

Concurrency Theory

Winter 2025/26

Lecture 9: Properties of Hennessy-Milner Logic

Thomas Noll, Peter Thiemann
Programming Languages Group
University of Freiburg

<https://proglang.github.io/teaching/25ws/ct.html>

Thomas Noll, Peter Thiemann

Winter 2025/26

Outline of Lecture 9

- 1 Closure under Negation
- 2 HML and Process Traces
- 3 HML and Strong Bisimilarity
- 4 HML and Weak Bisimilarity

Closure under Negation I

Observation: Negation is *not* one of the HML constructs.

Reason: HML is closed under complement.

Closure under Negation I

Observation: Negation is *not* one of the HML constructs.

Reason: HML is closed under complement.

Lemma 9.1

For every $F \in \text{HMF}$ there exists $F^c \in \text{HMF}$ such that $\llbracket F^c \rrbracket = S \setminus \llbracket F \rrbracket$ for every LTS $(S, Act, \longrightarrow)$.

Closure under Negation I

Observation: Negation is *not* one of the HML constructs.

Reason: HML is closed under complement.

Lemma 9.1

For every $F \in \text{HMF}$ there exists $F^c \in \text{HMF}$ such that $\llbracket F^c \rrbracket = S \setminus \llbracket F \rrbracket$ for every LTS $(S, Act, \longrightarrow)$.

Proof.

Definition of F^c :

$$\begin{array}{ll} tt^c := ff & ff^c := tt \\ (F_1 \wedge F_2)^c := F_1^c \vee F_2^c & (F_1 \vee F_2)^c := F_1^c \wedge F_2^c \\ (\langle \alpha \rangle F)^c := [\alpha]F^c & ([\alpha]F)^c := \langle \alpha \rangle F^c \end{array}$$

Closure under Negation II

Proof (Lemma 9.1; continued).

We show $\llbracket F^c \rrbracket = S \setminus \llbracket F \rrbracket$ by induction on the structure of $F \in \text{HMF}$:

- $F = \text{tt}$ ($F = \text{ff}$ analogously):

$$\llbracket F^c \rrbracket = \llbracket \text{ff} \rrbracket \stackrel{\text{Def. 8.2}}{=} \emptyset = S \setminus S \stackrel{\text{Def. 8.2}}{=} S \setminus \llbracket \text{tt} \rrbracket = S \setminus \llbracket F \rrbracket$$

Closure under Negation II

Proof (Lemma 9.1; continued).

We show $\llbracket F^c \rrbracket = S \setminus \llbracket F \rrbracket$ by induction on the structure of $F \in \text{HMF}$:

- $F = F_1 \wedge F_2$ ($F = F_1 \vee F_2$ analogously):

$$\begin{aligned}\llbracket F^c \rrbracket &= \llbracket F_1^c \vee F_2^c \rrbracket \\ &\stackrel{\text{Def. 8.2}}{=} \llbracket F_1^c \rrbracket \cup \llbracket F_2^c \rrbracket \\ &\stackrel{\text{ind. hyp.}}{=} S \setminus \llbracket F_1 \rrbracket \cup S \setminus \llbracket F_2 \rrbracket \\ &\stackrel{\text{de Morgan}}{=} S \setminus (\llbracket F_1 \rrbracket \cap \llbracket F_2 \rrbracket) \\ &\stackrel{\text{Def. 8.2}}{=} S \setminus \llbracket F \rrbracket\end{aligned}$$

Closure under Negation II

Proof (Lemma 9.1; continued).

We show $\llbracket F^c \rrbracket = S \setminus \llbracket F \rrbracket$ by induction on the structure of $F \in \text{HMF}$:

- $F = \langle \alpha \rangle F_0$ ($F = [\alpha]F_0$ analogously):

$$\begin{aligned}\llbracket F^c \rrbracket &= \llbracket \llbracket \alpha \rrbracket F_0^c \rrbracket \\ &\stackrel{\text{Def. 8.2}}{=} \llbracket \cdot\alpha\cdot \rrbracket (\llbracket F_0^c \rrbracket) \\ &\stackrel{\text{Def. 8.2}}{=} \{s \in S \mid \forall s' \in S : s \xrightarrow{\alpha} s' \Rightarrow s' \in \llbracket F_0^c \rrbracket\} \\ &\stackrel{\text{ind. hyp.}}{=} \{s \in S \mid \forall s' \in S : s \xrightarrow{\alpha} s' \Rightarrow s' \in S \setminus \llbracket F_0 \rrbracket\} \\ &= \{s \in S \mid \forall s' \in \llbracket F_0 \rrbracket : s \not\xrightarrow{\alpha} s'\} \\ &= S \setminus \{s \in S \mid \exists s' \in \llbracket F_0 \rrbracket : s \xrightarrow{\alpha} s'\} \\ &\stackrel{\text{Def. 8.2}}{=} S \setminus \langle \cdot\alpha\cdot \rangle (\llbracket F_0 \rrbracket) \\ &\stackrel{\text{Def. 8.2}}{=} S \setminus \llbracket F \rrbracket\end{aligned}$$



Outline of Lecture 9

1 Closure under Negation

2 HML and Process Traces

3 HML and Strong Bisimilarity

4 HML and Weak Bisimilarity

Lemma 9.2 (HML and process traces)

Let (S, Act, \rightarrow) be an LTS, and let $s, t \in S$ satisfy the same HMF (i.e., for all $F \in \text{HMF}$: $s \models F \iff t \models F$). Then $\text{Tr}(s) = \text{Tr}(t)$.

Lemma 9.2 (HML and process traces)

Let (S, Act, \rightarrow) be an LTS, and let $s, t \in S$ satisfy the same HMF (i.e., for all $F \in \text{HMF}$: $s \models F \iff t \models F$). Then $\text{Tr}(s) = \text{Tr}(t)$.

Proof.

Let $s, t \in S$ such that for all $F \in \text{HMF}$: $s \models F \iff t \models F$.

Assumption: $\text{Tr}(s) \neq \text{Tr}(t)$.

Then there exists $n \geq 1$ and $w = \alpha_1 \dots \alpha_n \in \text{Act}^+$ with $w \in \text{Tr}(s) \setminus \text{Tr}(t)$ (or vice versa).

Hence, for $F := \langle \alpha_1 \rangle \dots \langle \alpha_n \rangle tt \in \text{HMF}$: $s \models F$ but $t \not\models F$. \square

Remark: The converse does *not* hold.

Example 9.3

- Let
 - $P := a.(b.\text{nil} + c.\text{nil}) \in Prc$ and
 - $Q := a.b.\text{nil} + a.c.\text{nil} \in Prc$.
- Then $Tr(P) = Tr(Q) = \{\varepsilon, a, ab, ac\}$.

Remark: The converse does *not* hold.

Example 9.3

- Let
 - $P := a.(b.\text{nil} + c.\text{nil}) \in Prc$ and
 - $Q := a.b.\text{nil} + a.c.\text{nil} \in Prc$.
- Then $Tr(P) = Tr(Q) = \{\varepsilon, a, ab, ac\}$.
- Let $F := [a](\langle b \rangle \text{tt} \wedge \langle c \rangle \text{tt}) \in HMF$.
- Then $P \models F$ but $Q \not\models F$.

Remark: The converse does *not* hold.

Example 9.3

- Let
 - $P := a.(b.\text{nil} + c.\text{nil}) \in Prc$ and
 - $Q := a.b.\text{nil} + a.c.\text{nil} \in Prc$.
- Then $Tr(P) = Tr(Q) = \{\varepsilon, a, ab, ac\}$.
- Let $F := [a](\langle b \rangle \text{tt} \wedge \langle c \rangle \text{tt}) \in HMF$.
- Then $P \models F$ but $Q \not\models F$.
- Thus: HML can distinguish **branching behaviour** of processes (just as bisimulation can...).

Outline of Lecture 9

- 1 Closure under Negation
- 2 HML and Process Traces
- 3 HML and Strong Bisimilarity
- 4 HML and Weak Bisimilarity

- Strong bisimilarity (and observation congruence) are based on mutual mimicking of processes.
- They possess the required properties of behavioural equivalences.
- In particular, \sim and \approx^c are deadlock-sensitive CCS congruences.
- Hennessy-Milner Logic (HML) is a logic for expressing properties of processes.

- Strong bisimilarity (and observation congruence) are based on mutual mimicking of processes.
- They possess the required properties of behavioural equivalences.
- In particular, \sim and \approx^c are deadlock-sensitive CCS congruences.
- Hennessy-Milner Logic (HML) is a logic for expressing properties of processes.

Aim

Study the connection between strong bisimilarity and satisfaction of HML formulae.

Definition 9.4 (Finitely branching LTS)

- A process $P \in Prc$ is **finitely branching** if the set $\{P' \in Prc \mid P \xrightarrow{\alpha} P'\}$ is finite for every $\alpha \in Act$.
- A labelled transition system is **finitely branching** if each state is finitely branching.

Definition 9.4 (Finitely branching LTS)

- A process $P \in Prc$ is **finitely branching** if the set $\{P' \in Prc \mid P \xrightarrow{\alpha} P'\}$ is finite for every $\alpha \in Act$.
- A labelled transition system is **finitely branching** if each state is finitely branching.

Example 9.5

- (1) The process $A_{rep} = a.nil \parallel A_{rep}$ (“ A replicated”) is not finitely branching.
By induction on n , one can prove that for each $n \in \mathbb{N}$:

$$A_{rep} \xrightarrow{a} \underbrace{a.nil \parallel \cdots \parallel a.nil}_{n \text{ times}} \parallel nil \parallel A_{rep}$$

Finitely Branching Transition Systems

Definition 9.4 (Finitely branching LTS)

- A process $P \in Prc$ is **finitely branching** if the set $\{P' \in Prc \mid P \xrightarrow{\alpha} P'\}$ is finite for every $\alpha \in Act$.
- A labelled transition system is **finitely branching** if each state is finitely branching.

Example 9.5

- (1) The process $A_{rep} = a.\text{nil} \parallel A_{rep}$ ("A replicated") is not finitely branching.
By induction on n , one can prove that for each $n \in \mathbb{N}$:

$$A_{rep} \xrightarrow{a} \underbrace{a.\text{nil} \parallel \cdots \parallel a.\text{nil}}_{n \text{ times}} \parallel \text{nil} \parallel A_{rep}$$

- (2) Also the "process" $A^{<\omega} = \sum_{i \in \mathbb{N}} a^i$ with $a^0 = \text{nil}$ and $a^{i+1} = a.a^i$ is not finitely branching:
for each $i \geq 1$, $A^{<\omega} \xrightarrow{a} a^{i-1}$.

Relationship Between HML and Strong Bisimilarity I

Theorem 9.6 (Hennessy-Milner Theorem)

Let $(S, Act, \{\xrightarrow{a} \mid a \in Act\})$ be a finitely branching LTS and $s, t \in S$. Then:

$$s \sim t \quad \text{iff} \quad \text{for every } F \in \text{HMF} : (s \models F \iff t \models F).$$

Relationship Between HML and Strong Bisimilarity I

Theorem 9.6 (Hennessy-Milner Theorem)

Let $(S, Act, \{\xrightarrow{a} \mid a \in Act\})$ be a finitely branching LTS and $s, t \in S$. Then:

$$s \sim t \quad \text{iff} \quad \text{for every } F \in \text{HMF} : (s \models F \iff t \models F).$$

Proof.

“ \Rightarrow ”: Assume $s \sim t$ and $s \models F$ for some $F \in \text{HMF}$.
We show $t \models F$ by structural induction on F .

Relationship Between HML and Strong Bisimilarity I

Theorem 9.6 (Hennessy-Milner Theorem)

Let $(S, Act, \{\xrightarrow{a} \mid a \in Act\})$ be a finitely branching LTS and $s, t \in S$. Then:

$$s \sim t \quad \text{iff} \quad \text{for every } F \in \text{HMF} : (s \models F \iff t \models F).$$

Proof.

“ \Rightarrow ”: Assume $s \sim t$ and $s \models F$ for some $F \in \text{HMF}$.

We show $t \models F$ by structural induction on F . Interesting cases:

- $F = \langle \alpha \rangle F'$:

- Since $s \models F$, there ex. $s' \in S$ such that $s \xrightarrow{\alpha} s'$ and $s' \models F'$.
- Since $s \sim t$, there ex. $t' \in S$ such that $t \xrightarrow{\alpha} t'$ and $s' \sim t'$.
- By induction hypothesis, $t' \models F'$. Thus, $t \models \langle \alpha \rangle F' = F$.

Relationship Between HML and Strong Bisimilarity I

Theorem 9.6 (Hennessy-Milner Theorem)

Let $(S, Act, \{\xrightarrow{a} | a \in Act\})$ be a finitely branching LTS and $s, t \in S$. Then:

$$s \sim t \quad \text{iff} \quad \text{for every } F \in \text{HMF} : (s \models F \iff t \models F).$$

Proof.

“ \Rightarrow ”: Assume $s \sim t$ and $s \models F$ for some $F \in \text{HMF}$.

We show $t \models F$ by structural induction on F . Interesting cases:

- $F = \langle \alpha \rangle F'$:
 - Since $s \models F$, there ex. $s' \in S$ such that $s \xrightarrow{\alpha} s'$ and $s' \models F'$.
 - Since $s \sim t$, there ex. $t' \in S$ such that $t \xrightarrow{\alpha} t'$ and $s' \sim t'$.
 - By induction hypothesis, $t' \models F'$. Thus, $t \models \langle \alpha \rangle F' = F$.
- $F = [\alpha]F'$: Assume that $t \xrightarrow{\alpha} t'$ for some $t' \in S$.
 - Since $s \sim t$, there ex. $s' \in S$ such that $s \xrightarrow{\alpha} s'$ and $s' \sim t'$.
 - Since $s \models [\alpha]F'$, also $s' \models F'$.
 - By induction hypothesis, $t' \models F'$. Thus, $t \models [\alpha]F' = F$.



Relationship Between HML and Strong Bisimilarity II

Theorem (Hennessy-Milner Theorem)

Let $(S, Act, \{\xrightarrow{a} \mid a \in Act\})$ be a finitely branching LTS and $s, t \in S$. Then:

$$s \sim t \quad \text{iff} \quad \text{for every } F \in \text{HMF} : (s \models F \iff t \models F).$$

Proof.

“ \Leftarrow ”: Define $u \equiv v$ iff $(u \models F \iff v \models F)$ for every $F \in \text{HMF}$, and let $s \equiv t$.

Relationship Between HML and Strong Bisimilarity II

Theorem (Hennessy-Milner Theorem)

Let $(S, Act, \{\xrightarrow{a} \mid a \in Act\})$ be a finitely branching LTS and $s, t \in S$. Then:
 $s \sim t$ iff for every $F \in HMF : (s \models F \iff t \models F)$.

Proof.

“ \Leftarrow ”: Define $u \equiv v$ iff $(u \models F \iff v \models F)$ for every $F \in HMF$, and let $s \equiv t$.

We prove $s \sim t$ by showing that \equiv is a strong bisimulation.

To this aim, let $u \equiv v$ and $u \xrightarrow{\alpha} u'$ for some $u' \in S$.

We have to show that ex. $v' \in S$ with $v \xrightarrow{\alpha} v'$ and $u' \equiv v'$.

Relationship Between HML and Strong Bisimilarity II

Theorem (Hennessy-Milner Theorem)

Let $(S, Act, \{\xrightarrow{a} \mid a \in Act\})$ be a finitely branching LTS and $s, t \in S$. Then:
 $s \sim t \text{ iff for every } F \in HMF : (s \models F \iff t \models F)$.

Proof.

“ \Leftarrow ”: Define $u \equiv v$ iff $(u \models F \iff v \models F)$ for every $F \in HMF$, and let $s \equiv t$.

We prove $s \sim t$ by showing that \equiv is a strong bisimulation.

To this aim, let $u \equiv v$ and $u \xrightarrow{\alpha} u'$ for some $u' \in S$.

We have to show that ex. $v' \in S$ with $v \xrightarrow{\alpha} v'$ and $u' \equiv v'$.

- Assume that there is no such v' .
- Let $\{v' \in S \mid v \xrightarrow{\alpha} v'\} = \{v'_1, \dots, v'_n\}$ with $n \in \mathbb{N}$ (finitely branching!).
- By the previous assumption, $u' \not\equiv v'_i$ for each $i \in [n]$.
- Thus, for each $i \in [n]$ there ex. $F_i \in HMF$ with $u' \models F_i$ and $v'_i \not\models F_i$.
(If $u' \not\models F_i$ and $v'_i \models F_i$, then consider F_i^c [Lemma 9.1] instead.)
- Hence, for $G := \langle \alpha \rangle (F_1 \wedge \dots \wedge F_n)$: $u \models G$ but $v \not\models G$, which contradicts $u \equiv v$.

Proving non-bisimilarity

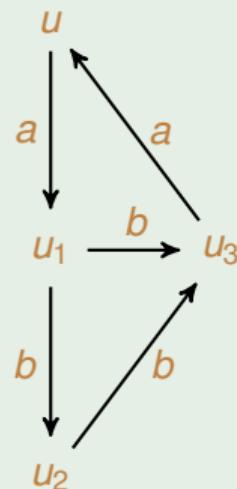
Showing $P \not\sim Q$ thus amounts to finding a single HML-formula F with $P \models F$ and $Q \not\models F$.

Proving Non-Bisimilarity

Proving non-bisimilarity

Showing $P \not\sim Q$ thus amounts to finding a single HML-formula F with $P \models F$ and $Q \not\models F$.

Example 9.7

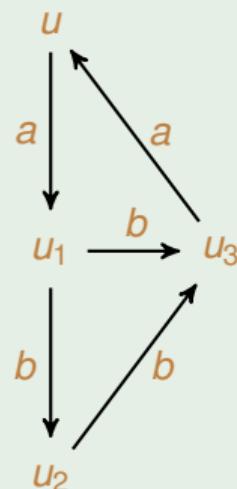


Proving Non-Bisimilarity

Proving non-bisimilarity

Showing $P \not\sim Q$ thus amounts to finding a single HML-formula F with $P \models F$ and $Q \not\models F$.

Example 9.7



Distinguishing formulae

(satisfied by row state / violated by column state):

	u	u_1	u_2	u_3
u	—	$\langle a \rangle tt$	$\langle a \rangle tt$	$\langle a \rangle \langle b \rangle tt$
u_1	$[a]ff$	—	—	$[a]ff$
u_2	$[a]ff$	—	—	$[a]ff$
u_3	$\langle a \rangle \langle a \rangle tt$	$\langle a \rangle tt$	$\langle a \rangle tt$	—

Counterexample for Non-Finitely Branching Processes

Lemma 9.8

Let $A^{<\omega} = \sum_{i \in \mathbb{N}} a^i$ (see Example 9.5) and $A^\omega = a.A^\omega$. Then $A^{<\omega}$ and $A^{<\omega} + A^\omega$

- (1) are not strongly bisimilar, but
- (2) satisfy the same HML formulae.

Counterexample for Non-Finitely Branching Processes

Lemma 9.8

Let $A^{<\omega} = \sum_{i \in \mathbb{N}} a^i$ (see Example 9.5) and $A^\omega = a.A^\omega$. Then $A^{<\omega}$ and $A^{<\omega} + A^\omega$

- (1) are not strongly bisimilar, but
- (2) satisfy the same HML formulae.

Proof.

- (1) Assume that $A^{<\omega} \sim A^{<\omega} + A^\omega$. Then $A^{<\omega} + A^\omega \xrightarrow{a} A^\omega$ must be mimicked by $A^{<\omega} \xrightarrow{a} a^{i-1}$ for some $i \geq 1$. But obviously $A^\omega \not\sim a^{i-1}$.

Counterexample for Non-Finitely Branching Processes

Lemma 9.8

Let $A^{<\omega} = \sum_{i \in \mathbb{N}} a^i$ (see Example 9.5) and $A^\omega = a.A^\omega$. Then $A^{<\omega}$ and $A^{<\omega} + A^\omega$

- (1) are not strongly bisimilar, but
- (2) satisfy the same HML formulae.

Proof.

- (1) Assume that $A^{<\omega} \sim A^{<\omega} + A^\omega$. Then $A^{<\omega} + A^\omega \xrightarrow{a} A^\omega$ must be mimicked by $A^{<\omega} \xrightarrow{a} a^{i-1}$ for some $i \geq 1$. But obviously $A^\omega \not\sim a^{i-1}$.
- (2) By structural induction on $F \in \text{HMF}$, using the following lemma. □

Counterexample for Non-Finitely Branching Processes

Lemma 9.8

Let $A^{<\omega} = \sum_{i \in \mathbb{N}} a^i$ (see Example 9.5) and $A^\omega = a.A^\omega$. Then $A^{<\omega}$ and $A^{<\omega} + A^\omega$

- (1) are not strongly bisimilar, but
- (2) satisfy the same HML formulae.

Proof.

- (1) Assume that $A^{<\omega} \sim A^{<\omega} + A^\omega$. Then $A^{<\omega} + A^\omega \xrightarrow{a} A^\omega$ must be mimicked by $A^{<\omega} \xrightarrow{a} a^{i-1}$ for some $i \geq 1$. But obviously $A^\omega \not\sim a^{i-1}$.
- (2) By structural induction on $F \in \text{HMF}$, using the following lemma. □

Lemma 9.9

For every $F \in \text{HMF}$, $A^\omega \models F$ iff $a^k \models F$, where k is the modal depth^a of F .

^athe maximal number of nested occurrences of modal operators in F

Counterexample for Non-Finitely Branching Processes

Lemma 9.8

Let $A^{<\omega} = \sum_{i \in \mathbb{N}} a^i$ (see Example 9.5) and $A^\omega = a.A^\omega$. Then $A^{<\omega}$ and $A^{<\omega} + A^\omega$

- (1) are not strongly bisimilar, but
- (2) satisfy the same HML formulae.

Proof.

- (1) Assume that $A^{<\omega} \sim A^{<\omega} + A^\omega$. Then $A^{<\omega} + A^\omega \xrightarrow{a} A^\omega$ must be mimicked by $A^{<\omega} \xrightarrow{a} a^{i-1}$ for some $i \geq 1$. But obviously $A^\omega \not\sim a^{i-1}$.
- (2) By structural induction on $F \in \text{HMF}$, using the following lemma. □

Lemma 9.9

For every $F \in \text{HMF}$, $A^\omega \models F$ iff $a^k \models F$, where k is the modal depth^a of F .

^athe maximal number of nested occurrences of modal operators in F

Outline of Lecture 9

- 1 Closure under Negation
- 2 HML and Process Traces
- 3 HML and Strong Bisimilarity
- 4 HML and Weak Bisimilarity

Recap: Weak Bisimulation

Definition (Weak transition relation; Definition 5.8)

For $\alpha \in Act$, $\xrightarrow{\alpha} \subseteq Prc \times Prc$ is given by

$$\xrightarrow{\alpha} := \begin{cases} \left(\xrightarrow{\tau} \right)^* \circ \xrightarrow{\alpha} \circ \left(\xrightarrow{\tau} \right)^* & \text{if } \alpha \neq \tau \\ \left(\xrightarrow{\tau} \right)^* & \text{if } \alpha = \tau. \end{cases}$$

where $\left(\xrightarrow{\tau} \right)^*$ denotes the reflexive and transitive closure of relation $\xrightarrow{\tau}$.

Definition (Weak bisimulation; Definition 5.9)

(Milner 1989)

A binary relation $\rho \subseteq Prc \times Prc$ is a **weak bisimulation** whenever for every $(P, Q) \in \rho$ and $\alpha \in Act$ (including $\alpha = \tau$):

- (1) if $P \xrightarrow{\alpha} P'$, then there exists $Q' \in Prc$ such that $Q \xrightarrow{\alpha} Q'$ and $P' \rho Q'$, and
- (2) if $Q \xrightarrow{\alpha} Q'$, then there exists $P' \in Prc$ such that $P \xrightarrow{\alpha} P'$ and $P' \rho Q'$.

Recap: Weak Bisimulation

Definition (Weak transition relation; Definition 5.8)

For $\alpha \in Act$, $\xrightarrow{\alpha} \subseteq Prc \times Prc$ is given by

$$\xrightarrow{\alpha} := \begin{cases} \left(\xrightarrow{\tau} \right)^* \circ \xrightarrow{\alpha} \circ \left(\xrightarrow{\tau} \right)^* & \text{if } \alpha \neq \tau \\ \left(\xrightarrow{\tau} \right)^* & \text{if } \alpha = \tau. \end{cases}$$

where $\left(\xrightarrow{\tau} \right)^*$ denotes the reflexive and transitive closure of relation $\xrightarrow{\tau}$.

Definition (Weak bisimulation; Definition 5.9)

(Milner 1989)

A binary relation $\rho \subseteq Prc \times Prc$ is a **weak bisimulation** whenever for every $(P, Q) \in \rho$ and $\alpha \in Act$ (including $\alpha = \tau$):

- (1) if $P \xrightarrow{\alpha} P'$, then there exists $Q' \in Prc$ such that $Q \xrightarrow{\alpha} Q'$ and $P' \rho Q'$, and
- (2) if $Q \xrightarrow{\alpha} Q'$, then there exists $P' \in Prc$ such that $P \xrightarrow{\alpha} P'$ and $P' \rho Q'$.

Introducing Weak Modalities

Goal: Modify HML by turning strong modal operators into weak ones
(see J. Parrow et al.: *Weak Nominal Modal Logic*, FORTE 2017)

Introducing Weak Modalities

Definition 9.10 (Syntax and semantics of weak modalities)

- $wHMF$ is obtained from HMF of (Definition 8.1) by replacing $\langle \alpha \rangle$ and $[\alpha]$ with

$$\langle\langle \alpha \rangle\rangle F \quad \text{and} \quad [[\alpha]] F$$

for $\alpha \in Act$ and $F \in wHMF$.

Introducing Weak Modalities

Definition 9.10 (Syntax and semantics of weak modalities)

- $wHMF$ is obtained from HMF of (Definition 8.1) by replacing $\langle \alpha \rangle$ and $[\alpha]$ with

$$\langle\langle \alpha \rangle\rangle F \quad \text{and} \quad [[\alpha]] F$$

for $\alpha \in Act$ and $F \in wHMF$.

- Modifying Definition 8.2, for an LTS $(S, Act, \longrightarrow)$ and $F \in wHMF$ we let

$$[[\langle\langle \alpha \rangle\rangle F]] := \langle\langle \cdot \alpha \cdot \rangle\rangle ([[F]]) \qquad [[[\alpha]] F]] := [[\cdot \alpha \cdot]] ([[F]])$$

where $\langle\langle \cdot \alpha \cdot \rangle\rangle, [[\cdot \alpha \cdot]] : 2^S \rightarrow 2^S$ are given by

$$\begin{aligned}\langle\langle \cdot \alpha \cdot \rangle\rangle(T) &:= \{s \in S \mid \exists s' \in T : s \xrightarrow{\alpha} s'\} \\ [[\cdot \alpha \cdot]](T) &:= \{s \in S \mid \forall s' \in S : s \xrightarrow{\alpha} s' \text{ implies } s' \in T\}\end{aligned}$$

Again, we write $s \models F$ iff $s \in [[F]]$, and $F, G \in wHMF$ are equivalent (written $F \equiv G$) iff they are satisfied by the same processes in every LTS.

Relationship Between HML with Weak Modalities and Weak Bisimilarity

Theorem 9.11 (Hennessy-Milner Theorem for weak bisimulation)

Let $(S, Act, \{\xrightarrow{a} | a \in Act\})$ be a finitely branching LTS and $s, t \in S$. Then:

$$s \approx t \quad \text{iff} \quad \text{for every } F \in wHMF : (s \models F \iff t \models F).$$

Proof.

see J. Parrow et al.: *Weak Nominal Modal Logic*, FORTE 2017



Relationship Between HML with Weak Modalities and Weak Bisimilarity

Theorem 9.11 (Hennessy-Milner Theorem for weak bisimulation)

Let $(S, Act, \{\xrightarrow{a} | a \in Act\})$ be a finitely branching LTS and $s, t \in S$. Then:

$$s \approx t \quad \text{iff} \quad \text{for every } F \in wHMF : (s \models F \iff t \models F).$$

Proof.

see J. Parrow et al.: *Weak Nominal Modal Logic*, FORTE 2017



Example 9.12 (Counterexample to congruence of weak bisimilarity)

- Lecture 6: $\tau.a.nil + b.nil \not\approx a.nil + b.nil$
- Distinguishing $wHMF$ formula: $\langle\langle\tau\rangle\rangle[[b]]ff$
- Check using CAAL