HOMEWORK 7

MATH 2001

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ABSTRACT. This is the first homework assignment. The problems are from Hammack [Ham18, Ch.5]:

- **Chapter 5** , Exercises: 1, 2, 3, 4, 5, 16, 17, 18, 19, 20

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Chapter 5

Ch.5, Exercise 1. Suppose $n \in \mathbb{Z}$. If n^2 is even, then n is even.

Solution to Ch.5, Exercise 1.

Proposition Suppose $n \in \mathbb{Z}$. If n^2 is even, then n is even.

Proof. (Contrapositive) Suppose n is odd, then $n = 2a + 1$ for some $a \in \mathbb{Z}$ by definition. Thus $n^2 = 4a^2 + 4a + 1 = 2(a^2 + 2a) + 1$, so $n^2 = 2b + 1$ where $b = a^2 + 2a \in \mathbb{Z}$. Therefor n^2 is odd by definition.

□

Ch.5, Exercise 2. Suppose $n \in \mathbb{Z}$. If n^2 is odd, then n is odd.

Solution to Ch.5, Exercise 2.

Proposition Suppose $n \in \mathbb{Z}$. If n^2 is odd, then n is odd.

Proof. (Contrapositive) Suppose n is even, then $n = 2a, a \in \mathbb{Z}$ by definition. Thus $n^2 = 4a^2 = 2(2a^2)$, so $n^2 = 2b$ where $b = 2a^2 \in \mathbb{Z}$. Therefor n^2 is even by definition.

□

Ch.5, Exercise 3. Suppose $a, b \in \mathbb{Z}$. If $a^2(b^2 - 2b)$ is odd, then a and b are odd.

Solution to Ch.5, Exercise 3.

Proposition Suppose $a, b \in \mathbb{Z}$. If $a^2(b^2 - 2b)$ is odd, then a and b are odd.

Proof. (Contrapositive) Suppose a and b are even, then $a = 2c$ and $b = 2d$ for $c, d \in \mathbb{Z}$ by definition. Thus $a^2(b^2 - 2b) = 4c^2(4d^2 - 4d) = 2(2c^2(4d^2 - 4d))$, so $a^2(b^2 - 2b) = 2q$ where $q = 2c^2(4d^2 - 4d) \in \mathbb{Z}$. Therefor $a^2(b^2 - 2b)$ is even by definition.

□

Ch.5, Exercise 4. Suppose $a, b, c \in \mathbb{Z}$. If a does not divide bc, then a does not divide b.

Solution to Ch.5, Exercise 4.

Proposition Suppose $a, b, c \in \mathbb{Z}$. If a does not divide bc, then a does not divide b.

Proof. (Contrapositive) Suppose a divide b, then $b = qa$ for some $q \in \mathbb{Z}$. Thus $bc = qac = (qc)a$, so $bc = ra$ where $r = qc$. There for a divide bc by definition.

□

Ch.5, Exercise 5. Suppose $x \in \mathbb{R}$. If $x^2 + 5x < 0$ then $x < 0$.

Solution to Ch.5, Exercise 5.

Proposition Suppose $x \in \mathbb{R}$. If $x^2 + 5x < 0$ then $x < 0$.

Proof. (Contrapositive) Suppose $x \geq 0$. Then $x^2 \geq 0$ and $5x \geq 0$, so $x^2 + 5x \geq 0$.

□

Ch.5, Exercise 16. Suppose $x, y \in \mathbb{Z}$. If $x + y$ is even, then x and y have the same parity.

Solution to Ch.5, Exercise 16.

Proposition Suppose $x, y \in \mathbb{Z}$. If $x + y$ is even, then x and y have the same parity.

Proof. (Contrapositive) Suppose x and y have different parity (x is even, y is odd), then $x = 2a, a \in \mathbb{Z}$ and $y = 2b + 1, b \in \mathbb{Z}$ by definition. Thus $x + y = 2a + 2b + 1 = 2(a + b) + 1$, so $x + y = 2c + 1$

where $c = a + b \in \mathbb{Z}$. Therefor $x + y$ is odd. We get same proof when x is odd and y is even.

$\square$

Ch.5, Exercise 17. If n is odd, then $8|(n^2 - 1)$.

Solution to Ch.5, Exercise 17.

Proposition If n is odd, then $8|(n^2 - 1)$

Proof. Suppose n is odd, then $n = 2a + 1$ for some $a \in \mathbb{Z}$ by definition. Thus $n^2 - 1 = 4a^2 + 4a = 4(a^2 + a) = 4(a(a + 1))$. Because $a(a + 1)$ is even, we have $a(a + 1) = 2b$ for some $b \in \mathbb{Z}$. Thus $n^2 - 1 = 4(2b) = 8b$. Therefor $8|(n^2 - 1)$ by definition.

$\square$

Ch.5, Exercise 18. If $a, b \in \mathbb{Z}$, then $(a + b)^3 \equiv a^3 + b^3 (mod 3)$.

Solution to Ch.5, Exercise 18.

Proposition If $a, b \in \mathbb{Z}$, then $(a + b)^3 \equiv a^3 + b^3 (mod 3)$.

Proof. Suppose $a, b \in \mathbb{Z}$, then $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = (a^3 + b^3) + 3(a^2b + ab^2)$. Thus $(a + b)^3 (mod 3) = (a^3 + b^3)(mod 3) + 3(a^2b + ab^2)(mod 3) = (a^3 + b^3)(mod 3) + 0 = (a^3 + b^3)(mod 3)$ by definition. Therefor $(a + b)^3 \equiv a^3 + b^3 (mod 3)$.

$\square$

Ch.5, Exercise 19. Let $a, b, c \in \mathbb{Z}$ and $n \in \mathbb{N}$. If $a \equiv b(mod n)$ and $a \equiv c(mod n)$, then $c \equiv b(mod n)$.

Solution to Ch.5, Exercise 19.

Proposition Let $a, b, c \in \mathbb{Z}$ and $n \in \mathbb{N}$. If $a \equiv b(mod n)$ and $a \equiv c(mod n)$, then $c \equiv b(mod n)$.

Proof. Suppose $a \equiv b(mod n)$ and $a \equiv c(mod n)$, then $a(mod n) = b(mod n)$ and $a(mod n) = c(mod n)$ by definition. Thus $a(mod n) = c(mod n)$. Therefor $a \equiv c(mod n)$ by definition.

□

Ch.5, Exercise 20. If $a \in \mathbb{Z}$ and $a \equiv 1(mod 5)$, then $a^2 \equiv 1(mod 5)$.

Solution to Ch.5, Exercise 20.

Proposition If $a \in \mathbb{Z}$ and $a \equiv 1(mod 5)$, then $a^2 \equiv 1(mod 5)$.

Proof. Suppose $a \equiv 1(mod 5)$, then $a(mod 5) = 1(mod 5) = 1$ by definition. Thus $a = 5b + 1$ for some $b \in \mathbb{Z}$ by definition. Thus $a^2 = 25b^2 + 10b + 1 = 5(5b^2 + 2b) + 1$, so $a^2 = 5c + 1$ where $c = 5b^2 + 2b \in \mathbb{Z}$. Thus $a^2(mod 5) = 1(mod 5) = 1$. Therefor $a \equiv 1(mod 5)$ by definition.

□

References

[Ham18] Richard Hammack, *Book of Proof*, 3 ed., Creative Commons, 2018.

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