Coupling constants in nuclear physics

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1 Introduction

Understanding the interaction between nucleons and external fields is essential in nuclear physics. We'll explore two coupling mechanisms that arise in quantum nuclear interactions:

1. **Relativistic phonon nuclear coupling** $(a \cdot cp)$ – where phonons couple to nucleons through momentum exchange (see Hagelstein 2023 for more detail).
2. **Electric dipole coupling** $(d \cdot E)$ – where an electric field couples to nucleons through electric dipole moments.
3. **Magnetic dipole coupling** $(\mu \cdot B)$ - where a magnetic field couples to nucleonics through magnetic dipole moments.

This document explores these couplings, derives their respective coupling constants, and compares their strength.

2 Relativistic phonon nuclear coupling $(a \cdot cp)$

2.1 p in $a \cdot cp$

For a nucleus of mass M moving within a solid, the momentum-based coupling energy scales as:

$$E \sim pc \tag{1}$$

From kinetic energy considerations:

$$\frac{p^2}{2M} = E \quad \Rightarrow \quad p = \sqrt{2ME} \tag{2}$$

If E represents phonon oscillations:

$$E = n\hbar\omega_A \tag{3}$$

where n is the phonon occupation number. Distributing this energy over N atoms:

$$p = \sqrt{\frac{2M}{N}} \sqrt{\hbar\omega_A} \sqrt{n} \tag{4}$$

Note that later we'll drop the $\sqrt{n}$ because it will be picked up in the Hamiltonian operator $(b^\dagger + b)$.

2.2 a in $a \cdot cp$

From Eq. 470 in section 6.11 of <u>Models for nuclear fusion in the solid state</u>, the a component of $a \cdot cp$ can be approximated as:

$$a \sim \frac{1}{2} \frac{\Delta E}{Mc^2} \frac{\pi}{m_p c} \tag{5}$$

where:

- ΔE is the nuclear transition energy,
- m_p is the mass of a single proton within the nucleus,
- π represents the relative momentum of that nucleon.

Since angular momentum is approximately $\hbar$, and using the Fermi scale $l_F = 10^{-15}$ m, we estimate:

$$\pi \sim \frac{\hbar}{l_F} \tag{6}$$

Substituting this into the expression for a gives:

$$a \sim \frac{1}{2} \frac{\Delta E}{Mc^2} \frac{\bar{\lambda}_c}{l_F} \tag{7}$$

where $\bar{\lambda}_c = \hbar/m_p c$ is the reduced Compton wavelength, approximately:

$$\bar{\lambda}_c \approx 2 \times 10^{-16} \text{ m.} \tag{8}$$

To account for hindrance effects in nuclear transitions, we introduce a suppression factor O, where $O \sim 0.01$. This modifies the expression to:

$$a \sim \frac{1}{2} \frac{\Delta E}{Mc^2} \frac{\bar{\lambda}_c}{l_F} O \tag{9}$$

which simplifies to:

$$a \sim \frac{\Delta E}{Mc^2} \times 10^{-3} \tag{10}$$

The final value of a depends on both the nuclear transition type and the specific nucleus under consideration.

2.3 Overall coupling constant

Let's consider a single two level system (TLS) interacting with a single phonon mode. The Hamiltonian can be written as:

$$H = \frac{\Delta E}{2}\sigma_z + \hbar\omega_A\left(b^\dagger b + \frac{1}{2}\right) + U\left(b^\dagger + b\right)\sigma_x \tag{11}$$

where ΔE is the transition energy between the 2 levels of the TLS, $\hbar\omega_A$ is the energy of each quantum of the field, and U is the coupling constant between the TLS and the field. The σ operators are the Pauli matrices and $b^\dagger$, b are the field creation and annihilation operators respectively. Note that usually a is used for the field operators, but in these notes we use b to avoid confusion with $a \cdot cp$.

The $a \cdot cp$ coupling constant U can be defined by combining Eq. 4 (without the $\sqrt{n}$) with Eq. 10:

$$U = c\sqrt{\frac{2M}{N}}\sqrt{\hbar\omega_A} \times \frac{\Delta E}{Mc^2} \times 10^{-3} \tag{12}$$

Normalising U to $\hbar\omega_A$ gives:

$$\frac{U}{\hbar\omega_A} = \sqrt{\frac{2Mc^2}{N}}\frac{1}{\sqrt{\hbar\omega_A}} \times \frac{\Delta E}{Mc^2} \times 10^{-3} \tag{13}$$

Rearranging and simplifying leads to:

$$\frac{U}{\hbar\omega_A} = \sqrt{\frac{2}{N}}\sqrt{\frac{\Delta E}{Mc^2}}\sqrt{\frac{\Delta E}{\hbar\omega_A}} \times 10^{-3} \tag{14}$$

which can also be written as:

$$\frac{U}{\hbar\omega_A} = \sqrt{\frac{2}{N}}\sqrt{\frac{\hbar\omega_A}{Mc^2}}\frac{\Delta E}{\hbar\omega_A} \times 10^{-3} \tag{15}$$

2.4 Example for a 24MeV palladium transition with acoustic phonons

- $\Delta E \approx 24 \times 10^6$ eV

- $Mc^2 \approx 10^{11}$ eV

- $\hbar\omega_A \approx 10^{-8}$ eV

- $N \approx 10^{18}$

$$\frac{U}{\hbar\omega_A} \approx \sqrt{\frac{2}{10^{18}}} \sqrt{\frac{24 \times 10^6}{10^{11}}} \sqrt{\frac{24 \times 10^6}{10^{-8}}} \times 10^{-3} \tag{16}$$

$$\approx 10^{-6} \tag{17}$$

2.5 Dicke enhancement

For more detail on the Dicke model, see the notes I made on the subject.

For an ensemble of N nuclei interacting collectively with a phonon field, coupling is enhanced by $\sqrt{N}$, leading to:

$$\frac{U}{\hbar\omega_A} \sim 10^3 \tag{18}$$

Based on this, we can be far into the "deep strong coupling" regime where $U/\hbar\omega_A > 1$.

For more detail on the Deep strong coupling, see the notes I made on the subject.

3 Electric dipole coupling (E1 transitions)

3.1 E in $d \cdot E$

Electric field strength due to phonons follows from force relations:

$$F = \frac{dp}{dt} = ZeE \tag{19}$$

For oscillatory motion:

$$\frac{dp}{dt} \sim \omega_A p \Rightarrow E = \frac{\omega_A p}{Ze} \tag{20}$$

Substituting our previous result for p:

$$E = \frac{\omega_A \sqrt{2M\hbar\omega_A n}}{Ze\sqrt{N}} \tag{21}$$

3.2 d in $d \cdot E$

We can connect two key expressions related to electric dipole interactions:

1. Radiation from an electric dipole – describes how an oscillating electric dipole emits radiation.

2. Radiative decay rates from Weisskopf – provides an estimate for transition rates.

3.2.1   Radiation from an electric dipole

The radiative decay rate due to dipole radiation is given by:

$$\gamma_{\mathrm{rad}} = \frac{4}{3} \frac{1}{4\pi\epsilon_0\hbar} \frac{\omega^3}{c^3} d^2 \tag{22}$$

Rewriting in terms of the fine-structure constant α:

$$\gamma_{\mathrm{rad}} = \frac{4}{3} \frac{1}{e^2} \alpha \frac{\omega^3}{c^2} d^2 \tag{23}$$

where the fine-structure constant is defined as:

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \tag{24}$$

3.2.2   Weisskopf estimate for E1 transition

Weisskopf's formula for radiative decay is given by (Eq. A.190 of Dommelen's book):

$$\gamma_{\mathrm{rad}} = \frac{2(L+1)}{L\left[(2L+1)!!\right]^2} \alpha(kR)^{2L}\omega \left(\frac{3}{L+3}\right)^2 \tag{25}$$

where:

- !! is the double factorial.
- L is the multipolarity ($L = 1$ for dipole, $L = 2$ for quadrupole).
- k is the wavenumber of the emitted radiation.
- R is the nuclear radius, given by:

$$R = R_0 A^{1/3} \tag{26}$$

where R_0 is the radius of a single nucleon and A is the number of nucleons. Note that there exist other forms of Weisskopf's formula that are more convenient for numerical evaluation but they obscure the physical constants.

For a dipole transition $(L = 1)$, this simplifies to:

$$\gamma_{\text{rad}} = \frac{2 \times 2}{1 \times [3!!]^2} \alpha \frac{\omega^2}{c^2} R_0^2 A^{2/3} \omega \left(\frac{3}{4}\right)^2 \tag{27}$$

Rewriting more compactly:

$$\gamma_{\text{rad}} = \frac{9}{4 \times (3!!)^2} \alpha \frac{\omega^3}{c^2} R_0^2 A^{2/3} \tag{28}$$

3.2.3 Equating the two expressions

From the previous derivations, we equate:

$$\frac{9}{4 \times (3!!)^2} \alpha \frac{\omega^3}{c^2} R_0^2 A^{2/3} = \frac{4}{3} \frac{1}{e^2} \alpha \frac{\omega^3}{c^2} d^2 \tag{29}$$

Rearranging:

$$\frac{27}{16 \times (3!!)^2} A^{2/3} e^2 R_0^2 = d^2 \tag{30}$$

Taking the square root:

$$d = \frac{\sqrt{27}}{4 \times (3!!)} A^{1/3} e R_0 \tag{31}$$

which simplifies to:

$$d = \frac{\sqrt{27}}{2880} A^{1/3} e R_0 \tag{32}$$

Approximating numerically:

$$d \approx 2 \times 10^{-3} A^{1/3} e R_0 \tag{33}$$

3.3 Overall coupling constant

If we again use this simple Hamiltonian in which a single TLS interacts with a single phonon mode:

$$H = \frac{\Delta E}{2}\sigma_z + \hbar\omega_A \left(a^\dagger a + \frac{1}{2}\right) + U\left(b^\dagger + b\right)\sigma_x \tag{34}$$

then the $d \cdot E$ coupling constant U can be defined by combining Eq. 21 (without the $\sqrt{n}$) with Eq. 33:

$$\frac{U}{\hbar\omega_A} = \frac{1}{\hbar\omega_A} \frac{\omega_A\sqrt{2M\hbar\omega_A}}{Ze\sqrt{N}} \times 2 \times 10^{-3} A^{1/3} eR_0 \tag{35}$$

Rearranging,

$$\frac{U}{\hbar\omega_A} = \frac{\sqrt{2}}{Z\sqrt{N}} \sqrt{\frac{Mc^2}{\hbar\omega_A}} \frac{\hbar\omega_A R_0}{\hbar c} A^{1/3} \times 2 \times 10^{-3} \tag{36}$$

We recognize $\hbar c/R_0$ as the localization energy of a nucleon, which we call E_L. Thus, we obtain:

$$\frac{U}{\hbar\omega_A} = \frac{\sqrt{2}}{Z\sqrt{N}} \sqrt{\frac{Mc^2}{\hbar\omega_A}} \frac{\hbar\omega_A}{E_L} A^{1/3} \times 2 \times 10^{-3} \tag{37}$$

which can also be written as:

$$\frac{U}{\hbar\omega_A} = \frac{\sqrt{2}}{Z\sqrt{N}} \sqrt{\frac{Mc^2}{E_L}} \sqrt{\frac{\hbar\omega_A}{E_L}} A^{1/3} \times 2 \times 10^{-3} \tag{38}$$

3.4 Example for a 24MeV palladium transition with acoustic phonons

Given:

- $A \approx 106$
- $N \approx 10^{18}$
- $Z \approx 46$
- $Mc^2 \approx 10^{11}$ eV
- $\hbar\omega_A \approx 10^{-8}$ eV

First, let's calculate the localization energy:

$$E_L = \frac{\hbar c}{R_0} = \frac{6.6 \times 10^{-34} \text{ Js} \times 3 \times 10^8 \text{ m/s}}{10^{-15} \text{ m}} \tag{39}$$

$$= 2 \times 10^{-10} \text{ J} = 1.2 \times 10^9 \text{ eV} \approx 10^9 \text{ eV} \tag{40}$$

Now, substituting these numbers gives:

$$\frac{U}{\hbar \omega_A} \approx \frac{\sqrt{2}}{106 \times 10^9} \times \sqrt{\frac{10^{11}}{10^{-8}}} \times \frac{10^{-8}}{10^9} \times 2 \times 10^{-3} \times 106^{1/3} \tag{41}$$

Approximating:

$$\approx \frac{\sqrt{2}\sqrt{10}}{46 \times 10^9} \times 10^9 \times 10^{-17} \times 2 \times 10^{-3} \times 106^{1/3} \tag{42}$$

$$\approx 0.1 \times 2 \times 10^{-20} \times 106^{1/3} \tag{43}$$

$$\approx 10^{-20} \tag{44}$$

3.5 Dicke enhancement

For more detail on the Dicke model, see the notes I made on the subject.

For an ensemble of N nuclei interacting collectively with a phonon field, coupling is enhanced by $\sqrt{N}$, leading to:

$$\frac{U}{\hbar \omega_A} \sim 10^{-12} \tag{45}$$

and so even with Dicke enhancement, electric dipole coupling from phonons remains in the weak coupling regime.

4 Magnetic dipole coupling (M1 transitions)

4.1 B in $\mu \cdot B$

We assume there is an externally driven oscillatory magnetic field B with frequency ω in some volume V. Since field energy density $\sim \frac{1}{\mu_0} B^2$ then:

$$\frac{1}{\mu_0} B^2 V = n \hbar \omega \tag{46}$$

where n is the field occupation number.

We can therefore write:

$$B = \sqrt{\frac{\mu_0 n \hbar \omega}{V}} \tag{47}$$

4.2 μ in $\mu \cdot B$

In order to calculate the dipole moment μ associated with the $\mu \cdot B$ coupling, we'll pursue a similar analysis as we did for E1 transitions, namely:

We can connect two key expressions related to magnetic dipole interactions:

1. Radiation from a magnetic dipole – describes how an oscillating magnetic dipole emits radiation.
2. Radiative decay rates from Weisskopf – provides an estimate for transition rates.

4.2.1 Radiation from a magnetic dipole

The radiative decay rate due to dipole radiation is given by:

$$\gamma_{\text{rad}} = \frac{\mu_0}{12\pi\hbar} \frac{\omega^3}{c^3} \mu^2 \tag{48}$$

Rewriting in terms of the fine-structure constant α:

$$\gamma_{\text{rad}} = \frac{\alpha\omega^3}{3e^2c^4} \mu^2 \tag{49}$$

where the fine-structure constant is defined as:

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \tag{50}$$

4.2.2 Weisskopf estimate for M1 transition

Weisskopf's formula for radiative decay is given by (Eq. A.192 of Dommelen's book):

$$\gamma_{\text{rad}} = 10\frac{2(L+1)}{L[(2L+1)!!]^2}\alpha(kR)^{2L}\omega\left(\frac{3}{l+3}\right)^2\left(\frac{\hbar}{m_pcR}\right)^2 \tag{51}$$

where:

- !! is the double factorial.
- L is the multipolarity ($L = 1$ for dipole, $L = 2$ for quadrupole).
- k is the wavenumber of the emitted radiation.
- m_p is the proton mass
- R is the nuclear radius, given by:

$$R = R_0 A^{1/3} \tag{52}$$

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where R_0 is the radius of a single nucleon and A is the number of nucleons. Note that there exist other forms of Weisskopf's formula that are <u>more convenient for numerical evaluation</u> but they obscure the physical constants.

The last term can be related to the reduced Compton wavelength $\bar{\lambda}_c = \hbar/m_p c$:

$$\gamma_{\text{rad}} = 10 \frac{2(L+1)}{L[(2L+1)!!]^2} \alpha (kR)^{2L} \omega \left(\frac{3}{l+3} \right)^2 \left(\frac{\bar{\lambda}_c}{R} \right)^2 \tag{53}$$

It's instructive to compare the radiation rate for a magnetic dipole vs electric dipole:

$$\gamma_{\text{rad,B}} = \gamma_{\text{rad,E}} \times 10 \left(\frac{\bar{\lambda}_c}{R} \right)^2 \tag{54}$$

Given that $R_0 \sim 10^{-15}$ m and $\bar{\lambda}_c \approx 2 \times 10^{-16}$ m then:

$$\gamma_{\text{rad,B}} = \gamma_{\text{rad,E}} \times 2.5 \left(\frac{1}{A} \right)^{2/3} \tag{55}$$

For $A \approx 100$, $\gamma_{\text{rad,B}} = 0.1\gamma_{\text{rad,E}}$.

For a dipole transition ($L = 1$), Weisskopf's formula simplifies to:

$$\gamma_{\text{rad}} = \frac{20 \times 2}{1 \times [3!!]^2} \alpha \frac{\omega^2}{c^2} R^2 \omega \left(\frac{3}{4} \right)^2 \left(\frac{\bar{\lambda}_c}{R} \right)^2 \tag{56}$$

Rewriting more compactly:

$$\gamma_{\text{rad}} = \frac{20}{(3!!)^2} \alpha \frac{\omega^3}{c^2} \bar{\lambda}_c^2 \tag{57}$$

4.2.3 Equating the two expressions

$$\frac{20}{(3!!)^2} \alpha \frac{\omega^3}{c^2} \bar{\lambda}_c^2 = \frac{\alpha \omega^3}{3e^2 c^4} \mu^2 \tag{58}$$

$$\frac{20}{(720)^2} \left(\frac{\hbar}{m_p c} \right)^2 3e^2 c^2 = \mu^2 \tag{59}$$

$$\frac{60}{(720)^2} \left(\frac{e\hbar}{m_p} \right)^2 = \mu^2 \tag{60}$$

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$$\frac{\sqrt{60}}{720}\frac{e\hbar}{m_p} = \mu \tag{61}$$

And so

$$\mu \approx 0.02\mu_N \tag{62}$$

Where $\mu_N = e\hbar/m_p \approx 5 \times 10^{-27}$ J/T is the nuclear magneton.

4.3 Overall coupling constant

If we again use this simple Hamiltonian in which a single TLS interacts with a single mode but this time it's not a phonon mode but a photon mode:

$$H = \frac{\Delta E}{2}\sigma_z + \hbar\omega_A\left(a^\dagger a + \frac{1}{2}\right) + U\left(b^\dagger + b\right)\sigma_x \tag{63}$$

then a $\mu \cdot B$ coupling constant U can be defined by simply multiplying Eq. 62 by Eq. 47 (without the $\sqrt{n}$) :

$$U \approx 0.02\frac{\mu_N B}{\sqrt{n}} \tag{64}$$

$$U \approx 0.02\mu_N\sqrt{\frac{\mu_0\hbar\omega}{V}} \tag{65}$$

4.4 Example with low frequency solenoid

For 1 MHz field oscillations (~ 4 neV), and using a volume $V = 0.001$ m^{-3}

$$\begin{aligned}
U &\approx 0.02 \times 5 \times 10^{-27} \times \sqrt{\frac{4\pi \times 10^{-7} \times 4 \times 10^{-9} \times 1.6 \times 10^{-19}}{0.001}} \\
\frac{U}{\hbar\omega} &\approx 0.02 \times 5 \times 10^{-27} \times \sqrt{\frac{4\pi \times 10^{-7}}{0.001 \times 4 \times 10^{-9} \times 1.6 \times 10^{-19}}} \\
\frac{U}{\hbar\omega} &\approx 3 \times 10^{-16}
\end{aligned} \tag{66}$$

4.5 Dicke enhancement

For more detail on the Dicke model, see the <u>notes</u> I made on the subject.

For an ensemble of $N \sim 10^{18}$ nuclei interacting collectively with this low frequency magnetic field, coupling is enhanced by $\sqrt{N}$, leading to:

$$\frac{U}{\hbar\omega_A} \sim 3 \times 10^{-7} \tag{67}$$

and so even with Dicke enhancement, magnetic dipole coupling remains in the weak coupling regime.