GAME THEORY

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Part III. Two-Person General-Sum Games

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1. Bimatrix Games — Safety Levels

The simplest case to consider beyond two-person zero-sum games are the two-person non-zero-sum games. Examples abound in Economics: the struggle between labor and management, the competition between two producers of a single good, the negotiations between buyer and seller, and so on. Good reference material may be found in books of Owen and Straffin already cited. The material treated in the rest of the course is much more oriented to economic theory. For a couple of good references with emphasis on applications in economics, consult the books, *Game Theory for Applied Economists* by Robert Gibbons (1992), Princeton University Press, and *Game Theory with Economic Applications* by H. Scott Bierman and Luis Fernandez (1993), Addison-Wesley Publishing Co. Inc.

1.1 General-Sum Strategic Form Games. Two-person general-sum games may be defined in extensive form or in strategic form. The *normal* or *strategic* form of a two-person game is given by two sets X and Y of pure strategies of the players, and two real-valued functions $u_1(x, y)$ and $u_2(x, y)$ defined on $X \times Y$, representing the payoffs to the two players. If I chooses $x \in X$ and II chooses $y \in Y$, then I receives $u_1(x, y)$ and II receives $u_2(x, y)$.

A finite two-person game in strategic form can be represented as a matrix of ordered pairs, sometimes called a bimatrix. The first component of the pair represents Player I's payoff and the second component represents Player II's payoff. The matrix has as many rows as Player I has pure strategies and as many columns as Player II has pure strategies. For example, the bimatrix

$$\begin{pmatrix} (1,4) & (2,0) & (-1,1) & (0,0) \\ (3,1) & (5,3) & (3,-2) & (4,4) \\ (0,5) & (-2,3) & (4,1) & (2,2) \end{pmatrix} \tag{1}$$

represents the game in which Player I has three pure strategies, the rows, and Player II has four pure strategies, the columns. If Player I chooses row 3 and Player II column 2, then I receives -2 (i.e. he loses 2) and Player II receives 3.

An alternative way of describing a finite two person game is as a pair of matrices. If m and n representing the number of pure strategies of the two players, the game may be represented by two $m \times n$ matrices $\boldsymbol{A}$ and $\boldsymbol{B}$. The interpretation here is that if Player I chooses row i and Player II chooses column j, then I wins a_{ij} and II wins b_{ij}, where a_{ij} and b_{ij} are the elements in the ith row, jth column of $\boldsymbol{A}$ and $\boldsymbol{B}$ respectively. Note that $\boldsymbol{B}$ represents the winnings of Player II rather than her losses as would be the case for a zero-sum game. The game of bimatrix (1) is represented as $(\boldsymbol{A}, \boldsymbol{B})$, where

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 3 & 5 & 3 & 4 \\ 0 & -2 & 4 & 2 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{B} = \begin{pmatrix} 4 & 0 & 1 & 0 \\ 1 & 3 & -2 & 4 \\ 5 & 3 & 1 & 2 \end{pmatrix} \tag{2}$$

Note that the game is zero-sum if and only if the matrix $\boldsymbol{B}$ is the negative of the matrix $\boldsymbol{A}$, i.e. $\boldsymbol{B} = -\boldsymbol{A}$.

1.2 General-Sum Extensive Form Games. The extensive form of a game may be defined in the same manner as it was defined in Part II. The only difference is that since now the game is not zero-sum, the payoff cannot be expressed as a single number. We must indicate at each terminal vertex of the Kuhn tree how much is won by Player I and how much is won by Player II. We do this by writing this payoff as an ordered pair of real numbers whose first component indicates the amount that is paid to Player I, and whose second component indicates the amount paid to player II. The following Kuhn tree is an example.

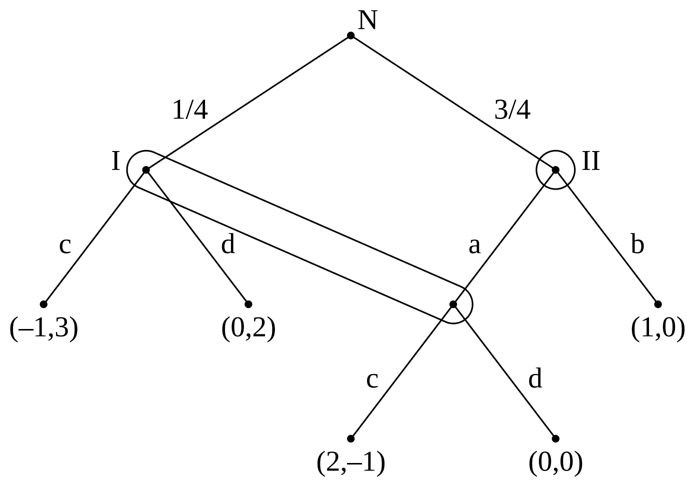


Figure 1.1.

If the first move by chance happens to go down to the right, if Player II chooses a, and if Player I happens to choose c, the payoff is $(2, -1)$, which means that Player I wins 2 and Player II loses 1. Note that the second component represents Player II's winnings rather than losses. In particular, a game is a zero-sum game if and only if the components of each payoff vector sum to zero.

1.3 Reducing Extensive Form to Strategic Form. The problem of reducing a general sum game in extensive form to one in strategic form is solved in a completely similar manner as for the case of a zero-sum game. The only difference is that the payoffs are ordered pairs. If there are random moves, the outcome is a random distribution over these ordered pairs which is replaced by the average of the ordered pairs. This is done by taking the corresponding average over each component of the pair separately.

As an illustration, consider the game of Figure 1. Player I has two pure strategies, $X = \{c, d\}$, and Player II has two pure strategies, $Y = \{a, b\}$. The corresponding strategic form of this game is given by the 2×2 bimatrix,

$$\begin{array}{c c} & \begin{array}{cc} a & \qquad b \end{array} \\ \begin{array}{c} c \\ d \end{array} & \begin{pmatrix} (5/4, 0) & (2/4, 3/4) \\ (0, 2/4) & (3/4, 2/4) \end{pmatrix} \end{array}. \tag{3}$$

For example, the component of the first row, first column is computed as follows. One-fourth of the time nature goes left, and I uses c, resulting in a payoff of $(-1,3)$. Three-fourths of the time nature goes right, Player II uses a and Player I uses c, giving a payoff of $(2,-1)$. Therefore, the average payoff is $(1/4)(-1,3)+(3/4)(2,-1) = (-1/4+6/4, 3/4-3/4) = (5/4,0)$. The other components of the bimatrix are computed similarly.

1.4 Overview. The analysis of two-person games is necessarily more complex for general-sum games than for zero-sum games. When the sum of the payoffs is no longer zero (or constant), maximizing one's own payoff is no longer equivalent to minimizing the opponent's payoff. The minimax theorem does not apply to bimatrix games. One can no longer expect to play "optimally" by simply looking at one's own payoff matrix and guarding against the worst case. Clearly, one must take into account the opponent's matrix and the reasonable strategy options of the opponent. In doing so, we must remember that the opponent is doing the same. The general-sum case requires other more subtle concepts of solution.

The theory is generally divided into two branches, the *noncooperative theory* and the *cooperative theory*. In the noncooperative theory, either the players are unable to communicate before decisions are made, or if such communication is allowed, the players are forbidden or are otherwise unable to make a binding agreement on a joint choice of strategy. The main noncooperative solution concept is the *strategic equilibrium*. This theory is treated in the next two chapters. In the cooperative theory, it is assumed that the players are allowed to communicate before the decisions are made. They may make threats and counterthreats, proposals and counterproposals, and hopefully come to some compromise. They may jointly agree to use certain strategies, and it is assumed that such an agreement can be made binding.

The cooperative theory itself breaks down into two branches, depending on whether or not the players have comparable units of utility and are allowed to make monetary *side payments* in units of utility as an incentive to induce certain strategy choices. The corresponding solution concept is called the *TU cooperative value* if side payments are allowed, and the *NTU cooperative value* if side payments are forbidden or otherwise unattainable. The initials TU and NTU stand for "transferable utility" and "non-transferable utility" respectively.

1.5 Safety Levels. One concept from zero-sum games carries over and finds important use in general sum games. This is the safety level, or the amount that each player can guarantee winning on the average. In a bimatrix game with $m \times n$ matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, Player I can guarantee winning on the average at least

$$v_I = \max_{\boldsymbol{p}} \min_j \sum_{i=1}^{m} p_i a_{ij} = \mathrm{Val}(\boldsymbol{A}). \tag{4}$$

This is called the **safety level of Player I**. (This is by definition the lower value of $\boldsymbol{A}$, which by the minimax theorem is also the upper value or the value of $\boldsymbol{A}$. So we may write $v_I = \mathrm{Val}(\boldsymbol{A})$.) Player I can achieve this payoff without considering the payoff matrix of

Player II. A strategy, $\boldsymbol{p}$, that achieves the maximum in (4) is called a **maxmin strategy for Player I**.

Similarly, the **safety level of Player II** is

$$v_{II} = \max_{\boldsymbol{q}} \min_{i} \sum_{j=1}^{n} b_{ij} q_j = \operatorname{Val}(\boldsymbol{B}^{\mathsf{T}}), \tag{5}$$

since Player II can guarantee winning this amount on the average. Any strategy $\boldsymbol{q}$, that achieves the maximum in (5) is a **maxmin strategy for Player II**. (Note as a technical point that v_{II} is the value of $\boldsymbol{B}^{\mathsf{T}}$, the transpose of $\boldsymbol{B}$. It is not the value of $\boldsymbol{B}$. This is because $\operatorname{Val}(\boldsymbol{B})$ is defined as the value of a game where the components represent winnings of the row chooser and losses of the column chooser.) An example should clarify this.

Consider the example of the following game.

$$\begin{pmatrix} (2,0) & (1,3) \\ (0,1) & (3,2) \end{pmatrix} \qquad \text{or} \qquad \boldsymbol{A} = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}.$$

From the matrix $\boldsymbol{A}$, we see that Player I's maxmin strategy is $(3/4, 1/4)$ and his safety level is $v_I = 3/2$. From the matrix $\boldsymbol{B}$, we see the second column dominates the first. (Again these are II's winnings; she is trying to maximize) Player II guarantees winning at least $v_{II} = 2$ by using her maxmin strategy, namely column 2. Note that this is the value of $\boldsymbol{B}^{\mathsf{T}}$ (whereas $\operatorname{Val}(\boldsymbol{B}) = 1$).

Note that if both players use their maxmin strategies, then Player I only gets v_I, whereas Player II gets $(3/4)3 + (1/4)2 = 11/4$. This is pleasant for Player II. But if Player I looks at $\boldsymbol{B}$, he can see that II is very likely to choose column 2 because it strictly dominates column 1. Then Player I would get 3 which is greater than v_I, and Player II would get $v_{II} = 2$.

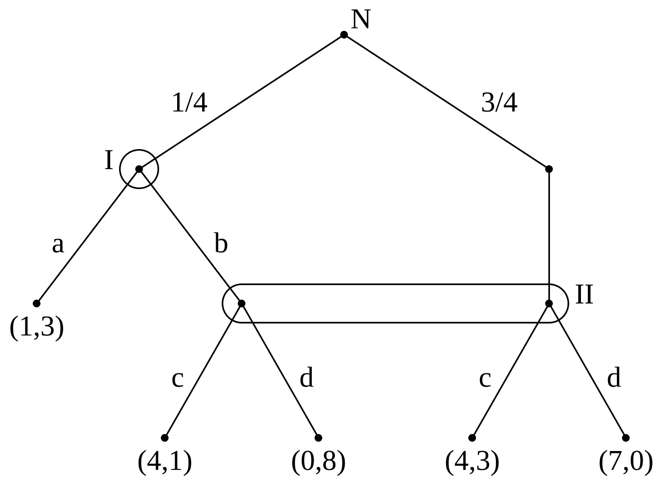
The payoff (3,2) from the second row, second column, is rather stable. If each believes the other is going to choose the second strategy, then each would choose the second strategy. This is one of the main viewpoints of noncooperative game theory, where such a strategy pair is called a strategic equilibrium.

In TU cooperative game theory, where the units used to measure I's payoff are assumed to be the same as the units used to measure Player II's payoff, the players will jointly agree on (3,2), because it gives the largest sum, namely 5. However, in the agreement the players must also specify how the 5 is to be divided between the two players. The game is not symmetric; Player II has a threat to use column 1 and Player I has no similar threat. We will see later some of the suggestions on how to split the 5 between the players.

The NTU theory is more complex since it is assumed that the players measure their payoffs in noncomparable units. Side payments are not feasible or allowed. Any deviation from the equilibrium (3,2) would have to be an agreed upon mixture of the other three payoffs. (The only one that makes sense to mix with (3,2) is the payoff (1,3)).

1.6 Exercises.

1. Convert the following extensive form game to strategic form.



2. Find the safety levels and maxmin strategies for the players in the bimatrix games,

(a) $\begin{pmatrix} (1,1) & (5,0) \\ (0,5) & (4,4) \end{pmatrix}$.

(b) $\begin{pmatrix} (3,10) & (1,5) \\ (2,0) & (4,20) \end{pmatrix}$.

3. Contestants I and II in a game show start the last round with winnings of \$400 and \$500 dollars respectively. Each must decide to pass or gamble, not knowing the choice of the other. A player who passes keeps the money he/she started with. If Player I gambles, he wins \$200 with probability 1/2 or loses his entire \$400 with probability 1/2. If Player II gambles, she wins or loses \$200 with probability 1/2 each. These outcomes are independent. Then the contestant with the higher amount at the end wins a bonus of \$400.
(a) Draw the Kuhn tree.
(b) Put into strategic form.
(c) Find the safety levels.

4. *A Coordination Game.* The following game is a coordination game. The safety levels and maxmin strategies for the players indicate that the first row, first column would be chosen giving both players 4. Yet if they could coordinate on the second row, second column, they would receive 6 each.

$$\begin{pmatrix} (4,4) & (4,0) \\ (0,4) & (6,6) \end{pmatrix}$$

Suppose you, as row chooser, are playing this game once against a person chosen at random from this class. Which row would you choose? or, if you prefer, which mixed strategy would you use? Your score on this question depends on what the other students in the class do. You must try to predict what they are going to do. Do not reveal your answer to this question to the other students in the class.

2. Noncooperative Games

Two-person general-sum games and n-person games for $n > 2$ are more difficult to analyze and interpret than the zero-sum two-person games of Part II. The notion of "optimal" behavior does not extend to these more complex situations. In the noncooperative theory, it is assumed that the players cannot overtly cooperate to attain higher payoffs. If communication is allowed, no binding agreements may be formed. One possible substitute for the notion of a "solution" of a game is found in the notion of a strategic equilibrium.

2.1 Strategic Equilibria. A finite n-person game in strategic form is given by n nonempty finite sets, X_1, X_2, ..., X_n, and n real-valued functions u_1, u_2, ..., u_n, defined on $X_1 \times X_2 \times \cdots \times X_n$. The set X_i represents the pure strategy set of player i and $u_i(x_1, x_2, \ldots, x_n)$ represents the payoff to player i when the pure strategy choices of the players are x_1, x_2, ..., x_n, with $x_j \in X_j$ for $j = 1, 2, \ldots, n$.

Definition. *A vector of pure strategy choices $(x_1, x_2, \ldots, x_n)$ with $x_i \in X_i$ for $i = 1, \ldots, n$ is said to be a pure strategic equilibrium, or PSE for short, if for all $i = 1, 2, \ldots, n$, and for all $x \in X_i$,*

$$u_i(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \geq u_i(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n). \tag{1}$$

Equation (1) says that if the players other than player i use their indicated strategies, then the best player i can do is to use x_i. Such a pure strategy choice of player i is called a **best response** to the strategy choices of the other players. The notion of strategic equilibrium may be stated: a particular selection of strategy choices of the players forms a PSE if each player is using a best response to the strategy choices of the other players.

Consider the following examples with two players,

$$\text{(a)} \quad \begin{pmatrix} (3,3) & (0,0) \\ (0,0) & (5,5) \end{pmatrix} \qquad\qquad \text{(b)} \quad \begin{pmatrix} (3,3) & (4,3) \\ (3,4) & (5,5) \end{pmatrix}$$

In (a), the first row, first column, denoted $\langle 1, 1\rangle$, is a strategic equilibrium with equilibrium payoff $(3, 3)$. If each believes the other is going to choose the first strategy, neither player will want to change to the second strategy. The second row, second column, $\langle 2, 2\rangle$, is also a strategic equilibrium. Since its equilibrium payoff is $(5, 5)$, both players prefer this equilibrium. In (b), the first row, first column, $\langle 1, 1\rangle$, is still an equilibrium according to the definition. Neither player can gain by changing strategy. On the other hand, neither player can be hurt by changing, and if they both change, they both will be better off. So the equilibrium $\langle 1, 1\rangle$ is rather unstable.

Example (a) is of a game in which the players receive the same payoff, but are not allowed to communicate. If they were allowed to communicate, they would choose the joint action giving the maximum payoff. Other examples of this nature occur in the class of rendezvous games, in which two players randomly placed on a graph, each not knowing the position of the other, want to meet in minimum time. See the book of Alpern and Gal (2003).

If players in a noncooperative game are allowed to communicate and do reach some informal agreement, it may expected to be a strategic equilibrium. Since no binding agreements may be made, the only agreements that may be expected to occur are those that are **self-enforcing**, in which no player can gain by unilaterally violating the agreement. Each player is maximizing his return against the strategy the other player announced he will use.

It is useful to extend this definition to allow the players to use mixed strategies. We denote the set of probabilities over k points by $\mathcal{P}_k$:

$$\mathcal{P}_k = \{\boldsymbol{p} = (p_1, \ldots, p_k) : \ p_i \geq 0 \text{ for } i = 1, \ldots, k, \text{ and } \textstyle\sum_1^k p_i = 1\}. \tag{2}$$

Let m_i denote the number of pure strategy choices of player i, so that the set X_i has m_i elements. Then the set of mixed strategies of player i is just $\mathcal{P}_{m_i}$. It is denoted by X_i^* where $X_i^* = \mathcal{P}_{m_i}$.

We denote the set of elements of X_i by the first m_i integers, $X_i = \{1, 2, \ldots, m_i\}$. Suppose that for $i = 1, 2, \ldots, n$, Player i uses $\boldsymbol{p}_i = (p_{i1}, p_{i2}, \ldots, p_{im_i}) \in X_i^*$. Then the average payoff to player j is

$$g_j(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n) = \sum_{i_1=1}^{m_1} \cdots \sum_{i_n=1}^{m_n} p_{1i_1} \cdots p_{ni_n} u_j(i_1, \ldots, i_n). \tag{3}$$

Then the analogous definition of equilibrium using mixed strategies is as follows.

Definition. *A vector of mixed strategy choices $(\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n)$ with $\boldsymbol{p}_i \in X_i^*$ for $i = 1, \ldots, n$ is said to be a strategic equilibrium, or SE for short, if for all $i = 1, 2, \ldots, n$, and for all $\boldsymbol{p} \in X_i^*$,*

$$g_i(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{i-1}, \boldsymbol{p}_i, \boldsymbol{p}_{i+1}, \ldots, \boldsymbol{p}_n) \geq g_i(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{i-1}, \boldsymbol{p}, \boldsymbol{p}_{i+1}, \ldots, \boldsymbol{p}_n). \tag{4}$$

Any mixed strategy $\boldsymbol{p}_i$ that satisfies (4) for all $\boldsymbol{p} \in X_i^*$ is a **best response** of player i to the mixed strategies of the other players. Thus, a particular selection of mixed strategy choices of the players forms an SE if and only if each player is using a best response to the strategy choices of the other players. No player can gain by unilaterally changing strategy. Note that a PSE is a special case of an SE.

This notion of best response represents a practical way of playing a game: *Make a guess at the probabilities that you think your opponents will play their various pure strategies, and choose a best response to this.* This is an example of the famous **Bayesian** approach to decision making. Of course in a game, this may be a dangerous procedure. Your opponents may be better at this type of guessing than you.

The first question that arises is "Do there always exist strategic equilibria?". This question was resolved in 1951 by John Nash in the following theorem which generalizes von Neumann's minimax theorem. In honor of this achievement, strategic equilibria are also called **Nash equilibria**.

Theorem. *Every finite n-person game in strategic form has at least one strategic equilibrium.*

A proof of this theorem using the Brouwer Fixed Point Theorem is given in Appendix 3. This proof is an existence proof and gives no indication of how to go about finding equilibria. However, in the case of bimatrix games where $n = 2$, the Lemke-Howson algorithm may be used to compute strategic equilibria in a finite number of steps using a simplex-like pivoting algorithm (see Parthasarathy and Raghavan (1971) for example).

One of the difficulties of the noncooperative theory is that there are usually many equilibria giving different payoff vectors as we shall see in the following examples. Another difficulty is that even if there is a unique strategic equilibrium, it may not be considered as a reasonable solution or a predicted outcome. In the rest of this section we restrict attention to $n = 2$, the two-person case.

2.2 Examples. *Example 1.* **A Coordination Game.** Consider the game with bimatrix

$$\begin{pmatrix} (3,3) & (0,2) \\ (2,1) & (5,5) \end{pmatrix}$$

and corresponding payoff matrices

$$\boldsymbol{A} = \begin{pmatrix} 3 & 0 \\ 2 & 5 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{B} = \begin{pmatrix} 3 & 2 \\ 1 & 5 \end{pmatrix}$$

The corresponding maxmin (MM) strategies are (1/2,1/2) for Player I and (3/5,2/5) for Player II. The safety levels are $(v_I, v_{II}) = (5/2, 13/5)$.

Here there are two obvious pure strategic equilibria (PSE's) corresponding to the payoffs (3,3) and (5,5). Both players prefer the second SE because it gives them both 5 instead of 3. If they could coordinate their actions, this outcome would be expected. However, if they cannot communicate and if both players believe the other is going to choose the first strategy, then they are both going to choose the first strategy and receive the payoff 3. One cannot say the outcome (3,3) is irrational. If that's the way things have always been, then one who tries to change things hurts oneself. This phenomenon occurs often, usually with many players. To try to change the structure of a language or the typewriter keyboard or the system of measurement requires a lot of people to change simultaneously before any advantage is realized.

There is a third less obvious equilibrium point that sometimes occurs in these games. If each player has an equalizing strategy for the other player's matrix, then that pair of strategies forms an equilibrium. This is because if an opponent uses a strategy that makes it not matter what you do, then anything you do is a best response, in particular the equalizing strategy on the opponent's matrix. (Recall that an equalizing strategy is one that gives the same average payoff to the opponent no matter what the opponent does.)

Let us find this **equalizing strategic equilibrium** for the above game. Note that each player uses the matrix of his opponent. Player I has the equalizing strategy $\boldsymbol{p} =$

$(4/5, 1/5)$ for $\boldsymbol{B}$, and Player II has the equalizing strategy $\boldsymbol{q} = (5/6, 1/6)$ for $\boldsymbol{A}$. If the players use these strategies, the average payoff is $(5/2, 13/5)$, the same as the safety levels.

Is it possible that the average payoff from a strategic equilibrium is less than the safety level for one of the players? The answer is no. (See Exercise 1.) Therefore the strategic equilibrium $(\boldsymbol{p}, \boldsymbol{q})$ is as poor a strategic equilibrium as you can get. Moreover, it is extremely unstable. It is true that it does neither player any good to deviate from his/her equilibrium strategy, but on the other hand it does not harm a player to change to another strategy.

In the above example, the payoffs for the three SE's are all different. The players have the same preferences as to which of the three outcomes they would prefer. In the next example, the players have different preferences between the two pure strategic equilibria.

Example 2. **The Battle of the Sexes.** Suppose the matrices are

$$\begin{array}{cc} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} (2,1) & (0,0) \\ (0,0) & (1,2) \end{pmatrix} \end{array} \quad \text{so that} \quad \boldsymbol{A} = \begin{array}{cc} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \quad \text{and} \quad \boldsymbol{B} = \begin{array}{cc} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \end{array}.$$

The name of this game arises as a description of the game played between a husband and wife in choosing which movie to see, a or b. They prefer different movies, but going together is preferable to going alone. Perhaps this should be analyzed as a cooperative game, but we analyze it here as a noncooperative game.

The pure strategy vectors (a, a) and (b, b) are both PSE's but Player I prefers the first and Player II the second.

First note that the safety levels are $v_I = v_{II} = 2/3$, the same for both players. Player I's MM strategy is (1/3,2/3), while Player II's MM strategy is (2/3,1/3). There is a third strategic equilibrium given by the equalizing strategies $\boldsymbol{p} = (2/3, 1/3)$ and $\boldsymbol{q} = (1/3, 2/3)$. The equilibrium payoff for this equilibrium point, $(v_I, v_{II}) = (2/3, 2/3)$, is worse for both players than either of the other two equilibrium points.

Example 3. **The Prisoner's Dilemma.** It may happen that there is a unique SE but that there are other outcomes that are better for both players. Consider the game with bimatrix

$$\begin{array}{cc} & \begin{array}{cc} \text{cooperate} & \text{defect} \end{array} \\ \begin{array}{c} \text{cooperate} \\ \text{defect} \end{array} & \begin{pmatrix} (3,3) & (0,4) \\ (4,0) & (1,1) \end{pmatrix} \end{array}$$

In this game, Player I can see that no matter which column Player II chooses, he will be better off if he chooses row 2. For if Player I chooses row 2 rather than row 1, he wins 4 rather than 3 if Player II chooses column 1, and he wins 1 rather than 0 if she chooses column 2. In other words, Player I's second strategy of choosing the second row strictly dominates the strategy of choosing the first. On the other hand, the game is symmetric. Player II's second column strictly dominates her first. However, if both players use their dominant strategies, each player receives 1, whereas if both players use their dominated strategies, each player receives 3.

A game that has this feature, that both players are better off if together they use strictly dominated strategies, is called the Prisoner's Dilemma. The story that leads to this bimatrix and gives the game its name is as follows. Two well-known crooks are captured and separated into different rooms. The district attorney knows he does not have enough evidence to convict on the serious charge of his choice, but offers each prisoner a deal. If just one of them will turn state's evidence (i.e. rat on his confederate), then the one who confesses will be set free, and the other sent to jail for the maximum sentence. If both confess, they are both sent to jail for the minimum sentence. If both exercise their right to remain silent, then the district attorney can still convict them both on a very minor charge. In the numerical matrix above, we take the units of measure of utility to be such that the most disagreeable outcome (the maximum sentence) has value 0, and the next most disagreeable outcome (minimum sentence) has value 1. Then we let being convicted on a minor charge to have value 3, and being set free to have value 4.

This game has abundant economic application. An example is the manufacturing by two companies of a single good. Both companies may produce either at a high or a low level. If both produce at a low level, the price stays high and they both receive 3. If they both produce at the high level the price drops and they both receive 1. If one produces at the high level while the other produces at the low level, the high producer receives 4 and the low producer receives 0. No matter what the other producer does, each will be better off by producing at a high level.

2.3 Finding All PSE's. For larger matrices, it is not difficult to find all pure strategic equilibria. This may be done using an extension of the method of finding all saddle points of a zero-sum game. With the game written in bimatrix form, put an asterisk after each of Player I's payoffs that is a maximum of its column. Then put an asterisk after each of Player II's payoffs that is a maximum of its row. Then any entry of the matrix at which both I's and II's payoffs have asterisks is a PSE, and conversely.

An example should make this clear.

$$\begin{array}{c c} & \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{pmatrix} (2,1) & (4,3) & (7^*,2) & (7^*,4) & (0,5^*) & (3,2) \\ (4^*,0) & (5^*,4) & (1,6^*) & (0,4) & (0,3) & (5^*,1) \\ (1,3^*) & (5^*,3^*) & (3,2) & (4,1) & (1^*,0) & (4,3^*) \\ (4^*,3) & (2,5^*) & (4,0) & (1,0) & (1^*,5^*) & (2,1) \end{pmatrix} \end{array}$$

In the first column, Player I's maximum payoff is 4, so both 4's are given asterisks. In the first row, Player II's maximum is 5, so the 5 receives an asterisk. And so on.

When we are finished, we see two payoff vectors with double asterisks. These are the pure strategic equilibria, (C, b) and (D, e), with payoffs $(5, 3)$ and $(1, 5)$ respectively. At all other pure strategy pairs, at least one of the players can improve his/her payoff by switching pure strategies.

In a two-person zero-sum game, a PSE is just a saddle point. Many games have no PSE's, for example, zero-sum games without a saddle point. However, just as zero-sum games of perfect information always have saddle points, non-zero-sum games of perfect information always have at least one PSE that may be found by the method of backward induction.

2.4 Iterated Elimination of Strictly Dominated Strategies. Since in general-sum games different equilibria may have different payoff vectors, it is more important than in zero-sum games to find all strategic equilibria. *We may remove any strictly dominated row or column without losing any equilibrium points* (Exercise 7).

We, being rational, would not play a strictly dominated pure strategy, because there is a (possibly mixed) strategy that guarantees us a strictly better average payoff no matter what the opponent does. Similarly, if we believe the opponent is as rational as we are, we believe that he/she will not use a dominated strategy either. Therefore we may cancel any dominated pure strategy of the opponent before we check if we have any dominated pure strategies which may now be eliminated.

This argument may be iterated. If we believe our opponent not only is rational but also believes that we are rational, then we may eliminate our dominated pure strategies, then eliminate our opponent's dominated pure strategies, and then again eliminate any of our own pure strategies that have now become dominated. The ultimate in this line of reasoning is that if it is *common knowledge* that the two players are rational, then we may iteratively remove dominated strategies as long as we like. (A statement is "common knowledge" between two players if each knows the statement, and each knows the other knows the statement, and each knows the other knows the other knows the statement, ad infinitum.)

As an example of what this sort of reasoning entails, consider a game of Prisoner's Dilemma that is to be played sequentially 100 times. The last time this is to be played it is clear that rational players will choose to defect. The other strategy is strictly dominated. But now that we know what the players will do on the last game we can apply strict domination to the next to last game to conclude the players will defect on that game too. Similarly all the way back to the first game. The players will each receive 1 at each game. If they could somehow break down their belief in the other's rationality, they might receive 3 for each game.

Here is another game, called the **Centipede Game**, that illustrates this anomaly more vividly. This is a game of perfect information with no chance moves, so it is easy to apply the iterated removal of strictly dominated strategies. Here is the game in extensive form.

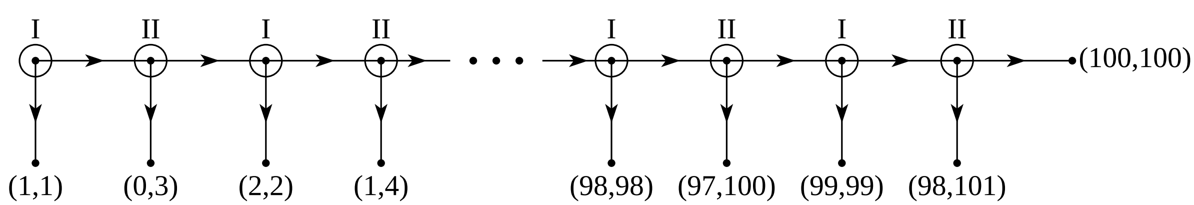


Figure 2.1 The Centipede Game.

Since this is a game of perfect information, it may be solved by backward induction. At the last move, Player II will certainly go down instead of across since that gives her 101 instead of 100. Therefore at the next to last move, Player I will go down rather than across since that gives him 99 instead of the 98. And so forth, back to the initial position, where Player I will go down rather than across because he receives 1 instead of 0. This is the unique PSE because all eliminated strategies were strictly dominated.

Empirical evidence acquired by playing similar games shows that this gives a poor prediction of how people actually play this game. See the book of David M. Kreps (1990) *Game Theory and Economic Modeling*, Oxford University Press, for a discussion.

2.5 Exercises.

1. **Strategic Equilibria Are Individually Rational.** A payoff vector is said to be *individually rational* if each player receives at least his safety level. Show that if $(\boldsymbol{p}, \boldsymbol{q})$ is a strategic equilibrium for the game with matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, then $\boldsymbol{p}^{\mathsf{T}}\boldsymbol{A}\boldsymbol{q} \geq v_I$ and $\boldsymbol{p}^{\mathsf{T}}\boldsymbol{B}\boldsymbol{q} \geq v_{II}$. Thus, the payoff vector for a strategic equilibrium is individually rational.

2. Find the safety levels, the MM-strategies, and find all SE's and associated vector payoffs of the following games in strategic form.

(a) $\begin{pmatrix} (0,0) & (2,4) \\ (2,4) & (3,3) \end{pmatrix}$. (b) $\begin{pmatrix} (1,4) & (4,1) \\ (2,2) & (3,3) \end{pmatrix}$. (c) $\begin{pmatrix} (0,0) & (0,-1) \\ (1,0) & (-1,3) \end{pmatrix}$.

3. **The Game of Chicken.** Two players speed head-on toward each other and a collision is bound to occur unless one of them chickens out at the last minute. If both chicken out, everything is okay (they both win 1). If one chickens out and the other does not, then it is a great success for the player with iron nerves (payoff $= 2$) and a great disgrace for the chicken (payoff $= -1$). If both players have iron nerves, disaster strikes (both lose 2).
(a) Set up the bimatrix of this game.
(b) What are the safety levels, what are the MM strategies, and what is the average payoff if the players use the MM strategies?
(c) Find all three SE's.

4. **An extensive form non-zero-sum game.** A coin with probability 2/3 of heads and 1/3 of tails is tossed and the outcome is shown to player I but not to player II. Player I then makes a claim which may be true or false that the coin turned up heads or that the coin turned up tails. Then, player II, hearing the claim, must guess whether the coin came up heads or tails. Player II wins \$3 if his guess is correct, and nothing otherwise. Player I wins \$3 if I has told the truth in his claim. In addition, Player I wins an additional \$6 if player II guesses heads.
(a) Draw the Kuhn tree.
(b) Put into strategic (bimatrix) form.
(c) Find all PSE's.

5. Find all PSE's of the following games in strategic form.

(a) $$\begin{pmatrix} (-3,-4) & (\ \ 2,-1) & (\ \ 0,\ \ 6) & (\ \ 1,\ \ 1) \\ (\ \ 2,\ \ 0) & (\ \ 2,\ \ 2) & (-3,\ \ 0) & (\ \ 1,-2) \\ (\ \ 2,-3) & (-5,\ \ 1) & (-1,-1) & (\ \ 1,-3) \\ (-4,\ \ 3) & (\ \ 2,-5) & (\ \ 1,\ \ 2) & (-3,\ \ 1) \end{pmatrix}.$$

(b) $$\begin{pmatrix} (\ \ 0,\ \ 0) & (\ \ 1,-1) & (\ \ 1,\ \ 1) & (-1,\ \ 0) \\ (-1,\ \ 1) & (\ \ 0,\ \ 1) & (\ \ 1,\ \ 0) & (\ \ 0,\ \ 0) \\ (\ \ 1,\ \ 0) & (-1,-1) & (\ \ 0,\ \ 1) & (-1,\ \ 1) \\ (\ \ 1,-1) & (-1,\ \ 0) & (\ \ 1,-1) & (\ \ 0,\ \ 0) \\ (\ \ 1,\ \ 1) & (\ \ 0,\ \ 0) & (-1,-1) & (\ \ 0,\ \ 0) \end{pmatrix}.$$

6. Consider the bimatrix game: $\begin{pmatrix} (0,0) & (1,2) & (2,0) \\ (0,1) & (2,0) & (0,1) \end{pmatrix}$.
(a) Find the safety levels for the two players.
(b) Find all PSE's.
(c) Find all SE's given by mixed equalizing strategies.

7. **Strategic Equilibria Survive Elimination of Strictly Dominated Strategies.** Suppose row 1 is strictly dominated (by a probability mixture of rows 2 through m, i.e. $a_{1j} < \sum_{i=2}^{m} x_i a_{ij}$ for all j where $x_i \geq 0$ and $\sum_2^m x_i = 1$), and suppose $(\boldsymbol{p}^*, \boldsymbol{q}^*)$ is a strategic equilibrium. Show that $p_1^* = 0$.

8. Consider the non-cooperative bimatrix game: $\begin{pmatrix} (3,4) & (2,3) & (3,2) \\ (6,1) & (0,2) & (3,3) \\ (4,6) & (3,4) & (4,5) \end{pmatrix}$.
(a) Find the safety levels, and the maxmin strategies for both players.
(b) Find as many strategic equilibria as you can.

9. A PSE vector of strategies in a game in extensive form is said to be a **subgame perfect equilibrium** if at every vertex of the game tree, the strategy vector restricted to the subgame beginning at that vertex is a PSE. If a game has perfect information, a subgame perfect equilibrium may be found by the method of backward induction. Figure 2.2 is an example of a game of perfect information that has a subgame perfect PSE and another PSE that is not subgame perfect.
(a) Solve the game for an equilibrium using backward induction.
(b) Put the game into strategic form.
(c) Find another PSE of the strategic form game, relate it to the extensive form game and show it is not subgame perfect.

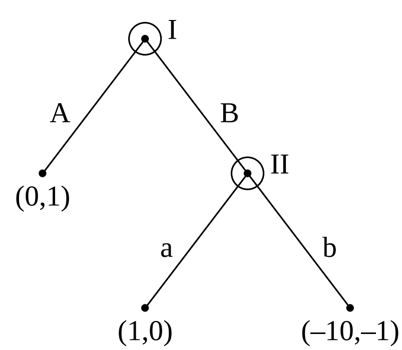


Figure 2.2 An Extensive Form Game.

10. Suppose you are playing the centipede game once as Player I against a person chosen at random from this class. At what point would you choose the option to go down ending the game, assuming the opponent has not already ended the game?

Now answer the same question assuming you are Player II.

Your score on this question depends on what the other students in the class do. Do not reveal your answer to this question to the other students in the class.

3. Models of Duopoly

The examples given of the noncooperative theory of equilibrium have generally shown the theory to have poor predictive power. This is mainly because there may be multiple equilibria with no way to choose among them. Alternately, there may be a unique equilibrium with a poor outcome, even one found by iterated elimination of dominated strategies as in the prisoner's dilemma or the centipede game. But there are some situations, such as the prisoner's dilemma played once, in which strategic equilibria are quite reasonable predictive indicators of behavior. We begin with a model of duopoly due to A. Cournot (1838).

3.1 The Cournot Model of Duopoly. There are two competing firms producing a single homogeneous product. These firms must choose how much of the good to produce. The cost of producing one unit of the good is a constant c, the same for both firms. If a Firm i produces the quantity q_i units of the good, then the cost to Firm i is cq_i, for $i = 1, 2$. (There is no setup cost.) The price of a unit of the good is negatively related to the total amount produced. If Firm 1 produces q_1 and Firm 2 produces q_2 for a total of $Q = q_1 + q_2$, the price is

$$P(Q) = \begin{cases} a - Q & \text{if } 0 \le Q \le a \\ 0 & \text{if } Q > a \end{cases} = (a - Q)^+ \tag{1}$$

for some constant a. (This is not a realistic assumption, but the price will be approximately linear near the equilibrium point, and that is the main thing.) We assume the firms must choose their production quantities simultaneously; no collusion is allowed.

The pure strategy spaces for this game are the sets $X = Y = [0, \infty)$. Note these are infinite sets, so the game is not a finite game. It would not hurt to restrict the strategy spaces to $[0, a]$; no player would like to produce more than a units because the return is zero. The payoffs for the two players are the profits,

$$u_1(q_1, q_2) = q_1 P(q_1 + q_2) - cq_1 = q_1(a - q_1 - q_2)^+ - cq_1 \tag{2}$$

$$u_2(q_1, q_2) = q_2 P(q_1 + q_2) - cq_2 = q_2(a - q_1 - q_2)^+ - cq_2 \tag{3}$$

This defines the strategic form of the game. We assume that $c < a$, since otherwise the cost of production would be at least as great as any possible return.

First, let us find out what happens in the monopolistic case when there is only one producer. That is, suppose $q_2 = 0$. Then the return to Firm 1 if it produces q_1 units is $u(q_1) = q_1(a - q_1)^+ - cq_1$. The firm will choose q_1 to maximize this quantity. Certainly the maximum will occur for $0 < q_1 < a$; in this case, $u(q_1) = q_1(a - c) - q_1^2$, and we may find the point at which the maximum occurs by taking a derivative with respect to q_1, setting it to zero and solving for q_1. The resulting equation is $u'(q_1) = a - c - 2q_1 = 0$, whose solution is $q_1 = (a - c)/2$. The monopoly price is $P((a - c)/2) = (a + c)/2$, and the monopoly profit is $u((a - c)/2) = (a - c)^2/4$.

To find a duopoly PSE, we look for a pure strategy for each player that is a best response to the other's strategy. We find simultaneously the value of q_1 that maximizes (2) and the value of q_2 that maximizes (3) by setting the partial derivatives to zero.

$$\frac{\partial}{\partial q_1} u_1(q_1, q_2) = a - 2q_1 - q_2 - c = 0 \tag{4}$$

$$\frac{\partial}{\partial q_2} u_2(q_1, q_2) = a - q_1 - 2q_2 - c = 0 \tag{5}$$

(u_1 is a quadratic function of q_1 with a negative coefficient, so this root represents a point of maximum.) Solving these equations simultaneously and denoting the result by q_1^* and q_2^*, we find

$$q_1^* = (a - c)/3 \qquad \text{and} \qquad q_2^* = (a - c)/3. \tag{6}$$

Therefore, (q_1^*, q_2^*) is a PSE for this problem.

In this SE, each firm produces less than the monopoly production, but the total produced is greater than the monopoly production. The payoff each player receives from this SE is

$$u_1(q_1^*, q_2^*) = \frac{a-c}{3}\left(a - \frac{a-c}{3} - \frac{a-c}{3}\right) - c\frac{a-c}{3} = \frac{(a-c)^2}{9}. \tag{7}$$

Note that the total amount received by the firms in this equilibrium is $(2/9)(a - c)^2$. This is less than $(1/4)(a - c)^2$, which is the amount that a monopoly would receive using the monopolistic production of $(a - c)/2$. This means that if the firms were allowed to cooperate, they could improve their profits by agreeing to share the production and profits. Thus each would produce less, $(a-c)/4$ rather than $(a-c)/3$, and receive a greater profit, $(a - c)^2/8$ rather than $(a - c)^2/9$.

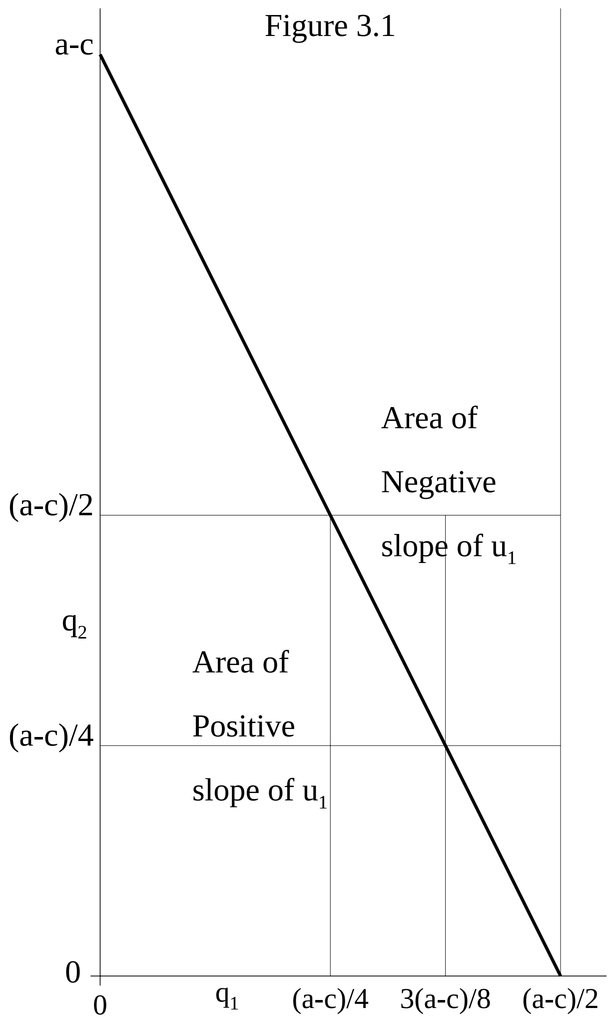
On the other hand, the duopoly price is $P(q_1^* + q_2^*) = (a+2c)/3$, which is less than the monopoly price, $(a+c)/2$ (since $c < a$). Thus, the consumer is better off under a duopoly than under a monopoly.

This PSE is in fact the unique SE. This is because it can be attained by *iteratively deleting strictly dominated strategies*. To see this, consider the points at which the function u_1 has positive slope as a function of $q_1 \geq 0$ for fixed $q_2 \geq 0$. The derivative (4) is positive provided $2q_1 + q_2 < a - c$. See Figure 3.1.

For all values of $q_2 \geq 0$, the slope is negative for all $q_1 > (a - c)/2$. Therefore, all $q_1 > (a - c)/2$ are strictly dominated by $q_1 = (a - c)/2$.

But since the game is symmetric in the players, we automatically have all $q_2 > (a-c)/2$ are strictly dominated and may be removed. When all such points are removed from consideration in the diagram, we see that for all remaining q_2, the slope is positive for all $q_1 < (a - c)/4$. Therefore, all $q_1 < (a - c)/4$ are strictly dominated by $q_1 = (a - c)/4$.

Again symmetrically eliminating all $q_2 < (a - c)/4$, we see that for all remaining q_2, the slope is negative for all $q_1 > 3(a - c)/8$. Therefore, all $q_1 > 3(a - c)/8$ are strictly



dominated by $q_1 = 3(a-c)/8$. And so on, chipping a piece off from the lower end and then one from the upper end of the interval of the remaining q_1 not yet eliminated. If this is continued an infinite number of times, all q_1 are removed by iterative elimination of strictly dominated strategies except the point q_1^*, and by symmetry q_2^* for Player II.

Note that the prisoner's dilemma puts in an appearance here. Instead of using the SE obtained by removing strictly dominated strategies, both players would be better off if they could cooperate and produce $(a-c)/4$ each.

3.2 The Bertrand Model of Duopoly. In 1883, J. Bertrand proposed a different model of competition between two duopolists, based on allowing the firms to set prices rather than to fix production quantities. In this model, demand is a function of price rather than price a function of quantity available.

First consider the case where the two goods are identical and price information to the consumer is perfect so that the firm that sets the lower price will corner the market. We use the same price/demand function (1) solved for demand Q in terms of price P,

$$Q(P) = \begin{cases} a-P & \text{if } 0 \le P \le a \\ 0 & \text{if } P > a \end{cases} = (a-P)^+. \tag{8}$$

The actual demand is $Q(P)$ where P is the lowest price. The monopoly behavior under this model is the same as for the Cournot model of the previous section. The monopolist sets the price at $(a+c)/2$ and produces the quantity $(a-c)/2$, receiving a profit of $(a-c)^2/4$.

Suppose firms 1 and 2 choose prices p_1 and p_2 respectively. We assume that if $p_1 = p_2$ the firms share the market equally. We take the cost for a unit production again to be $c > 0$ so that the profit is $p_i - c$ times the amount sold. Then the payoff functions are

$$u_1(p_1,p_2) = \begin{cases} (p_1-c)(a-p_1)^+ & \text{if } p_1 < p_2 \\ (p_1-c)(a-p_1)^+/2 & \text{if } p_1 = p_2 \\ 0 & \text{if } p_1 > p_2 \end{cases} \tag{9}$$

and

$$u_2(p_1,p_2) = \begin{cases} (p_2-c)(a-p_2)^+ & \text{if } p_2 < p_1 \\ (p_2-c)(a-p_2)^+/2 & \text{if } p_2 = p_1 \\ 0 & \text{if } p_2 > p_1 \end{cases} \tag{10}$$

Here there is a unique PSE but it is rather disappointing. Both firms charge the production cost, $p_1^* = p_2^* = c$, and receive a payoff of zero. This is the safety level for each player. It is easy to check that this is an equilibrium. No other pair of prices can be an equilibrium because either firm could capture the entire market by slightly undercutting the other's price.

This feature of capturing the entire market by undercutting the other's price is not entirely reasonable for a number of reasons. Usually the products of two firms are not entirely interchangeable so some consumers may prefer one product to the other even if it costs somewhat more. In addition there is the problem of the consumer getting the information on the prices, and there is the feature of brand loyalty by consumers. We may modify the model in an attempt to take this into account.

The Bertrand Model with Differentiated Products. Again we assume that the firms choose prices p_1 and p_2 and that the cost of unit production is $c > 0$. Since the profits per unit produced are $p_1 - c$ and $p_2 - c$, we may assume that the prices will satisfy $p_1 \ge c$ and $p_2 \ge c$. This time we assume that the demand functions of the products of the firms for given price selections are given by

$$\begin{aligned} q_1(p_1,p_2) &= (a - p_1 + bp_2)^+ \\ q_2(p_1,p_2) &= (a - p_2 + bp_1)^+, \end{aligned} \tag{11}$$

where $b > 0$ is a constant representing how much the product of one firm is a substitute for the product of the other. We assume $b \le 1$ for simplicity. These demand functions are unrealistic in that one firm could conceivably charge an arbitrarily high price and still have a positive demand provided the other firm also charges a high enough price. However, this function is chosen to represent a linear approximation to the "true" demand function, appropriate near the usual price settings where the equilibrium is reached.

Under these assumptions, the strategy sets of the firms are $X = [0,\infty)$ and $Y = [0,\infty)$, and the payoff functions are

$$\begin{aligned} u_1(p_1,p_2) &= q_1(p_1,p_2)(p_1-c) = (a-p_1+bp_2)^+(p_1-c) \\ u_2(p_1,p_2) &= q_2(p_1,p_2)(p_2-c) = (a-p_2+bp_1)^+(p_2-c). \end{aligned} \tag{12}$$

To find the equilibrium prices, we must find points (p_1^*, p_2^*) at which u_1 is maximized in p_1 and u_2 is maximized in p_2 simultaneously. Assuming $a - p_1 + bp_2 > 0$ and $a - p_2 + bp_1 > 0$, we find

$$\frac{\partial}{\partial p_1} u_1(p_1, p_2) = a - 2p_1 + bp_2 + c = 0$$
$$\frac{\partial}{\partial p_2} u_2(p_1, p_2) = a - 2p_2 + bp_1 + c = 0.$$

Again the functions are quadratic in the variable of differentiation with a negative coefficient, so the resulting roots represent maxima. Solving simultaneously and denoting the result by p_1^* and p_2^*, we find

$$p_1^* = p_2^* = \frac{a + c}{2 - b}.$$

3.3 The Stackelberg Model of Duopoly. In the Cournot and Bertrand models of duopoly, the players act simultaneously. H. von Stackelberg (1934) proposed a model of duopoly in which one player, called the dominant player or leader, moves first and the outcome of that player's choice is made known to the other player before the other player's choice is made. An example might be General Motors, at times big enough and strong enough in U.S. history to play such a dominant role in the automobile industry. Let us analyze the Cournot model from this perspective.

Firm 1 chooses an amount to produce, q_1, at a cost c per unit. This amount is then told to Firm 2 which then chooses an amount q_2 to produce also at a cost of c per unit. Then the price P per unit is determined by equation (1), $P = (a - q_1 - q_2)^+$, and the players receive $u_1(q_1, q_2)$ and $u_2(q_1, q_2)$ of equations (2) and (3).

Player I's pure strategy space is $X = [0, \infty)$. From the mathematical point of view, the only difference between this model and the Cournot model is that Firm 2's pure strategy space, Y, is now a set of functions mapping q_1 into q_2. However, this is now a game of perfect information that can be solved by backward induction. Since Firm 2 moves last, we first find the optimal q_2 as a function of q_1. That is, we solve equation (5) for q_2. This gives us Firm 2's strategy as

$$q_2(q_1) = (a - q_1 - c)/2. \tag{13}$$

Since Firm 1 now knows that Firm 2 will choose this best response, Firm 1 now wishes to choose q_1 to maximize

$$\begin{aligned} u_1(q_1, q_2(q_1)) &= q_1(a - q_1 - (a - q_1 - c)/2) - cq_1 \\ &= -\frac{1}{2}q_1^2 + \frac{a - c}{2}q_1. \end{aligned} \tag{14}$$

This quadratic function is maximized by $q_1 = q_1^* = (a - c)/2$. Then Firm 2's best response is $q_2^* = q_2(q_1^*) = (a - c)/4$.

Let us analyze this SE and compare its payoff to the payoff of the SE in the Cournot duopoly. Firm 1 produces the monopoly quantity and Firm 2 produces less than the

Cournot SE. The payoff to Firm 1 is $u_1(q_1^*, q_2^*) = (a-c)^2/8$ and the payoff to Firm 2 is $u_2(q_1^*, q_2^*) = (a-c)^2/16$. Therefore Firm 1's profits are greater than that given by the Cournot equilibrium, and Firm 2's are less. Note that the total amount produced is $(3/4)(a-c)$, which is greater than $(2/3)(a-c)$, the total amount produced under the Cournot equilibrium. This means the Stackelberg price is lower than the Cournot price, and the consumer is better off under the Stackelberg model.

The information that Firm 2 received about Firm 1's production has been harmful. Firm 1 by announcing its production has increased its profit. This shows that having more information may make a player worse off. More precisely, being given more information and having that fact be common knowledge may make you worse off.

3.4 Entry Deterrence. Even if a firm acts as a monopolist in a certain market, there may be reasons why it is in the best interests of the firm to charge less than the monopoly price, or equivalently, produce more than the monopoly production. One of these reasons is that the high price of the good may attract another firm to enter the market.

We can see this in the following example. Suppose the price/demand relationship can be expressed as

$$P(Q) = \begin{cases} 17 - Q & \text{if } 0 \le Q \le 17 \\ 0 & \text{otherwise,} \end{cases} \tag{15}$$

where Q represents the total amount produced, and P represents the price per unit amount. Suppose additionally, that the cost to the firm of producing q_1 items is $q_1 + 9$. That is, there is a fixed cost of 9 and a constant marginal cost of 1 per unit quantity. The profit to the firm of producing quantity q_1 of the good is

$$u(q_1) = (17 - q_1)q_1 - (q_1 + 9) = 16q_1 - q_1^2 - 9. \tag{16}$$

The value of q_1 that maximizes the profit is found by setting the derivative of $u(q_1)$ to zero and solving for q_1:

$$u'(q_1) = 16 - 2q_1 = 0.$$

So the monopoly production is

$$q_1 = 8,$$

the monopoly price is 9, and the monopoly profit is

$$u(8) = 9 \cdot 8 - 17 = 55.$$

Suppose now a competing firm observes this market and thinks of producing a small amount, q_2, to hopefully make a small profit. Suppose also that this firm also has the same cost, q_2+9, as the monopoly firm. On producing q_2 the price will drop to $P(8+q_2) = 9-q_2$, and the competing firm's profit will be

$$u_2 = (9 - q_2)q_2 - (q_2 + 9) = 8q_2 - q_2^2 - 9. \tag{17}$$

This is maximized at $q_2 = 4$ and the profit there is $u_2 = 7$. Since this is positive, the firm has an incentive to enter the market.

Of course, the incumbent monopolist can foresee this possibility and can calculate the negative effect it will have on the firm's profits. If the challenger enters the market with a production of 4, the price will drop to $P(8+4) = 5$, and the monopolist's profits will drop from 55 to $5 \cdot 8 - 17 = 23$. It seems reasonable that some preventative measures might be worthwhile.

If the monopolist produces a little more than the monopoly quantity, it might deter the challenger from entering the market. How much more should be produced? If the monopolist produces q_1, then the challenger's firm's profits may be computed as in (17) by

$$u_2(q_1, q_2) = (17 - q_1 - q_2)q_2 - (q_2 + 9).$$

This is maximized at $q_2 = (16 - q_1)/2$ for a profit of

$$u_2(q_1, (16 - q_1)/2) = (16 - q_1)^2/4 - 9.$$

The profit is zero if $(16 - q_1)^2 = 36$, or equivalently, if $q_1 = 10$.

This says that if the monopolist produces 10 rather than 8, then the challenger can see that it is not profitable to enter the market.

However, the monopolist's profits are reduced by producing 10 rather than 8. From (16) we see that the profits to the firm when $q_1 = 10$ are

$$u_1(10) = 7 \cdot 10 - 19 = 51$$

instead of 55. This is a relatively small amount to pay as insurance against the much bigger drop in profits from 55 to 18 the monopolist would suffer if the challenger should enter the market.

The above analysis assumes that the challenger believes that, even if the challenger should enter the market, the monopolist will continue with the monopoly production, or the pre-entry production. This would be the case if the incumbent monopolist were considered as the dominant player in a Stackelberg model. Note that the strategy pair, $q_1 = 10$ and $q_2 = 0$, does not form a strategic equilibrium in this Stackelberg model, since $q_1 = 10$ is not a best response to $q_2 = 0$. To analyze the situation properly, we should enlarge the model to allow the game to be played sequentially several times.

When analyzed as a Stackelberg duopoly, then at equilibrium, the dominant player produces 8, the weaker player produces 4, and the price drops to 5. The dominant player's profit is 23, and the weaker player's profit is 7.

If this problem were analyzed as a Cournot duopoly, we would find that, at equilibrium, each firm would produce $5\frac{1}{3}$, the price would drop to $6\frac{1}{3}$, and each firm would realize a profit of $19\frac{4}{9}$. This low profit is another reason that the incumbent firm should make strong efforts to deter the entry of a challenger.

3.5 Exercises.

1.(a) Suppose in the Cournot model that the firms have different production costs. Let c_1 and c_2 be the costs of production per unit for firms 1 and 2 respectively, where both c_1 and c_2 are assumed less than $a/2$. Find the Cournot equilibrium.

(b) What happens, if in addition, each firm has a set up cost. Suppose Player I's cost of producing x is $x+2$, and II's cost of producing y is $3y+1$. Suppose also that the price function is $p(x,y) = 17 - x - y$, where x and y are the amounts produced by I and II respectively. What is the equilibrium production, and what are the players' equilibrium payoffs?

2. Extend the Cournot model of Section 3.1 to three firms. Firm i chooses to produce q_i at cost cq_i where $c > 0$. The selling price is $P(Q) = (a-Q)^+$ where $Q = q_1 + q_2 + q_3$. What is the strategic equilibrium?

3. Modify the Bertrand model with differentiated products to allow sequential selection of the price as in Stackelberg's variation of Cournot's model. The dominant player announces a price first and then the subordinate player chooses a price. Solve by backward induction and compare to the SE for the simultaneous selection model.

4. Consider the Cournot duopoly model with the somewhat more realistic price function,

$$P(Q) = \begin{cases} \frac{1}{4}Q^2 - 5Q + 26 & \text{for } 0 \le Q \le 10, \\ 1 & \text{for } Q \ge 10. \end{cases}$$

This price function starts at 26 for $Q = 0$ and decreases down to 1 at $Q = 10$ and then stays there. Assume that the cost, c, of producing one unit is $c = 1$ for both firms. No firm would produce more than 10 because the return for selling a unit would barely pay for the cost of producing it. Thus we may restrict the productions q_1, q_2, to the interval $[0, 10]$.

(a) Find the monopoly production, and the optimal monopoly return.

(b) Show that if $q_2 = 5/2$, then $u_1(q_1, 5/2)$ is maximized at $q_1 = 5/2$. Show that this implies that $q_1 = q_2 = 5/2$ is an equilibrium production in the duopoly.

5. **An Advertising Campaign.** Two firms may compete for a given market of total value, V, by investing a certain amount of effort into the project through advertising, securing outlets, etc. Each firm may allocate a certain amount for this purpose. If firm 1 allocates $x > 0$ and firm 2 allocates $y > 0$, then the proportion of the market that firm 1 corners is $x/(x+y)$. The firms have differing difficulties in allocating these resources. The cost per unit allocation to firm i is c_i, $i = 1, 2$. Thus the profits to the two firms are

$$M_1(x,y) = V \cdot \frac{x}{x+y} - c_1 x$$
$$M_2(x,y) = V \cdot \frac{y}{x+y} - c_2 y$$

If both x and y are zero, the payoffs to both are zero.

(a) Find the equilibrium allocations, and the equilibrium profits to the two firms, as a function of V, c_1 and c_2.

(b) Specialize to the case $V = 1$, $c_1 = 1$, and $c_2 = 2$.

4. Cooperative Games

In one version of the noncooperative theory, communication between the players is allowed but the players are forbidden to make binding agreements. In the cooperative theory, we allow communication of the players and also allow binding agreements to be made. This requires some outside mechanism to enforce the agreements. In the noncooperative theory, the only believable outcome would be some Nash equilibrium because such an outcome is self-enforcing: neither player can gain by breaking the agreement. With the extra freedom to make enforceable binding agreements in the cooperative theory, the players can generally do much better. For example in the prisoner's dilemma, the only Nash equilibrium is for both players to defect. In the cooperative theory, they can reach a binding agreement to both use the cooperate strategy, and both players will be better off.

The cooperative theory is divided into two classes of problems depending on whether or not there is a mechanism for transfer of utility from one player to the other. If there such a mechanism, we may think of the transferable commodity as "money", and assume that both players have a linear utility for money. We may take the scaling of the respective utilities to be such that the utility of no money is 0 and the utility of one unit of money is 1. In Section 2, we treat the **transferable utility** (TU) case. In Section 3, we treat the **nontransferable utility** (NTU) case.

4.1 Feasible Sets of Payoff Vectors. One of the main features of cooperative games is that the players have freedom to choose a joint strategy. This allows any probability mixture of the payoff vectors to be achieved. For example in the battle of the sexes, the players may agree to toss a coin to decide which movie to see. (They may also do this in the noncooperative theory, but after the coin is tossed, they are allowed to change their minds, whereas in the cooperative theory, the coin toss is part of the agreement.) The set of payoff vectors that the players can achieve if they cooperate is called the feasible set. The distinguishing feature of the TU case is that the players may make **side payments** of utility as part of the agreement. This feature results in a distinction between the NTU feasible set and the TU feasible set.

When players cooperate in a bimatrix game with matrices $(\boldsymbol{A}, \boldsymbol{B})$, they may agree to achieve a payoff vector of any of the mn points, (a_{ij}, b_{ij}) for $i = 1, \ldots, m$ and $j = 1 \ldots, n$. They may also agree to any probability mixture of these points. The set of all such payoff vectors is the convex hull these mn points. Without a transferable utility, this is all that can be achieved.

Definition. *The NTU feasible set is the convex hull of the mn points, (a_{ij}, b_{ij}) for $i = 1, \ldots, m$ and $j = 1 \ldots, n$.*

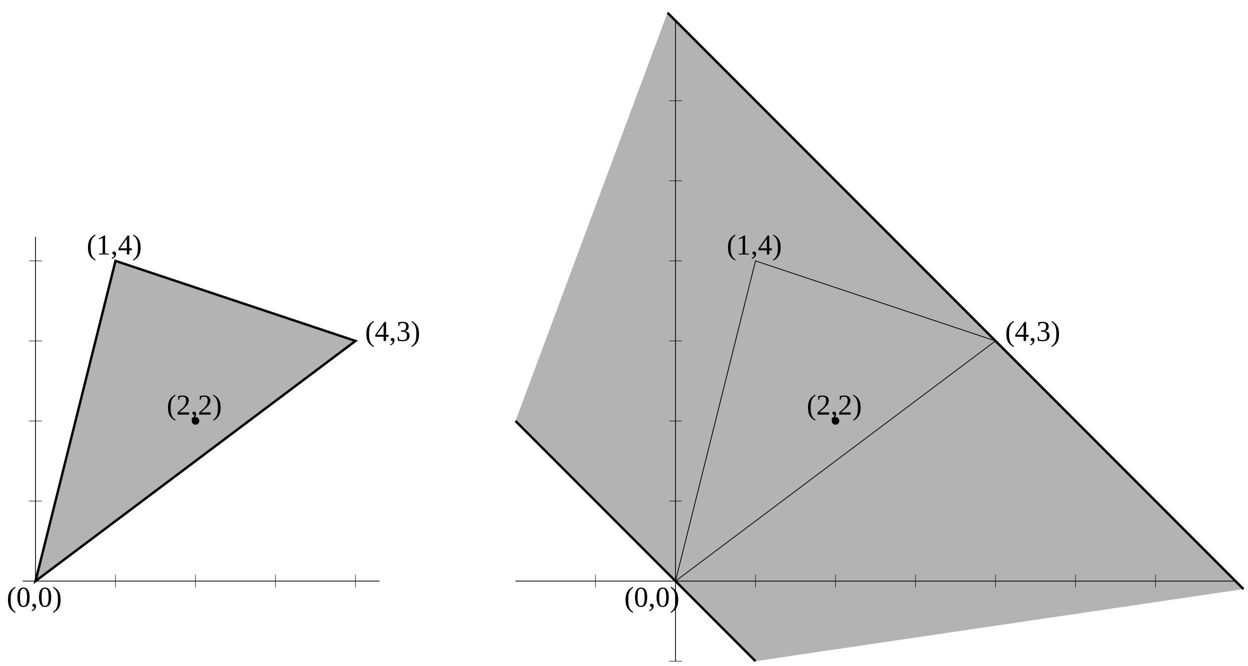
By making a side payment, the payoff vector (a_{ij}, b_{ij}) can be changed to $(a_{ij}+s, b_{ij}-s)$. If the number s is positive, this represents a payment from Player II to Player I. If s is negative, the side payment is from Player I to Player II. Thus the whole line of slope -1 through the point (a_{ij}, b_{ij}) is part of the TU feasible set. And we may take probability mixtures of these as well.

Definition. *The TU feasible set is the convex hull of the set of vectors of the form* $(a_{ij}+s, b_{ij}-s)$ *for* $i=1,\ldots,m$ *and* $j=1\ldots,n$ *and for arbitrary real numbers* s.

As an example, the bimatrix game

$$\begin{pmatrix} (4,3) & (0,0) \\ (2,2) & (1,4) \end{pmatrix} \tag{1}$$

has two pure strategic equilibria, upper left and lower right. This game has the NTU feasible and TU feasible sets given in Figure 4.1.



The NTU Feasible Set

The TU Feasible Set

Figure 4.1

If an agreement is reached in a cooperative game, be it a TU or an NTU game, it may be expected to be such that no player can be made better off without making at least one other player worse off. Such an outcome is said to be Pareto optimal.

Definition. *A feasible payoff vector,* (v_1, v_2), *is said to be* **Pareto optimal** *if the only feasible payoff vector* (v_1', v_2') *such that* $v_1' \geq v_1$ *and* $v_2' \geq v_2$ *is the vector* $(v_1', v_2') = (v_1, v_2)$.

In the example above, the Pareto feasible outcomes for the NTU game are simply the vectors on the line segment joining the points $(4,3)$ and $(1,4)$. The Pareto optimal outcomes for the TU game are the vectors on the line of slope -1 through the point $(4,3)$.

For more general convex feasible sets in the plane, the set of Pareto optimal points is the set of upper right boundary points.

4.2 Cooperative Games with Transferable Utility. In this section, we restrict attention to the transferable utility case and assume that the players are "rational" in the sense that, given a choice between two possible outcomes of differing personal utility, each player will select the one with the higher utility.

The TU-Problem: In the model of the game, we assume there is a period of *preplay negotiation*, during which the players meet to discuss the possibility of choosing a joint strategy together with some possible side payment to induce cooperation. They also discuss what will happen if they cannot come to an agreement; each may threaten to use some unilateral strategy that is bad for the opponent.

If they do come to an agreement, it may be assumed that the payoff vector is Pareto optimal. This is because if the players are about to agree to some feasible vector v and there is another feasible vector, v', that is better for one of the players without making any other player worse off, that player may propose changing to the vector v', offering to transfer some of his gain in utility to the other players. The other players, being rational would agree to this proposal.

In the discussion, both players may make some threat of what strategy they will take if an agreement is not reached. However, a threat to be believable must not hurt the player who makes it to a greater degree than the opponent. Such a threat would not be credible. For example, consider the following bimatrix game.

$$\begin{pmatrix} (5,3) & (0,-4) \\ (0,0) & (3,6) \end{pmatrix} \tag{2}$$

If the players come to an agreement, it will be to use the lower right corner because it has the greatest total payoff, namely 9. Player II may argue that she should receive at least half the sum, $4\frac{1}{2}$. She may even feel generous in "giving up" as a side payment some of the 6 she would be winning. However, Player I may threaten to use row 1 unless he is given at least 5. That threat is very credible since if Player I uses row 1, Player II's cannot make a counter-threat to use column 2 because it would hurt her more than Player I. The counter-threat would not be credible.

In this model of the preplay negotiation, the threats and counter-threats may be made and remade until time to make a decision. Ultimately the players announce what threats they will carry out if agreement is not reached. It is assumed that if agreement is not reached, the players will leave the negotiation table and carry out their threats. However, being rational players, they will certainly reach agreement, since this gives a higher utility. The threats are only a formal method of arriving at a reasonable amount for the side payment, if any, from one player to the other.

The TU problem then is to choose the threats and the proposed side payment judiciously. The players use threats to influence the choice of the final payoff vector. The problem is how do the threats influence the final payoff vector, and how should the players choose their threat strategies? For two-person TU-games, there is a very convincing answer.

The TU Solution: If the players come to an agreement, then rationality implies that they will agree to play to achieve the largest possible total payoff, call it σ,

$$\sigma = \max_i \max_j (a_{ij} + b_{ij}) \tag{3}$$

as the payoff to be divided between them. That is they will jointly agree to use some row i_0 and column j_0 such that $a_{i_0 j_0} + b_{i_0 j_0} = \sigma$. Such a joint choice $\langle i_0, j_0 \rangle$, is called their **cooperative strategy**. But they must also agree on some final payoff vector (x^*, y^*), such that $x^* + y^* = \sigma$, as the appropriate division of the total payoff. Such a division may then require a **side payment** from one player to the other. If $x^* > a_{i_0 j_0}$, then Player I would receive a side payment of the difference, $x^* - a_{i_0 j_0}$, from Player II. If $x^* < a_{i_0 j_0}$, then Player II would receive a side payment of the difference, $a_{i_0 j_0} - x^*$, from Player I.

Suppose now that the players have selected their **threat strategies**, say $\boldsymbol{p}$ for Player I and $\boldsymbol{q}$ for Player II. Then if agreement is not reached, Player I receives $\boldsymbol{p}^{\mathsf{T}}\boldsymbol{A}\boldsymbol{q}$ and Player II receives $\boldsymbol{p}^{\mathsf{T}}\boldsymbol{B}\boldsymbol{q}$. The resulting payoff vector,

$$\boldsymbol{D} = \boldsymbol{D}(\boldsymbol{p}, \boldsymbol{q}) = (\boldsymbol{p}^{\mathsf{T}}\boldsymbol{A}\boldsymbol{q}, \boldsymbol{p}^{\mathsf{T}}\boldsymbol{B}\boldsymbol{q}) = (D_1, D_2) \tag{4}$$

is in the NTU feasible set and is called the **disagreement point** or **threat point**. Once the disagreement point is determined, the players must agree on the point (x, y) on the line $x + y = \sigma$ to be used as the cooperative solution. Player I will accept no less than D_1 and Player II will accept no less than D_2 since these can be achieved if no agreement is reached. But once the disagreement point has been determined, the game becomes symmetric. The players are arguing about which point on the line interval from $(D_1, \sigma - D_1)$ to $(\sigma - D_2, D_2)$ to select as the cooperative solution. No other considerations with respect to the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ play any further role. Therefore, the midpoint of the interval, namely

$$\boldsymbol{\varphi} = (\varphi_1, \varphi_2) = \left(\frac{\sigma - D_2 + D_1}{2}, \frac{\sigma - D_1 + D_2}{2} \right) \tag{5}$$

is the natural compromise. Both players suffer equally if the agreement is broken. The point, $\boldsymbol{\varphi}$, may be determined by drawing the line from $\boldsymbol{D}$ with 45° slope until it hits the line $x + y = \sigma$ as in Figure 4.2.

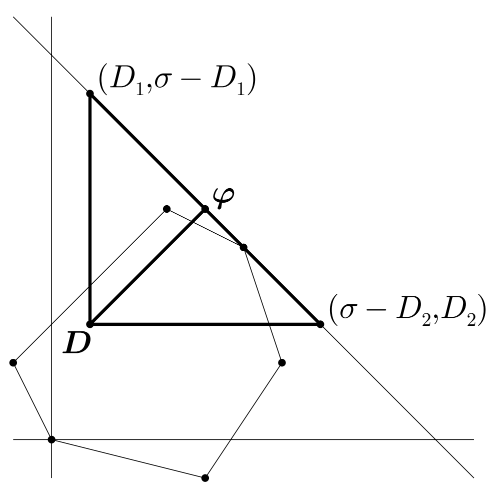


Figure 4.2

We see from (5) what criterion the players should use to select the threat point. Player I wants to maximize $D_1 - D_2$ and Player II wants to minimize it. This is in fact a zero-sum game with matrix $\boldsymbol{A} - \boldsymbol{B}$:

$$D_1 - D_2 = \boldsymbol{p}^{\mathsf{T}}\boldsymbol{A}\boldsymbol{q} - \boldsymbol{p}^{\mathsf{T}}\boldsymbol{B}\boldsymbol{q} = \boldsymbol{p}^{\mathsf{T}}(\boldsymbol{A} - \boldsymbol{B})\boldsymbol{q}. \tag{6}$$

Let $\boldsymbol{p}^*$ and $\boldsymbol{q}^*$ denote optimal strategies of the game $\boldsymbol{A} - \boldsymbol{B}$ for Players I and II respectively, and let δ denote the value,

$$\delta = \mathrm{Val}(\boldsymbol{A} - \boldsymbol{B}) = \boldsymbol{p}^{*\mathsf{T}}(\boldsymbol{A} - \boldsymbol{B})\boldsymbol{q}^*. \tag{7}$$

If Player I uses $\boldsymbol{p}^*$ as his threat, then the best Player II can do is to use $\boldsymbol{q}^*$, and conversely. When these strategies are used, the disagreement point becomes $\boldsymbol{D}^* = (D_1^*, D_2^*) = \boldsymbol{D}(\boldsymbol{p}^*, \boldsymbol{q}^*)$. Since $\delta = \boldsymbol{p}^{*\mathsf{T}}\boldsymbol{A}\boldsymbol{q}^* - \boldsymbol{p}^{*\mathsf{T}}\boldsymbol{B}\boldsymbol{q}^* = D_1^* - D_2^*$, we have as the TU solution:

$$\boldsymbol{\varphi}^* = (\varphi_1^*, \varphi_2^*) = \left(\frac{\sigma + \delta}{2}, \frac{\sigma - \delta}{2}\right). \tag{8}$$

Suppose the players have decided on $\langle i_0, j_0\rangle$ as the cooperative strategy to be used, where $a_{i_0j_0} + b_{i_0j_0} = \sigma$. To achieve the payoff (8), this requires a side payment of $(\sigma+\delta)/2 - a_{i_0j_0}$ from Player II to Player I. If this quantity is negative, the payment of $a_{i_0j_0} - (\sigma+\delta)/2$ goes from Player I to Player II.

Examples. 1. Consider the TU game with bimatrix

$$\begin{pmatrix} (0,0) & (6,2) & (-1,2) \\ (4,-1) & (3,6) & (5,5) \end{pmatrix}.$$

This is the matrix upon which Figure 4.2 is based. But we shall see that the optimal disagreement point is in a somewhat different place than the one in the figure.

The maximum value of $a_{ij} + b_{ij}$ occurs in the second row third column, so the cooperative strategy is $\langle 2, 3\rangle$, giving a total payoff of $\sigma = 10$. If they come to an agreement, Player I will select the second row, Player II will select the third column and both players will receive a payoff of 5. They must still decide on a side payment, if any.

They consider the zero-sum game with matrix,

$$\boldsymbol{A} - \boldsymbol{B} = \begin{pmatrix} 0 & 4 & -3 \\ 5 & -3 & 0 \end{pmatrix}.$$

The first column is strictly dominated by the last. The threat strategies are then easily determined to be

$$\boldsymbol{p}^* = (.3, .7)^{\mathsf{T}}$$
$$\boldsymbol{q}^* = (0, .3, .7)^{\mathsf{T}}$$

The value of this game is: $\delta = \text{Val}\begin{pmatrix} 4 & -3 \\ -3 & 0 \end{pmatrix} = -9/10$. Therefore from (8), the TU-value is

$$\boldsymbol{\varphi}^* = ((10 - .9)/2, (10 + .9)/2) = (4.55, 5.45).$$

To arrive at this payoff from the agreed payoff vector, $(5,5)$, requires a side payment of 0.45 from Player I to Player II.

We may also compute the disagreement point, $\boldsymbol{D}^* = (D_1^*, D_2^*)$.

$$\begin{aligned} D_1^* &= \boldsymbol{p}^{*\,T}\boldsymbol{A}\boldsymbol{q}^* = .3(6 \cdot .3 - .7) + .7(3 \cdot .3 + 5 \cdot .7) = 3.41 \\ D_2^* &= \boldsymbol{p}^{*\,T}\boldsymbol{B}\boldsymbol{q}^* = .3(2 \cdot .3 + 2 \cdot .7) + .7(6 \cdot .3 + 5 \cdot .7) = 4.31 \end{aligned}$$

It is easy to see that the line from $\boldsymbol{D}^*$ to $\boldsymbol{\varphi}^*$ is 45°, because $D_2^* - D_1^* = \varphi_2^* - \varphi_1^* = 0.9$.

2. It is worthwhile to note that there may be more than one possible cooperative strategy yielding σ as the sum of the payoffs. The side payment depends on which one is used. Also there may be more than one possible disagreement point because there may be more than one pair of optimal strategies for the game $\boldsymbol{A} - \boldsymbol{B}$. However, all such disagreement points must be on the same 45° line, since the point $\boldsymbol{\varphi}$ depends on the disagreement point only through the value, δ, and all disagreement points have the same TU-value.

Here is an example containing both possibilities.

$$\begin{pmatrix} (1,5) & (2,2) & (0,1) \\ (4,2) & (1,0) & (2,1) \\ (5,0) & (2,3) & (0,0) \end{pmatrix}$$

There are two cooperative strategies giving total payoff $\sigma = 6$, namely $\langle 1,1 \rangle$ and $\langle 2,1 \rangle$. The matrix $\boldsymbol{A} - \boldsymbol{B}$ is

$$\begin{pmatrix} -4 & 0 & -1 \\ 2 & 1 & 1 \\ 5 & -1 & 0 \end{pmatrix}$$

which has a saddle-point at $\langle 2,3 \rangle$. Thus $\boldsymbol{D} = (2,1)$ is the disagreement point, and the value is $\delta = 1$. Thus the TU cooperative value is $\boldsymbol{\varphi} = (7/2, 5/2)$.

However, there is another saddle-point at $\langle 2,2 \rangle$ that, of course, has the same value $\delta = 1$. But this time the disagreement point is $\boldsymbol{\varphi} = (1,0)$. All such disagreement points must be on the 45° line through $\boldsymbol{\varphi}$.

If $\langle 2,1 \rangle$ is used as the cooperative strategy, the resulting vector payoff of $(4,2)$ requires that Player I pay 1/2 to Player I. If $\langle 1,1 \rangle$ is used as the cooperative strategy, the resulting vector payoff of $(5,1)$ requires that Player I pay 3/2 to Player II.

4.3 Cooperative Games with Non-Transferable Utility.

We now consider games in which side payments are forbidden. It may be assumed that the utility scales of the players are measured in noncomparable units. The players may argue, threaten, and come to a binding agreement as before, but there is no monetary unit with which the players can agree to make side payments. The players may barter goods that they own, but this must be done within the game and reflected in the bimatrix of the game.

We approach NTU games through the **Nash Bargaining Model**. This model is based on two elements assumed to be given and known to the players. One element is a compact (i.e. bounded and closed), convex set, S, in the plane. One is to think of S as the set of vector payoffs achievable by the players if they agree to cooperate. It is the analogue of the NTU-feasible set, although it is somewhat more general in that it does not have to be a polyhedral set. It could be a circle or an ellipse, for example. We refer to S as the **NTU-feasible set**.

The second element of the Nash Bargaining Model is a point, $(u^*, v^*) \in S$, called the **threat point** or **status-quo point**. Nash viewed the bargaining model as a game between two players who come to a market to barter goods. For him, the players have the option of not entering into any trade agreement at all, and it was natural for him to take the status-quo point as $(u^*, v^*) = (0, 0) \in S$. The subsequent theory requires that (u^*, v^*) be an arbitrary point of S.

Given an NTU-feasible set, S, and a threat point, $(u^*, v^*) \in S$, the problem is to decide on a feasible outcome vector for this game that will somehow reflect the value of the game to the players. That is, we want to find a point, $(\bar{u}, \bar{v}) = \boldsymbol{f}(S, u^*, v^*)$, to be considered a "fair and reasonable outcome" or "solution" of the game for an arbitrary compact convex set S and point $(u^*, v^*) \in S$. In the approach of Nash, "fair and reasonable" is defined by a few axioms. Then it is shown that these axioms lead to a *unique* solution, $\boldsymbol{f}(S, u^*, v^*)$. Here are the axioms.

Nash Axioms for $\boldsymbol{f}(S, u^*, v^*) = (\bar{u}, \bar{v})$.

(1) **Feasibility.** $(\bar{u}, \bar{v}) \in S$.

(2) **Pareto Optimality.** There is no point $(u, v) \in S$ such that $u \geq \bar{u}$ and $v \geq \bar{v}$ except $(\bar{u}, \bar{v})$ itself.

(3) **Symmetry.** If S is symmetric about the line $u = v$, and if $u^* = v^*$, then $\bar{u} = \bar{v}$.

(4) **Independence of irrelevant alternatives.** If T is a closed convex subset of S, and if $(u^*, v^*) \in T$ and $(\bar{u}, \bar{v}) \in T$, then $\boldsymbol{f}(T, u^*, v^*) = (\bar{u}, \bar{v})$.

(5) **Invariance under change of location and scale.** If $T = \{(u', v') : u' = \alpha_1 u + \beta_1, v' = \alpha_2 v + \beta_2 \text{ for } (u, v) \in S\}$, where $\alpha_1 > 0$, $\alpha_2 > 0$, β_1, and β_2 are given numbers, then

$$\boldsymbol{f}(T, \alpha_1 u^* + \beta_1, \alpha_2 v^* + \beta_2) = (\alpha_1 \bar{u} + \beta_1, \alpha_2 \bar{v} + \beta_2).$$

Analysis of the Axioms. It is useful to review the axioms to see which might be weakened or changed to allow other "solutions".

The first axiom is incontrovertible. The agreed outcome must be feasible.

The second axiom reflects the rationality of the players. If the players work together and reach agreement, they would not accept (u, v) as the outcome if they could also achieve $(\hat{u}, \hat{v})$ with $\hat{u} > u$ and $\hat{v} > v$. However, the second axiom is slightly stronger than this. It says that they would not accept (u, v) if they could achieve $(\hat{u}, \hat{v})$ with $\hat{u} \geq u$ and $\hat{v} > v$ (or $\hat{u} > u$ and $\hat{v} \geq v$). This plays no role in the main case of the theorem (when there is a $(u, v) \in S$ such that $u > u^*$ and $v > v^*$). But suppose S consists of the line from (0,0) to (0,1), inclusive, and $(u^*, v^*) = (0, 0)$. Player 1 can achieve a payoff of 0 without entering into any agreement. So to agree to the point (0,1) requires a weak kind of altruistic behavior on his part. It is true that this agreement would not hurt him, but still this weak altruism does not follow from the assumed rationality of the players.

The third axiom is a fairness axiom. If the game is symmetric in the players, there is nothing in the game itself to distinguish the players so the outcome should be symmetric.

The fourth axiom is perhaps the most controversial. It says that if two players agree that $(\bar{u}, \bar{v})$ is a fair and reasonable solution when S is the feasible set, then points in S far away from $(\bar{u}, \bar{v})$ and (u^*, v^*) are irrelevant. If S is reduced to a subset $T \subset S$, then as long as T still contains $(\bar{u}, \bar{v})$ and (u^*, v^*), the players would still agree on $(\bar{u}, \bar{v})$. But let S be the triangle with vertices (0,0), (0,4) and (2,0), and let the threat point be (0,0). Suppose the players agree on (1,2) as the outcome. Would they still agree on (1,2) if the feasible set were T, the quadralateral with vertices (0,0), (0,2), (1,2) and (2,0)? Conversely, if they agree on (1,2) for T, would they agree on (1,2) for S? The extra points in S cannot be used as threats because it is assumed that all threats have been accounted for in the assumed threat point. These extra points then represent unattainable hopes or ideals, which player 2 admits by agreeing to the outcome (1,2).

The fifth axiom, just reflects the understanding that the utilities of the players are separately determined only up to change of location and scale. Thus, if one of the players decides to change the location and scale of his utility, this changes the numbers in the bimatrix, but does not change the game. The agreed solution should undergo the same change.

Theorem. *There exists a unique function* $\boldsymbol{f}$ *satisfying the Nash axioms. Moreover, if there exists a point* $(u, v) \in S$ *such that* $u > u^*$ *and* $v > v^*$*, then* $\boldsymbol{f}(S, u^*, v^*)$ *is that point of* S *that maximizes* $(u - u^*)(v - v^*)$ *among points of* S *such that* $u \geq u^*$ *and* $v \geq v^*$.

Below we sketch the proof in the interesting case where there exists a point $(u, v) \in S$ such that $u > u^*$ and $v > v^*$. The uninteresting case is left to the exercises.

First we check that the point $(\bar{u}, \bar{v})$ in $S^+ = \{(u, v) \in S : u \geq u^*, v \geq v^*\}$ indeed satisfies the Nash axioms. The first 4 axioms are very easy to verify. To check the fifth axiom, note that

if $(u - u^*)(v - v^*)$ is maximized over S^+ at $(\bar{u}, \bar{v})$,

then $(\alpha_1 u - \alpha_1 u^*)(\alpha_2 v - \alpha_2 v^*)$ is maximized over S^+ at $(\bar{u}, \bar{v})$,

so $(\alpha_1 u + \beta_1 - \alpha_1 u^* - \beta_1)(\alpha_2 v + \beta_2 - \alpha_2 v^* - \beta_2)$ is maximized over S^+ at $(\bar{u}, \bar{v})$,

hence $(u' - \alpha_1 u^* - \beta_1)(v' - \alpha_2 v^* - \beta_2)$ is maximized over T^+ at $(\alpha_1 \bar{u} + \beta_1, \alpha_2 \bar{v} + \beta_2)$, where $T^+ = \{(u', v') \in S^+ : u' = \alpha_1 u + \beta_1, v' = \alpha_2 v + \beta_2\}$.

To see that the axioms define the point uniquely, we find what $(\bar{u},\bar{v})$ must be for certain special sets S, and extend step by step to all closed bounded convex sets. First note that if S is symmetric about the line $u=v$ and $(0,0)\in S$, then axioms (1), (2), and (3) imply that $\boldsymbol{f}(S,0,0)$ is that point $(z,z)\in S$ farthest up the line $u=v$. Axiom 4 then implies that if T is any closed bounded convex subset of the half plane $H_z=\{(u,v): u+v\le 2z\}$ where $z>0$, and if $(z,z)\in T$ and $(0,0)\in T$, then $\boldsymbol{f}(T,0,0)=(z,z)$, since such a set T is a subset of a symmetric set with the same properties.

Now for an arbitrary closed convex set S and $(u^*,v^*)\in S$, let $(\hat{u},\hat{v})$ be the point of S^+ that maximizes $(u-u^*)(v-v^*)$. Define α_1, β_1, α_2, and β_2 so that $\begin{cases}\alpha_1 u^*+\beta_1=0\\ \alpha_1\hat{u}+\beta_1=1\end{cases}$ and $\begin{cases}\alpha_2 v^*+\beta_2=0\\ \alpha_2\hat{v}+\beta_2=1\end{cases}$, and let T be as in axiom 5. According to the invariance of $(\hat{u},\hat{v})$ under change of location and scale, the point $(1,1)=(\alpha_1\hat{u}+\beta_1,\alpha_2\hat{v}+\beta_2)$ maximizes $u\cdot v$ over T^*. Since the slope of the curve $uv=1$ at the point $(1,1)$ is -1, the set T, being convex, is a subset of H_1, and so by the last sentence of the previous paragraph, $\boldsymbol{f}(T,0,0)=(1,1)$. By axiom 5, $\boldsymbol{f}(T,0,0)=(\alpha_1\bar{u}+\beta_1,\alpha_2\bar{v}+\beta_2)$ where $\boldsymbol{f}(S,u^*,v^*)=(\bar{u},\bar{v})$. Since $(\alpha_1\bar{u}+\beta_1,\alpha_2\bar{v}+\beta_2)=(1,1)=(\alpha_1\hat{u}+\beta_1,\alpha_2\hat{v}+\beta_2)$, we have $\bar{u}=\hat{u}$ and $\bar{v}=\hat{v}$, so that $(\hat{u},\hat{v})=\boldsymbol{f}(S,u^*,v^*)$. ∎

Here is a geometric interpretation. Consider the curves (hyperbolas) $(u-u^*)(v-v^*)=c$ for a constant c. For large enough c, this curve will not intersect S. Now bring c down until the curve just osculates S. The NTU-solution is the point of osculation.

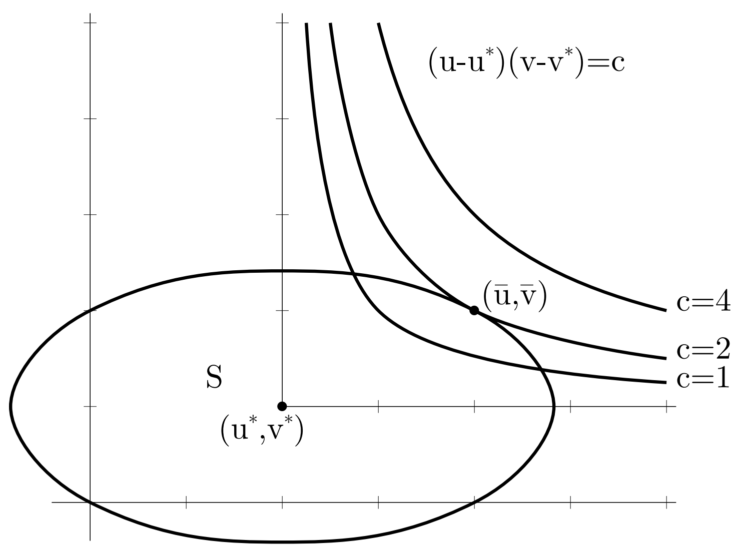


Figure 4.3

Moreover, at the point $(\bar{u},\bar{v})$, of osculation, the slope of the curve is the negative of the slope of the line from (u^*,v^*) to $(\bar{u},\bar{v})$. (Check this.)

Examples. 1. Let S be the triangle with vertices $(0,0)$, $(0,1)$ and $(3,0)$, and let the threat point be $(0,0)$. The Pareto optimal boundary is the line from $(0,1)$ to $(3,0)$ of slope $-1/3$. The curve of the form $u\cdot v=c$ that osculates this line must have slope $-1/3$ at the point of osculation. So the slope of the line from (0,0) to $(\bar{u}, \bar{v})$ must be 1/3. This intersects the Pareto boundary at the midpoint, $(3/2, 1/2)$. This is therefore the NTU-solution.

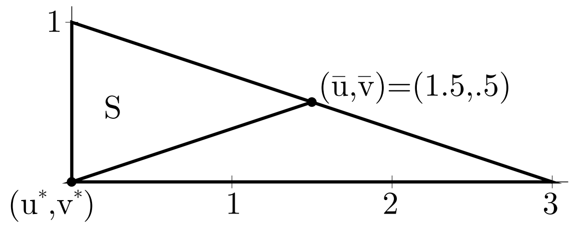


Figure 4.4

2. Let the NTU-feasible set be the ellipse, $S = \{(x, y) : (x-2)^2 + 4(y-1)^2 \le 8\}$. Let the threat point be $(u^*, v^*) = (2, 1)$. The situation looks much like that of Figure 4.2. The point $(x, y) \in S$ that maximizes the product, $(x-2)(y-1)$, is the NTU-solution. This point must be on the Pareto optimal boundary consisting of the arc of the ellipse from $(2, 1+\sqrt{2})$ to $(2+2\sqrt{2}, 1)$. On this arc, $y - 1 = \sqrt{2-(x-2)^2/4}$, so we seek $x \in [2, 2+2\sqrt{2}]$ to maximize $(x-2)(y-1) = (x-2)\sqrt{2-(x-2)^2/4}$. The derivative of this is $\sqrt{2-(x-2)^2/4} - (x-2)^2/4\sqrt{2-(x-2)^2/4}$. Setting this to zero reduces to $(x-2)^2 = 4$, whose roots are 2 ± 2. Since $x \in [2, 2+2\sqrt{2}]$, we must have $x = 4$, and $y = 2$. Therefore $(\bar{u}, \bar{v}) = (4, 2)$ is the NTU-solution of the game.

3. Consider the game with bimatrix (1), whose NTU-feasible set is given in Figure 4.1(a). What should be taken as the threat point? If we take the view of Nash, that either player may refuse to enter into agreement, thus leaving the players at the status-quo point $(0, 0)$, then we should add this strategy of non-cooperation to the player's pure strategy sets. The resulting bimatrix of the game is really

$$\begin{pmatrix} (4,3) & (0,0) & (0,0) \\ (2,2) & (1,4) & (0,0) \\ (0,0) & (0,0) & (0,0) \end{pmatrix} \tag{9}$$

This has the same NTU-feasible set as Figure 4.1(a). We may take the threat point to be $(u^*, v^*) = (0, 0)$. The Pareto optimal boundary is the line segment from (1,4) to (4,3). This line has slope $-1/3$. However, the line of slope 1/3 from the origin intersects the extension of this line segment at a point to the right of (4,3). This means that $x \cdot y$ increases as (x, y) travels along this line segment from (1,4) to (4,3). The NTU-solution is therefore (4,3).

The Lambda-Transfer Approach. There is another approach, due to Lloyd Shapley, to solving the NTU-problem that has certain advantages over the Nash approach. First, it relates the solution to the corresponding solution to the TU-problem. Second, it avoids the difficult-to-justify fourth axiom. Third, the threat point arises naturally as a function of the bimatrix and does not have to be specified a priori. Fourth, it extends to more general problems, but when specialized to the problems with status-quo point (0,0), it gives the same answer as the Nash solution.

The main difficulty with the NTU-problems is the lack of comparability of the utilities. If we pretend the utilities are measured in the same units and apply the TU-theory to arrive at a solution, it may happen that the TU-solution is not in the NTU-feasible set. If it did happen to be in the NTU-feasible set, the players can use it as the NTU-solution since it can be achieved without any transfer of utility. But what can be done if the TU-solution is not in the NTU-feasible set?

Recall that the utilities are not measured in the same units. Someone might suggest that an increase of one unit in Player 1's utility is worth an increase of λ units in Player 2's utility, where $\lambda > 0$. If that were so, we could analyze the game as follows. If the original bimatrix is $(\boldsymbol{A}, \boldsymbol{B})$, we first consider the game with bimatrix $(\lambda\boldsymbol{A}, \boldsymbol{B})$, solve it for the TU-solution, and then divide Player 1's payoff by λ to put it back into Player 1's original units. This is called the **λ-transfer game**. By the methods of Section 4.2, the TU-solution to the λ-transfer game is $\boldsymbol{\varphi}(\lambda) = (\varphi_1(\lambda), \varphi_2(\lambda))$, where

$$\varphi_1(\lambda) = \frac{\sigma(\lambda) + \delta(\lambda)}{2\lambda}, \qquad \varphi_2(\lambda) = \frac{\sigma(\lambda) - \delta(\lambda)}{2} \tag{10}$$

where $\sigma(\lambda) = \max_{ij}\{\lambda a_{ij} + b_{ij}\}$ and $\delta(\lambda) = \mathrm{Val}(\lambda\boldsymbol{A} - \boldsymbol{B})$. If the point $\boldsymbol{\varphi}(\lambda)$ is in the NTU-feasible set, it could, with the justification given earlier, be used as the NTU-solution.

It turns out that there generally exists a *unique* value of λ, call it λ^*, such that $\boldsymbol{\varphi}(\lambda^*)$ is in the NTU-feasible set. This $\boldsymbol{\varphi}(\lambda^*)$ can be used as the NTU-solution. The value of λ^* is called the **equilibrium exchange rate**.

Though it is easy, in principle with the help of a computer, to approximate the value of $\boldsymbol{\varphi}(\lambda^*)$, it is difficult to find $\mathrm{Val}(\lambda\boldsymbol{A} - \boldsymbol{B})$ analytically. This problem disappears for bimatrix games, $(\boldsymbol{A}, \boldsymbol{B})$, when the matrices $\boldsymbol{A}$ and $-\boldsymbol{B}$ have saddle points in the same position in the matrix. Such bimatrix games, when played as NTU games, are said to be **fixed threat point games**. This is because whatever be the value of λ, the matrix game, $\lambda\boldsymbol{A} - \boldsymbol{B}$, which is used to determine the threat point, has a saddle point at that same position in the matrix. Thus, the threat strategy and the threat point of the λ-transfer game will not depend on λ. For example, in the bimatrix (10), the fixed threat point is in the lower right corner, because both $\boldsymbol{A}$ and $-\boldsymbol{B}$ have saddle points there.

Here is another example. Consider the bimatrix,

$$\begin{pmatrix} (-1,1) & (1,3) \\ (0,0) & (3,-1) \end{pmatrix} \tag{11}$$

Both $\boldsymbol{A}$ and $-\boldsymbol{B}$ have saddlepoints in the lower left corner. Therefore, $\lambda\boldsymbol{A} - \boldsymbol{B} = \begin{pmatrix} -\lambda-1 & \lambda-3 \\ 0 & 3\lambda+1 \end{pmatrix}$ has a saddle point in the lower left corner, whatever be the value of $\lambda \geq 0$. So, (0,0) is a fixed threat point of the game. To find the NTU-solution, we change scale along the x-axis until we find a λ^* such that the diagonal line, $x = y$, exits S at a point of slope -1 (or more precisely, S is contained below the line of slope -1 at the point of exit). Then we change scale back, and find that the line of slope λ^* from (0,0) exits the set S at a point of slope $-\lambda^*$. The point of exit is of course the NTU-solution, and the method of finding it is the same as for the Nash solution!

In the example, $\lambda^* = 2$, and the NTU-solution is $(1.25, 2.5)$.

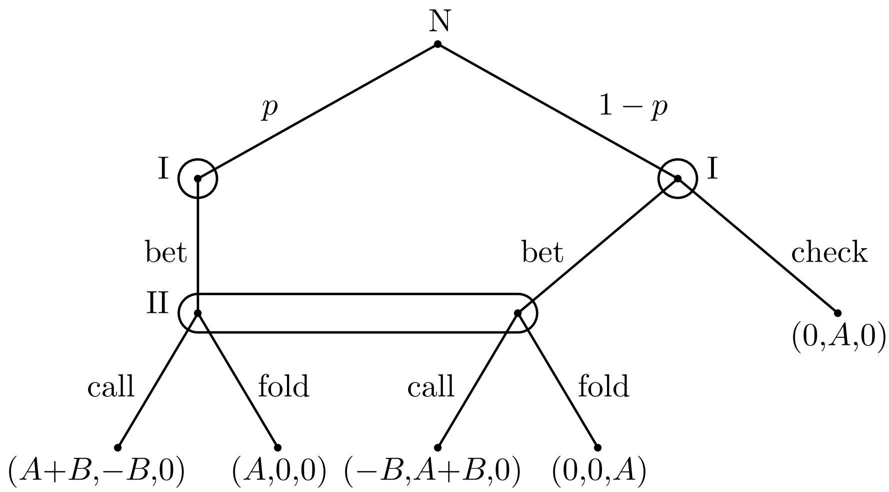
The argument, that the NTU-solution for the λ-transfer game with a fixed threat point is the Nash solution, is perfectly general and works even if the fixed threat point is not the origin. The NTU-solution is found as the point of exit from S of a line of slope λ^* from the fixed threat point, such that all of S is contained below the line of slope $-\lambda^*$ through the point of exit.

4.4 End-Game with an All-In Player. Poker is usually played with the "table-stakes" rule. This rule states that each player risks only the amount of money before him at the beginning of the hand. This means that a player can lose no more than what is in front of him on the table. It also means that a player cannot add to this amount during the play of a hand.

When a player puts all the money before him into the pot, he is said to be "all-in". When a player goes all-in, his money and an equal amount matched by each other player still contesting the pot is set aside and called the main pot. All further betting, if any, occurs only among the remaining players who are not all in. These bets are placed in a side pot. Since the all-in player places no money in the side pot, he/she is not allowed to win it. The winner of the main pot is the player with the best hand and the winner of the side pot is the non-all-in player with the best hand. Betting in the side pot may cause a player to fold. When this happens, he/she relinquishes all rights to winning the main pot as well as the side pot. This gives the all-in player a subtle advantage. Occasionally, a player in the side pot will fold with a winning hand which allows the all-in player to win the main pot, which he would not otherwise have done. This possibility leads to some interesting problems.

As an example, consider End-Game in which there is an all-in player. Let's call the all-in player Player III. He can do nothing. He can only watch and hope to win the main pot. Let's assume he has four kings showing in a game of 5 card stud poker. Unfortunately for him, Player II has a better hand showing, say four aces. However, Player I has a possible straight flush. If he has the straight flush, he beats both players; otherwise he loses to both players. How should this hand be played? Will the all-in player with the four kings ever win the main pot?

We set up an idealized version of the problem mathematically. Let A denote the size of the main pot, and let p denote the probability that Player I has the winning hand. (Assume that the value of p is common knowledge to the players.) As in End-Game, Player I acts first by either checking or betting a fixed amount B. If Player I checks, he wins A if he has the winning hand, and Player II wins A otherwise. Player III wins nothing. If Player I bets, Player II may call or fold. If Player II calls and Player I has a winning hand, Player I wins $A + B$, Player II loses B, and Player III wins nothing. If Player II calls and Player I does not have the winning hand, Player I loses B and Player II wins $A + B$ and Player III wins nothing. If Player II folds and Player I has the winning hand, Player I



wins A and the others win nothing. But if Player II folds and Player I does not have the winning hand, Player III wins A and the others win nothing.

As in Basic Endgame, Player I never gains by checking with a winning hand, so Player I has two pure strategies, the *honest* strategy (bet with a winning hand, fold with a losing hand) and the *bluff* strategy (bet with a winning or losing hand). Player II also has two pure strategies, *call*, and *fold*. Player III has no strategies so the payoffs of the three players may be written in a 2 by 2 table as follows.

$$\begin{array}{l c} & \begin{array}{cc} \quad\quad\quad\quad\quad\quad\quad \textit{call} \quad\quad\quad\quad\quad\quad\quad & \quad\quad\quad \textit{fold} \end{array} \\ \begin{array}{l} \textit{honest} \\ \textit{bluff} \end{array} & \begin{pmatrix} (p(A+B), -pB+(1-p)A, 0) & (pA, (1-p)A, 0) \\ (p(A+B)-(1-p)B,\ -pB+(1-p)(A+B),\ 0) & (pA, 0, (1-p)A) \end{pmatrix} \end{array}$$

Ordinarily, A will be somewhat larger than B, and p will be rather small. The analysis does not depend very much on the actual values of these three numbers, so it will be easier to understand if we take specific values for them. Let's take $A = 100$, $B = 10$ and $p = 1/10$. Then the payoff matrix becomes

$$\begin{array}{l c} & \begin{array}{cc} \textit{call} & \quad\quad \textit{fold} \end{array} \\ \begin{array}{l} \textit{honest} \\ \textit{bluff} \end{array} & \begin{pmatrix} (11, 89, 0) & (10, 90, 0) \\ (2, 98, 0) & (10, 0, 90) \end{pmatrix} \end{array}$$

This is a constant-sum game for three players, but since Player III has no strategy choices and since coalitions among players are strictly forbidden by the rules of poker, it is best to consider this as non-constant-sum game between Players I and II. Removing the payoff for Player III from consideration, the matrix becomes

$$\begin{array}{l c} & \begin{array}{cc} \textit{call} & \quad \textit{fold} \end{array} \\ \begin{array}{l} \textit{honest} \\ \textit{bluff} \end{array} & \begin{pmatrix} (11, 89) & (10, 90) \\ (2, 98) & (10, 0) \end{pmatrix} \end{array} \tag{12}$$

First note that row 1 is weakly dominates row 2, and if row 2 is removed, column 2 dominates column 1. This gives us an equilibrium at (row 1, column 2), with payoff

(10,90). Player I cannot gain by betting with a losing hand. Even if the bluff is successful and Player II folds, Player III will still beat him. Worse, Player II may call his bluff and he will lose the bet.

So it seems that Player I might as well be honest. The result is that Player I never bluffs and Player II always folds when Player I bets, and the payoff is (10,90,0). This is the accepted and time-honored course of action in real games in poker rooms around the world. But let's look at it more closely.

As long as Player II uses column 2, it doesn't hurt Player I to bluff. In fact, there are more equilibria. The strategy pair $(1-p,p)$ for Player I and column 2 for Player II is a equilibrium provided $p \le 1/99$ (Exercise 7(a)). The equilibrium with $p = 1/99$ has payoff $(10, 89\frac{1}{11}, \frac{10}{11})$. This takes payoff from Player II and gives it to Player III, something Player II wants to avoid. She may be willing to concede a small fracion of her winnings to avoid this.

Player I can play this equilibrium without loss. In fact, he can try for more by bluffing with probability greater than $p = 1/99$. If he does this, Player II's best reply is column 1. If Player I uses $p = 1/9$ and Player II uses column 2, Player I's average payoff is still $11(8/9) + 2(1/9) = 10$. A value of p between 1/99 and 1/9 encourages Player II to play column 1, and then Player I's average payoff will be greater than 10. So Player I wants more than 10, and Player II wants Player I not to use row 2.

Noncooperative games that are repeated in time in which side payments are not allowed may be considered as NTU cooperative games, in which the results of play over time take the place of preplay negotiations. If this game is analyzed as an NTU game by the methods of this section, the resulting payoff is $(10\frac{5}{6}, 89\frac{1}{6}, 0)$ (Exercise 7(b)). This may be achieved by Player I always playing row 1, and Player II playing column 1 with probability 5/6. In other words, Player II calls 5 times out of 6, even though she knows that Player I never bluffs. (Another way of achieving this payoff is to have Player II call all the time, and to have Player I bluff with probability 1/54.)

4.5 Exercises.

1. For the following bimatrix games, draw the NTU and TU feasible sets. What are the Pareto optimal outcomes?

$$\text{(a)} \quad \begin{pmatrix} (0,4) & (3,2) \\ (4,0) & (2,3) \end{pmatrix} \qquad\qquad \text{(b)} \quad \begin{pmatrix} (3,1) & (0,2) \\ (1,2) & (3,0) \end{pmatrix}$$

2. Find the cooperative strategy, the TU solution, the side payment, the optimal threat strategies, and the disagreement point for the two matrices (1) and (2) of Sections 4.1 and 4.2.

3. Find the cooperative strategy, the TU solution, the side payment, the optimal threat strategies, and the disagreement point for the following matrices of Exercise 2.5.5. For (a), you may want to use the Matrix Game Solver on the web at http://www.math.ucla.edu/~tom/gamesolve.html.

(a) $$\begin{pmatrix} (-3,-4) & (\ \ 2,-1) & (\ \ 0,\ \ 6) & (\ \ 1,\ \ 1) \\ (\ \ 2,\ \ 0) & (\ \ 2,\ \ 2) & (-3,\ \ 0) & (\ \ 1,-2) \\ (\ \ 2,-3) & (-5,\ \ 1) & (-1,-1) & (\ \ 1,-3) \\ (-4,\ \ 3) & (\ \ 2,-5) & (\ \ 1,\ \ 2) & (-3,\ \ 1) \end{pmatrix}.$$

(b) $$\begin{pmatrix} (\ \ 0,\ \ 0) & (\ \ 1,-1) & (\ \ 1,\ \ 1) & (-1,\ \ 0) \\ (-1,\ \ 1) & (\ \ 0,\ \ 1) & (\ \ 1,\ \ 0) & (\ \ 0,\ \ 0) \\ (\ \ 1,\ \ 0) & (-1,-1) & (\ \ 0,\ \ 1) & (-1,\ \ 1) \\ (\ \ 1,-1) & (-1,\ \ 0) & (\ \ 1,-1) & (\ \ 0,\ \ 0) \\ (\ \ 1,\ \ 1) & (\ \ 0,\ \ 0) & (-1,-1) & (\ \ 0,\ \ 0) \end{pmatrix}.$$

4. Let $S = \{(x, y) : y \geq 0 \quad \text{and} \quad y \leq 4 - x^2\}$ be the NTU-feasible set.

(a) Find the NTU-solution if $(u^*, v^*) = (0, 0)$.

(b) Find the NTU-solution if $(u^*, v^*) = (0, 1)$.

5. Find the NTU-solution and the equilibrium exchange rate for the following fixed threat point games.

(a) $\begin{pmatrix} (6,3) & (0,0) & (0,0) \\ (1,8) & (4,6) & (0,0) \\ (0,0) & (0,0) & (0,0) \end{pmatrix}$. (b) $\begin{pmatrix} (1,0) & (-1,1) & (0,0) \\ (3,3) & (-2,9) & (2,7) \end{pmatrix}$.

6. Find the NTU-solution and the equilibrium exchange rates of the following games without a fixed threat point.

(a) $\begin{pmatrix} (5,2) & (0,0) \\ (0,0) & (1,4) \end{pmatrix}$. (b) $\begin{pmatrix} (3,2) & (0,5) \\ (2,1) & (1,0) \end{pmatrix}$.

7. (a) In Endgame with an all-in player with the values $A = 100$, and $B = 10$, show that the strategy pair $(1-p, p)$ for Player I and column 2 for Player II is an equilibrium pair with payoff $(10, 90(1-p), 90p)$ provided $0 \leq p \leq 1/99$. Show that these are the only equilibria. Show that if $p = 1/99$, Players I and II only get their safety levels.

(b) Find the TU solution of this game. Show that the TU solution is in the NTU feasible set and so is also the NTU solution.