Version 2: Derivation of the Power Transmitted by Various Waveguides and Transmission Lines

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In this document, we compare the formalism of Jackson [9] used by G. Rybka [12] to derive the power transmitted by a parallel-wire transmission line to the formalism of Collin [5] and Buts [3] for general waveguides. The parallel-wire transmission line, square waveguide, and circular waveguide are treated in detail. The parallel-strip transmission line is treated approximately. Circular and helical electron trajectories are investigated, and the Doppler shift created by helical trajectories is derived. Finally, we write an equation to calculate the spectral components of the power transmitted from an electron with an arbitrary trajectory.

CONTENTS

I. OVERVIEW OF WAVEGUIDES, TE, TM, AND TEM MODES

Hollow waveguides that extend in the z-direction to infinity will support transverse electric (TE) and transverse magnetic (TM) modes. In TE modes the z-component of the electric field is zero, and for the TM modes the z-component of the magnetic field is zero. The energy carried by TE and TM modes propagate at the group velocity, which is slower than c, and the transverse electric and magnetic fields are not necessarily in phase. Hollow waveguides only allow certain frequencies propagate along the waveguide. The minimum frequency, or cutoff frequency ω_{mn}, usually corresponds to a wavelength twice the largest dimension multiplied by some geometric factor. Frequencies below this will attenuate as $e^{-\omega_{mn}z/c}$, where z is the distance from the source. For a square waveguide, the attenuation goes as $e^{-z\pi/a}$, where a is the largest dimension.

A pair of conductors or a waveguide that is not hollow will support TEM modes, in which both the electric and magnetic fields are purely transverse to the axis of the conductors. This type of waveguide is called a transmission line, and includes the two-wire example. TEM modes have no cutoff frequency and the fields propagate at the speed of light. The electric and magnetic fields remain in phase at all times (for perfectly conducting waveguides). In principal, TE and TM modes could also propagate, but if the separation of the conductors is small compared to the wavelength of the radiation, the radiation frequency is below the TE and TM mode cutoff frequencies and modes other than the TEM are attenuated.

If the waveguide is not infinite, but is long compared to all other dimensions, radiation propagating down the waveguide will be a mode of the waveguide.

In Project8, the wavelength λ of radiation produced by an electron circling in a 1 T field will be about 1.1 cm. The cyclotron radius R_c will be about 0.5 mm. Therefore, using a wire spacing d approximately 2 mm will pick up radiation in the near zone, where $R_c << d << \lambda$. This also ensures that only TEM modes propagate, since all other modes will die away as

$e^{-z\pi/d}$. Using a 5 cm long antenna, other modes will be negligible a few wire spacings from the electron.

For general waveguides, Maxwell's equations separate into transverse field components, $\mathbf{E}_t$ and $\mathbf{H}_t$, and fields parallel to the wire axis, E_z and H_z, which are zero for TEM modes. The fields are assumed to have periodic time and z-dependence: $\mathbf{E}(x,y,z,t) = \mathbf{E}_\lambda(x,y)e^{\pm ikz - i\omega t}$. For TEM modes, the transverse electric and magnetic fields are connected by simplified Maxell's equations (Jackson [9] eqns. 8.23–8.25):

$$\frac{\partial \mathbf{E}_t}{\partial z} = -i\omega \hat{\mathbf{z}} \times \mathbf{B}_t \tag{1}$$

$$\frac{\partial \mathbf{B}_t}{\partial z} = -i\mu\epsilon\omega \hat{\mathbf{z}} \times \mathbf{E}_t \tag{2}$$

Then the solutions, propagating as $e^{\pm ikz}$, are (eqn. 8.28 from Jackson):

$$k = \omega\sqrt{\mu\epsilon}$$

$$\mathbf{H}_t = \pm\sqrt{\frac{\epsilon}{\mu}}\hat{\mathbf{z}} \times \mathbf{E}_t = \pm\frac{1}{Z_0}\hat{\mathbf{z}} \times \mathbf{E}_t, \tag{3}$$

The quantity $\sqrt{\frac{\mu}{\epsilon}}$ is the free-space wave impedance, denoted as Z_0. These transverse fields can be generalized to TE or TM modes by replacing Z_0 with the TE or TM wave impedance Z_λ, which is frequency-dependent.

Finding the TEM transverse fields is easy, they are solutions to the analogous two dimensional electrostatic problem. For example, the parallel-wire transmission line solutions are the same as two infinite line charges and line currents.

II. THE JACKSON FIELD METHOD

Following the derivation of Gray Rybka [12] and Jackson [9], we can calculate the power from a current density inside a waveguide. The most general current density for a single electron is:

$$\mathbf{J}(\mathbf{r},t) = q\mathbf{v}_0(t)\delta[\mathbf{r} - \mathbf{r}_0(t)] \tag{4}$$

Close to the current density, the fields are complicated, but far from the current density (greater than a few dimensions of the waveguide) only modes of the waveguide propagate. We wish to calculate the power transmitted down the waveguide, so we must find the modes of the waveguide and the amplitude of each mode excited by the source.

Far from the source, where evanescent modes have died away, the total field propagating in the $\pm$ z-directions can be written as a sum over all of the waveguide modes (Jackson Eq. 8.141 and 8.129):

$$\mathbf{E}^{\pm}(x,y,z,t) = \sum_{\lambda} A_{\lambda}^{\pm} e^{\pm ik_{\lambda}z} \left(\mathbf{E}_{t\lambda}(x,y) \pm E_{z\lambda}(x,y)\hat{\mathbf{z}}\right) * c(t) \tag{5}$$

$$\mathbf{H}^{\pm}(x,y,z,t) = \sum_{\lambda} A_{\lambda}^{\pm} e^{\pm ik_{\lambda}z} \left(\pm\frac{1}{Z_{\lambda}}\hat{\mathbf{z}} \times \mathbf{E}_{t\lambda}(x,y) + H_{z\lambda}(x,y)\hat{\mathbf{z}}\right) * c(t). \tag{6}$$

The time dependence and frequency of oscillation ω of the fields are determined by the source. In this expansion, the transverse and longitudinal field components must be normalized according to Jackson eqns. 8.131, 8.132, and 8.134. The transverse normalization is:

$$\int_A \mathbf{E}_{t\lambda}(x,y) \cdot \mathbf{E}_{t\mu}(x,y)da = \delta_{\lambda\mu} \tag{7}$$
$$\int_A \mathbf{H}_{t\lambda}(x,y) \cdot \mathbf{H}_{t\mu}(x,y)da = \frac{1}{Z_{\lambda}^2}\delta_{\lambda\mu}$$

The amplitude and time dependance of each mode can be calculated from the Lorentz reciprocity principle [6] and is given by Jackson eqn. 8.146, generalized for arbitrary time dependence :

$$A_{\lambda}^{\pm} * c(t) = -\frac{Z_{\lambda}}{2}\int_V \mathbf{J}(r,t) \cdot \mathbf{E}_{\lambda}(x,y)e^{\mp ik_{\lambda}z}d^3x \tag{8}$$

Once the modes of a waveguide ($\mathbf{E}_{\lambda}(x,y)$) are found, calculating the power transmitted from a given current density reduces to evaluating these coefficients. Notice that the positive z-direction coefficient depends on the current density coupling to the field propagating in the negative z-direction. This expression for the coefficient is valid for any waveguide.

Integrating over volume cancels the delta functions in the current density. Then the coefficients are:

$$A_{\lambda}^{\pm} * c(t) = -\frac{qZ_{\lambda}}{2}\left[\mathbf{v}_t(t) \cdot \mathbf{E}_{t\lambda}(r_0(t)) \pm v_z(t) \cdot E_{z\lambda}(r_0(t))\right] e^{\mp ik_{\lambda}z_0(t)} \tag{9}$$

where the field is evaluated at the position of the electron. As we will see later in this document, the main time dependence comes from the velocity of the electron; if the electron has cyclotron frequency ω, the field amplitude oscillates with ω. In addition, if the electric field $E_{t\lambda}(r_0(t))$ changes over the trajectory of the electron, higher harmonics of the cyclotron frequency are excited. We will see that analytically the Doppler shift comes from a Fourier transform over the $e^{\mp ik_{\lambda}z_0(t)}$ term.

If the electric field is approximately constant over the current density, it can be Taylor expanded about the center of the orbit. In the first term of this multipole expansion (eqn. 9.69 of Jackson), the electric field can be evaluated at the center of the orbiting electron. This approximation is used by G. Rybka [12].

A. Calculating Power

The total energy W_t radiated in one direction by a charge is the time integral of the power radiated, where the power is written in either the time or frequency domain.

$$W_t = \int_{-\infty}^{\infty} P(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} P(\omega)d\omega = \int_{-\infty}^{\infty} P(f)df \tag{10}$$

Note that the frequency-domain power is not the Fourier transform of the time-domain power; the frequency-domain power is calculated from the frequency-domain fields.

In a Project8 style experiment, we will record some value as a function of time, either $I(t)$, $V(t)$, or $P(t)$ (in units of current/voltage/energy per second) and the analysis will include a Fourier transform. To detect a signal, $P(f)$ (in units of energy per Hertz) must be larger than the noise. In this section, we analytically calculate $P(t)$ and $P(f)$.

In general, the power transmitted in the $\pm$ z-direction is a spatial integral over the Poynting vector dotted into the $\pm$ z-direction. Therefore, only the transverse fields contribute and this can be calculated from the coefficients in Eq. 9.

$$P^{\pm}(t) = \pm \int_A \mathbf{E}_t^{\pm}(r,t) \times \mathbf{H}_t^{\pm}(r,t) \cdot \hat{\mathbf{z}} da = \sum_{\lambda} \frac{1}{Z_\lambda}[A_\lambda^{\pm} * c(t)]^2 \tag{11}$$

where we have used the fact that the transverse fields are orthonormal. From here on, we drop the direction superscripts and write the power radiated in one direction as $P(t)$ when the power radiated in each direction is the same. When a doppler shift is included, the power in each direction is not the same, and in this case we keep the direction superscripts.

To find the power in frequency space, we take a Fourier transform of the fields in Eq. 5:

$$E^{\pm}(r,\omega) = \sum_{\lambda} A_\lambda^{\pm} e^{\pm ik_\lambda z} \left(\mathbf{E}_{t\lambda}(r_t) \pm E_{z\lambda}(r_t)\hat{\mathbf{z}}\right) * C(\omega) \tag{12}$$

$$H^{\pm}(r,\omega) = \sum_{\lambda} A_\lambda^{\pm} e^{\pm ik_\lambda z} \left(\frac{1}{Z_\lambda}\hat{\mathbf{z}} \times \mathbf{E}_{t\lambda}(r_t) + H_{z\lambda}(r_t)\hat{\mathbf{z}}\right) * C(\omega) \tag{13}$$

where

$$A_\lambda * C(\omega) = \int_{-\infty}^{\infty} A_\lambda * c(t) e^{i\omega t} dt \tag{14}$$

$$A_\lambda * c(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A_\lambda * C(\omega) e^{-i\omega t} d\omega \tag{15}$$

$$\tag{16}$$

Therefore, the spectral components of the power radiated in the $\pm$ z-direction is:

$$P^{\pm}(\omega) = \pm \int_A \mathbf{E}_t^{\pm}(r,\omega) \times \mathbf{H}_t^{\pm}(r,\omega) \cdot \hat{\mathbf{z}} da = \sum_\lambda \frac{1}{Z_\lambda} [A^{\pm} * C(\omega_m)]^2 \tag{17}$$

From here on, we drop the direction superscripts and write the power radiated in one direction as $P(\omega)$ when it is the same in each direction.

1. *Time-averaged Power*

The time-averaged power $\bar{P}$ (in units of energy per time) is the integral of the power emitted over one cycle divided by the period. This quantity was calculated by Gray for the two-wire transmission line:

$$\bar{P} = \frac{1}{\tau} \int_0^\tau P(t) dt \tag{18}$$

Jackson often writes the field time-dependence as $e^{i\omega t}$, where it is assumed that the real part will be taken. In this special case, the time-averaged power can be calculated with

$$\bar{P} = \frac{1}{2} \sum_\lambda \frac{1}{Z_\lambda} |A_\lambda|^2 \tag{19}$$

because the time average of $\cos^2 \omega t$ is $1/2$. If only one harmonic is excited, the time-averaged power is the same as the power radiated at that single harmonic.

2. *Periodic fields with Finite Duration*

Following Section V of Collin [5], we represent periodic fields (with period $\tau = 2\pi/\omega_c$) as a Fourier series. The time-dependence of the fields can be written as:

$$c(t) = \sum_{m=-\infty}^{\infty} C(\omega_m) e^{i\omega_m t} \tag{20}$$

where $\omega_m = m\omega_c$. The Fourier components of the fields are

$$C(\omega_m) = \frac{1}{\tau}\int_0^\tau c(t)e^{-i\omega_m t}dt \tag{21}$$

Then the field is:

$$\mathbf{E}(r,t) = \sum_\lambda \sum_{m=-\infty}^{\infty} (A_\lambda^\pm e^{\pm ik_\lambda z})\,(\mathbf{E}_{t\lambda}(r_t) \pm E_{z\lambda}(r_t)) * C(\omega_m)e^{i\omega_m t} \tag{22}$$

If the time duration T is much longer than the period, $T > 10\tau$, each harmonic is well separated and the field coefficients in frequency space are:

$$A_\lambda * C(\omega) = \sum_{m=-\infty}^{\infty} A_\lambda C(\omega_m)\delta(f - f_m) \tag{23}$$

The frequency-space power is:

$$P(\omega) = \sum_\lambda \frac{1}{Z_\lambda} \sum_{m=-\infty}^{\infty} [A * C(\omega_m)]^2 \delta(f - f_m)^2 \tag{24}$$

To find the time-average power, we integrate over all frequencies to find the total energy radiated and divided by the time duration T.

$$\bar{P} = \frac{W_t}{T} = \sum_\lambda \frac{1}{Z_\lambda} \sum_{m=-\infty}^{\infty} [A * C(\omega_m)]^2 \tag{25}$$

Note that in this definition of the Fourier transform, integrating over $d\omega$ cancels a factor of 2π.

III. WAVEGUIDE EXAMPLES

A. Example: Square Waveguide with Higher Harmonics

The time-average power radiated into the dominant mode (TE_{10}) of a square waveguide by a charge circulating within it was calculated from a vector potential by Collin [5]. The charge and resulting field have period τ, and harmonics higher than the cyclotron frequency are excited. Using the method outlined above, we calculate the time-averaged power from the spectral components of the power with the Jackson method and compare to Collin's results.

1. *Electric Fields*

Far from the electron, the electric field in the waveguide is the sum of all propagating modes as in Eq. 5. For simplicity, we follow Collin and only consider the dominant mode, the TE_{10} mode, which is given by Jackson eqn. 8.46. Renormalized as in Eq. 7, the transverse field of the TE_{10} mode is:

$$\mathbf{E}_t(r_t) = \sqrt{\frac{2}{ab}} \sin\left(\frac{\pi x}{a}\right) \hat{\mathbf{y}}. \tag{26}$$

Here, a and b are the dimensions of the square waveguide, with a being the larger dimension. One corner of the waveguide sits at the origin.

For an electron in circular orbit around the point x_0, y_0 with radius R_c, the position and velocity are:

$$\mathbf{r}_0(t) = [x_0 + R_c \cos(\omega_c t)]\hat{\mathbf{x}} + [y_0 + R_c \sin(\omega_c t)]\hat{\mathbf{y}} \tag{27}$$

$$\mathbf{v}(t) = -R_c\omega_c \sin(\omega_c t)\hat{\mathbf{x}} + R_c\omega_c \cos(\omega_c t)\hat{\mathbf{y}} \tag{28}$$

$$\tag{29}$$

Using the transverse electric field and the electron velocity, the coefficient of the TE_{10} mode can be calculated from Eq. 9. The coefficient is:

$$A_{10} * c(t) = -\frac{qZ_{10}}{2} v_y E_y = \frac{qZ_{10}}{2} \omega_c R_c \cos(\omega_c t) \sqrt{\frac{2}{ab}} \sin\left(\frac{\pi(x_0 + R_c \cos\omega_c t)}{a}\right) \tag{30}$$

2. *Power for Circulating Charge*

The total time-averaged power radiated in either direction of a rectangular waveguide in the TE_{10} mode calculated from the instantaneous power (Eq. 18) is:

$$\bar{P}_{10} = \frac{1}{\tau}\int_0^\tau \frac{1}{Z_{10}}[A * c(t)]^2 dt = \frac{Z_{10}}{4} q^2 \omega_c^2 R_c^2 \frac{2}{ab}\frac{1}{\tau}\int_0^\tau \cos^2(\omega_c t)\sin^2\left(\frac{\pi(x_0 + R_c\cos\omega_c t)}{a}\right) dt \tag{31}$$

This can be Taylor expanded about the center of the orbit. After integrating over $\cos^2 \omega_c t$, this simplifies to:

$$\bar{P} = \frac{Z_0 \omega^3 q^2 R^2}{8ck_\lambda} \frac{2}{ab} \sin^2(\frac{\pi x_0}{a}) \tag{32}$$

This is the first term in a Taylor series expansion of the total power.

To find the power in frequency space, we use the Fourier series method of Collin outlined in Section II A 2. First, we find the Fourier coefficients of the fields as in Eq. 21:

$$\begin{aligned} A_{10} * C(\omega_m) =& \ \tfrac{1}{\tau}\int_0^\tau A_{10} * c(t) e^{-i\omega_m t} dt \\ =& \ \tfrac{Z_{10}}{2} q\omega_c R_c \sqrt{\tfrac{2}{ab}} \tfrac{1}{\tau}\int_0^\tau \cos(\omega_c t) \sin\left(\tfrac{\pi(x_0 + R_c \cos\omega_c t)}{a}\right) e^{-i\omega_m t} dt \end{aligned} \tag{33}$$

Collin performs the same integral without approximating the field at the center of the orbit. Using his result for the integral, his equation 46, we calculate the total time-averaged power from the frequency space power (from Eq. 25) for $\lambda = \mathrm{TE}_{10}$:

$$\bar{P}_{10} = \frac{1}{Z_{10}} \sum_{m=-\infty}^{\infty} [A^{\pm} * C(\omega_m)]^2 = \sum_{m=-\infty}^{\infty} \frac{Z_0 m \omega_c^3 R_c^2 q^2}{4ck_\lambda} \frac{2}{ab} J_m'(\frac{\pi v}{\omega_c a})^2 \begin{cases} \sin^2(\frac{\pi x_0}{a}) & \text{m odd} \\ \cos^2(\frac{\pi x_0}{a}) & \text{m even} \end{cases} \tag{34}$$

in agreement with Collin. The derivative of the Bessel function comes from integrating the field over the orbit, so clearly the field changing over the orbit introduces higher harmonics within the TE_{10} mode.

If $R_c << a$, the derivative of the Bessel function is approximately 1/2 for the first positive and negative harmonic. This is the same as approximating the field at the center of the orbit. If we square the Bessel functions and sum the m=+1 and m=−1 terms of Eq. 34, we recover the time-averaged power calculated in the time domain of Eq. 32. Alternatively, if we approximate the field in Eq. 33 at the center of the orbit, only the first harmonic contributes, and we also recover Eq. 32. Obviously, keeping higher-order terms in the Taylor series expansion of the field in Eq. 33 or the Bessel function in Eq. 34 would produce the higher harmonics.

B. Example: Circular Waveguide with Doppler Shift

The spectral components of the power transmitted in a circular waveguide of radius b has been calculated by Buts [3] and includes the Doppler shift in frequency. Be aware that Buts uses a different definition of the Fourier transform than this document and uses Gaussian units instead of SI.

1. *Electric Fields*

The dominant mode of a circular waveguide is the TE_{11} mode[6, 7], so we investigate all TE modes. The normalized TE-mode transverse electric fields are:

$$\mathbf{E}_{t,nj}(r_t) = \frac{cN_j\sqrt{2}}{\omega_j}\left[\frac{in}{\rho}J_n(\frac{\omega_j\rho}{c})\hat{\rho} - \frac{\omega_j}{c}J_n'(\frac{\omega_j\rho}{c})\hat{\phi}\right]e^{-in\phi} \tag{35}$$

where ω_j are the cutoff frequencies, defined by the jth zero of the derivative of the Bessel function, $J_n'(\frac{\omega_j b}{c}) = 0$. The normalization N_j is defined by Buts.

For an electron in circular orbit around the origin with radius R_c and with a velocity in the +z-direction, the position and velocity are:

$$\mathbf{r}_0(t) = R_c\hat{\rho} + \omega_c t\hat{\phi} + \beta ct\hat{\mathbf{z}} \tag{36}$$

$$\mathbf{v}(t) = R_c\omega_c\hat{\phi} + \beta c\hat{\mathbf{z}} \tag{37}$$

The coefficients are:

$$A^{\pm}_{nj} * c(t) = -\frac{qZ_{nj}}{2}v_\phi E_\phi e^{\mp ik\beta ct} = \frac{qZ_{nj}}{2}R_c\omega_c N_j\sqrt{2}J_n'\left(\frac{\omega_j R_c}{c}\right)e^{-i(n\omega_c \pm k\beta c)t} \tag{38}$$

If the electron orbit is centered around the axis of the waveguide the radius and therefore the Bessel function are constants. All time dependence is in the exponent. In this example, the higher harmonics of the cyclotron frequency occur for higher modes, when n$>$ 1.

2. *Power from a Charge in a Helical Orbit*

Using the method of Jackson, the first step in calculating $P(\omega)$ is to Fourier transform $*c(t)$, remembering that k is a function of ω. We look at the power in both directions, with the top sign corresponding to the +z-direction. Buts performs this integral in his equation 3.52, and the result is:

$$\int_{-\infty}^{\infty} e^{it(\omega - n\omega_c \mp \beta\sqrt{\omega^2-\omega_j^2})}dt = 2\pi\sum\frac{\delta(\omega-\omega_j^*)}{|1\mp\beta\beta_{ph}|} \tag{39}$$

The normalization of the delta function comes from the derivative of the integrand evaluated at the roots ω_j^*, where β_{ph} is the phase velocity is units of c: $\beta_{ph} = \omega_j^*/kc$. The sum is over the two roots of the equation:

$$\omega - n\omega_c \mp \beta\sqrt{\omega^2-\omega_j^2} = 0 \tag{40}$$

The roots are the same regardless of the sign of $\beta \cdot \mathbf{k}$, and are:

$$\omega_j^* = \frac{n\omega_c \pm \beta\sqrt{n^2\omega_c^2 + (\beta^2 - 1)\omega_j^2}}{1 - \beta^2} \tag{41}$$

Therefore, it appears the radiation in both directions includes frequencies at both roots, or a doppler shift both up and down in frequency. Two effects are at work in shifting the frequency. First, the velocity of the particle introduces a Doppler shift; if $\beta \to 0$, this reduces to $n\omega_c$. Second, the cutoff frequencies affect the allowed frequencies; if $\omega_j \to 0$ we recover the standard Doppler shift equation.

The spectral components of the power calculated with Jackson's method are:

$$P(f) = \sum_{nj} \frac{1}{Z_{nj}} [A_{nj} * C(f)]^2 = \sum_{nj} \frac{N_j^2 q^2 R_c^2 \omega_c^2}{2\epsilon_0 c} \frac{\beta_{ph}}{(\beta\beta_{ph} \mp 1)^2} \left[J_n'(\frac{\omega_j R_c}{c}) \right]^2 \delta(f - f_j^*)^2 \tag{42}$$

Therefore, the total energy radiated is:

$$W_t = \int P(f) df = \sum_{nj} \frac{N_j^2 q^2 R_c^2 \omega_c^2}{2\epsilon_0 c} \frac{\beta_{ph}}{(\beta\beta_{ph} \mp 1)^2} \left[J_n'(\frac{\omega_j R_c}{c}) \right]^2 \tag{43}$$

This has units of power, not energy, and does not match that calculated by Buts.

Buts has (in Gaussian units):

$$W_t = \sum_{nj} P(\omega_j^*) = \sum_{nj} \frac{N_j^2 q^2 \pi R_c \omega_c \beta_{ph}}{4c\sqrt{\beta_{ph}^2 - 1}|\beta\beta_{ph} - 1|} \left[J_n'(\frac{\omega_j R_c}{c}) \right]^2 \tag{44}$$

This has units of ergs/cm, not energy.

Reproducing Buts' calculation in SI units, I get:

$$W_t = \sum_{nj} \frac{N_j^2 q^2 R_c^2 \omega_c^2}{4\epsilon_0 c} \frac{\beta_{ph}}{(\beta\beta_{ph} \mp 1)^2} \left[J_n'(\frac{\omega_j R_c}{c}) \right]^2 \tag{45}$$

This is twice what I calculate with the Jackson method, and has units of power.

The discrepancy is being investigated.

IV. TRANSMISSION LINES

As mentioned in the introduction, transmission lines consisting of two conductors will support TEM modes if:

- $d << \lambda$: the conductor separation is much smaller than the wavelength

- $L >> \lambda$: the conductors are much longer than the wavelength
- $R \simeq 0$: perfect, lossless conductors, or low-loss conductors
- $G \simeq 0$: perfect, lossless dielectrics, or low-loss dielectrics

Although the signal on transmission lines is carried by the fields between the conductors, an equivalent circuit which obeys the transmission line equations can describe the same signal in terms voltage and current. The transmitted power calculated from the Poynting vector is the same as that calculated from the transmission line equations, $P(t) = I(t)V(t) = V^2(t)/Z_c$. We use this equivalency to calculate the voltage from the power.

A. Transmission Line Equations

The transmission line equations are coupled ODE for the current and voltage:

$$\frac{\partial V}{\partial z} = -RI(z) - i\omega LI(z) \tag{46}$$

$$\frac{\partial I}{\partial z} = -GV(z) - i\omega CV(z) \tag{47}$$

The line is characterized by:

- L = inductance per unit length of the conductor pair (H/cm)
- C = capacitance per unit length of the conductor pair (F/cm)
- R = resistance per unit length of the conductor pair (Ω/cm)
- G = conductance per unit length of the conductor pair (S/m)

A lossless line is constructed of a perfect conductor (R=0) and a perfect dielectric (G=0).

The equations can be written as second order ODEs:

$$\frac{\partial^2 V}{\partial z^2} = \gamma^2 V(z) \tag{48}$$

$$\frac{\partial^2 I}{\partial z^2} = \gamma^2 I(z) \tag{49}$$

with $\gamma = \alpha + i\beta = \sqrt{(R + i\omega L)(G + i\omega C)}$. The solutions are:

$$V(z,t) = (V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z})e^{i\omega t} \tag{50}$$

$$I(z,t) = (I_0^+ e^{-\gamma z} - I_0^- e^{+\gamma z})e^{i\omega t} \tag{51}$$

These are waves traveling in the ± z direction with attenuation constant α and phase constant β.

The ratio of voltage to current is the characteristic impedance of the transmission line, Z_c.

$$Z_c = \frac{V^+}{I^+} = \sqrt{\frac{(R + i\omega L)}{(G + i\omega C)}} \tag{52}$$

In general this is a complex quantity, meaning the current and voltage can be out of phase.

1. Low-loss Lines

Low-loss lines are a special case of the general transmission line, where $R << \omega L$ and $G << \omega C$. To first order, the phase contant $\beta = \omega\sqrt{LC}$. This is linear in frequency, analogous to the loss-free field case, and $LC = \mu\epsilon = 1/v^2$. The characteristic impedance simplifies to:

$$Z_c = \sqrt{\frac{L}{C}} = \frac{\epsilon}{C} Z_0 \tag{53}$$

and is purely real so the current and voltage are in phase. This is the free-space wave impedance Z_0 multiplied by a geometric factor. Care should be taken not to confuse wave impedance Z_0 and characteristic impedance Z_c. The attenuation constant in a low-loss line is non-zero, $\alpha = (R/Z_c + GZ_c)/2$. In the UW prototype, the copper will have some resistance R and the dielectric will have some shunt conductance G.

The resistance per unit length $R = \rho/\delta * K$ is the resistivity ρ divided by the skin depth δ multiplied by a geometric factor K that depends on the transmission line. The skin depth is frequency dependent: $\delta = \sqrt{2\rho/\omega\mu}$, where μ is the permeability of copper. At 27 GHz, the skin depth is 0.15 μm for copper, much smaller than the proposed wire radius of 50 μm. The resistivity is temperature dependent: we use $\rho = 2.15$ nΩ*m for copper at 80 K. The resistance for each line is defined later in this document.

If the conductors (with capacitance C) are surrounded by dielectrics with permittivity $\epsilon = \epsilon' - i\epsilon''$, the shunt conductance per unit length $G = \omega\epsilon''/\epsilon' C = \omega \tan\delta C$. The UW prototype is supported by—but not surrounded by—a Rogers Duroid 5870 dielectric with $\tan\delta = 0.0012$. The capacitance for each line is defined later in this document.

If the radiation must travel a distance z_{ant} to reach the antenna readout end of the line, the voltage is attenuated by:

$$V(z_{ant}) = V_0 e^{-\alpha z_{ant}} \tag{54}$$

$$P = V_0^2/Z_c e^{-2\alpha z_{ant}} \tag{55}$$

For the transmission lines of total length 7.2 cm considered for the UW prototype, we will operate in this low-loss regime. Since both the skin depth and the shunt conductance are frequency-dependent, the attenuation coefficient is also, but changes little over the frequency range of interest.

2. *Reflections*

Real transmission lines are not infinite; at the end reflections can occur. If waves are initially launched in both directions from a stationary electron source at the origin, the waves have the same amplitude:

$$V(z,t) = (V_0 e^{-\gamma z} + V_0 e^{+\gamma z})e^{i\omega t} \tag{56}$$

$$I(z,t) = (I_0 e^{-\gamma z} - I_0 e^{+\gamma z})e^{i\omega t} \tag{57}$$

If the +z direction end of the transmission line is not infinite or matched, the wave in the +z direction will travel a distance $2z_{\text{ref}}$ and become a wave traveling in the –z-direction:

$$V_{\text{ref}}(z,t) = \Gamma_L V_0 e^{+\gamma z} e^{-\alpha 2 z_{\text{ref}}} e^{i\omega(t-2z_{\text{ref}}/v)} \tag{58}$$

$$I_{\text{ref}}(z,t) = -\Gamma_L I_0 e^{+\gamma z} e^{-\alpha 2 z_{\text{ref}}} e^{i\omega(t-2z_{\text{ref}}/v)} \tag{59}$$

This includes a complex reflection coefficient Γ_L, a change of direction, attenuation, and a time delay. The amplitude and phase change of the reflected wave depends on how the line is terminated with load impedance Z_L. Written in terms of the normalized load impedance Z_L/Z_c, the reflection coefficient is:

$$\Gamma_L = V_0/V_{\text{ref}} = \frac{\bar{Z}_L - 1}{\bar{Z}_L + 1} \tag{60}$$

A few cases are important.

- $Z_L = Z_c$: matched load, $\Gamma_L = 0$ and no reflections occur

- $Z_L = 0$: shorted wires, $\Gamma_L = -1$, full reflection 180° out of phase
- $Z_L = \infty$: open wires, $\Gamma_L = 1$, full reflection with no phase delay

At the readout end of the antenna (in the -z direction) the combined waves are:

$$V(z,t) = V_0 e^{+\gamma z}(1 + \Gamma_L e^{-\alpha 2 z_{\text{ref}}} e^{-i\omega 2 z_{\text{ref}}/v}) e^{i\omega t} \tag{61}$$

$$I(z,t) = -I_0 e^{+\gamma z}(1 + \Gamma_L e^{-\alpha 2 z_{\text{ref}}} e^{-i\omega 2 z_{\text{ref}}/v}) e^{i\omega t} \tag{62}$$

Clearly the time delay introduces an interference term. The reflected wave can constructively interfere with another wave emitted $2z_{\text{ref}}/v$ later to create 4 times the nominal power, or destructively interfere such that no power is transmitted.

B. Parallel Wire Transmission Line

To find the transverse fields of the parallel-wire transmission line, we solve the electrostatic problems of uniform line charges and current charges of infinite length in the z-direction. For all transmission lines, the induced charges and currents fulfill the boundary conditions, that $E_z = 0$ and $B_\perp = 0$ at the conductor surface. Referring back to Eq. 3, the electric and magnetic fields are not independent, nor are the current and charge. The ratio of charge to current, λ/i, is $\sqrt{\mu\epsilon} = \sqrt{LC}$ at all points along the wire. These electrostatic examples assume infinite wires of uniform charge and current. In the transmission line, the charge and current will not be uniform, but will be periodic and opposite sign on each wire, canceling all longitudinal fields. The transverse fields are the same as the two dimensional solution of this electrostatic problem.

1. Electric Fields

The electric field from two oppositely charged conducting cylinders (two wires) is solved in Section 4.6.3 of Haus and Mechler [8]. The charge per unit length on the right wire is λ, and on the left wire is $-\lambda$. The field at a point P depends on the separation of the wires to the point, r_1 and r_2, as in Fig. 1. The potential is the superposition of two wires:

$$\Phi = \frac{-\lambda}{2\pi\epsilon} \ln \frac{r_1}{r_2}, \tag{63}$$

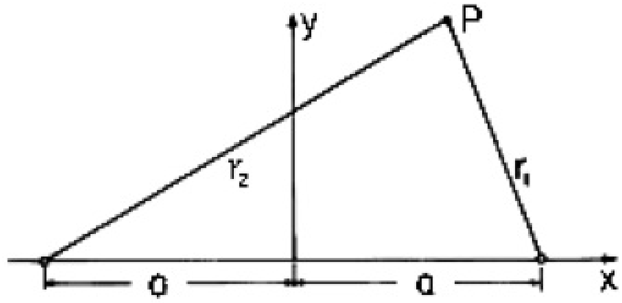

FIG. 1. The field at a point P a distance r_1 from a wire with charge per unit length $+\lambda$ and a distance r_2 from a wire with charge per unit length $-\lambda$ is the superposition of the field from each wire.

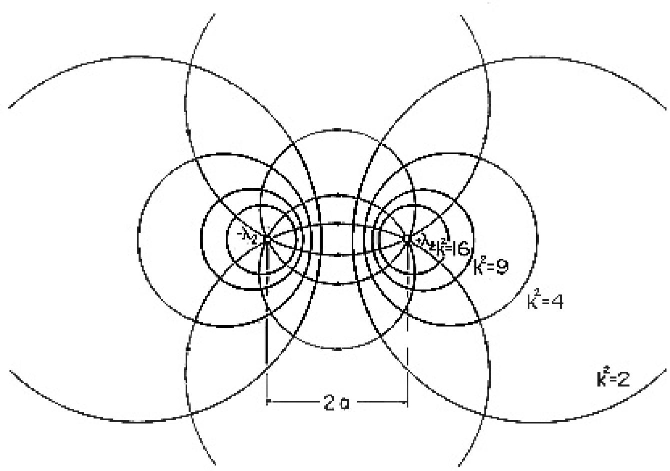

FIG. 2. Electric field and equipotential lines resulting from two wires separated by distance 2a. The right wire has a charge per unit length of $+\lambda$, the left wire has charge per unit length of $-\lambda$.

and the electric field is the superposition of two wires:

$$\mathbf{E}(x, y) = -\boldsymbol{\nabla} V = \frac{\lambda}{2\pi\epsilon}\left(\frac{\hat{\mathbf{r}}_1}{r_1} - \frac{\hat{\mathbf{r}}_2}{r_2}\right). \tag{64}$$

Figure 2 shows the electric field and potential. The equipotentials are circles around the wires, so generalizing from wires to cylinders aligned with an equipotential line is simple. Figure 3 shows two cylinders with radius R separated by a distance $2l$ are equivalent to wires separated by a distance $2a$, where $a^2 = l^2 - R^2$. The electric field is the same, except the distance to the point P is calculated from the equivalent wire position $\pm a$ instead of the center of the cylinder at $\pm l$.

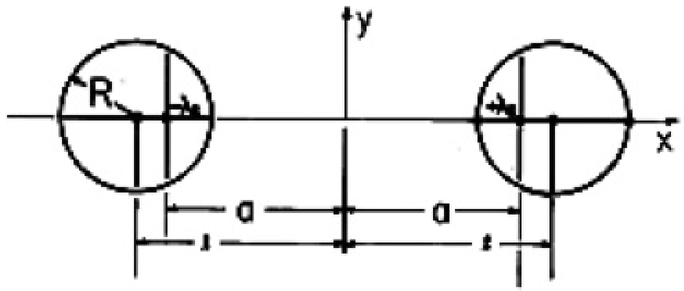

FIG. 3. Two cylinders, radius R, separated by a distance $2l$, create fields equivalent to two wires with zero radius separated by distance $2a$, where $a^2 = l^2 - R^2$.

2. Magnetic Fields

The magnetic field from two wires carrying current i is treated in Section 8.6.1 of Haus and Melcher [8]. The setup is the same as Fig. 1, with a current in the +z-direction in the right wire (out of the page) and a current in the −z-direction in the left wire. The vector potential is the superposition of two wires; it points opposite the direction of the current:

$$A_z = -\frac{\mu i}{2\pi}\ln\frac{r_1}{r_2} \tag{65}$$

The magnetic field is:

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} = \frac{\mu i}{2\pi}\left(\frac{\hat{\mathbf{z}} \times \hat{\mathbf{r}}_1}{r_1} - \frac{\hat{\mathbf{z}} \times \hat{\mathbf{r}}_2}{r_2}\right) \tag{66}$$

Figure 4 shows the resulting magnetic field. Again, the surfaces of constant A_z are circles, so we can easily calculate the field from cylinders with radius R by aligning the cylinder with a surface of constant A_z. Replace the cylinders at a position of $\pm l$ with wires at at position of $\pm a$, where $a^2 = l^2 - R^2$

3. Equivalent Circuit Parameters

The capacitance per unit length of these cylinders, expanded for small wire radius R, is also calculated in Section 4.6.3 of Haus and Mechler [8].

$$C = \frac{\lambda}{V} = \frac{\pi\epsilon}{\ln\left[\frac{l}{R} + \sqrt{(\frac{l}{R})^2 - 1}\right]} \simeq \frac{\pi\epsilon}{\ln\frac{2l}{R}} \tag{67}$$

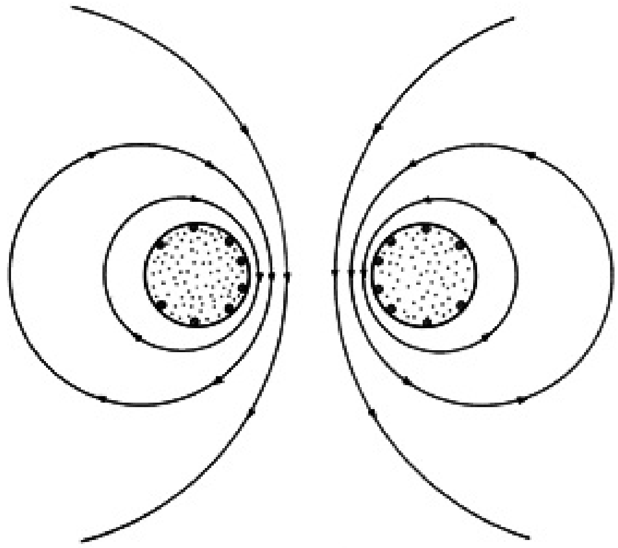

FIG. 4. Magnetic field and surfaces of constant A_z resulting from two current-carrying cylinders separated by distance $2l$, equivalent to two wires separated by $2a$. The right wire has a current of i out of the page, the left wire has a current of $-i$.

For low-loss parallel-wires, the characteristic impedance is:

$$Z_c = \sqrt{\frac{L}{C}} = \frac{\epsilon}{C} Z_0 = \frac{Z_0}{\pi} \ln \left[\frac{l}{R} + \sqrt{(\frac{l}{R})^2 - 1} \right] \simeq \frac{Z_0}{\pi} \ln \frac{2l}{R} \tag{68}$$

The inductance per unit length of the cylinder, evaluated for small wire radius, is also discussed in Section 8.6.1 of Haus and Melcher [8].

$$L = \frac{\mu}{\pi} \ln \left[\frac{l}{R} + \sqrt{\left(\frac{l}{R}\right)^2 - 1} \right] \simeq \frac{\mu}{\pi} \ln \frac{2l}{R} \tag{69}$$

The resistance per unit length of the cylinder is given by Collin [7] as:

$$R = \frac{\rho}{\pi \delta R} \frac{l/R}{[(l/R)^2 - 1]^{1/2}} \tag{70}$$

4. Power from a Charge in a Helical Orbit

Now we consider the power transmitted by a parallel-wire transmission line, where only TEM modes propagate. The electric fields in Eq. 5 simplify and we drop the sum over λ

and replace Z_0 with $\sqrt{\frac{\mu}{\epsilon}}$. Also, $\mathbf{H}_t$ can be written in terms of $\mathbf{E}_t$.

$$\mathbf{E}(x,y,z,t) = (A^+e^{ikz} + A^-e^{-ikz})\mathbf{E}_t(x,y) * c(t) \tag{71}$$

$$\mathbf{H}(x,y,z,t) = (A^+e^{ikz} - A^-e^{-ikz})\sqrt{\frac{\epsilon}{\mu}}\hat{\mathbf{z}} \times \mathbf{E}_t(x,y) * c(t). \tag{72}$$

For parallel wires, the transverse components are given by the electrostatic case and must be re-normalized. Using the equality of the work per unit length to assemble the line of charge (the capacitance) and the energy stored in the fields as suggested by Gray, we find the normalization:

$$\frac{\epsilon}{2}\int_A \mathbf{E}_t(x,y) \cdot \mathbf{E}_t(x,y)da = \frac{\lambda^2}{2C} \tag{73}$$

Therefore, the normalized E-field is:

$$\mathbf{E}_t(x,y) = \frac{1}{2\sqrt{\pi \ln\frac{2l}{R}}}\left(\frac{\hat{\mathbf{r}}_1}{r_1} - \frac{\hat{\mathbf{r}}_2}{r_2}\right) \tag{74}$$

Assuming the particle trajectory is a helix, we can write the particle position and velocity as complex quantities:

$$\mathbf{r}(t) = (x_0 + R_c e^{-i\omega_c t})\hat{\mathbf{x}} + (y_0 + iR_c e^{-i\omega_c t})\hat{\mathbf{y}} + \beta c t\hat{\mathbf{z}} \tag{75}$$

$$\mathbf{v}(t) = -i\omega_c R_c e^{-i\omega_c t}(\hat{\mathbf{x}} + i\hat{\mathbf{y}}) + \beta c\hat{\mathbf{z}} \tag{76}$$

Using the coordinates system defined in Fig. 1 and the effective cylinder separation defined in Fig. 3, we write the coefficients (given by Eq. 9 and Taylor expanding the field at the center of the orbit):

$$\begin{aligned} A^{\pm} * c(t) &= -\tfrac{qZ_0}{2}\mathbf{v}_t \cdot \mathbf{E}_t e^{\mp ik\beta ct} \\ &= \tfrac{qZ_0}{2} iR_c\omega_c \tfrac{1}{2\sqrt{\pi\ln\frac{2l}{R}}} e^{-i(\omega_c \pm k\beta c)t}\left[\tfrac{(x_0-a)+iy_0}{r_1^2} - \tfrac{(x_0+a)+iy_0}{r_2^2}\right] \end{aligned} \tag{77}$$

If the current density is small compared to the wire separation ($R_c << d$), approximating the fields at the center of the orbit should be reasonable.

Taking the real part, squaring, and time averaging over the $\cos^2((\omega_c \pm k\beta c)t)$ and $\sin^2((\omega_c \pm k\beta c)t)$ terms, the time average power is:

$$\bar{P} = \frac{1}{\tau}\int_0^{\tau}\frac{1}{Z_0}[A^{\pm} * c(t)]^2 dt = \frac{1}{32}\sqrt{\frac{\mu}{\epsilon}}\frac{q^2R_c^2\omega_c^2}{\pi\ln\frac{2l}{R}}\frac{[(r_2^2(x_0-a) - r_1^2(x_0+a))^2 - y_0^2(r_1^2 - r_2^2)^2]}{r_1^4 r_2^4} \tag{78}$$

Anywhere on the $x_0 = 0$ line, $r_1 = r_2$ and this reduces to:

$$\bar{P}_{x_0=0} = \frac{1}{8}\sqrt{\frac{\mu}{\epsilon}}\frac{q^2 R_c^2 \omega_c^2}{\pi \ln\frac{2l}{R}}\frac{a^2}{r^4} \tag{79}$$

At the position $x_0, y_0 = 0$ where $r = a$ we can compare to Gray. We evaluate the time average power using his definition of the wire spacing, $a = d/2$, and neglect the wire radius so that $l = a$.

$$\bar{P}_{x_0,y_0=0} = \frac{q^2 R_c^2 \omega_c^2}{2\pi\epsilon_0 c d^2 \ln\frac{d}{R}} \tag{80}$$

This is four times larger than Gray's calculation (Gray says this is most likely due his using $a = d$ instead of $a = d/2$).

For large wire spacings, the maximum power occurs when the electron is as close to one wire as possible, when $x_0 = l - R - R_c$. The other wire is neglected. This is two times smaller than Gray's calculation:

$$\bar{P}_{max} = \frac{q^2 R_c^2 \omega_c^2}{32\pi\epsilon_0 c (R_c + R)^2 \ln\frac{d}{R}} \tag{81}$$

As already mentioned, the total time-average power is the sum over all frequencies. The equations in this sections are evaluated at the center of the electron orbit, so are the first terms in a Taylor series expansion. These equations should tell us the approximate power radiated into the first harmonic.

Now, to calculate the spectral components of the power, $P(\omega)$, including the Doppler shift, we take the Fourier transform of Eq. 77 (where the field is approximated at the center of the orbit). To evaluate the power transmitted in the ±z-directions, we keep the direction superscripts. Remembering that k is a function of ω, the integral results in a delta function:

$$A^{\pm} * C(\omega) \propto \int_{-\infty}^{\infty} e^{-i(\omega_c \pm \omega\beta - \omega)t} dt = \frac{2\pi}{|1 \pm \beta|}\delta(\omega - \omega^*) \tag{82}$$

The frequencies that propagate, ω^*, are roots of the equation $\omega_c \pm \omega\beta - \omega = 0$. The choice of sign depends on the direction of the radiation. The power in the ±z-direction has frequency of $\omega^* = \omega_c/(1 \mp \beta)$, where $\omega_c = qB_0/\gamma mc$ and the particle velocity is in the +z-direction.

Plugging everything in:

$$P(\omega) = \frac{1}{16}\sqrt{\frac{\mu}{\epsilon}}\frac{q^2 R_c^2 \omega_c^2}{\pi \ln\frac{2l}{R}}\frac{[(r_2^2(x_0 - a) - r_1^2(x_0 + a))^2 - y_0^2(r_1^2 - r_2^2)^2]}{r_1^4 r_2^4}\frac{4\pi^2}{|1 \pm \beta|^2}\delta(\omega - \frac{\omega_c}{1 \mp \beta})^2 \tag{83}$$

The doppler shift is calculated somewhat differently in the code and is discussed in [10].

5. *Comparison to Free Radiation*

The Larmor formula gives the total instantaneous power radiated by a point charge in free space. The radiation is circularly polarized and goes like $\sin^2\theta$, where θ is the angle between the instantaneous acceleration and the direction. Since the magnitude of the acceleration for an electron in circular orbit is constant, the total time-average power is the same as the total instantaneous power. In SI units, this is:

$$P_L = \frac{q^2 a^2}{6\pi\epsilon_0 c^3} = \frac{q^2\omega^4 R_c^2}{6\pi\epsilon_0 c^3} \tag{84}$$

Comparing this to the time-averaged power radiated in both directions by an electron orbiting between two wires:

$$\frac{\bar{P}_{x_0,y_0=0}}{P_L} = \frac{6c^2}{d^2 \ln\frac{d}{R}\omega^2} = \frac{3\lambda^2}{d^2\ln\frac{d}{R}2\pi^2} \tag{85}$$

since $\omega\lambda = 2\pi c$. Plugging in the values $R = 50\mu$m and $\lambda = 1.1$cm, more power is collected by the wires from both directions than is radiated into free space when $d < 2.2$mm. At the minimum practical wire spacing, $2(R+R_c) = 1.1$ mm, the total power collected by the wires is 6.1 times the power radiated into free space.

C. Parallel Strip Transmission Line

An alternate choice for the UW Prototype antennna is parallel strips. Parallel strips that are wider than their separation have fields at their center that are approximately the same as an infinite parallel plate capacitor. For the ideal infinite parallel plate capacitor of length l and separation g, the electric field is:

$$E = \frac{V}{g}\hat{\mathbf{y}} \tag{86}$$

The capacitance is:

$$C = \epsilon\frac{l}{g} \tag{87}$$

And the field normalization can be computed from:

$$\frac{\epsilon}{2}\int_A \mathbf{E}_t(x,y)\cdot\mathbf{E}_t(x,y)da = \frac{1}{2}CV^2 \tag{88}$$

The normalized field between the ideal plates is:

$$E = \frac{1}{\sqrt{lg}}\hat{\mathbf{y}} \tag{89}$$

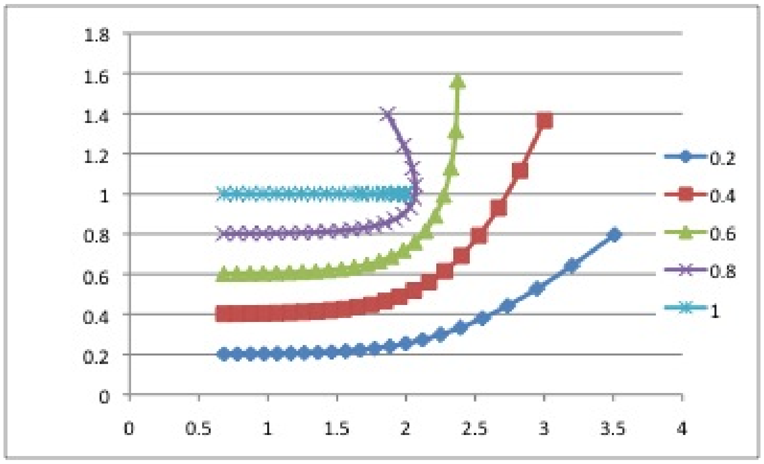

FIG. 5. Equipotential lines surrounding a finite parallel-strip capacitor with $l = 2g$ and the origin at the center of the strips. The potential is approximately linear within the central region.

1. Electric and Magnetic Fields

The fringing fields of a finite parallel-strip capacitor can be solved using conformal mapping, but the functional form is complicated. A simple set of transcendental equations describe the lines of force and equipotential near the edge of the strips[4].

$$x = \frac{g}{\pi}(\phi + e^{\phi}\cos\psi + 1) + l \tag{90}$$

$$y = \frac{g}{\pi}(\psi + e^{\phi}\sin\psi) \tag{91}$$

where x and y are coordinates measured from the center of the strips, ψ are the equipotential lines, and ϕ are the lines of force. The equipotential lines are shown in Fig. 5.

Since the fields extend through all space and are not confined within the strips, the real finite strips store more energy than the idealized plates without fringe fields. Therefore the real capacitance is larger than the ideal capacitance. The real capacitance is[11]:

$$C = \epsilon\frac{K}{K'} \tag{92}$$

where K and K' are the complete elliptic integrals of the second kind with modulus $\sqrt{b}$ and $\sqrt{1-b}$ respectively. The parameter b is a function of l and g and is defined in [11]. The real capacitance can be expressed as a correction times the ideal capacitance, $C = nC_0 = n\epsilon\frac{l}{g}$. Then the normalized electric field is:

$$E = \frac{1}{\sqrt{nlg}}\hat{\mathbf{y}} \tag{93}$$

where n is a correction for fringing. If $l = 2g$, $n = 1.8$. The magnetic field would be in the x-direction.

2. Equivalent Circuit Parameters

For the parallel strips, the capacitance per unit length and characteristic impedance are:

$$C = \frac{n\epsilon l}{g} \tag{94}$$

$$Z_c = \frac{\epsilon}{C} Z_0 = \frac{g}{nl} Z_0 \tag{95}$$

The resistance per unit length can be approximated by assuming current only flows in the skin depth δ:

$$R = \frac{\rho}{\delta l} \tag{96}$$

3. Charge in Helical Orbit

The coefficients of the efield are:

$$\begin{aligned} A^{\pm} * c(t) &= -\tfrac{qZ_0}{2} \mathbf{v}_t \cdot \mathbf{E}_t e^{\mp ik\beta ct} \\ &= \tfrac{qZ_0}{2} R_c \omega_c \tfrac{1}{\sqrt{nlg}} e^{-i(\omega_c \pm k\beta c)t} \end{aligned} \tag{97}$$

The time-averaged power is:

$$\bar{P} = \frac{1}{8} \sqrt{\frac{\mu}{\epsilon}} \frac{q^2 R_c^2 \omega_c^2}{nlg} \tag{98}$$

4. Comparison to Free Radiation

$$\frac{\bar{P}_{pp}}{P_L} = \frac{6\pi c^2}{8nlg\omega^2} = \frac{3\lambda^2}{16nlg\pi} \tag{99}$$

If we maintain $l = 2g$, the power collected by the parallel strips is greater than that radiated into free space when the gap is less than 1.4 mm. At the minimum practical wire spacing, $2(R + R_c) = 1.1$ mm, the total power collected by the wires is 2.0 times the power radiated into free space.

D. Coaxial Cable

1. Electric and Magnetic Fields

The normalized electric field is:

$$E = \frac{\sqrt{C/\epsilon}}{2\pi}\frac{\hat{\mathbf{r}}}{r} = \frac{1}{\sqrt{2\pi \ln(b/a)}}\frac{\hat{\mathbf{r}}}{r} \tag{100}$$

2. Equivalent Circuit Parameters

Taken from Collin [7].

$$C = \frac{\epsilon 2\pi}{\ln(b/a)} \tag{101}$$

$$Z_c = \frac{\epsilon Z_0}{C} = \frac{Z_0 \ln(b/a)}{2\pi} \tag{102}$$

$$R = \frac{\rho}{\delta 2\pi}\left(\frac{1}{a} + \frac{1}{b}\right) \tag{103}$$

V. CALCULATING SPECTRAL COMPONENTS OF POWER FROM A CHARGE IN AN ARBITRARY ORBIT

In the UW prototype, the electron trajectory is a complicated function and the spectral components of the power cannot be calculated analytically. In the examples of this document, we calculate the coefficients as a function of time and fourier transform:

$$A^{\pm} * C(\omega) = \int_{-\infty}^{\infty} A_\lambda * c(t) e^{i\omega t} dt = \frac{qZ_\lambda}{2}\int_{-\infty}^{\infty} \mathbf{v}_t(t)\cdot \mathbf{E}_t\left(r(t)\right) e^{i(\omega t \mp kz(t))} dt \tag{104}$$

The time dependence can be quite complicated, with higher harmonics excited by the charge passing through a varying electric field, and the Doppler shift changing with time since β varies with pitch angle. Using a monte carlo that tracks a particle in time to calculate $\mathbf{v}_t(t)$ and $\mathbf{r}(t)$, we can calculate the coefficients $A * c(t)$ at discrete time steps throughout the track and Fourier transform it with FFTW to find the coefficients $A * C(\omega)$ in frequency space. Then the power can be calculated from the coefficients.

$$P(\omega) = \frac{1}{Z_\lambda}[A * C(\omega)]^2 \tag{105}$$

This method is outlined in [10].

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