HOMEWORK 10

MATH 2001

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ABSTRACT. This is the first homework assignment. The problems are from Hammack [**?**, Ch. 1, §1.1]:

- **Chapter 12 Section 12.1**, Exercises: 4, 6. **Section 12.2**, Exercises: 5, 10. **Section 12.3**, Exercises: 1, 2.

CONTENTS

Chapter 12 Section 12.1 1
Ch.12, §12.1, Exercise 4 1
Ch.12, §12.1, Exercise 6 2
Ch.12, §12.2, Exercise 5 3
Ch.12, §12.2, Exercise 10 3
Ch.12, §12.3, Exercise 1 4
Ch.12, §12.3, Exercise 2 4

CHAPTER 12 SECTION 12.1

Ch.12, §12.1, Exercise 4. There are eight different functions $f : \{a, b, c\} \to \{0, 1\}$. List them all. Diagrams will suffice.

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Solution to Ch.1, §1.1, Exercise 2.

$$f = \{(a,0),(b,0),(c,0)\}$$

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□

Ch.12, §12.1, Exercise 6. Suppose $f : \mathbb{Z} \to \mathbb{Z}$ is defined as $f = \{(x, 4x+5) : x \in \mathbb{Z}\}$. State the domain, codomain and range of f. Find $f(10)$.

Solution to Ch.12, §12.1, Exercise 6.

The domain of the function is $\{x : x \in \mathbb{Z}\}$.

The codomain of the function is $\{x : x \in \mathbb{Z}\}$.

The range of the function is $\{x : x = 4n, n \in \mathbb{Z}\}$.

□

Ch.12, §12.2, Exercise 5. A function $f : \mathbb{Z} \to \mathbb{Z}$ is defined as $f(n) = 2n + 1$. Verify whether this function is injective and whether it is surjective.

Solution to Ch.12, §12.2, Exercise 5.

Proposition. The function is injective but not surjective.

Step 1. Prove the function is injective. *Proof.* Assume $f(x) = f(y)$. We have $2x + 1 = 2y + 1$, thus $x = y$.

Step 2. Prove the function is not surjective.

Proof. There exist element 2 which $f(x) = 2x + 1 \neq 2$ for every $x \in \mathbb{Z}$. The function is injective but not surjective. □

Ch.12, §12.2, Exercise 10. Prove that the function $f : \mathbb{R} - 1 \to \mathbb{R} - 1$ defined by $f(x) = \left(\frac{x+1}{x-1}\right)^3$ is bijective.

Solution to Ch.12, §12.2, Exercise 10.

Proposition. The $f(x)$ is bijective.

Step 1. Prove $f(x)$ is injective.

Proof. Assume $f(x) = f(y)$, we have $\left(\frac{x+1}{x-1}\right)^3 = \left(\frac{y+1}{y-1}\right)^3$, taking the cubic root on both side get $\left(\frac{x+1}{x-1}\right) = \left(\frac{y+1}{y-1}\right)$. Therefor we have:

$$(x+1) * (y-1) = (y+1) * (x-1)$$

$$xy + y - x = xy + x - y$$

$$y - x = x - y$$

$$2y = 2x$$

$$x = y$$

Step 2. Prove $f(x)$ is sujective.

Proof. Suppose $f(x) = c$, we have $f(x) = \left(\frac{x+1}{x-1}\right)^3 = c$. Sove for x, we get $x = \frac{1+\sqrt[3]{c}}{1-\sqrt[3]{c}}, x \in \mathbb{R}$. It follows that $f(x)$ is surjective. The $f(x)$ is surjective and injective, so it is bijective. □

Ch.12, §12.3, Exercise 1. Prove that if six numbers are chosen at random, then at least two of them will have the same remainder when divided by 5.

Solution to Ch.1, §1.1, Exercise 38. Suppose the set A that made of the remanders of any number n where $n \in \mathbb{Z}$ divided by 5. Such a set contains the elements $A = \{0, 1, 2, 3, 4\}$. Thus, we have $|A| = 5$. The set B consist six random numbers, so $|B| = 6$. Consider the function $f : B \to A$, such that $f(x) = x \ mod\ 5$. Because $|B| > |A|$, the function is not injective by pigeohole principle. □

Ch.12, §12.3, Exercise 2. Prove that if a is a natural number, then there exist two unequal natural numbers k and l for which $a^k - a^l$ is divisible by 10.

Solution to Ch.1, §1.1, Exercise 40. Assume a set $A = \{x \in \mathbb{N} : x = a^n\}$ where $a, n \in \mathbb{N}$. The a^k and a^l would be the two different elements of the set A. Suppose the set $B = \{y \in \mathbb{N} : y = m \ mod\ 10\}$, then $|B| = 10$. Consider a function $f : A \to B$, such that $f(z) = z \ mod\ 10$. Based on pigeonhole princeple f is not injective, because $|A| > |B|$. Therefor, there exist different k and l, such that $a^k \ mod\ 10 = a^l \ mod\ 10$. Thus there exist $(a^k - a^l) \div 10 = n$ where $n \in \mathbb{N}$ based on definition. □

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