Problem 1

Check the distributive laws for $\cup$ and $\cap$ and De Morgan's laws.

Theorem 1 ($\cap$ distributes over $\cup$)**.** *Set intersection distributes across set union. That is, for any sets A, B, and C, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$*

Proof. Given that sets are said to be equal iff they are subsets of each other, the goal can be shown directly by demonstrating both of the sets in question are subsets of each other. Symbolically, that would be:

- $[A \cap (B \cup C)] \subset [(A \cap B) \cup (A \cap C)]$
- $[(A \cap B) \cup (A \cap C)] \subset [A \cap (B \cup C)]$

First, let's consider the nature of $A \cap (B \cup C)$: either the set is empty, or there exists at least one element in the set. We will address these possibilities independently.

If $A \cap (B \cup C) = \emptyset$, then $[A \cap (B \cup C)] \subset [(A \cap B) \cup (A \cap C)]$ since the empty set is a subset of all sets.

If $\exists x \in A \cap (B \cup C)$, then both $x \in A$ and $x \in B \cup C$ by the definition of set intersection. Further, by the definition of set union, $x \in B$ or $x \in C$. Without loss of generality, assume $x \in B$. Since both $x \in A$ and $x \in B$, the definition of set intersection gives us $x \in A \cap B$. By disjunctive introduction, $x \in A \cap B \vee x \in A \cap C$, and by the definition of set union, $x \in (A \cap B) \cup (A \cap C)$. So $\exists x \in A \cap (B \cup C) \implies x \in (A \cap B) \cup (A \cap C)$, and by the definition of subset $[A \cap (B \cup C)] \subset [(A \cap B) \cup (A \cap C)]$.

Thus, regardless if $A \cap (B \cup C) = \emptyset$ or $\exists x \in A \cap (B \cup C)$, it is certain that $[A \cap (B \cup C)] \subset [(A \cap B) \cup (A \cap C)]$.

Second, let's consider the nature of $(A \cap B) \cup (A \cap C)$: either the set is empty, or there exists at least one element in the set. We will address these possibilities independently.

If $(A \cap B) \cup (A \cap C) = \emptyset$, then $[(A \cap B) \cup (A \cap C)] \subset [A \cap (B \cup C)]$ since the empty set is a subset of all sets.

If $\exists x \in (A \cap B) \cup (A \cap C)$, then either $x \in A \cap B$ or $x \in A \cap C$ by the definition of set union. Without loss of generality, assume $x \in A \cap B$. By the definition of set intersection, $x \in A$ and $x \in B$. By disjunctive introduction, $x \in B \vee x \in C$, and by the definition of set union, $x \in B \cup C$. By the definition of set intersection, $x \in A \cap (B \cup C)$. So $\exists x \in (A \cap B) \cup (A \cap C) \implies x \in A \cap (B \cup C)$, and by definition of subset $[(A \cap B) \cup (A \cap C)] \subset [A \cap (B \cup C)]$.

Thus, regardless if $(A \cap B) \cup (A \cap C) = \emptyset$ or $\exists x \in (A \cap B) \cup (A \cap C)$, it is certain that $[(A \cap B) \cup (A \cap C)] \subset [A \cap (B \cup C)]$.

Since both $[A \cap (B \cup C)] \subset [(A \cap B) \cup (A \cap C)]$ and $[(A \cap B) \cup (A \cap C)] \subset [A \cap (B \cup C)]$ it is therefore the case that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. ∎

Theorem 2 ($\cup$ distributes over $\cap$)**.** *Set union distributes across set intersection. That is, given sets A, B, and C, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$*

Proof. Given the definition of set equality, it is sufficent to demonstrate equality by showing that each set is a subset of the other. Symbolically:

- $[A \cup (B \cap C)] \subset [(A \cup B) \cap (A \cup C)]$
- $[(A \cup B) \cap (A \cup C)] \subset [A \cup (B \cap C)]$

First, let's consider the nature of $A \cup (B \cap C)$. In the trivial case that the set has no memebers (i.e., it is equal to the empty set), then $[A \cup (B \cap C)] \subset [(A \cup B) \cap (A \cup C)]$ since the empty set is a subset of all other sets. If instead $\exists x \in A \cup (B \cap C)$, then either $x \in A$ or $x \in (B \cap C)$ (or both) by the definition of set union.

When $x \in A$, then $x \in A \vee x \in B$ and $x \in A \vee x \in C$ by disjunctive introduction, which gives $x \in A \cup B$ and $x \in A \cup C$ by the definition of set union.By the definition of set intersection, $x \in [(A \cup B) \cap (A \cup C)]$. So assuming $x \in A$ gives $[A \cup (B \cap C)] \subset [(A \cup B) \cap (A \cup C)]$.

When $x \in (B \cap C)$, then $x \in B$ and $x \in C$ by the definition of set intersection. This gives both $x \in A \cup B$ and $x \in A \cup C$ by disjunctive introduction and the definition of set union. By the definition of set intersection, we get that $x \in (A \cup B) \cap (A \cup C)$. So assuming $x \in (B \cap C)$ gives $[A \cup (B \cap C)] \subset [(A \cup B) \cap (A \cup C)]$.

So $\exists x \in A \cup (B \cap C)$ also gives $[A \cup (B \cap C)] \subset [(A \cup B) \cap (A \cup C)]$ regardless of the particular conditions when $x \in A \cup (B \cap C)$. Thus it is true that $[A \cup (B \cap C)] \subset [(A \cup B) \cap (A \cup C)]$ in all circumstances.

Second, let's consider the nature of $(A \cup B) \cap (A \cup C)$. In the trivial case that the set has no members (i.e., it is equal to the empty set), then $[(A \cup B) \cap (A \cup C)] \subset [A \cup (B \cap C)]$ since the empty set is a subset of all other sets. However, there is more to consider if the set is non-empty.

When $(A \cup B) \cap (A \cup C) \neq \emptyset$, then $\exists x \in (A \cup B) \cap (A \cup C)$ by the negation of the definition of the empty set. By the definition of set intersection, $x \in A \cup B \wedge x \in A \cup C$. By the definition of set union, $(x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$. We can proceed by separately considering the cases that $x \in A$, $x \notin A \wedge x \in B$, and $x \notin A \wedge x \in C$. Note that either $x \notin A \wedge x \notin B$ or $x \notin A \wedge x \notin C$ would directly contradict $(x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$.

If $x \in A$, then $(x \in A) \vee (x \in B \cap C)$ by disjunctive introduction. By the definition of set union, $x \in A \cup (B \cap C)$.

Next, let's say $x \notin A \wedge x \in B$. Now it would also seem either $x \in C$ or $x \notin C$.

Suppose $x \notin C$. With what is given, we would directly have that $x \notin A \wedge x \notin C$. But, as noted above, this contradicts what has already been established. So it is not the case that $x \notin C$.

So we have reached $x \notin A \wedge x \in B \wedge x \in C$. By the definition of set intersection, we have $x \notin A \wedge x \in B \cap C$. By conjunctive elimination, we have $x \in B \cap C$. By disjunctive introduction, we have $(x \in A) \vee (x \in B \cap C)$. By the definition of set union, we have $x \in A \cup (B \cap C)$.

The demonstration from $x \notin A \wedge x \in C$ is equivalently constructed as the demonstration from $x \notin A \wedge x \in B$.

Since all cases descending from $\exists x \in (A \cup B) \cap (A \cup C)$ lead either to contradiction or a demonstration that $x \in A \cup (B \cap C)$, it is certain that $(A \cup B) \cap (A \cup C) \neq \emptyset$ implies $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$. Thus, when taken with the trivial case that was previously shown, it has been demonstrated that $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$.

Therefore, since the relevant sets have been shown to be subsets of each other, by the definition of set equality we have that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. ■

Theorem 3 (De Morgan's First Law). *Set difference across set union is equivalent to set intersection of set differences. That is, given sets A, B, and C,* $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

Proof. Given the definition of set equality, it is sufficent to demonstrate equality by showing that each set is a subset of the other. Symbolically:

- $[A \setminus (B \cup C)] \subset [(A \setminus B) \cap (A \setminus C)]$
- $[(A \setminus B) \cap (A \setminus C)] \subset [A \setminus (B \cup C)]$

First, let's consider the nature of $A \setminus (B \cup C)$. In the trivial case that the set has no memebers (i.e., it is equal to the empty set), then $[A \setminus (B \cup C)] \subset [(A \setminus B) \cap (A \setminus C)]$ since the empty set is a subset of all other sets. If instead $\exists x \in A \setminus (B \cup C)$, then it is certain that $x \in A$ and $x \notin (B \cup C)$ by the definition of set difference. We can proceed by exploring whether or not $x \in B$ or $x \in C$.

Suppose $x \in B$. By disjunctive introduction, $(x \in B) \vee (x \in C)$. By the definition of set union, $x \in (B \cup C)$. But this contradicts $x \notin (B \cup C)$, which has already been shown from our assumptions. So our supposition is false, and it is certain that $x \notin B$. A similar consturction starting with a supposition of $x \in C$ yields a similar contradiction, so we also have that $x \notin C$.

To recap, we have that $x \in A$, $x \notin (B \cup C)$, $x \notin B$, and $x \notin C$. So we have both $(x \in A) \wedge (x \notin B)$ and $(x \in A) \wedge (x \notin C)$. By the definition of set difference, we have both $x \in (A \setminus B)$ and $x \in (A \setminus C)$. By the definition of set intersection, we have that $x \in (A \setminus B) \cap (A \setminus C)$. Thus we have shown $A \setminus (B \cup C) \subset (A \setminus B) \cap (A \setminus C)$ in all circumstances.

Second, let's consider the nature of $(A \setminus B) \cap (A \setminus C)$. In the trivial case that the set has no memebers (i.e., it is equal to the empty set), then $(A \setminus B) \cap (A \setminus C) \subset A \setminus (B \cup C)$ since the empty set is a subset of all other sets. If instead $\exists x \in (A \setminus B) \cap (A \setminus C)$, then it is certain that both $x \in (A \setminus B)$ and $x \in (A \setminus C)$ by the definition of set intersection. By the definition of set difference, we have $(x \in A) \wedge (x \notin B)$ and $(x \in A) \wedge (x \notin C)$. So we have $x \in A$, $x \notin B$, and $x \notin C$.

Suppose $x \in (B \cup C)$. Then either $x \in B$ or $x \in C$ or both by the definition of set union. But each of these possibilities contradicts what we have already established, so the supposition is false and $x \notin (B \cup C)$.

Since we have both $x \in A$ and $x \notin (B \cup C)$, then we have $x \in A \setminus (B \cup C)$ by the definition of set difference. Thus we have shown $(A \setminus B) \cap (A \setminus C) \subset A \setminus (B \cup C)$ in all circumstances.

Since we have demonstrated each set is a subset of the other, we have therefore demonstrated that $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$. ■

Theorem 4 (De Morgan's Second Law)**.** *Set difference across set intersection is equivalent to set union of set differences. That is, given sets A, B, and C, $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$*

Proof. Given the definition of set equality, it is sufficent to demonstrate equality by showing that each set is a subset of the other. Symbolically:

- $[A \setminus (B \cap C)] \subset [(A \setminus B) \cup (A \setminus C)]$
- $[(A \setminus B) \cup (A \setminus C)] \subset [A \setminus (B \cap C)]$

First, let's consider the nature of $A \setminus (B \cap C)$. In the trivial case that the set has no members (i.e., it is equal to the empty set), then $[A \setminus (B \cap C)] \subset [(A \setminus B) \cup (A \setminus C)]$ since the empty set is a subset of all other sets. If instead $\exists x \in A \setminus (B \cap C)$, then $x \in A$ and $x \notin (B \cap C)$ by the definition of set difference. We can proceed by examining whether or not $x \in B$ and $x \in C$.

If $x \notin B$, then combining with the already established fact that $x \in A$ gives $x \in A \setminus B$ by the definition of set intersection. By disjunctive introduction, we have that $(x \in A \setminus B) \vee (x \in A \setminus C)$. By the definition of set union, we have that $x \in (A \setminus B) \cup (A \setminus C)$. So $\exists x \in A \setminus (B \cap C) \wedge x \notin B$ is sufficent to demonstrate that $A \setminus (B \cap C) \subset (A \setminus B) \cup (A \setminus C)$. A similar construction assuming $x \notin C$ yields $\exists x \in A \setminus (B \cap C) \wedge x \notin C \implies x \in (A \setminus B) \cup (A \setminus C)$.

The previous cases admit the possibility that the alternate set in question may or may not contain member x (i.e., the previous demonstration in the case that $x \notin B$ holds for both $x \in C$ and $x \notin C$). However, suppose both $x \in B$ and $x \in C$. Then by the definition of set intersection, $x \in (B \cap C)$. But this contradicts $x \notin (B \cap C)$, which has already been established directly from our assumptions. So it is not the case that both $x \in B$ and $x \in C$.

Since all cases eminating from $\exists x \in A \setminus (B \cap C)$ yield either contradictions or demonstrations that $x \in (A \setminus B) \cup (A \setminus C)$, and since the case that $A \setminus (B \cap C) = \emptyset$ has also given the particular result, we can say that $A \setminus (B \cap C) \subset (A \setminus B) \cup (A \setminus C)$ in all circumstances.

Second, let's consider the nature of $(A \setminus B) \cup (A \setminus C)$. In the trivial case that the set has no members (i.e., it is equal to the empty set), then $(A \setminus B) \cup (A \setminus C) \subset A \setminus (B \cap C)$ since the empty set is a subset of all other sets. If instead $\exists x \in (A \setminus B) \cup (A \setminus C)$, then either $x \in A \setminus B$ or $x \in A \setminus C$ or both by the definition of set union.

If $x \in A \setminus B$, then $x \in A$ and $x \notin B$ by the definition of set difference. Suppose $x \in (B \cap C)$. Then $x \in B$ and $x \in C$ by the definition of set intersection.

But this contradicts that $x \notin B$, which has already been established by the assumptions we have made for this case. So $x \notin (B \cap C)$. Since we have already established that $x \in A$, we have that $x \in A \setminus (B \cap C)$ by the definition of set difference. So $\exists x \in (A \setminus B) \cup (A \setminus C) \wedge x \in (A \setminus B) \implies x \in A \setminus (B \cap C)$. A similar construction demonstrates that the case when $x \in A \setminus C$ also yields $\exists x \in (A \setminus B) \cup (A \setminus C) \wedge x \in (A \setminus C) \implies x \in A \setminus (B \cap C)$. Neither of these cases precludes the possibility that both $x \in A \setminus B$ and $x \in A \setminus C$, but the possibility that neither is true is precluded by the assumption $\exists x \in (A \setminus B) \cup (A \setminus C)$.

Since all cases eminating from $\exists x \in (A \setminus B) \cup (A \setminus C)$ yield either contradictions or demonstrations that $x \in A \setminus (B \cap C)$, and since the case that $(A \setminus B) \cup (A \setminus C) = \emptyset$ has also given the following result, we can say that $(A \setminus B) \cup (A \setminus C) \subset A \setminus (B \cap C)$ in all circumstances.

Since we have demonstrated each set is a subset of the other, we have therefore demonstrated that $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$. ■