Strong-field quantum phonodynamics

Matt Lilley

April 10, 2025

1 Introduction

I had a chat with my friend Chris Ridgers about his work on laser plasma interactions in a regime described as "Strong-field QED". The laser field is so intense that perturbation theory no longer works. Chris remarked on how part of the work involved using the "low" energy laser photons to cause an electron to emit a photon with many orders or magnitude more energy. Superficially this sounded similar to the phonon-nuclear system in which we use a low energy phonon to mediate a nuclear energy transition. We also have a highly excited phonon field that makes the system non-perturbative. Chris described various mathematical tools used in his work and I wondered whether we might be able to use some of them in our work.

In this document, I'll try and describe a regime that we might call "Strong-field quantum phonodynamics" or "Strong-field QPD" and compare/contrast it with what people sometimes refer to as "Deep strong coupling".

Unlike "Strong-field QED" that has as well defined parameter range given in terms of the "normalised vector potential" $a_0 \equiv eE/m_e\omega c \gg 1$, for "Strong field QPD" I've not yet come up with a single parameter. Let's see whether one comes out of the analysis in this document.

We'll explore two different models here:

- Many two-level-system (TLS) coupled to a single field (the Dicke model)
- Many TLS coupled to two different fields

2 Dicke model

For more detail on the Dicke model, see the notes I made on the subject.

2.1 Describing the states

The Dicke model describes a system where we have N identical TLS coupled to a single mode (i.e. single frequency/wavelength) of a quantised field. The Dicke

Hamiltonians can be written as:

$$H = \Delta E J_z + \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) + 2U\left(a^\dagger + a\right)(J_+ + J_-) \tag{1}$$

where ΔE is the transition energy between the 2 levels of the TLS, $\hbar\omega$ is the energy of each quantum of the field, and U is the coupling constant between the TLS and the field. The $a^\dagger$, a are the field creation and annihilation operators respectively and the J operators are total pseudo angular momentum operators:

$$J_+ + J_- = J_x = \frac{1}{2}\sum_{i=1}^{N}\sigma_{ix} \quad J_z = \frac{1}{2}\sum_{i=1}^{N}\sigma_{iz} \tag{2}$$

Here σ are the Pauli spin matrices the i in σ_i means that this operator only acts on TLS number i .

When written in this way, states are described in terms of 3 numbers $|n, j, m\rangle$ where n describes the number of field quanta, j describes the total pseudo angular momentum number (which is conserved) and m describes the z component of the total pseudo angular momentum (which can change). This notation allows us to conveniently describe situations where excitations are "delocalised" among the TLS. A delocalised excitation means that the excitation is shared among many TLS in such a way that you don't know which TLS holds the excitation at any moment.

By far the most significant kind of delocalised states are called "Dicke states" which have the largest $j = j_{\max} = N/2$. These states are capable of accelerated emission rates due to superradiance. Dicke starts are symmetric in the sense that if you swap any of the TLS around, the state remains unchanged. For example, consider a single excitation in 4 TLS - the Dicke state written in $|n, \pm, \pm, \pm, \pm\rangle$ notation looks like:

$$\Psi = \frac{1}{\sqrt{4}}\left(|n,+,-,-,-\rangle + |n,-,+,-,-\rangle + |n,-,-,+,-\rangle + |n,-,-,-,+\rangle\right) \tag{3}$$

Notice that if you swap any two TLS, the state looks the same.

The above state can instead be described by $j_{\max} = 4/2 = 2$ and $m = 1 \times 1/2 + 3 \times -1/2 = -1$

$$\Psi = |n, 2, -1 > \tag{4}$$

2.2 Coupling strength

The strength of the interaction between the TLS and the field is not only determined by the constant U but also how many field quanta n we have. This is because of how the field operators work:

$$a^\dagger|n,j,m\rangle = \sqrt{n+1}|n+1,j,m\rangle \tag{5}$$

$$a|n,j,m\rangle = \sqrt{n}|n-1,j,m\rangle \tag{6}$$

$$a^\dagger a|n,j,m\rangle = n|n,j,m\rangle \tag{7}$$

The more field quanta we have (the stronger the field), the larger the coupling terms will be.

The coupling is also affected by the number of TLS we have. This is because of how the the ladder operators J_+ and J_- create and destroy excitations of the TLS. This causes a raising and lowering of the m value like this:

$$J_+|n,j,m\rangle = \sqrt{j(j+1)-m(m+1)}|n,j,m+1\rangle \tag{8}$$

$$J_-|n,j,m\rangle = \sqrt{j(j+1)-m(m-1)}|n,j,m-1\rangle \tag{9}$$

These ladder operators are conceptually similar to the creation and annihilation operators of the field (see Eqs. 5 and 6). The details are however more complicated due to the addition rules of angular momentum. Despite the complexity, we know that the maximum total angular momentum $j_{\mathrm{max}} = N/2$ (from N TLS with pseudo angular momentum 1/2). We can therefore see that the number of TLS is going to have an effect on the coupling between TLS and field. The effect scales like at least $\sqrt{N}$ and at most N (see Dicke model notes for more detail).

The coupling term in Eq. 1 therefore conservatively scales like $U\sqrt{N}\sqrt{n}$ and optimistically as $UN\sqrt{n}$.

2.3 Deep strong coupling

In the notes on Deep strong coupling, we described a regime in which all the terms in the Hamiltonian in Eq. 1 were of the same order. In other words:

$$\Delta E \sim n\hbar\omega \sim U\sqrt{N}\sqrt{n} \tag{10}$$

where we have taken the conservative Dicke scaling of $\sqrt{N}$. This regime is characterised by a superradiant phase transition.

For a given $\Delta E, \hbar\omega$, Eq. 10 fixes the strength of the field n and the number of TLS N. For nuclear transitions mediated by phonons via a relativistic phonon nuclear coupling, we found that we could **not** satisfy Eq. 10.

Mathematically, the reason Eq. 10 is hard to satisfy is because $n\hbar\omega \sim U\sqrt{N}\sqrt{n}$ limits how large n can be because n grows faster than $\sqrt{n}$.

What happens if we look for a regime in which n is not constrained?

2.4 Strong field

Let's imagine that the field is so strong, i.e. n is so large, that the following is satisfied:

$$\Delta E \lesssim U\sqrt{N}\sqrt{n} \ll n\hbar\omega \tag{11}$$

In this regime, the field retains its "identity" but the TLS gets significantly altered by the field and the coupling. We should also expect to have a free exchange of energy between the TLS and the field in this regime but we shouldn't expect a formal phase transition.

Eq. 11 can in principle be satisfied for any type of coupling by just increasing the field strength. In practice, there will be physical limits on how strong a field can be. Let's try and work through a couple of examples using relativistic phonon nuclear coupling and magnetic coupling.

2.4.1 Relativistic phonon nuclear coupling

From notes on Coupling constants in nuclear physics, we derived the relativistic phonon nuclear coupling as:

$$\frac{U}{\hbar\omega_A} = \sqrt{\frac{2}{N}}\sqrt{\frac{\Delta E}{Mc^2}}\sqrt{\frac{\Delta E}{\hbar\omega_A}} \times 10^{-3} \tag{12}$$

where N is the number of nuclei involved in the phonon motion, and M is the mass of the nucleus and ω_A is the acoustic phonon mode frequency. The first condition in Eq. 11 gives:

$$\begin{aligned}
\Delta E &\lesssim U\sqrt{N}\sqrt{n} \\
\frac{\Delta E}{\hbar\omega_A} &\lesssim \sqrt{2}\sqrt{\frac{\Delta E}{Mc^2}}\sqrt{\frac{\Delta E}{\hbar\omega_A}} \times 10^{-3}\sqrt{n} \\
1 &\lesssim \sqrt{2}\sqrt{\frac{\hbar\omega_A}{Mc^2}} \times 10^{-3}\sqrt{n} \\
\frac{1}{2}\frac{Mc^2}{\hbar\omega_A} \times 10^{6} &\lesssim n \\
\frac{1}{2}Mc^2 \times 10^{6} &\lesssim n\hbar\omega_A
\end{aligned} \tag{13}$$

For palladium nuclear transitions mediated by acoustic phonons, $Mc^2 \approx 10^{11}$ eV and $\hbar\omega_A \approx 10^{-8}$ eV . This tells us that the number of phonons must be at least:

$$n \gtrsim 5 \times 10^{24} \tag{14}$$

And the total energy of the phonons is at least:

$$\frac{1}{2} \times 10^{11} \times 10^{6} \times 1.6 \times 10^{-19} = 8\,\mathrm{mJ} \tag{15}$$

This does not seem like an outrageous amount of energy.

The second condition in Eq. 11 gives:

$$\begin{aligned} U\sqrt{N}\sqrt{n} &\ll n\hbar\omega_A \\ \sqrt{2}\sqrt{\frac{\Delta E}{Mc^2}}\sqrt{\frac{\Delta E}{\hbar\omega_A}} \times 10^{-3}\sqrt{n} &\ll n \\ 2\frac{\Delta E}{Mc^2}\frac{\Delta E}{\hbar\omega_A} \times 10^{-6} &\ll n \end{aligned} \tag{16}$$

For a 24 MeV palladium transition mediated by acoustic phonons we have:

$$n \gg 10^{6} \tag{17}$$

So, we can describe some different regions of n for this palladium example:

- $n < 10^6$ - weak field, the coupling term is greater than the total field energy but is a lot less than the TLS energy.
- $10^6 < n < 5 \times 10^{24}$ - intermediate field, the field and the TLS dominate over the coupling.
- $n > 5 \times 10^{24}$ - strong field, the field dominates but now the coupling is bigger than the TLS energy

If the field is strong and the field quanta have a low energy such that $U\sqrt{N} > \hbar\omega$, then there can be a free exchange of energy between the field and the TLS because even though an individual low energy energy quanta cannot "hold" all the energy of a TLS transition, the coupling term can. Another way of thinking about this is that incredibly unlikely transitions like the downconversion of nuclear energy into phonon energy or the upconversion of phonon energy to nuclear energy become possible.

From a practical point of view, 8 mJ of phonon energy is not a lot for a macroscopic solid sample. There is however an interesting question to ask - could we see these strong field effects with a single nucleus?

2.4.1.1 Ion traps

Although the phonon energy doesn't seem that large - Eq. 15 gave 8 mJ - it can be a bit deceiving. For example, one might imagine giving 8 mJ of energy to a single nucleus via the electric field of an ion trap such as a Paul trap like the one used in Cai 2021. Such a setup might allow us to explore relativistic phonon nuclear coupling of a single nucleus. However, we must consider both the physical limits on the size of the electric field that would drive the oscillatory motion and the size of the experiment over which acceleration of the nucleus occurs.

From notes on Coupling constants in nuclear physics, we derived the magnitude of an oscillating electric field associated with phonon motion to be:

$$E = \frac{\omega_A \sqrt{2Mn\hbar\omega_A}}{Ze\sqrt{N}} \tag{18}$$

The electric field depends on the frequency and so in principle we can bring the field down to manageable levels. However, the smaller the field, the greater than size of the experiment because smaller field has to act over a greater distance in order to produce phonon energies required from Eq. 13.

A simple test of practicality is to substitute the largest practical E field of $\sim 10^{11}\mathrm{Vm}^{-1}$. It should be noted that this field is used for acceleration of charged particles and not for trapping of ions. It will however give us a good bound for how large a hypothetical trap would need to be.

For a single palladium nucleus with $Z \sim 50$, the size of the trap d can be estimated from force multiplied by distance:

$$\begin{aligned}
ZeEd &= 8 \times 10^{-3} \\
d &= \frac{8 \times 10^{-3}}{ZeE} \\
d &\sim \frac{5 \times 10^{16}}{50 \times 10^{11}} \\
d &\sim 10^{4} \text{ m}
\end{aligned} \tag{19}$$

A monstrously large ion trap!

A smaller electric field would only make the situation worse and so we can conclude that the strong field regime cannot be accessed for a single nucleus in an ion trap.

In some ways this might not be so surprising because for a single nucleus, we're requiring $\frac{1}{2}Mc^2 \times 10^6$ eV to be transferred to it - one million times more energy than it's rest mass energy!!!

One might hope to alleviate the problems described above by adding more nuclei to the trap and thus bringing down the energy per nucleus. However, in order to avoid collective non-neutral plasma effects coming into play we'd need to limit $N < 1000$. This would still make the trap ~ 10 m which is orders of magnitude larger that typical trap sizes.

From this perspective to seems like we cannot study free energy exchange between phonons and the nucleus with small numbers of nuclei.

2.4.2 Magnetic dipole coupling

From notes on Coupling constants in nuclear physics, we derived the magnetic dipole coupling as:

$$U \approx 0.02\frac{\mu_N B}{\sqrt{n}} = 0.02\mu_N\sqrt{\frac{\mu_0\hbar\omega}{V}} \tag{20}$$

where $\mu_N = e\hbar/m_p \approx 5\times10^{-27}$ J/T is the nuclear magneton

The first condition in Eq. 11 gives:

$$\begin{aligned} \Delta E &\lesssim U\sqrt{N}\sqrt{n} \\ \Delta E &\lesssim 0.02\mu_N B\sqrt{N} \\ 1 &\lesssim 0.02\sqrt{N}\frac{\mu_N B}{\Delta E} \end{aligned} \tag{21}$$

We can turn this into a condition for the number of nuclei N by using the largest reasonable magnetic field strength of $B\sim 0.1$ T. Let's look at the 14 keV nuclear transition of ^{57}Fe .

$$\begin{aligned} N &\gtrsim \left(50\frac{\Delta E}{\mu_N B}\right)^2 \\ &\gtrsim 2500\times\left(\frac{14\times10^3\times1.6\times10^{-19}}{5\times10^{-27}\times0.1}\right)^2 \\ &\gtrsim 5\times10^{28} \end{aligned} \tag{22}$$

This works out at about 5000 kg of iron - at the very least!

The second condition in Eq. 11 gives:

$$\begin{aligned} U\sqrt{N}\sqrt{n} &\ll n\hbar\omega \\ 0.02\mu_N B\sqrt{N} &\ll \frac{1}{\mu_0}B^2 V \\ N &\ll \left(50\frac{1}{\mu_0\mu_N}BV\right)^2 \end{aligned} \tag{23}$$

Again we'll use $B = 0.1$ T . For the volume let's use the volume of 5×10^{28} iron atoms. The density of iron is about 8000 kg/m^{-3} so $V \approx 0.6$ m^3 . Substituting the numbers gives:

$$N \ll 2 \times 10^{65} \tag{24}$$

This condition is well satisfied.

If we were to use quanta with energy 4 neV which correspond to a frequency $f \approx 1$ MHz and wavelength $\lambda \approx 300$ m (which would define a coherence domain for the Dicke model) then we could in principle reach the strong field regime using magnetic coupling. It would be a LOT of iron of course and the solenoid would be very large too.

3 Two field Dicke model

When we looked at the Dicke model, we concluded that strong field would only be possible for a macroscopically large number of TLS. This was based on the the idea that the coupling energy must match the full transition energy of the TLS:

$$\Delta E \lesssim U\sqrt{N}\sqrt{n} \tag{25}$$

This condition meant that the phonon energy required would be far too much to give a single nucleus. It is possible to lower the energy requirement, by extending the Dicke model with an additional quantised field. The Hamiltonian for this two field Dicke model is:

$$H = \Delta E J_z + \hbar\omega_A \left(a^\dagger a + \frac{1}{2}\right) + \hbar\omega_p \left(b^\dagger b + \frac{1}{2}\right) + 2U_A \left(a^\dagger + a\right)(J_+ + J_-) + 2U_p \left(b^\dagger + b\right)(J_+ + J_-) \tag{26}$$

Conceptually we might think about the TLS as a nucleus, and the fields as a phonon field (A) and a photon field (p). Eq. 26 would then allow us to start exploring the possibility of energy moving from a low energy phonon field to a higher energy photon field with the TLS being the mediator. This would normally not be possible because of the mismatch between the energy quanta of the two fields. However, with a strong field it might be possible.

The photon quanta need not have the same energy as the TLS energy - e.g. it could be the case that:

$$\hbar\omega_A \ll \hbar\omega_p \ll \Delta E \tag{27}$$

We might then think about strong field as:

$$\hbar\omega_p \sim U_A\sqrt{N}\sqrt{n_A} \ll n_A\hbar\omega_A \tag{28}$$

instead of:

$$\Delta E \lesssim U_A\sqrt{N}\sqrt{n_A} \ll n_A\hbar\omega_A \tag{29}$$

Eq. 28 would lower the phonon energy requirement and might make single particle phonon nuclear coupling observable. It's not clear how observable it might be at this stage, but we can look at the phonon energy requirements as a first step.

3.0.1 Relativistic phonon nuclear coupling

From notes on Coupling constants in nuclear physics, we derived the relativistic phonon nuclear coupling as:

$$\frac{U}{\hbar\omega_A} = \sqrt{\frac{2}{N}}\sqrt{\frac{\Delta E}{Mc^2}}\sqrt{\frac{\Delta E}{\hbar\omega_A}} \times 10^{-3} \tag{30}$$

where N is the number of nuclei involved in the phonon motion, and M is the mass of the nucleus and ω_A is the acoustic phonon mode frequency. The first condition in Eq. 28 gives:

$$\begin{aligned}
\hbar\omega_p &\lesssim U_A\sqrt{N}\sqrt{n_A} \\
\frac{\hbar\omega_p}{\hbar\omega_A} &\lesssim \sqrt{2}\sqrt{\frac{\Delta E}{Mc^2}}\sqrt{\frac{\Delta E}{\hbar\omega_A}} \times 10^{-3}\sqrt{n_A} \\
\frac{\hbar\omega_p}{\Delta E} &\lesssim \sqrt{2}\sqrt{\frac{\hbar\omega_A}{Mc^2}} \times 10^{-3}\sqrt{n_A} \\
\frac{\hbar\omega_p}{\Delta E} &\lesssim \sqrt{2} \times 10^{-3}\sqrt{\frac{n_A\hbar\omega_A}{Mc^2}}
\end{aligned} \tag{31}$$

$$\frac{1}{2}Mc^2 \times 10^6 \times \left(\frac{\hbar\omega_p}{\Delta E}\right)^2 \lesssim n_A\hbar\omega_A$$

For palladium with $Mc^2 \approx 10^{11}$ eV and a nuclear transition at $\Delta E \sim 10$ MeV , if we imagine x-ray photons with energy $\hbar\omega_p \sim 10$ keV then the phonon energy required would be:

$$n\hbar\omega_A \gtrsim \frac{1}{2} \times 10^{11} \times 10^6 \times \left(\frac{10^3}{10^6}\right)^2 = \frac{1}{2} \times 10^{11} = \frac{1}{2}Mc^2 \tag{32}$$

This is 6 orders of magnitude less phonon energy that we needed before. Instead of 8 mJ, we'd need 8 nJ.

The reduction means that for a single nucleus ion trap experiment, we'd not need a 10km long trap but a 1cm trap. That calculation was however made with the largest electric field possible (10^{11} Vm^{-1}). For typical Paul traps, electric fields go up to 10^6 Vm^{-1}. That would again make the experiment balloon by $\times 10^5$ to 1km.

We can work out what energy of radiation $\hbar\omega_p$ would be possible for a physically realistic Paul trap by solving:

$$\begin{aligned} ZeEd &= n_A \hbar\omega_A \\ ZeEd &= \frac{1}{2} Mc^2 \times 10^6 \times \left(\frac{\hbar\omega_p}{\Delta E}\right)^2 \\ \frac{2ZeEd}{Mc^2 \times 10^6} &= \left(\frac{\hbar\omega_p}{\Delta E}\right)^2 \\ \sqrt{\frac{2ZeEd}{Mc^2 \times 10^6}}\Delta E &= \hbar\omega_p \end{aligned} \tag{33}$$

If we plug in the relevant numbers, $Mc^2 \sim 10^{11}$ eV , $Z \sim 50$, $E \sim 10^6$ Vm^{-1}, $d \sim 0.01$ m and $\Delta E \sim 10$ MeV we get

$$\hbar\omega_p = \sqrt{\frac{2 \times 50 \times 1.6 \times 10^{-19} \times 10^6 \times 0.01}{10^{11} \times 1.6 \times 10^{-19} \times 10^6}} \times 10 \times 10^6 = 31 \text{ eV} \tag{34}$$

And so if we wanted to explore relativistic phonon nuclear coupling with a single nucleus inside a Paul trap of typical parameters, then we'd be looking for emission of ultraviolet radiation.

The next question is how often would we expect to see this radiation.