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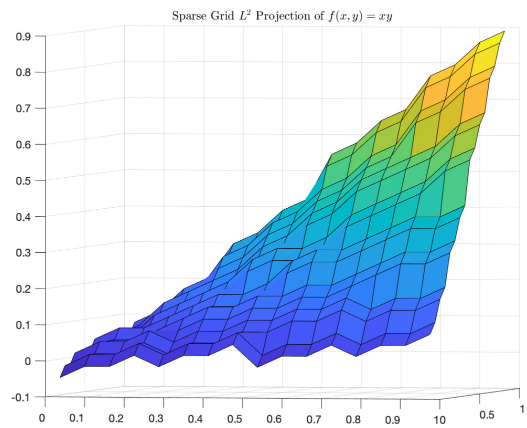
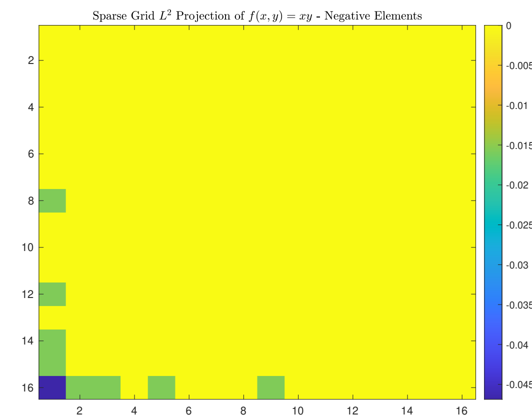
Positive L^2 projection onto sparse DG spaces

Problem: Even with $k = 0$ sparse DG space, that is piecewise constants, positivity of the L^2 projection is not preserved.

For example, with $d = 2$, $N = 4$, below is the L^2 projection of $f(x, y) = xy$ onto $\hat{\mathbf{V}}_N^0$ where $\Omega = [0, 1]^2$.





While this cannot happen on the full grid DG space with $k = 0$, it can arise for $k = 1$ and higher. This is usually solved by using slope limiters – allowing the function to be positive and maintain local mass conservation.

Since the sparse DG space does not access to the local basis functions on the finest level, slope limiting does not have an obvious extension to sparse DG. Moreover, mass is not conserved on the finest mesh since the sparse DG space cannot create the local basis functions on the finest mesh.

To guarantee positivity, we instead turn to a variational approach, as this should be unchanged by the sparse DG space. Given a function $w \in L^2(\Omega)$, define

$$K(w) = \left\{ v_h \in \hat{\mathbf{V}}_n^k \text{ such that } v_h \geq 0 \text{ on } \Omega, \int_\Omega v_h \, \mathrm{d}x = \int_\Omega w \, \mathrm{d}x \right\}.$$

Note that $K(v)$ is a closed and convex subset (but not a subspace) of $\hat{\mathbf{V}}_n^k$. The positive sparse DG projection of w is defined by

$$w_h^{\text{pos}} = \underset{v_h \in K(w)}{\operatorname{argmin}} \frac{1}{2} \| w - v_h \|_{L^2(\Omega)}^2$$

While the problem of finding w_h^{pos} is nonlinear, it is a convex optimization problem.

Below is some graphs of w_h^{pos} where $w = f(x, y)$.