Coupling constants in nuclear physics

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1 Introduction

Understanding the interaction between nucleons and external fields is essential in nuclear physics. We'll explore two coupling mechanisms that arise in quantum nuclear interactions:

1. **Relativistic phonon nuclear coupling ($a \cdot cp$)** – where phonons couple to nucleons through momentum exchange (see Hagelstein 2023 for more detail).
2. **Electric dipole coupling ($d \cdot E$)** – where an electric field couples to nucleons through electric dipole moments.
3. **Magnetic dipole coupling ($\mu \cdot B$) -** where a magnetic field couples to nucleonics through magnetic dipole moments.

This document explores these couplings, derives their respective coupling constants, and compares their strength.

2 Relativistic phonon nuclear coupling ($a \cdot cp$)

2.1 p in $a \cdot cp$

For a nucleus of mass M moving within a solid, the momentum-based coupling energy scales as:

$$E \sim pc \tag{1}$$

From kinetic energy considerations:

$$\frac{p^2}{2M} = E \quad \Rightarrow \quad p = \sqrt{2ME} \tag{2}$$

If E represents phonon oscillations:

$$E = n\hbar\omega_A \tag{3}$$

where n is the phonon occupation number. Distributing this energy over N atoms:

$$p = \sqrt{\frac{2M}{N}}\sqrt{\hbar\omega_A}\sqrt{n} \tag{4}$$

Note that later we'll drop the $\sqrt{n}$ because it should be picked up in the Hamiltonian operator $(b^\dagger + b)$.

2.2 a in $a \cdot cp$

From Eq. 470 in section 6.11 of Models for nuclear fusion in the solid state, the a component of $a \cdot cp$ can be approximated as:

$$a \sim \frac{1}{2}\frac{\Delta E}{Mc^2}\frac{\pi}{mc} \tag{5}$$

where:

- ΔE is the nuclear transition energy,
- m is the mass of a single nucleon within the nucleus,
- π represents the relative momentum of that nucleon.

Since angular momentum is approximately $\hbar$, and using the Fermi scale $l_F = 10^{-15}$ m, we estimate:

$$\pi \sim \frac{\hbar}{l_F} \tag{6}$$

Substituting this into the expression for a gives:

$$a \sim \frac{1}{2}\frac{\Delta E}{Mc^2}\frac{\bar{\lambda}_c}{l_F} \tag{7}$$

where $\bar{\lambda}_c = \hbar/mc$ is the reduced Compton wavelength, approximately:

$$\bar{\lambda}_c \approx 2 \times 10^{-16} \text{ m.} \tag{8}$$

To account for hindrance effects in nuclear transitions, we introduce a suppression factor O, where $O \sim 0.01$. This modifies the expression to:

$$a \sim \frac{1}{2}\frac{\Delta E}{Mc^2}\frac{\bar{\lambda}_c}{l_F}O \tag{9}$$

which simplifies to:

$$a \sim \frac{\Delta E}{Mc^2} \times 10^{-3} \tag{10}$$

The final value of a depends on both the nuclear transition type and the specific nucleus under consideration.

2.3 Overall coupling constant

Let's consider a single TLS interacts with a single phonon mode. The Hamiltonian can be written as:

$$H = \frac{\Delta E}{2}\sigma_z + \hbar\omega_A \left(b^\dagger b + \frac{1}{2}\right) + U\left(b^\dagger + b\right)\sigma_x \tag{11}$$

where ΔE is the transition energy between the 2 levels of the TLS, $\hbar\omega_A$ is the energy of each quantum of the field, and U is the coupling constant between the TLS and the field. The σ operators are the Pauli matrices and $b^\dagger$, b are the field creation and annihilation operators respectively. Note that usually a is used for the field operators, but in these notes we use b to avoid confusion with $a \cdot cp$.

The $a \cdot cp$ coupling constant U can be defined by combining Eq. 4 (without the $\sqrt{n}$) with Eq. 10:

$$U = c\sqrt{\frac{2M}{N}}\sqrt{\hbar\omega_A} \times \frac{\Delta E}{Mc^2} \times 10^{-3} \tag{12}$$

Normalising U to $\hbar\omega_A$ gives:

$$\frac{U}{\hbar\omega_A} = \sqrt{\frac{2Mc^2}{N}}\frac{1}{\sqrt{\hbar\omega_A}} \times \frac{\Delta E}{Mc^2} \times 10^{-3} \tag{13}$$

Rearranging and simplifying leads to:

$$\frac{U}{\hbar\omega_A} = \sqrt{\frac{2}{N}}\sqrt{\frac{\Delta E}{Mc^2}}\sqrt{\frac{\Delta E}{\hbar\omega_A}} \times 10^{-3} \tag{14}$$

which can also be written as:

$$\frac{U}{\hbar\omega_A} = \sqrt{\frac{2}{N}}\sqrt{\frac{\hbar\omega_A}{Mc^2}}\frac{\Delta E}{\hbar\omega_A} \times 10^{-3} \tag{15}$$

2.4 Example for palladium with acoustic phonons

- $\Delta E \approx 24 \times 10^6$ eV
- $Mc^2 \approx 10^{11}$ eV
- $\hbar\omega_A \approx 10^{-8}$ eV
- $N \approx 10^{18}$

$$\frac{U}{\hbar\omega_A} \approx \sqrt{\frac{2}{10^{18}}}\sqrt{\frac{24 \times 10^6}{10^{11}}}\sqrt{\frac{24 \times 10^6}{10^{-8}}} \times 10^{-3} \tag{16}$$

$$\approx 10^{-6} \tag{17}$$

2.5 Dicke enhancement

For an ensemble of N nuclei interacting collectively with a phonon field, coupling is enhanced by $\sqrt{N}$, leading to:

$$\frac{U}{\hbar\omega_A} \sim 10^3 \tag{18}$$

Based on this, we can be far into the "deep strong coupling" regime where $U/\hbar\omega_A > 1$.

3 Electric dipole coupling (E1 transitions)

3.1 E in $d \cdot E$

Electric field strength due to phonons follows from force relations:

$$F = \frac{dp}{dt} = ZeE \tag{19}$$

For oscillatory motion:

$$\frac{dp}{dt} \sim \omega_A p \Rightarrow E = \frac{\omega_A p}{Ze} \tag{20}$$

Substituting our previous result for p:

$$E = \frac{\omega_A\sqrt{2M\hbar\omega_A n}}{Ze\sqrt{N}} \tag{21}$$

3.2 d in $d \cdot E$

We can connect two key expressions related to electric dipole interactions:

1. Radiation from an electric dipole – describes how an oscillating electric dipole emits radiation.

2. Radiative decay rates from Weisskopf – provides an estimate for transition rates.

3.2.1 Radiation from an electric dipole

The radiative decay rate due to dipole radiation is given by:

$$\gamma_{\mathrm{rad}} = \frac{4}{3}\frac{1}{4\pi\epsilon_0\hbar}\frac{\omega^3}{c^3}d^2 \tag{22}$$

Rewriting in terms of the fine-structure constant α:

$$\gamma_{\mathrm{rad}} = \frac{4}{3}\frac{1}{e^2}\alpha\frac{\omega^3}{c^2}d^2 \tag{23}$$

where the fine-structure constant is defined as:

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \tag{24}$$

3.2.2 Weisskopf estimate for E1 transition

Weisskopf's formula for radiative decay is given by (Eq. A.190 of Dommelen's book):

$$\gamma_{\mathrm{rad}} = \frac{2(L+1)}{L\left[(2L+1)!!\right]^2}\alpha(kR)^{2L}\omega\left(\frac{3}{L+3}\right)^2 \tag{25}$$

where:

- L is the multipolarity ($L = 1$ for dipole, $L = 2$ for quadrupole).
- k is the wavenumber of the emitted radiation.
- R is the nuclear radius, given by:

$$R = R_0 A^{1/3} \tag{26}$$

where R_0 is the radius of a single nucleon and A is the number of nucleons. Note that there exist other forms of Weisskopf's formula that are more convenient for numerical evaluation but they obscure the physical constants.

For a dipole transition ($L = 1$), this simplifies to:

$$\gamma_{\text{rad}} = \frac{2 \times 2}{1 \times [3!!]^2} \alpha \frac{\omega^2}{c^2} R_0^2 A^{2/3} \omega \left(\frac{3}{4}\right)^2 \tag{27}$$

Rewriting more compactly:

$$\gamma_{\text{rad}} = \frac{9}{4 \times (3!!)^2} \alpha \frac{\omega^3}{c^2} R_0^2 A^{2/3} \tag{28}$$

3.2.3 Equating the two expressions

From the previous derivations, we equate:

$$\frac{9}{4 \times (3!!)^2} \alpha \frac{\omega^3}{c^2} R_0^2 A^{2/3} = \frac{4}{3} \frac{1}{e^2} \alpha \frac{\omega^3}{c^2} d^2 \tag{29}$$

Rearranging:

$$\frac{27}{16 \times (3!!)^2} A^{2/3} e^2 R_0^2 = d^2 \tag{30}$$

Taking the square root:

$$d = \frac{\sqrt{27}}{4 \times (3!!)} A^{1/3} e R_0 \tag{31}$$

which simplifies to:

$$d = \frac{\sqrt{27}}{2880} A^{1/3} e R_0 \tag{32}$$

Approximating numerically:

$$d \approx 2 \times 10^{-3} A^{1/3} e R_0 \tag{33}$$

3.3 Overall coupling constant

If we again use this simple Hamiltonian in which a single TLS interacts with a single phonon mode:

$$H = \frac{\Delta E}{2} \sigma_z + \hbar \omega_A \left(a^\dagger a + \frac{1}{2}\right) + U \left(b^\dagger + b\right) \sigma_x \tag{34}$$

then a $d \cdot E$ coupling constant U can be defined by combining Eq. 21 (without the $\sqrt{n}$) with Eq. 33:

$$\frac{U}{\hbar\omega_A} = \frac{1}{\hbar\omega_A}\frac{\omega_A\sqrt{2M\hbar\omega_A}}{Ze\sqrt{N}} \times 2 \times 10^{-3} A^{1/3} e R_0 \tag{35}$$

Rearranging,

$$\frac{U}{\hbar\omega_A} = \frac{\sqrt{2}}{Z\sqrt{N}}\sqrt{\frac{Mc^2}{\hbar\omega_A}\frac{\hbar\omega_A R_0}{\hbar c}} A^{1/3} \times 2 \times 10^{-3} \tag{36}$$

We recognize $\hbar c/R_0$ as the localization energy of a nucleon, which we call E_L. Thus, we obtain:

$$\frac{U}{\hbar\omega_A} = \frac{\sqrt{2}}{Z\sqrt{N}}\sqrt{\frac{Mc^2}{\hbar\omega_A}\frac{\hbar\omega_A}{E_L}} A^{1/3} \times 2 \times 10^{-3} \tag{37}$$

which can also be written as:

$$\frac{U}{\hbar\omega_A} = \frac{\sqrt{2}}{Z\sqrt{N}}\sqrt{\frac{Mc^2}{E_L}}\sqrt{\frac{\hbar\omega_A}{E_L}} A^{1/3} \times 2 \times 10^{-3} \tag{38}$$

Note how the expressions for $a \cdot cp$ and $d \cdot E$ have an interesting reciprocal relationship if we see that E_L plays the role of ΔE.

3.4 Example of Pd with Acoustic Phonons

Given:

- $A \approx 106$
- $N \approx 10^{18}$
- $Z \approx 106$
- $Mc^2 \approx 10^{11}$ eV
- $\hbar\omega_A \approx 10^{-8}$ eV

First, let's calculate the localization energy:

$$E_L = \frac{\hbar c}{R_0} = \frac{6.6 \times 10^{-34} \times 3 \times 10^8}{10^{-15}} \tag{39}$$

$$= 2 \times 10^{-10} \text{ J} = 1.2 \times 10^9 \text{ eV} \approx 10^9 \text{ eV} \tag{40}$$

Now, substituting these numbers gives:

$$\frac{U}{\hbar\omega_A} \approx \frac{\sqrt{2}}{106 \times 10^9} \times \sqrt{\frac{10^{11}}{10^{-8}}} \times \frac{10^{-8}}{10^9} \times 2 \times 10^{-3} \times 106^{1/3} \tag{41}$$

Approximating:

$$\approx \frac{\sqrt{2}\sqrt{10}}{106 \times 10^9} \times 10^9 \times 10^{-17} \times 2 \times 10^{-3} \times 106^{1/3} \tag{42}$$

$$= \sqrt{20} \times 2 \times 10^{-20} \times 106^{-2/3} \tag{43}$$

$$\approx 4 \times 10^{-21} \tag{44}$$

3.5 Dicke enhancement

For an ensemble of N nuclei interacting collectively with a phonon field, coupling is enhanced by $\sqrt{N}$, leading to:

$$\frac{U}{\hbar\omega_A} \sim 4 \times 10^{-12} \tag{45}$$

and so even with Dicke enhancement, dipole coupling remains in the weak coupling regime.

4 Magnetic dipole coupling (M1 transitions)

4.1 B in $\mu \cdot B$

We assume there is an externally driven oscillatory magnetic field B with frequency ω in some volume V. Since field energy density $\sim \mu_0 B^2$ then:

$$\frac{1}{\mu_0} B^2 V = n\hbar\omega \tag{46}$$

where n is the field occupation number.

We can therefore write:

$$B = \sqrt{\frac{\mu_0 n\hbar\omega}{V}} \tag{47}$$

4.2 μ in $\mu \cdot B$

In order to calculate the dipole moment μ associated with the $\mu \cdot B$ coupling, we'll pursue a similar analysis as we did for E1 transitions, namely:

We can connect two key expressions related to magnetic dipole interactions:

1. Radiation from a magnetic dipole – describes how an oscillating magnetic dipole emits radiation.
2. Radiative decay rates from Weisskopf – provides an estimate for transition rates.

4.2.1 Radiation from a magnetic dipole

The radiative decay rate due to dipole radiation is given by:

$$\gamma_{\text{rad}} = \frac{\mu_0}{12\pi\hbar}\frac{\omega^3}{c^3}\mu^2 \tag{48}$$

Rewriting in terms of the fine-structure constant α:

$$\gamma_{\text{rad}} = \frac{\alpha\omega^3}{3e^2c^4}\mu^2 \tag{49}$$

where the fine-structure constant is defined as:

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \tag{50}$$

4.2.2 Weisskopf estimate for M1 transition

Weisskopf's formula for radiative decay is given by (Eq. A.192 of Dommelen's book):

$$\gamma_{\text{rad}} = 10\frac{2(L+1)}{L[(2L+1)!!]^2}\alpha(kR)^{2L}\omega\left(\frac{3}{l+3}\right)^2\left(\frac{\hbar}{m_pcR}\right)^2 \tag{51}$$

where:

- L is the multipolarity ($L = 1$ for dipole, $L = 2$ for quadrupole).
- k is the wavenumber of the emitted radiation.
- m_p is the proton mass
- R is the nuclear radius, given by:

$$R = R_0A^{1/3} \tag{52}$$

where R_0 is the radius of a single nucleon and A is the number of nucleons. Note that there exist other forms of Weisskopf's formula that are more convenient for numerical evaluation but they obscure the physical constants.

The last term can be related to the reduced Compton wavelength $\bar{\lambda}_c = \hbar/mc$:

$$\gamma_{\mathrm{rad}} = 10\frac{2(L+1)}{L[(2L+1)!!]^2}\alpha(kR)^{2L}\omega\left(\frac{3}{l+3}\right)^2\left(\frac{\bar{\lambda}_c}{R}\right)^2 \tag{53}$$

It's instructive to compare the radiation rate for a magnetic dipole vs electric dipole:

$$\gamma_{\mathrm{rad,B}} = \gamma_{\mathrm{rad,E}} \times 10\left(\frac{\bar{\lambda}_c}{R}\right)^2 \tag{54}$$

Given that $R_0 \sim 10^{-15}$ m and $\bar{\lambda}_c \approx 2 \times 10^{-16}$ m then:

$$\gamma_{\mathrm{rad,B}} = \gamma_{\mathrm{rad,E}} \times 2.5\left(\frac{1}{A}\right)^{2/3} \tag{55}$$

For $A \approx 100$, $\gamma_{\mathrm{rad,B}} = 0.1\gamma_{\mathrm{rad,E}}$.

For a dipole transition ($L = 1$), Weisskopf's formula simplifies to:

$$\gamma_{\mathrm{rad}} = \frac{20 \times 2}{1 \times [3!!]^2}\alpha\frac{\omega^2}{c^2}R^2\omega\left(\frac{3}{4}\right)^2\left(\frac{\bar{\lambda}_c}{R}\right)^2 \tag{56}$$

Rewriting more compactly:

$$\gamma_{\mathrm{rad}} = \frac{20}{(3!!)^2}\alpha\frac{\omega^3}{c^2}\bar{\lambda}_c^2 \tag{57}$$

4.2.3 Equating the two expressions

$$\frac{20}{(3!!)^2}\alpha\frac{\omega^3}{c^2}\bar{\lambda}_c^2 = \frac{\alpha\omega^3}{3e^2c^4}\mu^2 \tag{58}$$

$$\frac{20}{(720)^2}\left(\frac{\hbar}{m_pc}\right)^2 3e^2c^2 = \mu^2 \tag{59}$$

$$\frac{60}{(720)^2}\left(\frac{e\hbar}{m_p}\right)^2 = \mu^2 \tag{60}$$

$$\frac{\sqrt{60}}{720}\frac{e\hbar}{m_p} = \mu \tag{61}$$

And so

$$\mu \approx 0.02\mu_N \tag{62}$$

Where $\mu_N = e\hbar/m_p \approx 5 \times 10^{-27}$ J/T is the nuclear magneton.

4.3 Overall coupling constant

If we again use this simple Hamiltonian in which a single TLS interacts with a single mode but this time it's not a phonon mode but a magnon mode:

$$H = \frac{\Delta E}{2}\sigma_z + \hbar\omega_A \left(a^\dagger a + \frac{1}{2}\right) + U\left(b^\dagger + b\right)\sigma_x \tag{63}$$

then a $\mu \cdot B$ coupling constant U can be defined by simply multiplying Eq. 62 by Eq. 47 (without the $\sqrt{n}$) :

$$U \approx 0.02\frac{\mu_N B}{\sqrt{n}} \tag{64}$$

$$U \approx 0.02\mu_N\sqrt{\frac{\mu_0\hbar\omega}{V}} \tag{65}$$

4.4 Example with low frequency solenoid

For 1 MHz field oscillations ($\sim$ 4 neV), and using a volume $V = 0.001$ m^{-3}

$$\begin{aligned} U &\approx 0.02 \times 5 \times 10^{-27} \times \sqrt{\frac{4\pi \times 10^{-7} \times 4 \times 10^{-9} \times 1.6 \times 10^{-19}}{0.001}} \\ \frac{U}{\hbar\omega} &\approx 0.02 \times 5 \times 10^{-27} \times \sqrt{\frac{4\pi \times 10^{-7}}{0.001 \times 4 \times 10^{-9} \times 1.6 \times 10^{-19}}} \\ \frac{U}{\hbar\omega} &\approx 2.8 \times 10^{-16} \end{aligned} \tag{66}$$