Triangles

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1 Triangles

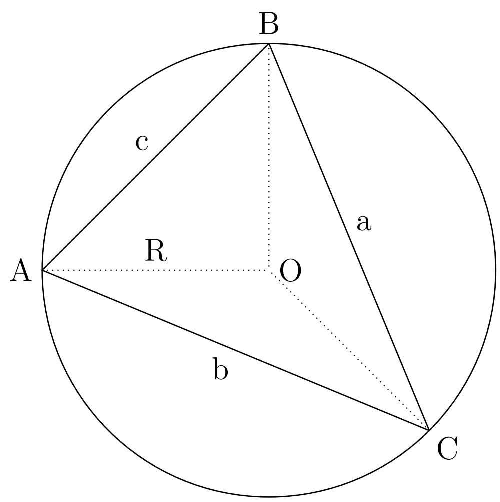
We state the following two theorems without proof:

1.1 Law of Sines

A triangle satisfies the following formula:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R,$$

where a, b, c are the sides, A, B, C are the opposite angles, and R is the circumradius of the triangle (radius of the circle that goes around the triangle).



1.2 Law of Cosines

The Law of Cosines is an extension of the Pythagorean theorem. It is

$$c^2 = a^2 + b^2 - 2ab\cos C$$

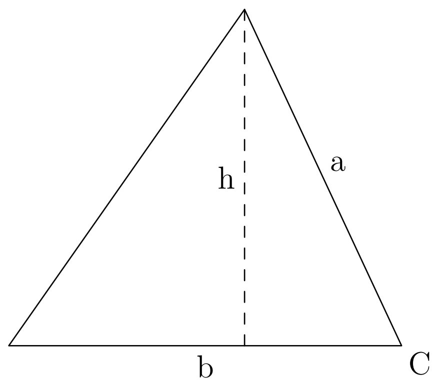
For example, in an equilateral triangle with side length one, we have

$$1^2 = 1^2 + 1^2 - 2 \cdot 1 \cdot 1 \cos 60^\circ = 2 - 2 \cdot \frac{1}{2} = 1$$

which is true.

1.3 Area Formulas

The simplest area formula is $\frac{1}{2}bh$ where b is "base" (one side of the triangle) and h the height to that base. However, notice that $h = a \sin C$ (from the definition of sin). So another formula for the area is $\frac{ab \sin C}{2}$. If you have two sides and the angle in between, this is the best way to compute the area.



Going even further, from the identity $\sin^2 \theta + \cos^2 \theta$ and the Law of Cosines,

$$\sin C = \sqrt{1 - \cos^2 C} = \sqrt{1 - \left(\frac{a^2 + b^2 - c^2}{2ab}\right)^2}$$

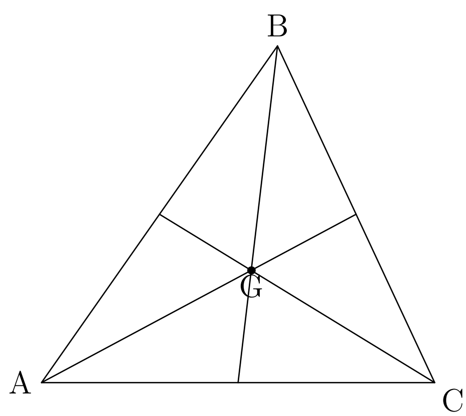
Plugging this into the above area formula and simplifying terms, we get Heron's Formula:

$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)}$$

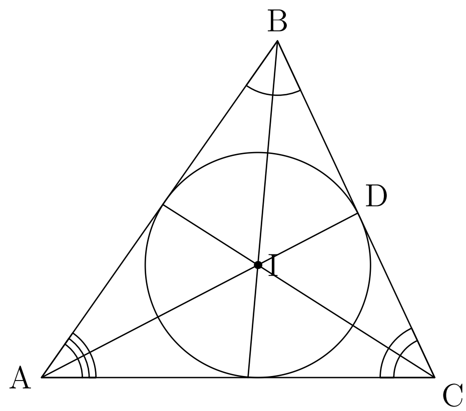
where $[ABC]$ means the area of ABC and s is the semi-perimeter, $s = \frac{a+b+c}{2}$.

1.4 Special Lines & Points

The *median* goes from one vertex of the triangle to the midpoint of the opposite side. The three medians intersect at one point, the *centroid*. Putting it on a coordinate grid (or the complex plane), you can show that the centroid (typically labelled G) is the average of the three vertices. Therefore, it divides the median in a $2:1$ ratio.



The *angle bisector* does exactly what it sounds like: it divides an angle of the triangle in half. The angle bisectors meet at the *incenter* (typically labelled I), which happens to be the center of the circle contained ("inscribed") in the triangle.



The angle bisector cuts the opposite side so that the ratio between the two pieces is equal to the ratio of the two sides. I.e.

$$\frac{AB}{AC} = \frac{DB}{DC}.$$

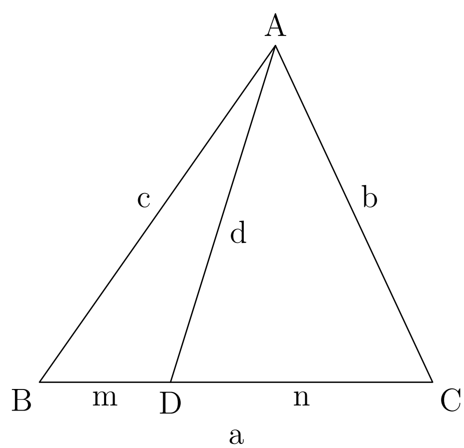
This can be proved with the Law of Sines.

1.5 Stewart's Theorem

In any triangle ABC with cevian AD, the following formula holds:

$$b^2m + c^2n = amn + d^2a.$$

This is sometimes written as $man + dad = bmb + cnc$ (which inspires the mnemonic "a man and his dad put a bomb in their sink"). This can be proved with the Law of Cosines on the angles $\angle BDA, \angle CDA$.



2 Problems

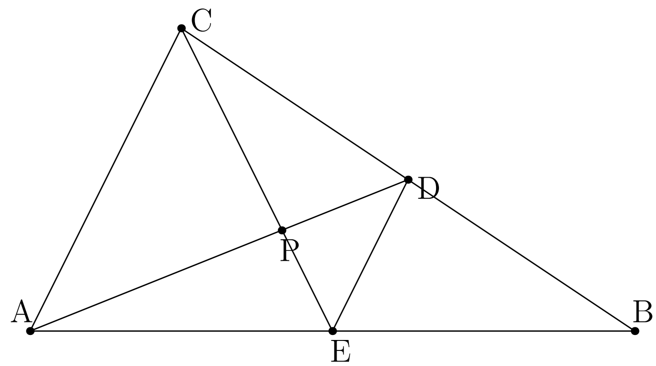
Problem 1: Triangle ABC has $\angle A = 60°$ and $\angle B = 75°$. Given the length of BC (a) is $5\sqrt{6}$, what is the length of AB (c)?

Problem 2: Triangle ABC has $\angle C = 60°$, $AC = 3$ and $BC = 4$. What is the length of AB?

Problem 3: A triangle has sides of length $13, 14$, and 15. What is the area of the triangle?

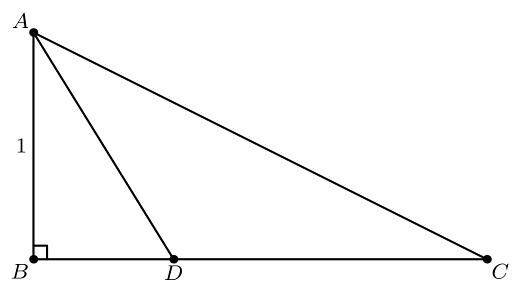
Problem 4: Triangle ABC has sides of lengths $AB = 3, BC = 5, AC = 7$. What is the measure of angle $\angle ABC$ ($\angle B$)?

Problem 5: (2013 AMC 10B #16) In triangle ABC, medians AD and CE intersect at P, $PE = 1.5$, $PD = 2$, and $DE = 2.5$. What is the area of $AEDC$?



Problem 6: (2019 AMC 10A #21) A sphere with center O has radius 6. A triangle with sides of length $15, 15$, and 24 is situated in space so that each of its sides is tangent to the sphere. What is the distance between O and the plane determined by the triangle? (Hint: find the inradius and then use the Pythagorean Theorem.)

Problem 7: (2009 AMC 10B #20) Triangle ABC has a right angle at B, $AB = 1$, and $BC = 2$. The bisector of $\angle BAC$ meets $\overline{BC}$ at D. What is BD?



Problem 8: (2002 AMC 12B #23) In $\triangle ABC$, we have $AB = 1$ and $AC = 2$. Side $\overline{BC}$ and the median from A to $\overline{BC}$ have the same length. What is BC?