MA 101 | Mid-Semester Examination
Mathematics I | 14:00–16:00 IST
IIT GUWAHATI | September 19, 2023

Model Answers of Mid-Semester Examination

1. (5 points) Prove or disprove the following statement.
If $A \subset \mathbb{R}$ is such that every continuous function $f : A \to \mathbb{R}$ is bounded, then A is closed and bounded.
(If you want to prove the statement, then give a rigorous proof. If you want to disprove it, give an example of a set A, which is either not closed or unbounded, but every continuous function on A is bounded.)

Solution: Consider the function $f : A \to \mathbb{R}$ given by $f(x) = x$. Then, f is continuous. Since every continuous function on A is bounded,

$$\begin{aligned} & |f(x)| \leq M, \text{for some } M \text{ and for all } x \in A \\ \Rightarrow \quad & |x| \leq M, \forall x \in A. \end{aligned}$$

Thus, A is bounded.
Suppose A is not closed. Then $\exists$ a sequence $\{x_n\} \subset A$ such that $x_n \to a$, but $a \notin A$. Consider the function $f : A \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{x-a}.$$

Then, f is continuous, but $\lim_{n\to\infty} f(x_n) = \infty$, a contradiction. Thus A must be closed.

Common Mistakes and Comments: While proving the given statement you cannot assume anything about A, other than the given fact. As soon as you assume something about A, it becomes proof by example, which is not valid. So if you have written in your proof "Let $A = \ldots$", then you have not got any marks.

Also, a closed bounded set does not necessarily mean a close bounded interval. For example, the set $\left\{\frac{1}{n} : n = 1, 2, \ldots\right\} \cup 0$ is closed and bounded but not an interval. Many of you seem to have this misconception.

2. (5 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that both f and $|f|$ are differentiable. If $f(x_0) = 0$ for some $x_0 \in \mathbb{R}$, then show that $f'(x_0) = 0$.

Solution: Since $|f|$ is differentiable at x_0, $\lim_{h\to 0} \frac{|f(x_0+h)|}{h}$ exists. [Note that $f(x_0) = 0$.]
Now, as f is differentiable,

$$\lim_{h\to 0} \frac{|f(x_0+h)|}{|h|} = |f'(x_0)|.$$

Here we use the fact that $g(x) = |x|$ is continuous to push the limit inside the modulus.
Now, if $f'(x_0) \neq 0$, then

$$\lim_{h\to 0} \frac{\frac{|f(x_0+h)|}{h}}{\left|\frac{f(x_0+h)}{h}\right|} = \lim_{h\to 0} \frac{|h|}{h} \text{ exists, by quotient rule.}$$

This is a contradiction. Thus, $f'(x_0) = 0$.

Common Mistakes and Comments: Derivative of $|f|$ is different from absolute value of derivative of f.

Sign of derivative of f at a point does not tell anything about increasing/decreasing nature of the function in an interval.

3. (3 points) Evaluate

$$\lim_{x\to 0} \frac{\int_0^x (x-t)\sin(t^2)dt}{x^4}.$$

Solution:

$$\begin{aligned}\text{Let } F(x) &= \int_0^x (x-t)\sin(t^2)dt.\\ &= x\int_0^x \sin(t^2)dt - \int_0^x t\sin(t^2)dt.\end{aligned}$$

By the Fundamental Theorem of Calculus and product rule, we have

$$F'(x) = \int_0^x \sin(t^2)dt + x\sin(x^2) - x\sin(x^2) = \int_0^x \sin(t^2)dt.$$

Again, by the Fundamental Theorem of Calculus,

$$F''(x) = \sin(x^2).$$

Now, by using L'Hospital's rule twice,

$$\lim_{x\to 0} \frac{\int_0^x (x-t)\sin(t^2)dt}{x^4} = \lim_{x\to 0} \frac{\sin(x^2)}{12\cdot x^2} = \frac{1}{12}.$$

Common Mistakes and Comments: If you have used Taylor series expansion in your proof, then you have got atmost 1 mark even if the answer is correct. This is because in order to justify integrating a Taylor series term by term or taking limit in a Taylor series, you need an advanced concept of uniform convergence of sequence of functions, which is not done in the course.

4. (5 points) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. If for every $c \in (a, b)$, f is Riemann integrable on $[c, b]$, then using Riemann's criterion show that f is Riemann integrable on $[a, b]$.

Solution: Since $f : [a, b] \to \mathbb{R}$ is bounded, let $M = \sup_{x\in[a,b]} f(x)$ and $m = \inf_{x\in[a,b]} f(x)$.
Let $\epsilon > 0$. Then by Archimedean property, $\exists\ N \in \mathbb{N}$ such that $\frac{1}{N} < \min\{(b-a), \frac{\epsilon}{2(M-m)}\}$.
Now since f is Riemann integrable on $[c, b]$ for every $c \in (a, b)$, f is Riemann integrable on $[a + \frac{1}{N}, b]$. So by Riemann's criterion, there exists a partition P' of $[a + \frac{1}{N}, b]$ s.t.

$$U(f, P') - L(f, P') < \frac{\epsilon}{2}$$

.
Now consider the partition $P = \{a, P'\}$ of $[a, b]$. Then,

$$
\begin{aligned}
U(f,P) - L(f,P) &= \Big[\sup_{x\in[a,\,a+\frac{1}{N}]} f(x) - \inf_{x\in[a,\,a+\frac{1}{N}]} f(x)\Big]\frac{1}{N} + U(f,P') - L(f,P') \\
&< (M-m)\frac{1}{N} + \frac{\epsilon}{2}. \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}
$$

Thus, by Riemann's criterion, f is Riemann integrable on $[a,\,b]$.
Note that in the above calculation we have assumed that f is not constant. If f is constant then the conclusion is trivial.

Common Mistakes and Comments: The question asks to use the Riemann's criterion. Hence, marks are only awarded if a valid proof is given using Riemann's criterion only. Attempted proofs using any other criterion do not carry any marks.

5. (a) (3 points) Let $f, g : [a,\,b] \to \mathbb{R}$ be continuous functions, which are also differentiable on $(a,\,b)$. Show that there exists a point $c \in (a,\,b)$ such that

$$(f(b) - f(a))\, g'(c) = (g(b) - g(a))\, f'(c).$$

Solution: Let us define $\phi : [a,\,b] \to \mathbb{R}$ by $\phi(x) = (f(b) - f(a))\, g(x) - (g(b) - g(a))\, f(x)$. Then ϕ is continuous function on $[a,\,b]$ and differentiable on (a,b). Moreover, $\phi(a) = \phi(b)$. Then by using Rolle's theorem, there exists $c \in (a,b)$ such that

$$\phi'(c) = 0 \implies (f(b) - f(a))\, g'(c) = (g(b) - g(a))\, f'(c).$$

(b) (2 points) Let $f : (-1,\,1) \to \mathbb{R}$ be a differentiable function. If $f(\frac{1}{n}) = 0$ for all $n \geq 2$, then show that $f'(0) = 0$.

Solution: As f is differentiable on $(-1,1)$, f is continuous on $(-1,1)$. Thus,

$$f(0) = \lim_{n\to\infty} f(\frac{1}{n}) = 0.$$

Now, as f is differentiable at 0,

$$f'(0) = \lim_{n\to\infty} \frac{f(\frac{1}{n}) - f(0)}{\frac{1}{n} - 0}$$

Note that $\frac{f(\frac{1}{n})-f(0)}{\frac{1}{n}} = 0, \forall n \geq 2$. Therefore, $f'(0) = 0$.

Common Mistakes and Comments: In part(a), most of you have used MVT on functions f and g seperately and then concluded. However, this is not correct as c's corresponding to f and g can be different.

6. Let (x_n) be a bounded sequence of real numbers.

(a) (2 points) Show that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $x_n < \limsup x_n + \varepsilon$ for all $n \geq N$.

Solution: Let $y_n = \sup\{x_n, x_{n+1}, \cdots\}$, $\forall n \geq 1$.
Then $\limsup x_n = \lim_{n\to\infty} y_n$. Then for every $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that $y_n < \limsup x_n + \epsilon, \forall n \geq N \implies x_n \leq y_n < \limsup x_n + \epsilon, \forall n \geq N$

(b) (3 points) Show that for every $\varepsilon > 0$, $x_n > \limsup x_n - \varepsilon$ for infinitely many n's.

Solution: Fix $\epsilon > 0$. If possible assume that $x_n > \limsup x_n - \epsilon$ for only finitely many n's. Thus $\{n \geq 1 : x_n > \limsup x_n - \epsilon\}$ is ether empty or having finitely many numbers. Let $\{n \geq 1 : x_n > \limsup x_n - \epsilon\}$ has k elements, say $n_1 < n_2 < \cdots < n_k$. Then,

$$\begin{aligned} & x_n \leq \limsup x_n - \epsilon, \forall n \geq n_k + 1. \\ \Rightarrow\quad & y_n = \sup\{x_n, x_{n+1}, \cdots\} \leq \limsup x_n - \epsilon, \forall n \geq n_k + 1. \\ \Rightarrow\quad & \limsup x_n \leq \limsup x_n - \epsilon, \text{ which is a contradiction.} \end{aligned}$$

One can similarly get a contradiction if $\{n \geq 1 : x_n > \limsup x_n - \epsilon\} = \phi$. Thus our assumption is wrong and given statement is correct.

(c) (2 points) Show that there exists a subsequence of (x_n) that converges to $\limsup x_n$.

Solution: Combining (a) and (b), there exists n_k's in $\mathbb{N}$ such that $n_1 < n_2 < \ldots$ and

$$\limsup x_n - \frac{1}{k} < x_{n_k} < \limsup x_n + \frac{1}{k} \text{ for all } k \geq 1.$$

Then the subsequence (x_{n_k}) converges to $\limsup x_n$.

Common Mistakes and Comments: In part (a), many of you have concluded that $x_n \leq \limsup x_n$, which is not correct; consider the sequence $(\frac{1}{n})$.

In part (c), most of you have tried to use Bolzano-Weierstrass theorem. However, this theorem gives only existence of converging subsequence and nothing can be said about its' limit. Some of you have tried to use parts (a) and (b) to conclude that this limit should be $\limsup x_n$, but that does not work, as part (b) does not say that $x_n > \limsup x_n - \epsilon$ for all large enough n. It only says that the inequality is true for infinitely many n's, thus it is possible that the inequality is not true for infinitely many n's. For example, it may be true for all even indices, but not true for all odd indices.

The sequence (y_n) given by $y_n = \sup\{x_n, x_{n+1}, \ldots\}$ is not a subsequence of (x_n).

From parts (a) and (b), you can conclude that for a given $\epsilon > 0$, there exists a subsequence (x_{n_k}) such that $|x_{n_k} - \limsup x_n| < \epsilon$ for all large enough k. But, this does not give a proof of part (c) as the subsequence might change if you change ϵ. This is the reason you need to use "$\frac{1}{k}$" argument as given in the model solution.