

# Concurrency Theory

Winter 2025/26

## Lecture 6: Properties of Weak Bisimulation

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<https://proglang.github.io/teaching/25ws/ct.html>

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# Outline of Lecture 6

- 1 Recap: Game Characterisation and Variants of Strong Bisimulation
- 2 Properties of Weak Bisimilarity
- 3 Weak Bisimilarity vs. Trace Equivalence
- 4 Deadlock Sensitivity
- 5 Congruence Property
- 6 Observation Congruence
- 7 Game Characterisation of Weak Bisimilarity

# Rules of the Bisimulation Game

## Rules

In each round, the current configuration  $(s, t)$  is changed as follows:

- (1) the **attacker** chooses one of the two processes in the current configuration, say  $t$ , and makes an  $\xrightarrow{\alpha}$ -move for some  $\alpha \in Act$  to  $t'$ , say, and
- (2) the **defender** must respond by making an  $\xrightarrow{\alpha}$ -move in the other process  $s$  of the current configuration under the same action  $\alpha$ , yielding  $s \xrightarrow{\alpha} s'$ .

The pair of processes  $(s', t')$  becomes the new current configuration.

The game continues with another round.

## Results

- (1) If one player cannot move, the other player wins:
  - attacker cannot move if  $s \not\xrightarrow{\alpha}$  and  $t \not\xrightarrow{\alpha}$
  - defender cannot move if no matching transition available
- (2) If the game is played *ad infinitum*, the defender wins.

# Game Characterisation of Bisimulation

Theorem (Game characterisation of bisimulation)

(Stirling 1995, Thomas 1993)

- (1)  $s \sim t$  iff *the defender has a universal winning strategy* from configuration  $(s, t)$ .
- (2)  $s \not\sim t$  iff *the attacker has a universal winning strategy* from configuration  $(s, t)$ .

*(By means of a universal winning strategy, a player can always win, regardless of how the other player selects their moves.)*

Proof.

by relating winning strategy of defender/attacker to existence/non-existence of strong bisimulation relation



Thus, a bisimulation game can be used to prove bisimilarity as well as non-bisimilarity.<sup>1</sup> It often provides elegant arguments for  $s \not\sim t$ .

<sup>1</sup>Later we will present yet another method to check this.

# Strong Simulation

**Observation:** sometimes, the concept of strong bisimulation is **too strong** (example: extending a system by new features).

## Definition (Strong simulation)

- Relation  $\rho \subseteq Prc \times Prc$  is a **strong simulation** if, whenever  $(P, Q) \in \rho$  and  $P \xrightarrow{\alpha} P'$ , there exists  $Q' \in Prc$  such that  $Q \xrightarrow{\alpha} Q'$  and  $P' \rho Q'$ .
- $Q$  **strongly simulates**  $P$ , denoted  $P \sqsubseteq Q$ , if there exists a strong simulation  $\rho$  such that  $P \rho Q$ . Relation  $\sqsubseteq$  is called **strong similarity**.
- $P$  and  $Q$  are **strongly simulation equivalent** if  $P \sqsubseteq Q$  and  $Q \sqsubseteq P$ .

**Thus:** If  $Q$  strongly simulates  $P$ , then whatever transition  $P$  takes,  $Q$  can match it while retaining all of  $P$ 's options.

**But:**  $P$  does not need to be able to match each transition of  $Q$ !

# Strong Simulation and Bisimilarity

Lemma (Bisimilarity implies simulation equivalence)

If  $P \sim Q$ , then  $P \sqsubseteq Q$  and  $Q \sqsubseteq P$ .

Proof.

A strong bisimulation  $\rho \subseteq Prc \times Prc$  for  $P \sim Q$  is a strong simulation for both directions. □

**Caveat:** The converse does not generally hold!

# Weak Bisimulation

## Definition (Weak bisimulation)

(Milner 1989)

A binary relation  $\rho \subseteq Prc \times Prc$  is a **weak bisimulation** whenever for every  $(P, Q) \in \rho$  and  $\alpha \in Act$  (including  $\alpha = \tau$ ):

- (1) if  $P \xrightarrow{\alpha} P'$ , then there exists  $Q' \in Prc$  such that  $Q \xrightarrow{\alpha} Q'$  and  $P' \rho Q'$ , and
- (2) if  $Q \xrightarrow{\alpha} Q'$ , then there exists  $P' \in Prc$  such that  $P \xrightarrow{\alpha} P'$  and  $P' \rho Q'$ .

## Definition (Weak bisimilarity)

Processes  $P$  and  $Q$  are **weakly bisimilar**, denoted  $P \approx Q$ , iff there is a weak bisimulation  $\rho$  with  $P \rho Q$ .

Thus,

$$\approx = \bigcup \{ \rho \subseteq Prc \times Prc \mid \rho \text{ is a weak bisimulation} \}.$$

Relation  $\approx$  is called **weak bisimilarity** or **observational equivalence**.

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5 Congruence Property

6 Observation Congruence

7 Game Characterisation of Weak Bisimilarity

## Example 6.1 (A polling process)

(Koomen 1982)

A process that is willing to receive on port  $a$  or port  $b$ , and then terminates:

$$A? = a.\text{nil} + \tau.B?$$

$$B? = b.\text{nil} + \tau.A?$$

# Divergence I

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- But note that  $A? \xrightarrow{\tau} B? \xrightarrow{\tau} A?$  forms a  $\tau$ -loop, whereas  $a.\text{nil} + b.\text{nil}$  does not have a loop (not even a  $\tau$ -loop).

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- Thus,  $\approx$  assumes that if a process can escape from a  $\tau$ -loop, it eventually will do so.<sup>a</sup>  
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Divergence ("livelock") is a  $\tau$ -loop.
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Divergence (“livelock”) is a  $\tau$ -loop.
- Also note that  $\text{Div} \approx \text{nil}$  where  $\text{Div} = \tau.\text{Div}$ .
- Thus, a deadlocked process is weakly bisimilar to a process that can only diverge.
- This is justified by the fact that “observations” can only be made by interacting with the process.

# Divergence II

## Example 6.2 (A simple communication protocol)

**Observation:** (unbounded) retransmission after communication failures can also cause  $\tau$ -loops.

$Spec = acc.\overline{del}.Spec;$

$Impl = (Send \parallel Med \parallel Rec) \setminus \{send, trans, ack, error\};$

$Send = acc.Sending;$

$Sending = \overline{send}.Wait;$

$Wait = ack.Send + error.Sending;$

$Med = send.Trans;$

$Trans = \overline{trans}.Med + \tau.Err;$

$Err = \overline{error}.Med;$

$Rec = trans.Del;$

$Del = \overline{del}.Ack;$

$Ack = \overline{ack}.Rec;$

(analyse using CAAL tool)

# Properties of Weak Bisimilarity

## Lemma 6.3 (Properties of $\approx$ )

- (1)  $P \sim Q$  implies  $P \approx Q$ .
- (2)  $\approx$  is an equivalence relation (i.e., reflexive, symmetric, and transitive).
- (3)  $\approx$  is the largest weak bisimulation.
- (4)  $\approx$  abstracts from  $\tau$ -loops.

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- (4)  $\approx$  abstracts from  $\tau$ -loops.

## Proof.

- (1) straightforward (as  $\xrightarrow{\alpha} \subseteq \xrightleftharpoons{\alpha}$ )
- (2) similar to Lemma 4.7(1) for  $\sim$
- (3) similar to Lemma 4.7(2) for  $\sim$
- (4) see Examples 6.1 and 6.2



## Lemma 6.4 (Milner's $\tau$ -laws)

$$\alpha.\tau.P \approx \alpha.P$$

$$P + \tau.P \approx \tau.P$$

$$\alpha.(P + \tau.Q) \approx \alpha.(P + \tau.Q) + \alpha.Q$$

## Lemma 6.4 (Milner's $\tau$ -laws)

$$\begin{aligned}\alpha.\tau.P &\approx \alpha.P \\ P + \tau.P &\approx \tau.P \\ \alpha.(P + \tau.Q) &\approx \alpha.(P + \tau.Q) + \alpha.Q\end{aligned}$$

## Proof.

by constructing appropriate weak bisimulation relations (left as an exercise)



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# Weak Bisimilarity vs. Trace Equivalence

## Definition 6.5 (Observational trace language)

The **observational trace language** of  $P \in Prc$  is defined by:

$$ObsTr(P) := \{ \hat{w} \in (A \cup \bar{A})^* \mid \text{there ex. } P' \in Prc : P \xrightarrow{w} P' \}$$

where  $\hat{w}$  is obtained from  $w$  by removing all  $\tau$ -actions.

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## Definition 6.6 (Observational trace equivalence)

$P, Q \in Prc$  are **observationally trace equivalent** if  $ObsTr(P) = ObsTr(Q)$ .

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## Definition 6.6 (Observational trace equivalence)

$P, Q \in Prc$  are observationally trace equivalent if  $ObsTr(P) = ObsTr(Q)$ .

## Theorem 6.7

$P \approx Q$  implies that  $P$  and  $Q$  are observationally trace equivalent. The reverse does not hold.

Proof.

similar to Theorem 4.9



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# Observational Deadlocks

## Definition 6.8 (Observational deadlock)

Let  $P, Q \in Prc$  and  $w \in Act^*$  such that

$P \xrightarrow{w} Q$  and there exists no  $Q' \in Prc$  and  $\lambda \in A \cup \bar{A}$  such that  $Q \xrightarrow{\lambda} Q'$ .

Then  $Q$  is called an **observational  $\hat{w}$ -deadlock** of  $P$ .

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## Definition 6.9 (Observational deadlock sensitivity)

Relation  $\equiv \subseteq Prc \times Prc$  is **observationally deadlock sensitive** whenever  $P \equiv Q$  implies for every  $v \in (A \cup \bar{A})^*$ :

$P$  has an observational  $v$ -deadlock    iff     $Q$  has an observational  $v$ -deadlock.

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## Theorem 6.10

$\approx$  is observationally deadlock sensitive.

Proof.

similar to Theorem 4.13 for  $\approx$

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# Congruence

Lemma 6.11 (Partial CCS congruence property of  $\approx$ )

Whenever  $P, Q \in Prc$  such that  $P \approx Q$ ,

$$\begin{array}{lll} \alpha.P & \approx & \alpha.Q & \text{for every } \alpha \in Act \\ P \parallel R & \approx & Q \parallel R & \text{for every } R \in Prc \\ P \setminus L & \approx & Q \setminus L & \text{for every } L \subseteq A \\ P[f] & \approx & Q[f] & \text{for every } f : A \rightarrow A \end{array}$$

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Proof.

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Proof.

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What about choice?

- $\tau.a.nil \approx a.nil$  (cf. Example 5.11(1)) and  $b.nil \approx b.nil$  (reflexivity).

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- But  $\tau.a.nil + b.nil \not\approx a.nil + b.nil$ . (Why?)

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Proof.

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- $\tau.a.nil \approx a.nil$  (cf. Example 5.11(1)) and  $b.nil \approx b.nil$  (reflexivity).
- But  $\tau.a.nil + b.nil \not\approx a.nil + b.nil$ . (Why?)
- Thus, weak bisimilarity is **not a CCS congruence**, which motivates a slight adaptation of  $\approx$ .



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# Observation Congruence

Definition 6.12 (Observation congruence)

(Milner 1989)

$P, Q \in Prc$  are **observationally congruent**, denoted  $P \approx^c Q$ , if for every  $\alpha \in Act$  (including  $\alpha = \tau$ ):

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## Remark

- $\approx^c$  differs from  $\approx$  only in that  $\approx^c$  requires  **$\tau$ -moves** by  $P$  or  $Q$  to be mimicked by at least one  $\tau$ -move in the other process.
- Moreover, this only applies to the **first step**; the successors just have to satisfy  $P' \approx Q'$  (and not necessarily  $P' \approx^c Q'$ ).

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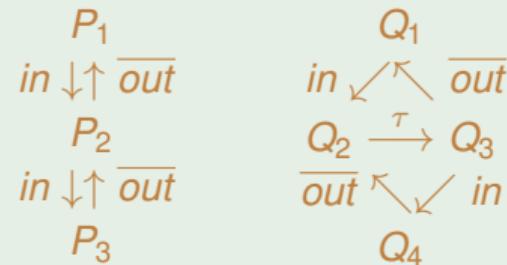
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- Moreover, this only applies to the **first step**; the successors just have to satisfy  $P' \approx Q'$  (and not necessarily  $P' \approx^c Q'$ ).
- Thus: if  $P \not\xrightarrow{\tau}$  and  $Q \not\xrightarrow{\tau}$ , then  $P \approx^c Q$  iff  $P \approx Q$ .

# Examples

## Example 6.13

(1) Sequential and parallel two-place buffer:

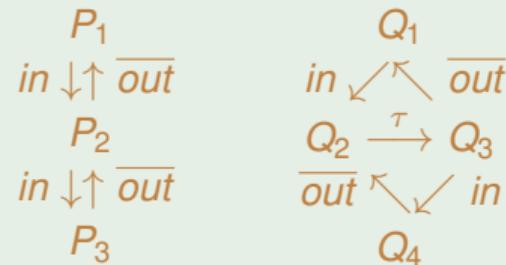


$P_1 \approx^c Q_1$  since  $P_1 \approx Q_1$  (cf. Example 5.11(3)) and neither  $P_1$  nor  $Q_1$  has initial  $\tau$ -steps.

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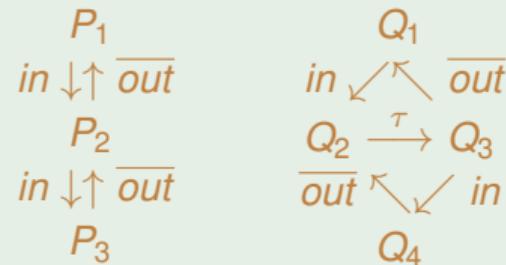
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thus the counterexample to congruence of  $\approx$  for  $+$  does not apply.

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thus the counterexample to congruence of  $\approx$  for  $+$  does not apply.
- (3)  $a.\tau.\text{nil} \approx^c a.\text{nil}$  (since  $\tau.\text{nil} \approx \text{nil}$ ).

# Properties of Observation Congruence

## Lemma 6.14

For every  $P, Q \in Prc$ ,

- (1)  $P \sim Q$  implies  $P \approx^c Q$ , and  $P \approx^c Q$  implies  $P \approx Q$
- (2)  $\approx^c$  is an equivalence relation
- (3)  $\approx^c$  is a CCS congruence
- (4)  $\approx^c$  is observationally deadlock-sensitive
- (5)  $P \approx^c Q$  if and only if  $P + R \approx Q + R$  for every  $R \in Prc$
- (6)  $P \approx Q$  if and only if  $(P \approx^c Q \text{ or } P \approx^c \tau.Q \text{ or } \tau.P \approx^c Q)$

Proof.

omitted



# Properties of Observation Congruence

## Lemma 6.14

For every  $P, Q \in Prc$ ,

- (1)  $P \sim Q$  implies  $P \approx^c Q$ , and  $P \approx^c Q$  implies  $P \approx Q$
- (2)  $\approx^c$  is an equivalence relation
- (3)  $\approx^c$  is a CCS congruence
- (4)  $\approx^c$  is observationally deadlock-sensitive
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Proof.

omitted



**Note:** (5) states that  $\approx^c$  is the “minimal fix” to establish congruence of  $\approx$ .

# Outline of Lecture 6

- 1 Recap: Game Characterisation and Variants of Strong Bisimulation
- 2 Properties of Weak Bisimilarity
- 3 Weak Bisimilarity vs. Trace Equivalence
- 4 Deadlock Sensitivity
- 5 Congruence Property
- 6 Observation Congruence
- 7 Game Characterisation of Weak Bisimilarity

# Weak Bisimilarity as a Game

## Rules

In each round, the current configuration  $(s, t)$  is changed as follows:

- (1) the **attacker** chooses one of the processes in the current configuration, say  $t$ , and makes an  $\xrightarrow{\alpha}$ -move for some  $\alpha \in Act$  to  $t'$ , say,

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The game continues with another round.

## Results

- (1) If one player cannot move, the other player wins.
  - attacker cannot move if  $s \not\xrightarrow{\alpha}$  and  $t \not\xrightarrow{\alpha}$
  - defender cannot move if no matching transition available
- (2) If the game can be played *ad infinitum*, the defender wins.

# Game Characterisation of Weak Bisimilarity

Theorem 6.15 (Game characterisation of weak bisimilarity) ( Stirling 1995, Thomas 1993)

- (1)  $s \approx t$  iff *the defender has a universal winning strategy* from configuration  $(s, t)$ .
- (2)  $s \not\approx t$  iff *the attacker has a universal winning strategy* from configuration  $(s, t)$ .

*(By means of a universal winning strategy, a player can always win, regardless of how the other player selects their moves.)*

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Proof.

by relating winning strategy of defender/attacker to existence/non-existence of weak bisimulation relation

