Phase in electromagnetic coupling

January 8, 2026

1 Phase in electomagnetic coupling

These notes build a clean toy-model Hamiltonian for two identical two-level systems (TLS) coupled to a single quantised electromagnetic (EM) mode, focusing on:

- How to write the interaction for **electric** and **magnetic** dipole transitions.
- Why and how **phase factors** like $e^{\pm i\phi}$ appear in cavity/waveguide Hamiltonians.
- The difference between:
  - **spatial phase** (from e^{ikx} in a travelling wave), and
  - **quadrature rotation** (coupling to X vs P of the oscillator).
- A conceptual clarification of "phase" in EM waves (travelling vs standing waves).

Throughout, we use a *single mode* (one harmonic oscillator) and two TLS labelled donor D and acceptor A.

1.1 The basic ingredients

1.1.1 TLS Hamiltonian

For TLS $j \in \{D, A\}$ with transition frequency ω_j,

$$H_{\mathrm{TLS}} = \sum_{j=D,A} \frac{\hbar\omega_j}{2}\,\sigma_z^{(j)}. \tag{1}$$

- $\sigma_z^{(j)}$ is the Pauli z operator acting on TLS j.
- $\sigma_\pm^{(j)}$ are raising/lowering operators, with $\sigma_x^{(j)} = \sigma_+^{(j)} + \sigma_-^{(j)}$.

1.1.2 Single quantised mode Hamiltonian

One EM mode of frequency ω_c is described by bosonic operators $a, a^\dagger$:

$$H_{\text{mode}} = \hbar\omega_c\, a^\dagger a. \tag{2}$$

Total free Hamiltonian:

$$H_0 = H_{\text{mode}} + H_{\text{TLS}}. \tag{3}$$

1.2 Dipole interactions: the starting point

A TLS transition can couple to:

- the **electric field** via an electric transition dipole $\hat{\mathbf{d}}$, and/or
- the **magnetic field** via a magnetic transition dipole $\hat{\boldsymbol{\mu}}$.

In the dipole approximation, at the TLS position $\mathbf{r}_j$:

$$H_{\text{int}} = -\sum_{j=D,A} \left(\hat{\mathbf{d}}^{(j)} \cdot \hat{\mathbf{E}}(\mathbf{r}_j) + \hat{\boldsymbol{\mu}}^{(j)} \cdot \hat{\mathbf{B}}(\mathbf{r}_j) \right). \tag{4}$$

1.2.1 Transition operators (minimal, aligned case)

To keep the algebra transparent, assume each transition dipole is aligned with the local field polarisation so dot products reduce to scalars:

- $\hat{d}^{(j)} = d\,\sigma_x^{(j)}$
- $\hat{\mu}^{(j)} = \mu\,\sigma_x^{(j)}$

Then Eq. 4 becomes

$$H_{\text{int}} = -\sum_{j=D,A} \sigma_x^{(j)} \left(d\,\hat{E}(\mathbf{r}_j) + \mu\,\hat{B}(\mathbf{r}_j) \right). \tag{5}$$

(General vector/complex matrix-element versions give the same structure but with additional projection factors and complex conjugation; the key ideas below do not rely on the simplification.)

1.3 Quantising the fields for a single mode

A single mode at position $\mathbf{r}$ can always be written as

$$\hat{\mathbf{E}}(\mathbf{r}) = \mathbf{E}_{\text{zpf}}(\mathbf{r})\, a + \mathbf{E}^*_{\text{zpf}}(\mathbf{r})\, a^\dagger, \qquad \hat{\mathbf{B}}(\mathbf{r}) = \mathbf{B}_{\text{zpf}}(\mathbf{r})\, a + \mathbf{B}^*_{\text{zpf}}(\mathbf{r})\, a^\dagger. \tag{6}$$

- “zpf” means zero-point field amplitude (including polarisation and spatial mode function).

- The complex nature of $\mathbf{E}_{\text{zpf}}(\mathbf{r})$ or $\mathbf{B}_{\text{zpf}}(\mathbf{r})$ is exactly where phase factors come from.

A useful scalar shorthand at each TLS position is:

$$\hat{B}(\mathbf{r}_j) = B_{\text{zpf}}\big(u_j a + u_j^* a^\dagger\big), \qquad u_j \equiv (\text{complex mode factor at } \mathbf{r}_j). \tag{7}$$

Write $u_j = |u_j| e^{i\theta_j}$:

$$u_j a + u_j^* a^\dagger = |u_j| \left(e^{i\theta_j} a + e^{-i\theta_j} a^\dagger\right). \tag{8}$$

This algebraic form is the origin of Hamiltonians containing $(e^{i\phi} a + e^{-i\phi} a^\dagger)$.

1.4 Phase factors from spatial separation in a travelling wave (B-only)

1.4.1 Travelling-wave mode function

For a running (travelling) wave along x, a standard mode factor is

$$u(x) = e^{ikx}. \tag{9}$$

Then the magnetic field operator is

$$\hat{B}(x) = B_{\text{zpf}} \left(e^{ikx} a + e^{-ikx} a^\dagger\right). \tag{10}$$

1.4.2 Two identical TLS at positions x_D and x_A

Insert Eq. 10 into the magnetic-only part of Eq. 5 (set $d = 0$):

$$H_{\text{int}}^{(B)} = -\mu B_{\text{zpf}} \sum_{j=D,A} \sigma_x^{(j)} \left(e^{ikx_j} a + e^{-ikx_j} a^\dagger\right). \tag{11}$$

Define the couplings $V_j \equiv \mu B_{\text{zpf}}$ (or include $|u(x_j)|$ if the mode has spatial envelope). Then

$$H_{\text{int}}^{(B)} = -\sum_{j=D,A} V_j \, \sigma_x^{(j)} \left(e^{ikx_j} a + e^{-ikx_j} a^\dagger\right). \tag{12}$$

1.4.3 Only the relative phase matters

You can rephase the oscillator without changing H_{mode}:

- Transform $a \to a\, e^{-ikx_D}$ (and $a^\dagger \to a^\dagger e^{ikx_D}$).

This makes the donor term purely $(a + a^\dagger)$, and the acceptor term keeps the **relative** phase

$$\Delta\phi = k(x_A - x_D). \tag{13}$$

So you obtain the familiar-looking structure

$$H_{\mathrm{int}}^{(B)} = -(a + a^\dagger)\, V_D\, \sigma_x^{(D)} - \left(e^{i\Delta\phi} a + e^{-i\Delta\phi} a^\dagger\right)\, V_A\, \sigma_x^{(A)}. \tag{14}$$

Key point: in this B-only travelling-wave case, $\Delta\phi$ is simply the **spatial phase** sampled by the two TLS.

1.4.4 Special case: $\lambda/4$ separation

If $x_A - x_D = \lambda/4$, then $k(x_A - x_D) = 2\pi(\lambda/4)/\lambda = \pi/2$, so $\Delta\phi = \pi/2$.

This is the cleanest route to a $\pi/2$ factor with B-only coupling: a travelling wave and a quarter-wavelength separation.

1.5 Common confusion: is this "E vs B quadratures"?

1.5.1 The simple answer

- In the travelling-wave, B-only story above, the phase $\Delta\phi$ comes from e^{ikx}, i.e. different complex amplitudes at different positions.
- Writing $(e^{i\Delta\phi} a + e^{-i\Delta\phi} a^\dagger)$ as a "rotated quadrature" is **just a re-expression** of that same spatial phase.

There is no need to invoke electric coupling to explain the B-only relative phase.

1.6 Quadratures: what the $e^{\pm i\phi}$ combination really means

Define oscillator quadratures

$$X \equiv a + a^\dagger, \qquad P \equiv i(a^\dagger - a). \tag{15}$$

Then the identity

$$e^{i\phi}a + e^{-i\phi}a^\dagger = X\cos\phi + P\sin\phi \tag{16}$$

shows that $(e^{i\phi}a + e^{-i\phi}a^\dagger)$ is a **rotated quadrature**.

1.6.1 What does $\phi = \pi/2$ mean?

Set $\phi = \pi/2$ in Eq. 16:

$$e^{i\pi/2}a + e^{-i\pi/2}a^\dagger = P \times (\text{sign}). \tag{17}$$

So "$\pi/2$ in the Hamiltonian" is shorthand for "coupling to the orthogonal quadrature".

In the travelling-wave case, that orthogonal-quadrature appearance is simply because sampling the mode at $x = \lambda/4$ rotates the combination of a and $a^\dagger$.

1.7 How coupling to both E and B rotates the quadrature at a single point

This is a different mechanism: it can produce an effective phase angle even without spatial separation.

1.7.1 Step 1: In one common convention, E and B correspond to different quadratures

A standard quantisation choice uses the vector potential $\hat{\mathbf{A}} \propto X$, then:

- $\hat{\mathbf{B}} = \nabla \times \hat{\mathbf{A}} \propto X$
- $\hat{\mathbf{E}} = -\partial_t \hat{\mathbf{A}} \propto P$

At a fixed position $\mathbf{r}_0$ (suppress vector details):

$$\hat{E}(\mathbf{r}_0) = E_{\text{zpf}}\, P, \qquad \hat{B}(\mathbf{r}_0) = B_{\text{zpf}}\, X. \tag{18}$$

1.7.2 Step 2: Couple a single TLS to both

Use Eq. 5 at $\mathbf{r}_0$:

$$H_{\text{int}}(\mathbf{r}_0) = -\sigma_x \left(d\,\hat{E}(\mathbf{r}_0) + \mu\,\hat{B}(\mathbf{r}_0) \right). \tag{19}$$

Insert Eq. 18:

$$H_{\text{int}}(\mathbf{r}_0) = -\sigma_x \left(g_E\, P + g_B\, X \right), \tag{20}$$

where

- $g_E \equiv dE_{\text{zpf}}$
- $g_B \equiv \mu B_{\text{zpf}}$

1.7.3 Step 3: Rewrite as a rotated quadrature

Define

$$V = \sqrt{g_B^2 + g_E^2}, \qquad \phi = \arctan\left(\frac{g_E}{g_B}\right). \tag{21}$$

Then

$$g_B X + g_E P = V\left(X\cos\phi + P\sin\phi\right). \tag{22}$$

Using Eq. 16:

$$H_{\text{int}}(\mathbf{r}_0) = -V\,\sigma_x\left(e^{i\phi}a + e^{-i\phi}a^\dagger\right). \tag{23}$$

Interpretation: coupling to both E and B gives two independent linear couplings to the same oscillator, hence a rotated quadrature.

1.7.4 Important caveat (identical TLS at the same point)

If two TLS are identical and co-located (and oriented the same way), they have the same ratio g_E/g_B and therefore the same ϕ. In that case:

- there is no *relative* phase between them arising from this mechanism alone,
- you can rephase the mode to remove the common ϕ from both simultaneously.

To get a *relative* phase from this mechanism, something must make g_E/g_B differ between the TLS (different position in a standing wave where E and B vary differently, different orientation, etc.).

1.8 Conceptual interlude: are E and B "in phase" by Maxwell?

A major conceptual hurdle is distinguishing:

- **travelling waves** (propagating plane waves), and
- **standing waves** (superpositions of counter-propagating waves, e.g. cavity modes).

1.8.1 Travelling plane wave: E and B are in phase

A standard vacuum plane wave propagating in $+x$:

$$E_y(x,t) = E_0 \cos(kx - \omega t), \qquad B_z(x,t) = \frac{E_0}{c} \cos(kx - \omega t). \tag{24}$$

At fixed x, the maxima/minima occur at the same times: no relative $\pi/2$ time shift.

A compact statement in complex-amplitude form is:

$$\mathbf{B}_0 = \frac{1}{\omega}\, \mathbf{k} \times \mathbf{E}_0, \tag{25}$$

so both share the same space–time factor $e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$.

1.8.2 Why the magnetic field "subtracts" in a standing-wave construction

A standing wave is built from two travelling waves of opposite propagation direction. For the same E polarisation, reversing propagation flips the sign of B because

- $\mathbf{B} \propto \hat{\mathbf{k}} \times \mathbf{E}$,
- $\hat{\mathbf{k}} \to -\hat{\mathbf{k}}$ implies $\mathbf{B} \to -\mathbf{B}$.

This is why, when you add counter-propagating waves, the E fields add but the B fields subtract.

1.8.3 Standing wave: E and B can be $\pi/2$ out of phase in time at a point

Adding the two waves yields (one common form):

$$E_y(x,t) = 2E_0 \cos(kx)\cos(\omega t), \qquad B_z(x,t) = 2\frac{E_0}{c} \sin(kx)\sin(\omega t). \tag{26}$$

At a fixed x where both $\cos(kx)$ and $\sin(kx)$ are nonzero:

- $E \propto \cos(\omega t)$
- $B \propto \sin(\omega t)$

This is a genuine $\pi/2$ time shift.

1.8.4 Common confusion: "cos vs sin means $\pi/2$"

- Yes: $\sin(\omega t) = \cos(\omega t - \pi/2)$.
- But you must compare **two signals with the same argument** at the same location/time origin.

In a travelling wave, you can write both E and B using $\cos(kx - \omega t)$ (or both using sin) consistently; switching cos $\leftrightarrow$ sin for one but not the other would correspond to shifting reference phase for one field only, which is not the physical plane-wave relation.

1.9 Putting it all together: what "phase" means in the Hamiltonian

There are two distinct (but algebraically similar) appearances of phases:

1.9.1 1) Spatial phase (travelling wave, even with B-only)

- The mode function has $u(x) = e^{ikx}$.
- Different TLS positions sample different complex coefficients.
- This yields a relative phase $\Delta\phi = k(x_A - x_D)$ in the coupling, Eq. 13.
- $\lambda/4$ separation gives $\Delta\phi = \pi/2$.

1.9.2 2) Quadrature rotation (coupling to both E and B at one point)

- In a common quantisation convention, $B \sim X$ and $E \sim P$.
- Coupling to both gives $g_B X + g_E P = V(X\cos\phi + P\sin\phi)$ with $\phi = \arctan(g_E/g_B)$.
- This produces an $e^{\pm i\phi}$ structure even without spatial separation, Eq. 23.
- For identical TLS at the same point, this does not create a relative phase by itself.

1.10 Takeaways

- A travelling-wave spatial mode $u(x) = e^{ikx}$ gives **position-dependent complex coupling**; the relative phase is $\Delta\phi = k\Delta x$ (Eq. 13).
- A $\pi/2$ phase factor in $(e^{i\phi}a + e^{-i\phi}a^\dagger)$ corresponds to coupling to the **orthogonal oscillator quadrature** (Eq. 16).
- With **B-only coupling**, a $\pi/2$ *relative* phase arises cleanly from $\lambda/4$ **spatial separation** in a travelling wave (Eq. 14).
- Coupling to both **E and B** at a single point yields a **quadrature rotation** with $\phi = \arctan(g_E/g_B)$ (Eq. 21), even without spatial separation.
- Maxwell's equations allow E and B to be **in phase in travelling waves** (Eq. 24) but often $\pi/2$ **out of phase in standing waves** (Eq. 26).