

# Coupling constants in nuclear physics

Matt Lilley

March 5, 2025

## 1 Introduction

Understanding the interaction between nucleons and external fields is essential in nuclear physics. We'll explore two coupling mechanisms that arise in quantum nuclear interactions:

1. **Relativistic phonon nuclear coupling** ( $a \cdot cp$ ) – where phonons couple to nucleons through momentum exchange (see [Hagelstein 2023](#) for more detail).
2. **Electric dipole coupling** ( $d \cdot E$ ) – where an electric field couples to nucleons through dipole moments.

This document explores both couplings, derives their respective coupling constants, and compares their strength.

## 2 Relativistic phonon nuclear coupling ( $a \cdot cp$ )

### 2.1 $p$ in $a \cdot cp$

For a nucleus of mass  $M$  moving within a solid, the momentum-based coupling energy scales as:

$$E \sim pc \tag{1}$$

From kinetic energy considerations:

$$\frac{p^2}{2M} = E \quad \Rightarrow \quad p = \sqrt{2ME} \tag{2}$$

If  $E$  represents phonon oscillations:

$$E = n\hbar\omega_A \tag{3}$$

where  $n$  is the phonon occupation number. Distributing this energy over  $N$  atoms:

$$p = \sqrt{\frac{2M}{N}} \sqrt{\hbar\omega_A} \sqrt{n} \quad (4)$$

Note that later we'll drop the  $\sqrt{n}$  because it should be picked up in the Hamiltonian operator  $(b^\dagger + b)$ .

## 2.2 $a$ in $a \cdot cp$

From Eq. 470 in section 6.11 of Models for nuclear fusion in the solid state, the  $a$  component of  $a \cdot cp$  can be approximated as:

$$a \sim \frac{1}{2} \frac{\Delta E}{Mc^2} \frac{\pi}{mc} \quad (5)$$

where: -  $\Delta E$  is the nuclear transition energy, -  $m$  is the mass of a single nucleon within the nucleus, -  $\pi$  represents the relative momentum of that nucleon.

Since angular momentum is approximately  $\hbar$ , and using the Fermi scale  $l_F = 10^{-15}$  m, we estimate:

$$\pi \sim \frac{\hbar}{l_F} \quad (6)$$

Substituting this into the expression for  $a$  gives:

$$a \sim \frac{1}{2} \frac{\Delta E}{Mc^2} \frac{\bar{\lambda}_c}{l_F} \quad (7)$$

where  $\bar{\lambda}_c = \hbar/mc$  is the reduced Compton wavelength, approximately:

$$\bar{\lambda}_c \approx 2 \times 10^{-16} \text{ m}. \quad (8)$$

To account for hindrance effects in nuclear transitions, we introduce a suppression factor  $O$ , where  $O \sim 0.01$ . This modifies the expression to:

$$a \sim \frac{1}{2} \frac{\Delta E}{Mc^2} \frac{\bar{\lambda}_c}{l_F} O \quad (9)$$

which simplifies to:

$$a \sim \frac{\Delta E}{Mc^2} \times 10^{-3} \quad (10)$$

The final value of  $a$  depends on both the nuclear transition type and the specific nucleus under consideration.

### 2.3 Overall coupling constant

Let's consider a single TLS interacts with a single phonon mode. The Hamiltonian can be written as:

$$H = \frac{\Delta E}{2} \sigma_z + \hbar \omega_A \left( b^\dagger b + \frac{1}{2} \right) + U (b^\dagger + b) \sigma_x \quad (11)$$

where  $\Delta E$  is the transition energy between the 2 levels of the TLS,  $\hbar \omega_A$  is the energy of each quantum of the field, and  $U$  is the coupling constant between the TLS and the field. The  $\sigma$  operators are the Pauli matrices and  $b^\dagger, b$  are the field creation and annihilation operators respectively. Note that usually  $a$  is used for the field operators, but in these notes we use  $b$  to avoid confusion with  $a \cdot cp$ .

then an  $a \cdot cp$  coupling constant  $U$  can be defined by combining Eq. 4 (without the  $\sqrt{n}$ ) with Eq. 10:

$$U = c \sqrt{\frac{2M}{N}} \sqrt{\hbar \omega_A} \times \frac{\Delta E}{Mc^2} \times 10^{-3} \quad (12)$$

Normalising  $U$  to  $\hbar \omega_A$  gives:

$$\frac{U}{\hbar \omega_A} = \sqrt{\frac{2Mc^2}{N}} \frac{1}{\sqrt{\hbar \omega_A}} \times \frac{\Delta E}{Mc^2} \times 10^{-3} \quad (13)$$

Rearranging and simplifying leads to:

$$\frac{U}{\hbar \omega_A} = \sqrt{\frac{2}{N}} \sqrt{\frac{\Delta E}{Mc^2}} \sqrt{\frac{\Delta E}{\hbar \omega_A}} \times 10^{-3} \quad (14)$$

which can also be written as:

$$\frac{U}{\hbar \omega_A} = \sqrt{\frac{2}{N}} \sqrt{\frac{\hbar \omega_A}{Mc^2}} \frac{\Delta E}{\hbar \omega_A} \times 10^{-3} \quad (15)$$

### 2.4 Example for palladium with acoustic phonons

- $\Delta E \approx 24 \times 10^6$  eV
- $Mc^2 \approx 10^{11}$  eV
- $\hbar \omega_A \approx 10^{-8}$  eV
- $N \approx 10^{18}$

$$\frac{U}{\hbar\omega_A} \approx \sqrt{\frac{2}{10^{18}}} \sqrt{\frac{24 \times 10^6}{10^{11}}} \sqrt{\frac{24 \times 10^6}{10^{-8}}} \times 10^{-3} \quad (16)$$

$$\approx 10^{-6} \quad (17)$$

## 2.5 Dicke enhancement

For an ensemble of  $N$  nuclei interacting collectively with a phonon field, coupling is enhanced by  $\sqrt{N}$ , leading to:

$$\frac{U}{\hbar\omega_A} \sim 10^3 \quad (18)$$

Based on this, we can be far into the “deep strong coupling” regime where  $U/\hbar\omega_A > 1$ .

## 3 Electric dipole coupling (E1 transitions)

### 3.1 $E$ in $d \cdot E$

Electric field strength due to phonons follows from force relations:

$$F = \frac{dp}{dt} = ZeE \quad (19)$$

For oscillatory motion:

$$\frac{dp}{dt} \sim \omega_A p \Rightarrow E = \frac{\omega_A p}{Ze} \quad (20)$$

Substituting our previous result for  $p$ :

$$E = \frac{\omega_A \sqrt{2M\hbar\omega_A n}}{Ze\sqrt{N}} \quad (21)$$

### 3.2 $d$ in $d \cdot E$

We can connect two key expressions related to dipole interactions:

1. Radiation from a dipole – describes how an oscillating dipole emits radiation.
2. Radiative decay rates from Weisskopf – provides an estimate for transition rates.

### 3.2.1 Radiation from a dipole

The radiative decay rate due to dipole radiation is given by:

$$\gamma_{\text{rad}} = \frac{4}{3} \frac{1}{4\pi\epsilon_0\hbar} \frac{\omega^3}{c^3} d^2 \quad (22)$$

Rewriting in terms of the fine-structure constant  $\alpha$ :

$$\gamma_{\text{rad}} = \frac{4}{3} \frac{1}{e^2} \alpha \frac{\omega^3}{c^2} d^2 \quad (23)$$

where the fine-structure constant is defined as:

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \quad (24)$$

### 3.2.2 Weisskopf estimate for E1 transition

Weisskopf's formula for radiative decay is given by (Eq. 16.35 of Bielajew's book or Eq. A.190 of Dommelen's book):

$$\gamma_{\text{rad}} = \frac{8\pi(L+1)}{L[(2L+1)!!]^2} \alpha (kR)^{2L} \omega \left( \frac{3}{L+3} \right)^2 \quad (25)$$

where: -  $L$  is the multipolarity ( $L = 1$  for dipole,  $L = 2$  for quadrupole). -  $k$  is the wavenumber of the emitted radiation. -  $R$  is the nuclear radius, given by:

$$R = R_0 A^{1/3} \quad (26)$$

where  $R_0$  is the radius of a single nucleon and  $A$  is the number of nucleons. Note that there exist other forms of Weisskopf's formula that are more convenient for numerical evaluation but they obscure the physical constants.

For a dipole transition ( $L = 1$ ), this simplifies to:

$$\gamma_{\text{rad}} = \frac{8\pi \times 2}{1 \times [3!!]^2} \alpha \frac{\omega^2}{c^2} R_0^2 A^{2/3} \omega \left( \frac{3}{4} \right)^2 \quad (27)$$

Rewriting more compactly:

$$\gamma_{\text{rad}} = \frac{9\pi}{(3!!)^2} \alpha \frac{\omega^3}{c^2} R_0^2 A^{2/3} \quad (28)$$

### 3.2.3 Equating the two expressions

From the previous derivations, we equate:

$$\frac{9\pi}{(3!!)^2} \alpha \frac{\omega^3}{c^2} R_0^2 A^{2/3} = \frac{4}{3} \frac{1}{e^2} \alpha \frac{\omega^3}{c^2} d^2 \quad (29)$$

Rearranging:

$$\frac{27\pi}{4 \times (3!!)^2} A^{2/3} e^2 R_0^2 = d^2 \quad (30)$$

Taking the square root:

$$d = \frac{\sqrt{27\pi}}{2 \times (3!!)} A^{1/3} e R_0 \quad (31)$$

which simplifies to:

$$d = \frac{\sqrt{27\pi}}{1440} A^{1/3} e R_0 \quad (32)$$

Approximating numerically:

$$d \approx 6 \times 10^{-3} A^{1/3} e R_0 \quad (33)$$

### 3.3 Overall coupling constant

If we again use this simple Hamiltonian in which a single TLS interacts with a single phonon mode:

$$H = \frac{\Delta E}{2} \sigma_z + \hbar \omega_A \left( a^\dagger a + \frac{1}{2} \right) + U (b^\dagger + b) \sigma_x \quad (34)$$

then a  $d \cdot E$  coupling constant  $U$  can be defined by combining Eq. 21 (without the  $\sqrt{n}$ ) with Eq. 33:

$$\frac{U}{\hbar \omega_A} = \frac{1}{\hbar \omega_A} \frac{\omega_A \sqrt{2M\hbar\omega_A}}{Ze\sqrt{N}} \times 6 \times 10^{-3} A^{1/3} e R_0 \quad (35)$$

Rearranging,

$$\frac{U}{\hbar \omega_A} = \frac{\sqrt{2}}{Z\sqrt{N}} \sqrt{\frac{Mc^2}{\hbar \omega_A}} \frac{\hbar \omega_A R_0}{\hbar c} A^{1/3} \times 6 \times 10^{-3} \quad (36)$$

We recognize  $\hbar c/R_0$  as the localization energy of a nucleon, which we call  $E_L$ . Thus, we obtain:

$$\frac{U}{\hbar\omega_A} = \frac{2\pi\sqrt{2}}{Z\sqrt{N}} \sqrt{\frac{Mc^2}{\hbar\omega_A} \frac{\hbar\omega_A}{E_L}} A^{1/3} \times 6 \times 10^{-3} \quad (37)$$

which can also be written as:

$$\frac{U}{\hbar\omega_A} = \frac{2\pi\sqrt{2}}{Z\sqrt{N}} \sqrt{\frac{Mc^2}{E_L}} \sqrt{\frac{\hbar\omega_A}{E_L}} A^{1/3} \times 6 \times 10^{-3} \quad (38)$$

Note how the expressions for  $a \cdot cp$  and  $d \cdot E$  have an interesting reciprocal relationship if we see that  $E_L$  plays the role of  $\Delta E$ .

### 3.4 Example of Pd with Acoustic Phonons

Given: -  $A \approx 106$  -  $N \approx 10^{18}$  -  $Z \approx 106$  -  $Mc^2 \approx 10^{11}$  eV -  $\hbar\omega_A \approx 10^{-8}$  eV

First, let's calculate the localization energy:

$$E_L = \frac{\hbar c}{R_0} = \frac{6.6 \times 10^{-34} \times 3 \times 10^8}{10^{-15}} \quad (39)$$

$$= 2 \times 10^{-10} \text{ J} = 1.2 \times 10^9 \text{ eV} \approx 10^9 \text{ eV} \quad (40)$$

Now, substituting these numbers gives:

$$\frac{U}{\hbar\omega_A} \approx \frac{2\pi\sqrt{2}}{106 \times 10^9} \times \sqrt{\frac{10^{11}}{10^{-8}}} \times \frac{10^{-8}}{10^9} \times 6 \times 10^{-3} \times 106^{1/3} \quad (41)$$

Approximating:

$$\approx \frac{2\pi\sqrt{2}\sqrt{10}}{106 \times 10^9} \times 10^9 \times 10^{-17} \times 6 \times 10^{-3} \times 106^{1/3} \quad (42)$$

$$\approx 8\pi \times 6 \times 10^{-20} \times 106^{-2/3} \quad (43)$$

$$\approx 7 \times 10^{-20} \quad (44)$$

### 3.5 Dicke enhancement

For an ensemble of  $N$  nuclei interacting collectively with a phonon field, coupling is enhanced by  $\sqrt{N}$ , leading to:

$$\frac{U}{\hbar\omega_A} \sim 7 \times 10^{-11} \quad (45)$$

and so even with Dicke enhancement, dipole coupling remains in the weak coupling regime.