# HOMEWORK 8

# MATH 2001

QI WANG

ABSTRACT. This is the first homework assignment. The problems are from Hammack [Ham18, Ch. 1, §1.1]:

- **Chapter 7** Exercises: 19, 20, 21
- **Chapter 8** Exercises: 2, 6, 10, 12, 14, 16, 18

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## Chapter 7

**Ch.7, Exercise 19.** If $n \in \mathbb{Z}$, then $2^0 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1$.

*Solution to Ch.7, Exercise 19.*

**Proposition** If $n \in \mathbb{Z}$, then $2^0 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1$.

*Proof*:

$n = 0 :\ 2^0 = 2^1 - 1 = 1$

$n = 1 :\ 2^0 + 2^1 = 2^2 - 1 = 3$

$n = 2 :\ 2^0 + 2^1 + 2^2 = 2^3 - 1 = 7$

*Assume*:

$n = k - 1 :\ 2^0 + 2^1 + 2^2 + \cdots + 2^{k-1} = 2^k - 1$

*Induction Proof*:

$n = k :\ 2^0 + 2^1 + 2^2 + \cdots + 2^{k-1} + 2^k = 2^k - 1 + 2^k = 2^{k+1} - 1$

$\square$

**Ch.7, Exercise 20.** There exists an $n \in \mathbb{N}$ for which $11|(2^n - 1)$.

*Solution to Ch.7, Exercise 20.*

**Proposition** There exists an $n \in \mathbb{N}$ for which $11|(2^n - 1)$.

*Proof* (direct) Because zero divides by eleven equals to zero, we have $11|0$. Let $2^n - 1 = 0$, so we get $n = 0$. Thus, there exist an $n$ when $n = 0$ for $11|(2^n - 1)$.

$\square$

**Ch.8, Exercise 2.** Prove that $\{6n : n \in \mathbb{Z}\} = \{2n : n \in \mathbb{Z}\} \wedge \{3n : n \in \mathbb{Z}\}$.

*Solution to Ch.8, Exercise 2.*

**Proposition** Prove that $\{6n : n \in \mathbb{Z}\} = \{2n : n \in \mathbb{Z}\} \wedge \{3n : n \in \mathbb{Z}\}$.

*Proof*

**Step 1**: Suppose $a \in \{6n : n \in \mathbb{Z}\}$, we have $a = 6n = 2(3n)$. Thus $a = 2(b)$ where $b = 3n \in \mathbb{Z}$, so $a \in \{2n : n \in \mathbb{Z}\}$. We also have $a = 6n = 3(2n)$, so $a = 3c$ where $c = 2n \in \mathbb{Z}$. Thus $a \in \{3n : n \in \mathbb{Z}\}$. Therefor $\{6n : n \in \mathbb{Z}\} \subseteq \{2n : n \in \mathbb{Z}\} \wedge \{3n : n \in \mathbb{Z}\}$.

**Step 2**: Suppose $a \in \{2n : n \in \mathbb{Z}\} \wedge \{3n : n \in \mathbb{Z}\}$. Then, we have $a = 2b$ and $a = 3c$ where $b, c \in \mathbb{Z}$. Thus $2|a$ and $3|a$, so $(2*3)|a = 6|a$. Therefore $a = 6d$ for $d \in \mathbb{Z}$. Thus $a \in \{6n : n \in \mathbb{Z}\}$, so $\{2n : n \in \mathbb{Z}\} \wedge \{3n : n \in \mathbb{Z}\} \subseteq \{6n : n \in \mathbb{Z}\}$.

$\square$

**Ch.8, Exercise 6.** Suppose $x, y \in \mathbb{R}$. Then $x^3 + x^2y = y^2 + xy$ if and only if $y = x^2$ or $y = -x$.

*Solution to Ch.1, §1.1, Exercise 30.*

**Proposition** Suppose $x, y \in \mathbb{R}$. Then $x^3 + x^2y = y^2 + xy$ if and only if $y = x^2$ or $y = -x$.

*Proof* Rearranging the equation $x^3 + x^2y = y^2 + xy$, we get $x^3 - xy = y^2 - x^2y$. Thus $x(x^2 - y) = y(y - x^2)$, so we have $x(x^2 - y) = -y(x^2 - y)$. Therefore we solve the equation get either $y = x^2$ or $y = -x$. $\square$

**Ch.8, Exercise 10.** If $a \in \mathbb{Z}$, then $a^3 \equiv a(mod 3)$.

*Solution to Ch.8, Exercise 12.*

**Proposition** If $a \in \mathbb{Z}$, then $a^3 \equiv a(mod\ 3)$.

*Proof (proof by case)*

**case 1:** Suppose $a \equiv 1(mod\ 3)$. We get $a = 3n + 1, n \in \mathbb{Z}$. Thus

$a^3 = 27n^3 + 27n^2 + 9b + 1 = 3(9b^3 + 9b^2 + 3b) + 1$ where $9b^3 + 9b^2 + 3b \in \mathbb{Z}$, so $a^3 \equiv 1(mod\ 3)$. Therefor $a^3 \equiv a(mod\ 3)$.

**case 2:** Suppose $a \equiv 2(mod\ 3)$. We get $a = 3n + 2, n \in \mathbb{Z}$. Thus $a^3 = 27n^3 + 54n^2 + 36b + 8 = 3(9b^3 + 18b^2 + 12b + 2) + 2$ where $9b^3 + 18b^2 + 12b + 2 \in \mathbb{Z}$, so $a^3 \equiv 2(mod\ 3)$. Therefor $a^3 \equiv a(mod\ 3)$.

**case 3:** Suppose $a \equiv 0(mod\ 3)$. We get $a = 3n, n \in \mathbb{Z}$. Thus $a^3 = 27n^3 = 3(9b^3)$ where $9b^3 \in \mathbb{Z}$, so $a^3 \equiv 0(mod\ 3)$. Therefor $a^3 \equiv a(mod\ 3)$.

For each case $a^3 \equiv a(mod\ 3)$. □

**Ch.8, Exercise 12.** There exist a positive real number $x$ for which $x^2 < \sqrt{x}$.

*Solution to Ch.8, Exercise 12.*

**Proposition** There exist a positive real number $x$ for which $x^2 < \sqrt{x}$.

*Proof* Assume $x = 0.25$, then $x^2 = 0.0625$ and $\sqrt{x} = 0.5$.

□

**Ch.8, Exercise 14.** Suppose $a \in \mathbb{Z}$. Then $a^2 | a$ if and only if $a \in \{-1, 0, 1\}$.

*Solution to Ch.8, Exercise 14.*

**Proposition** Suppose $a \in \mathbb{Z}$. Then $a^2 | a$ if and only if $a \in \{-1, 0, 1\}$.

*Proof* Assume $a^2 | a$, we get $a/a^2 = n$, where $n \in \mathbb{Z}$. By solving the equation, we get $a = 1, a = -1$ or $a = 0$.

□

**Ch.8, Exercise 16.** Suppose $a, b \in \mathbb{Z}$. If $ab$ is odd, then $a^2 + b^2$ is even.

*Solution to Ch.8, Exercise 16.*

**Proposition** Suppose $a, b \in \mathbb{Z}$. If $ab$ is odd, then $a^2 + b^2$ is even.

*Proof* Suppose $a$ and $b$ are odd. We have $a = 2n + 1$ and $b = 2m + 1$. Thus $a^2 + b^2 = 4n^2 + 4n + 1 + 4m^2 + 4m + 1 = 2(2n^2 + 2n + 2m^2 + 2m + 1)$ where $2n^2 + 2n + 2m^2 + 2m + 1 \in \mathbb{Z}$. Therefore $a^2 + b^2$ is even.

$\square$

**Ch.8, Exercise 18.** There is a set $X$ for which $\mathbb{N} \in X$ and $\mathbb{N} \subseteq X$.

*Solution to Ch.8, Exercise 18.*

**Proposition** There is a set $X$ for which $\mathbb{N} \in X$ and $\mathbb{N} \subseteq X$.

*Proof* There is $\wp\mathbb{N}$ which $\mathbb{N} \in \wp\mathbb{N}$ and $\mathbb{N} \subseteq \wp\mathbb{N}$.

$\square$

## References

[Ham18] Richard Hammack, *Book of Proof*, 3 ed., Creative Commons, 2018.

University of Colorado, Department of Mathematics, Campus Box 395, Boulder, CO 80309-0395

*Email address*: `casa@math.colorado.edu`