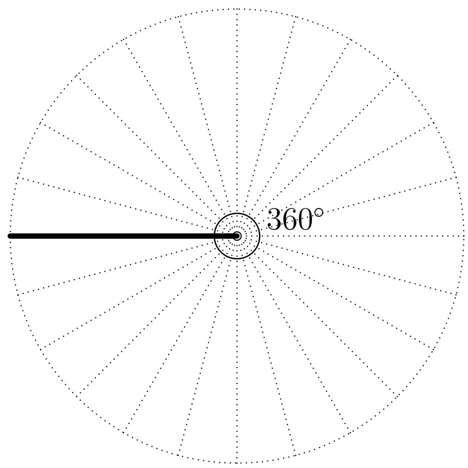
Circles

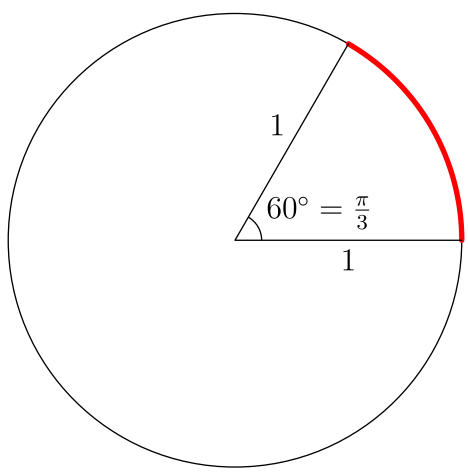
James Camacho

1 Angles

A “degree” is equal to 1 out of 360 of a full rotation, or 1/360 of a circle:

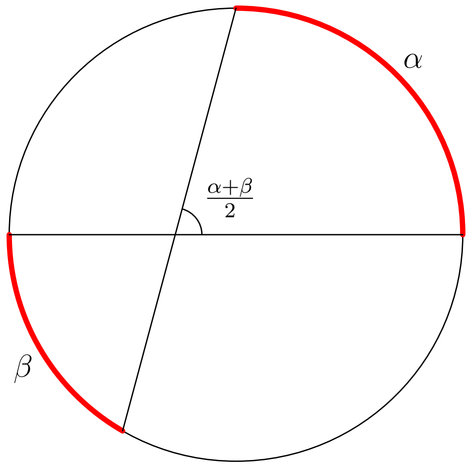


Some people prefer to use radians. If you consider a circle of radius 1, then the equivalent value in radians would be the arc length measured out by the angle:

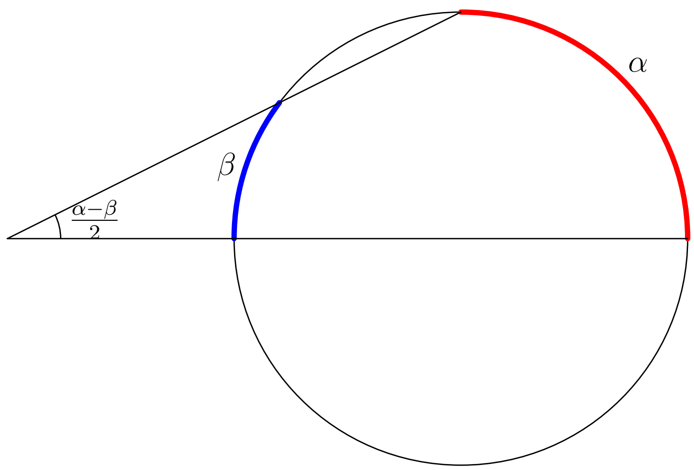


because 60° is one-sixth of a full circle, and a circle with radius one has circumference 2π. Throughout this text I will use degrees as radians only become more useful when you’re doing calculus.

The angle measure of lines intersecting within a circle but *not* at the center is a little trickier to find, but it’s equal to the average of the measures of the arcs:



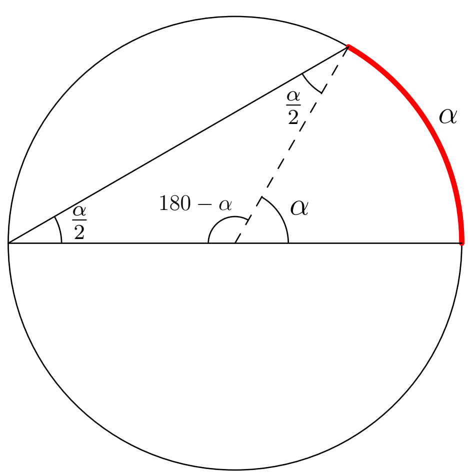
Similarly, if the point is *outside* the circle, the angle is half the difference between the measure of the arcs (i.e. the smaller arc is treated as a negative angle):



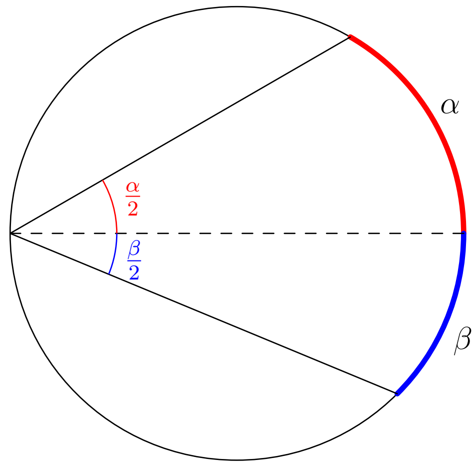
The proof will be left to the exercises, but it relies on this result:

Theorem. The angle of a point on a circle is equal to half the measure of the arc it intersects.

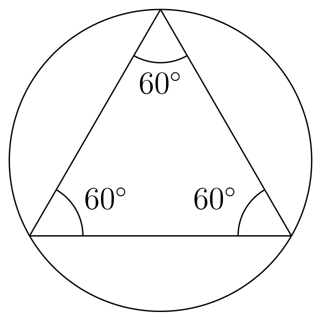
Proof. When one of the chords is a diameter, we can draw the following picture:



If neither are a diameter, we can draw a diameter and break it up into two pieces, each of which are half their respective arcs:



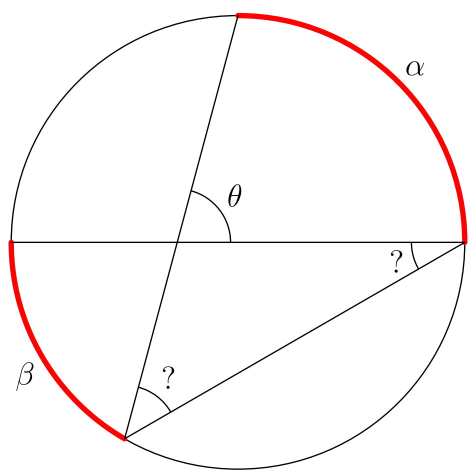
This suggests a way to find the angles of any regular polygon. For example, an equilateral triangle has three equal angles of measure 60° as each arc is $\frac{1}{3}$ of the circle or 120°:



2 Exercises

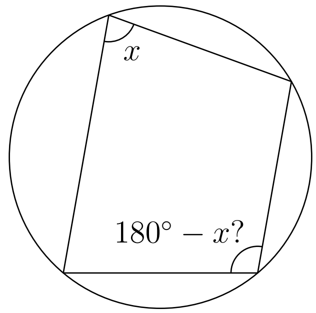
Problem 1: Write the following angles in terms of radians: $180^\circ, 90^\circ, 75^\circ$. If $^\circ$ were treated as a variable, what would it's value be?

Problem 2: Fill in the rest of the diagram to prove that inside-point angle formula:

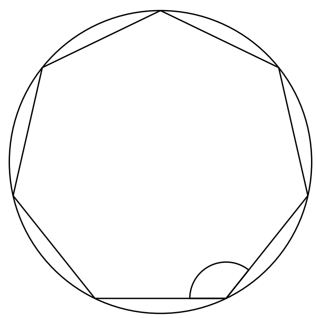


Try to find a similar way to prove the outside-point angle formula.

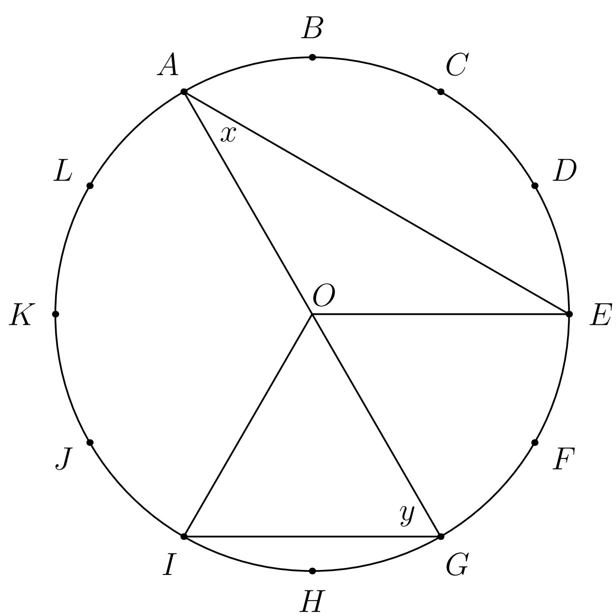
Problem 3: Show that any cyclic quadrilateral has opposite angles summing to $180°$. A *cyclic* quadrilateral is one that can be inscribed in a circle.



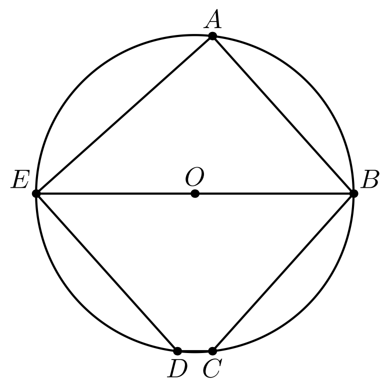
Problem 4: What is the degree measure for each angle of a regular pentagon? What about a regular dodecagon (12 sides)? Can you find a formula for the each angle of a regular n-gon?



Problem 5: (2014 AMC 8 #15) The circumference of the circle with center O is divided into 12 equal arcs, marked the letters A through L as seen below. What is the number of degrees in the sum of the angles x and y?

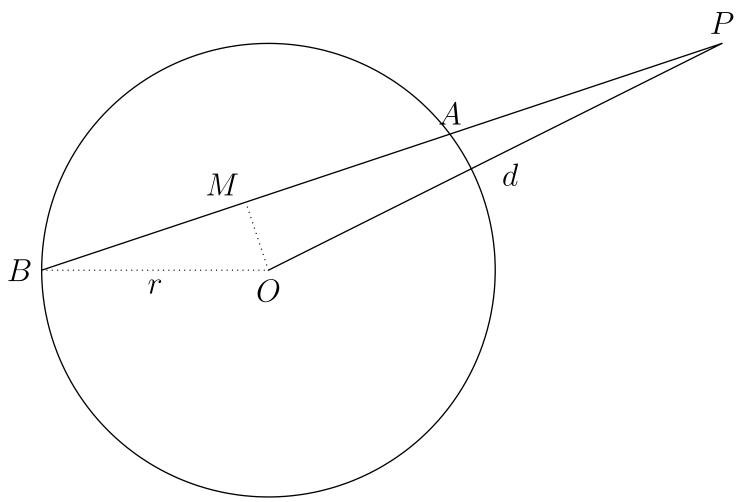


Problem 6: (2011 AMC 10B #17) In the given circle, the diameter $\overline{EB}$ is parallel to $\overline{DC}$, and $\overline{AB}$ is parallel to $\overline{ED}$. The angles AEB and ABE are in the ratio $4:5$. What is the degree measure of angle BCD?



3 Power of a Point

A point P that is d away from the center of a circle with radius r has *power* equal to $d^2 - r^2$. This power is also equal to the product of the distances to the circumference of the circle on any line (e.g. $PA \cdot PB$ in the diagram).



This is true because of the Pythagorean theorem. We have $AM = BM$ because triangle AOB is isoceles ($AO = BO = r$), so

$$PA \cdot PB = (PM - AM)(PM + AM) = (PM)^2 - (AM)^2.$$

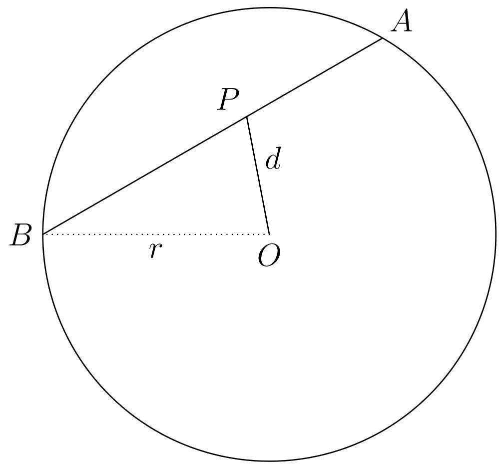
Now from the Pythagorean theorem,

$$(PM)^2 = d^2 - (MO)^2,$$
$$(MO)^2 = r^2 - (AM)^2.$$

So

$$PA \cdot PB = (PM)^2 - (AM)^2 = d^2 - (r^2 - (AM)^2) - (AM)^2 = d^2 - r^2.$$

The power of a point applies equally well when a point is inside the circle.



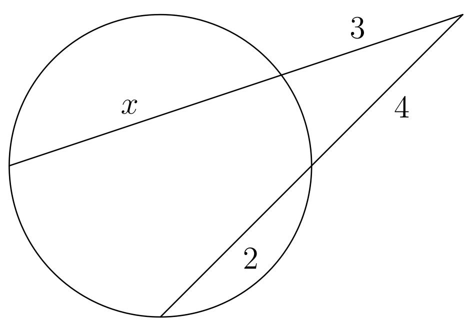
In this case the power will be negative (because $d < r$), which seems impossible as the product of two distances should be positive. But really, one of the distances is a "negative" distance as the rays $\overleftrightarrow{PB}, \overrightarrow{PA}$ are pointing in opposite directions. In my opinion, it's best to just take absolute values so everything's positive. You get

$$PA \cdot PB = r^2 - d^2.$$

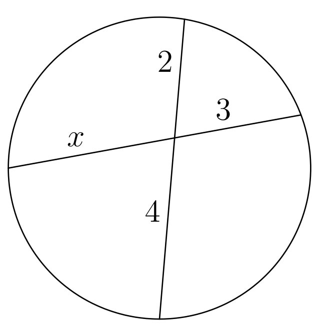
4 Exercises.

(All taken or adapted from https://artofproblemsolving.com/wiki/index.php/Power_of_a_Point_Theorem.)

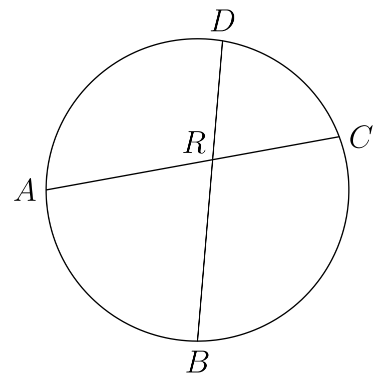
Problem 7: Find x.



Problem 8: Find x.



Problem 9: If the ratios $AR : RC = 3 : 5$ and $BR : RD = 5 : 27$, find the ratio $AC : BD$.



(Figure not drawn to scale.)

Problem 10: Chords AB and CD of a given circle are perpendicular to each other and intersect at a right angle at point E. Given that $BE = 16$, $DE = 4$, and $AD = 5$, find CE.

Problem 11: Two tangents from an external point P are drawn to a circle and intersect it at A and B. A third tangent meets the circle at T, and the tangents $\overrightarrow{PA}$ and $\overrightarrow{PB}$ at points Q and R, respectively (this means that T is on the minor arc AB). If $AP = 20$, find the perimeter of $\triangle PQR$.

Problem 12: Circle O intersects square $ABCD$, cutting each side into three equal segments. If the side length of $ABCD$ is $6\sqrt{2}$, find the radius of the circle. Express your answer in simplest radical form.