Coupling constants in nuclear physics

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1 Introduction

Understanding the interaction between nucleons and external fields is essential in nuclear physics. We'll explore two coupling mechanisms that arise in quantum nuclear interactions:

1. **Relativistic phonon nuclear coupling** $(a \cdot cp)$ – where phonons couple to nucleons through momentum exchange.
2. **Electric dipole coupling** $(d \cdot E)$ – where an electric field couples to nucleons through dipole moments.

This document explores both couplings, derives their respective coupling constants, and compares their strength.

2 Relativistic phonon nuclear coupling $(a \cdot cp)$

2.1 p in $a \cdot cp$

For a nucleus of mass M moving within a solid, the momentum-based coupling energy scales as:

$$E \sim pc \tag{1}$$

From kinetic energy considerations:

$$\frac{p^2}{2M} = E \quad \Rightarrow \quad p = \sqrt{2ME} \tag{2}$$

If E represents phonon oscillations:

$$E = n\hbar\omega_A \tag{3}$$

where n is the phonon occupation number. Distributing this energy over N atoms:

$$p = \sqrt{\frac{2M}{N}} \sqrt{\hbar\omega_A} \sqrt{n} \tag{4}$$

Note that later we'll drop the $\sqrt{n}$ because it should be picked up in the Hamiltonian operator $(b^\dagger + b)$.

2.2 a in $a \cdot cp$

From Eq. 470 in section 6.11 of Models for nuclear fusion in the solid state, the a component of $a \cdot cp$ can be approximated as:

$$a \sim \frac{1}{2} \frac{\Delta E}{Mc^2} \frac{\pi}{mc} \tag{5}$$

where: - ΔE is the nuclear transition energy, - m is the mass of a single nucleon within the nucleus, - π represents the relative momentum of that nucleon.

Since angular momentum is approximately $\hbar$, and using the Fermi scale $l_F = 10^{-15}$ m, we estimate:

$$\pi \sim \frac{\hbar}{l_F} \tag{6}$$

Substituting this into the expression for a gives:

$$a \sim \frac{1}{2} \frac{\Delta E}{Mc^2} \frac{\bar{\lambda}_c}{l_F} \tag{7}$$

where $\bar{\lambda}_c = \hbar/mc$ is the reduced Compton wavelength, approximately:

$$\bar{\lambda}_c \approx 2 \times 10^{-16} \text{ m}. \tag{8}$$

To account for hindrance effects in nuclear transitions, we introduce a suppression factor O, where $O \sim 0.01$. This modifies the expression to:

$$a \sim \frac{1}{2} \frac{\Delta E}{Mc^2} \frac{\bar{\lambda}_c}{l_F} O \tag{9}$$

which simplifies to:

$$a \sim \frac{\Delta E}{Mc^2} \times 10^{-3} \tag{10}$$

The final value of a depends on both the nuclear transition type and the specific nucleus under consideration.

2.3 Overall coupling constant

If we use this simple Hamiltonian in which a single TLS interacts with a single phonon mode:

$$H = \frac{\Delta E}{2}\sigma_z + \hbar\omega_A\left(b^\dagger b + \frac{1}{2}\right) + U\left(b^\dagger + b\right)\sigma_x \tag{11}$$

then an $a \cdot cp$ coupling constant U can be defined by combining Eq. 4 (without the $\sqrt{n}$) with Eq. 10:

$$U = c\sqrt{\frac{2M}{N}}\sqrt{\hbar\omega_A} \times \frac{\Delta E}{Mc^2} \times 10^{-3} \tag{12}$$

Normalising U to $\hbar\omega_A$ gives:

$$\frac{U}{\hbar\omega_A} = \sqrt{\frac{2Mc^2}{N}}\frac{1}{\sqrt{\hbar\omega_A}} \times \frac{\Delta E}{Mc^2} \times 10^{-3} \tag{13}$$

Rearranging and simplifying leads to:

$$\frac{U}{\hbar\omega_A} = \sqrt{\frac{2}{N}}\sqrt{\frac{\Delta E}{Mc^2}}\sqrt{\frac{\Delta E}{\hbar\omega_A}} \times 10^{-3} \tag{14}$$

which can also be written as:

$$\frac{U}{\hbar\omega_A} = \sqrt{\frac{2}{N}}\sqrt{\frac{\hbar\omega_A}{Mc^2}}\frac{\Delta E}{\hbar\omega_A} \times 10^{-3} \tag{15}$$

2.4 Example for palladium with acoustic phonons

- $\Delta E \approx 24 \times 10^6$ eV
- $Mc^2 \approx 10^{11}$ eV
- $\hbar\omega_A \approx 10^{-8}$ eV
- $N \approx 10^{18}$

$$\frac{U}{\hbar\omega_A} \approx \sqrt{\frac{2}{10^{18}}}\sqrt{\frac{24 \times 10^6}{10^{11}}}\sqrt{\frac{24 \times 10^6}{10^{-8}}} \times 10^{-3} \tag{16}$$

$$\approx 10^{-6}). \tag{17}$$

2.5 Dicke enhancement

For an ensemble of N nuclei interacting collectively with a phonon field, coupling is enhanced by $\sqrt{N}$, leading to:

$$\frac{U}{\hbar\omega_A} \sim 10^3 \tag{18}$$

Based on this, we can be far into the "deep strong coupling" regime where $U/\hbar\omega_A > 1$.

3 Electric dipole coupling (E1 transitions)

3.1 E in $d \cdot E$

Electric field strength due to phonons follows from force relations:

$$F = \frac{dp}{dt} = ZeE \tag{19}$$

For oscillatory motion:

$$\frac{dp}{dt} \sim \omega_A p \Rightarrow E = \frac{\omega_A p}{Ze} \tag{20}$$

Substituting our previous result for p:

$$E = \frac{\omega_A \sqrt{2M\hbar\omega_A n}}{Ze\sqrt{N}} \tag{21}$$

3.2 d in $d \cdot E$

We can connect two key expressions related to dipole interactions:

1. Radiation from a dipole – describes how an oscillating dipole emits radiation.

2. Radiative decay rates from Weisskopf – provides an estimate for transition rates.

3.2.1 Radiation from a dipole

The radiative decay rate due to dipole radiation is given by:

$$\gamma_{\text{rad}} = \frac{4}{3}\frac{1}{4\pi\epsilon_0\hbar}\frac{\omega^3}{c^3}d^2 \tag{22}$$

Rewriting in terms of the fine-structure constant α:

$$\gamma_{\text{rad}} = \frac{4}{3}\frac{1}{e^2}\alpha\frac{\omega^3}{c^2}d^2 \tag{23}$$

where the fine-structure constant is defined as:

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \tag{24}$$

3.2.2 Weisskopf estimate for E1 transition

Weisskopf's formula for radiative decay is given by (Eq. 16.35 of Bielajew's book or Eq. A.190 of Dommelen's book):

$$\gamma_{\text{rad}} = \frac{8\pi(L+1)}{L\left[(2L+1)!!\right]^2}\alpha(kR)^{2L}\omega\left(\frac{3}{L+3}\right)^2 \tag{25}$$

where: - L is the multipolarity ($L=1$ for dipole, $L=2$ for quadrupole). - k is the wavenumber of the emitted radiation. - R is the nuclear radius, given by:

$$R = R_0 A^{1/3} \tag{26}$$

where R_0 is the radius of a single nucleon and A is the number of nucleons. Note that there exist other forms of Weisskopf's formula that are more convenient for numerical evaluation but they obscure the physical constants.

For a dipole transition ($L=1$), this simplifies to:

$$\gamma_{\text{rad}} = \frac{8\pi\times 2}{1\times[3!!]^2}\alpha\frac{\omega^2}{c^2}R_0^2A^{2/3}\omega\left(\frac{3}{4}\right)^2 \tag{27}$$

Rewriting more compactly:

$$\gamma_{\text{rad}} = \frac{9\pi}{(3!!)^2}\alpha\frac{\omega^3}{c^2}R_0^2A^{2/3} \tag{28}$$

3.2.3 Equating the two expressions

From the previous derivations, we equate:

$$\frac{9\pi}{(3!!)^2}\alpha\frac{\omega^3}{c^2}R_0^2A^{2/3} = \frac{4}{3}\frac{1}{e^2}\alpha\frac{\omega^3}{c^2}d^2 \tag{29}$$

Rearranging:

$$\frac{27\pi}{4 \times (3!!)^2} A^{2/3} e^2 R_0^2 = d^2 \tag{30}$$

Taking the square root:

$$d = \frac{\sqrt{27\pi}}{2 \times (3!!)} A^{1/3} e R_0 \tag{31}$$

which simplifies to:

$$d = \frac{\sqrt{27\pi}}{1440} A^{1/3} e R_0 \tag{32}$$

Approximating numerically:

$$d \approx 6 \times 10^{-3} A^{1/3} e R_0 \tag{33}$$

3.3 Overall coupling constant

If we again use this simple Hamiltonian in which a single TLS interacts with a single phonon mode:

$$H = \frac{\Delta E}{2} \sigma_z + \hbar\omega_A \left(a^\dagger a + \frac{1}{2} \right) + U \left(b^\dagger + b \right) \sigma_x \tag{34}$$

then a $d \cdot E$ coupling constant U can be defined by combining Eq. 21 (without the $\sqrt{n}$) with Eq. 33:

$$\frac{U}{\hbar\omega_A} = \frac{1}{\hbar\omega_A} \frac{\omega_A \sqrt{2M\hbar\omega_A}}{Ze\sqrt{N}} \times 6 \times 10^{-3} A^{1/3} e R_0 \tag{35}$$

Rearranging,

$$\frac{U}{\hbar\omega_A} = \frac{\sqrt{2}}{Z\sqrt{N}} \sqrt{\frac{Mc^2}{\hbar\omega_A}} \frac{\hbar\omega_A R_0}{\hbar c} A^{1/3} \times 6 \times 10^{-3} \tag{36}$$

We recognize $\hbar c / R_0$ as the localization energy of a nucleon, which we call E_L. Thus, we obtain:

$$\frac{U}{\hbar\omega_A} = \frac{2\pi\sqrt{2}}{Z\sqrt{N}} \sqrt{\frac{Mc^2}{\hbar\omega_A}} \frac{\hbar\omega_A}{E_L} A^{1/3} \times 6 \times 10^{-3} \tag{37}$$

which can also be written as:

$$\frac{U}{\hbar\omega_A} = \frac{2\pi\sqrt{2}}{Z\sqrt{N}}\sqrt{\frac{Mc^2}{E_L}}\sqrt{\frac{\hbar\omega_A}{E_L}}A^{1/3} \times 6 \times 10^{-3} \tag{38}$$

Note how the expressions for $a \cdot cp$ and $d \cdot E$ have an interesting reciprocal relationship if we see that E_L plays the role of ΔE.

3.4 Example of Pd with Acoustic Phonons

Given: - $A \approx 106$ - $N \approx 10^{18}$ - $Z \approx 106$ - $Mc^2 \approx 10^{11}$ eV - $\hbar\omega_A \approx 10^{-8}$ eV

First, let's calculate the localization energy:

$$E_L = \frac{\hbar c}{R_0} = \frac{6.6 \times 10^{-34} \times 3 \times 10^8}{10^{-15}} \tag{39}$$

$$= 2 \times 10^{-10} \text{ J} = 1.2 \times 10^9 \text{ eV} \approx 10^9 \text{ eV} \tag{40}$$

Now, substituting these numbers gives:

$$\frac{U}{\hbar\omega_A} \approx \frac{2\pi\sqrt{2}}{106 \times 10^9} \times \sqrt{\frac{10^{11}}{10^{-8}}} \times \frac{10^{-8}}{10^9} \times 6 \times 10^{-3} \times 106^{1/3} \tag{41}$$

Approximating:

$$\approx \frac{2\pi\sqrt{2}\sqrt{10}}{106 \times 10^9} \times 10^9 \times 10^{-17} \times 6 \times 10^{-3} \times 106^{1/3} \tag{42}$$

$$\approx 8\pi \times 6 \times 10^{-20} \times 106^{-2/3} \tag{43}$$

$$\approx 7 \times 10^{-20} \tag{44}$$

3.5 Dicke enhancement

For an ensemble of N nuclei interacting collectively with a phonon field, coupling is enhanced by $\sqrt{N}$, leading to:

$$\frac{U}{\hbar\omega_A} \sim 7 \times 10^{-11} \tag{45}$$

and so even with Dicke enhancement, dipole coupling remains in the weak coupling regime.