Phase in electromagnetic coupling

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1 Phase in Electomagnetic coupling

These notes build up a clean "mental model" for how phases arise in light–matter coupling when two **identical** two-level systems (TLS) interact with a **single quantised electromagnetic mode**, including both **electric** and **magnetic** dipole contributions. The focus is on:

- how we write H_{int} from general E and B couplings,
- how this reduces to coupling to a **single oscillator quadrature**,
- where the **phase factors** come from,
- how phases can be **gauged away** (factored out),
- and when a **relative phase difference** between donor/acceptor is physically meaningful.

1.1 1. The setup: two identical TLS and a single mode

We have two identical TLS labelled $j \in \{D, A\}$ (donor/acceptor), transition frequency ω_0, and a single bosonic mode (e.g. cavity/waveguide mode) with frequency ω_c:

$$H_0 = \hbar\omega_c a^\dagger a + \sum_{j=D,A} \frac{\hbar\omega_0}{2}\, \sigma_z^{(j)}. \tag{1}$$

We will mostly care about the **interaction** Hamiltonian.

1.2 2. General dipole interaction with E and B

In the dipole approximation (TLS size $\ll \lambda$), the interaction for TLS j is:

$$H_{\text{int}}^{(j)} = -\hat{\mathbf{d}}^{(j)} \cdot \hat{\mathbf{E}}(\mathbf{r}_j) - \hat{\boldsymbol{\mu}}^{(j)} \cdot \hat{\mathbf{B}}(\mathbf{r}_j), \tag{2}$$

where:

- $\hat{\mathbf{d}}^{(j)}$ is the electric dipole operator of TLS j,
- $\hat{\boldsymbol{\mu}}^{(j)}$ is the magnetic dipole operator,
- $\mathbf{r}_j$ is the TLS position.

Total interaction is:

$$H_{\text{int}} = \sum_{j=D,A} H_{\text{int}}^{(j)}. \tag{3}$$

1.2.1 2.1 Identical TLS operators

For identical TLS, the transition operators have the same matrix elements, but geometry/projections can differ with position/orientation.

A common simplification is to take:

$$\hat{\mathbf{d}}^{(j)} = \mathbf{d}\,(\sigma_+^{(j)} + \sigma_-^{(j)}) = \mathbf{d}\,\sigma_x^{(j)}, \qquad \hat{\boldsymbol{\mu}}^{(j)} = \boldsymbol{\mu}\,(\sigma_+^{(j)} + \sigma_-^{(j)}) = \boldsymbol{\mu}\,\sigma_x^{(j)}. \tag{4}$$

More generally, $\mathbf{d}$ and $\boldsymbol{\mu}$ can be complex vectors, but the structural results below remain the same.

1.3 3. One quantised mode: field operators and complex mode functions

For a **single mode**, the electric and magnetic field operators at position $\mathbf{r}$ can always be written as:

$$\hat{\mathbf{E}}(\mathbf{r}) = \mathbf{E}_{\text{zpf}}(\mathbf{r})\,a + \mathbf{E}_{\text{zpf}}^*(\mathbf{r})\,a^\dagger, \qquad \hat{\mathbf{B}}(\mathbf{r}) = \mathbf{B}_{\text{zpf}}(\mathbf{r})\,a + \mathbf{B}_{\text{zpf}}^*(\mathbf{r})\,a^\dagger. \tag{5}$$

Here $\mathbf{E}_{\text{zpf}}(\mathbf{r})$ and $\mathbf{B}_{\text{zpf}}(\mathbf{r})$ are **complex vector-valued mode amplitudes** (including polarisation and spatial dependence), and "zpf" means "zero-point field".

A travelling-wave-like mode often has:

$$\mathbf{E}_{\text{zpf}}(\mathbf{r}) \propto e^{i\mathbf{k}\cdot\mathbf{r}}, \qquad \mathbf{B}_{\text{zpf}}(\mathbf{r}) \propto e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{6}$$

A lossless standing-wave cavity mode can often be chosen real up to sign (so phases are 0 or π).

1.4 4. The simplest case: magnetic dipole only (B-only)

Assume $\mathbf{d} = 0$ and only magnetic coupling:

$$H_{\text{int}}^{(j)} = -\hat{\boldsymbol{\mu}}^{(j)} \cdot \hat{\mathbf{B}}(\mathbf{r}_j). \tag{7}$$

Insert Eqs. 4 and 5:

$$H_{\text{int}}^{(j)} = -\sigma_x^{(j)} \Big[\boldsymbol{\mu} \cdot \mathbf{B}_{\text{zpf}}(\mathbf{r}_j)\, a + \boldsymbol{\mu} \cdot \mathbf{B}_{\text{zpf}}^{*}(\mathbf{r}_j)\, a^{\dagger} \Big]. \tag{8}$$

Define the complex scalar coupling coefficient:

$$g_j \equiv -\frac{1}{\hbar}\, \boldsymbol{\mu} \cdot \mathbf{B}_{\text{zpf}}(\mathbf{r}_j), \qquad g_j^{*} = -\frac{1}{\hbar}\, \boldsymbol{\mu} \cdot \mathbf{B}_{\text{zpf}}^{*}(\mathbf{r}_j). \tag{9}$$

Then:

$$H_{\text{int}}^{(j)} = \hbar\, \sigma_x^{(j)} \big(g_j a + g_j^{*} a^{\dagger} \big). \tag{10}$$

1.4.1 4.1 Writing the phase explicitly

Write g_j in polar form:

$$g_j = |g_j| e^{i\theta_j}. \tag{11}$$

Then:

$$g_j a + g_j^{*} a^{\dagger} = |g_j| \big(e^{i\theta_j} a + e^{-i\theta_j} a^{\dagger} \big), \tag{12}$$

so:

$$H_{\text{int}}^{(j)} = \hbar |g_j|\, \sigma_x^{(j)} \big(e^{i\theta_j} a + e^{-i\theta_j} a^{\dagger} \big). \tag{13}$$

1.4.2 4.2 Where does θ_j come from?

It is simply:

$$\theta_j = \arg\big(\boldsymbol{\mu} \cdot \mathbf{B}_{\text{zpf}}(\mathbf{r}_j) \big). \tag{14}$$

For a travelling wave with $\mathbf{B}_{\text{zpf}}(\mathbf{r}) \propto e^{i\mathbf{k}\cdot\mathbf{r}}$ and fixed polarisation,

$$\theta_j \simeq \mathbf{k} \cdot \mathbf{r}_j + (\text{constant}). \tag{15}$$

1.5 5. Factoring out a global phase (why only relative phase matters)

The free Hamiltonian $\hbar\omega_c a^\dagger a$ is invariant under:

$$a \to a\, e^{-i\chi}, \qquad a^\dagger \to a^\dagger e^{i\chi}. \tag{16}$$

This is a "phase convention" for the mode. Use it to make one emitter's phase vanish.

Take $\chi = \theta_D$. Then donor's factor becomes:

$$e^{i\theta_D} a + e^{-i\theta_D} a^\dagger \to a + a^\dagger. \tag{17}$$

Acceptor keeps only the **relative phase**:

$$\Delta\phi \equiv \theta_A - \theta_D. \tag{18}$$

So the two-emitter B-only interaction can be written as:

$$H_{\text{int}} = \hbar|g_D|\sigma_x^{(D)}(a + a^\dagger) + \hbar|g_A|\sigma_x^{(A)}\big(e^{i\Delta\phi} a + e^{-i\Delta\phi} a^\dagger\big). \tag{19}$$

1.6 6. When do you get a nontrivial relative phase $\Delta\phi$?

1.6.1 6.1 Travelling-wave / running-wave mode

If $\mathbf{B}_{\text{zpf}}(\mathbf{r}) \propto e^{i\mathbf{k}\cdot\mathbf{r}}$ then:

$$\Delta\phi = \mathbf{k}\cdot(\mathbf{r}_A - \mathbf{r}_D). \tag{20}$$

In 1D, $\Delta\phi = k(x_A - x_D)$, and $\Delta\phi = \pi/2$ corresponds to separation $\Delta x = \lambda/4$.

1.6.2 6.2 Ideal standing-wave cavity mode

If the mode function can be chosen real everywhere, then g_j is real up to a sign, so:

- phases are typically 0 or π,
- you cannot get a continuous $\Delta\phi$ unless you use a running-wave basis or the mode is intrinsically complex (lossy/open cavity, quasi-normal mode, etc.).

1.6.3 6.3 Extra ways to get complex spatial phase

Even for "one mode", $\mathbf{B}_{\text{zpf}}(\mathbf{r})$ can be intrinsically complex in:

- leaky/open cavities (quasi-normal modes),
- waveguides/rings (naturally travelling-wave eigenmodes),
- strongly confined/evanescent structures where phase varies over shorter effective wavelengths.

1.7 7. Quadratures: what they are and why they appear

For one harmonic oscillator mode, define quadratures:

$$X \equiv a + a^\dagger, \qquad P \equiv i(a^\dagger - a). \tag{21}$$

A useful identity is:

$$e^{i\phi}a + e^{-i\phi}a^\dagger = X\cos\phi + P\sin\phi. \tag{22}$$

So the "phase factor" form is exactly "a rotated quadrature":

$$X_\phi \equiv X\cos\phi + P\sin\phi. \tag{23}$$

1.7.1 7.1 Important conceptual separation

- In the travelling-wave B-only case, $\Delta\phi$ comes from the **complex mode function** e^{ikx}.
- Rewriting it as quadratures via Eq. 22 is **pure algebra**, not new physics.
- It does **not** mean the acceptor is coupling to E instead of B.

1.8 8. Coupling to both E and B at a single point: how a quadrature angle arises *without spatial separation*

Now allow both dipoles:

$$H_{\text{int}}^{(j)} = -\hat{\mathbf{d}}^{(j)} \cdot \hat{\mathbf{E}}(\mathbf{r}_j) - \hat{\boldsymbol{\mu}}^{(j)} \cdot \hat{\mathbf{B}}(\mathbf{r}_j). \tag{24}$$

At a single point, in many common quantisation conventions (notably those built from $\mathbf{A}$), one finds that for a single mode:

- $\hat{\mathbf{B}}(\mathbf{r})$ is proportional to one quadrature (often X),
- $\hat{\mathbf{E}}(\mathbf{r})$ is proportional to the orthogonal quadrature (often P).

A toy model is:

$$\hat{\mathbf{B}}(\mathbf{r}_j) = \mathbf{B}_{\text{zpf}}(\mathbf{r}_j)\, X, \qquad \hat{\mathbf{E}}(\mathbf{r}_j) = \mathbf{E}_{\text{zpf}}(\mathbf{r}_j)\, P. \tag{25}$$

Project onto dipoles and define real scalar couplings:

$$g_{B,j} \equiv \boldsymbol{\mu} \cdot \mathbf{B}_{\text{zpf}}(\mathbf{r}_j), \qquad g_{E,j} \equiv \mathbf{d} \cdot \mathbf{E}_{\text{zpf}}(\mathbf{r}_j). \tag{26}$$

Then for TLS j:

$$H_{\text{int}}^{(j)} = -\sigma_x^{(j)} \big(g_{B,j} X + g_{E,j} P \big). \tag{27}$$

Now combine into one rotated quadrature using:

$$V_j = \sqrt{g_{B,j}^2 + g_{E,j}^2}, \qquad \phi_j = \arctan\left(\frac{g_{E,j}}{g_{B,j}} \right), \tag{28}$$

giving:

$$g_{B,j} X + g_{E,j} P = V_j \big(X \cos\phi_j + P \sin\phi_j \big) = V_j \big(e^{i\phi_j} a + e^{-i\phi_j} a^\dagger \big). \tag{29}$$

So:

$$H_{\text{int}}^{(j)} = -V_j\, \sigma_x^{(j)} \big(e^{i\phi_j} a + e^{-i\phi_j} a^\dagger \big). \tag{30}$$

1.8.1 8.1 What is the meaning of ϕ_j here?

It is the **quadrature angle** selected by the ratio of electric to magnetic coupling at that point:

- $g_{E,j} = 0 \Rightarrow \phi_j = 0$ (pure X coupling),
- $g_{B,j} = 0 \Rightarrow \phi_j = \pi/2$ (pure P coupling),
- both nonzero $\Rightarrow$ rotated quadrature.

1.8.2 8.2 Does this produce a *relative* phase between two identical TLS?

If the TLS are identical and sit in identical conditions, they have the same g_E/g_B, hence the same ϕ_j. In that case, a global rephasing can remove the common phase from both simultaneously.

A relative phase arises only if:

$$\phi_A \neq \phi_D, \tag{31}$$

which requires a difference in:

- position-dependent field ratios $E_{\text{zpf}}(\mathbf{r})/B_{\text{zpf}}(\mathbf{r})$,
- orientation/projection of dipoles on local polarisation,
- or (more exotic) intrinsic complex relative phase between electric and magnetic transition moments.

1.9 9. Combining both mechanisms (general picture)

In the most general single-mode case, each emitter's interaction can be written as:

$$H_{\text{int}}^{(j)} = \hbar\, \sigma_x^{(j)} \Big(\alpha_j a + \alpha_j^* a^\dagger\Big), \tag{32}$$

where the complex coefficient α_j contains *everything*:

- electric contribution via $\mathbf{E}_{\text{zpf}}(\mathbf{r}_j)$ and $\mathbf{d}$,
- magnetic contribution via $\mathbf{B}_{\text{zpf}}(\mathbf{r}_j)$ and $\boldsymbol{\mu}$,
- spatial dependence (e.g. $e^{i\mathbf{k}\cdot\mathbf{r}_j}$),
- polarisation/projection factors.

Write $\alpha_j = |\alpha_j| e^{i\theta_j}$ and you get:

$$H_{\text{int}}^{(j)} = \hbar |\alpha_j| \sigma_x^{(j)} \big(e^{i\theta_j} a + e^{-i\theta_j} a^\dagger\big). \tag{33}$$

Then only the **relative phase** $\Delta\phi = \theta_A - \theta_D$ survives after choosing the mode phase convention.

1.10 10. What is physically observable about these phases?

1.10.1 10.1 Global phase is not observable in the single-mode Hamiltonian

You can always choose the phase of a to remove one emitter's phase entirely. That's why the literature often writes the donor with $(a + a^\dagger)$ and puts the phase on the acceptor.

1.10.2 10.2 Relative phase becomes meaningful when there is interference

A relative phase matters when the physics compares "paths" or "collective combinations", e.g.:

- superradiant vs subradiant (bright/dark) collective states in two-emitter coupling,

- direct donor–acceptor coupling terms,
- multiple modes / continuum (propagation and retardation),
- coherent drives with a fixed phase reference.

Even in a single-mode model, the relative phase influences which collective TLS superposition couples most strongly to the field.

1.11 11. Summary "rules of thumb"

- **Single travelling wave**: E and B are in phase in time for that wave.
- **Standing wave**: locally, E and B are often $\pi/2$ out of phase in time (energy sloshing).
- **B-only coupling**: phases arise from the complex spatial mode function $u(\mathbf{r})$ (e.g. e^{ikx}).
- **E + B coupling**: even at one point, the TLS couples to a linear combination of quadratures; this is a "quadrature rotation" with angle $\phi = \arctan(g_E/g_B)$.
- **Global phase** can be removed by redefining a; only **relative phase** between emitters or channels matters.
- **Relative phase difference** requires emitters to experience different complex coefficients: either different spatial phase (travelling wave) or different g_E/g_B ratio (varying geometry/orientation/field ratios).

1.12 12. Minimal "canonical forms" to remember

1.12.1 12.1 B-only (travelling wave)

$$H_{\text{int}} = \hbar \sum_{j=D,A} \sigma_x^{(j)} \big(g_0 e^{i\mathbf{k}\cdot\mathbf{r}_j} a + g_0 e^{-i\mathbf{k}\cdot\mathbf{r}_j} a^\dagger \big). \tag{34}$$

After gauging away the donor phase:

$$H_{\text{int}} = \hbar g_0 \sigma_x^{(D)} (a + a^\dagger) + \hbar g_0 \sigma_x^{(A)} \big(e^{i\Delta\phi} a + e^{-i\Delta\phi} a^\dagger \big), \quad \Delta\phi = \mathbf{k}\cdot(\mathbf{r}_A - \mathbf{r}_D). \tag{35}$$

1.12.2 12.2 E + B at one point (quadrature rotation)

$$H_{\text{int}}^{(j)} = -\sigma_x^{(j)} \big(g_{B,j} X + g_{E,j} P \big) = -V_j \sigma_x^{(j)} \big(e^{i\phi_j} a + e^{-i\phi_j} a^\dagger \big), \quad \phi_j = \arctan\left(\frac{g_{E,j}}{g_{B,j}} \right). \tag{36}$$
