, \|\hat{u} - u^*\|^2_{\rho,\hat{x}} \leqq 2 \int_0^{2\pi N} d\hat{y}\, \|\hat{u} - \hat{\Pi}_0 \hat{u}\|^2_{\rho,\hat{x}} + 2 \int_0^{2\pi N} d\hat{y}\, \|\hat{\Pi}_0(\hat{u} - \hat{P}_0 \hat{u})\|^2_{\rho,\hat{x}}$$

$$\leqq 2 \int_0^{2\pi N} d\hat{y}\, \|\hat{u} - \hat{\Pi}_0 \hat{u}\|^2_{\rho,\hat{x}} + 2 C_\rho \int_0^1 d\hat{x}\, \|\hat{u} - \hat{P}_0 \hat{u}\|^2_{0,\hat{y}}.$$

Similarly, if $\hat{v}\in H_p^\sigma(0,2\pi N)$, we have

$$\|\hat{P}_0\hat{v}\|_{\sigma,\hat{y}}\leqq C_\sigma\|\hat{P}_0\hat{v}\|_{0,\hat{y}}\leqq C_\sigma\|\hat{v}\|_{0,\hat{y}}$$

by Lemma 1.2 and the definition of $\hat{P}_0$. Then

$$\begin{aligned}\int_0^1 d\hat{x}\,\|\hat{u}-u^*\|_{\sigma,\hat{y}}^2 &\leqq 2\int_0^1 d\hat{x}\,\|\hat{u}-\hat{P}_0\hat{u}\|_{\sigma,\hat{y}}^2+2\int_0^1 d\hat{x}\,\|\hat{P}_0(\hat{u}-\hat{\Pi}_0\hat{u})\|_{\sigma,\hat{y}}^2\\ &\leqq 2\int_0^1 d\hat{x}\,\|\hat{u}-\hat{P}_0\hat{u}\|_{\sigma,\hat{y}}^2+2C_\sigma\int_0^{2\pi N} d\hat{y}\,\|\hat{u}-\hat{\Pi}_0\hat{u}\|_{\rho,\hat{x}}^2.\end{aligned}$$

The result follows by estimates (2.12) and (2.13). □

By (2.10), Lemmata 2.2 and 2.3, and by (2.2), (2.3), we get the following estimate.

Theorem 2.1. *Assume $u\in H^{r,s}(Q)$ with $r^{-1}+s^{-1}<2$, and set $\bar{r}=\min(r,k+1)$. There exists a constant C independent of h, N and u, such that*

$$\|u-\mathscr{P}u\|_{0,Q}\leqq C(h^{\bar{r}}+N^{-s})\left[\int_0^{2\pi} dy\,|u|_{\bar{r},x}^2+\int_0^1 dx\,|u|_{s,y}^2\right]^{\frac{1}{2}}. \quad \square \tag{2.15}$$

3. H^1-Error Estimates

An H^1-error estimate for the interpolation operator $\mathscr{P}$ can be derived using the results stated in the previous section. Suppose $u\in H_p^{r,s}(Q)$ with $r\geqq 1$, $s\geqq 1$, $r^{-1}+s^{-1}<2$. For any $u^*\in\hat{V}$ we have

$$\begin{aligned}\|D_x(u-\mathscr{P}u)\|_{0,Q}^2 &= N^{-1}\sum_K h_K^{-1}\|D_{\hat{x}}(\hat{u}-\hat{\mathscr{P}}\hat{u})\|_{0,\hat{K}}^2\\ &\leqq N^{-1}\sum_K h_K^{-1}\|I-\hat{\mathscr{P}}\|_{\mathscr{L}(H_p^{r,s}(\hat{K}),H^1(\hat{K}))}^2\cdot\|\hat{u}-u^*\|_{r,s,\hat{K}},\end{aligned}\tag{3.1}$$

and

$$\begin{aligned}\|D_y(u-\mathscr{P}u)\|_{0,Q}^2 &= N\sum_K h_K\|D_{\hat{x}}(\hat{u}-\hat{\mathscr{P}}\hat{u})\|_{0,\hat{K}}^2\\ &\leqq N\sum_K h_K\|I-\hat{\mathscr{P}}\|_{\mathscr{L}(H^{r,s}(\hat{K}),H^1(\hat{K}))}\cdot\|\hat{u}-u^*\|_{r,s,\hat{K}}.\end{aligned}\tag{3.2}$$

We note that for any $\hat{w}\in\hat{V}$ the inequality $\|\hat{w}\|_{1,\hat{K}}\leqq C\|\hat{w}\|_{0,\hat{K}}$ holds with a constant C independent of h and N. This follows from Lemma 2.1 and the equivalence of all norms over $\mathbb{P}_k(0,1)$. Then by Lemma 2.2 we get

$$\|I-\hat{\mathscr{P}}\|_{\mathscr{L}(H_p^{r,s}(\hat{K}),H^1(\hat{K}))}\leqq C. \tag{3.3}$$

Hence, using Lemma 2.3, inequalities (2.2) and (2.5) and Theorem 2.1, and assuming (in the next Theorem only) that τ_h be a quasi-uniform decomposition of Ω [5], we can conclude with the following result.

Theorem 3.1. *Assume $u\in H_p^{r,s}(Q)$ with $r\geqq 1$, $s\geqq 1$ and $r^{-1}+s^{-1}<2$. There exists a constant C independent of h, N and u such that*

$$\|u-\mathscr{P}u\|_{1,Q} \leqq C\{h^{\bar{r}-1}(1+hN)+N^{1-s}(1+h^{-1}N^{-1})\}$$
$$\cdot\left\{\int_0^{2\pi} dy\,|u|_{\bar{r},x}^2+\int_0^1 dx\,|u|_{s,y}^2\right\}^{\frac{1}{2}}. \quad \square \tag{3.4}$$

The previous inequality exhibits a dependence on the coupled term hN. In some situations it is reasonable to assume that h and N^{-1} are of the same order. This amounts to require that the distances between the interpolation points in the x-direction and in the y-direction be comparable. In the case of rectangular finite elements, this assumption is the usual hypothesis of *regular decomposition* ([5]).

Corollary 3.1. *In the same hypotheses of Theorem* 3.1, *assume in addition that there exist two constants* $C_1>0$ *and* $C_2>0$ *such that* $C_1 \leqq hN \leqq C_2$. *Then*

$$\|u-\mathscr{P}u\|_{1,Q} \leqq C(h^{\bar{r}-1}+N^{1-s})\left(\int_0^{2\pi} dy\,|u|_{\bar{r},x}^2+\int_0^1 dx\,|u|_{s,y}^2\right)^{\frac{1}{2}}. \quad \square \tag{3.5}$$

However, it is possible to achieve an "uncoupled estimate" without assumptions on the behaviour of hN, by requiring a mild extra regularity to u.

Theorem 3.2. *Assume* $u\in H_p^{r,s}(Q)$ *with* $r>\frac{3}{2}$, $s>\frac{3}{2}$ *and* $D_x u\in H^{0,s-1}(Q)$, $D_y u\in H^{r-1,0}(Q)$. *Set* $\bar{r}=\min(r,k+2)$. *Then there exists a constant* C *independent of* h *and* N *such that*

$$\|u-\mathscr{P}u\|_{1,Q} \leqq C(h^{\bar{r}-1}+N^{1-s})\left(\int_0^{2\pi} dy\,|u|_{\bar{r},x}^2+\int_0^1 dx\,|u|_{s,y}^2\right.$$
$$\left.+\int_0^{2\pi} dy\,|D_y u|_{\bar{r}-1,x}^2+\int_0^1 dx\,|D_x u|_{s-1,y}^2\right)^{\frac{1}{2}}. \tag{3.6}$$

Proof. (i) We have

$$\|D_y(u-\mathscr{P}u)\|_{0,k}^2 \leqq 2\int_0^1 dx(\|u-P^N u\|_{1,y}^2+\|P^N(u-\Pi^k u)\|_{1,y}^2).$$

By (1.14) we get

$$\int_0^1 dx\,\|u-P^N u\|_{1,y}^2 \leqq CN^{2(1-s)}\int_0^1 dx\,|u|_{s,y}^2.$$

On the other hand, by the continuity of P^N in $H_p^1(0,2\pi)$, by Lemma 1.5 and estimate (1.6) it follows

$$\int_0^1 dx\,\|P^N(u-\Pi^k u)\|_{1,y}^2 \leqq \int_0^1 dx\,\|u-\Pi^k u\|_{1,y}^2$$
$$\leqq \int_0^{2\pi} dy\,\|D_y u-\Pi^k D_y u\|_{0,x}^2 \leqq Ch^{2(\bar{r}-1)}\int_0^{2\pi} dy\,|D_y u|_{\bar{r}-1,s}^2.$$

ii) We consider now the term $D_x(u-\mathscr{P}u)$. We could argue as previously, using the continuity of Π^k in $H^1(0,1)$, but this property does not hold for more dimensional finite elements. So, we follow a different strategy, which can be trivially extended to more dimensions. For any $u^*\in\hat{V}$ we have

$$\|D_x(u-\mathscr{P}u)\|_{0,Q}^2 = N^{-1}\sum_K h_K^{-1}\|D_{\hat{x}}(I-\hat{\mathscr{P}})(\hat{u}-u^*)\|_{0,\hat{K}}^2. \tag{3.7}$$

Assume for the moment that the following estimates hold:

$$\|D_{\hat{x}}\hat{\mathscr{P}}(\hat{u}-u^*)\|_{0,\hat{K}} \leqq C\,\|D_{\hat{x}}(\hat{u}-u^*)\|_{\bar{r}-1,s-1,\hat{K}}, \tag{3.8}$$

$$\inf_{u^*\in\hat{V}}\|D_x(\hat{u}-u^*)\|_{\bar{r}-1,s-1,\hat{K}}^2 \leqq C\left[\int_0^{2\pi N} d\hat{y}\,|\hat{u}|_{\bar{r},\hat{x}}^2 + \int_0^1 d\hat{x}\,|D_{\hat{x}}\hat{u}|_{s-1,\hat{y}}^2\right] \tag{3.9}$$

for a constant C independent of N. Then (3.6) follows, using (2.2) and (2.3).

We prove now (3.8). We set for convenience $\hat{w}=\hat{u}-u^*$, and we note that $D_{\hat{x}}\hat{\mathscr{P}}\hat{w}=\hat{P}D_{\hat{x}}\hat{\Pi}\hat{w}$, since $D_{\hat{x}}\hat{\Pi}\hat{w}$ is continuous on the closure of $\hat{K}$ by the assumptions on u. Then by (1.12)

$$\int_0^1 d\hat{x}\int_0^{2\pi N}|\hat{P}D_{\hat{x}}\hat{\Pi}\hat{w}|^2 d\hat{y} = 2\pi\sum_{j=0}^{2N}\int_0^1 d\hat{x}\,|D_{\hat{x}}\hat{\Pi}\hat{w}(\hat{x},N y_j)|^2. \tag{3.10}$$

Set $\hat{\phi}(\hat{x})=\hat{w}(\hat{x},N y_j)$ for a fixed j. We have, for any $p\in\mathbb{P}_0(0,1)$

$$\begin{aligned}\|D_{\hat{x}}\hat{\Pi}\hat{\phi}\|_{0,\hat{x}} &= \|D_{\hat{x}}\hat{\Pi}(\hat{\phi}+p)\|_{0,\hat{x}}\\ &\leqq \|\hat{\Pi}\|_{\mathscr{L}(H^\rho(0,1),H^1(0,1))}\|\hat{\phi}+p\|_{\rho,\hat{x}}\end{aligned} \tag{3.11}$$

where $\rho>\frac{1}{2}$. On the other hand, if $\rho\geqq 1$, we get by (1.5)

$$\begin{aligned}\inf_{p\in\mathbb{P}_0}\|\hat{\phi}+p\|_{\rho,\hat{x}}^2 &\leqq \inf_{p\in\mathbb{P}_0}\|\hat{\phi}+p\|_{0,\hat{x}}^2+\|D_{\hat{x}}\hat{\phi}\|_{\rho-1,\hat{x}}^2\\ &\leqq C\,|\hat{\phi}|_{1,\hat{x}}^2+\|D_{\hat{x}}\hat{\phi}\|_{\rho-1,\hat{x}}^2.\end{aligned}$$

Hence, by (3.11)

$$\int_0^1 d\hat{x}\,|D_{\hat{x}}\hat{\Pi}\hat{w}(\hat{x},N y_j)|^2 \leqq C\,\|D_{\hat{x}}\hat{w}(\cdot,N y_j)\|_{\rho-1,\hat{x}}^2. \tag{3.13}$$

Finally the term on the right can be bounded by

$$C\,\|D_{\hat{x}}\hat{w}\|_{H^{\bar{r}-1,s-1}((0,1)\times(N_{y_j},N_{y_{j+1}}))}$$

due to [10], Theorem 2.1, taking $\rho=1+(\bar{r}-1)(s-3/2)/(s-1)$. This concludes the proof of (3.8).

We prove now (3.9). Take $u^*=\hat{\Pi}_0\hat{P}_0\hat{u}$ as in the proof of Lemma 2.3. Then

$$\begin{aligned}\|D_x(\hat{u}-\hat{\Pi}_0\hat{P}_0\hat{u})\|_{\bar{r}-1,0}^2 &\leqq 2\int_0^{2\pi N} d\hat{y}(\|\hat{u}-\hat{\Pi}_0\hat{u}\|_{\bar{r},\hat{x}}^2+\|D_{\hat{x}}\hat{\Pi}_0(\hat{u}-\hat{P}_0\hat{u})\|_{\bar{r}-1,\hat{x}}^2)\\ &\leqq 2\int_0^{2\pi N} d\hat{y}\,|\hat{u}|_{\bar{r},\hat{x}}^2 + C_{\bar{r}}\int_0^{2\pi N} d\hat{y}\,\|D_{\hat{x}}\hat{\Pi}_0(\hat{u}-\hat{P}_0\hat{u})\|_{0,\hat{x}}^2\end{aligned}$$

due to (2.11) and to the uniform equivalence of norms over $\mathbb{P}_k(0, 1)$. Arguing as in the proof of (3.11)–(3.12), we get

$$\|D_{\hat{x}}\hat{\Pi}_0(\hat{u}-\hat{P}_0\hat{u})\|_{0,\hat{x}} \leqq C\,\|D_{\hat{x}}(\hat{u}-\hat{P}_0\hat{u})\|_{0,\hat{x}},$$

so that, by (2.12) we obtain

$$\int_0^{2\pi N} d\hat{y}\,\|D_{\hat{x}}\hat{\Pi}_0(\hat{u}-\hat{P}_0\hat{u})\|^2_{0,\hat{x}} \leqq C\int_0^1 d\hat{x}\,|D_{\hat{x}}\hat{u}|^2_{s-1,\hat{y}}.$$

On the other hand, since $\hat{P}_0$ commutes with $D_{\hat{x}}$, we have

$$\|D_{\hat{x}}(\hat{u}-\hat{\mathscr{P}}\hat{u})\|^2_{0,s-1,\hat{K}} \leqq 2\,\|D_{\hat{x}}\hat{u}-\hat{P}_0 D_{\hat{x}}\hat{u}\|^2_{0,s-1,\hat{K}} + 2\,\|\hat{P}_0 D_{\hat{x}}(\hat{u}-\hat{\Pi}_0\hat{u})\|^2_{0,s-1,\hat{K}}.$$

By Lemma 2.1 we have

$$\|\hat{P}_0 D_{\hat{x}}(\hat{u}-\hat{\Pi}_0\hat{u})\|_{0,s-1,\hat{K}} \leqq C\,\|D_{\hat{x}}\hat{u}-D_{\hat{x}}\hat{\Pi}_0\hat{u}\|^2_{0,\hat{K}},$$

hence we can conclude the proof of (3.9) by (2.12) and (2.11). □

Corollary. *Assume $u \in H_p^{r,r}(Q)$, $r > 3/2$. Then there exists a constant C indepedent of h, N and u such that*

$$\|u-\mathscr{P}u\|_{1,Q} \leqq C(h^{\bar{r}-1}+N^{1-r})\,\|u\|_{r,Q}. \quad \square \tag{3.14}$$

Remark 3.1. In some applications it is of interest to consider finite elements interpolation combined with trigonometric L^2 projection. For the analysis of this approximation one is led to deal with the operator $\mathscr{P}^* = \Pi^k P_0^N$ instead of $\mathscr{P}$. It is an easy matter to check that all the results established for $\mathscr{P}$ still holds for $\mathscr{P}^*$, going back to the proofs given until now, and using (1.12) instead of (1.8).

Since $\mathscr{P}^*$ is defined for any function $u \in H_x^\rho(L_y^2)$, $\rho > \frac{1}{2}$, the condition $r^{-1} + s^{-1} < 2$ can be relaxed into $r > \frac{1}{2}$, $s \geqq 0$ in Theorem 2.1, and dropped in Theorem 3.1. Finally, in Theorem 3.2 the condition $s > 3/2$ can be weakened in $s \geqq 1$. □

Remark 3.2. The orthogonal projection operator from $L^2(Q)$ onto $\mathscr{V}_{h,N}$ has been studied in [11]. □

Remark 3.3. The results given in Theorems 2.1, 3.1 and 3.2 can be extended to the case of finite elements in more dimensions. Ω will be now a polygonal domain in $\mathbb{R}^d$ (d > 1), $\mathscr{T}_h$ a regular decomposition of Ω in triangles of diameter less than h, Π^k is the standard Lagrange interpolation operator associated to $\mathscr{T}_h$ by polynomials of degree k. Theorems 2.1 and 3.1 still hold with the obvious change in notations assuming now $dr^{-1}+s^{-1}<2$. Indeed, with this assumption, the Sobolev imbedding $H^{r,s}(Q) \subseteq C^0(\bar{Q})$ holds, so the operator $\mathscr{P}$ is defined and continuous on $H^{r,s}(Q)$. In the proof of Theorem 3.2, we have to choose $\rho > \max(1, d/2)$ to get estimate (3.13). Hence, (3.6) holds assuming $r > 1 + d/2$ and $s > 3/2$. □

4. An Example

To clarify how to apply some of the results given in this paper, we present an application to the convergence analysis of a combined finite element and pseudospectral method to approximate a linear model problem.

Let f, $\underline{b}$ and b_0 be some functions in Q, and consider the problem

$$\begin{aligned} &-\Delta u+\operatorname{div}\underline{b}u+b_0 u=f \quad \text{in } Q,\\ &u(0,y)=u(1,y)=0 \quad \forall\, y\in[0,2\pi],\\ &u(x,0)=u(x,2\pi),\ u_x(x,0)=u_x(x,2\pi) \quad \forall\, x\in[0,1]. \end{aligned} \tag{4.1}$$

Assuming that

$$\tfrac{1}{2}\operatorname{div}\underline{b}+b_0\geqq\beta>0 \quad \text{in } Q, \tag{4.2}$$

it is classical to deduce that problem (3.1) has a unique solution in $V=\{\phi\in H_p^{1,1}(Q)|\phi(0,y)=\phi(1,y)=0\ \forall\, y\in[0,2\pi]\}$ if $f\in V'$. A combined finite element-spectral approximation of (4.1) is a global Galerkin procedure in the space $\mathscr{V}_{h,N}^0=V_h^0\otimes S_N$ (where $V_h^0=V_h\cap H_0^1(0,1)$), which is a subspace of V. Then it consists in looking for a solution $U\in\mathscr{V}_{h,N}^0$ such that

$$\iint_Q \{\nabla U\cdot\overline{\nabla v}+(\operatorname{div}\underline{b}\,U+b_0 U)\bar{v}\}\,dx\,dy=\iint_Q f\bar{v}\,dx\,dy \qquad \forall\, v\in\mathscr{V}_{h,N}^0.$$

The analysis of this scheme is standard using energy methods. However, in computations one cannot use the Fast Fourier Transform algorithm in the y-variable. Hence one cannot exploit one of the main features of spectral methods. For this reason we consider a combined finite element-pseudospectral approximation of (4.1).

For convenience of notations we set $Lv=\operatorname{div}\underline{b}v+b_0 v$, and $\tilde{L}v=\operatorname{div}P^N(\underline{b}v)+P^N(b_0 v)$; for any $x\in(0,1), Lv(x,\cdot)\in S_N$. The problem we study is:

$$\begin{aligned} &\text{find } U\in\mathscr{V}_{h,N}^0: \quad \forall\chi\in V_h^0\\ &\int_0^1\left\{\frac{\partial U}{\partial x}(y_\ell)\chi'-\left(\frac{\partial^2 U}{\partial y^2}-\tilde{L}U\right)(y_\ell)\chi\right\}dx=\int_0^1 f(y_\ell)\chi\,dx, \qquad 0\leqq\ell\leqq 2N. \end{aligned} \tag{4.3}$$

This corresponds to *collocate* the elliptic equation at the points $\{y_\ell\}$ and then to take Galerkin finite element approximation with respect to the x-variable.

Let ψ be any function of S_N, and set $w(x,y)=\chi(x)\psi(y)$. Multiplying by $\psi(y_\ell)$ the ℓ-th equation of (4.3), adding up on ℓ and using (1.11), the problem (4.3) can be written equivalently as follows:

$$\begin{aligned} &\text{find } U\in\mathscr{V}_{h,N}^0: \quad \forall\, w\in\mathscr{V}_{h,N}^0\\ &\int_0^1\left\{\left(\frac{\partial U}{\partial x},\frac{\partial w}{\partial x}\right)_N-\left(\frac{\partial^2 U}{\partial y^2}-\tilde{L}U,w\right)_N\right\}dx=\int_0^1(f,w)_N\,dx. \end{aligned}$$

Integrating by parts and using (1.12) we have, finally, the problem:

$$\begin{aligned} &\text{find } U\in\mathscr{V}_{h,N}^0: \quad \forall\, w\in\mathscr{V}_{h,N}^0\\ &\iint_Q \nabla U\cdot\overline{\nabla w}\,dx\,dy+\iint_Q \tilde{L}U\bar{w}\,dx\,dy=\int_0^1(f,w)_N\,dx. \end{aligned} \tag{4.4}$$

Let us assume that $f \in H_p^{r-2,s-2}(Q)$, $D_x f \in L_x^2(H_{p,y}^{s-3})$ and $D_y f \in L_y^2(H_x^{r-3})$ for some $r \geqq 2$ and $s > 5/2$. Then by standard arguments one can show that $u \in H_p^{r,s}(Q)$, $D_x u \in H_p^{0,s-1}(Q)$ and $D_y u \in H^{r-1,0}(Q)$. Assume also for the sake of simplicity that $\underline{b}$ and b_0 are infinitely differentiable over $\bar{Q}$ and satisfy (4.2). The following result holds.

Theorem 4.1. *With the assumptions on $f, \underline{b}$ and b_0 made above we get the estimate:*

$$\|U-u\|_{1,Q} \leqq C\{(h^{r-1}+N^{1-s})(\|u\|_{r,s,Q}+\|D_x u\|_{0,s-1,Q} + \|D_y u\|_{r-1,0,Q})+N^{2-s}\|f\|_{0,s-2,Q}\}. \tag{4.5}$$

Proof. Define $u^* = \mathscr{P}u$; by (4.1) we get

$$\iint_Q \nabla u^* \cdot \nabla \bar{w}\, dx\, dy + \iint_Q \tilde{L}u^* \bar{w}\, dx\, dy = \iint_Q f\bar{w}\, dx\, dy + \iint_Q (\tilde{L}u^* - Lu)\bar{w}\, dx\, dy + \iint_Q \nabla(u^*-u)\nabla\bar{w}\, dx\, dy. \tag{4.6}$$

Setting $e = u^* - U$, $E[v, z] = \int_0^1 (v, z)_N\, dx - \iint_Q v\bar{z}\, dx\, dy$, and subtracting (4.4) from (4.6) we get

$$\iint_Q \nabla e \cdot \overline{\nabla w}\, dx\, dy + \iint_Q \tilde{L}ew\, dx\, dy = -E[f, w] + \iint_0 (\tilde{L}u^* - Lu)\bar{w}\, dx\, dy + \iint_Q \nabla(u^*-u)\cdot\nabla\bar{w}\, dx\, dy.$$

Taking $w = e$, integrating by parts and using the relation $\frac{1}{2}\iint_Q (\operatorname{div}\underline{b})|e|^2\, dx\, dy = \iint_Q \operatorname{div}(\underline{b}e)\bar{e}\, dx\, dy$ we obtain:

$$\begin{aligned}
&\|\nabla e\|_{0,Q}^2 + \beta\|e\|_{0,Q}^2 \\
&\quad \leqq \iint_Q |\nabla e|^2\, dx\, dy + \iint_Q (\tfrac{1}{2}\operatorname{div}\underline{b} + b_0)|e|^2\, dx\, dy \\
&\quad \leqq \Big|\iint_Q \{(\underline{b}e - P^N\underline{b}e)\cdot\overline{\nabla e} + (b_0 e - P^N b_0 e)\bar{e}\}\, dx\, dy\Big| + |E[f, e]| \\
&\qquad + \Big|\iint_Q \tilde{L}(u^*-u)\bar{e}\, dx\, dy\Big| + \Big|\iint_Q (\tilde{L}-L)u\bar{e}\, dx\, dy\Big| + \Big|\iint_Q \nabla(u^*-u)\cdot\nabla\bar{e}\, dx\, dy\Big|.
\end{aligned} \tag{4.7}$$

By (1.4) the first term of the right hand side can be bounded by $CN^{-1}\|e\|_{1,Q}^2$ and then it can be dropped assuming N sufficiently large. We evaluate now the remaining terms of the right hand side of (4.7). By (1.8) it follows

$$\begin{aligned}
|E[f, e]| &= \left|\int_0^1 dx\left\{(f, e)_N - \int_0^{2\pi} f\bar{e}\, dy\right\}\right| = \left|\int_0^1 dx \int_0^{2\pi} (P^N f - f)\bar{e}\, dy\right| \\
&\leqq CN^{2-s}\|f\|_{0,s-2,Q}\cdot\|e\|_{0,Q}.
\end{aligned} \tag{4.8}$$

Integrating by parts we get

$$\begin{aligned}
\Big|\iint_Q \tilde{L}(u^*-u)\bar{e}\, dx\, dy\Big| &= \Big|\iint_Q \{P^N\underline{b}(u^*-u)\cdot\nabla\bar{e} + P^N b_0(u^*-u)\bar{e}\}\, dx\, dy\Big| \\
&\leqq C_1(\underline{b}, b_0)\|u^*-u\|_{1,Q}\|e\|_{1,Q}.
\end{aligned} \tag{4.9}$$

The last inequality holds by (1.8). Using again integration by parts we have

$$|\iint_Q (\tilde{L}-L)\, u\bar{e}\, dx\, dy| = |\iint_Q \{-(P^N-I)\underline{b}u\cdot\overline{\nabla e}+(P^N-I)\, b_0 u\cdot\bar{e}\}\, dx\, dy|$$
$$\leqq \|(P^N-I)\underline{b}u\|_{0,Q}\|e\|_{1,Q}+\|(P^N-I)\, b_0 u\|_{0,Q}\|e\|_{0,Q}. \quad (4.10)$$

Finally, by Cauchy-Schwarz inequality it follows

$$|\iint_Q \nabla(u^*-u)\cdot\nabla\bar{e}\, dx\, dy| \leqq \|u^*-u\|_{1,Q}\|e\|_{1,Q}. \quad (4.11)$$

Now the estimate (4.5) holds due to Theorem 3.2, to (4.7), (4.8), (4.9), (4.10), (4.11) and to the triangle's inequality

$$\|U-u\|_{1,Q} \leqq \|e\|_{1,Q}+\|u^*-u\|_{1,Q}. \quad \square$$

Remark 4.1. In the commonest case when $r=s$, assuming $f\in H^{r-2}(Q)$ with $r>5/2$, inequality (4.5) becomes

$$\|u-U\|_{1,Q} \leqq C\{(h^{r-1}+N^{1-r})\|f\|_{r-2,Q}+N^{2-r}\|f\|_{0,r-2,Q}\}. \quad (4.11)$$

So, for instance, if linear finite elements are used, the balance in the right hand side of (4.11) is achieved with $h\simeq N^{-1}$ if $r=3$, $h\simeq N^{-2}$ if $r=4$, and so on. $\square$

We conclude by presenting some numerical results for the scheme (4.3), in which linear finite elements are used. The corresponding algebraic system has been solved by an ADI-procedure, which allows to preserve the computational advantages of both Finite Element and Spectral Methods. In (4.3) we take $b=(\cos x, y)^T$, $b_0\equiv 5$, while the exact solution is $u=\exp(\sin x)\cdot(y^3-y^2)$. We report the relative error in the H^1-norm.

h^{-1}	N	Error	h^{-1}	N	Error
5	4	$0.15E-1$	19	4	$0.15E-1$
	8	$0.49E-2$		8	$0.70E-3$
	16	$0.48E-2$		16	$0.44E-3$
9	4	$0.12E-1$	29	4	$0.15E-1$
	8	$0.19E-2$		8	$0.42E-3$
	16	$0.18E-2$		16	$0.20E-3$

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