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Combined Finite Element and Spectral Approximation of the Navier-Stokes Equations

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Summary. We present a method for the numerical approximation of Navier-Stokes equations with one direction of periodicity. In this direction a Fourier pseudospectral method is used, in the two others a standard F.E.M. is applied. We prove optimal rate of convergence where the two parameters of discretization intervene independently.

Approximation des équations de Navier-Stokes par une méthode éléments finis-spectrale Fourier

Resumé. On présente une méthode d'approximation numérique des équations de Navier-Stokes possédant une direction de périodicité. Dans cette direction une méthode pseudospectrale basée sur des développements en série de Fourier est utilisée, dans les deux autres on applique une méthode d'éléments finis standard. On montre que la convergence est optimale et que les deux paramètres de discrétisation peuvent être choisis de façon indépendante.

Subject Classifications: AMS(MOS): 65N30; CR: 5.17.

I. Introduction

Let Ω be a bounded domain of $\mathbb{R}^2$ with boundary Γ and set $I=(0,2\pi)$. The aim of this paper is to present the analysis of an approximation of a branch of non-singular solutions of the stationary Navier-Stokes equations for an incompressible viscous fluid confined in $Q=\Omega\times I$ with periodic boundary conditions over I.

In a previous paper [4], the authors have studied numerical methods which can be efficiently applied to such a problem. They are based on the coupling of a finite element scheme over Ω and a spectral Fourier approximation in the periodicity direction. A general way of deriving error estimates depending on the mesh size h and the largest wave number N was presented there.

In this paper. the strategy is used in the error analysis of a coupled method which is first order accurate in h and infinite order accurate in N. The effect of numerical quadrature for evaluating the various integrals appearing in the method is considered. The effect of taking derivatives in the periodic direction by a pseudo spectral algorithm is also taken into account.

The stability and convergence results are proven under no unrealistic restriction on the choice of h and N. The condition $N^{-1} \geqq h$ looks natural for a numerical scheme which is more accurate in the periodicity direction than in the others.

In order to simplify the analysis of the approximation we shall consider only convex polyhedral domains Ω. The case of general curved domains is a simple consequence of the present results and those exposed in [7].

Other combined finite element/spectral methods have been investigated in [3-12].

Throughout the paper C will denote different constants independent of h and N.

II. Preliminaries and Notations

For any open set $\vartheta \in \mathbb{R}^n$ and any real $s \geqq 0$, we denote by $H^s(\vartheta)$ the classical Sobolev space, equipped with the norm $\|\cdot\|_{s,\vartheta}$ and the semi-norm $|\cdot|_{s,\vartheta}$ (see [11]). We note $L^2(\vartheta) = H^0(\vartheta)$.

Let us set

$$
\begin{aligned}
&L = \{f \in H^0(Q) \mid \int_\vartheta f = 0\}; \\
&V = [H^{1;1}_{0;p}(Q)]^3;
\end{aligned}
\tag{2.1}
$$

where

$$
H^{1;1}_{0;p}(Q) = \left\{ v \in H^1(Q) \left| \begin{array}{ll} v(x,0) = v(x,2\pi) & \forall x \in \Omega \\ v(\cdot, y)_{|\Gamma} = 0 & \forall y \in I \end{array} \right. \right\}.
$$

We want to study a fluid whose behavior is governed by the stationary, incompressible Navier-Stokes equations.

Denoting by $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ the fluid velocity, by p its pressure, by $\mathbf{f} = (f_1, f_2, f_3)$ the body forces and by R the Reynolds number of the fluid, the equations can be written as follows:

Find $\tilde{\mathbf{u}}$ in V and p in L such that:

$$
-\Delta \tilde{\mathbf{u}} = R(\mathbf{f} - (\tilde{\mathbf{u}} \cdot \nabla)\tilde{\mathbf{u}} - \nabla p) \quad \text{in } Q, \tag{2.2}
$$

$$
\nabla \cdot \tilde{\mathbf{u}} = 0 \quad \text{in } Q. \tag{2.3}
$$

Setting $\mathbf{u} = \frac{1}{R}\tilde{\mathbf{u}}$, and using the divergence free conditions for $\mathbf{u}$ we derive a new equivalent form for the Navier Stokes equations.

Find $\mathbf{u} = (u_1, u_2, u_3)$ in V and p in L such that:

$$-\Delta \mathbf{u}+\nabla p=\mathbf{f}-\lambda \sum_{i=1}^{3}\left(\frac{\partial(u_i \mathbf{u})}{\partial x_i}\right) \quad \text{in } Q, \tag{2.4}$$

$$\nabla \cdot \mathbf{u}=0 \qquad \text{in } Q, \tag{2.5}$$

(where we have set $\lambda=R^2$).

We can note that this problem has an equivalent variational formulation that is:

Find $\mathbf{u}=(u_1, u_2, u_3)$ in V and p in L such that:

$$(\nabla \mathbf{u}, \nabla \mathbf{v})-(p, \nabla \cdot \mathbf{v})=(\mathbf{f}, \mathbf{v})+\lambda \sum_{i=1}^{3}\left((u_i \mathbf{u}), \frac{\partial \mathbf{v}}{\partial x_i}\right) \quad \text{for any } \mathbf{v}\in V, \tag{2.6}$$

$$(\nabla \cdot \mathbf{u}, q)=0 \quad \text{for any } q \text{ in } L. \tag{2.7}$$

(Hereafter $(.,.)$ denotes the $L^2(Q)$ inner product).

In the following we shall suppose there exists a branch of non singular solutions of this problem, in a sense precised later.

III. Numerical Approximation of Problem (2.6)–(2.7)

Let $(\mathscr{C}h)_h$ be a regular family of finite triangulation of $\bar{\Omega}$ such that $\bar{\Omega}=\bigcup_{K\in\mathscr{C}h} K$ (see [6] for more details).

We shall examine here a first order approximation in h. To this end let us define, for each h, the finite demensional spaces:

$$\mathscr{S}_{k,h}=\{\phi\in L^2(\Omega) \mid \forall K\in\mathscr{C}h, \phi_{|K}\in\mathbb{P}_k\}, \tag{3.1}$$

$$\tilde{\mathscr{S}}_{1,h}=\{\phi\in L^2(\Omega) \mid \forall K\in\mathscr{C}h, \phi_{|K}\in\mathbb{P}_1 \oplus \{\lambda_1 \lambda_2 \lambda_3\}\}, \tag{3.2}$$

where $\mathbb{P}_k$ denotes the space of all polynomials defined on $\mathbb{R}^2$ of degree less than or equal to k and $\lambda_1, \lambda_2, \lambda_3$ are the barycenter coordinates over K.

For any integer $N>0$, we now introduce the space S_N defined by

$$S_N=\operatorname{span}\left\{\phi_l(y)=\frac{1}{\sqrt{2\pi}}\exp(ily) \mid -N\leqq l\leqq N\right\}. \tag{3.3}$$

Let δ be the couple (h, N^{-1}); the spaces V and L will be approximated by:

$$V_\delta=W_\delta\cap V \quad \text{with} \quad W_\delta=(\mathscr{S}_{2,h}\otimes S_N)^2\times(\tilde{\mathscr{S}}_{1,h}\otimes S_N); \tag{3.4}$$

$$L_\delta=(\mathscr{S}_{0,h}\otimes S_N)\cap L. \tag{3.5}$$

For convenience we recall the definition of nonisotropic Sobolev spaces $H^{r,s}(Q)$ that will be used in the sequel. For any positive real r, s:

$$H^{r,s}(Q)=H^0(I; H^r(\Omega))\cap H^s(I; H^0(\Omega)).$$

Here, for any integer $p \geqq 0$ and any Hilbert space X, we set

$$H^p(I;X) = \{v \mid \int_I \|D^k v\|_X^2 < +\infty \text{ for } 0 \leqq k \leqq p).$$

($\|\cdot\|_X$ denote the norm over X).

For a positive non integer value of p, the Hilbert space $H^p(I;X)$ is defined by interpolation (for more details see [11]).

The space $H^{r,s}(Q)$ is provided with the norm

$$\|v\|_{r,s} = (\|v\|^2_{H^0(I;H^r(\Omega))} + \|v\|^2_{H^s(I;H^0(\Omega))})^{1/2}.$$

Let $C_p^\infty(I; C^\infty(\Omega))$ the space of C^∞ functions v defined over Q, periodic over I. We define $H_p^{r,s}(Q)$ as being the closure of $C_p^\infty(I; C^\infty(\Omega))$ in $H^{r,s}(Q)$.

The spaces V_δ and L_δ verify the following properties:

Theorem 3.1. *Let $s \geqq 0$ and $r \in (0,1)$; for any p in $H_p^{r,s}(Q) \cap L$ we have*

$$\inf_{q \in L_\delta} \|p - q\|_{0,Q} \leqq C(h^r + N^{-s}) \|p\|_{r,s}, \tag{3.6}$$

and for any

$$\mathbf{u} \in Y_{r,s} = [H_p^{r+1,s+1}(Q) \cap H^r(\Omega; H_p^1(I)) \cap H^1(\Omega; H_p^s(1))]$$

we have

$$\inf_{\mathbf{v} \in V_\delta} \|\mathbf{u} - \mathbf{v}\|_V \leqq C(h^r + N^{-s}) \|\mathbf{u}\|_{Y_{r,s}}. \tag{3.7}$$

Proof. We shall first prove (3.6)

Let π_0 denote the orthogonal projection upon $\mathscr{S}_{0,h}$ in the $L^2(\Omega)$ inner product. It is well known that, for any $u \in H^r(\Omega)$

$$\|u - \pi_0 u\|_{0,\Omega} < C h^r \|u\|_{r,\Omega} \qquad 0 \leqq r \leqq 1. \tag{3.8}$$

Similary, P_0 will denote the orthogonal projection operator upon S_N in the $L^2(I)$ inner product. For any w in $H_p^s(I)$

$$\left(\text{with } H_p^s(I) = \left\{ v \in H^s(I) \,\middle|\, \frac{d^r v}{dx^r}(0) = \frac{d^r v}{dx^r}(2\pi)\; 0 \leqq r \leqq s-1 \right\}\right)$$

we have (see [5])

$$\|w - P_0 w\|_{\sigma,I} \leqq C N^{\sigma - s} \|w\|_{s,I} \qquad s \geqq \sigma \geqq 0. \tag{3.9}$$

It is an easy matter to verify that the projection operator upon $\mathscr{S}_{0,h} \otimes S_N$ in the $L^2(Q)$ inner product is actually $\pi_0 \circ P_0 = P_0 \circ \pi_0$.

Moreover the image of L under the $L^2(Q)$-projection operator is L_δ.

Hence, we can write:

$$\begin{aligned}\inf_{q \in L_\delta} \|p - q\|_{0,Q} &= \|p - P_0 \circ \pi_0 p\|_{0,Q} \leqq \|p - P_0 p\|_{0,Q} + \|P_0(p - \pi_0 p)\|_{0,Q} \\ &= [\int_\Omega \|p - P_0 p\|^2_{0,I}]^{1/2} + [\int_\Omega \|P_0(p - \pi_0 p)\|^2_{0,I}]^{1/2}.\end{aligned}$$

Since

$$\int_{\Omega} \|P_0(p-\pi_0 p)\|_{0,I}^2 = \int_{\Omega} \|p-\pi_0 p\|_{0,I}^2 = \int_{I} \|p-\pi_0 p\|_{0,\Omega}^2,$$

by (3.8) and (3.9) we obtain

$$\inf_{q \in L_\delta} \|p-q\|_{0,Q} \leqq C(N^{-s}\|p\|_{0,s} + h^r \|p\|_{r,0}),$$

and (3.6) follows.

Let us now consider (3.7). We introduce the orthogonal projection operator $\underset{\sim}{\pi}_0$ upon $\mathscr{S}_{2,h} \cap H_0^1(\Omega)$ (resp. $\tilde{\pi}_0$ upon $\tilde{\mathscr{S}}_{1,h} \cap H_0^1(\Omega)$) in the $H_0^1(\Omega)$ inner product.

We get (see, [6]):

$$\|u - \underset{\sim}{\pi}_0 u\|_{1,\Omega} \leqq C h^{r-1} \|u\|_{r,\Omega} \quad \text{for any } u \in H^r(\Omega) \cap H_0^1(\Omega),\ 1 \leqq r \leqq 3, \tag{3.10}$$

$$\|u - \tilde{\pi}_0 u\|_{1,\Omega} \leqq C h^{r-1} \|u\|_{r,\Omega} \quad \text{for any } u \in H^r(\Omega) \cap H_0^1(\Omega),\ 1 \leqq r \leqq 2. \tag{3.11}$$

Finally, let $\mathscr{P}$ denote the projection operator over V_δ with respect to the inner product of V. We have

$$\mathscr{P}\mathbf{u} = ((P_0 \circ \underset{\sim}{\pi}_0)\, u_1,\ (P_0 \circ \underset{\sim}{\pi}_0)\, u_2,\ (P_0 \circ \tilde{\pi}_0)\, u_3). \tag{3.12}$$

It is an easy matter to check that any operator of derivation commutes with P_0, and that $\dfrac{\partial}{\partial x_3}$ commutes with $\underset{\sim}{\pi}_0$ and $\tilde{\pi}_0$. Hence (3.7) can be obtained by the same arguments used in proving (3.6). □

Looking at (3.6) and (3.7) we can note that, in the applications one will generally take h and N such that h^r and N^{-s} be asymptotically equivalent. Since, for a smooth function, s can be arbitrarily large while r is bounded, a necessary condition is

$$h < N^{-1}. \tag{3.13}$$

In the following we shall assume that (3.13) holds.

The discrete problem associated with problem (2.6), (2.7), will be: find (u_δ, p_δ) in $V_\delta \times L_\delta$ such that

$$(\nabla \mathbf{u}_\delta, \nabla \mathbf{v}_\delta) + (p_\delta, \nabla \cdot \mathbf{v}_\delta) = (\mathbf{f}, \mathbf{v}_\delta)_\delta + \lambda \sum_{i=1}^{3} \left(u_{\delta i} \mathbf{u}_\delta, \frac{\partial \mathbf{v}_\delta}{\partial x_i} \right)_\delta \tag{3.14}$$

$$\text{for any } \mathbf{v}_\delta \text{ in } V_\delta,$$

$$(\nabla \cdot \mathbf{u}_\delta, q_\delta) = 0 \quad \text{for any } q_\delta \text{ in } L_\delta. \tag{3.15}$$

where $(\cdot,\cdot)_\delta$ is a discrete inner product we shall define next. Note that a Galerkin method for the approximation of (2.6), (2.7) would involve the same formulation as in (3.14), (3.15) but with an exact inner product in (3.14). Here we consider the use of a quadrature scheme in order to evaluate the right hand side in that equation.

Remark 3.1. This paper analyse the approximation properties of problem (3.14), (3.15). In practice a quadrature scheme is generally used to evaluate the left hand side of (3.14). The properties of such an approximate problem are deduced from the forthcoming results as can be seen in [6]. $\square$

The quadrature scheme we consider involves two systems of points:

- a system over Ω corresponding to a system $\{(\hat{\xi}_m, \hat{\omega}_m) | m \in \mathcal{M}(\hat{K}) \subset \mathbb{N}\}$ of nodes and weights of quadrature over a master triangle $\hat{K}$.
- over I we want to use a pseudo spectral method, hence we consider the trapezoidal rule with nodes $\gamma_j = j\frac{\pi}{N}$, $0 \leqq j \leqq 2N-1$, and the corresponding constant weight π/N.

We first deduce the numerical quadrature formula over Ω. We have $\bar{\Omega} = \bigcup_{K \in \mathscr{C}h} K$ and each triangle K is the image of $\hat{K}$ via an invertible mapping J_K:

$$\hat{x} \in \hat{K} \to J_K(\hat{x}) = B_K \hat{x} + b_K. \tag{3.16}$$

We can assume that the (constant) Jacobian of B_K of the mapping J_K is positive, hence, if $\hat{\varphi}(\hat{x}) = \varphi(x)$ we have

$$\int_K \varphi(x)\, dx = \det B_K \int_{\hat{K}} \hat{\varphi}(\hat{x})\, d\hat{x}. \tag{3.17}$$

We assume that

$$\forall \hat{\varphi} \in \mathbb{P}_4(\hat{K}) \int_{\hat{K}} \hat{\varphi}(\hat{x})\, d\hat{x} = \sum_{m \in \mathcal{M}(\hat{K})} \hat{\omega}_m \hat{\varphi}(\hat{\xi}_m). \tag{3.18}$$

Setting $\omega_{m,K} = \det(B_K)\, \hat{\omega}_m$, $\xi_{m,K} = J_K(\hat{\xi}_m)$, the integration scheme over Q is given by:

$$\int_Q \varphi(\mathbf{x})\, d\mathbf{x} \sim \frac{\pi}{N} \sum_{K \in \mathscr{C}h} \sum_{m \in \mathcal{M}(K)} \sum_{j=0}^{2N-1} \varphi(\xi_{m,K}, \gamma_j)\, \omega_{m,K}.$$

For φ and ψ in $C^0(Q)$, we define the approximate inner product by:

$$(\varphi, \psi)_\delta = \frac{\pi}{N} \sum_{K \in \mathscr{C}h} \sum_{m \in \mathcal{M}(K)} \sum_{j=0}^{2N-1} \varphi(\xi_{m,K}, \gamma_j)\, \psi(\xi_{m,K}, \gamma_j)\, \omega_{m,K}. \tag{3.20}$$

we have $(\varphi, \psi)_\delta = (\varphi, \psi)$ for any φ, ψ such that $\varphi\psi \in \mathscr{S}_{4,h} \otimes S_{2N}$. $\square$

Remark 3.2. For any $\varphi \in C^0(\bar{I})$, let $P_c\varphi$ denote the function of S_N defined by $P_c\varphi(\gamma_j) = \varphi(\gamma_j)$, $0 \leqq j \leqq 2N-1$. Since

$$\frac{\pi}{N} \sum_{j=0}^{2N-1} \varphi \frac{\partial \psi}{\partial x_3}(\gamma_j) = -\int_I \frac{\partial}{\partial x_3}(P_c\varphi)\, \psi\, dx_3 \quad \text{if } \psi \in S_N,$$

the scheme (3.14) implies a pseudo-spectral treatment of the derivative in the direction of the periodicity. $\square$

IV. Error Estimate for the Approximate Problem

Let us denote by $T\colon V'\times L'\to V\times L$ (resp T_δ: $V'\times L'\to V_\delta\times L_\delta$) the operator that solves the Stokes problem (resp. the discrete Stokes problem) i.e.

$$u=(\mathbf{u},p)=T((\mathbf{g},\eta)) \;\Rightarrow\; \begin{cases} -\Delta\mathbf{u}+\nabla p=\mathbf{g}\\ \nabla\cdot\mathbf{u} \qquad =0\text{'} \end{cases}$$

$$u_\delta=(\mathbf{u}_\delta,p_\delta)=T_\delta((\mathbf{g},\eta)) \;\Leftrightarrow\; \begin{cases} (\nabla\mathbf{u}_\delta,\nabla\mathbf{v})-(p_\delta,\nabla\cdot\mathbf{v})=\langle\mathbf{g},\mathbf{v}\rangle & \nabla\mathbf{v}\in V_\delta\\ (q,\nabla\cdot\mathbf{u}_\delta) \qquad\qquad\qquad =0 & \forall q\in L_\delta. \end{cases} \tag{4.1}$$

T is uniquelly defined and continuous. Moreover T has the following regularity properties.

Lemma 4.1. *The operator T is continuous from $(H_p^{r,s}(Q))^3\times L'$ into*

$$(H_p^{r+2,s+2}(Q))^3\times H_p^{r+1,s+1}(Q))$$

for any $r\in(-1,0)$ and $s\geqq r$. Further more if $s\geqq 0$ the range of T lies in

$$(H_p^{s+1}(I,H_0^1(\Omega)))^3\times(H_p^s(I,L^2(\Omega))).$$

Proof. For $(\mathbf{g},\eta)$ in $V'\times L'$, let us set $(\mathbf{u},p)=T((\mathbf{g},\eta))$ and let us define for any function v and any integer k:

$$\mathring{v}(x_1,x_2,k)=\int_0^{2\pi} v(x_1,x_2,x_3)\,e^{-ikx_3}dx_3 \tag{4.2}$$

(if v only depends on x_3 $\mathring{v}(k)$ is the k-th Fourier coefficient of v).

From (4.1) we then get:

$$k^2\mathring{u}_1-\frac{\partial^2\mathring{u}_1}{\partial x_1^2}-\frac{\partial^2\mathring{u}_1}{\partial x_2^2}+\frac{\partial\mathring{p}}{\partial x_1}=\mathring{g}_1, \tag{4.3.1}$$

$$k^2\mathring{u}_2-\frac{\partial^2\mathring{u}_2}{\partial x_1^2}-\frac{\partial^2\mathring{u}_2}{\partial x_2^2}+\frac{\partial\mathring{p}}{\partial x_2}=\mathring{g}_2, \tag{4.3.2}$$

$$k^2\mathring{u}_3-\frac{\partial^2\mathring{u}_3}{\partial x_1^2}-\frac{\partial^2\mathring{u}_3}{\partial x_2^2}-ik\mathring{p}=\mathring{g}_3, \tag{4.3.3}$$

$$\frac{\partial\mathring{u}_1}{\partial x_1}+\frac{\partial\mathring{u}_2}{\partial x_2}-ik\mathring{u}_3=0. \tag{4.3.4}$$

Let us suppose now that $\mathbf{g}$ is in $(H_p^{0,s}(Q))^3$ for some $s>0$. Then it follows that

$$\sum_{k=-\infty}^{+\infty} k^{2s}\|\mathring{\mathbf{g}}(\cdot,\cdot,k)\|_{0,\Omega}^2<+\infty. \tag{4.4}$$

Multiplying (4.3.j) by $k^{(2s+2)}\bar{\mathring{u}}_j(1\leqq j\leqq 3)$ and summing up, one easily derives the following equation:

$$\int_\Omega \sum_{j=1}^{3} \sum_{k=-\infty}^{+\infty} k^{2s+4}|\mathring{u}_j|^2 - k^{2s+2}\left(\frac{\partial^2 \mathring{u}_j}{\partial x_1^2}+\frac{\partial^2 \mathring{u}_j}{\partial x_2^2}\right)\bar{u}_j dx_1 dx_2$$
$$=\int_\Omega \sum_{k=-\infty}^{+\infty} \left(\sum_{j=1}^{3} k^{2s+2}\mathring{g}_j\bar{\mathring{u}}_j - \sum_{j=1}^{2} k^{2s+2}\frac{\partial \mathring{p}}{\partial x_j}\bar{\mathring{u}}_j + ik^{2s+3}\mathring{p}\bar{\mathring{u}}_3\right).$$

Integrating by parts and using (4.3.4) we get

$$\sum_{j=1}^{3} \sum_{k=-\infty}^{+\infty} \int_\Omega k^{2s+4}|\mathring{u}_j|^2 + k^{2s+2}\left(\left|\frac{\partial \mathring{u}_j}{\partial x_1}\right|^2 + \left|\frac{\partial \mathring{u}_j}{\partial x_2}\right|^2\right) dx_1 dx_2$$
$$\sum_{j=1}^{3} \sum_{k=-\infty}^{+\infty} K^{2s+2}\mathring{g}_j\bar{\mathring{u}}_j dx_1 dx_2. \tag{4.5}$$

Due to (4.4) one has:

$$\sum_{j=1}^{3} \sum_{k=-\infty}^{+\infty} \tfrac{1}{2}k^{2s+4}\|\mathring{u}_j(.,.,k)\|_{0,\Omega}^2 + k^{2s+2}\|\mathring{u}_j(.,.,k)\|_{1,\Omega}^2 < +\infty.$$

Hence $\mathbf{g}\in(H_p^{0,s}(Q))^3$ implies $\mathbf{u}\in(H_p^{0,s+2}(Q))^3\cap(H_p^{s+1}(I;H_0^1(\Omega))$. Multiplying now (4.3.j) by

$$k^{2s}\left(\frac{\partial^2 \bar{\mathring{u}}_j}{\partial x_1^2}+\frac{\partial^2 \bar{\mathring{u}}_j}{\partial x_2^2}\right) \qquad (1\leqq j\leqq 3)$$

and summing up one derives, after integrating by parts and using once more (4.3.4), that:

$$\mathbf{u}\in(H_p^s(I;H^2(\Omega))^3 \quad \text{hence} \quad \mathbf{u}\in(H_p^{2,s+2}(Q))^3.$$

From the definition of T, the lemma holds for $r=s=-1$. The regularity of $\mathbf{u}$ announced is then achieved by interpolation technique. The regularity of p is obtained in a similar way. □

Now, from the compactness of the imbedding

$$H_p^{r+2,s+2}(Q)\hookrightarrow H_p^{1,1}(Q) \qquad \text{for } r>-1;\ s>-1$$

we obtain:

Corollary 4.1. *The operator T is compact from $(H_p^{r,s}(Q))^3\times L'$ into $V\times L$ for any $r>-1, s>-1$.* □

Before studying the operator T_δ we need some preliminary result. Let b_i, $i\in I\subset\mathbb{N}$, be the vertices of the triangulation or the midpoints of each side of K, $K\in\mathscr{C}h$. Let Φ_i be the basis functions of $\mathscr{S}_{2,h}$ associated with each b_i, $i\in I$ (i.e. $\Phi_i(b_j)=\delta_{ij}$). Finally let Δ_i denote the support of Φ_i.

Let $v\in H^l(\Omega)\cap H_0^1(\Omega)$, for $1\leqq l\leqq 2$, we define, for each $i\in I^0=\{j\in I\,|\,b_j\in\Omega\}$ an element p_i of

$$H_i=\{\varphi\in C^0(\Delta_i)\,|\,\varphi_{|K}\in\mathbb{P}_2(K),\ \forall K\in\mathscr{C}h\}$$

by:

$$\int_{\Delta_i}(v-p_i)q=0 \qquad \text{for any } q \text{ in } H_i,$$

we then set:

$$\underset{\sim}{v}_h = \sum_{i \in I^0} p_i(b_i)\, \Phi_i.$$

We define an element $\tilde{v}_h$ of $\tilde{\mathscr{S}}_{1,h} \cap H_0^1(\Omega)$ in the same way, with now c_j, $j \in J \subset \mathbb{N}$, in place of b_i, where c_j are either the vertices or the barycenter of each K, $K \in \mathscr{C}h$ (We recall that the set $\{c_j\}_{j \in J}$ is $\tilde{\mathscr{S}}_{1,h}$-unisolvent).

The following result is proved in [1] and it does not require the uniform regularity of $\mathscr{C}h$

Lemma 4.2. *For any v in $H^l(\Omega) \cap H^1(\Omega)(1 \leqq l \leqq 2)$, it follows that*

$$\sum_{K \in \mathscr{C}h} \left| \|v - \underset{\sim}{v}_h\|_{l,K}^2 + (h^{1-l} \|v - \underset{\sim}{v}_h\|_{1,K})^2 + (h^{-l} \|v - \underset{\sim}{v}_h\|_{0,K})^2 \right| \leqq C \|v\|_{l,\Omega}^2;$$

$$\sum_{K \in \mathscr{C}h} \left| \|v - \tilde{v}_h\|_{l,K}^2 + (h^{1-l} \|v - \tilde{v}_h\|_{1,K})^2 + (h^{-l} \|v - \tilde{v}_h\|_{0,K})^2 \right| \leqq C \|v\|_{l,\Omega}^2. \quad \square \tag{4.6}$$

Lemma 4.3. *Let q_δ belong to L_δ; there exists a function $\mathbf{v}_\delta$ in V_δ such that*

$$(\operatorname{div} \mathbf{v}_\delta - q_\delta, s_\delta) = 0 \quad \textit{for any } s_\delta \textit{ in } L_\delta, \tag{4.7}$$

$$|\mathbf{v}_\delta|_{1,Q} \leqq \mathbf{C} \|q_\delta\|_{0,Q}. \tag{4.8}$$

Proof. This proof is similar to the one of Lemma 2.5 of [9] p. 76. The divergence operator is an isomorphism from the orthogonal complement (in V for the inner product $(\nabla \mathbf{v}, \nabla \mathbf{u})$) of $\{\mathbf{u} \in V | \operatorname{div} \mathbf{u} = 0\}$ onto L (see Lemma 3.2 of [9] p. 33).

Hence there exists exactly one function $\mathbf{v}$ such that

$$\operatorname{div} \mathbf{v} = q_\delta \quad \text{and} \quad \|\mathbf{v}\|_V \leqq C \|q_\delta\|_{0,Q}.$$

Let us set $\mathbf{w} = P_0 \mathbf{v}$ and $\mathbf{w}_\delta = (\underset{\sim}{w}_{1_h}, \underset{\sim}{w}_{2_h}, \tilde{w}_{3_h})$. There exists a unique function $\mathbf{v}_\delta \in V_\delta$ such that: for any z in I and any K of $\mathscr{C}h$ with vertices a_1, a_2, a_3

$$\mathbf{v}_\delta(a_i, z) = \mathbf{w}_\delta(a_i, z) \quad 1 \leqq i \leqq 3,$$

$$\int_{[a_i, a_j]} (\mathbf{v}_\delta - P_0 \mathbf{v})_l(\cdot, z) = 0 \quad 1 \leqq i < j \leqq 3, \ l = 1, 2,$$

$$\int_K (\mathbf{v}_\delta - P_0 \mathbf{v})_3(\cdot, z) = 0. \tag{4.9}$$

Moreover, it is an easy matter to verify that, by (4.9)

$$\int_Q \operatorname{div}(\mathbf{v} - \mathbf{v}_\delta)\, s_\delta = 0 \quad \text{for any } s_\delta \textit{ in } L_\delta,$$

so (4.7) is proved.

Let us set $\mathbf{e}_\delta = \mathbf{v}_\delta - \mathbf{w}_\delta \in V_\delta$ and $\mathbf{e} = P_0 \mathbf{v} - \mathbf{w}_\delta$. Then

$$|\mathbf{v}_\delta|_{1,Q} \leqq |\mathbf{w}_\delta|_{1,Q} + |\mathbf{e}_\delta|_{1,Q}.$$

From (3.9) and (4.6) we have

$$|\mathbf{v}_\delta|_{1,Q} \leqq \|\mathbf{v}\|_{1,Q} + |\mathbf{e}_\delta|_{1,Q}. \tag{4.10}$$

In order to get (4.8) it is sufficient to prove that $|\mathbf{e}_\delta|_{1,Q} \leqq C\|\mathbf{v}\|_{1,Q}$. From (4.9) we deduce that, for any $z \in I$:

$$\begin{aligned} &\mathbf{e}_\delta(a_i, z) = 0 \quad 1 \leqq i \leqq 3, \\ &\int_{[a_i, a_j]} (\mathbf{e}_\delta - \mathbf{e})_l(\cdot, z) = 0 \quad 1 \leqq i < j \leqq 3, \ l = 1, 2, \\ &\int_K (\mathbf{e}_\delta - \mathbf{e})_3(\cdot, z) = 0 \end{aligned} \tag{4.11}$$

Let us focuse on $e_\delta = (\mathbf{e}_\delta)_1$, the other components can be treated in the same way.

From (4.11) we can verify that in each trangle K, e_δ is of the form

$$e_\delta = \sum_{1 \leqq i < j \leqq 3} e_\delta(a_{ij}, \cdot)\, p_{ij}, \tag{4.12}$$

where a_{ij} is the midpoint of the side $[a_j, a_j]$ and $p_{ij} = 4\lambda_i \lambda_j$. Therefore, setting $\mathbb{K} = K \times I$, we get

$$\begin{aligned} |e_\delta|_{1,\mathbb{K}} \leqq \sum_{1 \leqq i < j \leqq 3} (&\|e_\delta(a_{ij}, \cdot)\|_{0,I} |p_{ij}|_{1,K} \\ &+ |e_\delta(a_{ij}, \cdot)|_{1,I} \|p_{ij}\|_{0,K}). \end{aligned} \tag{4.13}$$

Denoting by $\|\cdot\|$, the matrix norm associated with the euclidean norm of $\mathbb{R}^2$, and using the same arguments as in [9], Lemma 2.5, p. 76 we get

$$\begin{aligned} |e_\delta|_{1,\mathbb{K}} \leqq C[&\|B_K^{-1}\| (\|e\|_{0,\mathbb{K}}^2 + \|B_K\|^2 \|e\|_{1,\mathbb{K}}^2)^{1/2} \\ &+ (\|e\|_{1,\mathbb{K}}^2 + \|B_K\|^2 \|e\|_{H^1(I;\, H^1(K))}^2)^{1/2}]. \end{aligned}$$

Besides that, one has the following inverse inequality (see [5])

$$\|\varphi\|_{\sigma, I} \leqq C N^{\sigma - \nu} \|\varphi\|_{\nu, I} \quad \text{for any } \varphi \text{ in } S_N, \ 0 \leqq \nu \leqq \sigma. \tag{4.14}$$

Hence:

$$|e_\delta|_{1,\mathbb{K}} \leqq C(h^{-1} \|e\|_{0,\mathbb{K}} + \|e\|_{1,\mathbb{K}} + hN \|e\|_{1,\mathbb{K}}).$$

Finally, from (4.6), (3.9) and (3.12) we have

$$|e_\delta|_{1,Q} \leqq C \|P_0 v\|_{1,Q} \leqq \|v\|_{1,Q}. \quad \square$$

Remark 4.2. With the choice (3.4) for V_δ, the condition (4.11) defines $(\mathbf{e}_\delta)_3$ locally (i.e. it depends on v in the vicinity of K only). This would not be the case with the choice

$$V_\delta = (\mathscr{S}_{2,h}^0 \otimes S_n)^3 \cap V.$$

Theorem 4.1. *The operator T_δ is uniquely defined by* (4.2); *moreover we get the estimate*

$$\|T - T_\delta\|_{\mathscr{L}((H_p^{r,s})^3 \times L';\, V \times L)} \leqq C(N^{-(s+1)} + h^{r+1}); \quad r \in [-1, 0], \ s \geqq r. \tag{4.15}$$

Proof. Arguing as in [9] p. 69 we can prove that Lemma 4.3 ensures that the discrete inf-sup condition holds (see [2] corollary 2.1). Then (4.15) is achieved due to Lemma 4.1 and Theorem 3.1. □

Let us now introduce the mapping $H: \mathbb{R} \times (V \times L) \to V' \times \{0\}$ and $H_\delta: \mathbb{R} \times (V \times L) \to V'_\delta \times \{0\}$

$$H(\lambda, (\mathbf{u}, p)) = \left(\mathbf{f} - \lambda \sum_{i=1}^{3} \frac{\partial(u_i \mathbf{u})}{\partial x_i}, 0\right), \tag{4.16}$$

$$\langle H_\sigma(\lambda, (\mathbf{u}_\delta, p_\delta)), \boldsymbol{\varphi}_\delta \rangle = \left(\mathbf{f} - \lambda \sum_{i=1}^{3} \frac{u_{\delta_i} \mathbf{u}_\delta}{\partial x_i}, \boldsymbol{\varphi}_\delta\right)_\delta$$
$$\text{for any } \boldsymbol{\varphi}_\delta \text{ in } V_\delta. \tag{4.17}$$

It is an easy matter to check that H and H_δ are C^∞ mappings.

With these notations the exact problem (2.6), (2.7) and the discrete one (3.14), (3.15) can be written respectively as follows:

$$\begin{aligned} &\text{find } x = (\lambda, (\mathbf{u}, p)) \in X \text{ such that:} \\ &F(x) \equiv (\mathbf{u}, p) + TH(\lambda, (\mathbf{u}, p)) = 0, \end{aligned} \tag{4.18}$$

$$\begin{aligned} &\text{find } x_\delta = (\lambda, (\mathbf{u}_\delta, p_\delta)) \in X_\delta \text{ such that:} \\ &F_\delta(x_\delta) \equiv (\mathbf{u}_\delta, p_\delta) + T_\delta H_\delta(\lambda, (\mathbf{u}_\delta, p_\delta)) = 0, \end{aligned} \tag{4.19}$$

where $X = \mathbb{R} \times (V \times L)$ and $X_\delta = \mathbb{R} \times (V_\delta \times L_\delta)$.

In Sect. 1, we assumed the existence of a branch of non singular solutions for the problem (4.18); we can now give a sense to that hypothesis as follows.

There exists a compact interval $\Lambda \subset \mathbb{R}^+$ and a continuous function $\lambda \hookrightarrow y(\lambda) = ((\mathbf{u}(\lambda), p(\lambda))$ from Λ into $Y = V \times L$ such that

$$F(\lambda, y(\lambda)) = 0, \tag{4.20}$$

$$D_u F[\lambda, y(\lambda)] \text{ is an isomorphism of } Y. \tag{4.21}$$

Here $D_u F[\lambda, y]$ denotes the Frechet derivative of $F(\mu, u)$ with respect to u computed at the point (λ, y).

In the following we shall suppose that there exists two positive constant ρ and σ with $2\rho^{-1} + \sigma^{-1} < 2$ such that:

$$\sup_{\lambda \in \Lambda} \{ \|\mathbf{u}(\lambda)\|_{H_p^{\rho,\sigma}(Q)} + \|p(\lambda)\|_{H_p^{\rho-1,\sigma-1}(Q)} \} \leqq C.$$

Let $\pi_\delta: X \to X_\delta$ be defined by:

$$\pi_\delta(\mu, (\mathbf{v}, q)) = (\mu, (\mathscr{P}\mathbf{v}, \pi_0 \circ P_0(q))) \quad \text{for any } (\mu, (\mathbf{v}, q)) \in X. \tag{4.22}$$

Due to Theorem 3.1, we have:

$$\forall x \in X \quad \lim_{\delta \to 0} \|x - \pi_\delta x\|_X = 0. \tag{4.23}$$

We will prove now that F_δ is a "good approximation of F" (in the sense of [8], Sect. 3). Indeed we have the following result:

Lemma 4.4. *Assume that $f \in H^{\bar\rho,\bar\sigma}(Q)$ for some reals $\bar\rho$ and $\bar\sigma$ such that $2\bar\rho^{-1} + \bar\sigma^{-1} < 2$. For any fixed x, ξ in X, we have*

$$\lim_{\delta\to 0} \|F(x) - F_\delta(x)\|_X = \lim_{\delta\to 0} \|F'[x](\xi) - F'_\delta[\pi_\delta x](\pi_\delta \xi)\|_X = 0. \tag{4.24}$$

Moreover, for any $k=1,2$ there exist three positive constants ε, C_1, C_2 such that

$$\forall x_\delta \in X_\delta, \quad \|x_\delta - \pi_\delta(\lambda, y(\lambda))\|_X \leqq \varepsilon \;\Rightarrow\; \|F_\delta^{(k)}(x_\delta)\|_X \leqq C_k. \tag{4.25}$$

Finally, the following stability condition is verified.

$$\lim_{\delta\to 0} \sup_{\xi\in X, \|\xi\|X=1} \|F'[\lambda, y(\lambda)](\xi) - F'_\delta[\pi_\delta(\lambda, y(\lambda))](\pi_\delta\xi)\|_X = 0. \tag{4.26}$$

Proof. We give first an error bound for the term $H - H_\delta$. We denote by $\hat{\mathbf{K}}$ (resp. $\mathbf{K}$) the reference simplex $\hat K \times (0, 2\pi N)$ (resp. the simplex $K \times (0, 2\pi)$).

The mapping: $\mathbf{x} \in \mathbf{K} \xmapsto{F_{\mathbf{K}}} \hat{\mathbf{x}} \in \hat{\mathbf{K}}$ defined by:

$$\hat{\mathbf{x}} = (J_K^{-1}(x_1, x_2), N x_3) = F_{\mathbf{K}}(\mathbf{x}), \tag{4.27}$$

is an affine, invertible mapping from $\mathbf{K}$ onto $\hat{\mathbf{K}}$.

According to the formulae (3.18), (3.19), (3.20), let us define, for any $\hat\psi \in C^\infty(\hat{\mathbf{K}})$ and any ψ in $C^\infty(\mathbf{K})$:

$$E_{\hat{\mathbf{K}}}(\hat\psi) = \left| \int_{\hat{\mathbf{K}}} \hat\psi - \pi \sum_{m\in\mathcal{M}(\hat K)} \sum_{j=0}^{2N-1} \hat\psi(\hat\xi_m, N v_j)\,\omega_m \right|, \tag{4.28}$$

$$E_{\mathbf{K}}(\psi) = \left| \int_{\mathbf{K}} \psi - \frac{\pi}{N} \sum_{m\in\mathcal{M}(\hat K)} \sum_{j=0}^{2N-1} \psi(\xi_{m,K}, v_j)\,\omega_{m,K} \right|. \tag{4.29}$$

If $\psi(\mathbf{x}) = \hat\psi(F_{\mathbf{K}}\mathbf{x})$ we get

$$E_{\mathbf{K}}(\psi) = \det(B_K) N^{-1} E_{\hat{\mathbf{K}}}(\hat\psi). \tag{4.30}$$

For any real r, $s \geqq 0$ such that $2r^{-1} + s^{-1} < 2$, the Sobolev imbedding theorem (see e.g. [4], Lemma 1.3 and Remark 3.3) implies there exists a positive constant C, independent of δ such that, for any $\hat\psi \in C^\infty(\hat{\mathbf{K}})$

$$\|\hat\psi\|_{L^\infty(\hat K \times (v_j, v_{j+1}))} \leqq C \|\hat\psi\|_{H^{r,s}(\hat K \times (v_j, v_{j+1}))}$$

for any $j = 0, \ldots, 2N-1$.

Hence, it is an easy matter to verify that, for any $\hat\psi$, $\hat\chi$ in $C^\infty(\hat K)$:

$$E_{\hat{\mathbf{K}}}(\hat\psi\hat\chi) \leqq C \sum_{j=0}^{2N-1} \|\hat\psi\hat\chi\|_{L^\infty(\hat K \times (v_j, v_{j+1}))} \leqq C \|\hat\psi\|_{H^{r,s}(\hat{\mathbf{K}})} \|\hat\chi\|_{H^{r,s}(\hat{\mathbf{K}})}. \tag{4.31}$$

From the uniform equivalence of norms over $\mathbb{P}^l(\hat{K})$ and from Lemma 2.1 of [4] we have, if $\hat{\chi}\in\mathbb{P}^l(\hat{K})\otimes S_N$ for any positive integer l and N

$$\|\hat{\chi}\|_{H^{r,s}(\hat{K})}\leqq C\,\|\hat{\chi}\|_{0,\hat{K}}. \tag{4.32}$$

Moreover, for any $\hat{\psi}$ in $C^\infty(\hat{\mathbf{K}})$, there exists a constant C, independent of δ and $\hat{\psi}$, such that (see [4] Lemma 2.3)

$$\inf_{\psi^*\in\mathbb{P}^l\otimes S_N}\|\hat{\psi}-\psi^*\|_{H^{r,s}(\hat{\mathbf{K}})}$$

$$\leqq C\left(\int_0^{2\pi N}|\hat{\psi}|^2_{r,\hat{K}}+\int_{\hat{K}}|\hat{\psi}|^2_{s,(0,2\pi N)}\right)^{1/2}\qquad s\geqq 0,\ 0\leqq r\leqq l+1. \tag{4.33}$$

Now, using (3.18) and (4.30) we deduce that, for any $\psi\in C^\infty(\mathbf{K})$, $\chi\in\mathbb{P}^3\otimes S_N$, $\psi^*\in\mathbb{P}^1\otimes S_N$:

$$\begin{aligned}E_{\mathbf{K}}(\psi\chi)&=\det(B_K)N^{-1}E_{\hat{K}}(\hat{\psi}\hat{\chi})\\&=\det(B_K)N^{-1}E_{\hat{K}}((\hat{\psi}-\psi^*)\hat{\chi}).\end{aligned}$$

Thus, from (4.31), (4.32) and (4.33) it follows that:

$$\begin{aligned}E_{\mathbf{K}}(\psi\chi)^2\leqq C\Bigg(&N^{-2s}\int_K\|\psi\|^2_{s,I}dx_1\,dx_2\\&+h^{2r}\int_0^{2\pi}\|\psi\|^2_{r,K}dx_3\Bigg)\|\chi\|^2_{0,\mathbf{K}}.\end{aligned} \tag{4.34}$$

For any fixed $(\mathbf{u},\mathbf{v},\varphi)$ in V_δ^3, a repeated use of (4.34) with $\chi=\dfrac{\partial\varphi_j}{\partial x_i}$ and $\psi=u_iv_j$ $(1\leqq i,j\leqq 3)$ gives:

$$\sum_{i=1}^3\int_Q u_i\mathbf{v}\frac{\partial\varphi}{\partial x_i}-\left(u_i\mathbf{v},\frac{\partial\varphi}{\partial x_i}\right)_\delta$$

$$\leqq C\,\|\varphi\|_V\left[\sum_{K\in\mathscr{C}h}\left(\sum_{i,j=1}^3 N^{-2s}\|u_iv_j\|^2_{L^2(K;H^s(I))}+h^{2r}\|u_iv_j\|^2_{L^2(I;H^r(K))}\right)^{1/2}\right]. \tag{4.35}$$

For any $1\leqq i,\ j\leqq 3$, $u_iv_j\in\mathbb{P}_6\times S_{2N}$ over each $\mathbf{K}$, hence by both finite element and Fourier inverse inequalities (see 4.14 and [6] Theorem 3.2.6) we get:

$$\begin{aligned}&\|u_iv_j\|_{L^2(K;H^s(I))}\leqq CN^{s-1/4}\|u_iv_j\|_{L^2(K;H^{1/4}(I))}\\&\|u_iv_j\|_{L^2(I;H^r(K))}\leqq Ch^{1/4-r}\|u_iv_j\|_{L^2(I;H^{1/4}(K))}.\end{aligned}$$

So, we obtain:

$$\left|\int_Q\sum_{i=1}^3 u_i\mathbf{v}\frac{\partial\varphi}{\partial x_i}-\left(\sum_{i=1}^3 u_i\mathbf{v},\frac{\partial\varphi}{\partial x_i}\right)_\delta\right|$$

$$\leqq C(N^{-1/4}+h^{1/4})\,\|\varphi\|_V\sum_{i,j=1}^3\|u_iv_j\|_{1/4,1/4}. \tag{4.36}$$

Due to the classical Sobolev imbeddings we have

$$\|u_i v_j\|_{0,Q} \leqq C\,\|u_i\|_{3/4,Q}\,\|v_j\|_{3/4,Q},$$
$$\|u_i v_j\|_{1,Q} \leqq C\,\|u_i\|_{7/4,Q}\,\|v_j\|_{7/4,Q}.$$

hence, by interpolation, one has

$$\|u_i v_j\|_{1/4,1/4} = \|u_i v_j\|_{1/4,Q} \leqq C\,\|u_i\|_{1,Q}\,\|v_j\|_{1,Q}, \tag{4.37}$$

and finally

$$\left|\sum_{i=1}^{3} \int_Q u_i \mathbf{v} \frac{\partial \varphi}{\partial x_i} - \left(u_i \mathbf{v}, \frac{\partial \varphi}{\partial x_i}\right)_\delta\right|$$
$$\leqq C(N^{-1/4} + h^{1/4})\,\|\varphi\|_V\,\|\mathbf{u}\|_V\,\|v\|_V. \tag{4.38}$$

Working out similarly we can easily check that, for any φ in V_δ it follows

$$|\int_Q \mathbf{f}\varphi - (\mathbf{f},\varphi)_\delta| \leqq C(N^{-\bar{\sigma}} + h^{\bar{\rho}})\,\|\mathbf{f}\|_{\bar{\rho},\bar{\sigma}}\,\|\varphi\|_V. \tag{4.39}$$

From (4.38) and (4.39) we derive easily that, for any $(\mathbf{u},\mathbf{v})\in V^2$, $\lambda\in\mathbb{R}$, $p\in L$ we have:

$$\lim_{\delta\to 0} \|T_\delta H(\pi_\delta(\lambda,(\mathbf{u},p))) - T_\delta H_\delta(\pi_\delta(\lambda,(\mathbf{u},p)))\|_X = 0, \tag{4.40}$$

$$\lim_{\delta\to 0} \|T_\delta D_{\mathbf{u}} H[\pi_\delta(\lambda,(\mathbf{u},p))](\mathscr{P}\mathbf{v})$$
$$- T_\delta D_{\mathbf{u}} H_\delta[\pi_\delta(\lambda,(\mathbf{u},p))](\mathscr{P}\mathbf{v})\|_X = 0. \tag{4.41}$$

Using the continuity of G and T, and (4.15) one can easily prove

$$\lim_{\delta\to 0} \|TH(\lambda,(\mathbf{u},p)) - T_\delta H(\pi_\delta(\lambda,(\mathbf{u},p)))\|_X = 0; \tag{4.42}$$

$$\lim_{\delta\to 0} \|TD_{\mathbf{u}} H[\lambda,(\mathbf{u},p)]\,\mathbf{v} - T_\delta D_{\mathbf{u}} H[\pi_\delta(\lambda,(\mathbf{u},p))](\mathscr{P}\mathbf{v})\|_X = 0. \tag{4.43}$$

Due to (4.40) and (4.42) we derive (see (4.18) for the definition of x)

$$\lim_{\delta\to 0} \|F(x) - F_\delta(\pi_\delta(x))\|_X = 0;$$

(4.41) and (4.43) imply that

$$\lim_{\delta\to 0} \|D_{\mathbf{u}} F[x](\mathbf{v}) - D_{\mathbf{u}} F_\delta[\pi_\delta x](\mathscr{P}\mathbf{v})\|_X = 0.$$

Using the same kind of arguments we can complete the proof to get

$$\lim \|F'[x](\xi) - F'_\delta[\pi_\delta x](\pi_\delta \xi)\|_X = 0.$$

Finally, from (4.38) and (4.39) we easily obtain (4.25) and (4.26). □

Lemma 4.4 assures that the hypotheses of Theorem 3.2 of [8] are now fulfiled, hence we can state the main result of this paper:

Theorem 4.2. *There exist* $\bar{C}>0$ *and, for any* δ *small enough, a unique branch* $\lambda\in\Lambda\mapsto(\mathbf{u}_\delta(\lambda),\, p_\delta(\lambda))$ *of solutions of the discrete problem* (4.19) *such that:*

$$\|(\lambda,(\mathbf{u}_\delta(\lambda),p_\delta(\lambda)))-\pi_\delta(\lambda,(\mathbf{u}(\lambda),p(\lambda))\|\leqq\bar{C}\quad \textit{for any } \lambda\in\Lambda.$$

Furthermore if $r\leqq\min(\rho,2)$ *and* $f\in H_p^{\sigma-1,r-1}(Q)$ *we have*

$$\sup_{\lambda\in\Lambda}\|\mathbf{u}(\lambda)-\mathbf{u}_\delta(\lambda)\|_V+\|p(\lambda)-p_\delta(\lambda)\|_V\leqq C(N^{1-\sigma}+h^{r-1}). \tag{4.44}$$

Proof. By the Theorem 3.2 of [8] quoted above, we have that the first assertion holds together with the inequality:

$$\|\mathbf{u}(\lambda)-\mathbf{u}_\delta(\lambda)\|_V+\|p(\lambda)-p_\delta(\lambda)\|_L\leqq C\,\|F_\delta(\pi_\delta(\lambda,y(\lambda))\|_X.$$

(see (4.20), (4.21) for the definition of $y(\lambda)$).

For typographical convenience only, we drop out the dependence of $\mathbf{u}$ and p on λ.

Since $F(\lambda,(\mathbf{u},p))=0$, we obtain:

$$\begin{aligned}\|F_\delta(\pi_\delta(\lambda,(\mathbf{u},p)))\|_X\leqq C(&\|\mathbf{u}-\mathscr{P}\mathbf{u}\|_V+\|p-(\pi_0\circ P_0)p\|_L\\ &+\|TH(\lambda,(\mathbf{u},p))-T_\delta H_\delta(\pi_\delta(\lambda,(\mathbf{u},p)))\|_X).\end{aligned} \tag{4.45}$$

Let us examine the last term on the right hand side

$$\begin{aligned}\|TH(\lambda,(\mathbf{u},p))-T_\delta H_\delta(\pi_\delta(\lambda,(\mathbf{u},p)))\|_X\leqq{}&\|(T-T_\delta)H(\lambda,(\mathbf{u},p))\|_X\\ &+\|T_\delta H(\lambda,(\mathbf{u},p))-T_\delta H(\pi_\delta(\lambda,(\mathbf{u},p)))\|_X\\ &+\|T_\delta(H-H_\delta)(\pi_\delta(\lambda,(\mathbf{u},p)))\|_X.\end{aligned} \tag{4.46}$$

Since $F(\lambda,(\mathbf{u},p))=0$ we have $(\mathbf{u},p)=-TH(\lambda,(\mathbf{u},p))$ hence (4.15) implies that

$$\|(T-T_\delta)H(\lambda,(\mathbf{u},p))\|_X\leqq C(N^{1-\sigma}+h^{r-1})(\|\mathbf{u}\|_{r,\sigma}+\|p\|_{r-1,\sigma-1}). \tag{4.47}$$

By the continuity of T_δ and the locally lipschitz continuity of H we get:

$$\|T_\delta H(\lambda,(\mathbf{u},p))-T_\delta H(\pi_\delta(\lambda,(\mathbf{u},p))\|_X\leqq C(N^{1-\sigma}+h^{r-1})\,\|\mathbf{u}\|_{r,\sigma}^2+\|p\|_{r-1,\sigma-1}). \tag{4.48}$$

Using now (4.35) we easily derive

$$\begin{aligned}&\|T_\delta(H-H_\delta)(\pi_\delta(\lambda,(\mathbf{u},p)))\|_X\\ &\leqq C(N^{1-\sigma}+h^{r-1})\Big[\sum_{K\in\mathscr{C}h}\Big(\sum_{1\leqq i,j\leqq 3}\|\mathscr{P}u_i\mathscr{P}u_j\|_{H^{r,\sigma}(\mathbb{K})}^2\Big)^{1/2}+\|f\|_{\alpha,\beta}\Big].\end{aligned} \tag{4.49}$$

where $\alpha=\max(r-1,\bar{\rho})$ and $\beta=\max(\sigma-1,\bar{\sigma})$.

Since $\dfrac{2}{\rho}+\dfrac{1}{\sigma}<2$, $\sigma>1$ and $r=\min(\rho,2)$ we have $\dfrac{2}{r}+\dfrac{1}{\sigma}<2$ hence $H^{r,\sigma}(\mathbb{K})$ is an algebra so we get

$$\begin{aligned}&\|T_\delta(H-H_\delta)(\pi_\delta(\lambda,(\mathbf{u},p)))\|_X\\ &\qquad\leqq C(N^{1-\sigma}+h^{r-1})\Big[\sum_{K\in\mathscr{C}h}\|\mathscr{P}\mathbf{u}\|_{H^{r,\sigma}(\mathbb{K})}^2+\|f\|_{\alpha,\beta}\Big].\end{aligned} \tag{4.50}$$

From (4.45), (4.46), (4.47), (4.48) and (4.50) we finally obtain

$$\|\mathbf{u}-\mathbf{u}_\delta\|_V+\|p-p_\delta\|_L \leqq C(N^{1-\sigma}+h^{r-1})[(1+\|u\|_{\rho,\sigma})^2+\|p\|_{\rho-1,\sigma-1} \\ +\|f\|_{\alpha,\beta}+\sum_{K\in\mathscr{C}h}\|u\|_{H^{r,\sigma}(\mathbb{K})}].$$

Besides that, let us set $\mathscr{Q}\mathbf{u}=(\underset{\sim}{u}_1,\underset{\sim}{u}_2,\tilde{u}_3)$ (see Lemma 4.2). One has, for any K in $\mathscr{C}h$

$$\begin{aligned}\|\mathscr{P}u\|_{H^{r,\sigma}(\mathbb{K})}&=\int_I\|\mathscr{P}u(\cdot,\cdot,z)\|_{r,K}^2\,dz+\int_K\|\mathscr{P}u(x,y,\cdot)\|_{\sigma,I}^2\,dx\,dy\\ &\leqq\int_I\{\|(\mathscr{P}\mathbf{u}-\mathscr{Q}\mathbf{u})(\cdot,\cdot,z)\|_{r,K}^2+\|\mathscr{Q}\mathbf{u}(\cdot,\cdot,z)\|_{r,K}^2\}\,dz\\ &\quad+\int_K\{\|(\mathscr{P}\mathbf{u}-P_0\mathbf{u})(x,y,\cdot)\|_{\sigma,I}^2+\|P_0\mathbf{u}(x,y,\cdot)\|_{\sigma,I}^2\}\,dx\,dy.\end{aligned}$$

From (4.6) and (3.9) one has, for any (x,y,z) in Q:

$$\|\mathscr{Q}\mathbf{u}(\cdot,\cdot,z)\|_{r,K}\leqq C\,\|\mathbf{u}(\cdot,\cdot,z)\|_{r,K},$$

$$\|P_0\mathbf{u}(x,y,\cdot)\|_{\sigma,I}\leqq C\,\|\mathbf{u}(x,y,\cdot)\|_{\sigma,I}.$$

The inverse inequalities give:

$$\begin{aligned}\|(\mathscr{P}\mathbf{u}-\mathscr{Q}\mathbf{u})(\cdot,\cdot,z)\|_{r,K}^2&\leqq Ch^{2(1-r)}\|(\mathscr{P}\mathbf{u}-\mathscr{Q}\mathbf{u})(\cdot,\cdot,z)\|_{1,K}^2\\ &\leqq Ch^{2(1-r)}(\|(\mathscr{P}\mathbf{u}-\mathbf{u})(\cdot,\cdot,z)\|_{1,K}^2+\|(\mathbf{u}-\mathscr{Q}\mathbf{u})(\cdot,\cdot,z)\|_{1,K}^2),\end{aligned}$$

and

$$\begin{aligned}\|(\mathscr{P}\mathbf{u}-P_0\mathbf{u})(x,y,\cdot)\|_{\sigma,I}^2&\leqq CN^{2(\sigma-1)}\|(\mathscr{P}\mathbf{u}-P_0\mathbf{u})(x,y,\cdot)\|_{1,I}^2\\ &\leqq CN^{2(\sigma-1)}(\|(\mathscr{P}\mathbf{u}-\mathbf{u})(x,y,\cdot)\|_{1,I}^2+\|(\mathbf{u}-P_0\mathbf{u})(x,y,\cdot)\|_{1,I}^2).\end{aligned}$$

From (3.7), (3.12), (3.9) and (4.6) we can conclude that

$$\|\mathscr{P}\mathbf{u}\|_{H^{r,\sigma}(\mathbb{K})}^2\leqq C\,\|u\|_{H^{r,\sigma}(\mathbb{K})}^2.$$

Now (4.44) is an easy consequence of (4.51), and the theorem is completely proved. □

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