Dicke model

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1 Introduction

The purpose of this document is to describe the Dicke model.

2 Rabi model

Before we can talk about the Dicke model, we must first familiarise ourselves with a single two level system (TLS) interacting with a single mode (i.e. single frequency/wavelength) of a quantised field. This is often called the Rabi model and its Hamiltonian can be written as:

$$H_{\mathrm{Rabi}} = \frac{\Delta E}{2}\sigma_z + \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) + U\left(a^\dagger + a\right)\left(\sigma_+ + \sigma_-\right) \tag{1}$$

where ΔE is the transition energy between the 2 levels of the TLS, $\hbar\omega$ is the energy of each quantum of the field, and U is the coupling constant between the TLS and the field. The σ operators are the Pauli spin matrices and $a^\dagger$, a are the field creation and annihilation operators respectively.

It's worth noting that we're using the Pauli spin matrices as a mathematical tool to describe two levels. Just keep in mind that we're not really talking about spin angular momentum here.

The TLS has just 2 states denoted by $|\pm\rangle$ but the quantised field has infinitely many states denoted by the number of quanta $|n\rangle$. The combined state of the system can therefore be represented by $|n,\pm\rangle$. Each quantum state can be conceptually thought of as a pendulum whose frequency is related to the energy of the state (see e.g. Briggs et.al).

The strength of the interaction between the TLS and the field is not only determined by the constants U but also how many field quanta n we have. This is because of how the field operators work:

$$a^\dagger|n,\pm\rangle = \sqrt{n+1}|n+1,\pm\rangle \tag{2}$$

$$a|n, \pm\rangle = \sqrt{n}|n-1, \pm\rangle \tag{3}$$

$$a^\dagger a|n, \pm\rangle = n|n, \pm\rangle \tag{4}$$

The more field quanta we have (the stronger the field), the larger the coupling terms will be compared to the TLS term which remains unchanged.

3 Dicke model

The Dicke model describes a system where we have N identical TLS coupled to a single mode (i.e. single frequency/wavelength) of a quantised field. The Dicke Hamiltonians is a simple extension of the Rabi Hamiltonian in Eq. 1 in the sense that we just add N copies of the TLS terms as seen below:

$$H_{\text{Dicke}} = \frac{\Delta E}{2}\sum_{i=1}^{N}\sigma_z^{(i)} + \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) + U\sum_{i=1}^{N}(a^\dagger + a)(\sigma_+^{(i)} + \sigma_-^{(i)}) \tag{5}$$

The states of this system are described by $|n, \pm, \pm, \pm, \pm, ...\rangle$.

It's worth emphasising that there is no spatial dependence of the field in Eq. 5. One way to understand this physically is that all the TLS are very close together in the sense that they are located in a region of space that is much smaller than the wavelength of the mode. In that situation, all the TLS will experience the same strength of field at any given moment - in other words the field appears constant in space. This is the how Dicke originally presented his ideas in his 1954 paper. It's also possible to use this Hamiltonian to describe many TLS arranged in a very special way so that they are placed at integer multiples of the mode wavelength.

Much like how the strength of the field changes the size of the coupling term in the Rabi model (and also in the Dicke model), the number of TLS also has an effect - this is the origin of what's called Dicke superradiance. It is however difficult to see this right now because Eq. 5 is not written in a convenient form. We're going to need to rewrite it by appealing to the physics of spin.

3.1 Pseudo-spin

We noted earlier that the use of the Pauli spin matrices is just a mathematical tool to describe two levels. Although we're not describing spin here, we are working with what is often called "pseudo spin". The rules of angular momentum work just as well for pseudo angular momentum. In particular, the rules of angular momentum addition and conservation.

This means that we can treat all the TLS together as if they are a single object with many levels whose energies are determined by the addition rules of pseudo angular momentum. The Hamilton then looks like:

$$H = \Delta E J_z + \hbar\omega \left(a^\dagger a + \frac{1}{2} \right) + 2U \left(a^\dagger + a \right) (J_+ + J_-) \tag{6}$$

where the total pseudo total angular momentum operators (J) for N TLS are:

$$J_+ + J_- = J_x = \frac{1}{2} \sum_{i=1}^{N} \sigma_{ix} \quad J_z = \frac{1}{2} \sum_{i=1}^{N} \sigma_{iz} \tag{7}$$

and noting that i in σ_i means that this operator only acts on TLS number i .

When written in this way, each state can now be described in terms of 3 numbers $|n, j, m\rangle$ where j describes the total pseudo angular momentum number (which is conserved) and m describes the z component of the total pseudo angular momentum (which can change). This notation allows us to conveniently describe situations where excitations are "delocalised" among the TLS. By far the most significant kind of delocalised states are called "Dicke states" which have the largest $j = j_{\max} = N/2$. Dicke states are symmetric in the sense that if you swap any of the TLS around, the state remains unchanged. For example, consider a single excitation in 4 TLS - the Dicke state looks like:

$$\Psi_0 = \frac{1}{\sqrt{4}} \left(|0, +, -, -, -\rangle + |0, -, +, -, -\rangle + |0, -, -, +, -\rangle + |0, -, -, -, +\rangle \right) \tag{8}$$

Notice that if you swap any two TLS, the state looks the same.

The above state can instead be described by $j_{\max} = 4/2 = 2$ and $m = 1 \times 1/2 + 3 \times -1/2 = -1$

$$\Psi_0 = |0, 2, -1 > \tag{9}$$

In this way of describing the system, the ladder operators J_+ and J_- create and destroy excitations of the TLS. This causes a raising and lowering of the m value like this:

$$J_+ |n, j, m\rangle = \sqrt{j(j+1) - m(m+1)} |n, j, m+1\rangle \tag{10}$$

$$J_- |n, j, m\rangle = \sqrt{j(j+1) - m(m-1)} |n, j, m-1\rangle \tag{11}$$

These ladder operators are conceptually similar to the creation and annihilation operators of the field (see Eqs. 2 and 3). The details are however more complicated due to the addition rules of angular momentum. Despite the complexity, we know that the maximum total angular momentum $j_{\max} = N/2$ (from N TLS with pseudo angular momentum 1/2). We can therefore see that the number of TLS is going to have an effect on the coupling between TLS and field. Exactly what effect depends on the details. Let's explore some of those details next.

3.2 Superradiance

Dicke states with $j = j_{\max}$ are significant because of the acceleration properties that they offer; something people often describe as superradiance.

For the case with **all TLS excited** , $m = N/2$:

$$J_{-}|n, N/2, N/2\rangle = \sqrt{N}|n, N/2, N/2-1\rangle \tag{12}$$

For the first de-excitation, the coupling terms gets enhanced by $\sqrt{N}$. This might not seem surprising at first glance because we have N TLS excited and so we should expect the rate of emission (which go as the square of the coupling) to be enhanced by N.

For the case of **a single excitation** , $m = -N/2 + 1$:

$$J_{-}|n, N/2, -N/2+1\rangle = \sqrt{N}|n, N/2, -N/2\rangle \tag{13}$$

For this singe de-excitation the coupling term also gets enhanced by $\sqrt{N}$. This is more surprising because the rate of emission (which go as the square of the coupling) to be enhanced by N even though there is only a single excitation.

For the case of **50% excitation**, $m = 0$:

$$J_{-}|n, N/2, 0\rangle = \sqrt{N^2+N}|n, N/2, -1\rangle \tag{14}$$

For the first de-excitation, the coupling terms gets enhanced by $\sim \sqrt{N^2}$ for large N. In other words, the rate of emission (which go as the square of the coupling) to be enhanced by N^2 - this is where the super in superradiance comes from.