Concurrency Theory

Winter 2025/26

Lecture 14: Interleaving Semantics of Petri Nets

Thomas Noll, Peter Thiemann
Programming Languages Group
University of Freiburg

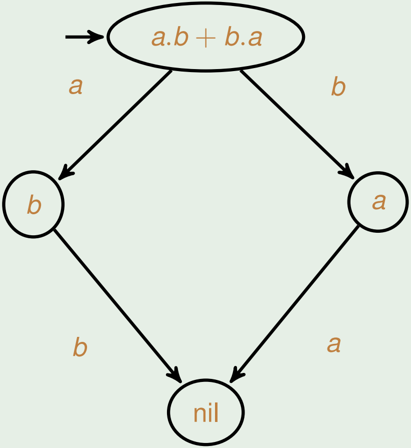
https://proglang.github.io/teaching/25ws/ct.html

Thomas Noll, Peter Thiemann

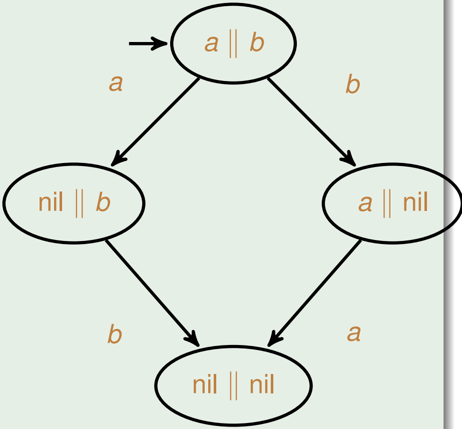
Winter 2025/26

Motivation

Example 14.1 (LTSs of CCS processes)



LTS of $a.b.\text{nil} + b.a.\text{nil}$



LTS of $a.\text{nil} \parallel b.\text{nil}$

Carl Adam Petri (1926–2010)

Semantics: Executions and Traces

Models of computation in the 1960s: lambda-calculus, finite automata, Turing machines, ...

States: current configurations of the machine

- One or more initial states
- Possibly some distinguished final states

Transitions: moves between configurations

Lambda calculus	$(\lambda x.xx)(\lambda y.y)$	$\longrightarrow$	$(\lambda y.y)(\lambda z.z)$
Turing machine	$0010q_1011$	$\longrightarrow$	$001q_201011$
Finite automaton	q_1	$\xrightarrow{a}$	q_2
Pushdown automaton	$(q_1, XYYZ)$	$\xrightarrow{a}$	$(q_2, XYXYYZ)$

Petri's Question

C.A. Petri points out a discrepancy between
how Theoretical Physics and Theoretical Computer Science
described systems in 1962:

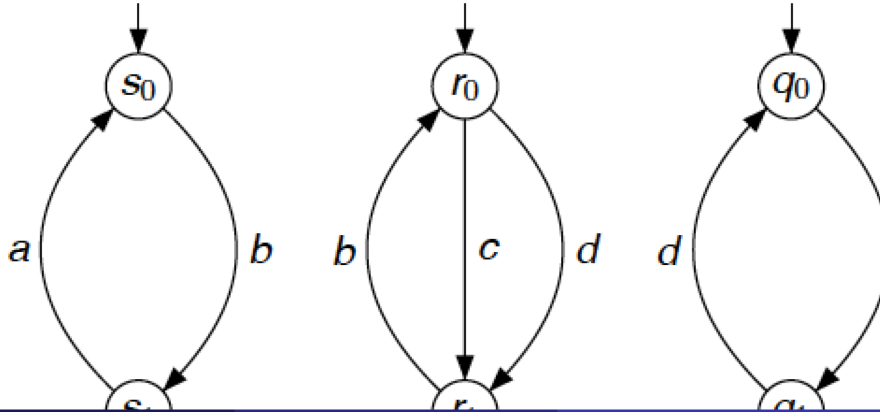
> Theoretical Physics describes systems as a collection of interacting particles (subsystems), without a notion of global clock or simultaneity

> Theoretical Computer Science describes systems as sequential virtual machines going through a temporally ordered sequence of global states

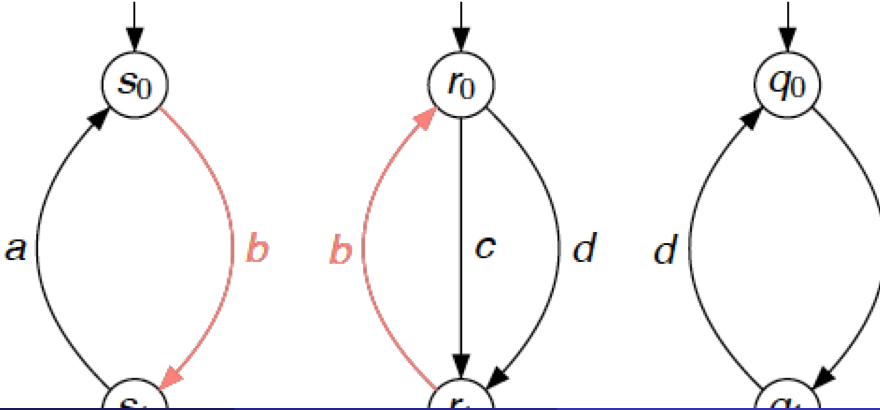
Petri's question:

> Which kind of abstract machine should be used to describe the physical implementation of a Turing machine?

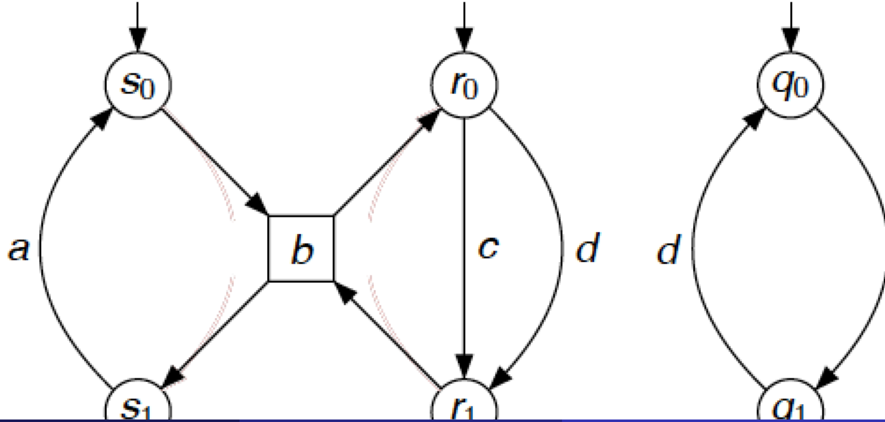
A graphical representation of interacting finite auton



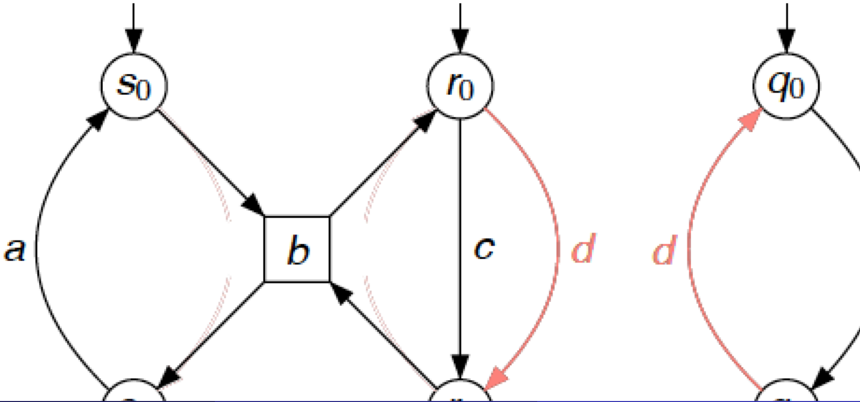
A graphical representation of interacting finite autom



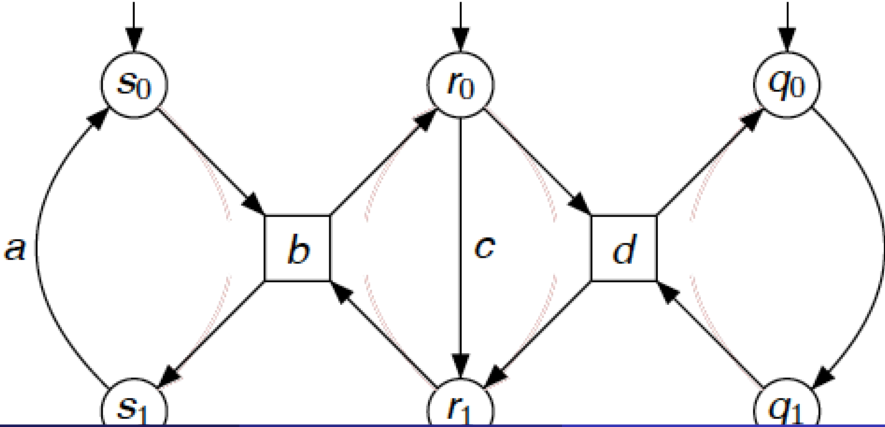
A graphical representation of interacting finite autom



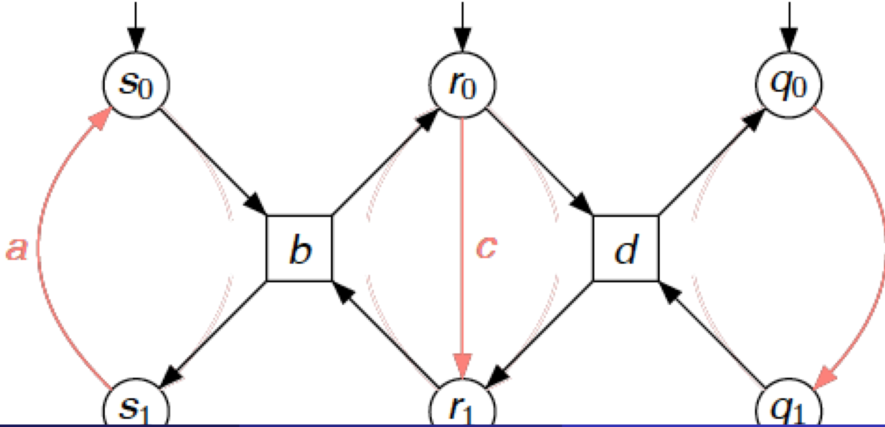
A graphical representation of interacting finite autor



A graphical representation of interacting finite autom

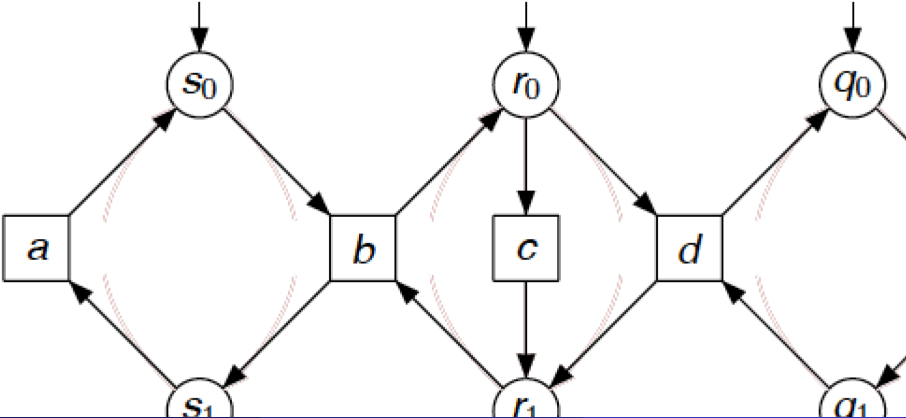


A graphical representation of interacting finite autom

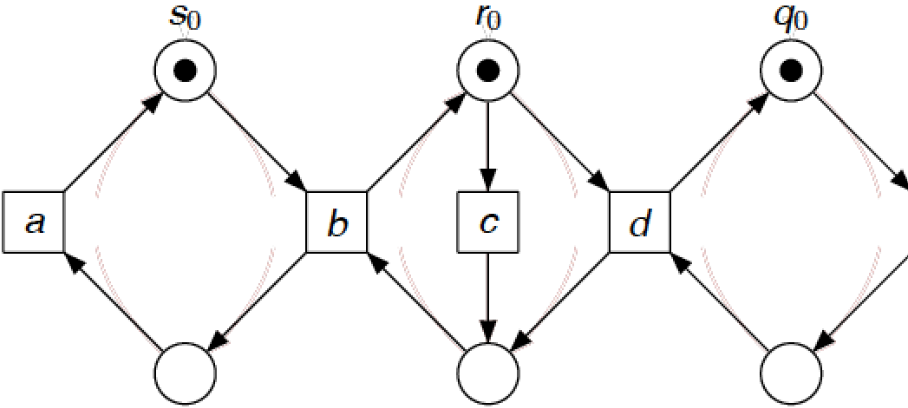


Finite Automata and Petri Nets VII

A graphical representation of interacting finite aut



A graphical representation of interacting finite automat



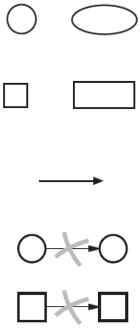
Components of a Net

A Petri net is a structure with two kinds of elements: places and transitions. These are connected by arcs.

A place is represented by a circle or ellipse. A place p always models a passive component: p can store, accumulate or show things.

A transition is represented by a square or rectangle. A transition t always models an active component: t can produce things, consume, transport, or change them.

Places and transitions are connected to each other by directed arcs. Graphically, an arc is represented by an arrow. An arc models an abstract, sometimes only notional relation between components. Arcs run from places to transitions or vice versa.



Nets

Definition 14.2 (Petri net)

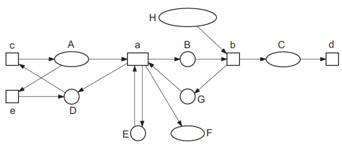
A Petri net N is a triple (P, T, F) where:

- P is a finite set of places,
- T is a finite set of transitions with $P \cap T = \emptyset$, and
- $F \subseteq (P \times T) \cup (T \times P)$ are the arcs.[a]

Places and transitions are generically called nodes.

[a] F is also called the flow relation.

Example 14.3



$$P = \{A, B, C, \ldots\}$$
$$T = \{a, b, c, \ldots\}$$
$$F = \{(A, a), (a, B), (B, b), \ldots\}$$

Pre- and Post-Sets

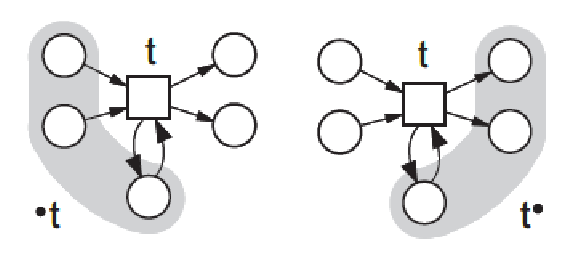
Definition 14.4 (Pre- and post-sets)

Let node $x \in P \cup T$.

- The pre-set of x is defined by $\quad {}^\bullet x := \{y \mid (y, x) \in F\}$.
- The post-set of x is defined by $\quad x^\bullet = \{y \mid (x, y) \in F\}$.

Two nodes $x, y \in P \cup T$ form a loop if $x \in {}^\bullet y$ and $y \in {}^\bullet x$.

Example 14.5



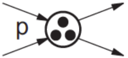
Markings

Definition 14.6 (Marking)

- A marking M of a net $N = (P, T, F)$ is a mapping $M : P \to \mathbb{N}$.
- For net $N = (P, T, F)$ and marking M_0, the quadruple (P, T, F, M_0) is called an elementary system net with initial marking M_0.

Intuition:

- A marking can be seen as a multiset of places.
- It defines a distribution of tokens across places.
- Tokens are depicted as black dots.



$M(p) = 3$

Remark: In generic (= non-elementary) system nets, several types (colours) of tokens can be distinguished.

Transition Firing

Definition 14.7 (Enabling and occurrence of a transition)

Let (P, T, F, M_0) be an elementary system net and $M : P \to \mathbb{N}$.

- Marking M enables a transition $t \in T$ if $M(p) \geq 1$ for each place $p \in {}^\bullet t$.
- Transition $t \in T$ can occur in marking M if t is enabled in M.
- Its occurrence or firing leads to marking M', denoted by the step relation $M \xrightarrow{t} M'$ and defined for each place $p \in P$ by

$$M'(p) := M(p) - F(p, t) + F(t, p)$$

where we represent relation F by its characteristic function.

Intuition: Transition t is enabled whenever every $p \in {}^\bullet t$ holds at least one token.

On t's occurrence, one token is removed from each place in ${}^\bullet t$, and one token is put in each place in $t^\bullet$:

$$M'(p) = \begin{cases} M(p) - 1 & \text{if } p \in {}^\bullet t \text{ and } p \notin t^\bullet \\ M(p) + 1 & \text{if } p \in t^\bullet \text{ and } p \notin {}^\bullet t \end{cases}$$

Transition Occurrence

Definition (Enabling and occurrence of a transition)

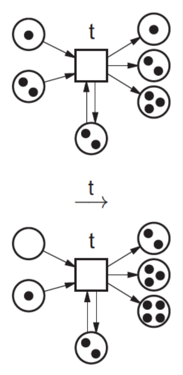
Let (P, T, F, M_0) be an elementary system net and $M : P \to \mathbb{N}$.

- Marking M enables a transition $t \in T$ if $M(p) \geq 1$ for each place $p \in {}^\bullet t$.
- Transition $t \in T$ can occur in marking M if t is enabled in M.
- Its occurrence or firing leads to marking M', denoted by the step relation $M \xrightarrow{t} M'$ and defined for each place $p \in P$ by

$$M'(p) := M(p) - F(p, t) + F(t, p)$$

where we represent relation F by its characteristic function

Example 14.8



The Interleaving Semantics of Petri Nets I

Goal: Establish an execution semantics by mapping a Petri net to a labelled transition system

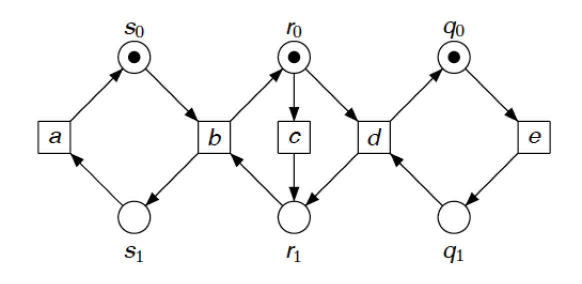
States: markings (i.e., distributions of tokens over the net)

Transitions: $M \xrightarrow{t} M'$ ("steps")

Sequential runs: $M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} M_2 \xrightarrow{t_3} \ldots$ (step sequences)

The Interleaving Semantics of Petri Nets II

Example 14.9

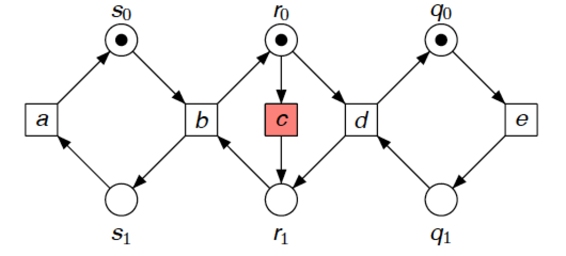


$$\begin{array}{c} s_1 \\ r_1 \\ q_1 \end{array} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(As the marking for s_0 is the complement of s_1, the marking for s_0 is omitted. The same applies to the places r_0 and q_0.)

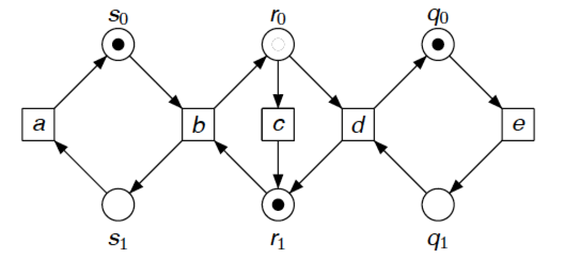
The Interleaving Semantics of Petri Nets III

Example 14.9 (continued)



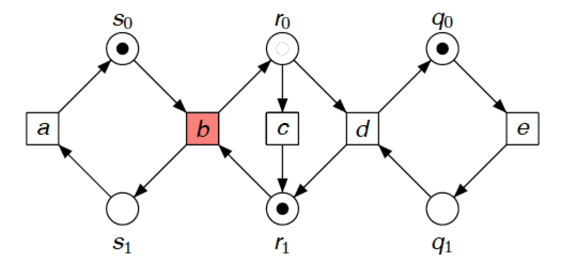
$$\begin{array}{c} s_1 \\ r_1 \\ q_1 \end{array} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{c}$$

Example 14.9 (continued)



$$\begin{array}{c} s_1 \\ r_1 \\ q_1 \end{array} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{c} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

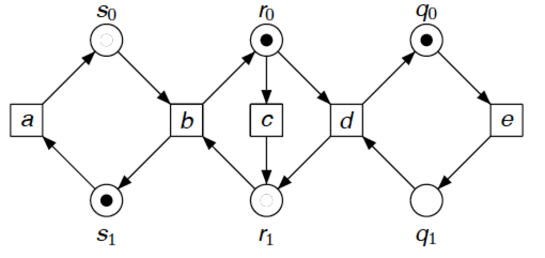
Example 14.9 (continued)



$$
\begin{array}{c}
s_1 \\ r_1 \\ q_1
\end{array}
\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\xrightarrow{c}
\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
\xrightarrow{b}
$$

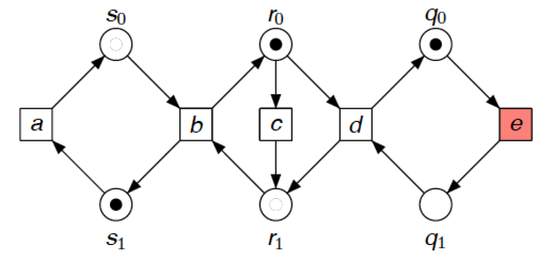
The Interleaving Semantics of Petri Nets VI

Example 14.9 (continued)



$$\begin{array}{c} s_1 \\ r_1 \\ q_1 \end{array} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{c} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \xrightarrow{b} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

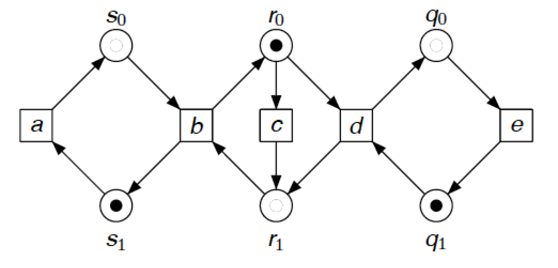
Example 14.9 (continued)



$$\begin{array}{c} s_1 \\ r_1 \\ q_1 \end{array} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{c} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \xrightarrow{b} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{e}$$

The Interleaving Semantics of Petri Nets VIII

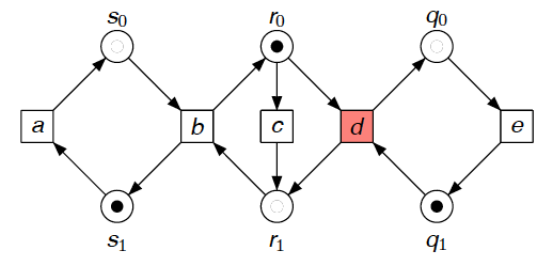
Example 14.9 (continued)



$$\begin{array}{c} s_1 \\ r_1 \\ q_1 \end{array} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{c} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \xrightarrow{b} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{e} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The Interleaving Semantics of Petri Nets IX

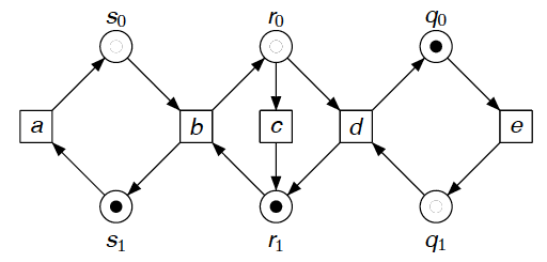
Example 14.9 (continued)



$$\begin{array}{c} s_1 \\ r_1 \\ q_1 \end{array} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{c} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \xrightarrow{b} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{e} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \xrightarrow{d}$$

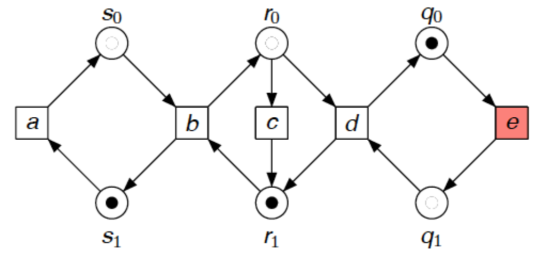
The Interleaving Semantics of Petri Nets X

Example 14.9 (continued)



$$
\begin{array}{c} s_1 \\ r_1 \\ q_1 \end{array}
\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\xrightarrow{c}
\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
\xrightarrow{b}
\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\xrightarrow{e}
\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}
\xrightarrow{d}
\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}
$$

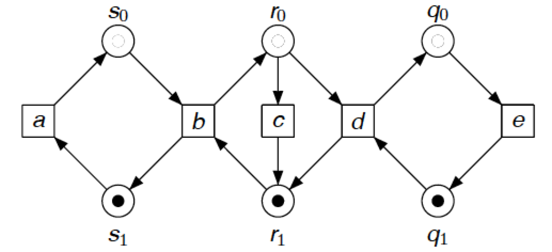
Example 14.9 (continued)



$$\begin{array}{c} s_1 \\ r_1 \\ q_1 \end{array} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{c} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \xrightarrow{b} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{e} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \xrightarrow{d} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \xrightarrow{e}$$

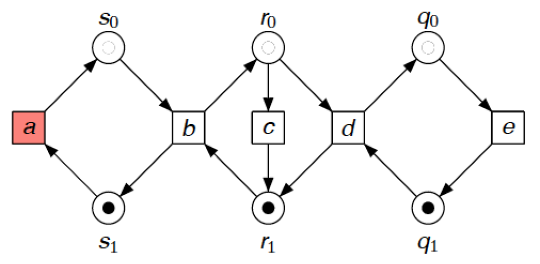
The Interleaving Semantics of Petri Nets XII

Example 14.9 (continued)



$$\begin{array}{c} s_1 \\ r_1 \\ q_1 \end{array} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{c} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \xrightarrow{b} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{e} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \xrightarrow{d} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \xrightarrow{e} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

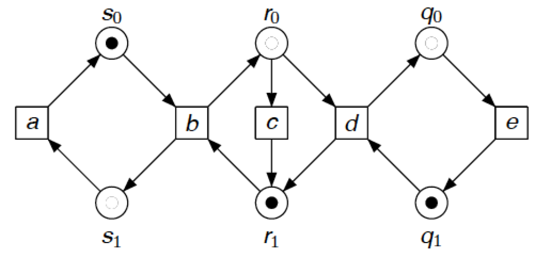
Example 14.9 (continued)



$$\begin{array}{c} s_1 \\ r_1 \\ q_1 \end{array} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{c} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \xrightarrow{b} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{e} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \xrightarrow{d} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \xrightarrow{e} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \xrightarrow{a}$$

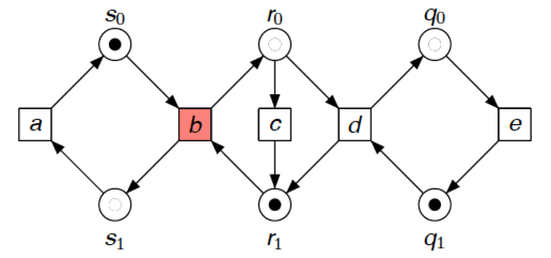
The Interleaving Semantics of Petri Nets XIV

Example 14.9 (continued)



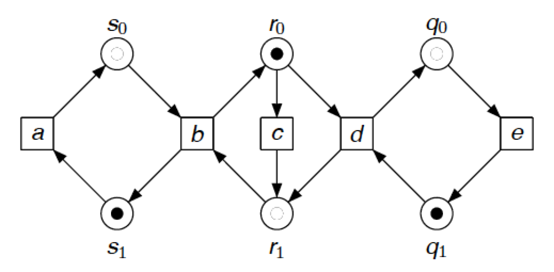
$$\begin{array}{c} s_1 \\ r_1 \\ q_1 \end{array} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{c} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \xrightarrow{b} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{e} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \xrightarrow{d} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \xrightarrow{e} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \xrightarrow{a}$$

Example 14.9 (continued)



$$
\begin{array}{c}
s_1 \\ r_1 \\ q_1
\end{array}
\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\xrightarrow{c}
\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
\xrightarrow{b}
\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\cdots
\xrightarrow{e}
\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\xrightarrow{a}
\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}
\xrightarrow{b}
$$

Example 14.9 (continued)



$$\begin{array}{c} s_1 \\ r_1 \\ q_1 \end{array} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{c} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \xrightarrow{b} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdots \xrightarrow{a} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \xrightarrow{b} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Reachable Markings

Definition 14.10 (Step sequence)

Let (P, T, F, M_0) be an elementary system net.

- A sequence of transitions $\sigma = t_1\,t_2 \ldots t_n \in T^*$ is a step sequence if there exist markings $M_1, \ldots, M_n$ such that

$$M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} \cdots \xrightarrow{t_{n-1}} M_{n-1} \xrightarrow{t_n} M_n.$$

- Marking M_n is then reached by the occurrence of σ, denoted $M_0 \xrightarrow{\sigma} M_n$.
- M is a reachable marking if there exists a step sequence σ such that $M_0 \xrightarrow{\sigma} M$.

Example 14.11

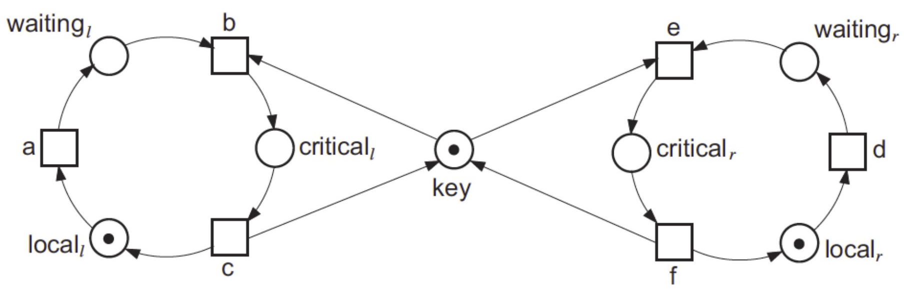
In the previous example,

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{cbedeab} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Mutual Exclusion I

Example 14.12

Two processes cycling through the states *local*, *waiting* and *critical*:

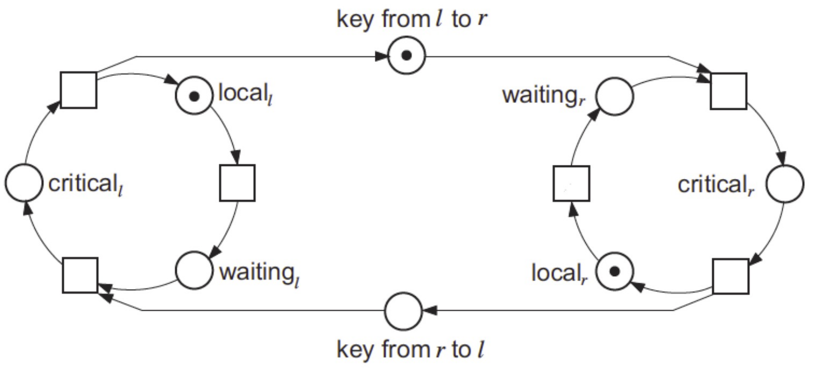


left process semaphore right process

Between transitions *b* and *e*, a conflict can arise infinitely often.
No strategy has been modelled to solve this conflict.

Example 14.13

A strategy where processes are acquiring access in an alternating fashion:



One-Bounded Elementary System Nets

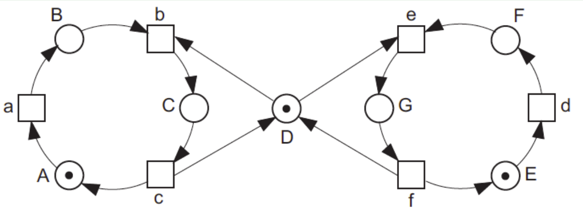
Definition 14.14 (One-boundedness)

An elementary system net $N = (P, T, F, M_0)$ is called one-bounded if for each reachable marking M and place $p \in P$,

$$M(p) \leq 1.$$

Remark: Markings of one-bounded elementary system nets can be described as a set of places.

Example 14.15



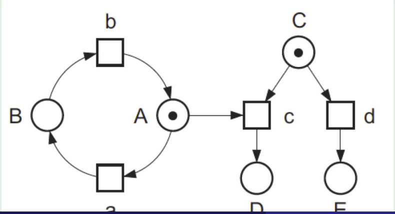
Sequential Runs

Definition 14.16 (Sequential run)

Let $N = (P, T, F, M_0)$ be an elementary system net.

- A sequential run of N is a sequence $M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} \cdots$ of steps of N starting with the initial marking M_0.
- A run can be finite or infinite.
- A finite run $M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_1} \cdots \xrightarrow{t_n} M_n$ is complete if M_n does not enable any transition.

Example 14.17



A sample complete run:

$$AC \xrightarrow{a} BC \xrightarrow{b} AC \xrightarrow{c} D$$

A sample incomplete run:

$$AC \xrightarrow{d} AE \xrightarrow{a} BE$$

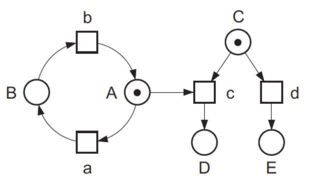
Marking Graph

Definition 14.18 (Marking graph)

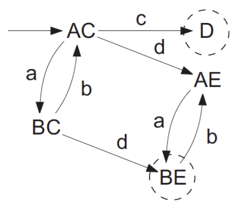
The marking graph of a net *N* has as nodes the reachable markings of *N* and as edges the corresponding steps of *N*.[a]

[a] Since firing an (enabled) transition in a marking yields a unique successor marking, marking graphs are a deterministic LTS.

Example 14.19



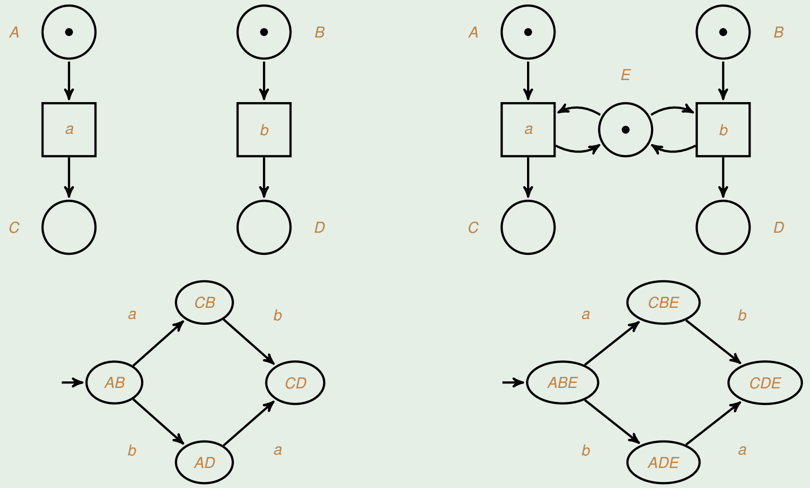
A sample elementary system net



... and its marking graph

Interleaving semantics

Example 14.20 (Petri nets and their marking graphs)



Thus: Marking graphs are isomorphic even though the nets behave differently

Summary

- A Petri net consists of places, transitions and arcs.
- An elementary system net is a Petri net plus a marking.
- Firing a single transition in a marking is a step.
- A sequential run is a sequence of steps starting in the initial marking.
- The marking graph has as nodes the reachable markings of the net and as edges its reachable steps.
- The marking graph represents the interleaving semantics of a net.