LDG Lewis Project

PDE: $$\frac{\partial f}{\partial t} = \frac{1}{v^2}\frac{\partial}{\partial v}\left[C(v)\frac{\partial f}{\partial v}\right] + \frac{1}{\sin(\theta)}\frac{\partial}{\partial \theta}\left[\sin(\theta)\, A(v)\frac{\partial f}{\partial \theta}\right]$$

$$= \frac{1}{v^2}\frac{\partial}{\partial v}\left[C(v)\frac{\partial f}{\partial v}\right] + \frac{1}{v\sin(\theta)}\frac{\partial}{\partial \theta}\left[\sin(\theta)\, A(v)\, v^2\left(\frac{1}{v}\frac{\partial f}{\partial \theta}\right)\right]$$

$$= \operatorname{div}\left(C(v)\frac{\partial f}{\partial v}\hat{v} + A(v)\, v^2\left(\frac{1}{v}\frac{\partial f}{\partial \theta}\right)\hat{\theta} \right)$$

$$= \operatorname{div}\left(B(v,\theta)\nabla f \right)$$

where $B(v,\theta)$ is a diffusion tensor defined by the following mapping on the basis vectors $\hat{v}, \hat{\theta}$

$$B(v,\theta)[\hat{v}] = C(v)\,\hat{v}$$

$$B(v,\theta)[\hat{\theta}] = v^2 A(v)\, v.$$

We can decompose B as A^2 where

$$A(v,\theta)[\hat{v}] = \sqrt{C(v)}\,\hat{v}$$

$$A(v,\theta)[\hat{\theta}] = v\sqrt{A(v)}\,\hat{\theta}.$$

Then we arrive at

$$\frac{\partial f}{\partial t} = \operatorname{div}\left(A^2(v,\theta)\nabla f\right)$$

LDG w/ spherical coordinates in 2D

$$\frac{\partial f}{\partial t} = \operatorname{div}\left(A^2(v,\theta)\nabla f\right) \quad \left\{ \frac{\partial f}{\partial t} = \operatorname{div}(A w) \right.$$

$$\Lambda(v,\theta)(\hat{v}) = \sqrt{C(v)}\,\hat{v}$$

$$\Lambda(v,\theta)(\hat{\theta}) = v\sqrt{A(v)}\,\hat{\theta}$$

$$w = \Lambda\nabla f$$

$$w = w_v\hat{v} + w_\theta\hat{\theta}$$

$$\Lambda w = \sqrt{C(v)}\,w_v\hat{v} + \sqrt{A(v)}\,w_\theta\hat{\theta}$$

$$\frac{\partial f}{\partial t} = \operatorname{div}(\Lambda w)$$

Test by z and throw derivative on z.

$$\left(\frac{\partial f}{\partial t}, z\right)_\Omega = (\operatorname{div}(\Lambda w), z)_\Omega = -(\Lambda w, \nabla z)_\Omega + \underbrace{\langle \widehat{\Lambda w}, [z]\rangle_{\mathcal{E}_I}}$$

Ignoring flux terms for simplicity

Write each term as Kronecker product of DIV, GRAD, or MASS

$$(\Lambda w, w)_\Omega = \int\int \sqrt{C(v)}\,w_v(v,\theta)\frac{\partial z}{\partial v}(v,\theta)\sin(\theta)\,v^2\,dv\,d\theta$$

$$v \quad \theta$$

$$G_1 = \underset{(\text{gfmz}, \text{flux}, \text{j+c})}{DIV\left(\sqrt{\lambda(v)}, -1, v^2\right)} \otimes \underset{(\text{gfmc}, \text{j+c})}{MASS\left(1, \sin(\theta)\right)} = M_\theta$$

$$+ \underbrace{\int_v \int_\theta v \sqrt{\lambda(v)}\, w_\theta(v,\theta)\, \frac{1}{v}\, \frac{\partial z}{\partial \theta}(v,\theta)\, \sin(\theta)\, v^2\, d\theta}$$

$$= v$$

$$M_{1,v} = MASS\left(\sqrt{\lambda(v)}, v\right) \otimes DIV\left(1, -1, \sin(\theta)\right) = G_2$$

$$\underbrace{\left(\frac{\partial f}{\partial t}, z\right)_\Omega = \int_v \int_\theta \frac{\partial f}{\partial t}\, z(v,\theta)\, \sin(\theta)\, v^2\, dv\, d\theta}$$

$$M_v = MASS\left(1, v^2\right) \otimes MASS\left(1, \sin(\theta)\right) = M_\theta$$

Matrix Eqn

$$M \frac{\partial f}{\partial t} = \tilde{S}_1 w_v + \tilde{S}_2 w_\theta. \qquad M = M_v \otimes M_\theta$$

$$\tilde{S}_1 = G_1 \otimes M_\theta, \quad \tilde{S}_2 = M_{1,v} \otimes G_2$$

$$w = A\nabla\xi \qquad \tau = \tau_v \hat{v} + \tau_\theta \hat{\theta}$$

$$(w, \tau)_\Omega = \underbrace{\int_v\int_\theta w_v \tau_v \sin(\theta) v^2 \, dv \, d\theta}_{\mathrm{MASS}(1, v^2) \otimes \mathrm{MASS}(1, \sin(\theta))} + \underbrace{\int_v\int_\theta w_\theta \tau_\theta \sin(\theta) v^2 \, dv \, d\theta}_{\text{same as}}$$

$$(A\nabla\xi, \tau)_\Omega = \underbrace{\int_v\int_\theta \sqrt{A(v)} \frac{\partial \xi}{\partial v} \tau_v \sin(\theta) v^2 \, dv \, d\theta}_{-G_1^T \; = \; \mathrm{GRAD}(\sqrt{A(v)}, -1, v^2) \otimes \mathrm{MASS}(1, \sin(\theta)) = M_\theta}$$

$$+ \underbrace{\int_v\int_\theta v\sqrt{A(v)} \underbrace{\frac{1}{v} \frac{\partial \xi}{\partial \theta} \tau_\theta \sin(\theta) v^2}_{=v} \, dv \, d\theta}_{M_{\ldots} \; = \; \mathrm{MASS}(v\sqrt{A(\ldots)}, v) \otimes \mathrm{GRAD}(\ldots) = \ldots^T}$$

$M_{1,v} = MASS(v\sqrt{\lambda(v)}, v) \otimes GRAD(1, -1, \sin(\theta)) = -G_2'$

Matrix Equation

$$\tau_v^T M w_v + \tau_\theta^T M w_\theta = \tau_v^T S_1 f + \tau_\theta^T S_2 f \quad \forall \tau_v, \tau_\theta$$

Full Matrix System (w/o BCs)

$$M \frac{\partial f}{\partial t} = \tilde{S}_1 w_\theta + \tilde{S}_2 w_v$$

$$\underbrace{\tau_v^T M w_v + \tau_\theta^T M w_\theta = \tau_v^T S_1 f + \tau_\theta^T S_2 f}$$ ← We can split this as 2 eqns. set τ_v or $\tau_\theta = 0$ to isolate.

$\Rightarrow$

$$M w_v = S_1 f$$
$$M w_\theta = S_2 f$$

$M w_v = S_1 f \Rightarrow (M_v \otimes M_\theta) w_v = S_1 f$

$M w_v = S_2 f \Rightarrow (M_v \otimes M_\theta) w_\theta = S_2 f$

$S_1 = -G_1^T \otimes M_\theta$

$S_2 = M_{1,v} \otimes -G_2^T$

↑ because of $MASS(v\sqrt{\lambda(v)}, v)$

Thus $\tilde{S}_1 w_v = \tilde{S}_1 M^{-1} S_1 f$

$$\tilde{S}_2 w_\theta = \tilde{S}_1 M^{-1} S_2 \xi.$$

$$\tilde{S}_1 M S_1 = (G_1 \otimes M_\theta)(M_v^{-1} \otimes M_\theta^{-1})(-G_1^T \otimes M_\theta)$$

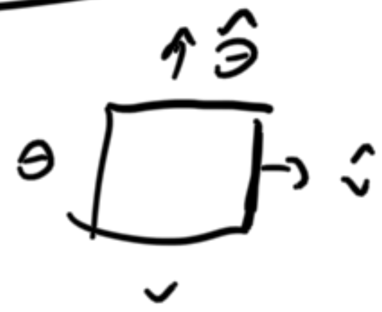
$$= -\underbrace{(G_1 M_v^{-1} G_1^T) \otimes M_\theta}_{\text{symmetric positive definite}}$$

$$\tilde{S}_2 M S_2 = (M_{1,v} \otimes G_2)(M_v^{-1} \otimes M_\theta^{-1})(M_{1,v} \otimes -G_2^T)$$

This requires two mass matrices rechained

$$= -\underbrace{(M_{1,v} M_v^{-1} M_{1,v}) \otimes G_2 M_\theta^{-1} G_2^T}_{\text{symmetric positive def}}$$

With Dirichlet BCs



$$M \frac{\partial \xi}{\partial t} = \tilde{S}_1 w_v + \tilde{S}_2 w_\theta$$

$$\tau_v^T M w_v + \tau_\theta^T M w_2 = \tau_v^T S_1 \xi + \tau_\theta^T S_2 \xi + \tau_v^T BC_1 + \tau_\theta^T BC_2$$

Here $\tau_v^T BC_1 + \tau_\theta^T BC_2$ is the action $\tau \to \langle \xi_D, \tau \cdot n \rangle_{\partial\Omega}$.

Note $\langle \xi_D, \tau \cdot n \rangle_{\partial\Omega} = \langle \xi_D, \tau \cdot \hat{v} \rangle_{\partial\Omega_v \times \Omega_\theta}$

$$+\langle f_D, \tau\cdot\hat{\theta}\rangle_{\Omega_v\times\partial\Omega_\theta}$$

$$= \langle f_D, \tau_v\rangle_{\partial\Omega_v\times\Omega_\theta} + \langle f_D, \tau_\theta\rangle_{\Omega_v\times\partial\Omega_\theta}$$

$$= \tau_v^T BC_1 + \tau_\theta^T BC_2$$

Thus $$M w_v = S_1 f + BC_1$$

$$M w_\theta = S_2 f + BC_2$$

$$\Rightarrow \quad \tilde{S}_1 w_v = \tilde{S}_1 M^{-1} S_1 f + \tilde{S}_1 M^{-1} BC_1$$

$$\tilde{S}_2 w_\theta = \tilde{S}_2 M^{-1} S_2 f + \tilde{S}_2 M^{-1} BC_2$$

$$\tilde{S}_1 M^{-1} BC_1 = (G_1 \otimes M_\theta)(M_v^{-1} \otimes M_\theta^{-1}) BC_1$$

$$= (G_1 M_v^{-1} \otimes I) BC_1$$

$$\tilde{S}_2 M^{-1} BC_2 = (M_{v} \otimes G_2)(M_v^{-1} \otimes M_\theta^{-1}) BC_2$$

$$= (M_v M_v^{-1} \otimes G_2 M_\theta^{-1}) BC_2$$

Needs to be implemented

Currently implemented in boundary_condition_vector.m