Counting

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1 Introduction

How many ways are there to arrange 5 differently colored balls in a row? Well, there are 5 ways to determine which ball is first in the row, 4 ways to determine which of the remaining four balls is next, and so forth down to just 1 choice for which of the remaining one balls is placed last. The answer is $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways to arrange the balls. In general there are

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 1$$

(pronounced "n factorial") ways to arrange n different balls in a row, or n different people in a line.

Example 1: Alice, Bob, Charlie, David, Ethan, and Frank are sitting down in a row of six seats at the movie theater. How many ways can they arrange themselves if Alice, Bob, and Charlie have to be in three neighboring seats?

Solution 1: First imagine there were only four seats and four people: David, Ethan, Frank, and Alice-Bob-Charlie. There would be $4! = 24$ ways to arrange the four people. However, one of these "people" is really three separate people, and their seat is really a group of three adjacent seats. There are $3! = 6$ ways to arrange Alice, Bob, and Charlie within their group of seats, so the total number of ways to arrange the people is $4! \cdot 3! = 24 \cdot 6 = 144$.

Now, how many ways are there to pick 2 balls to buy out of five different balls on the store shelf? There are 5 ways to choose the first ball to buy, and 4 ways to pick which of the remaining four balls to buy, making a total of $5 \cdot 4$ ways to choose the balls. However, it doesn't matter whether we grab the red ball first and then the blue ball, or the blue ball first and then the red ball—we'll still have grabbed the same two balls. So we have to divide by this double counting. We get

$$\binom{5}{2} = \frac{5 \cdot 4}{2} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(2)(3 \cdot 2 \cdot 1)}$$

which is equal to $\frac{5!}{2!(5-2)!}$ (Note: the binomial coefficient on the left is pronounced "5 choose 2," and literally means "how many ways can we pick 2 balls from 5 balls?"). In general, using the same argument, you can find that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

(or in English: "n choose k equals n factorial divided by k factorial times n-k factorial"). These values are known as the binomial coefficients, because they are exactly the coefficients in the expansion of $(1+x)^n$. The Binomial Theorem says

$$(1+x)^n = \binom{n}{0}x^0 + \binom{n}{1}x^1 + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n.$$

Remember that $x^0 = 1$. For example, with $n = 2$ we get

$$(1+x)^2 = (1+x)(1+x) = 1 + 2x + x^2$$

and the Binomial Theorem gives us

$$(1+x)^2 = \binom{2}{0}x^0 + \binom{2}{1}x^1 + \binom{2}{2}x^2 = 1 + 2x + x^2.$$

So why does the Binomial Theorem work? It's because $(1+x)^n$ has the n products

$$\underbrace{(1+x)(1+x)\cdots(1+x)}_{n \text{ times}}.$$

To get a term with x^k we need to choose the x from k of these products, and the 1 from every other one, which happens in exactly $\binom{n}{k}$ ways. For example, when $n = 3$ and $k = 1$, we have the product

$$(1+x)(1+x)(1+x)$$

and we can make the following choices to get one power of x: $(x, 1, 1), (1, x, 1)$, or $(1, 1, x)$. So the coefficient of x should be 3, which is equal to $\binom{3}{1}$.

Remarks. Binomial coefficients can be calculated a little faster with the formula

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}{k \cdot (k-1) \cdot (k-2) \cdots 1}$$

where there are k terms in both the numerator and denominator, and then cancelling out terms before multiplying. E.g.

$$\binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = \frac{7 \cdot 5}{1} = 35.$$

Also, there is the equality

$$\binom{n}{k} = \binom{n}{n-k}$$

because choosing k items out of n items is the same as choosing all but the other $n-k$ items.

Example 2: Call a *mountain number* a five-digit number that contains five distinct digits from $\{1, 2, 3, \ldots, 9\}$, such that the first three digits form an increasing sequence and the last three form a decreasing sequence. For example, 12543 and 18943 are mountain numbers but 12540, 13322, and 14352 are not. How many mountain numbers are there?

Solution 2: There are $\binom{9}{5} = \binom{9}{4} = 126$ ways to pick five distinct digits from $\{1, 2, 3, \ldots, 9\}$. The largest digit has to be the middle digit. Of the remaining four digits, we need to pick two of them to go on the left of the middle digit, in which case the other two will go on the right. The digits on the left/right form increasing/decreasing sequences, so once we pick which digits are on the left, the order of all the digits is fixed. There are $\binom{4}{2} = 6$ ways to pick the two numbers to put to the left, so the total number of mountain numbers is

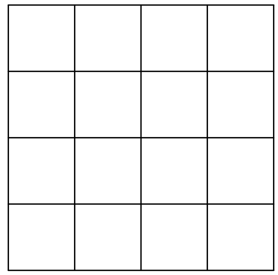
$$126 \cdot 6 = 756.$$

1.1 Problems

Problem 1: There are ten violinists in an orchestra. How many ways can we divide them into a first violin and a second violin section, each with five people and one designated as the section leader?

Problem 2: There are six people playing a game of basketball. How many ways are there two divide them into two equally sized teams? *Caution:* You have to be careful, because if A, B, C are on Team 1, it's the same as A, B, C being on Team 2. The teams are indistinguishable (as opposed to the sections in the previous problem).

Problem 3: In how many rectangles of any size are there in the 4×4 grid below?



Hint: Try choosing one pair of horizontal and one pair of vertical lines to define the boundaries of the rectangle.

2 Stars and Bars

This trick is referred to by many other names including "sticks and stones", "pirates and gold", and "balls and urns". Here's the problem: suppose you have \$6 to hand out among your 3 friends. How many ways can you distribute the money if each friend gets a whole number of dollars? You could give your first friend all \$6. Or you could give one friend \$3, another friend \$2, and the last one \$1. Or maybe you split it \$2 per person.

Look at the problem like this:

$$\circ\circ\circ|\circ|\circ\circ.$$

The circles represent dollars, and the lines are dividers between the people. The first friend will get all the money to the left of the first divider, the second friend will get everything in between the first and second dividers, and the last friend gets everything to the right of the last divider. There are 6 circles (dollars) and 2 dividers, and the problem is now equivalent to the number of ways to arrange these eight objects. But that's just the number of ways to pick 2 positions to put the dividers in—the circles will go in the remaining 6 positions. So we get

$$\binom{8}{2} = 28.$$

In general if there are n equivalent objects (dollars, stars, stones, gold, same-colored balls, etc.) to divide among k people, there are $k-1$ dividers. This means there are a total of $n+k-1$ objects to arrange, so we need to choose $k-1$ of $n+k-1$ positions to be where the dividers are placed. So, there are a total of

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$

ways to distribute the n objects among k people.

Warning! Some of the people could get zero dollars. You have to be careful when the problem specifies everyone receives at least one dollar (or similar), as this problem demonstrates:

Example 3: How many ways are there to roll three six-sided dice to get a sum of 8?

Solution 3: We need to distribute 8 pips among 3 different dice. However, no die can have 0 pips, they each need at least one. So, we subtract off one pip per die. Now we're left with 5 pips to distribute among 3 dice, where the dice can have any number of pips from 0–5. Using stars and bars, we get

$$\binom{5+3-1}{3-1} = \binom{7}{2} = 21.$$

Remark. We have to be careful there aren't too many pips on one die. In this case we could only distribute a maximum of 5 pips to one die, bringing its total to 6, but sometimes we're left with 8 or 9 pips to distribute. That becomes a much harder problem to solve, and we'll encounter it again when we learn about *generating functions.*

2.1 Problems

Problem 4: How many different quadruples (a_1, a_2, a_3, a_4) of positive integers satisfy the equation

$$a_1 + a_2 + a_3 + a_4 = 14?$$

Problem 5: A crew of five pirates find a chest containing 15 doubloons. Given that the captain gets a majority of the doubloons, and that each other pirate gets at least one doubloon, how many ways are there for the pirates to split their bounty? (*Hint*: Try breaking it up into cases based on the number of doubloons the captain gets.)

Problem 6: (2016 AMC 10A #20) For some particular value of N, when $(a+b+c+d+1)^N$ is expanded and like terms are combined, the resulting expression contains exactly 1001 terms that include all four variables $a, b, c,$ and d, each to some positive power. What is N?

(A) 9 **(B)** 14 **(C)** 16 **(D)** 17 **(E)** 19

3 Pascal's Triangle and Path Walking

$$\begin{array}{ccccccccccccc}
&&&&&&1&&&&&& \\
&&&&&1&&1&&&&& \\
&&&&1&&2&&1&&&& \\
&&&1&&3&&3&&1&&& \\
&&1&&4&&6&&4&&1&& \\
&1&&5&&10&&10&&5&&1& \\
1&&6&&15&&20&&15&&6&&1
\end{array}$$

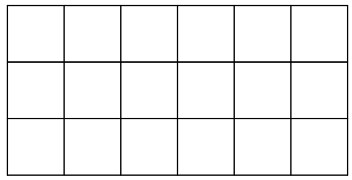
(source)

Pascal's triangle starts at the top (the 0th row) with 1. The values in each subsequent row are the sum of the two values above them, or if it happens to be on the edge just the one value above it. The numbers in Pascal's triangle are actually just the binomial coefficients!

$$\begin{array}{ccccccccccccc}
&&&&&&\binom{0}{0}&&&&&& \\
&&&&&\binom{1}{0}&&\binom{1}{1}&&&&& \\
&&&&\binom{2}{0}&&\binom{2}{1}&&\binom{2}{2}&&&& \\
&&&\binom{3}{0}&&\binom{3}{1}&&\binom{3}{2}&&\binom{3}{3}&&& \\
&&\binom{4}{0}&&\binom{4}{1}&&\binom{4}{2}&&\binom{4}{3}&&\binom{4}{4}&& \\
&\binom{5}{0}&&\binom{5}{1}&&\binom{5}{2}&&\binom{5}{3}&&\binom{5}{4}&&\binom{5}{5}& \\
\binom{6}{0}&&\binom{6}{1}&&\binom{6}{2}&&\binom{6}{3}&&\binom{6}{4}&&\binom{6}{5}&&\binom{6}{6}
\end{array}$$

Why does this happen? Imagine taking a path from the top down to the kth spot in the nth row. You'll have gone down a total of n times, and taken the right fork a total of k times. But we know there are $\binom{n}{k}$ ways to choose which of the moves downward are to the right, and the rest will be to the left. Alternatively, if you're in the kth spot in the nth row you can come from one of two spots: the spot above and to the left, or the spot above and two the right. So the number of ways to get to your end spot is the sum of the ways to get to the two spots above it.

Example 4: How many ways are there to walk from the bottom left corner to the top right corner along the grid lines, if you can only move up and to the right?



Solution 4: You move to the right a total of 6 times, and up a total of 3 times. So, we need to choose 3 of the 9 moves to be moves upward, and the rest will be moves to the right. This can be done in $\binom{9}{3} = 84$ ways.

3.1 Hockey Stick Principle

Pascal's triangle can be used to prove several important identities. For example, there are 2 different ways to move each time you go down a row, so you would expect the sum of terms on the nth row to be 2^n. Or,

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

This is in fact true, which you can also see from the Binomial Theorem:

$$2^n = (1+1)^n = \binom{n}{0}1^0 + \binom{n}{1}1^1 + \binom{n}{2}1^2 + \cdots + \binom{n}{n}1^n$$

but 1^k is always equal to 1.

A more difficult identity to see is the Hockey Stick Principle. Look at one diagonal of Pascal's triangle:

$$\begin{array}{ccccccccccccccccc}
&&&&&&&&1&&&&&&&& \\
&&&&&&&1&&1&&&&&&& \\
&&&&&&1&&2&&1&&&&&& \\
&&&&&1&&3&&3&&1&&&&& \\
&&&&1&&4&&6&&4&&1&&&& \\
&&&1&&5&&10&&10&&5&&1&&& \\
&&1&&6&&15&&20&&15&&6&&1&& \\
&1&&7&&21&&35&&35&&21&&7&&1& \\
1&&8&&28&&56&&70&&56&&28&&8&&1
\end{array}$$

source

The blue numbers add up to the red one, because

$$1 + 3 = 4, 4 + 6 = 10, 10 + 10 = 20, 20 + 15 = 35, 35 + 21 = 56.$$

Putting it in terms of binomial coefficients, we have

$$\binom{2}{0} + \binom{3}{1} + \binom{4}{2} + \cdots + \binom{7}{5} = \binom{8}{5}.$$

In general, the Hockey Stick Principle gives us the equation

$$\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+k}{k} = \binom{n+k+1}{k}.$$

Remember the identity $\binom{a}{b} = \binom{a}{a-b}$ from before? The identity above is equivalent to

$$\binom{n}{n} + \binom{n+1}{n} + \binom{n+2}{n} + \cdots + \binom{n+k}{n} = \binom{n+k+1}{n+1}.$$

These identities are super useful! Even if you don't remember them exactly, remember how to draw the hockey stick on Pascal's triangle to rederive them (I've done this so many times on tests).

Example 5: A group of four pirates find 11 coins in a treasure chest. How many ways are there to divide up the coins if the captain must get at least 4 coins?

Solution 5: If the captain gets 4 coins, then there are 7 coins left to divide among the other 3 pirates. Stars and bars tells us this can be done in $\binom{7+3-1}{3-1} = \binom{9}{2}$ ways. If the captain gets 5 coins, then there are 6 coins left so there are $\binom{6+3-1}{3-1} = \binom{8}{2}$ ways. Continuing up we get a total of

$$\binom{9}{2} + \binom{8}{2} + \cdots + \binom{2}{2}$$

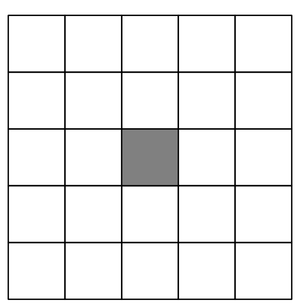
ways to divide up the coins. The Hockey Stick Principle tells us

$$\binom{9}{2} + \binom{8}{2} + \cdots + \binom{2}{2} = \binom{10}{3} = 120.$$

3.2 More Path Walking

Occasionally there are roadblocks in the path. Take this MATHCOUNTS problem from many years back:

Example 6: How many ways can you move from the bottom left square to the top right square, if you can only move one square to the right or up each turn and cannot land in the grey square?



There are two ways to go about this. You can label the number of paths from the bottom left square to every other square as follows:

1	5	9	17	34
1	4	4	8	17
1	3		4	9
1	2	3	4	5
1	1	1	1	1

The number in each square is computed similarly to how you make Pascal's Triangle—it's just the sum of the numbers in the square directly below or to the left of it. There are only 24 squares so it shouldn't take more than 30 seconds to draw out and get the answer of 34. This is a perfectly acceptable way to solve the problem. However, there is a slightly more clever way to solve it:

Solution 6: Pretend the grey roadblock weren't there. We're moving to the right 4 times and up 4 times, so there would be $\binom{4+4}{4} = 70$ paths we could take. However, we have to subtract off any path that goes through the grey square. There are $\binom{2+2}{2} = 6$ ways to get from the bottom left square to the grey square, and $\binom{2+2}{2} = 6$ ways to get from the grey square to the top right square, so there are $6 \cdot 6 = 36$ paths that go through the grey square. Subtracting these off, we get $70 - 36 = 34$.

What if there were two greyed out squares? It gets much more complicated especially if it's possible to move from one grey square to another (i.e. one grey square is to the right/above another). At that point I would go with the first solution which would most likely be faster.

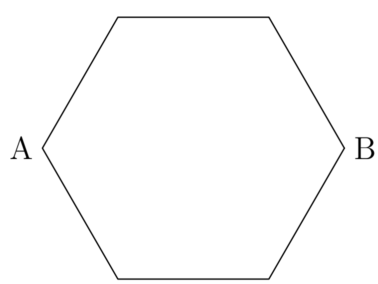
3.3 Problems

Problem 7: Evaluate

$$\binom{10}{5} - \binom{8}{3} - 2\binom{8}{4} - \binom{8}{5}.$$

Problem 8: Jim's mother has 17 identical marbles in a bowl. Jim grabs some number of the marbles and puts them in his pocket to share with his two friends at school that day. When he arrives at school he splits up the marbles in his pocket so that everyone has at least two marbles. How many ways can Jim grab and redistribute the marbles?

Problem 9: (Adapted from 2020 Mock AMC 12) Misha starts on Point A on a regular hexagon, and is trying to reach Point B, which is diametrically opposite of Point A. Given that one move consists of a move to an adjacent vertex, how many ways are there for him to be at Point B after 9 moves? What about after 11 moves?



Problem 10: (2019 AIME II #2) Lily pads $1, 2, 3, \ldots$ lie in a row on a pond. A frog makes a sequence of jumps starting on pad 1. From any pad k the frog jumps to either pad $k + 1$ or pad $k + 2$ chosen randomly with probability $\frac{1}{2}$ and independently of other jumps. The probability that the frog visits pad 7 is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

Problem 11: (1986 AIME # 11) The polynomial $1-x+x^2-x^3+\cdots+x^{16}-x^{17}$ may be written in the form $a_0+a_1y+a_2y^2+\cdots+a_{16}y^{16}+a_{17}y^{17}$, where $y=x+1$ and the a_i's are constants. Find the value of a_2. *Hint:* Try expressing x in terms of y and expanding out the original polynomial.

Problem 12: (Putnam 2020 A2, very hard) Let

$$S_k=\sum_{j=0}^{k}2^{k-j}\binom{k+j}{j}.$$

Find the value of S_5. Note: the notation $\sum_{j=0}^{k}$ means to take the sum from $j=0$ all the way up to k of the expression on the right. So,

$$S_5=2^5\binom{5}{0}+2^4\binom{6}{1}+2^3\binom{7}{2}+2^2\binom{8}{3}+2^1\binom{9}{4}+2^0\binom{10}{5}.$$

Hint: Try finding an expression for S_{k+1} in terms of S_k.

4 Vandermonde & Bijections

Consider the summation

$$\sum_{j=0}^{k}\binom{n}{j}\binom{m}{k-j}=\binom{n}{0}\binom{m}{k}+\binom{n}{1}\binom{m}{k-1}+\binom{n}{2}\binom{m}{k-2}+\cdots+\binom{n}{k}\binom{m}{0}.$$

How can you evaluate this? Well, suppose you have n blue stones and m red stones in a line:

$$\underbrace{\circ\circ\circ\cdots\circ}_{n\text{ stones}}\ \underbrace{\circ\circ\cdots\circ}_{m\text{ stones}}$$

The first term in the summation is the number of ways to pick 0 of the blue stones, and k of the red stones. The second term is the number of ways to pick 1 blue stone and $k-1$ red stones. And so forth. So the sum is the number of ways to pick some number of blue stones, and then another number of red stones, to get a total of k stones. But that's the same as just picking k stones out of all $n+m$ stones! So the sum is $\binom{n+m}{k}$. This is known as the Vandermonde identity:

$$\sum_{j=0}^{k}\binom{n}{j}\binom{m}{k-j}=\binom{n}{0}\binom{m}{k}+\binom{n}{1}\binom{m}{k-1}+\binom{n}{2}\binom{m}{k-2}+\cdots+\binom{n}{k}\binom{m}{0}=\binom{n+m}{k}.$$

Example 7: (2020 AIME I #7) A club consisting of 11 men and 12 women needs to choose a committee from among its members so that the number of women on the committee is one more than the number of men on the committee. The committee could have as few as 1 member or as many as 23 members. What is the number of committees that can be formed?

Solution 7: If we pick 0 men, we need to pick 1 woman. That can be done in $\binom{11}{0}\binom{12}{1}$ ways. If we pick 1 man, we need to pick 2 women. That can be done in $\binom{11}{1}\binom{12}{2}$ ways. Continuing on, we get the total number of ways to form committees is

$$\binom{11}{0}\binom{12}{1}+\binom{11}{1}\binom{12}{2}+\cdots+\binom{11}{11}\binom{12}{12}.$$

Remembering the identity

$$\binom{12}{j} = \binom{12}{12-j}$$

we can rewrite the sum as

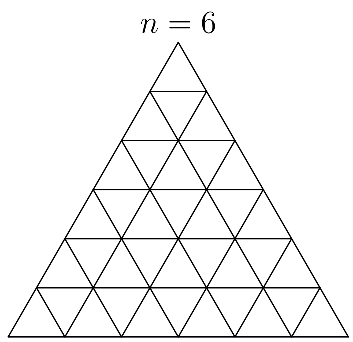
$$\binom{11}{0}\binom{12}{11} + \binom{11}{1}\binom{12}{10} + \cdots + \binom{11}{11}\binom{12}{0}.$$

This is a Vandermonde sum where $n = 11, m = 12$, and $k = 11$. So

$$\binom{11}{0}\binom{12}{11} + \binom{11}{1}\binom{12}{10} + \cdots + \binom{11}{11}\binom{12}{0} = \binom{23}{11}.$$

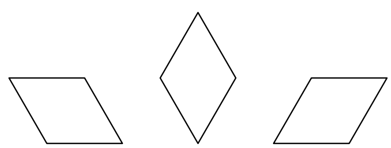
The Vandermonde identity comes from a "bijection" between what the sum represents and a much simpler idea. There are many problems that involve bijections out there. Most often you guess what the answer should be based on simpler cases, and then find the bijection that matches your answer. I'll just go over a few more examples of bijections.

Example 8: (From Yufei Zhao's handout.) Consider an equilateral triangle tiled with n^2 smaller equilateral triangles as in the following diagram:

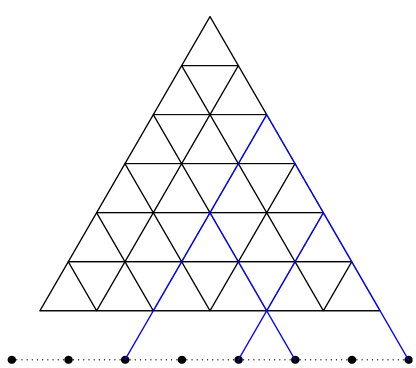


How many parallelograms of any size are there in it?

Solution 8: First, each parallelogram will be pointed in one of three ways:



By symmetry there are an equal number of parallelograms pointing in each direction, so let's just find the number pointed up-down like in the middle, and then multiply by 3 in the end. Here's the bijection: Extend the sides of a parallelogram one row *past* the bottom of the parallelogram.



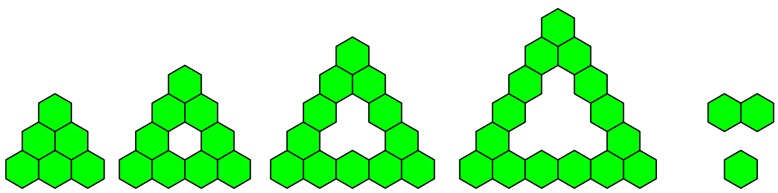
Every parallelogram will hit four separate points. Also, any four separate points will form a unique parallelogram by drawing lines going to the right from the left two points and lines going to the left from the right two points. So, the number of up-down parallelograms is the number of ways to pick four separate points out of the $n + 2$ points on dotted line, or $\binom{n+2}{4}$. Taking into account the other two types of parallelograms, we get the answer

$$3 \cdot \binom{n+2}{4}.$$

4.1 Problems

Problem 13: Farmer John has 8 cows. On a given day he will take at most 4 cows out of their pens for fresh air. He then milks 3 of the remaining cows. How many different ways can he perform this daily routine?

Problem 14: (Utah Math Olympiad 2013 #6, hard) How many ways can one tile the border of a triangular grid of hexagons of length n completely using only 1×1 and 1×2 hexagon tiles? Express your answer in terms of a well-known sequence, and prove that your answer holds true for all positive integers $n \geq 3$ (examples of such grids for $n = 3$, $n = 4$, $n = 5$, and $n = 6$ are shown below).

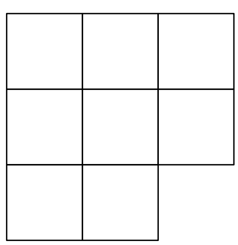


Hint: How many ways can you tile an $n \times 1$ rectangle with 1×1 and 2×1 tiles? It's the Fibonacci sequence. I'll refer you to this link to see why.

5 Hook Length Formula

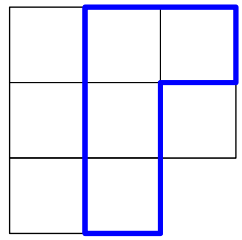
The Hook Length formula isn't super *useful*—I've only encountered it a few times—but it is a fun result. It's best to demonstrate it with an example.

Example 9: How many ways are there to write the numbers 1–8 in the following grid so that any number to the right is larger than the ones to the left, and any numbers below are larger than the numbers above? One filled in example is shown to the right.



1	4	7
2	5	8
3	6	

Solution 9: Look at each "hook" in the diagram. I've highlighted one in blue:



If we randomly arranged numbers in this hook, the smallest number would be in the top left only one-fourth of the time (because there are four numbers in the hook). However, the problem statement says that the numbers are increasing going to the right and going down, so the smallest number in the hook has to be in the top left. This is the idea behind the Hook Length formula. If we go through every hook (diagram with hook lengths below), we get only one out of every

$$5 \cdot 4 \cdot 4 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1$$

arrangements are valid. There are eight numbers, so there are a total of 8! arrangements. Therefore, the number of valid arrangements is

$$\frac{8!}{5 \cdot 4 \cdot 4 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1} = 42.$$

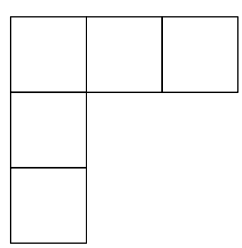
Hook lengths (written in top-left corner of each hook).

5	4	2
4	3	1
2	1	

In general, the Hook Length formula says that the number of ways to arrange n numbers in a similar shape is $n!$ divided by the product of the hook lengths. You can count the hook length by going to a square and finding how many squares are to the right/below it (and then adding 1 for the square you're on). Let's see how we can use this to solve a familiar problem from a previous section:

Example 10: Call a *mountain number* a five-digit number that contains five distinct digits from $\{1, 2, 3, \ldots, 9\}$, such that the first three digits form an increasing sequence and the last three form a decreasing sequence. For example, 12543 and 18943 are mountain numbers but 12540, 13322, and 14352 are not. How many mountain numbers are there?

Solution 10: Suppose the numbers were 1–5. Draw the following figure:



The hook lengths are $5, 2, 2, 1, 1$, so the Hook Length formula says there are

$$\frac{5!}{5 \cdot 2 \cdot 2 \cdot 1 \cdot 1} = 6$$

ways to arrange the numbers 1–5 (note: this time the numbers are *decreasing* instead of *increasing* when they go to the right/down, but that doesn't change the problem at all). This diagram just represents a mountain number... try flattenning it into a line. So there are 6 mountain numbers using the digits 1–5. Of course, the numbers could be any five from 1–9 (and there are $\binom{9}{5}$ ways to pick the five), so there are a total of

$$6 \cdot \binom{9}{5} = 756$$

mountain numbers.

5.1 Problems

Problem 15: How many ways are there to write the numbers 1–9 in a 3×3 grid of squares if the numbers must increase across each row and down each column?

Problem 16: The nth *Catalan number*, C_n, is defined as

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Show that this is always an integer.
Hint: Rewrite it as

$$C_n = \frac{(2n)!}{(n+1)(n!)(n!)} = \frac{(2n)!}{[(n+1)\cdot(n)\cdot(n-1)\cdots(1)]\cdot[(n)\cdot(n-1)\cdot(n-2)\cdots(1)]}$$

Problem 17: (2016 USAMO #2) Prove that for any positive integer k,

$$(k^2)! \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$$

is an integer.

The $\prod$ symbol (capital π, means "product") is similar to the $\sum$ symbol but for multiplication. It means take the product of that term on the right as j ranges from 0 to $k-1$. So

$$(k^2)! \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!} = (k^2)! \cdot \frac{0!}{k!} \cdot \frac{1!}{(k+1)!} \cdot \frac{2!}{(k+2)!} \cdots \frac{(k-1)!}{(2k-1)!}$$

Also, 0! is defined to be 1 (which makes sense because there is one way to arrange zero objects).
Hint: Consider a $k \times k$ grid of squares.