Final Exam

Intro to Discrete Math
MATH 2001
Spring 2020

Saturday May 2, 2020

PRINT YOUR NAME: ____

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total:	80	

- For the exam, you may use only the textbook, your lecture notes, your homework, and your previous exams, from this course.
- You may not use any other resources whatsoever.
- You may not discuss the exam with anyone, in any way, under any circumstances.
- **You must explain your answers, and you will be graded on the clarity of your solutions.**
- You must write your solutions in LaTeX, and you must upload your .pdf *and* .tex files to canvas.
- The exam is due at 11:59 PM Saturday May 2, 2020.

1. (10 points) **True or false.**

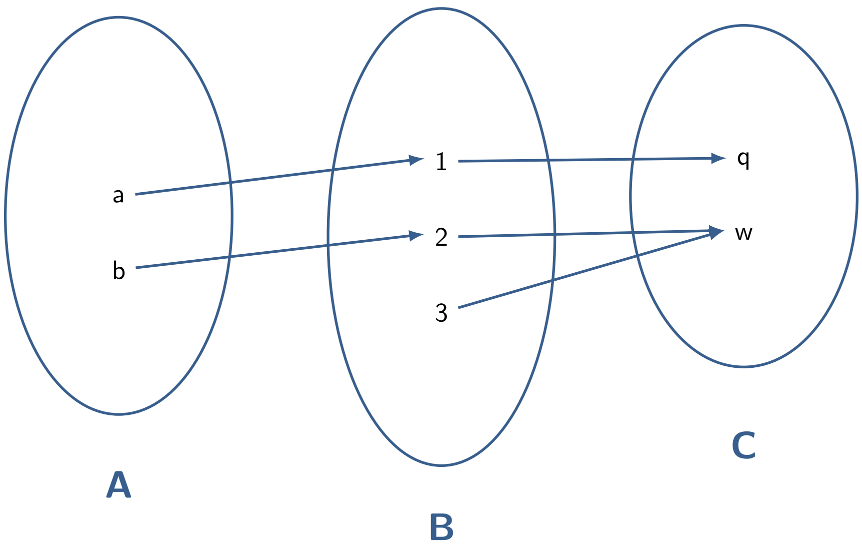
Let $f : A \to B$ and $g : B \to C$ be maps of sets. If $g \circ f$ is injective, then g is injective.

If true, give a proof. If false, provide a counter example.

Proof.

False.

Example:



$A = \{a, b\}, B = \{1, 2, 3\}, C = \{q, w\}$

The map $g \circ f$ is injective, but g is not.

$\square$

1
10 points

2. (10 points) **True or False.**

Suppose $f : A \to B$ is a surjective map of sets. Then there exists an injective map of sets $s : B \to A$.

If true, give a proof. If false, provide a counter example.

Proof.

True.

Assume $f : A \to B$ is a surjective map of sets. Then for all $b \in B$ we have $b = f(a), a \in A$. Assume $c = f(a), d = f(b)$ where $c, d \in B$ and $a, b \in A$. Then there is a map of sets $s : B \to A$ which $s = f^{-1}$. Then we have $s(c) = s(f(a)) = a$ and $s(d) = s(f(b)) = b$. Assume $a = b$, then we have $f(a) = f(b)$, it follows that $c = d$. There for s is injective. $\square$

2
10 points

3. (10 points) **True or False.**

A map of sets $f : A \to B$ is surjective if and only if for all maps of sets $g_1 : B \to C$ and $g_2 : B \to C$, we have that $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$.

If true, give a proof. If false, provide a counter example.

Proof.

True.

Step1: Prove if $f : A \to B$ is surjective, then for all $g_1 : B \to C$ and $g_2 : B \to C$ we have that $g_1 \circ f = g_2 \circ f \implies g_1 = g_2$.

(contrapositive) Assume $g_1 : B \to C, g_2 : B \to C$ and $g_1 \neq g_2$, then there exist $g_1(b) \neq g_2(b), b \in B$ by definition. Suppose $f : A \to B$ is surjective, then $b = f(a)$ for some $a \in A$. Thus we have $g_1(f(a)) \neq g_2(f(a))$, in other word $g_1 \circ f \neq g_2 \circ f$.(contrapositive proof)

Step2: Prove if for all maps of set $g_1 : B \to C$ and $g_2 : B \to C$, we have $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$, then $f : A \to B$ is surjective.

(contrapositive) Assume a map of sets $f : A \to B$. Suppose there exist $g_1 : B \to C$ and $g_2 : B \to C$ such that $g_1 \circ f = g_2 \circ f$ and $g_1 \neq g_2$. We have $g_1(f(a)) = g_2(f(a))$ for all $a \in A$. We also get $g_1(b) \neq g_2(b)$ for some $b \in B$. Thus it follows that $b \neq f(a)$. Therefor f is not surjective by pigeonhole principle.

□

3
10 points

4. Define a relation on $\mathbb{N} \times \mathbb{N}$ by the rule that for all $(a, b), (c, d) \in \mathbb{N} \times \mathbb{N}$,

$$(a, b) \sim (c, d) \iff a + d = b + c.$$

(a) (5 points) **What is this relation as a subset of** $(\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{N})$**?**

Proof.

$$\{((a, b), (a, b)) | a, b, c, d \in N, a + d = b + c\} \subseteq (\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{N})$$

□

(b) (5 points) **Show that the relation is an equivalence relation.**

Proof. Assume $(a, b), (c, d) \in \mathbb{N} \times \mathbb{N}$ and $(a, b) \sim (c, d)$. Then we have $a + d = b + c$ by definition, so $c + b = d + a$. It follows that $(c, d) \sim (a, b)$. Thus the relation is symetric. Suppose $c = a$ and $b = d$, then we have $a + b = b + a$, it follows that $(a, b) \sim (a, b)$. Thus it is reflexive. Assume $(c, d) \sim (e, f)$, then we get $c + f = d + e$, thus $d - c = f - e$. We also get $d - c = b - a$ from previous step, so $f - e = b - a$, then $a + f = b + e$, it follows that $(a, b) \sim (e, f)$ by definition. Therefor the relation is transitive.

□

4
10 points

5. (10 points) **True or False.** Let $Z = (\mathbb{N} \times \mathbb{N})/\sim$ be the set of equivalence classes in $\mathbb{N} \times \mathbb{N}$ for the equivalence relation given in the previous problem.

There is a bijective map of sets

$$f : Z \longrightarrow \mathbb{Z}$$

given by the rule $[(a,b)] \mapsto a - b$.

You must prove that the statement in italics is true or that it is false.

Proof. Ture

Assume the map of sets $f : Z \to \mathbb{Z}$ where $Z = (\mathbb{N} \times \mathbb{N})/\sim$ is the set of equivalence classes, such that $f([(a,b)]) = a - b, [(a,b)] \in Z$. Suppose the equivalence classes $[(a,b)]$ and $[(c,d)]$, such that $f([(a,b)]) = f([(c,d)])$, we have that $a - b = c - d$, so $a + d = b + c$, it follows that $(a,b) \sim (c,d)$ and $[(a,b)] = [(c,d)]$. There for f is injective. For every $k \in \mathbb{Z}$, we can find a equivalence class $[(m+k,m)]$, such that $f([(m+k,m)]) = m + k - m = k$. Therefor f is surjective.

$\square$

5
10 points

6. (10 points) **True of False.** Let S be a set and let $\mathscr{C} \subseteq \mathscr{P}(S)$ be a collection of subsets of S with the following property: For any set I, if for each $i \in I$ we are given a set $C_i \in \mathscr{C}$, and for all $i, j \in I$ we have $C_i \subseteq C_j$ or $C_j \subseteq C_i$, then $\bigcup_{i \in I} C_i \in \mathscr{C}$.

There exists $C \in \mathscr{C}$ such that for all $C' \in \mathscr{C}$, if $C \subseteq C'$, then $C = C'$.

If the statement in italics is true, give a proof. Otherwise, provide a counter example.

[Hint: Think in terms of POSETs.]

Proof. True

Assume set I is the index set of $\mathscr{C}$. Let $C_i \in \mathscr{C}$, we have $\{C_i | i \in I\}$ are the elements of $\mathscr{C}$. Suppose for all $i, j \in I$ we have $C_i \subseteq C_j$ or $C_j \subseteq C_i$ and $\bigcup_{i \in I} C_i \in \mathscr{C}$, then we get the partilly order relation R, which $aRb \iff a \subseteq b$, and the poset $(\mathscr{C}, \subseteq)$. Thus there is a element $C \in \mathscr{C}$ such that $C' \subseteq C$ for all $C' \in \mathscr{C}$ by definition of poset. Suppose $C \subseteq C'$, then we have $C = C'$.

□

6
10 points

7. (10 points) For an incidence plane (Π, Λ), given a line $\ell \in \Lambda$ and a point $P \in \Pi$ such that $P \notin \ell$, consider the following properties:

(P0) There *does not exist* $\ell' \in \Lambda$ such that $P \in \ell'$ and $\ell' \cap \ell = \emptyset$.

(P1) There *exists exactly one* $\ell' \in \Lambda$ such that $P \in \ell'$ and $\ell' \cap \ell = \emptyset$.

(P2) There *exists at least two* $\ell' \in \Lambda$ such that $P \in \ell'$ and $\ell' \cap \ell = \emptyset$.

Find an example of an incidence plane (Π, Λ) with exactly five points, such that for each of the properties above there exists a point and a line satisfying that property. In other words, find a single incidence plane (Π, Λ) with $|\Pi| = 5$, such that for $i = 0, 1, 2$, there exists $\ell_i \in \Lambda$ and $P_i \in \Pi$ with $P_i \notin \ell_i$, such that ℓ_i and P_i satisfy property (Pi).

7
10 points

8. A polynomial in one variable, x, with rational coefficients can, by definition, be written in the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_d x^d$$

where $a_0, a_1, \ldots, a_d \in \mathbb{Q}$, and d is some nonnegative integer. If in the expression above for $p(x)$ we have $a_d \neq 0$, then we say $p(x)$ has degree d.

We denote by $\mathbb{Q}[x]$ the set of all polynomials in x with rational coefficients. Denote by $\mathbb{Q}[x]_d \subseteq \mathbb{Q}[x]$ the subset consisting of those polynomials that have degree at most d, union the set containing the zero polynomial (the zero polynomial does not have a degree using our definition above).

(a) (5 points) Let d be a nonnegative integer. **Show that the set $\mathbb{Q}[x]_d$ is countable**.

[Hint: Show that $\mathbb{Q}[x]_d$ is in bijection with $\mathbb{Q}^{d+1}$.]

(b) (5 points) **True or False.** *The set $\mathbb{Q}[x]$ is countable.*

You must prove that the statement in italics is true or that it is false.

8
10 points