HOMEWORK 3

MATH 2001

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ABSTRACT. This is the third homework assignment. The problems are from Hammack [Ham18, Ch. 1, §1.3, §1.4, §1.5, §1.6 §1.7, §1.8]:

- **Chapter 1 Section 1.3**, Exercises: 2, 6, 10.
- **Chapter 1 Section 1.4**, Exercises: 2, 10, 14, 19.
- **Chapter 1 Section 1.5**, Exercises: 2, 4, 6.
- **Chapter 1 Section 1.6**, Exercises: 2.
- **Chapter 1 Section 1.7**, Exercises: 4, 5, 6.
- **Chapter 1 Section 1.8**, Exercises: 2, 11, 12.

CONTENTS

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Chapter 1 Section 1.3

Ch.1, §1.3, Exercise 2. List all the subsets of the following sets: $\{1, 2, \emptyset\}$

Solution to Ch.1, §1.3, Exercise 2.

$$\emptyset, \{1\}, \{2\}, \{\emptyset\}, \{1,2\}, \{1,\emptyset\}, \{2,\emptyset\}, \{1,2,\emptyset\}$$

□

Ch.1, §1.3, Exercise 8. List all the subsets of the following sets: $\{1, 2, \emptyset\}$

Solution to Ch.1, §1.3, Exercise 8.

$$\emptyset, \{\mathbb{R}\}, \{\mathbb{Q}\}, \{\mathbb{N}\}, \{\mathbb{R},\mathbb{Q}\}, \{\mathbb{R},\mathbb{N}\}, \{\mathbb{Q},\mathbb{N}\}, \{\mathbb{R},\mathbb{Q},\mathbb{N}\}$$

□

Ch.1, §1.3, Exercise 12. Write out the following sets by listing their elements between braces. $\{X : X \subseteq \{3, 2, a, \} \text{ and } |X| = 1\}$

Solution to Ch.1, §1.3, Exercise 12.

$$\{3\}, \{2\}, \{a\}$$

□

Ch.1, §1.4, Exercise 2. Write out the following sets by listing their elements between braces. $\mathscr{P}(\{1, 2, 3, 4\})$

Solution to Ch.1, §1.4, Exercise 2.

$$\{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\},$$

$$\{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$$

□

Ch.1, §1.4, Exercise 10. Write out the following sets by listing their elements between braces. $\{X \in \mathscr{P}(\{1, 2, 3\}) : |X| \leqslant 1)$

Solution to Ch.1, §1.4, Exercise 10.

$$\emptyset, \{1\}, \{2\}, \{3\}$$

□

Ch.1, §1.4, Exercise 14. Suppose that $|X| = m$ and $|X| = n$. Find the following cardinalities: $|\mathscr{P}(\mathscr{P}(A))|$

Solution to Ch.1, §1.4, Exercise 14.

$$|\mathscr{P}(\boldsymbol{A})| = 2^m$$

$$|\mathscr{P}(\mathscr{P}(\boldsymbol{A}))| = 2^{2^m}$$

$\square$

Ch.1, §1.4, Exercise 19. Suppose that $|\boldsymbol{X}| = m$ and $|\boldsymbol{X}| = n$. Find the following cardinalities: $|\mathscr{P}(\mathscr{P}(\mathscr{P}(A \times \emptyset)))|$

Solution to Ch.1, §1.4, Exercise 19.

$$|\boldsymbol{A} \times \emptyset| = \emptyset$$

$$|\mathscr{P}(\emptyset)| = 2^0 = 1$$

$$|\mathscr{P}(\mathscr{P}(\emptyset))| = 2^1 = 2$$

$$|\mathscr{P}(\mathscr{P}(\mathscr{P}(\emptyset))| = 2^2 = 4$$

$\square$

Ch.1, §1.5, Exercise 2. Suppose that $\boldsymbol{A} = \{0, 2, 4, 6, 8\}$, $\boldsymbol{B} = \{1, 3, 5, 7\}$ and $\boldsymbol{C} = \{2, 8, 4\}$. Find:

(a) $\boldsymbol{A} \cup \boldsymbol{B}$

(b) $\boldsymbol{A} \cap \boldsymbol{B}$

(c) $\boldsymbol{A} - \boldsymbol{B}$

(d) $\boldsymbol{A} - \boldsymbol{C}$

(e) $\boldsymbol{B} - \boldsymbol{A}$

(f) $\boldsymbol{A} \cap \boldsymbol{C}$

(g) $\boldsymbol{B} \cap \boldsymbol{C}$

(h) $\boldsymbol{C} - \boldsymbol{A}$

(i) $\boldsymbol{C} - \boldsymbol{B}$

Solution to Ch.1, §1.5, Exercise 2.

(a) $\boldsymbol{A} \cup \boldsymbol{B}$

$$\boldsymbol{A} \cup \boldsymbol{B} = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$$

(b) $\boldsymbol{A} \cap \boldsymbol{B}$

$$\boldsymbol{A} \cap \boldsymbol{B} = \emptyset$$

(c) $\boldsymbol{A} - \boldsymbol{B}$

$$\boldsymbol{A} - \boldsymbol{B} = \{0, 2, 4, 6, 8\}$$

(d) $\boldsymbol{A} - \boldsymbol{C}$

$$\boldsymbol{A} - \boldsymbol{C} = \{0, 6\}$$

(e) $\boldsymbol{B} - \boldsymbol{A}$

$$\boldsymbol{B} - \boldsymbol{A} = \emptyset$$

(f) $\boldsymbol{A} \cap \boldsymbol{C}$

$$\boldsymbol{A} \cap \boldsymbol{C} = \{2, 8, 4\}$$

(g) $\boldsymbol{B} \cap \boldsymbol{C}$

$$\boldsymbol{B} \cap \boldsymbol{C} = \emptyset$$

(h) $\boldsymbol{C} - \boldsymbol{A}$

$$\boldsymbol{C} - \boldsymbol{A} = \emptyset$$

(i) $\boldsymbol{C} - \boldsymbol{B}$

$$\boldsymbol{C} - \boldsymbol{B} = \{2, 4, 8\}$$

$\square$

Ch.1, §1.5, Exercise 4. Suppose that $\boldsymbol{A} = \{a, b, c\}$, $\boldsymbol{B} = \{a, b\}$. Find:

(a) $(\boldsymbol{A} \times \boldsymbol{B}) \cap (\boldsymbol{B} \times \boldsymbol{B})$
(b) $(\boldsymbol{A} \times \boldsymbol{B}) \cup (\boldsymbol{B} \times \boldsymbol{B})$
(c) $(\boldsymbol{A} \times \boldsymbol{B}) - (\boldsymbol{B} \times \boldsymbol{B})$
(d) $(\boldsymbol{A} \cap \boldsymbol{B}) \times \boldsymbol{A}$
(e) $(\boldsymbol{A} \times \boldsymbol{B}) \cap \boldsymbol{B}$
(f) $\mathscr{P}(\boldsymbol{A}) \cap \mathscr{P}(\boldsymbol{B})$
(g) $\mathscr{P}(\boldsymbol{A}) - \mathscr{P}(\boldsymbol{B})$
(h) $\mathscr{P}(\boldsymbol{A} \cap \boldsymbol{B})$
(i) $\mathscr{P}(\boldsymbol{A}) \times \mathscr{P}(\boldsymbol{B})$

Solution to Ch.1, §1.5, Exercise 4.

(a) $(\boldsymbol{A} \times \boldsymbol{B}) \cap (\boldsymbol{B} \times \boldsymbol{B})$

$$\boldsymbol{A} \times \boldsymbol{B} = \{(b, a), (b, b), (c, a), (c, b), (d, a), (d, b)\}$$

$$\boldsymbol{B} \times \boldsymbol{B} = \{(a, a), (a, b), (b, a), (b, b)\}$$

$$(\boldsymbol{A} \times \boldsymbol{B}) \cap (\boldsymbol{B} \times \boldsymbol{B}) = \{(b, a), (b, b)\}$$

(b) $(\boldsymbol{A} \times \boldsymbol{B}) \cup (\boldsymbol{B} \times \boldsymbol{B})$

$$(\boldsymbol{A} \times \boldsymbol{B}) \cup (\boldsymbol{B} \times \boldsymbol{B}) =$$

$$\{(b, a), (b, b), (c, a), (c, b), (d, a), (d, b), (a, a), (a, b)\}$$

(c) $(\boldsymbol{A} \times \boldsymbol{B}) - (\boldsymbol{B} \times \boldsymbol{B})$

$$(\boldsymbol{A} \times \boldsymbol{B}) - (\boldsymbol{B} \times \boldsymbol{B}) = /(c,a),(c,b),(d,a),(d,b)/$$

(d) $(\boldsymbol{A} \cap \boldsymbol{B}) \times \boldsymbol{A}$

$$\boldsymbol{A} \cap \boldsymbol{B} = \{b\}$$

$$(\boldsymbol{A} \cap \boldsymbol{B}) \times \boldsymbol{A} = \{(b,a),(b,c),(b,d)\}$$

(e) $(\boldsymbol{A} \times \boldsymbol{B}) \cap \boldsymbol{B}$

$$\emptyset$$

(f) $\mathscr{P}(\boldsymbol{A}) \cap \mathscr{P}(\boldsymbol{B})$

$$\mathscr{P}(\boldsymbol{A}) = \{\emptyset, \{b\}, \{c\}, \{d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{b,c,d\}\}$$

$$\mathscr{P}(\boldsymbol{B}) = \{\emptyset, \{b\}, \{a\}, \{a,b\}\}$$

$$\mathscr{P}(\boldsymbol{A}) \cap \mathscr{P}(\boldsymbol{B}) = \{\emptyset, \{b\}\}$$

(g) $\mathscr{P}(\boldsymbol{A}) - \mathscr{P}(\boldsymbol{B})$

$$\mathscr{P}(\boldsymbol{A}) - \mathscr{P}(\boldsymbol{B}) = \{\{c\}, \{d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{b,c,d\}\}$$

(h) $\mathscr{P}(\boldsymbol{A} \cap \boldsymbol{B})$

$$\mathscr{P}(\boldsymbol{A} \cap \boldsymbol{B}) = \{\emptyset, \{b\}\}$$

(i) $\mathscr{P}(\boldsymbol{A}) \times \mathscr{P}(\boldsymbol{B})$

$$\mathscr{P}(\boldsymbol{A}) \times \mathscr{P}(\boldsymbol{B}) =$$

$$\emptyset, \emptyset\{\{b\},\{a\}\}, \{\{b\},\{b\}\}, \{\{b\},\{a,b\}\}, \emptyset, \{\{c\},\{a\}\}, \{\{c\},\{b\}\},$$

$$\{\{c\},\{a,b\}\}, \emptyset, \{\{d\},\{a\}\}, \{\{d\},\{b\}\}, \{\{d\},\{a,b\}\}, \emptyset, \{\{b,c\},\{a\}\},$$

$$\{\{b,c\},\{b\}\},\{\{b,c\},\{a,b\}\},\emptyset,\{\{b,d\},\{a\}\},\{\{b,d\},\{b\}\},$$

$$\{\{b,d\},\{a,b\}\},\emptyset,\{\{b,d\},\{a\}\},\{\{b,d\},\{b\}\},\{\{b,d\},\{a,b\}\},\emptyset,$$

$$\{\{b,c,d\},\{a\}\},\{\{b,c,d\},\{b\}\},\{\{b,c,d\},\{a,b\}\}$$

$\square$

Ch.1, §1.6, Exercise 2. Let $A = \{0,2,4,6,8\}$ and $\boldsymbol{B} = \{1,2,5,7\}$ have universal set $\boldsymbol{U} = \{0,1,2,\ldots,8\}$. Find:

(a) $\overline{\boldsymbol{A}}$

(b) $\overline{\boldsymbol{B}}$

(c) $\boldsymbol{A} \cap \overline{\boldsymbol{A}}$

(d) $\boldsymbol{A} \cup \overline{\boldsymbol{A}}$

(e) $\boldsymbol{A} - \overline{\boldsymbol{A}}$

(f) $\overline{\boldsymbol{A} \cup \boldsymbol{B}}$

(g) $\overline{\boldsymbol{A}} \cap \overline{\boldsymbol{B}}$

(h) $\overline{\boldsymbol{A} \cap \boldsymbol{B}}$

(i) $\overline{\overline{\boldsymbol{A}} \cap \boldsymbol{B}}$

Solution to Ch.1, §1.6, Exercise 2.

(a) $\overline{\boldsymbol{A}}$

$$\overline{\boldsymbol{A}} = \{1,3,5,7\}$$

(b) $\overline{\boldsymbol{B}}$

$$\overline{\boldsymbol{B}} = \{0,3,4,6,8\}$$

(c) $\boldsymbol{A} \cap \overline{\boldsymbol{A}}$

$$\boldsymbol{A} \cap \overline{\boldsymbol{A}} = \emptyset$$

(d) $\boldsymbol{A} \cup \overline{\boldsymbol{A}}$

$$\boldsymbol{A} \cup \overline{\boldsymbol{A}} = \boldsymbol{U}$$

(e) $\boldsymbol{A} - \overline{\boldsymbol{A}}$

$$\boldsymbol{A} - \overline{\boldsymbol{A}} = \boldsymbol{A}$$

(f) $\overline{\boldsymbol{A} \cup \boldsymbol{B}}$

$$\boldsymbol{A} \cup \boldsymbol{B} = \{0, 1, 2, 4, 5, 6, 7, 8\}$$

$$\overline{\boldsymbol{A} \cup \boldsymbol{B}} = \{3\}$$

(g) $\overline{\boldsymbol{A}} \cap \overline{\boldsymbol{B}}$

$$\overline{\boldsymbol{A}} \cap \overline{\boldsymbol{B}} = \{3\}$$

(h) $\overline{\boldsymbol{A} \cap \boldsymbol{B}}$

$$\boldsymbol{A} \cap \boldsymbol{B} = \{2\}$$

$$\overline{\boldsymbol{A} \cap \boldsymbol{B}} = \{0, 1, 3, 4, 5, 6, 7, 8\}$$

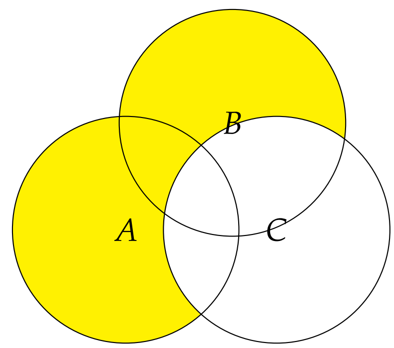
(i) $\overline{\overline{\boldsymbol{A}} \cap \boldsymbol{B}}$

$$\overline{\overline{\boldsymbol{A}} \cap \boldsymbol{B}} = \{1, 5, 7\}$$

$\square$

Ch.1, §1.7, Exercise 4. Draw a Venn Diagram for $(\boldsymbol{A} \cup \boldsymbol{B}) - \boldsymbol{C}$.

Solution to Ch.1, §1.5, Exercise 4.

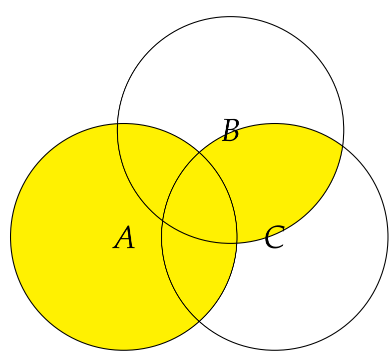


□

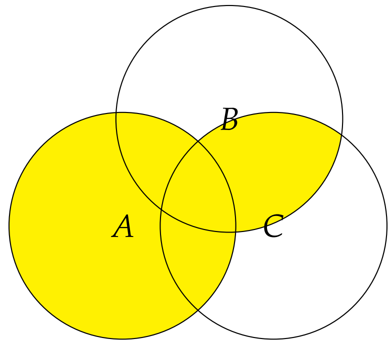
Ch.1, §1.7, Exercise 5. Draw a Venn Diagram for $\boldsymbol{A} \cup (\boldsymbol{B} \cap \boldsymbol{C})$ and $(\boldsymbol{A} \cup \boldsymbol{B}) \cap (\boldsymbol{A} \cup \boldsymbol{C})$. Base on your drawing, do you think $\boldsymbol{A} \cup (\boldsymbol{B} \cap \boldsymbol{C}) = (\boldsymbol{A} \cup \boldsymbol{B}) \cap (\boldsymbol{A} \cup \boldsymbol{C})$?

Solution to Ch.1, §1.7, Exercise 5.

$$\boldsymbol{A} \cup (\boldsymbol{B} \cap \boldsymbol{C})$$



$$(\boldsymbol{A} \cup \boldsymbol{B}) \cap (\boldsymbol{A} \cup \boldsymbol{C})$$

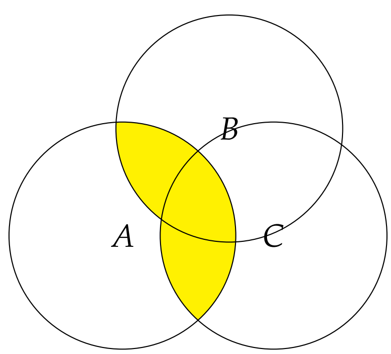


∎

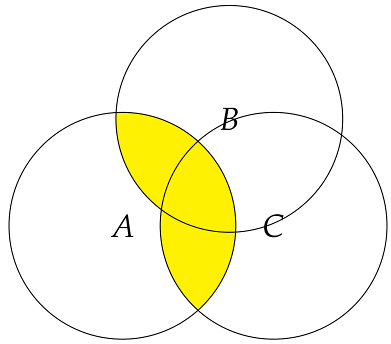
Ch.1, §1.7, Exercise 6. Draw a Venn Diagram for $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$. Base on your drawing, do you think $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$?

Solution to Ch.1, §1.7, Exercise 6.

$$A \cap (B \cup C)$$



$$(A \cap B) \cup (A \cap C)$$



$\square$

Ch.1, §1.8, Exercise 2. Suppose $\begin{cases} A_1 = \{0,2,4,8,10,12,14,16,18,20,22,24\}, \\ A_2 = \{0,3,6,9,12,15,21,24\}, \\ A_3 = \{0,4,8,12,16,20,24\}. \end{cases}$

(a) $\bigcup_{i=1}^{3} A_i$

(b) $\bigcap_{i=1}^{3} A_i$

Solution to Ch.1, §1.8, Exercise 2.

(a) $\bigcup_{i=1}^{3} A_i$

$$\{0,2,3,4,6,8,9,10,12,14,15,16,18,20,21,22,24\}$$

(b) $\bigcap_{i=1}^{3} A_i$

$$\{0,12,24\}$$

$\square$

Ch.1, §1.8, Exercise 11. $\bigcup_{\alpha\in I} A_\alpha \subseteq \bigcap_{\alpha\in I} A_\alpha$ always true for any collection of sets A_α with index set I?

Solution to Ch.1, §1.8, Exercise 11.

It is always true that $\bigcup_{\alpha \in I} \boldsymbol{A}_\alpha \subseteq \boldsymbol{A}_\alpha$ and $\boldsymbol{A}_\alpha \subseteq \bigcap_{\alpha \in I} \boldsymbol{A}_\alpha$, so $\bigcup_{\alpha \in I} \boldsymbol{A}_\alpha \subseteq \bigcap_{\alpha \in I} \boldsymbol{A}_\alpha$ is always true.

□

Ch.1, §1.8, Exercise 12. $\bigcup_{\alpha \in I} \boldsymbol{A}_\alpha = \bigcap_{\alpha \in I} \boldsymbol{A}_\alpha$, what do you think can be said about the relationships between the sets $\boldsymbol{A}_\alpha$?

Solution to Ch.1, §1.5, Exercise 4.

They are all equal.

□

References

[Ham18] Richard Hammack, *Book of Proof*, 3 ed., Creative Commons, 2018.

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