Problem 1

Determine which of the following statements are true for all sets A, B, C, and D. If a double implication fails, determine whether one or the other of the possible implications holds. If an equality fails, determine whether the statement becomes true if the "equals" symbol is replaced by one or the other of the inclusion symbols $\subset$ or $\supset$.

Problem 1(a)

$$A \subset B \land A \subset C \iff A \subset (B \cup C)$$

Theorem 1 (Double implication fails in Problem 1(a))**.** $A \subset B \land A \subset C \not\Longleftrightarrow A \subset (B \cup C)$

Proof. For double implication to hold, each predicate must imply the other. We will explore whether $A \subset (B \cup C) \implies A \subset B \land A \subset C$.

Consider $A \subset (B \cup C)$. Choose $A = \{m | m = 3k, k \in \mathbb{Z}\}$, $B = \{e | e = 2k, k \in \mathbb{Z}\}$, and $C = \{o | o = 2k + 1, k \in \mathbb{Z}\}$ (i.e., A is the set of multiples of 3, B is the set of all evens, and C is the set of all odds). Since all integers are even or odd, we can say that it is indeed the case that $A \subset (B \cup C)$ holds for our choices of A, B, and C. Suppose $A \subset (B \cup C) \implies A \subset B \land A \subset C$. Since we have already shown that the antecedent holds for our choices of A, B, and C, this would require the consequent to be true as well. So both $A \subset B$ and $A \subset C$. Considerring just $A \subset B$, this would mean that the set of multiples of 3 is a subset of the evens. This would imply that $\forall x \in A, x \in B$. Now note that $3 = x \in A$ because 3 is a multiple of 3. However, $3 = x \notin B$ as 3 is not even. This contradicts our requirement that for any $x \in A$, it is also the case that $x \in B$. So our supposition is false, and $A \subset (B \cup C) \not\Longrightarrow A \subset B \land A \subset C$.

Since we have demonstrated that one of the predicates does not imply the other predicate, we have therefore established that the double implication fails as well. ∎

Theorem 2 (LHS implies RHS in Problem 1(a))**.** $A \subset B \land A \subset C \implies A \subset (B \cup C)$

Proof. It is given that $A \subset B$ and $A \subset C$. Considerring just $A \subset B$, it is either the case that $A = \emptyset$ or $\exists x \in A \implies x \in B$. In the trivial case that $A = \emptyset$, $A \subset (B \cup C)$ since the empty set is a subset of all sets. In the case that A is not empty, then we can say that $\exists x \in A \implies (x \in B \lor x \in C)$ by disjunctive inclusion. By the definition of set union, we have that $x \in (B \cup C)$. So $A \neq \emptyset \implies A \subset (B \cup C)$. Since $A \subset (B \cup C)$ in all cases derived from our assumptions, we have therefore demonstrated that $A \subset B \land A \subset C \implies A \subset (B \cup C)$. ∎

Problem 1(b)

$$A \subset B \lor A \subset C \iff A \subset (B \cup C)$$

Theorem 3 (Double implication fails in Problem 1(b))**.** $A \subset B \vee A \subset C \not\Longleftrightarrow A \subset (B \cup C)$

Proof. For double implication to hold, each predicate must imply the other. We will explore whether $A \subset (B \cup C) \implies A \subset B \vee A \subset C$.

Consider $A \subset (B \cup C)$. Choose $A = \{m | m = 3k, k \in \mathbb{Z}\}$, $B = \{e | e = 2k, k \in \mathbb{Z}\}$, and $C = \{o | o = 2k + 1, k \in \mathbb{Z}\}$ (i.e., A is the set of multiples of 3, B is the set of all evens, and C is the set of all odds). Since all integers are even or odd, we can say that it is indeed the case that $A \subset (B \cup C)$ holds for our choices of A, B, and C. Suppose $A \subset (B \cup C) \implies A \subset B \vee A \subset C$. Since we have already shown that the antecedent holds for our choices of A, B, and C, this would require the consequent to be true as well. So either $A \subset B$ or $A \subset C$ or both.

First consider the possibility that $A \subset B$. This would mean that the set of multiples of 3 is a subset of the evens. This would imply that $\forall x \in A, x \in B$. Now note that $3 = x \in A$ because 3 is a multiple of 3. However, $3 = x \notin B$ as 3 is not even. This contradicts our requirement that for any $x \in A$, it is also the case that $x \in B$. So our supposition would be false in the case that $A \subset B$, and the implication would not hold.

Next consider the possibility that $A \subset C$. This would mean that the set of multiples of 3 is a subset of the odds. This would imply that $\forall x \in A, x \in C$. Now note that $6 = x \in A$ because 6 is a multiple of 3. However, $6 = x \notin C$ as 6 is not odd. This contradicts our requirement that for any $x \in A$, it is also the case that $x \in C$. So our supposition would be false in the case that $A \subset C$, and the implication would not hold.

Since both cases derived from our supposition reach contradictions, our supposition was false and the implication therefore does not hold. Since we have demonstrated that one of the predicates does not imply the other predicate, we have therefore established that the double implication fails as well. ∎

Theorem 4 (LHS implies RHS in Problem 1(b))**.** $A \subset B \vee A \subset C \implies A \subset (B \cup C)$

Proof. It is given that either $A \subset B$ or $A \subset C$ or both.

Consider the possibility that $A \subset B$. It is either the case that $A = \emptyset$ or $\exists x \in A \implies x \in B$. In the trivial case that $A = \emptyset$, $A \subset (B \cup C)$ since the empty set is a subset of all sets. In the case that A is not empty, then we can say that $\exists x \in A \implies (x \in B \vee x \in C)$ by disjunctive inclusion. By the definition of set union, we have that $x \in (B \cup C)$. So $A \neq \emptyset \implies A \subset (B \cup C)$ in this case. A similar construction can be made given $A \subset C$ to show that $A \subset (B \cup C)$ regardless of whether $A = \emptyset$ as well. Since $A \subset (B \cup C)$ in all cases derived from our assumptions, we have therefore demonstrated that $A \subset B \vee A \subset C \implies A \subset (B \cup C)$. ∎

Problem 1(c)

$$A \subset B \wedge A \subset C \iff A \subset (B \cap C)$$

Theorem 5 (Problem 1(c) holds)**.** $A \subset B \wedge A \subset C \iff A \subset (B \cap C)$

Proof. To demonstrate double implication, it is sufficient to demonstrate each implication.

First, consider $A \subset B \wedge A \subset C$. Now suppose $\neg[A \subset (B \cap C)]$. By the definition of the subset relation, we have that $\neg(x \in A \implies [x \in (B \cap C)])$. By the definition of set intersection, we have that $\neg(x \in A \implies [x \in B \wedge x \in C])$. By DeMorgan's Law, we have that $x \in A \wedge (x \notin B \vee x \notin C)$. By the distribution of $\wedge$ over $\vee$, we have $(x \in A \wedge x \notin B) \vee (x \in A \wedge x \notin C)$. By DeMorgan's Law, we have that $\neg(x \in A \implies x \in B) \vee \neg(x \in A \implies x \in B)$. By the definition of the subset relation, we have that $\neg(A \subset B) \vee \neg(A \subset C)$. However, $\neg(A \subset B)$ contradicts our assumption that $A \subset B$, and $\neg(A \subset C)$ contradicts our assumption that $A \subset C$. Since both cases of the conditional result in contradiction, the conditional as a whole also contradicts our assumptions. So our supposition was false, and $A \subset (B \cap C)$. Thus $A \subset B \wedge A \subset C \implies A \subset (B \cap C)$.

Second, consider $A \subset (B \cap C)$. By the definition of the subset relation, we see $x \in A \implies x \in (B \cap C)$. By the definition of set intersection, we have $x \in A \implies (x \in B \wedge x \in C)$. By the distribution of $\vee$ over $\wedge$, we have $(x \in A \implies x \in B) \wedge (x \in A \implies x \in C)$. By the definition of the subset relation, we have that $A \subset B \wedge A \subset C$. Thus $A \subset (B \cap C) \implies A \subset B \wedge A \subset C$.

Therefore, since both implications have been shown, it is certain that $A \subset B \wedge A \subset C \iff A \subset (B \cap C)$. ■

Problem 1(d)

$$A \subset B \vee A \subset C \iff A \subset (B \cap C)$$

Theorem 6 (Double implication fails in Problem 1(d))**.** $A \subset B \vee A \subset C \not\iff A \subset (B \cap C)$

Proof. For double implication to hold, each predicate must imply the other. We will explore whether $A \subset B \vee A \subset C \implies A \subset (B \cap C)$.

Consider $A \subset B \vee A \subset C$. Choose $A = \{m|m = 4k, k \in \mathbb{Z}\}$, $B = \{e|e = 2k, k \in \mathbb{Z}\}$, and $C = \{o|o = 2k+1, k \in \mathbb{Z}\}$ (i.e., A is the set of multiples of 4, B is the set of all evens, and C is the set of all odds). Since all multiples of 4 are even numbers, it is the case that $A \subset B$. By disjunctive introduction, we can say that $A \subset B \vee A \subset C$.

Now consider $B \cap C$. Given our choices for B and C, this would be the intersection of the evens and the odds. Since no even number is also an odd number (and vice-versa), we can say that $B \cap C = \emptyset$. Since our choice for A is the set of multiples of 4, we know that $\exists x \in A$ (for example, $4 = x \in A$). Further, we can say that $\neg[A \subset (B \cap C)]$ since $\exists x \in A \wedge x \notin (B \cap C)$.

Suppose $A \subset B \vee A \subset C \implies A \subset (B \cap C)$. It has already been established that $A \subset B \vee A \subset C$. By modus ponens, $A \subset (B \cap C)$. However, this contradicts $\neg[A \subset (B \cap C)]$, which has already been established. So our supposition is false and $A \subset B \vee A \subset C \not\iff A \subset (B \cap C)$. ■

Theorem 7 (RHS implies LHS in Problem 1(b))**.** $A \subset (B \cap C) \implies A \subset B \vee A \subset C$

Proof. Assume $A \subset (B \cap C)$. By the definition of the subset relation, $\forall x \in A, x \in (B \cap C)$. By the definition of set intersection, this can be extended to $\forall x \in A, x \in B \wedge x \in C$. By conjunctive elimination, we have that $\forall x \in A, x \in B$. By the definition of the subset relation, $A \subset B$. By disjunctive introduction, $A \subset B \vee A \subset C$. Therefore $A \subset (B \cap C) \implies A \subset B \vee A \subset C$. ∎

Problem 1(e)

$$A \setminus (A \setminus B) = B$$

Theorem 8 (Equality fails in Problem 1(e))**.** *Given two sets A and B, it is not the case that B is equal to the portion of A removing the other portion of A which does not overlap with B. Symbolically, that is:*

$$\neg[A \setminus (A \setminus B) = B]$$

Proof. For equality among sets to hold, it is necessary that both sets are demonstrably subsets of each other. Specifically, given particular choices for sets A and B, we will explore whether B is a subset of $A \setminus (A \setminus B)$.

Choose set A to be the set of all even integers. Choose set B to be the set of all odd integers. Since the set of all even integers does not have any members that are also in the set of all odd integers, removing B from A still leaves the whole of A. That is, for our particular choices for A and B, we have that $A \setminus B = A$. By substitution, removing from A the portion of A that does not overlap with B is simply removing all of A from itself. Removing such a set from itself must result in the empty set.

Continuing with our previous choices for sets A and B, suppose it really was the case that B is a subset of the portion of A removing the other portion of A which does not overlap with B. Since we have previously demonstrated that $A \setminus (A \setminus B)$ is the empty set, that would mean that the set of all odd integers is a subset of the empty set. However, that is absurd, as there certainly do exist odd integers, and that would contradict the notion that B is a subset of the empty set. Since we have reached a contradiction, we can say that we were incorrect in our supposition, and so it is not always the case that B is a subset of the portion of A removing the other portion of A which does not overlap with B. ∎

Theorem 9 (LHS is a subset of RHS in Problem 1(e))**.** *Given two sets A and B, it is the case that B is a subset of the portion of A removing the other portion of A which does not overlap with B. Symbolically, that is:*

$$A \setminus (A \setminus B) \subset B$$

Proof. Assume there are two arbitrary sets A and B. Note that removing the portion of A which overlaps with B is equivalent to taking the intesection of

A and the complement of B. Now removing from A the intersection of A and the complement of B leaves only the intersection of A and B. Finally, it is the case that the intersection of A and B is a subset of B given the definition of set intersection. ∎

Problem 1(f)

$$A \setminus (B \setminus A) = A \setminus B$$

Theorem 10 (Equality fails in Problem 1(f))**.** *Given sets A and B, it is not the case that the portion of A without the portion of B which does not overlap with A is equal to the portion of A which does not overlap with B. Symbolically, that is:*

$$\neg[A \setminus (B \setminus A) = A \setminus B]$$

Proof. Choose set A to be the set of integers. Choose set B to be the set of all odd integers specificaly. Since the set of all odd integers only includes members which are also in the set of all integers, removing A from B leaves the empty set. That is, for our particular choices of A and B, we have that $B \setminus A = \emptyset$. By substitution, removing from A the portion of B that does not overlap with A is simply removing the empty set from A. Removing the empty set leaves a set unchanged, so our particular choices for A and B actually give $A \setminus (B \setminus A) = A$.

Continuing with our previous choices for sets A and B, we can see that removing the the set of all odd integers (i.e., B) from the set of all integers (i.e., A) leaves the set of all even integers. Now suppose it really was the case that the portion of A without the portion of B which does not overlap with A is equal to the portion of A which does not overlap with B. Then that would mean that the set of all integers is equal to the set of just the even integers, which is absurd as there are certainly integers which are not even (specifically, there are odd integers). Since we have reached a contradiction, we can say that we were incorrect in our supposition, and so it is not always the case that A without the portion of B which does not overlap with A is equal to A without the overlapping portion of B. ∎

Theorem 11 (RHS is a subset of LHS in Problem 1(f))**.** *Given sets A and B, the portion of A which does not overlap with B is a subset of the portion of A without the portion of B which does not overlap with A. Symbolically, that is:*

$$A \setminus B \subset A \setminus (B \setminus A)$$

Proof. Assume there are two arbitrary sets A and B. Note that removing the portion of B which overlaps with A is equivalent to taking the intesection of B and the complement of A. Now removing from A the intersection of B and the complement of A would only remove portions of A that were also in the complement of A which also happened to be in B. As A and the complement of A necessarilly do not overlap at all, this would remove nothing from A regardless of the nature of B. Finally, taking the portion of A which does not overlap with B gives a subset of A, and, by substitution, a subset of the portion of A which does not overlap with the portion of B which does not itself overlap with A. ∎

Problem 1(g)

$$A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$$

Theorem 12 (Equality holds in Problem 1(g)). *Given sets A, B, and C, the intersection of A and the portion of B which does not overlap with C is equal to the portion of the intersection of A and B without the portion overlapping the intersection of A and C. Symbolically, that is:*

$$A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$$

Proof. For equality among sets to hold, it is necessary that both sets are demonstrably subsets of each other. Let A, B, and C be sets.

First, let's consider the intersection of A and the portion of B which does not overlap with C. Since the empty set is already known to be a subset of all sets, in order to demonstrate this first subset relationship it is only necessary to demonstrate that if some element were indeed in $A \cap (B \setminus C)$, then it would follow that the element must also be in $(A \cap B) \setminus (A \cap C)$.

Let x be some arbitrary member of $A \cap (B \setminus C)$. We know that x must be in both A and $B \setminus C$ due to the definition of set intersection. We can also say that x must be in B and that x must not be in C given the definition of set difference. So since x must be in both A and B, we know that x is in $A \cap B$ given the definition of set intersection. So all that remains to demonstrate the subset relationship is to show that x is not in $A \cap C$.

Let's suppose for a bit that x really was somehow in $A \cap C$. That would imply that both x was in A and that x was in C, again by the definition of set intersection. However, this contradicts the fact that x must not be in C, which we had already established before we made this absurd supposition. So it must be the case that x is not in $A \cap C$. As a result, we can say with certainty that the intersection of A and the portion of B which does not overlap with C is a subset of the portion of the intersection of A and B without the portion overlapping the intersection of A and C.

So we have demonstrated that

$$A \cap (B \setminus C) \subset (A \cap B) \setminus (A \cap C)$$

Second, let's consider the portion of the intersection of A and B without the portion overlapping the intersection of A and C. Again, since the empty set is already known to be a subset of all sets, we simply need to demonstrate that if some element were indeed to exist in $(A \cap B) \setminus (A \cap C)$, then it would follow that the element must also be in $A \cap (B \setminus C)$.

Let x be some arbitrary member of the intersection of A and B without the portion overlapping the intersection of A and C. We know that x must be in $A \cap B$ and that x must not be in $A \cap C$ due to the definition of set difference. Further, we know that x must be in both A and B given the definition of set intersection. So since x is in A, all that remains to demonstrate the subset relationship is to show that x must in $B \setminus C$ in order to show that x in $A \cap (B \setminus C)$.

Recall that x is not in $A \cap C$. In other words, it is not the case that both x is in A and x is in C. Using De Morgan's Law, we can translate this statement into the equivalent statement that either x is not in A or x is not in C (or both). Since we have already established that x must be in A, we can say that x must not be in C using the disjunctive syllogism. Combined with our previous demonstration that x must be in B, we can say that x must be in $B \setminus C$ using the definition of set difference.

So we have demonstrated that

$$(A \cap B) \setminus (A \cap C) \subset A \cap (B \setminus C)$$

Finally, since we have demonstrated that each set is a subset of the other, we can say that the sets are equal. ■

Problem 1(h)

$$A \cup (B \setminus C) = (A \cup B) \setminus (A \cup C)$$

Theorem 13 (Equality fails in Problem 1(h))**.** *Given sets A, B, and C, it is not the case that the union of A with the portion of B which does not overlap with C is equal to the set resulting from removing the union of A and C from the union of A and B. Symbolically, that is*

$$\neg[A \cup (B \setminus C) = (A \cup B) \setminus (A \cup C)]$$

Proof. Choose A to be the set of multiples of 3, B to be the set of multiples of 4, and C to be the set of multiples of 2. We can see that $B \setminus C$ is the empty set since all multiples of 4 are also multiples of 2. So $A \cup (B \setminus C)$ is just the multiples of 3 given the behavior of set union with the empty set. Note that $A \cup (B \setminus C)$ is not empty since there really do exist multiples of 3 (for example, 3 itself).

Given our choices, $A \cup B$ is the set of all multiples of 3 or 4, and $A \cup C$ is the set of all multiples of 3 or 2. Since any multiple of 3 or 4 (or both) would certainly be a multiple of either 3 or 2 or both, then $(A \cup B) \setminus (A \cup C)$ is the empty set.

Since $A \cup (B \setminus C)$ is a non-empty set and $(A \cup B) \setminus (A \cup C)$ is the empty set, it is certainly not the case that $A \cup (B \setminus C)$ is a subset of $(A \cup B) \setminus (A \cup C)$. Therefore, we can say that equality of the sets is not possible in this case. ■

Theorem 14 (RHS is a subset of LHS in Problem 1(h))**.** *Given sets A, B, and C, the set resulting from removing the union of A and C from the union of A and B is a subset of the union of A and the portion of B which does not overlap with C. Symbolically, that is*

$$(A \cup B) \setminus (A \cup C) \subset A \cup (B \setminus C)$$

Proof. Let A, B, and C be sets. Since the empty set is already known to be a subset of all sets, we will simply examine whether some arbitrary value which might exist in $(A \cup B) \setminus (A \cup C)$ would necessarilly also be in $A \cup (B \setminus C)$.

Let x be some arbitrary member of $(A \cup B) \setminus (A \cup C)$. We know that x is in $A \cup B$ and that x is not in $A \cup C$. Using the definition of set union, we can say both that x is in A or B and that it is not the case that x is in A or C (or both). De Morgan's Law tells use that the latter is equivalent to the statement that both x is not in A and x is not in C. This combined with the previously established statement that x is in A or x is in B (or both), the disjunctive syllogism informs us that it must be that x is in B. Taking this together with x not being in C, by definition we can say x is in $B \setminus C$. It follows that x is in $A \cup (B \setminus C)$ by disjunctive introduction.

So $(A \cup B) \setminus (A \cup C)$ is a subset of $A \cup (B \setminus C)$. ■

Problem 1(i)

$$(A \cap B) \cup (A \setminus B) = A$$

Theorem 15 (Equality holds in Problem 1(i))**.** *Given sets A and B, the union between the intersection of A and B with the portion of A not overlapping B is equal to A. Symbolically, that is*

$$(A \cap B) \cup (A \setminus B) = A$$

Proof. Let A and B be sets. We will demonstrate equality by demonstrating that $(A \cap B) \cup (A \setminus B)$ and A are subsets of each other.

First, we will address $(A \cap B) \cup (A \setminus B)$. Since the empty set is already known to be a subset of all sets, we will simply examine whether some arbitrary value which might exist in the set would necessarilly also be in A. So let x be in $(A \cap B) \cup (A \setminus B)$. We know that x is either in $A \cap B$ or in $A \setminus B$ due to the definition of set union. In the case that x is in $A \cap B$, we would know that x is in A and x is in B by the definition of set intersection, and so we immediately have that x is necessarilly in A. On the other hand, if x is in $A \setminus B$, then x is in A and x is not in B given the definition of set difference. Since we have shown that such an x is in A no matter what, it is certain that $(A \cap B) \cup (A \setminus B)$ is a subset of A.

Second, we will proceed from A. Again, since the empty set is known to be a subset of all sets, we only need to consider whether some arbitrary value which might exist in A would necessarilly also exist in $(A \cap B) \cup (A \setminus B)$. Let x be in A. Notice that it is either the case that x is also in B or it is not the case that x is in B. If x is indeed in B, then the fact that x is in both A and B means that x is in $A \cap B$ by the definition of set intersection, and by disjunctive introduction we know that x is also in $(A \cap B) \cup (A \setminus B)$. On the other hand, if x is not in B, then the definition of set difference tells us that x is in $A \setminus B$, and by extension x is also in $(A \cap B) \cup (A \setminus B)$. So A is a subset of $(A \cap B) \cup (A \setminus B)$. ■

Problem 1(j)

$$A \subset C \wedge B \subset D \implies (A \times B) \subset (C \times D)$$

Theorem 16 (Implication holds in Problem 1(j))**.** *Given sets A, B, C, and D, if A is a subset of C and B is a subset of D, then the cartesian product of A and B is a subset of the cartesian product of C and D. Symbolically, that is*

$$A \subset C \wedge B \subset D \implies (A \times B) \subset (C \times D)$$

Proof. Let A, B, C, and D be sets such that A is a subset of C and B is a subset of D. Consider the nature of $A \times B$. If the set is empty, then it is of course a subset of $C \times D$ since the empty set is a subset of all sets. If instead $A \times B$ is not empty, we can let (x, y) be some arbitrary member of $A \times B$ where x is in A and y is in B. Since we know that A is a subset of C and B is a subset of D, we can say that x is in C and that y is in D. Given the definition of cartesian product, we can say that (x, y) is also a member of $C \times D$. Therefore $A \times B$ must be a subset of $C \times D$. ■

Problem 1(k)

The converse of Problem 1(j).

Remark (The converse of Problem 1(j))**.** *The statement*

$$(A \times B) \subset (C \times D) \implies A \subset C \wedge B \subset D$$

is the converse of

$$A \subset C \wedge B \subset D \implies (A \times B) \subset (C \times D)$$

Theorem 17 (Implication fails in Problem 1(k))**.** *Given sets A, B, C, and D, it is not the case that the cartesian product of A and B being a subset of the cartesian product of C and D implies that both A is a subset of C and B is a subset of D. Symbolically, that is*

$$\neg[(A \times B) \subset (C \times D) \implies A \subset C \wedge B \subset D]$$

Proof. Choose A to be the empty set, B to be the odds, C to be the integers, and D to be the evens. Since A is the empty set, the cartesian product of A and B is also the empty set. Since the empty set is a subset of all sets, $A \times B$ is a subset of $C \times D$. Now suppose it was the case that $A \times B$ being a subset of $C \times D$ really did imply that both A was a subset of C and B was a subset of D. Given our particular choices, that would mean (among other things) that the set of all odds is a subset of the set of all evens, which is absurd. So given our choices it is not the case that $A \times B$ being a subset of $C \times D$ implies that both A is a subset of C and B is a subset of D. ■

Problem 1(l)

The converse of Problem 1(j), assuming that A and B are nonempty.

Remark (The converse of Problem 1(j) with nonempty A and B)**.** *The statement*

$$A \neq \emptyset \wedge B \neq \emptyset \wedge (A \times B) \subset (C \times D) \implies A \subset C \wedge B \subset D$$

is the converse of

$$A \subset C \wedge B \subset D \implies (A \times B) \subset (C \times D)$$

except with the added restriction that neither A nor B are empty.

Theorem 18 (Implication holds in Problem 1(l))**.** *Given sets A, B, C, and D, if A and B are nonempty and the cartesian product of A and B is a subset of the cartesian product of C and D, then both A is a subset of C and B is a subset of D. Symbolically, that is*

$$A \neq \emptyset \wedge B \neq \emptyset \wedge (A \times B) \subset (C \times D) \implies A \subset C \wedge B \subset D$$

Proof. Assume A, B, C, and D are sets such that neither A nor B are the empty set and the cartesian product of A and B is a subset of the cartesian product of C and D. Since neither A nor B are the empty set, we can say that the cartesian product of A and B is also not the empty set. Let (x, y) be in $A \times B$ where x is in A and y is in B. Since we have established that $A \times B$ is a subset of $C \times D$, we know that (x, y) is also in $C \times D$. Given the definition of the cartesian product, we can say that both x is in C and y is in D. As previously stated, x is an arbitrary member of A and y is an arbitrary member of B, so we can say that A is a subset of C and B is a subset of D. ∎

Problem 1(m)

$$(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$$

Theorem 19 (Equality fails in Problem 1(m))**.** *Given sets A, B, C, and D, it is not the case that the set union of the cartesian products of A with B and C with D is equal to the cartesian product of the set unions of A with C and B with D. Symbolically, that is*

$$\neg[(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)]$$

Proof. Choose A to be the set of just 1, B to be the set of just 2, C to be the set of just 3, and D to be the set of just 4. So the cartesian product of A and B is the set of just $(1, 2)$ and the cartesian product of C and D is the set of just $(3, 4)$. The union of $A \times B$ and $C \times D$ is the set $\{(1, 2), (3, 4)\}$. Now the union of A and C is the set $\{1, 3\}$, and the union of B and D is the set $\{2, 4\}$. The cartesian product of $A \cup C$ and $B \cup D$ is the set $\{(1, 2), (1, 4), (3, 2), (3, 4)\}$. Since $(A \cup C) \times (B \cup D)$ has members that $(A \times B) \cup (C \times D)$ does not, the two sets are not equal. ∎

Theorem 20 (LHS is a subset of RHS in Problem 1(m))**.** *Given sets A, B, C, and D, the set union of the cartesian products of A with B and C with D is a subset of the cartesian product of the set unions of A with C and B with D. Symbolically, that is*

$$(A \times B) \cup (C \times D) \subset (A \cup C) \times (B \cup D)$$

Proof. Assume A, B, C, and D are sets. If the set union of the cartesian products of A with B and C with D is an empty set, then we know it must be a subset of the cartesian product of the set unions of A with C and B with D since the empty set is a subset of all sets. So instead we will explore the possibility that there is an arbitrary member of $(A \times B) \cup (C \times D)$. This member would either be in $A \times B$ or $C \times D$ or both.

First, let (x, y) be in $A \times B$ where x is in A and y is in B. We can say that both x is in $A \cup C$ and y is in $B \cup D$ by disjunctive introduction. As a result, we can say that (x, y) is in the cartesian product of $A \cup C$ and $B \cup D$.

Second, let (x, y) be in $C \times D$ where x is in C and y is in D. We can say that both x is in $A \cup C$ and y is in $B \cup D$ by disjunctive introduction. As a result, we can say that (x, y) is in the cartesian product of $A \cup C$ and $B \cup D$. ∎

Problem 1(n)

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$$

Theorem 21 (Equality holds in Problem 1(n))**.** *Given sets A, B, C, and D, the intersection of the cartesian products of A with B and C with D is equal to the cartesian product of the intersections of A with C and B with D. Symbolically, that is*

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$$

Proof. For equality among sets to hold, it is necessary that both sets are demonstrably subsets of each other. Let A, B, and C be sets.

First consider $(A \times B) \cap (C \times D)$. In the trivial case that the set is empty, we can say that it is also a subset of $(A \cap C) \times (B \cap D)$ since the empty set is a subset of all sets. So instead, let (x, y) be some arbitrary member of the set. Given the definition of set intersection, we can say that (x, y) is in $A \times B$ and in $C \times D$. So we know that x is in A, that y is in B, that x is also in C, and that y is also in D. By the definition of set intersection, we can say that x is in $A \cap C$ and that y is in $B \cap D$. So taking the cartesian product of these two new sets, we can say that (x, y) is in $(A \cap C) \times (B \cap D)$. Thus we know for certain that $(A \times B) \cap (C \times D)$ is always a subset of $(A \cap C) \times (B \cap D)$.

Second consider $(A \cap C) \times (B \cap D)$. In the trivial case that the set is empty, we can say that it is a subset of $(A \times B) \cap (C \times D)$ since the empty set is a subset of all sets. Instead, let (x, y) be some arbitrary member of the set. So x is in $A \cap C$ and y is in $B \cap D$. By the definition of set intersection, we can say that x is in both A and C and that y is in both B and D. Constructing some new cartesian products, we can say that (x, y) is in $A \times B$ and that (x, y) is in $C \times D$. So by the definition of set intersection, we have that (x, y) is in $(A \times B) \cap (C \times D)$. Thus we know that $(A \cap C) \times (B \cap D)$ is a subset of $(A \times B) \cap (C \times D)$.

Therefore we have demonstrated that $(A \cap C) \times (B \cap D)$ is equal to $(A \times B) \cap (C \times D)$. ■

Problem 1(o)

$$A \times (B \setminus C) = (A \times B) \setminus (A \times C)$$

Theorem 22 (Equality holds in Problem 1(o))**.** *Given sets A, B, and C, the cartesian product of A with the portion of B that does not overlap with C is equal to the portion of the cartesian product of A with B which does not overlap with the cartesian product of A with C. Symbolically, that is*

$$(A \times (B \setminus C) = (A \times B) \setminus (A \times C)$$

Proof. For equality among sets to hold, it is necessary that both sets are demonstrably subsets of each other. Let A, B, and C be sets.

First consider $A \times (B \setminus C)$. In the trivial case that the set is empty, it is a subset of $(A \times B) \setminus (A \times C)$ since the empty set is a subset of all sets. So instead, let's explore the case that there is some arbitrary (x, y) in $A \times (B \setminus C)$. We know that x is in A and y is in $B \setminus C$, and given the definition of set difference, we can also say both that y is in B and that y is not in C. Constructing cartesian products, we can say that (x, y) is in $A \times B$, and we would like to explore whether or not (x, y) is also in $A \times C$.

Suppose (x, y) really is in $A \times C$. So we know both that x is in A and that y is in C. However this is absurd, as we have already established that y is not in C using only our initial assumptions. So our supposition is false, and (x, y) is not in $A \times C$. Combining this fact with the previously established (x, y) being in $A \times B$, by definition we know that (x, y) is in $(A \times B) \setminus (A \times C)$. Thus $A \times (B \setminus C)$ is a subset of $(A \times B) \setminus (A \times C)$.

Second consider $(A \times B) \setminus (A \times C)$. In the trivial case that the set is empty, it is a subset of $A \times (B \setminus C)$ since the empty set is a subset of all sets. So instead, we'll explore the case that there exists some arbitrary (x, y) in $(A \times B) \setminus (A \times C)$. We know that (x, y) is in $A \times B$ and not in $A \times C$ by the definition of set difference. So we have that x is in A and y is in B, but we need to determine if y is also in C.

Suppose y is indeed in C. Since we have already established that x is in A, we would see that (x, y) is in $A \times C$. However, this contradicts our previous statement that $(x, y$ is not in $A \times C$. So it must be that y is not in C. Taken together with the previously stated fact that y is in B, the definition of set difference gives us y in $B \setminus C$. So (x, y) is in $A \times (B \setminus C)$. Thus $(A \times B) \setminus (A \times C)$ is a subset of $A \times (B \setminus C)$.

Therefore $A \times (B \setminus C)$ is equal to $(A \times B) \setminus (A \times C)$. ■

Problem 1(p)

$$(A \setminus B) \times (C \setminus D) = (A \times C \setminus B \times C) \setminus (A \times D)$$

Theorem 23 (Equality holds in Problem 1(p))**.** *Given sets A, B, C, and D, the cartesian product of the portion of A which does not overlap with B with the portion of C which does not overlap with D is equal to the portion of the cartesian product of A with C which does not overlap with the cartesian product of B with C, except the portion that overlaps with the cartesian product of A with D. Symbolically, that is*

$$(A \setminus B) \times (C \setminus D) = (A \times C \setminus B \times C) \setminus (A \times D)$$

Proof. For equality among sets to hold, it is necessary that both sets are demonstrably subsets of each other. Let A, B, C, and D be sets.

First consider $(A \setminus B) \times (C \setminus D)$. In the trivial case that the set is empty, it is a subset of $(A \times C \setminus B \times C) \setminus (A \times D)$ since the empty set is a subset of all sets. Instead, let's consider the case that there exists some arbitrary (x, y) in $(A \setminus B) \times (C \setminus D)$. We know that x is in $A \setminus B$ and that y is in $C \setminus D$. Given the definition of set difference, we know that x is in A, that x is not in B, that y is in C, and that y is not in D. By the definition of the cartesian product, we have that (x, y) is in $A \times C$, but we need to determine if (x, y) is in either $B \times C$ or $A \times D$.

Suppose (x, y) is in $B \times C$. Then x is in B and y is in C. However, we have already established that x is not in B. So it is not the case that (x, y) is in $B \times C$. Since (x, y) is in $A \times C$, the definition of set difference gives us (x, y) in $A \times C \setminus B \times C$.

Now suppose (x, y) is in $A \times D$. Then x is in A and y is in D. However, we have already established that y is not in D, so it must be that $(x, y$ is not in $A \times D$. Combined with the previously established statement that (x, y) is in $A \times C \setminus B \times C$, we can say that (x, y) is in $(A \times C \setminus B \times C) \setminus (A \times D)$. Thus $(A \setminus B) \times (C \setminus D)$ is a subset of $(A \times C \setminus B \times C) \setminus (A \times D)$.

Second consider $(A \times C \setminus B \times C) \setminus (A \times D)$. In the trivial case that this set is empty, it must be a subset of $(A \setminus B) \times (C \setminus D)$ since the empty set is a subset of all sets. So we will instead consider the situation that there is some (x, y) in $(A \times C \setminus B \times C) \setminus (A \times D)$. Given the definition of set difference, we know that (x, y) is in $A \times C \setminus B \times C$, that (x, y) is not in $A \times D$, that (x, y) is in $A \times C$, and that (x, y) is not in $B \times C$. So we know that x is in A and that y is in C by the definition of cartesian product. We have yet to determine if x is in B or if y is in D.

Suppose x is in B. Then (x, y) is in $B \times C$ since we know that y is in C. However, we have already established that (x, y) is not in $B \times C$. So it cannot possibly be the case that x is in B. Further, we can say that x is in $A \setminus B$ by the definition of set difference.

Next, suppose y is in D. That would mean (x, y) is in $A \times D$. Yet that contradicts that (x, y) is not in $A \times D$, which was demonstrated from our assumptions. So our supposition must be false, and y is not in D. As we have already established that y is in C, we can say that y is in $C \setminus D$ by definition.

Finally, by constructing a cartesian product of other sets we have constructed, we can say that (x, y) is in $(A \setminus B) \times (C \setminus D)$. Thus $(A \times C \setminus B \times C) \setminus (A \times D)$ is a subset of $(A \setminus B) \times (C \setminus D)$.

Therefore $(A \setminus B) \times (C \setminus D)$ is equal to $(A \times C \setminus B \times C) \setminus (A \times D)$. ■

Problem 1(q)

$$(A \times B) \setminus (C \times D) = (A \setminus C) \times (B \setminus D)$$

Theorem 24 (Equality fails in Problem 1(q))**.** *Given sets of A, B, C, and D, it is not the case that the portion of the cartesian product of A with B wihtout the portion of the cartesian product of C with D is equal to the cartesian product of the portion of A which does not overlap with C with the portion of B which does not overlap with D. Symbolically, that is*

$$\neg[(A \times B) \setminus (C \times D) = (A \setminus C) \times (B \setminus D)]$$

Proof. Choose A to be the set $\{1\}$, choose B to be the set $\{2\}$, choose C to also be the set $\{1\}$, and choose D to be the empty set. So $A \setminus C$ is the empty set and $B \setminus D$ is the same as B. Now constructing some cartesian products, we have that $A \times B$ is $\{(1, 2)\}$, that $C \times D$ is the empty set, and that $(A \setminus C) \times (B \setminus D)$ is also the empty set. By the definition of set difference, we can see that $(A \times B) \setminus (C \times D)$ is $\{(1, 2)\}$. Since $\{(1, 2)\}$ is not empty, our choices have demonstrated that it is not always the case that $(A \times B) \setminus (C \times D)$ is equal to $(A \setminus C) \times (B \setminus D)$. ■

Theorem 25 (RHS is a subset of LHS in Problem 1(q))**.** *Given sets of A, B, C, and D, the cartesian product of the portion of A which does not overlap with C with the portion of B which does not overlap with D is a subset of the portion of the cartesian product of A with B wihtout the portion of the cartesian product of C with D. Symbolically, that is*

$$(A \setminus C) \times (B \setminus D) \subset (A \times B) \setminus (C \times D)$$

Proof. Assume A, B, C, and D are sets. Since $(A \setminus C) \times (B \setminus D)$ being empty immediately demonstrates that it would be a subset of $(A \times B) \setminus (C \times D)$, we will instead consider whether some arbitrary value in the set would also imply that such a value would also be in $(A \times B) \setminus (C \times D)$. So let (x, y) be in $(A \setminus C) \times (B \setminus D)$. We know that x must be in $A \setminus C$ and y must be in $B \setminus D$ given the nature of the cartesian product. Now the definition of set difference gives us that x is in A, that x is not in C, that y is in B, and that y is not in D. We can construct $A \times B$, which will have (x, y) as a member given the definition of the cartesian product. We can also construct $C \times D$, but we need to establish whether or not it contains (x, y).

Suppose (x, y) is in $C \times D$. Then x is in C and y is in D. However this is absurd, as we have already shown that x is not in C (without even going into the matter of the nature of y). So it must be the case that (x, y) is not in $C \times D$, and thus we also know that (x, y) is in $(A \times B) \setminus (C \times D)$ by definition.

Therefore $(A \setminus C) \times (B \setminus D)$ is a subset of $(A \times B) \setminus (C \times D)$. $\blacksquare$