Complex Numbers

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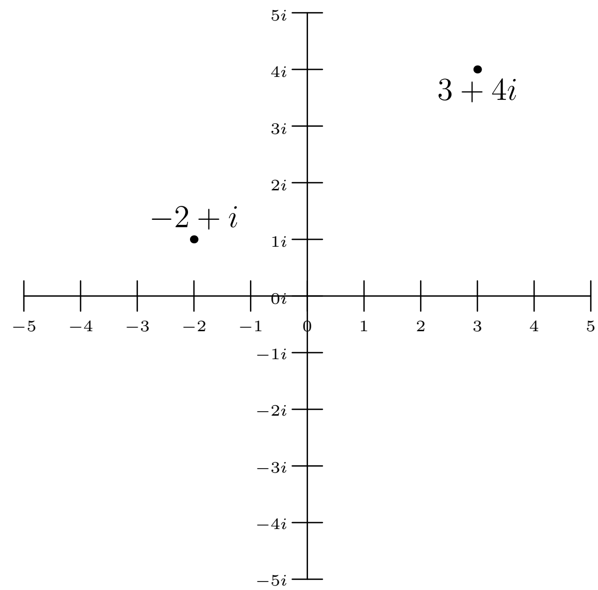
1 Intro To Complex Numbers

We define i to be $\sqrt{-1}$. Traditionally, square roots are always the positive root, e.g. $4^{\frac{1}{2}}$ may be equal to either -2 or 2, but $\sqrt{4}$ is defined to only be 2. But what does it mean for an imaginary number to be the "positive" root? It doesn't mean anything. So in this case we just define i to be the square root, but really we could just as easily define it to be $-i$ and nothing would change in math.

A complex number has a real and an imaginary part. Typically it is drawn on the *complex plane*:



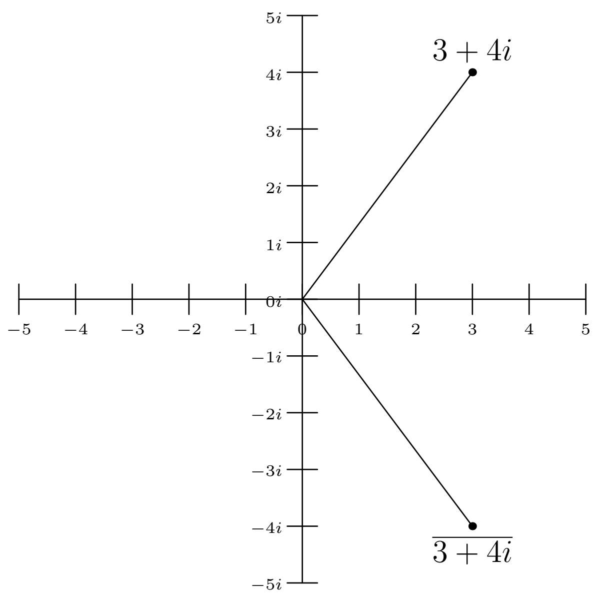
(I've put points at $3+4i$ and $-2+i$.) The *absolute value* (also known as the length or norm) of a complex number is defined as

$$|a+bi| = \sqrt{a^2+b^2}$$

which is really just the distance between $0+0i$ and $a+bi$ (use the Pythagorean Theorem). The *complex conjugate* (or just conjugate) is defined as

$$\overline{a+bi} = a - bi$$

Notice this is equivalent to flipping the complex number over the real axis (the "x-axis"):



Lots of times complex numbers are shown with vectors drawn to them (as I did above). The conjugate is most important because of the following:

$$(a+bi)\overline{(a+bi)} = |a+bi|^2$$

which can be shown by directly expanding out both sides:

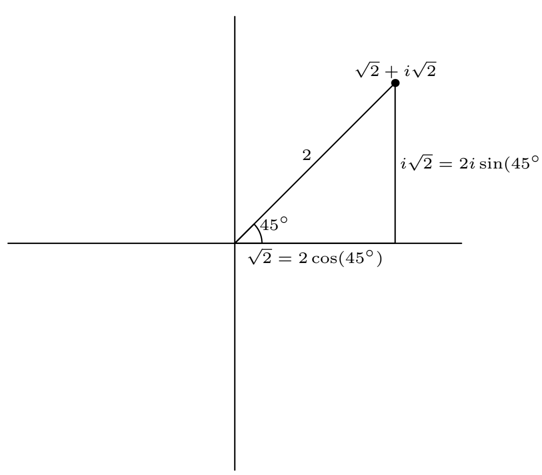
$$(a+bi)\overline{(a+bi)} = (a+bi)(a-bi) = a^2+b^2 = |a+bi|^2$$

from difference of squares. The conjugate is basically replacing i with $-i$, the other square root of -1. In this way, it makes sense the conjugation preserves multiplication and addition:

$$\overline{(a+bi)(c+di)} = \left(\overline{a+bi}\right)\left(\overline{c+di}\right),$$

$$\overline{(a+bi)+(c+di)} = \left(\overline{a+bi}\right) + \left(\overline{c+di}\right).$$

Now, this whole time we have been writing complex numbers using *rectangular* coordinates. You can also write them with *polar* coordinates. Here's an example with $\sqrt{2}+i\sqrt{2}$:



The angle is 45° and the absolute value of the complex number is 2 (use the Pythagorean theorem), so we can write

$$\sqrt{2} + i\sqrt{2} = 2\left(\cos 45^\circ + i \sin 45^\circ\right).$$

Recall that the cosine (cos) is the ratio between the side adjacent to the angle and the hypotenuse (so, the length of the real part divided by the absolute value), and the sine (sin) is the ratio between the side opposite the angle and the hypotenuse (so, the length of the imaginary part divided by the absolute value). In general, if the angle between the real axis and the complex number is θ and the absolute value is r, we can write

$$a + bi = r(\cos\theta + i\sin\theta),$$

which is sometimes written as $r\text{cis}(\theta)$ (cis stands for $\cos + i\sin$). However, it's more commonly written as

$$r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

It comes from the magical identity

$$e^{i\theta} = \cos\theta + i\sin\theta$$

(where $e \approx 2.73$ is Euler's number). The proof of this identity is beyond the scope of this book—it involves some calculus—but it is extremely useful. When written in this form, it is much easier to multiply complex numbers! For example (recall that $45^\circ = \pi/2$ radians),

$$(\sqrt{2} + i\sqrt{2})^4 = \left(\sqrt{2}e^{i\pi/2}\right)^4 = \sqrt{2}^4\left(e^{i\pi/2}\right)^4 = 2e^{4\cdot i\pi/2} = 2e^{2i\pi} = 2.$$

Multiplying complex numbers is just multiplying their absolute values and adding their angles!

$$r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = (r_1 r_2)e^{i(\theta_1+\theta_2)}.$$

This can also be used to prove the formula $|z| = z\overline{z}$ (where z is a complex number). If $z = re^{i\theta}$ then $\overline{z} = re^{-i\theta}$ so

$$z\overline{z} = r^2 e^{i\theta - i\theta} = r^2 e^0 = r^2 = |z|^2.$$

2 Problems

1. What is the value of $(1 + 2i)^2$?
2. Write $2e^{i\pi/3}$ in rectangular coordinates. Recall that $\pi = 180^\circ$.
3. Compute $\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^7$.
4. An nth root of unity is equal to

$$\omega = e^{2i\pi/n}$$

because

$$\omega^n = \left(e^{2i\pi/n}\right)^n = e^{n\cdot 2i\pi/n} = e^{2i\pi} = 1$$

(i.e ω is an nth root of 1, and unity means one).

(a) Find the value of

$$1 + \omega + \omega^2 + \omega^3 + \cdots + \omega^{n-1}.$$

Hint: Try multiplying by $1 - \omega$.

(b) Find in general the value of

$$1 + \omega^k + \omega^{2k} + \omega^{3k} + \cdots + \omega^{(n-1)k}.$$

What happens when $k = n$?

5. Show the following identities are true:

   (a) $\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}$

   (b) $\sin(\theta) = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

   Using these, find the value of

$$\cos(0^\circ) + \cos(10^\circ) + \cos(20^\circ) + \cdots + \cos(90^\circ)$$

3 Roots of Unity Filter

This is optional, but very cool. Consider a polynomial

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

In fact, it doesn't even have to be a polynomial, it could be a power series or a generating function (but those are beyond the scope of this handout). What if you wanted to find the sum of all the coefficients of a third power? I.e. you want to find

$$a_0 + a_3 + a_6 + \cdots$$

You can use a roots of unity filter. Let $\omega = e^{2i\pi/3}$ be a third root of unity. Exercise 4 above showed that

$$1 + \omega^k + \omega^{2k} = \begin{cases} 3 & 3|k \\ 0 & \text{otherwise.} \end{cases}$$

So,

$$\begin{aligned} P(1) + P(\omega) + P(\omega^2) &= a_0(1+1+1) + a_1(1+\omega^1+\omega^2) + a_2(1+\omega^2+\omega^4) + a_3(1+\omega^3+\omega^6) + \cdots \\ &= 3a_0 + 0a_1 + 0a_2 + 3a_3 + 0a_4 + 0a_5 + \cdots \\ &= 3(a_0 + a_3 + \cdots) \end{aligned}$$

So a roots of unity filter tells us that

$$a_0 + a_3 + a_6 + \cdots = \frac{P(1) + P(\omega) + P(\omega^2)}{3}$$

where ω is a third root of unity. Okay, let's use this in something a little more concrete.

Example 1: Compute the following sum:

$$\binom{11}{0} + \binom{11}{2} + \binom{11}{4} + \cdots + \binom{11}{10}$$

Solution 1: The binomial theorem tells us that

$$P(x) = (1+x)^{11} = \binom{11}{0} + \binom{11}{1}x + \binom{11}{2}x^2 + \cdots + \binom{11}{11}x^{11}.$$

Let $\omega = e^{2i\pi/2}$ be a second root of unity (in fact, ω is just -1, but we'll call it a second root of unity to keep in the spirit of roots of unity filters). We have

$$1 + \omega^k = \begin{cases} 2 & 2|k \\ 0 & \text{otherwise.} \end{cases}$$

So,

$$\frac{P(1)+P(\omega)}{2}$$

gives the sum of all the even coefficients, or

$$\binom{11}{0}+\binom{11}{2}+\binom{11}{4}+\cdots+\binom{11}{10}.$$

Computing directly,

$$\frac{P(1)+P(\omega)}{2}=\frac{P(1)+P(-1)}{2}=\frac{(1+1)^{11}+(1-1)^{11}}{2}=\frac{2^{11}}{2}=\boxed{1024}.$$

There are no problems for this section, but we'll look at roots of unity filters a lot more when we learn about generating functions.