LDG for Non-Cartesian PDE

$$\xi_t = -\mathrm{div}(A^2 \nabla \xi)$$

where $A\hat{\rho} = \sqrt{C_A}$

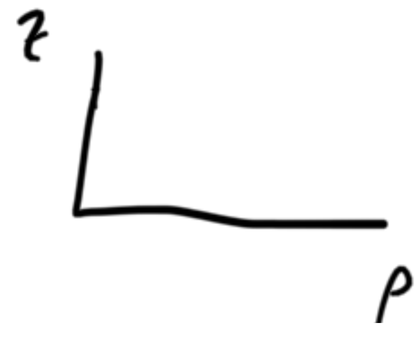
$$A\hat{z} = \sqrt{C_B}/\rho.$$

Mixed Formulation: $\xi_t = -\mathrm{div}(A\sigma)$

$$\sigma = A\nabla\xi$$

$$(\xi_t, v)_\Omega = -(\mathrm{div}(A\sigma), v)_\Omega = (A\sigma, \nabla v)_\Omega + \langle \widehat{A\sigma}, [v] \rangle_{\mathcal{E}_h}$$

$$\widehat{A\sigma} = \begin{cases} A\{\sigma\} & \text{if } e \in \mathcal{E}_h^i \leftarrow \text{interior edges} \\ 0 & \text{on } \partial\Omega \end{cases}$$



Assume C_A, C_B only depends on ρ.

Suppose $\sigma = \sigma_\rho \hat{\rho} + \sigma_\rho \hat{z}$

$$(A\sigma, \nabla v)_\Omega = \int_\rho \int_z (\sigma_\rho A \hat{\rho} + \sigma_z A \hat{z}) \cdot \left(\frac{\partial v}{\partial \rho} \hat{\rho} + \frac{1}{\rho} \sqrt{1-z^2}\, \hat{z} \right) \rho^2 \, d\rho \, dz$$

$$= \int_\rho \int_z \sqrt{C_A}\, \sigma_\rho \frac{\partial v}{\partial \rho} \underline{\rho^2 \, d\rho \, dz} + \int_\rho \int_z \frac{\sqrt{C_B}}{\rho} \sigma_z \frac{\partial v}{\partial z} \underline{\rho \sqrt{1-z^2} \, d\rho \, dz}$$

While the volume jacobian is $\rho^2 \, d\rho \, dz$, the effective jacobian once the gradient is given in the coordinate system, independent of the g/mes, is the same as the surface jacobians on the surface that is constant w.r.t the derivative

ρ derivative (first term): surface jacobian ρ^2

z derivative (second term): surface jacobian: $\rho\sqrt{1-z^2}$

Given $e \in \mathcal{E}_h^I$, then the normal on e is either $\pm\hat{\rho}$ or $\pm\hat{z}$.

$$\langle \hat{A\sigma}, [v] \rangle_e = \Big\{ \int_z (\{\sigma_\rho\} A \hat{\rho} + \{\sigma_z\} A \hat{z}) \cdot (v^+ \hat{\rho}^+ + v^- \hat{\rho}^-) \, \rho^2 \, dz$$

$$= \int_z \sqrt{c_a}\{\sigma_\rho\}\hat{\rho}\cdot[v]\ \underline{\rho^2\, dz} \qquad \text{if edge has } \rho \text{ constant}$$

$n = \pm\hat{\rho}$

$$\int_\rho \{\sigma_\rho\}\Delta\hat{\rho} + \{\sigma_z\}\Delta\hat{z}\cdot(v^+z^+ + v^-z^-)\ \rho\sqrt{1-z^2}\ d\rho$$

$$= \int_\rho \frac{\sqrt{c_a}}{\rho}\{\sigma_z\}\hat{z}\cdot[v]\ \underline{\rho\sqrt{1-z^2}\, d\rho} \qquad \text{if edge has } z \text{ constant}$$

$\hat{n} = \pm\hat{z}$