The 3-dimensional magnetic field calculation program package magfield3

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Introduction

Using this C program package one can perform magnetic field calculation for several number of coils that have different (local) symmetry axes. The coils that can be computed with the present program are all axisymmetric in the local coordinate system with the local symmetry axis, but of course the whole coil system (and therefore the magnetic field) is in this case not axially symmetric. In their local coordinate system the coils have rectangular cross section in the local $z-r$ meridian plane. The geometry of a coil can be represented by 8 numbers: 3 numbers for the spatial position of the coil centre, 2 numbers for the direction of the local symmetry axis, and 3 numbers for the geometrical parameters of the coil in its local coordinate system (length, inner and outer radia).

Although the coil system is not axially symmetric, we employ in our program some calculation methods for axisymmetric coils. Namely, in order to compute the magnetic field of a coil, we go first into the local coordinate system of the coil; in this local system we have axial symmetry, so we compute first the field here, and then we transform the field values back into the global coordinate system (see Fig. 1).

2 different methods are employed in this program for the magnetic field computation. One of them uses the elliptic integrals, namely the first, second and third complete elliptic integrals. In order to compute them, we use the numerical C programs from Ref. [1], sec. 6.11 (for the first and second complete elliptic integrals very accurate Chebyshev approximation formulae can also be found in Ref. [2]). Magnetic field expressions for solenoids, using the first, second and third complete elliptic integrals, can be found in Ref. [3]. Using the elliptic integrals one can calculate the magnetic field everywhere, even inside the coil windings. Unfortunately,

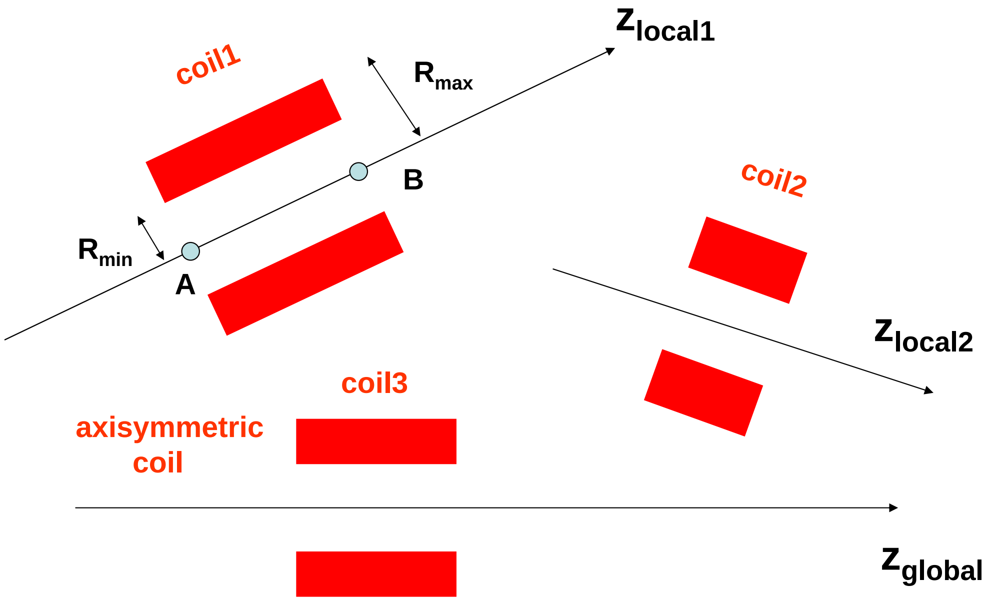

Figure 1: Coils with various local symmetry axes.

this method is rather slow. Therefore we use in our program a second, very efficient and fast calculation method, using Legendre polynomial expansion. There are 2 different types of Legendre polynomial expansions: if the field point (where we want to compute the magnetic field) is close to the local symmetry axis of the coil, we use the so called central Legendre polynomial series for the field calculation. If the field point is far from the coil, we use the remote Legendre polynomial series (for more information about these series see in the Appendix, and in Refs. [4, 5, 6]).

Input of coil parameters

The first step of the field calculation is the determination of the coil system. In the program package magfield3 the coil parameters are defined by the global integer Ncoil, which denotes the number of the coils (it should be of course positive), and by the global double matrix coil[Ncoilmax+1][14]. Here the global integer Ncoilmax denotes the maximal number of coils (Ncoil cannot be larger than Ncoilmax); Ncoilmax should be given by the user in a #define command, in the beginning of the calculation. The user has to define in the beginning of the computation the matrix values coil[i][j], i=1,. . . , Ncoil (index of the coils), j=0,. . ., 9, with the following definition (see also Fig. 1):

coil[i][0]: current density of coil i (in A/m^2); it is the total current (number of winding turns × current in the coil wire) divided by the coil cross section (which is : coil length × radial thickness of the coil); the current density is positive if looking from the point A towards point B (see below) the current flow direction is clockwise (for positive current density the magnetic field direction is from point A towards point B).

coil[i][j], j=1,2,3: Descartes coordinates of endpoint A of the coil, in m; this point is on the local symmetry axis, see Fig. 1;

coil[i][j], j=4,5,6: Descartes coordinates of endpoint B of the coil (in m), see Fig. 1;

coil[i][7], coil[i][8]: inner (R_{min}) and outer (R_{max}) radius of the coil winding (in m);

coil[i][9]: integration number (represented by double) for numerical integration in radial direction between inner and outer radia. This number is used only for the elliptic integral method. Usually coil[i][9]=20 gives a sufficient precision, if the coils are in radial direction not thick (one shoud use larger numbers for the coils with large radial thickness). If the user needs a fast computation with the elliptic integrals, without high accuracy, one can use here small numbers (even coil[i][9]=1 is allowed).

The user can decide how to determine these coil parameters. For example, one can define some kind of coil parameter set (which could be of course quite different from the parameters used above), these user-defined parameters could be written into a data file, and these parameters are then read from the data file. Then the user-defined parameter set should be converted into the above defined coil parameters. Obviously, different users prefer the employment of different coil parameters, and the program magfield3 provides this freedom of choice for the users. Nevertheless, if the user would like to employ as input the same coil parameters as defined in the program, then the conversion between different parameter sets is not needed; in that case the user can call in the beginning of the calculation the function input_coils(char *inputcoilfile), where the string inputcoilfile contains the name of the data file with the coil parameters (the structure of this file is explained in this function).

After the user has determined the input coil parameters Ncoil and coil[i][j], i=1,. . . ,Ncoil, j=0,. . .,9, then as a second step of the computation the function test_coils() has to be called. This function performs first a test for the coil parameters, and it computes some additional (redundant) coil parameters (length and axis directions of the coils). In addition, the number of the axisymmetric coils Ncoilaxisymm is calculated (a coil is defined as axisymmetric if its symmetry axis is the same as the z axis of the global coordinate system: using this definition we can save some computation time, if there are several axisymmetric coils). At the end of the function test_coils(), the coil parameters are written into the data file **magcoil.dat**. From now on, this data file defines the coil system for the program.

Calculation of the source coefficients

We can see from the Appendix that the Legendre polynomial expansion method needs the source points and the source coefficients. Therefore, the next step is to compute these numbers. For this purpose, the user has to call the function magsource(): then the program computes automatically all the necessary information needed for the Legendre polynomial calculation method. Inside the function magsource(), first the function magsource3_central() computes the central source points and source coefficients for all coils, then the function magsource3_remote() calculates the remote source points and source coefficients for all coils, and function magsource_axisymm() evaluates similar central and remote source points and coefficients separately for the axially symmetric coils (if Ncoilaxisymm>0). The source points, convergence radia and the source coefficients are written into the data files **magsource_central.dat**, **magsource_remote.dat** and **magsource_axisymm.dat**. The Legendre polynomial expansion method employs then the information contained in these files.

Calculating the source constants separataly for the axiymmetric coils saves us some computation time. Namely, for the axially symmetric coils in a fixed source point we have only one Legendre polynomial expansion that is common for all the axisymmetric coils. Therefore, the magnetic field computation time of the Legendre polynomial expansion is independent of the number of the axisymmetric coils (of course, the computation time of the source coefficients is not independent; but the latter is needed only once in the beginning of the calculation).

The coil parameter determination and the calling of the functions test_coils() and magsource() should be done within one program run. But then the magnetic field computation can be performed in a separate run.

Magnetic field calculation

Then the user can calculate the magnetic field in an arbitrary point P (defined by the Descartes coordinates P[1], P[2], P[3]).

There are 2 possibilities for this calculation. The user can call the function magfield_elliptic(P,B), which computes the magnetic field components B[1], B[2], B[3] using elliptic integrals. If the user wants to use only this method for the field calculation, it is not necessary to call the function magsource() (the function magfield_elliptic(P,B) does not use the source parameters, only the coil parameters from the data file **magcoil.dat**).

If the user wants to calculate the magnetic field in many points, he/she can save a lot of computation time by using instead the function magfield(P,B), which tries to perform the field calculation first with the Legendre polynomial expansion. The magnetic field is computed as the sum of the non-axisymmetric coils and the axisymmetric coils (as we have seen above, using this separation we can save some computation time). In both cases the program searches first for some central or remote source point with a fast Legendre polynomial convergence. The elliptic integral method is used only if for the given field point no source point with convergent (central or remote) Legendre polynomial expansion is found.

We have computed magnetic field components for various coil systems, both with the elliptic integral and with the Legendre polynomial expansion methods. The results of these completely different procedures agree completely, up to 13-15 digits (which is of course the maximal accuracy that one can achieve with double precision computations).

The magnetic field calculation functions use the 4 abovementioned data files **magcoil.dat**, **magsource_central.dat**, **magsource_remote.dat** and **magsource_axisymm.dat**. These files are created by calling the functions test_coils() and magsource(), and then the magnetic field computation could be performed in a separate program running (the above 4 data files are automatically read by the magnetic field calculation programs).

Symbolic integer constants

The user has to define by #define commands 4 symbolic integer constants: Ncoilmax, nmaxmag, Ncenmax and Ncenaxisymmmax. Here, Ncoilmax is not allowed to be smaller than the number of the coils Ncoil (otherwise we get an error message, and the program execution stops). The second number, nmaxmag denotes the maximal value of the Legendre polynomial series index n; nmaxmag=500 is recommended to be employed. By choosing a smaller value, for example 200, the user can make the source coefficient calculation faster; the speed of the magnetic field calculation is usually independent of nmaxmag, but in certain cases (for field points with large convergence ratio) it can be faster for larger nmaxmag value. Ncenmax is the maximal number of the central source points for all coils. A rough formula for this number is: 4 times the number of coils times the average length of the coils, divided by the average inner radius of the coils. If the user has chosen a Ncenmax value that is too small, a warning message from the function magsource3_central() is printed out (if the user calls magsource()), the value of the central source points for all coils (Ncen) is written out, and the program execution is stopped. Then the user should take the number Ncen as Ncenmax for the next programm running.

Finally, Ncenaxisymmmax is the maximal number of the central source points defined for the axisymmetric coils. This can be estimated as 4 times the length of the axisymmetric coil system along the z axis, divided by the average inner radius of the axisymmetric coils. If the user has chosen a Ncenaxisymmmax value that is too small, a warning message from the function magsource_axisymm() is printed out (if the user calls magsource()), and the program execution is stopped. Then the user should take a larger number Ncenaxisymmmax for the next programm running.

As we have mentioned already, the user can perform the source coefficient calculations (by calling the functions test_coils() and magsource()) and the magnetic field calculation in separate program running. It is suggested that the user should employ during the magnetic field computations the same values for the above 4 integer constants, as it was used during the coil parameter input and source coefficient calculations. This is strictly true for nmaxmag: the program execution stops if the user employs during the magnetic field computation a different value than it was used in the source coefficient calculation. In the case of the other 3 integer constants the program running stops (with an error message) only if the chosen values are too small.

Appendix

Central and remote Legendre polynomial expansions

The magnetic field calculation by Legendre polynomial expansion is much faster than with the elliptic integrals. In order to understand this method, let us introduce some definitions. Let us fix a point $(z = z_0, r = 0)$ on the z axis. It is possible to compute in this axis point the higher derivatives of the magnetic field with high accuracy. From these higher derivatives one can build some numbers B_n^{cen} $(n = 0, 1, 2, \ldots)$, where B_0^{cen} is the magnetic field in the $(z_0, 0)$ point, B_1^{cen} is proportional to the first z-derivative of the field here, B_2^{cen} is proportional to the second z-derivative, etc. So all these B_n^{cen} numbers are defined in the axis point $(z_0, 0)$. It is a special property of the axially symmetric coil system that the magnetic field in an arbitrary off-axis point (z, r), which is not far from the axis point $(z_0, 0)$, can be computed by using these numbers B_n^{cen}. We call the axis point $(z_0, 0)$ source point, and the numbers B_n^{cen} we call central source coefficients; namely, the B_n^{cen} numbers depend on the parameters (geometry and current) of the coils, which represent the source of the magnetic field.

Let us define the convergence radius ρ_{cen} as the minimal distance of the source point from the coil winding. For example, for a coil having $-L/2$ and $L/2$ minimal and maximal z coordinates and R_{min} inner radius, the ρ_{cen} values of the source points $(0, 0)$ and $(L, 0)$ are $\rho_{cen}(z_0 = 0) = R_{min}$ and $\rho_{cen}(z_0 = L) = \sqrt{(L/2)^2 + R_{min}^2}$, respectively. The axial component of the magnetic field in the field point (z, r) can be computed by the following series:

$$B_z = \sum_{n=0}^{\infty} B_n^{cen} \left(\frac{\rho}{\rho_{cen}}\right)^n P_n(u),$$

where ρ is the distance of the field point (z, r) from the source point $(z_0, 0)$ $(\rho = \sqrt{(z - z_0)^2 + r^2})$, $u = \cos\theta = (z - z_0)/\rho$, and $P_n(u)$ is the Legendre polynomial of order n (see Fig. 2). The radial component of the magnetic field can be computed as

$$B_r = -s \sum_{n=1}^{\infty} \frac{B_n^{cen}}{n+1} \left(\frac{\rho}{\rho_{cen}} \right)^n P_n'(u),$$

where $s = \sin\theta = \sqrt{1-u^2} = r/\rho$, and $P_n'(u)$ denotes the first derivative of the Legendre polynomial $P_n(u)$.

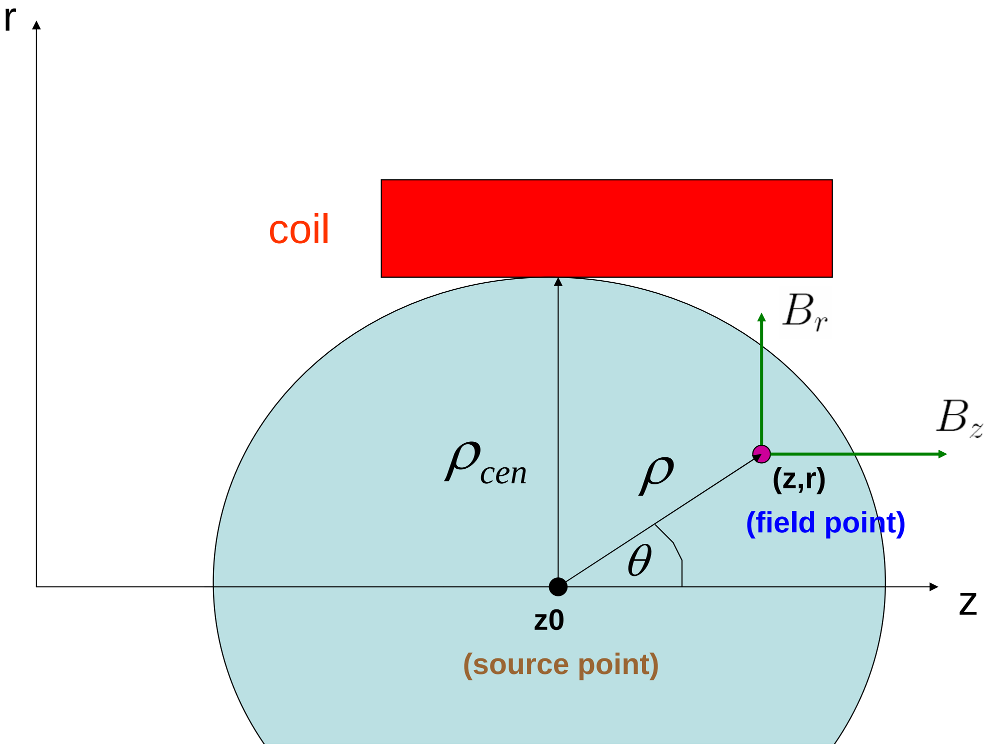

Figure 2: Central convergence circle: the central Legendre polynomial expansion is convergent for field points inside this circle.

Obviously, for numerical evaluation of the magnetic field components, one has to compute only a finite number of terms in these series. If the convergence ratio ρ/ρ_{cen} is small, the expansion converges fast, therefore only a small number of terms is sufficient to compute, in order to achieve some prescribed accuracy level for the magnetic field components. On the other hand, if the ratio ρ/ρ_{cen} is close to 1 (but smaller than 1), the convergence of the above formulae is slow, one needs many terms for the computation. If the ratio ρ/ρ_{cen} is larger than 1 (the field point is outside of the convergence circle with the source point as centre), the above series are not convergent, in this case one is not allowed to use these formulae for the field calculation.

The computation time for the case of 1 coil with the elliptic integral method is about 2 ms (the multiplication time of our computer is approximately 10 ns). On the other hand, using the central Legendre polynomial expansion, we get the following computation times for the various convergence ratios $rc = \rho/\rho_{cen}$ (n_{max} is the maximal number of n values that is needed for the series, in order to obtain the double precision accuracy):

$rc = 0.03$, $n_{max} = 15$ terms: t=5 μs;
$rc = 0.50$, $n_{max} = 42$ terms: t=15 μs;
$rc = 0.83$, $n_{max} = 140$ terms: t=45 μs;
$rc = 0.93$, $n_{max} = 350$ terms: t=110 μs.

Similar time values are obtained by using the remote Legendre polynomial expansion. It is obvious from these numbers that the Legendre polynomial expansion method is much faster than the elliptic integral one, even in the (usually rare) case of large convergence ratio values. For a fixed source point we can use the central Legendre polynomial expansion only for field points inside of the convergence circle. Nevertheless, we can define many source points, and their convergence circles provide us with a large region, where the central Legendre series method is applicable (Fig. 3 shows an example with 2 source points). For a fixed field point one should use a source point that results in the smallest ratio of convergence ρ/ρ_{cen}, in order to have fast convergence for the Legendre polynomial expansion. Fig. 4 shows that in the case of several coils (having the same symmetry axis) the overall central convergence radius is the minimum of the different convergence radia corresponding to the coils.

The above central Legendre polynomial expansion is appropriate for field points (z, r) that are not far from the source point $(z_0, 0)$. On the other hand, what happens if the field point is far from the coil and also from the axis of the coil ? In that case ρ/ρ_{cen} is larger than 1, and the above central Legendre polynomial expansion is not convergent. Fortunately, there is another type of Legendre polynomial expansion, which is useful for field points that are far from the coil (we call them remote points). Let us define the so called remote convergent radius as the maximal distance of the source point $(z_0, 0)$ from the coil winding. For example, for a coil having $-L/2$ and $L/2$ minimal and maximal z coordinates and R_{max} outer radius, the remote convergent radius ρ_{rem} of the source point $(0, 0)$ is $\rho_{rem} =$

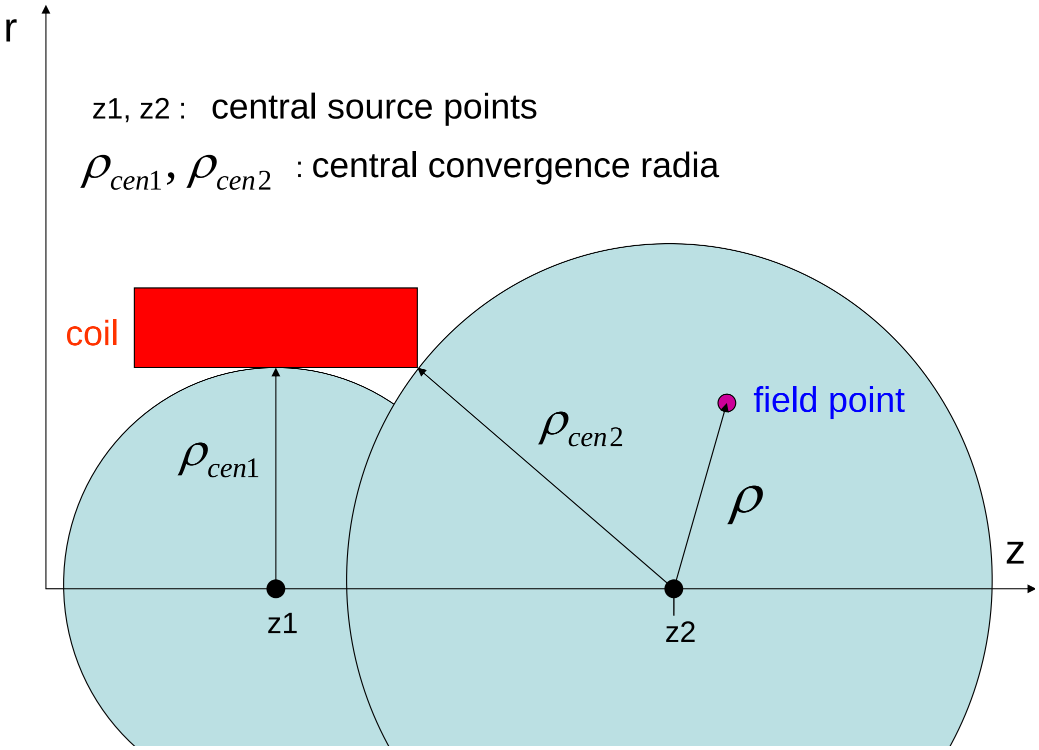
r
z1, z2 : central source points
ρ_{cen1}, ρ_{cen2} : central convergence radia
coil
ρ_{cen1}
ρ_{cen2}
field point
ρ
z
z1
z2

Figure 3: 2 central source points with the corresponding central convergence circles.

$\sqrt{(L/2)^2 + R_{max}^2}$ (see Fig. 5).

For field points whose distance ρ from the source point is larger than ρ_{rem}, we can apply the remote Legendre polynomial expansion formulae:

$$B_z = \sum_{n=2}^{\infty} B_n^{rem} \left(\frac{\rho_{rem}}{\rho} \right)^{n+1} P_n(u),$$

$$B_r = s \sum_{n=2}^{\infty} \frac{B_n^{rem}}{n} \left(\frac{\rho_{rem}}{\rho} \right)^{n+1} P_n'(u)$$

(where B_n^{rem} denote the remote source coefficients).

Similarly to the central expansion, if the convergence ratio ρ_{rem}/ρ is small (i.e. if the field point is very far from the coil), the expansion converges fast, therefore only a small number of terms is sufficient to take into account for the computation, in order to achieve some prescribed accuracy level for the magnetic field components. On the other hand, if the ratio ρ_{rem}/ρ is close to 1 (but smaller than 1), the convergence of the above

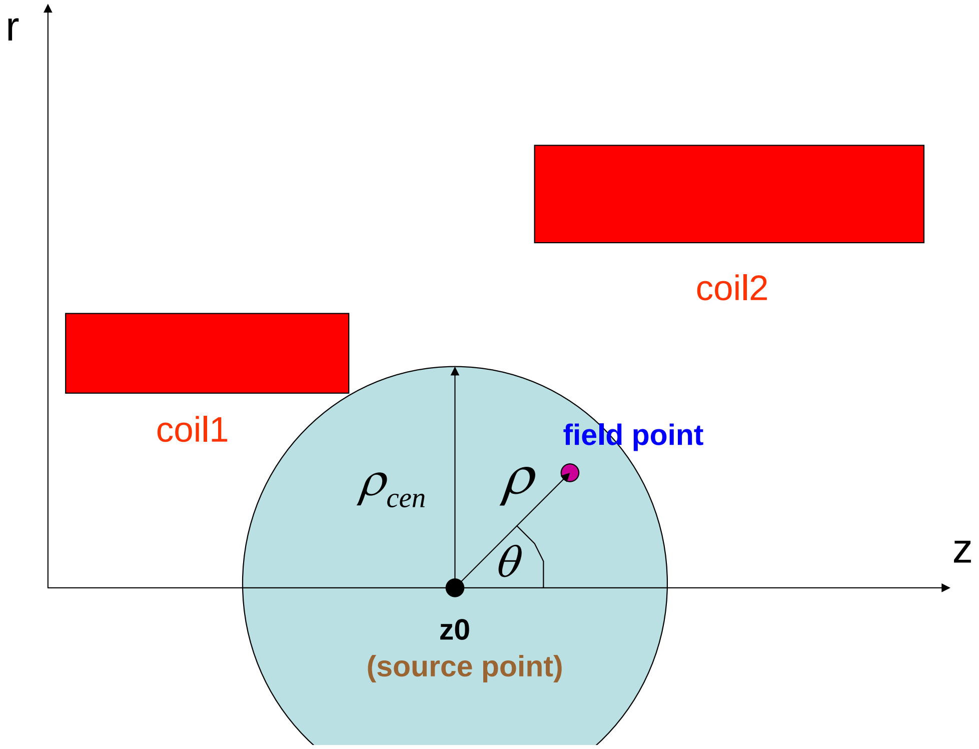

Figure 4: Central convergence circle with 2 coils.

formulae is slow, one needs many terms for the computation. If the ratio ρ_{rem}/ρ is larger than 1, the above series are not convergent, in this case one cannot use the remote Legendre polynomial expansion formulae for the field calculation.

The Legendre polynomials $P_n(u)$ and their derivatives $P'_n(u)$ can be calculated (up to large n indices) by recurrence relations (see Ref. [4]). For $n = 0$ and $n = 1$ we have:

$$P_0(u) = 1, \quad P_1(u) = u, \quad P'_0(u) = 0, \quad P'_1(u) = 1.$$

The recurrence relations for $n > 1$:

$$P_n(u) = ((2n-1)uP_{n-1} - (n-1)P_{n-2})/n,$$

$$P'_n(u) = ((2n-1)uP'_{n-1} - nP'_{n-2})/(n-1).$$

Using the remote Legendre polynomial expansion for the source point at the coil centre one needs only the terms with even n (the remote source

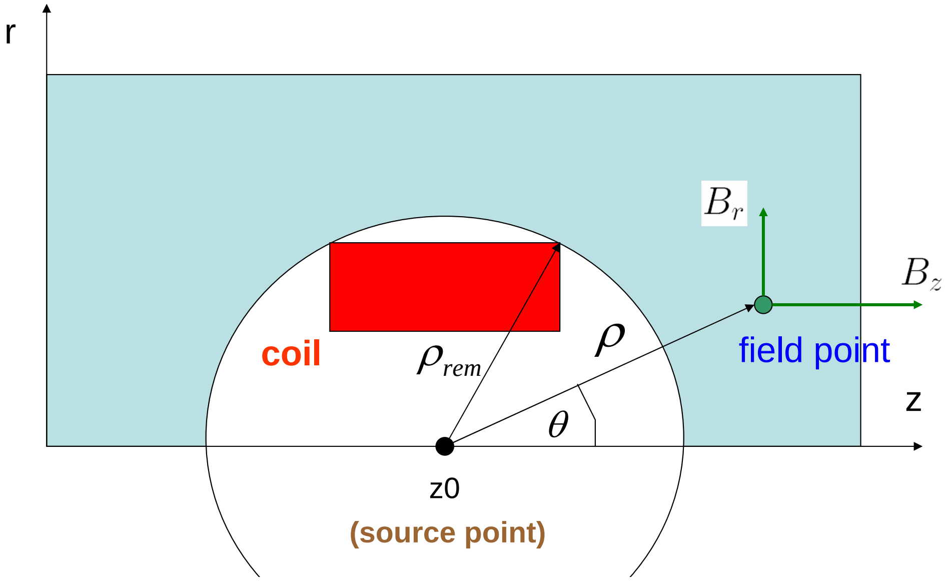

Figure 5: Remote convergence circle: the remote Legendre polynomial expansion is convergent for field points outside of this circle.

coefficients for odd n are in this case zero); one can save then a factor of 2 computer time by using the alternate recurrence formulae of Eq. 20 in Ref. [4].

Calculation of the central and remote source coefficients

Let us now calculate the central and remote source coefficients, that are needed for the above described Legendre polynomial series expansion calculation of the magnetic field. Let us assume that we have a coil with the z symmetry axis and with a rectangular cross section on the (z, r) merdian plane; the minimal and maximal z values of the coil are Z_{min} and Z_{max}, and the inner and outer radia are denoted by R_{min} and R_{max} (see Fig. 6). The current density of the coil is σ. A general coil winding point is denoted by (Z, R), where $Z_{min} \leq Z \leq Z_{max}$ and $R_{min} \leq R \leq R_{max}$. We fix a source point by the z_0 value, and we would like to compute the central and remote source coefficients B_n^{cen} and B_n^{rem} corresponding to this coil and source point. Let us start with the central source coefficients. First, we define the central convergence radius as the minimal distance between the source point $(z_0, 0)$ and the coil winding points (Z, R):

$$\rho_{cen} = \min_{(Z,R)} \sqrt{(Z - z_0)^2 + R^2}.$$

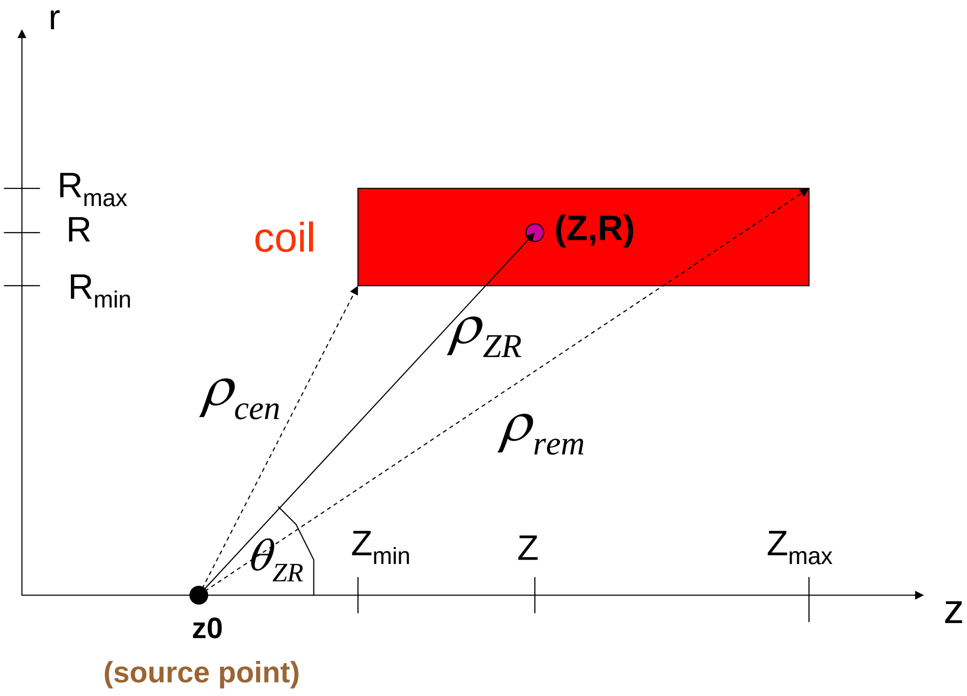

Figure 6: Notations for the source coefficient calculations.

The central source coefficient B_n^{cen} can be generally written as a 2 dimensional integral over the coil winding region:

$$B_n^{cen} = \int\limits_{R_{min}}^{R_{max}} dR \int\limits_{Z_{min}}^{Z_{max}} dZ \ \cdot b_n(Z, R),$$

where

$$b_n(Z, R) = \frac{1}{2} \mu_0 \, \sigma \frac{1 - u_{ZR}^2}{\rho_{cen}} \left(\frac{\rho_{cen}}{\rho_{ZR}} \right)^{n+1} P'_{n+1}(u_{ZR}).$$

Here ρ_{ZR} and u_{ZR} are defined as $\rho_{ZR} = \sqrt{(Z - z_0)^2 + R^2}$, $u_{ZR} = \cos \theta_{ZR} = (Z - z_0)/\rho_{ZR}$ (see Fig. 6).

The central source coefficients B_n^{cen} can be computed by 1 dimensional integrals

$$B_n^{cen} = \int_{R_{min}}^{R_{max}} dR \cdot \hat{b}_n(R),$$

whereas the

$$\hat{b}_n(R) = \int_{Z_{min}}^{Z_{max}} dZ \cdot b_n(Z, R)$$

integral can be evaluated analytically:

$$\hat{b}_0(R) = \frac{1}{2}\mu_0\sigma \left[\frac{Z - z_0}{\sqrt{R^2 + (Z - z_0)^2}} \right]_{Z_{min}}^{Z_{max}},$$

and for $n > 0$:

$$\hat{b}_n(R) = -\frac{\rho_{cen}}{n} \left[b_{n-1}(Z, R) \right]_{Z_{min}}^{Z_{max}}.$$

In order to compute the remote source coefficients, we define first the remote convergence radius as the maximal distance between the source point $(z_0, 0)$ and the coil winding points (Z, R):

$$\rho_{rem} = \max_{(Z,R)} \sqrt{(Z - z_0)^2 + R^2}.$$

The remote source coefficient B_n^{rem} (for $n \geq 2$) can be generally written as a 2 dimensional integral over the coil winding region:

$$B_n^{rem} = \int_{R_{min}}^{R_{max}} dR \int_{Z_{min}}^{Z_{max}} dZ \cdot b_n^*(Z, R),$$

where

$$b_n^*(Z, R) = \frac{1}{2}\mu_0\sigma \frac{1 - u_{ZR}^2}{\rho_{rem}} \left(\frac{\rho_{ZR}}{\rho_{rem}} \right)^n P'_{n-1}(u_{ZR}).$$

Similarly to the case of the central source coefficients, the B_n^{rem} remote source coefficients can be computed by 1 dimensional integration:

$$B_n^{rem} = \int_{R_{min}}^{R_{max}} dR \cdot \hat{b}_n^*(R),$$

where

$$\hat{b}_n^*(R) = \int_{Z_{min}}^{Z_{max}} dZ \cdot b_n^*(Z, R) = \frac{\rho_{rem}}{n+1} \left[b_{n+1}^*(Z, R) \right]_{Z_{min}}^{Z_{max}}.$$

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