```python
from typing import TypeVar, List, Tuple, Dict
Numeric = TypeVar('Numeric', int, float, complex)
Rationals = TypeVar('Rationals', int, float)
Integers = int
Num = int | float
IntList = List[int]
FloatList = List[float]
BoolList = List[float]
NumList = List[Num]
TupleListMST = List[Tuple[Num, int, int]]
EdgeTupleList = List[Tuple[int, int]]
EdgeTypeList = List[Tuple[int, int, int]]


def pairwise_func(seq):
    it = iter(seq)
    next(it)
    return zip(iter(seq), it)


def sort_3_elements(a, b, c):
    if b < a:
        a, b = b, a
    if c < b:
        b, c = c, b
        if b < a:
            a, b = b, a
    return a, b, c


################################
from sys import setrecursionlimit


setrecursionlimit(10000000)  # 10 million should be good enough for most problems


class UnionFindDisjointSets:
    """This Data structure is for non-directional disjoint sets."""

    def __init__(self, n):
        """Attributes declared here must be passed in or global if not used in classes."""
        self.num_sets = n        # optional information
        self.rank = [0] * n       # optional optimization
        self.set_sizes = [1] * n   # optional information
        self.parent = [i for i in range(n)]

    def find_set_recursive(self, u):
        """Recursively find which set u belongs to. Memoize on the way back up.

        Complexity: Time: O(\alpha(n)) -> O(1), inverse ackerman practically constant
                Space: Amortized O(1) stack space
        """
        root_parent = u if self.parent[u] == u else self.find_set_recursive(self.parent[u])
        self.parent[u] = root_parent
        return root_parent

    def find_set(self, x):
        """Iteratively find which set u belongs to. uses stack to memoize.

        Complexity: Time: O(\alpha(n)) -> O(1), inverse ackerman practically constant
                Space: O(\alpha(n)) stack space
        """
        xp, local_stack = x, []
        while xp != self.parent[xp]:
            local_stack.append(xp)
            xp = self.parent[xp]
        for c in local_stack:
            self.parent[c] = xp
        return xp

    def is_same_set(self, u, v):
        """Checks if u and v in same set. TIME and SPACE Complexity is the same as find_set"""
        return self.find_set(u) == self.find_set(v)

    def union_set(self, u, v):
        """Join the set that contains u with the set that contains v.

        Complexity: Time: O(\alpha(n)) -> O(1), inverse ackerman practically constant
```

```python
        Space: Amortized O(1) stack space
        """
        if not self.is_same_set(u, v):
            u_parent, v_parent = self.find_set(u), self.find_set(v)
            if self.rank[u_parent] > self.rank[v_parent]:   # u_parent shorter than v_parent
                u_parent, v_parent = v_parent, u_parent

            if self.rank[u_parent] == self.rank[v_parent]:     # an optional speedup
                self.rank[v_parent] = self.rank[v_parent] + 1
            self.parent[u_parent] = v_parent               # this line joins u with v
            self.num_sets -= 1
            self.set_sizes[v_parent] += self.set_sizes[u_parent]  # u = v so add them

    def size_of_u(self, u):  # optional information
        """Gives you the size of set u. TIME and SPACE Complexity is the same as find_set"""
        return self.set_sizes[self.find_set(u)]


##################################################

"""The following sections are here to support range query operations.
CODE USES ONLY MIN OR SUM QUERIES EXAMPLES
Other functions will most of the time just be replacing this format one of the following
    1.) name_of_variable = f(name_of_variable, data_structure[i])
    2.) name_of_variable = f( data_structure[i], data_structure[k])
    Where f(a, b) = a + b, a * b, gcd(a, b), max(a, b) to list a few examples

Requirements:
    function f(x) must commutative
    able to return the answer i to j
    when listed will support element or range updates
"""


from math import isqrt, gcd


# min use big value, max use smallest val, sum use 0, product use 1
# for gcd and others you will need to change the code so that you set it to the starting
# value this might mean longer implementations but that is the cost of the way I set it up
RQ_DEFAULT = 2 ** 32


class SquareRootDecomposition:
    """Block based query data structure based around square root sizes.

    Operations Supported:
        Update Element at position n.
        Range Query Operation: min, max, gcd, sum, product and commutative functions.
    """

    def __init__(self):
        """Attributes declared here must be passed in or global if not used in class."""
        self.size_of_blocks = self.num_of_blocks = self.data_range = 0
        self.decomposed_blocks = []
        self.data_copy = []

    def prepare_square_root_decomposition(self, new_data):
        """Prepare a new set of data for queries. O(f(x)) is complexity of calling f(x)

        Complexity per call: Time: O(n * O(f(x))), Space: O(n + sqrt(n)).
        """
        sqrt_len = isqrt(len(new_data)) + 1  # + 1 here to for to cover int(x) == floor(x)
        pad_val = RQ_DEFAULT  # DEFAULT INFO AT TOP OF CLASS
        self.size_of_blocks = self.num_of_blocks = sqrt_len
        self.decomposed_blocks, self.data_range = [pad_val] * sqrt_len, sqrt_len ** 2
        self.data_copy = [i for i in new_data] + [pad_val] * (self.data_range - len(new_data))
        for i in range(sqrt_len):
            self.update_block_n(i)

    def update_block_n(self, block_n):
        """For a given block n, compute the ranged operation on the block.

        Complexity per call: Time: O(sqrt(n) * O(f(x))), Space: (1)
        """
        start, end = block_n * self.size_of_blocks, (block_n + 1) * self.size_of_blocks
        result = RQ_DEFAULT  # DEFAULT INFO AT TOP OF CLASS
        for i in range(start, end):
            result = min(result, self.data_copy[i])
```

```python
        self.decomposed_blocks[block_n] = result

    def update_position_i_with_value(self, position_i, value):
        """Update position then follow it up with a block update.

        Complexity per call: Time: O(sqrt(n) * O(f(x))), Space: (1)
        """
        self.data_copy[position_i] = value                # O(1)
        self.update_block_n(position_i // self.num_of_blocks)   # but this costs O(sqrt(n))

    def range_query_i_to_j(self, left, right):
        """Compute a range query from left to right. Note right is inclusive here.

        Complexity per call: Time: O(sqrt(n) * O(f(x))), Space: (1)
        """
        block_size, result = self.size_of_blocks, RQ_DEFAULT  # DEFAULT INFO AT TOP OF CLASS
        if (right + 1) - left <= block_size:  # special case where right-left ~= block_size
            for i in range(left, right + 1):  # right + 1 since right is inclusive
                result = min(result, self.data_copy[i])
        else:
            left_block, right_block = left // block_size, right // block_size
            end1, start2 = (left_block + 1) * block_size, right_block * block_size
            for i in range(left, end1):
                result = min(result, self.data_copy[i])
            for i in range(start2, right + 1):  # right + 1 since right is inclusive
                result = min(result, self.data_copy[i])
            for i in range(left_block + 1, right_block):
                result = min(result, self.decomposed_blocks[i])
        return result


####################################################


from math import gcd


class SparseTable:
    """Logarithmic Datastructure based around levels of precomputed data.

    Operations Supported:
        DOES NOT SUPPORT DYNAMIC UPDATES must recompute structure after an update
        Range Query Operation: min, max, gcd, sum, product and commutative functions.
    """
    __slots__ = ("sparse_table", "k_value", "max_n")

    def __init__(self, max_n):
        """Attributes declared here must be passed in or global if not used in class."""
        self.k_value = max_n.bit_length()
        self.max_n = max_n
        self.sparse_table = [[0] * max_n for _ in range(self.k_value)]

    def prepare_sparse_table(self, new_array):
        """Prepare a new set of data for queries. O(f(x)) is complexity of calling f(x)

        Complexity per call: Time: O(O(f(x)) * n ln n), Space: O(n ln n), S(n log2 n).
        Variants shown:   f(x) might be O(ln n) like for gcd and lcm
          range min query: O(f(x)) = O(1) like most
          range sum query: O(f(x)) = O(1) like others [is commented out]
        """
        spare_table, i_end, local_max_n = self.sparse_table, self.k_value + 1, self.max_n + 1
        #   spare_table[0] = [el for el in array]  # USE IF max_n == len(array)
        for i, el in enumerate(new_array):
            spare_table[0][i] = el
        for i in range(1, i_end):  # start from 1 and compute sub problems lvl by lvl
            prev_i = i - 1  # compute once here for better readability and avoid n ln n
            j_end, prev_pow_2 = local_max_n - (1 << i), 1 << prev_i  # read comment above
            for j in range(j_end):
                spare_table[i][j] = min(spare_table[prev_i][j], spare_table[prev_i][j+prev_pow_2])
                # spare_table[i][j] = spare_table[prev_i][j] + spare_table[prev_i][j + prev_pow_2]

    def range_query_from_i_to_j(self, left, right):
        """Compute the range query over precomputed data.

        Complexity per call: Time: V1 = O(O(f(x)) * 1), V2 = O(O(f(x)) * ln n), Space: O(1)
        Variants shown:  f(x) might be O(ln n) like for gcd
          range min query: O(f(x)) = O(1) like most  V1
          range sum query: O(f(x)) = O(1) like others  V2
        Notes: V1 is min and max (there might be others) but for these you need only compute
```

```python
        two ranges so long as lower_bound == left and upper_bound == right, overlap is fine
        """
        result = RQ_DEFAULT  # see SquareRootDecomposition class for more info
        ind = (right - left + 1).bit_length()-1  # I think this is same as log2(r - l + 1)
        result = min(self.sparse_table[ind][left], self.sparse_table[ind][right - 2**ind + 1])
        for i in range(self.k_value-1, -1, -1):
            if (1 << i) <= right - left + 1:
                result = result + self.sparse_table[i][left]
                left = left + (1 << i)
        return result


####################################################


class FenwickTree:
    """Logarithmic Data structure based dynamic use of binary index tree. 1-based indexing.

    Operations Supported:
        Supports Dynamic updates on ranges and elements
        Range Query Operation: main use sum query, other functions exist tho like above
    """
    __slots__ = ("fenwick_tree", "fenwick_tree_size")

    def __init__(self, frequency_array=None):
        """Attributes declared here must be passed in or global if not used in class."""
        self.fenwick_tree = []
        self.fenwick_tree_size = 0
        if frequency_array:  # None is False, so we can have this optional check :)
            self.build_tree_from_frequency_array(frequency_array)
        # self.num_rows = self.num_cols = 0          # for 2d
        # self.fenwick_tree_2d = [[0] for _ in range(1)]  # for 2d

    def last_set_bit(self, a):
        """Will get the last set-bit of a used in 1-indexing. for 0-indexing use i & (i+1)"""
        return a & (-a)

    def build_tree_from_frequency_array(self, frequency_array):
        """Prepare a new set of data for queries. O(f(x)) is complexity of calling f(x)

        Complexity per call: Time: O(n * O(f(x)) ), T(n) usually, Space: O(n)

        Variant: just call updated on each element Not shown but is like O(n ln n)
        Variants shown:   f(x) might be O(ln n) like for gcd and lcm
          range sum query: O(f(x)) = O(1) like others [is commented out]
        """
        n = len(frequency_array)
        self.fenwick_tree_size = n
        self.fenwick_tree = [0] * (n + 1)
        for i, frequency in enumerate(frequency_array, 1):  # start from 1 in 1 based indexing
            self.fenwick_tree[i] += frequency
            pos = i + self.last_set_bit(i)
            if pos <= n:
                self.fenwick_tree[pos] += self.fenwick_tree[i]

    def build_tree_from_s(self, max_n, s):
        """Builds Fenwick tree given data s."""
        frequency_array = [0] * (max_n + 1)
        for i in s:
            frequency_array[i] += 1
        self.build_tree_from_frequency_array(frequency_array)

    def range_sum_from_i_to_j(self, left, right):
        """Returns the inclusive-exclusive range sum [i...j). version is 1-index based.
        rsq(i, j) = sum(1, right) - sum(1, left - 1)

        Complexity per call: Time: O(log n), Space: O(1).
        """
        if left > 1:  # when left isn't 1 we compute the formula above.
            return self.range_sum_from_i_to_j(1, right)-self.range_sum_from_i_to_j(1, left-1)
        sum_up_to_right = RQ_DEFAULT  # is 0, see SquareRootDecomposition class for more info
        while right:
            sum_up_to_right += self.fenwick_tree[right]
            right -= self.last_set_bit(right)
        return sum_up_to_right

    def update_index_by_delta(self, index, delta):
        """Updates the branch that follows index by delta. version is 1-index based.
        Branch defined as "for i in range(index,tree_size,f(i)=i & (-i))", for(start,end,step)
```

```python
        Complexity per call: Time: O(log n), Space: O(1).
        """
        while index <= self.fenwick_tree_size:
            self.fenwick_tree[index] += delta
            index += self.last_set_bit(index)

    # def update_index_by_delta_2d(self, row, col, delta):
    #     i, j, fenwick_2d = row, col, self.fenwick_tree_2d
    #     while i <= self.num_rows:
    #         while j <= self.num_cols:
    #             fenwick_2d[i][j], j = fenwick_2d[i][j] + delta, j + self.last_set_bit(j)
    #         j, i = col, i + self.last_set_bit(i)
    #
    # def sum_query_2d(self, row, col):
    #     i, j, result = row, col, 0
    #     while i:
    #         while j:
    #             result, j = result + self.fenwick_tree_2d[i][j], j - self.last_set_bit(j)
    #         j, i = col, i - self.last_set_bit(i)
    #     return result

    def select_k(self, k):  # TODO semi tested
        p = 2**(self.fenwick_tree_size.bit_length()-1)
        i = 0
        while p:
            fen_val = self.fenwick_tree[i + p]
            if k > fen_val:
                k -= fen_val
                i = i + p
            p //= 2
        return i + 1


class RangeUpdatePointQuery:
    def __init__(self, m):
        self.point_update_range_query = FenwickTree([0] * m)

    def update_range_i_to_j_by_delta(self, left, right, delta):  # TODO semi tested
        """For a range update we need only update all indices in [i,i+1...n] to have +delta +
        then all indices in [j+1,j+2...n] with -delta to negate range(j+1, n) that was changed

        Complexity per call: Time: O(log n), Space: O(1).
        """
        self.point_update_range_query.update_index_by_delta(left, delta)
        self.point_update_range_query.update_index_by_delta(right + 1, -delta)

    def point_sum_query_of_index(self, index):  # TODO semi tested
        """With restriction of only point sum query, will return range_sum(1, index).

        Complexity per call: Time: O(log n), Space: O(1).
        """
        return self.point_update_range_query.range_sum_from_i_to_j(1, index)


class RangeUpdateRangeQuery:
    def __init__(self, m):
        self.range_update_point_query = RangeUpdatePointQuery(m)
        self.point_update_range_query = FenwickTree([0] * m)

    def update_range_i_to_j_by_delta(self, left, right, delta):  # TODO semi tested
        """Updates range by delta like above but then also updates the augmented tree which is
        needed for dynamic range updates and queries for the same structure.

        Complexity per call: Time: O(log n), Space: O(1).
        """
        self.range_update_point_query.update_range_i_to_j_by_delta(left, right, delta)
        self.point_update_range_query.update_index_by_delta(left, delta * (left - 1))
        self.point_update_range_query.update_index_by_delta(right + 1, -delta * right)

    def range_sum_from_i_to_j(self, left, right):  # TODO semi tested
        """Similar to the Fenwick tree we use inclusion exclusion, but we need to use our aug
        tree to handle 3 cases to compute formula sum[0, i] = sum(rupq, i) * i - sum(purq, i)
             | original          | simplified        | conditions
        sum[0,i] =  | 0*i - 0            | 0                 |    i < left
                    | x*i - x(left - 1)  | x(i - (left - 1)) |  left <= i <= right
                    | 0*i - (x(left - 1) - x*right) | x(right - left + 1) |   i > right

        Complexity per call: Time: O(log n), Space: O(1).
```

```python
        """
        if left > 1:
            return self.range_sum_from_i_to_j(1, right)-self.range_sum_from_i_to_j(1, left-1)
        return (self.range_update_point_query.point_sum_query_of_index(right) * right
                - self.point_update_range_query.range_sum_from_i_to_j(1, right))


##################################################


class SegmentTree:
    def __init__(self, new_array):
        self.is_lazy = []
        self.lazy = []
        self.segment_tree = []
        self.array_a = []
        self.tree_size = 0
        self._version_1(new_array)
        # self._version_2(new_array)

    def _version_1(self, new_array):
        n = len(new_array)
        next_2_pow_n = n if 2**(n.bit_length()-1) == n else 2**n.bit_length()
        self.lazy = [-1] * (4 * next_2_pow_n)      # 4 * n used to avoid index out of bounds
        self.segment_tree = [0] * (4 * next_2_pow_n)  # 4 * n used to avoid index out of bound
        self.array_a = [el for el in new_array]
        if next_2_pow_n != n:                      # we add extra padding to the end to
            self.array_a.extend([RQ_DEFAULT]*(next_2_pow_n-n))  # make the len a power of 2
        self.tree_size = len(self.array_a)
        self.build_segment_tree_1()

    def _version_2(self, new_array):
        n = len(new_array)
        self.is_lazy = [False] * (4 * n)
        self.lazy = [0] * (4 * n)
        self.segment_tree = [0] * (4 * n)
        self.array_a = [el for el in new_array]
        self.tree_size = n
        self.build_segment_tree_2()

    def left_child(self, parent):
        """Macro function, gets left child in array based binary tree, hardcode to speedup."""
        return parent << 1

    def right_child(self, parent):
        """Macro function, gets right child in array based binary tree,"""
        return (parent << 1) + 1

    def conqur(self, a, b):
        """Macro function, handles to conqur conditional, paste in code for speedup."""
        return b if a == -1 else (a if b == -1 else min(a, b))

    def propagate(self, parent, left, right):
        lazy_p = self.lazy[parent]
        if lazy_p != -1:
            self.segment_tree[parent] = lazy_p
            if left != right:
                self.lazy[self.left_child(parent)] = self.lazy[self.right_child(parent)] = lazy_p
            else:
                self.array_a[left] = lazy_p
            self.lazy[parent] = -1

    def _build_segment_tree_1(self, parent, left, right):
        if left == right:
            self.segment_tree[parent] = self.array_a[left]
        else:
            mid = (left + right) // 2
            left_path, right_path = self.left_child(parent), self.right_child(parent)
            self._build_segment_tree_1(left_path, left, mid)
            self._build_segment_tree_1(right_path, mid + 1, right)
            left_val, right_val = self.segment_tree[left_path], self.segment_tree[right_path]
            self.segment_tree[parent] = self.conqur(left_val, right_val)
            # seg_tree = self.segment_tree  # uncomment if self.segment_tree gets too costly
            # seg_tree[parent] = self.conqur(seg_tree[left_path], seg_tree[right_path])

    def _build_segment_tree_2(self, parent, left, right):
        if left == right:
            self.segment_tree[parent] = left
        else:
```

```python
        mid = (left + right) // 2
        left_path, right_path = self.left_child(parent), self.right_child(parent)
        self._build_segment_tree_2(left_path, left, mid)
        self._build_segment_tree_2(right_path, mid + 1, right)
        left_i = self.segment_tree[left_path]
        right_i = self.segment_tree[right_path]
        self.segment_tree[parent] = (left_i if (self.array_a[left_i]
                                                <= self.array_a[right_i]) else right_i)

    def _range_min_query_1(self, parent, left, right, i, j):
        self.propagate(parent, left, right)
        if i > j:
            return -1
        if i <= left and j >= right:
            return self.segment_tree[parent]
        mid = (left + right) // 2
        left_min = self._range_min_query_1(self.left_child(parent), left, mid,
                                           i, min(mid, j))
        right_min = self._range_min_query_1(self.right_child(parent), mid + 1, right,
                                            max(i, mid + 1), j)
        return self.conqur(left_min, right_min)

    def _range_min_query_2(self, parent, left, right, i, j):
        if right < i or j < left:
            return -1, -1
        if self.is_lazy[parent]:
            return i, self.lazy[parent]
        if i <= left and j >= right:
            return self.segment_tree[parent], self.array_a[self.segment_tree[parent]]
        mid = (left + right) // 2
        left_ind, left_val = self._range_min_query_2(self.left_child(parent), left, mid, i, j)
        right_ind, right_val = self._range_min_query_2(self.right_child(parent),
                                                       mid+1, right, i, j)
        if left_ind == -1:
            return right_ind, right_val
        elif right_ind == -1:
            return left_ind, left_val
        else:
            return (left_ind, left_val) if left_val <= right_val else (right_ind, right_val)

    def _update_segment_tree_1(self, parent, left, right, i, j, value):
        self.propagate(parent, left, right)
        if i > j:
            return
        elif i <= left and j >= right:
            self.lazy[parent] = value
            self.propagate(parent, left, right)
        else:
            mid = (left + right) // 2
            left_path, right_path = self.left_child(parent), self.right_child(parent)
            self._update_segment_tree_1(left_path, left, mid, i, min(mid, j), value)
            self._update_segment_tree_1(right_path, mid + 1, right, max(i, mid + 1), j, value)
            left_subtree = (self.lazy[left_path] if self.lazy[left_path] != -1
                            else self.segment_tree[left_path])
            right_subtree = (self.lazy[right_path] if self.lazy[right_path] != -1
                             else self.segment_tree[right_path])
            self.segment_tree[parent] = (self.segment_tree[left_path]
                                         if left_subtree <= right_subtree
                                         else self.segment_tree[right_path])

    def _update_segment_tree_2(self, parent, left, right, i, j, new_value):
        if right < i or j < left:
            return self.segment_tree[parent]
        if i == left and j == right:
            if i == j:
                self.array_a[left] = new_value
                self.is_lazy[parent] = False
            else:
                self.lazy[parent] = new_value
                self.is_lazy[parent] = True
            self.segment_tree[parent] = i
            return i
        mid = (left + right) // 2
        left_path, right_path = self.left_child(parent), self.right_child(parent)
        if self.is_lazy[parent]:
            self.is_lazy[parent] = False
            self.is_lazy[left_path] = self.is_lazy[right_path] = True
            self.lazy[left_path] = self.lazy[right_path] = self.lazy[parent]
            self.segment_tree[left_path] = left
```

```python
            self.segment_tree[right_path] = mid
        left_ind = self._update_segment_tree_2(left_path, left, mid,
                                               max(i, left), min(j, mid), new_value)
        right_ind = self._update_segment_tree_2(right_path, mid+1, right,
                                                max(i, mid+1), min(j, right), new_value)
        self.segment_tree[parent] = (left_ind if (self.array_a[left_ind]
                                                  <= self.array_a[right_ind])
                                     else right_ind)
        return self.segment_tree[parent]

    def build_segment_tree_1(self):
        self._build_segment_tree_1(1, 0, self.tree_size - 1)

    def build_segment_tree_2(self):
        self._build_segment_tree_2(1, 0, self.tree_size - 1)

    def update_segment_tree_1(self, i, j, value):
        self._update_segment_tree_1(1, 0, self.tree_size - 1, i, j, value)

    def update_segment_tree_2(self, i, j, value):
        self._update_segment_tree_2(1, 0, self.tree_size - 1, i, j, value)

    def range_min_query_1(self, i, j):
        return self._range_min_query_1(1, 0, self.tree_size - 1, i, j)

    def range_min_query_2(self, i, j):
        return self._range_min_query_2(1, 0, self.tree_size - 1, i, j)[0]


#################################################

# from math import log2
from collections import deque
from heapq import heappush, heappop, heapify
from sys import setrecursionlimit

setrecursionlimit(10000000)  # 10 million should be good enough for most contest problems


INF: int = 2**31
# turn these into enums later
UNVISITED: int = -1
EXPLORED: int = -2
VISITED: int = -3
# turn these into enums later
TREE: int = 0
BIDIRECTIONAL: int = 1
BACK: int = 2
FORWARD: int = 3


class Graph:
    def __init__(self, v: int, e: int, r=None, c=None):
        self.num_edges: int = e
        self.num_nodes: int = v
        self.num_rows: int = r
        self.num_cols: int = c

        self.adj_list = []
        self.adj_list_trans = []   # for topological sort
        self.adj_matrix = []
        self.edge_list = []
        self.grid = []

        self.data_to_code = {}  # used for converting input into integer vertices
        self.code_to_data = []  # used for integer vertices back into original input

    def convert_data_to_code(self, data: object) -> int:
        """Converts data to the form: int u | 0 <= u < |V|, stores (data, u) pair."""
        if data not in self.data_to_code:
            self.data_to_code[data] = len(self.code_to_data)
            self.code_to_data.append(data)  # can be replaced with a count variable if needed
        return self.data_to_code[data]

    def add_edge_u_v_wt_into_directed_graph(self, u: object, v: object, wt: Num, data: Num):
        """A pick and choose function will convert u, v into index form then add it to the
        structure you choose.
        """
        u: int = self.convert_data_to_code(u)  # omit if u,v is in the form: int
```

```python
        v: int = self.convert_data_to_code(v)  # omit if u,v is in the form: int

        self.adj_list[u].append(v)
        # self.adj_list[u].append((v, wt))  # Adjacency list usage with weights
        self.adj_matrix[u][v] = wt          # Adjacency matrix usage
        self.edge_list.append((wt, u*self.num_nodes + v))   # Edge list usage space optimized
        # the following lines come as a pair-set used in max flow algorithm.
        self.edge_list.append([v, wt, data])
        self.adj_list[u].append(len(self.edge_list) - 1)

    def add_edge_u_v_wt_into_undirected_graph(self, u: object, v: object, wt: Num, data):
        """undirected graph version of the previous function. wt can be omitted."""
        self.add_edge_u_v_wt_into_directed_graph(u, v, wt, data)
        self.add_edge_u_v_wt_into_directed_graph(v, u, wt, data)

    def fill_grid_graph(self, new_grid: list[list[object]]):
        self.num_rows = len(new_grid)
        self.num_cols = len(new_grid[0])
        self.grid = [[self.convert_data_to_code(el) for el in row] for row in new_grid]


class GraphAlgorithms:
    def __init__(self, new_graph):
        self.path_i_j = []
        self.graph: Graph = new_graph
        self.min_spanning_cost: int = 0
        self.dfs_counter: int = 0
        self.dfs_root: int = 0
        self.root_children: int = 0
        self.region_num: int = 0

        self.dir_rc = [(1, 0), (0, 1), (-1, 0), (0, -1)]
        self.visited = []
        self.mst_node_list = []
        self.dist = []
        self.topo_sort_node_list = []
        self.parent = []
        self.low_values = []
        self.articulation_nodes = []
        self.bridge_edges = []
        self.directed_edge_type = []  # change last int to enum sometime
        self.component_region = []
        self.decrease_finish_order = []
        self.nodes_on_stack = []
        self.node_state = []  # change to enum type sometime
        self.bipartite_colouring = []
        self.last = []

    def flood_fill_via_dfs(self, row: int, col: int, old_val: object, new_val: object):
        """Computes flood fill graph traversal via recursive depth first search.

        Complexity: Time: O(|V| + |E|), Space: O(|V|): for grids |V|=row*col and |E|=4*|V|
        More uses: Region Colouring, Connectivity, Area/island Size, misc
        Input
          row, col: integer pair representing current grid position
          old_val, new_val: unexplored state, the value of an explored state
        """
        self.graph.grid[row][col] = new_val
        for row_mod, col_mod in self.dir_rc:
            new_row, new_col = row + row_mod, col + col_mod
            if (0 <= new_row < self.graph.num_rows
                    and 0 <= new_col < self.graph.num_cols
                    and self.graph.grid[new_row][new_col] == old_val):
                self.flood_fill_via_dfs(new_row, new_col, old_val, new_val)

    def flood_fill_via_bfs(self, start_row: int, start_col: int, old_val: object, new_val):
        """Computes flood fill graph traversal via breadth first search. Use on grid graphs.

        Complexity: Time: O(|V| + |E|), Space: O(|V|): for grids |V|=row*col and |E|=4*|V|
        More uses: previous uses tplus shortest connected pah
        """
        queue = deque([(start_row, start_col)])
        while queue:
            row, col = queue.popleft()
            for row_mod, col_mod in self.dir_rc:
                new_row, new_col = row + row_mod, col + col_mod
                if (0 <= new_row < self.graph.num_rows
                        and 0 <= new_col < self.graph.num_cols
                        and self.graph.grid[new_row][new_col] == old_val):
                    self.graph.grid[new_row][new_col] = new_val
                    queue.append((new_row, new_col))

    def min_spanning_tree_via_kruskals_and_heaps(self):  # tested
        """Computes mst of graph G stored in edge_list, space optimized via heap.

        Complexity per call: Time: O(|E|log |V|), Space: O(|E|) + Union_Find
        More uses: finding min spanning tree
        Variants: min spanning subgraph and forrest, max spanning tree, 2nd min best spanning
        Optimization: We use a heap to make space comp. O(|E|). instead of O(|E|log |E|)
        when using sort, however edge_list is CONSUMED. Also uses space optimization
        """
        vertices = self.graph.num_nodes
        heapify(self.graph.edge_list)
        ufds = UnionFindDisjointSets(vertices)
        min_spanning_tree = []
        while self.graph.edge_list and ufds.num_sets > 1:  # num_sets == 1 graph is Done
            wt, uv = heappop(self.graph.edge_list)   # uv is u*num_nodes + v
            v, u = divmod(uv, vertices)         # u, v = uv//n, uv%n
            if not ufds.is_same_set(u, v):
                min_spanning_tree.append((wt, u, v))
                ufds.union_set(u, v)
        self.mst_node_list = min_spanning_tree

    def prims_visit_adj_matrix(self, source: int):
        """Find min weight edge in adjacency matrix implementation of prims.

        Complexity per call: Time: O(|V|^2), T(|V| * 4|V|), Space: O(|V|), S(~5|V|)
        """
        vertices = self.graph.num_nodes
        not_seen_and_min_dist, mst_parent = {v: INF for v in range(vertices)}, [-1] * vertices
        min_spanning_tree, min_weight, not_seen_and_min_dist[source] = [], 0, 0
        for _ in range(vertices):
            u, best_dist = min(not_seen_and_min_dist.items(), key=lambda x: x[1])
            # if best_dist == INF:  # graph is disconnected case
            #    return       # uncomment when graph might be disconnected
            min_spanning_tree.append((min(u, mst_parent[u]), max(u, mst_parent[u])))
            min_weight += best_dist
            del not_seen_and_min_dist[u]  # remove u since it has been seen now
            for v, min_dist in not_seen_and_min_dist.items():
                if self.graph.adj_matrix[u][v] < min_dist:
                    not_seen_and_min_dist[v], mst_parent[v] = self.graph.adj_matrix[u][v], u
        # min_spanning_tree.sort()  # optional for when edges need to be ordered: nlog n cost
        self.mst_node_list = min_spanning_tree[1:]  # cut off the root node

    def min_spanning_tree_via_prims_adj_list(self, source: int):
        """Computes mst of graph G stored in adj_list. Uses edge compression u, v = u*|V| + v.

        Complexity: Time: O(|E|log |V|), Space: O(|E|)
        More uses: Gets a different min spamming tree than kruskal's
        """
        vertices = self.graph.num_nodes
        not_visited, mst_best_dist = [True] * vertices, [INF] * vertices
        heap = [(wt, v * vertices + source) for v, wt in self.graph.adj_list[source]]
        heapify(heap)
        for v, wt in self.graph.adj_list[source]:
            mst_best_dist[v] = wt
        min_spanning_tree, nodes_taken, msc, not_visited[source] = [], 0, 0, False
        while heap and nodes_taken < vertices - 1:  # max edges in mst == |V| - 1
            edge_wt, u_parent = heappop(heap)
            u, parent = divmod(u_parent, vertices)  # edge uncompress u, v = uv//|V|, uv%|V|
            if not_visited[u]:
                min_spanning_tree.append((min(u, parent), max(u, parent)))
                msc += edge_wt
                nodes_taken += 1
                not_visited[u] = False
                for v, wt in self.graph.adj_list[u]:
                    if wt < mst_best_dist[v] and not_visited[v]:
                        mst_best_dist[v] = wt
                        heappush(heap, (wt, v * vertices + u))  # edge compression v*|V| + u.
        # min_spanning_tree.sort()  # optional for when edges need to be ordered: nlog n cost
        self.mst_node_list = min_spanning_tree
        self.min_spanning_cost = msc if nodes_taken == vertices - 1 else -1

    def breadth_first_search_vanilla_template(self, source: int):
        """Template for distance based bfs traversal from node source.

        Complexity per call: Time: O(|V| + |E|), Space: O(|V|)
        More uses: connectivity, shortest path on monotone weighted graphs
```

```python
        """
        distance = [UNVISITED] * self.graph.num_nodes
        queue, distance[source] = deque([source]), 0
        while queue:
            u = queue.popleft()
            for v in self.graph.adj_list[u]:
                if distance[v] == UNVISITED:
                    distance[v] = distance[u] + 1
                    queue.append(v)
        self.dist = distance

    def topology_sort_via_tarjan_helper(self, u: int):
        """Recursively explore unvisited graph via dfs.

        Complexity per call: Time: O(|V|), Space: O(|V|) at deepest point
        """
        self.visited[u] = VISITED
        for v in self.graph.adj_list[u]:
            if self.visited[v] == UNVISITED:
                self.topology_sort_via_tarjan_helper(v)
        self.topo_sort_node_list.append(u)

    def topology_sort_via_tarjan(self, source=-1):
        """Compute a topology sort via tarjan method, on adj_list.

        Complexity per call: Time: O(|V| + |E|), Space: O(|V|)
        More Uses: produces a DAG, topology sorted graph, build dependencies
        """
        self.visited = [UNVISITED] * self.graph.num_nodes
        self.topo_sort_node_list = []
        if source != -1:
            self.topology_sort_via_tarjan_helper(source)
        else:
            for u in range(self.graph.num_nodes):
                if self.visited[u] == UNVISITED:
                    self.topology_sort_via_tarjan_helper(u)
        self.topo_sort_node_list = self.topo_sort_node_list[::-1]

    def topology_sort_via_kahns(self):
        """Compute a topology sort via kahn's method, on adj_list.

        Complexity per call: Time: O(|E|log|V|), Space: O(|V|)
        More uses: different ordering as tarjan's method
        bonus: heaps allow for custom ordering (i.e. use lowest indices first)
        """
        in_degree = [0] * self.graph.num_nodes
        for list_of_u in self.graph.adj_list:
            for v in list_of_u:
                in_degree[v] += 1
        topo_sort = []
        heap = [u for u, el in enumerate(in_degree) if el == 0]
        heapify(heap)
        while heap:
            u = heappop(heap)
            topo_sort.append(u)
            for v in self.graph.adj_list[u]:
                in_degree[v] -= 1
                if in_degree[v] <= 0:
                    heappush(heap, v)
        self.topo_sort_node_list = topo_sort

    def amortized_heap_fix(self, heap):
        """Should we need |V| space this will ensure that while still being O(log|V|)"""
        tmp = [-1] * self.graph.num_nodes
        for wt, v in heap:
            if tmp[v] == -1:
                tmp[v] = wt
        heap = [(wt, v) for v, wt in enumerate(tmp) if wt != -1]
        heapify(heap)

    def single_source_shortest_path_dijkstras(self, source: int, sink: int):
        """It is Dijkstra's pathfinder using heaps.

        Complexity per call: Time: O(|E|log |V|), Space: O(|V|)
        More uses: shortest path on state based graphs
        Input:
            source: can be a single nodes or list of nodes
            sink: the goal node
        """
```

```python
        distance, parents = [INF] * self.graph.num_nodes, [UNVISITED] * self.graph.num_nodes
        distance[source], parents[source] = 0, source
        heap = [(0, source)]
        while heap:
            cur_dist, u = heappop(heap)
            if distance[u] < cur_dist:
                continue
            # if u == sink: return cur_dist # uncomment this line for fast return
            for v, wt in self.graph.adj_list[u]:
                if distance[v] > cur_dist + wt:
                    distance[v] = cur_dist + wt
                    parents[v] = u
                    heappush(heap, (distance[v], v))
        self.dist = distance
        self.parent = parents

    def all_pairs_shortest_path_floyd_warshall_print(self, i, j):  # TODO RETEST
        """Generate the shortest path from i to j, used with Floyd Warshall."""
        path = []
        while i != j:
            path.append(j)
            j = self.parent[i][j]
        self.path_i_j = path[::-1]  # we need to reverse it ???

    def all_pairs_shortest_path_floyd_warshall(self):
        """Computes essentially a matrix operation on a graph.

        Complexity per call: Time: O(|V|^3), Space: O(|V|^2)
        More uses: Shortest path, Connectivity.
        Variants: Transitive closure, Maximin and Minimax path, Cheapest negative cycle,
            Finding diameter of a graph, Finding SCC of a directed graph.
        """
        matrix, matrix_size = self.graph.adj_matrix, self.graph.num_nodes
        parents = [[i] * matrix_size for i in range(matrix_size)]  # optional for path i to j
        for k in range(matrix_size):
            for i in range(matrix_size):
                for j in range(matrix_size):
                    matrix[i][j] = min(matrix[i][j], matrix[i][k] + matrix[k][j])
                    # if matrix[i][k] + matrix[k][j] < matrix[i][j]:  # for printing path i to j
                    #   matrix[i][j], parents[i][j] = matrix[i][k] + matrix[k][j], parents[k][j]
        self.parent = parents

    def apsp_floyd_warshall_variants(self):
        """Compressed Compilation of 5 variants of APSP. PICK AND CHOOSE IMPLEMENTATION.
        Contents:  # var 2 and 4 are untested right now
            Variant 1: Transitive Closure to check if i is directly or indirectly connected to j
            Variant 2: MiniMax and MaxiMin path problem. A[i][j] = INF if no edge exists
            Variant 3: Cheapest/Negative cycle, From APSP
            Variant 4: Diameter of a Graph, biggest shortest path in the graph, From APSP
            Variant 5: SCC on small graph, labeling and existence path from i->j and j->i exists

        Complexity per call: Time: O(V^3), T(V^3 + V^2), Space: O(V^2)
        """
        matrix, matrix_size = self.graph.adj_matrix, self.graph.num_nodes
        graph_diameter = 0
        for k in range(matrix_size):
            for i in range(matrix_size):
                for j in range(matrix_size):
                    matrix[i][j] |= matrix[i][k] and matrix[k][j]                # Variant 1
                    matrix[i][j] = min(matrix[i][j], max(matrix[i][k], matrix[k][j]))  # Variant 2
                    if matrix[j][j] < 0 and matrix[k][j] < INF and matrix[j][i] < INF:  # Variant 3
                        matrix[k][i] = -INF                                     # Variant 3
                    # following only requires loops i and j, Variant 4 IS LINE BELOW THIS COMMENT
                    graph_diameter = max(graph_diameter, matrix[i][j] if matrix[i][j] < INF else 0)
                    is_j_strongly_connected_to_i = matrix[i][j] and matrix[j][i]  # Variant 5

    def articulation_point_and_bridge_helper_via_dfs(self, u: int):
        """Recursion part of the dfs. It kind of reminds me of how Union find works.

        Complexity per call: Time: O(|E|), Space: O(|V|) or O(|E|) depending on point vs edge
        """
        self.visited[u] = self.dfs_counter
        self.low_values[u] = self.visited[u]
        self.dfs_counter += 1
        for v in self.graph.adj_list[u]:
            if self.visited[v] == UNVISITED:
                self.parent[v] = u
                if u == self.dfs_root:
                    self.root_children += 1
```

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```python
        self.articulation_point_and_bridge_helper_via_dfs(v)
        if self.low_values[v] >= self.visited[u]:
          self.articulation_nodes[u] = True
          if self.low_values[v] > self.visited[u]:
            self.bridge_edges.append((u, v))
        self.low_values[u] = min(self.low_values[u], self.low_values[v])
      elif v != self.parent[u]:
        self.low_values[u] = min(self.low_values[u], self.visited[v])

  def articulation_points_and_bridges_via_dfs(self):
    """Generates the name on an adj_list based graph.

    Complexity per call: Time: O(|E| + |V|), Space: O(|V|)
    More uses: finding the sets of single edge and vertex removals that disconnect the G.
    Bridges stored as edges and points are True values in articulation_nodes.
    """
    self.dfs_counter = 0
    for u in range(self.graph.num_nodes):
      if self.visited[u] == UNVISITED:
        self.dfs_root = u
        self.root_children = 0
        self.articulation_point_and_bridge_helper_via_dfs(u)
        self.articulation_nodes[self.dfs_root] = (self.root_children > 1)

  def cycle_check_on_directed_graph_helper(self, u: int):
    """Recursion part of the dfs. It is modified to list various types of edges.

    Complexity per call: Time: O(|E|), Space: O(|V|) at deepest call
    More uses: listing edge types: Tree, Bidirectional, Back, Forward/Cross edge. On top
    of listing Explored, Visited, and Unvisited.
    """
    self.visited[u] = EXPLORED
    for v in self.graph.adj_list[u]:
      edge_type: int = TREE
      if self.visited[v] == UNVISITED:
        edge_type = TREE
        self.parent[v] = u
        self.cycle_check_on_directed_graph_helper(v)
      elif self.visited[v] == EXPLORED:
        edge_type = BIDIRECTIONAL if v == self.parent[u] else BACK  # case graph not DAG
      elif self.visited[v] == VISITED:
        edge_type = FORWARD
      self.directed_edge_type.append((u, v, edge_type))
    self.visited[u] = VISITED

  def cycle_check_on_directed_graph(self):
    """Determines if a graph is cyclic or acyclic via dfs.

    Complexity per call: Time: O(|E| + |V|),
              Space: O(|E|) if you label each edge O(|V|) otherwise.
    More uses: Checks if graph is acyclic(DAG) which can open use for efficient algorithms
    """
    self.visited = [UNVISITED] * self.graph.num_nodes
    self.directed_edge_type = []  # can be ignored if needed
    for u in range(self.graph.num_nodes):
      if self.visited[u] == UNVISITED:
        self.cycle_check_on_directed_graph_helper(u)

  def strongly_connected_components_of_graph_kosaraju_helper(self, u: int, first: bool):
    """Pass one explore G and build stack, Pass two mark the SCC regions on transposition
    of G. # TODO RETEST
    Complexity per call: Time: O(|E| + |V|), Space: O(|V|)
    """
    self.visited[u] = VISITED
    self.component_region[u] = self.region_num
    neighbours = self.graph.adj_list[u] if first else self.graph.adj_list_trans[u]
    for v in neighbours:
      if self.visited[v] == UNVISITED:
        self.strongly_connected_components_of_graph_kosaraju_helper(v, first)
    if first:
      self.decrease_finish_order.append(u)

  def strongly_connected_components_of_graph_kosaraju(self):  # TODO RETEST
    """Marks the SCC of a directed graph using Kosaraju's method.

    Complexity per call: Time: O(|E| + |V|), Space: O(|V|)
    More Uses: Labeling and Identifying SCC regions(marks regions by numbers).
    """
    self.visited = [UNVISITED] * self.graph.num_nodes
```

```python
    self.component_region = [0] * self.graph.num_nodes
    for u in range(self.graph.num_nodes):
      if self.visited[u] == UNVISITED:
        self.strongly_connected_components_of_graph_kosaraju_helper(u, True)
    self.visited = [UNVISITED] * self.graph.num_nodes
    self.region_num = 1
    for u in reversed(self.decrease_finish_order):
      if self.visited[u] == UNVISITED:
        self.strongly_connected_components_of_graph_kosaraju_helper(u, False)
        self.region_num += 1

  def strongly_connected_components_of_graph_tarjans_helper(self, u: int):  # TODO RETEST
    """Recursive part of tarjan's, pre-order finds the SCC regions, marks regions postord.

    Complexity per call: Time: O(|E| + |V|), Space: O(|V|)
    """
    self.low_values[u] = self.node_state[u] = self.dfs_counter
    self.dfs_counter += 1
    self.nodes_on_stack.append(u)
    self.visited[u] = VISITED
    for v in self.graph.adj_list[u]:
      if self.node_state[v] == UNVISITED:
        self.strongly_connected_components_of_graph_tarjans_helper(v)
      if self.visited[v] == VISITED:
        self.low_values[u] = min(self.low_values[u], self.low_values[v])
    if self.low_values[u] == self.node_state[u]:
      self.region_num += 1
      while True:
        v: int = self.nodes_on_stack.pop()
        self.visited[v], self.component_region[v] = UNVISITED, self.region_num
        if u == v:
          break

  def strongly_connected_components_of_graph_tarjans(self):  # TODO RETEST
    """Marks the SCC regions of a directed graph using tarjan's method.

    Complexity per call: Time: O(|E| + |V|), Space: O(|V|)
    More uses: Labeling and Identifying SCC regions(marks regions by numbers).
    """
    max_v = self.graph.num_nodes
    self.visited, self.node_state = [UNVISITED] * max_v, [UNVISITED] * max_v
    self.low_values, self.component_region = [0] * max_v, [0] * max_v
    self.nodes_on_stack, self.region_num, self.dfs_counter = [], 0, 0
    for u in range(self.graph.num_nodes):
      if self.node_state[u] == UNVISITED:
        self.strongly_connected_components_of_graph_tarjans_helper(u)

  def bipartite_check_on_graph_helper(self, source: int, color: IntList):  # TODO RETEST
    """Uses bfs to check if the graph region connected to source is bipartite.

    Complexity per call: Time: O(|E| + |V|), Space: O(|V|)
    """
    queue = deque([source])
    color[source] = 0
    is_bipartite = True
    while queue and is_bipartite:
      u = queue.popleft()
      for v in self.graph.adj_list[u]:
        if color[v] == UNVISITED:
          color[v] = not color[u]
          queue.append(v)
        elif color[v] == color[u]:
          is_bipartite = False
          break
    return is_bipartite

  def bipartite_check_on_graph(self):  # TODO RETEST
    """Checks if a graph has the bipartite property.

    Complexity per call: Time: O(|E| + |V|), Space: O(|V|)
    More Uses: check bipartite property, labeling a graph for 2 coloring if it is bipart.
    """
    is_bipartite, color = True, [UNVISITED] * self.graph.num_nodes
    for u in range(self.graph.num_nodes):
      if color[u] == UNVISITED:
        is_bipartite = is_bipartite and self.bipartite_check_on_graph_helper(u, color)
        if not is_bipartite:
          break
    self.bipartite_colouring = color if is_bipartite else None
```

```python
def max_flow_find_augmenting_path_helper(self, source: int, sink: int):
    """Will check if augmenting path in the graph from source to sink exists via bfs.
    # TODO RETEST NEW
    Complexity per call: Time: O(|E| + |V|), Space O(|V|)
    Input
      source: the node which we are starting from.
      sink: the node which we are ending on.
    """
    distance, parents = [-1] * self.graph.num_nodes, [(-1, -1)] * self.graph.num_nodes
    queue, distance[source] = deque([source]), 0
    while queue:
        u = queue.popleft()
        if u == sink:
            self.dist, self.parent = distance, [el for el in parents]
            return True
        for idx in self.graph.adj_list[u]:
            v, cap, flow = self.graph.edge_list[idx]
            if cap - flow > 0 and distance[v] == -1:
                distance[v] = distance[u] + 1
                parents[v] = (u, idx)
                queue.append(v)
    self.dist, self.parent = [], []
    return False

def send_flow_via_augmenting_path(self, source: int, sink: int, flow_in: Num):
    """Function to recursively emulate sending a flow. returns min pushed flow.
    # TODO RETEST
    Complexity per call: Time: O(|V|), Space O(|V|)
    Uses: preorder finds the min pushed_flow post order mutates edge_list based on flow.
    Input:
      source: in this function it's technically the goal node
      sink: the current node we are observing
      flow_in: the smallest flow found on the way down
    """
    if source == sink:
        return flow_in
    u, edge_ind = self.parent[sink]
    _, edge_cap, edge_flow = self.graph.edge_list[edge_ind]
    pushed_flow = self.send_flow_via_augmenting_path(
        source, u, min(flow_in, edge_cap - edge_flow))
    self.graph.edge_list[edge_ind][2] = edge_flow + pushed_flow
    self.graph.edge_list[edge_ind ^ 1][2] -= pushed_flow
    return pushed_flow

def send_max_flow_via_dfs(self, u: int, sink: int, flow_in: Num):  # TODO RETEST
    """Function to recursively emulate sending a flow via dfs. Returns min pushed flow.

    Complexity per call: Time: O(|E| * |V|), Space O(|V|)
    More uses: a more efficient way of sending a flow
    Input:
      u: is the current node to be observed.
      sink: is the goal node (we might be able to just put it as instance var?).
      flow_in: the smallest flow found on the way down.
    """
    if u == sink or flow_in == 0:
        return flow_in
    start, end = self.last[u], len(self.graph.adj_list[u])
    for i in range(start, end):
        self.last[u], edge_ind = i, self.graph.adj_list[u][i]
        v, edge_cap, edge_flow = self.graph.edge_list[edge_ind]
        if self.dist[v] != self.dist[u] + 1:
            continue
        pushed_flow = self.send_max_flow_via_dfs(v, sink, min(flow_in, edge_cap-edge_flow))
        if pushed_flow != 0:
            self.graph.edge_list[edge_ind][2] = edge_flow + pushed_flow
            self.graph.edge_list[edge_ind ^ 1][2] -= pushed_flow
            return pushed_flow
    return 0

def max_flow_via_edmonds_karp(self, source: int, sink: int):
    """Compute max flow using edmonds karp's method.

    Complexity per call: Time: O(|V| * |E|^2), Space O(|V|)
    More Uses: max flow of the graph, min cut of the graph.
    """
    max_flow = 0
    while self.max_flow_find_augmenting_path_helper(source, sink):
        flow = self.send_flow_via_augmenting_path(source, sink, INF)
```

```python
        if flow == 0:
            break
        max_flow += flow
    return max_flow

def max_flow_via_dinic(self, source: int, sink: int):
    """Compute max flow using Dinic's method.

    Complexity per call: Time: O(|E| * |V|^2), Space O(|V|)
    More Uses: faster than the one above for most cases.
    """
    max_flow = 0
    while self.max_flow_find_augmenting_path_helper(source, sink):
        self.last = [0] * self.graph.num_nodes
        flow = self.send_max_flow_via_dfs(source, sink, INF)
        while flow != 0:
            max_flow += flow
            flow = self.send_max_flow_via_dfs(source, sink, INF)
    return max_flow


##################################################


from bisect import bisect_left, bisect_right
from collections import Counter
from functools import lru_cache
from itertools import takewhile, accumulate, repeat
from math import isqrt, log, gcd, prod, cos, sin, tau
from operator import mul as operator_mul
from random import choices
from array import array


class MathAlgorithms:
    def __init__(self):
        """Only take what you need. This list needs to be global or instance level or passed
        in. Attributes declared here must be passed in or global if not used in class format.
        """
        self.mod_p = 0

        self.sum_prime_factors = []
        self.num_divisors = []
        self.sum_divisors = []
        self.euler_phi = []
        self.sum_diff_prime_factors = []
        self.num_diff_prime_factors = []
        self.num_prime_factors = []

        self.binomial = {}
        self.fact = []
        self.inv_fact = []
        self.catalan_numbers = []

        self.primes_list = []
        self.primes_set = set()
        self.prime_factors = []
        self.factor_list = {}
        self.min_primes_list = []

        self.mrpt_bounds = []
        self.mrpt_tests = []

        self.fibonacci_list = []
        self.fibonacci_dict = {0: 0, 1: 1, 2: 1}

        self.fft_lengths = []
        self.fft_swap_indices = []
        self.fft_roots_of_unity = []

    def sieve_of_eratosthenes(self, n_inclusive: int) -> None:
        """Generates list of primes up to n via eratosthenes method.

        Complexity: Time: O(n lnln(n)), Space: post call O(n/ln(n)), mid-call O(n)
        Variants: number and sum of prime factors, of diff prime factors, of divisors, and phi
        """
        limit, prime_sieve = isqrt(n_inclusive) + 1, [True] * (n_inclusive + 1)
        prime_sieve[0] = prime_sieve[1] = False
        for prime in range(2, limit):
```

```python
        if prime_sieve[prime]:
            for composite in range(prime * prime, n_inclusive + 1, prime):
                prime_sieve[composite] = False
    self.primes_list = [i for i, is_prime in enumerate(prime_sieve) if is_prime]

def sieve_of_eratosthenes_optimized(self, n_inclusive: int) -> None:
    """Odds only optimized version of the previous method. Optimized to start at 3.

    Complexity: Time: O(max(n lnln(sqrt(n)), n)),
                Space: post call O(n/ln(n)), mid-call O(n/2)
    """
    sqrt_n, limit = ((isqrt(n_inclusive) - 3) // 2) + 1, ((n_inclusive - 3) // 2) + 1
    primes_sieve = [True] * limit
    for i in range(sqrt_n):
        if primes_sieve[i]:
            prime = 2*i + 3
            start = (prime*prime - 3)//2
            for j in range(start, limit, prime):
                primes_sieve[j] = False
    self.primes_list = [2] + [2*i + 3 for i, el in enumerate(primes_sieve) if el]

def block_sieve_odd(self, limit: int):
    """Block sieve that builds up block by block to the correct amount needed.

    Complexity: Time: O(max(n lnln(sqrt(n)), n)),
                Space: post call O(n/ln(n)), mid-call O(sqrt(n))
    """
    n, limit = limit, limit + 10
    end_sqrt, end_limit = isqrt(limit) + 1, (limit - 1) // 2
    sieve_and_block, primes, smaller_primes = [True] * (end_sqrt + 1), [2], []
    app, smaller_app = primes.append, smaller_primes.append
    for prime in range(3, end_sqrt, 2):
        if sieve_and_block[prime]:
            smaller_app([prime, (prime * prime - 1) // 2])
            for j in range(prime * prime, end_sqrt + 1, prime * 2):
                sieve_and_block[j] = False
    for low in range(0, end_limit, end_sqrt):
        for i in range(end_sqrt):
            sieve_and_block[i] = True
        for i, [p, idx] in enumerate(smaller_primes):
            for idx in range(idx, end_sqrt, p):
                sieve_and_block[idx] = False
            smaller_primes[i][1] = idx - end_sqrt + (0 if idx >= end_sqrt else p)
        if low == 0:
            sieve_and_block[0] = False
        for i in range(min(end_sqrt, (end_limit + 1) - low)):
            if sieve_and_block[i] and (low + i) * 2 + 1 <= n:
                app((low + i) * 2 + 1)
    self.primes_list = primes

def prime_sieve_super_fast(self, limit: int):
    """Optimized wheel sieve with bit compression.

    Complexity: Time: O(max(n lnln(sqrt(n)), n)),
                Space: post call O(n/ln(n)), mid-call S((n/3)/8)  4/32 == 1/8
    """
    res = [] if limit < 2 else [2] if limit == 2 else [2, 3]
    if limit > 4:
        n = limit + 1
        flag = n % 6 == 2
        sieve = array('L', repeat(0, (n // 3 + flag >> 5) + 1))
        for i in range(1, isqrt(n) // 3 + 1):
            if not ((sieve[i >> 5] >> (i & 31)) & 1):
                k = (3 * i + 1) | 1
                for j in range(k * k // 3, n // 3 + flag, 2 * k):
                    sieve[j >> 5] |= 1 << (j & 31)
                for j in range(k * (k - 2 * (i & 1) + 4) // 3, n // 3 + flag, 2 * k):
                    sieve[j >> 5] |= 1 << (j & 31)
        res.extend([(3 * i + 1) | 1 for i in range(1, n // 3 + (limit % 6 == 1))
                    if not ((sieve[i >> 5] >> (i & 31)) & 1)])
    self.primes_list = res

def sieve_of_min_primes(self, n_inclusive: int) -> None:
    """Stores the min or max prime divisor for each number up to n.

    Complexity: Time: O(max(n lnln(sqrt(n)), n)), Space: post call O(n)
    """
    min_primes = [0] * (n_inclusive + 1)
    min_primes[1] = 1
```

```python
    for prime in self.primes_list:
        min_primes[prime] = prime
        start, end, step = prime * prime, n_inclusive + 1, prime if prime == 2 else 2*prime
        for multiple in range(start, end, step):
            min_primes[multiple] = prime
    self.min_primes_list = min_primes

def sieve_of_eratosthenes_variants(self, n_inclusive: int) -> None:
    """Seven variants of prime sieve listed above.

    Complexity:
        function 1: Time: O(n log(n)), Space: O(n)
        function 2: Time: O(n lnln(n)), Space: O(n)
        function 3: Time: O(n lnln(n)), Space: O(n)
        function 4: Time: O(n lnln(n) log(n)), Space: O(n)
    """
    self.euler_phi_plus_sum_and_number_of_diff_prime_factors(n_inclusive)
    self.num_and_sum_of_divisors(n_inclusive)
    self.num_and_sum_of_prime_factors(n_inclusive)

def num_and_sum_of_divisors(self, limit: int) -> None:
    """Does a basic sieve. Complexity function 1."""
    limit += 1
    num_div = [1] * limit
    sum_div = [1] * limit
    for divisor in range(2, limit):
        for multiple in range(divisor, limit, divisor):
            num_div[multiple] += 1
            sum_div[multiple] += divisor
    self.num_divisors = num_div
    self.sum_divisors = sum_div

def euler_phi_plus_sum_and_number_of_diff_prime_factors(self, limit: int) -> None:
    """This is basically same as sieve just using different ops. Complexity function 2."""
    limit += 1
    num_diff_pf = [0] * limit
    sum_diff_pf = [0] * limit
    phi = [i for i in range(limit)]
    for prime in range(2, limit):  # name is prime since we only iterate if its prime
        if num_diff_pf[prime] == 0:
            for multiple in range(prime, limit, prime):
                num_diff_pf[multiple] += 1
                sum_diff_pf[multiple] += prime
                phi[multiple] = (phi[multiple]//prime) * (prime-1)
    self.num_diff_prime_factors = num_diff_pf
    self.sum_diff_prime_factors = sum_diff_pf
    self.euler_phi = phi

def num_and_sum_of_prime_factors(self, limit: int) -> None:
    """This uses similar idea to sieve but avoids divisions. Complexity function 3."""
    inclusive_limit = limit + 1
    num_pf = [0] * inclusive_limit
    sum_pf = [0] * inclusive_limit
    for prime in range(2, inclusive_limit):
        if num_pf[prime] == 0:  # or sum_pf if using that one
            exponent_limit = int(log(limit, prime)) + 2
            for exponent in range(1, exponent_limit):
                prime_to_exponent = prime**exponent
                for multiple in range(prime_to_exponent, inclusive_limit, prime_to_exponent):
                    sum_pf[multiple] += prime
                    num_pf[multiple] += 1
    self.num_prime_factors = num_pf
    self.sum_prime_factors = sum_pf

def num_and_sum_of_divisors_faster(self, limit: int) -> None:
    """runs in around x0.8-x0.5 the runtime of the slower one. Complexity function 4."""
    inclusive_limit = limit + 1
    num_divs = [1] * inclusive_limit
    sum_divs = [1] * inclusive_limit
    cur_pows = [1] * inclusive_limit
    for prime in range(2, inclusive_limit):
        if num_divs[prime] == 1:
            exponent_limit = int(log(limit, prime)) + 2
            for exponent in range(1, exponent_limit):
                prime_to_exponent = prime ** exponent
                for multiple in range(prime_to_exponent, inclusive_limit, prime_to_exponent):
                    cur_pows[multiple] += 1
            tmp = prime - 1  # this line and the line below used for sum_divs
            prime_powers = [prime ** exponent for exponent in range(0, exponent_limit + 1)]
```

```python
        for multiple in range(prime, inclusive_limit, prime):
          num_divs[multiple] *= cur_pows[multiple]
          sum_divs[multiple] *= ((prime_powers[cur_pows[multiple]] - 1) // tmp)
          cur_pows[multiple] = 1
    self.num_divisors = num_divs
    self.sum_divisors = sum_divs

  def is_prime_trivial(self, n: int) -> bool:
    """Tests if n is prime via divisors up to sqrt(n).

    Complexity per call: Time: O(sqrt(n)), T(sqrt(n)/3), Space: O(1)
    Optimizations: 6k+i method, since we checked 2 and 3 only need test form 6k+1 and 6k+5
    """
    if n < 4:  # base case of n in 0, 1, 2, 3
      return n > 1
    if n % 2 == 0 or n % 3 == 0:  # this check is what allows us to use 6k + i
      return False
    limit = isqrt(n) + 1
    for p in range(5, limit, 6):
      if n % p == 0 or n % (p+2) == 0:
        return False
    return True

  def is_composite(self, a: int, d: int, n: int, s: int) -> bool:
    """The witness test of miller rabin.

    Complexity per call: Time O(log^3(n)), Space: O(2**s, bits)
    """
    if 1 == pow(a, d, n):
      return False
    for i in range(s):
      if n-1 == pow(a, d * 2**i, n):
        return False
    return True

  def miller_rabin_primality_test(self, n: int, precision_for_huge_n=16) -> bool:
    """Probabilistic primality test with error rate of 4^(-k) past 341550071728321.

    Complexity per call: Time O(k log^3(n)), Space: O(2**s) bits
    Note: range(16) used to just do a small test to weed out lots of numbers.
    """
    if n < self.primes_list[-1]:
      return n in self.primes_set
    if any((n % self.primes_list[p] == 0) for p in range(16)) or n == 3215031751:
      return False  # 3215031751 is an edge case for this data set
    d, s = n-1, 0
    while d % 2 == 0:
      d, s = d // 2, s + 1
    if n < self.mrpt_bounds[-1]:
      for i, bound in enumerate(self.mrpt_bounds, 2):
        if n < bound:
          return not any(self.is_composite(self.mrpt_tests[j], d, n, s) for j in range(i))
    return not any(self.is_composite(prime, d, n, s)
                   for prime in choices(self.primes_list, k=precision_for_huge_n))

  def miller_rabin_primality_test_prep(self):
    """This function needs to be called before miller rabin"""
    self.mrpt_bounds = [1373653, 25326001, 118670087467,
                        2152302898747, 3474749660383, 341550071728321]
    self.mrpt_tests = [2, 3, 5, 7, 11, 13, 17]
    self.sieve_of_eratosthenes(1000)  # comment out if different size needed
    self.primes_set = set(self.primes_list)  # comment out if already have bigger size

  def prime_factorize_n(self, n: int) -> Dict:
    """A basic prime factorization of n function. without primes its just O(sqrt(n))

    Complexity: Time: O(sqrt(n)/ln(sqrt(n))), Space: O(log n)
    Variants: number and sum of prime factors, of diff prime factors, of divisors, and phi
    """
    limit, prime_factors = isqrt(n) + 1, []
    for prime in takewhile(lambda x: x < limit, self.primes_list):
      while n % prime == 0:
        n //= prime
        prime_factors.append(prime)
    if n > 1:  # n is prime or last factor of n is prime
      prime_factors.append(n)
    self.factor_list = Counter(prime_factors)
    return Counter(prime_factors)
```

```python
  def prime_factorize_n_log_n(self, n: int) -> Dict:
    """An optimized prime factorization of n function based on min primes already sieved.

    Complexity: Time: O(log n), Space: O(log n)
    Optimization: assign append to function and assign min_prime[n] to a value in the loop
    """
    prime_factors = []
    while n > 1:
      prime_factors.append(self.min_primes_list[n])
      n = n // self.min_primes_list[n]
    self.factor_list = Counter(prime_factors)
    return Counter(prime_factors)

  def factorial_prime_factors(self, limit: int) -> None:
    """This uses similar idea to sieve but avoids divisions. Complexity function 3."""
    end_point = bisect_right(self.primes_list, limit)
    prime_factors = [0] * end_point
    for i in range(end_point):
      prime, prime_amount, x = self.primes_list[i], 0, limit
      while x:
        x //= prime
        prime_amount += x
      prime_factors[i] = prime_amount
    self.num_prime_factors = prime_factors

  def prime_factorize_n_variants(self, n: int) -> int:
    """Covers all the variants listed above, holds the same time and space complexity"""
    limit = isqrt(n) + 1
    sum_diff_prime_factors, num_diff_prime_factors = 0, 0
    sum_prime_factors, num_prime_factors = 0, 0
    sum_divisors, num_divisors, euler_phi = 1, 1, n
    for prime in takewhile(lambda x: x < limit, self.primes_list):
      if n % prime == 0:
        mul, total = prime, 1  # for sum of divisors
        power = 0              # for num of divisors
        while n % prime == 0:
          n //= prime
          power += 1    # for num prime factors, num divisors, and sum prime factors
          total += mul     # for sum divisors
          mul *= prime     # for sum divisors
        sum_diff_prime_factors += prime
        num_diff_prime_factors += 1
        sum_prime_factors += (prime * power)
        num_prime_factors += power
        num_divisors *= (power + 1)
        sum_divisors *= total
        euler_phi -= (euler_phi//prime)
    if n > 1:
      num_diff_prime_factors += 1
      sum_diff_prime_factors += n
      num_prime_factors += 1
      sum_prime_factors += n
      num_divisors *= 2
      sum_divisors *= (n + 1)
      euler_phi -= (euler_phi // n)
    return num_diff_prime_factors

  def polynomial_function_f(self, x: int, c: int, m: int) -> int:
    """Represents the function f(x) = (x^2 + c) in pollard rho and brent, cycle finding"""
    return (x * x + c) % m  # paste this in code for speed up. is here for clarity only

  def pollard_rho(self, n: int, x0=2, c=1) -> int:
    """Semi fast integer factorization. Based on the birthday paradox and floyd cycle find

    Complexity per call: Time: O(min(max(p), n^0.25) * ln n), Space: O(log2(n) bits)
    """
    x, y, g = x0, x0, 1
    while g == 1:  # when g != 1 then we found a divisor of n shared with x - y
      x = self.polynomial_function_f(x, c, n)
      y = self.polynomial_function_f(self.polynomial_function_f(y, c, n), c, n)
      g = gcd(abs(x - y), n)
    return g

  def brent_pollard_rho(self, n: int, x0=2, c=1) -> int:
    """Faster version of above. Similar time complexity. uses faster cycle finder."""
    x, m = x0, 128  # 128 here is used as a small power of 2 vs using 100 more below
    g = q = left = 1
    xs = y = 0
    while g == 1:
```

```python
        y, k = x, 0
        for _ in range(1, left):
            x = (x * x + c) % n
        while k < left and g == 1:
            xs, end = x, min(m, left - k)    # here we are using a technique similar to cache
            for _ in range(end):       # and loop unrolling were we try for sets of cycles
                x = (x * x + c) % n      # if we over shoot we can just go back which is
                q = (q * abs(y - x)) % n  # technically what end computes
            k, g = k + m, gcd(q, n)
        left = left * 2
    if g == n:
        while True:
            xs = (xs * xs + c) % n
            g = gcd(abs(xs - y), n)
            if g != 1:
                break
    return g

def extended_euclid_recursive(self, a: int, b: int) -> Tuple[int, int, int]:
    """Solves coefficients of Bezout identity: ax + by = gcd(a, b), recursively

    Complexity per call: Time: O(log n), Space: O(log n) at the deepest call.
    """
    if 0 == b:
        return 1, 0, a
    x, y, d = self.extended_euclid_recursive(b, a % b)
    return y, x-y*(a//b), d

def extended_euclid_iterative(self, a: int, b: int) -> Tuple[int, int, int]:
    """Solves coefficients of Bezout identity: ax + by = gcd(a, b), iteratively.

    Complexity per call: Time: O(log n) about twice as fast in python vs above, Space:O(1)
    Optimizations and notes:
        divmod and abs are used to help deal with big numbers, remove if < 2**64 for speedup
    """
    last_remainder, remainder = abs(a), abs(b)
    x, y, last_x, last_y = 0, 1, 1, 0
    while remainder:
        last_remainder, (quotient, remainder) = remainder, divmod(last_remainder, remainder)
        x, last_x = last_x - quotient * x, x
        y, last_y = last_y - quotient * y, y
    return -last_x if a < 0 else last_x, -last_y if b < 0 else last_y, last_remainder

def safe_modulo(self, a: int, n: int) -> int:
    """Existence is much for c++ which doesn't always handle % operator nicely.
    use ((a % n) + n) % n for getting proper mod of a potential negative value
    use (a + b) % --> ((a % n) + (b % n)) % n for operations sub out + for * and -
    """
    return ((a % n) + n) % n

def modular_linear_equation_solver(self, a: int, b: int, n: int) -> List[int]:
    """Solves gives the solution x in ax = b(mod n).
    # TODO RETEST
    Complexity per call: Time: O(log n), Space: O(d)
    """
    x, y, d = self.extended_euclid_iterative(a, n)
    if 0 == b % d:
        x = (x * (b//d)) % n
        return [(x + i*(n//d)) % n for i in range(d)]
    return []

def linear_diophantine_1(self, a: int, b: int, c: int) -> Tuple[int, int]:
    """Solves for x, y in ax + by = c. From stanford icpc 2013-14
    # TODO RETEST
    Complexity per call: Time: O(log n), Space: O(1).
    Notes: order matters? 25x + 18y = 839 != 18x + 25y = 839
    """
    d = gcd(a, b)
    if c % d == 0:
        x = c//d * self.mod_inverse(a//d, b//d)
        return x, (c - a * x) // b
    return -1, -1

def linear_diophantine_2(self, a: int, b: int, c: int) -> Tuple[int, int]:
    """Solves for x0, y0 in x = x0 + (b/d)n, y = y0 - (a/d)n.
    derived from ax + by = c, d = gcd(a, b), and d|c.
    Can further derive into: n = x0 (d/b), and n = y0 (d/a).
    # TODO RETEST
    Complexity per call: Time: O(log n), Space: O(1).
```

```python
    Optimizations and notes:
        unlike above this function order doesn't matter if a != b
        for a speedup call math.gcd(a, b) at start and return accordingly on two lines
    """
    x, y, d = self.extended_euclid_iterative(a, b)
    return (-1, -1) if c % d != 0 else (x * (c // d), y * (c // d))

def mod_inverse(self, b: int, m: int) -> None | int:
    """Solves b^(-1) (mod m).

    Complexity per call: Time: O(log n), Space: O(1)
    """
    x, y, d = self.extended_euclid_iterative(b, m)
    return None if d != 1 else x % m  # -1 instead of None if we intend use return

def chinese_remainder_theorem_1(self, remainders: List[int], modulos: List[int]) -> int:
    """Steven's CRT version to solve x in x = r[0] (mod m[0]) ... x = r[n-1] (mod m[n-1]).
    # TODO RETEST
    Complexity per call: Time: O(n log n), Space: O(1)? O(mt) bit size:
    Optimizations:
        prod is used from math since 3.8,
        we use mod mt in the forloop since x might get pretty big.
    """
    mt, x = prod(modulos), 0
    for i, modulo in enumerate(modulos):
        p = mt // modulo
        x = (x + (remainders[i] * self.mod_inverse(p, modulo) * p)) % mt
    return x

def chinese_remainder_theorem_helper(self, mod1: int, rem1: int,
                                     mod2: int, rem2: int) -> Tuple[int, int]:
    """Chinese remainder theorem (special case): find z such that z % m1 = r1, z % m2 = r2
    Here, z is unique modulo M = lcm(m1, m2). Return (z, M).  On failure, M = -1.
    from: stanford icpc 2016
    # TODO RETEST
    Complexity per call: Time: O(log n), Space: O(1)
    """
    s, t, d = self.extended_euclid_iterative(mod1, mod2)
    if rem1 % d != rem2 % d:
        mod3, s_rem_mod, t_rem_mod = mod1*mod2, s*rem2*mod1, t*rem1*mod2
        return ((s_rem_mod + t_rem_mod) % mod3) // d, mod3 // d
    return 0, -1

def chinese_remainder_theorem_2(self, remainders: List[int],
                                modulos: List[int]) -> Tuple[int, int]:
    """Chinese remainder theorem: find z such that z % m[i] = r[i] for all i.  Note that
    solution is unique modulo M = lcm_i (m[i]).  Return (z, M). On failure, M = -1.
    Note that we do not require the r[i]'s to be relatively prime.
    from: stanford icpc 2016
    # TODO RETEST
    Complexity per call: Time: O(n log n), Space: O(1)? O(mt) bit size
    """
    z_m = remainders[0], modulos[0]
    for i, modulo in enumerate(modulos[1:], 1):
        z_m = self.chinese_remainder_theorem_helper(z_m[1], z_m[0], modulo, remainders[i])
        if -1 == z_m[1]:
            break
    return z_m

def fibonacci_n_iterative(self, n: int) -> None:
    """Classic fibonacci solver. Generates answers from 0 to n inclusive.

    Complexity per call: Time: O(n), Space: O(n).
    """
    fib_list = [0] * (n + 1)
    fib_list[1] = 1
    for i in range(2, n+1):
        fib_list[i] = fib_list[i - 1] + fib_list[i - 2]
    self.fibonacci_list = fib_list

@lru_cache(maxsize=None)
def fibonacci_n_dp_cached(self, n: int) -> int:
    """Cached Dynamic programming to get the nth fibonacci. Derived via Cassini's identity

    Complexity per call: Time: O(log n), Space: increase by O(log n).
    Optimization: can go back to normal memoization with same code but using dictionary.
    """
    if n < 3:
        return 1 if n else 0
```

```python
    f1, f2 = self.fibonacci_n_dp_cached(n // 2 + 1), self.fibonacci_n_dp_cached((n - 1) // 2)
    return f1 * f1 + f2 * f2 if n & 1 else f1 * f1 - f2 * f2

@lru_cache(maxsize=None)
def fibonacci_n_dp_cached_faster(self, n: int) -> int:
    """Same as above but runs in ~Time*0.75 i.e. Above takes 20 seconds this takes 15."""
    if n < 3:
        return 1 if n else 0
    k = (n + 1)//2 if n & 1 else n//2
    k1, k2 = self.fibonacci_n_dp_cached_faster(k), self.fibonacci_n_dp_cached_faster(k-1)
    return k1*k1 + k2*k2 if n & 1 else (2*k2 + k1) * k1

def generate_catalan_n(self, n: int) -> None:
    """Generate catalan up to n iteratively.

    Complexity per call: Time: O(n*O(multiplication)), Space: O(n * 2^(log n)).
    """
    catalan = [0] * (n+1)
    catalan[0] = 1
    for i in range(n-1):
        catalan[i + 1] = catalan[i] * (4*i + 2) // (i + 2)
    self.catalan_numbers = catalan

def generate_catalan_n_mod_inverse(self, n: int, p: int) -> None:
    """Generate catalan up to n iteratively cat n % p.

    Complexity per call: Time: O(n log n), Space: O(n * (2^(log n)%p)).
    Variants: use prime factors of the factorial to cancel out the primes
    """
    catalan = [0] * (n+1)
    catalan[0] = 1
    for i in range(n-1):
        catalan[i+1] = (((4*i + 2) % p) * (catalan[i] % p) * pow(i+2, p-2, p)) % p
    self.catalan_numbers = catalan

def catalan_via_prime_factors_slower(self, n: int, k: int, mod_m: int) -> int:
    """Compute the nth Catalan number mod_n via prime factor reduction of C(2n, n)/(n+1).
    Notes: The function "num_and_sum_of_prime_factors" needs to be modified for computing number
    of each prime factor in all the numbers between 1-2n or 1 to n.

    Complexity per call: Time: O(max(n lnln(sqrt(n)), n)), Space: O(n/ln(n)).
    Optimization: compute bottom! once and factorize k+1 instead of two calls to factorial
     factor
    """
    top, bottom, ans = n, k, 1  # n = 2n and k = n in C(n, k) = (2n, n)
    self.prime_sieve_super_fast(n)  # or any sieve
    self.factorial_prime_factors(top)
    top_factors = [amt for amt in self.num_prime_factors]
    self.factorial_prime_factors(bottom)  # will handle n!n! in one go stored in num
    for prime_ind, exponent in enumerate(self.num_prime_factors):
        top_factors[prime_ind] -= exponent
    self.factorial_prime_factors(bottom + 1)  # will handle n!n! in one go stored in num
    for prime_ind, exponent in enumerate(self.num_prime_factors):
        top_factors[prime_ind] -= exponent
    for prime_ind, exponent in enumerate(top_factors):  # remember use multiplication not add
        if exponent > 0:
            ans = (ans * pow(self.primes_list[prime_ind], exponent, mod_m)) % mod_m
    return ans

def c_n_k(self, n: int, k: int, inverse: bool) -> int:
    """Computes C(n, k) % p. From competitive programming 4.

    Complexity per call: Time: v1 = O(log n), v2 = O(1), Space: O(1).
    v1 is uncommented, v2 is the commented out line, and must be precomputed see below.
    """
    if n <= k or k == 0:  # base case: explicit
        return 0 if n < k else 1
    if inverse:
        return 0 if n < k else (self.fact[n] * self.inv_fact[k] * self.inv_fact[n-k]) % self.mod_p
    n_fact, k_fact, n_k_fact, p = self.fact[n], self.fact[k], self.fact[n - k], self.mod_p
    return (n_fact * pow(k_fact, p - 2, p) * pow(n_k_fact, p - 2, p)) % p

def binomial_coefficient_n_mod_p_prep(self, max_n: int, mod_p: int, inverse: bool):
    """Does preprocessing for binomial coefficients. From competitive programming 4.

    Complexity per call: Time: v1 O(n), v2 = O(n), Space: O(n).
    Optimization and notes: v2 -> uncomment lines for C(n, k) % p in O(1) time, see above.
    """
    factorial_mod_p = [1] * max_n
```

```python
    for i in range(1, max_n):
        factorial_mod_p[i] = (factorial_mod_p[i - 1] * i) % mod_p
    self.mod_p, self.fact = mod_p, factorial_mod_p
    if inverse:
        inverse_factorial_mod_p = [0] * max_n
        inverse_factorial_mod_p[-1] = pow(factorial_mod_p[-1], mod_p-2, mod_p)
        for i in range(max_n-2, -1, -1):
            inverse_factorial_mod_p[i] = (inverse_factorial_mod_p[i+1] * (i+1)) % mod_p
        self.inv_fact = inverse_factorial_mod_p

@lru_cache(maxsize=None)
def binomial_coefficient_dp_with_cache(self, n: int, k: int) -> int:
    """Uses the recurrence to calculate binomial coefficient. Cached for memoization.

    Complexity per call: Time: O(n*k), Space: O(n*k).
    """
    if n == k or 0 == k:
        return 1
    take_case = self.binomial_coefficient_dp_with_cache(n-1, k)
    skip_case = self.binomial_coefficient_dp_with_cache(n-1, k-1)
    return take_case + skip_case

def fft_prepare_swap_indices(self, a_len: int):
    """Gives us all the swap pairs needed to for an FFT call stored in fft_swap_indices.

    Complexity per call: Time: O(n), Space: O(n), S(n/2) | n == 2^i -> minimax value of i.
    """
    a_bit_len, a_bit_half = a_len.bit_length(), (a_len.bit_length()-1)//2
    swap_size = (1 << a_bit_half) - 1
    swap_size = swap_size << (a_bit_half-1) if a_bit_len & 1 else swap_size << a_bit_half
    swaps, k, ind = [None] * swap_size, 0, -1
    for i in range(1, a_len):
        bit_mask = a_len >> 1
        while k & bit_mask:
            k = k ^ bit_mask
            bit_mask = bit_mask >> 1
        k = k ^ bit_mask
        if i < k:
            swaps[ind := ind + 1] = (i, k)
    self.fft_swap_indices = swaps

def fft_prepare_lengths_list(self, a_len: int):
    """Function for all powers 2 from 2 - a_len, inclusive. O(log n) complexity."""
    self.fft_lengths = [2 ** power for power in range(1, a_len.bit_length())]

def fft_prepare_roots_helper(self, length: int, angle: float) -> List[complex]:
    """Precomputes roots of unity for a given length and angle. accumulate used here :).

    Complexity per call: Time: O(n), Space: O(n) | n == 2^i.
    """
    initial_root_of_unity = complex(1)
    multiplier = complex(cos(angle), sin(angle))
    return list(accumulate([multiplier] * length, operator_mul, initial=initial_root_of_unity))

def fft_prepare_roots_of_unity(self, invert: bool):
    """Precomputes all roots of unity for all lengths. Stores the result for later use.

    Complexity per call: Time: O(2^((log n)+1)-1) = O(n),
               Space: O(n), S(2^((log n)+1)-1) | n == len of our data, which is 2^i.
    """
    signed_tau: float = -tau if invert else tau
    self.fft_roots_of_unity = [self.fft_prepare_roots_helper(length//2 - 1, signed_tau/length)
                               for length in self.fft_lengths]

def fft_in_place_fast_fourier_transform(self, a_vector: List[complex], invert: bool):
    """Optimized in-place Cooley-Tukey FFT algorithm. Modifies a_vector.

    Complexity per call: Time: O(n log n), Space: O(1) | technically O(n) but we precompute.
    Optimizations: swap indices, lengths, and roots of unity all have be calculated beforehand.
     This allows us to only do those once in when doing multiplication
    """
    a_len = len(a_vector)
    for i, j in self.fft_swap_indices:
        a_vector[i], a_vector[j] = a_vector[j], a_vector[i]
    for k, length in enumerate(self.fft_lengths):  # is [2, 4, 8..., 2^(i-1), 2^i] | n == 2^i
        j_end = length // 2  # j_end is to avoid repeated divisions in the innermost loop
        for i in range(0, a_len, length):
            for j, w in enumerate(self.fft_roots_of_unity[k]):
                i_j, i_j_j_end = i + j, i + j + j_end
```

```python
            u, v = a_vector[i_j], w * a_vector[i_j_j_end]
            a_vector[i_j], a_vector[i_j_j_end] = u + v, u - v
        if invert:
          a_vector[:] = [complex_number/a_len for complex_number in a_vector]

    def fft_normalize(self, a_vector: List[int], base: int) -> List[int]:
        """Normalizes polynomial a for a given base. base 10 will result in a base 10 number

        Complexity per call: Time: O(n), Space: O(1) -> in fact we often reduce overall space.
        """
        carry, end = 0, len(a_vector) - 1
        for i in range(end + 1):
          carry, a_vector[i] = divmod(a_vector[i] + carry, base)
        while 0 == a_vector[end]:
          end -= 1
        return a_vector[:end+1][::-1]

    def fft_multiply_in_place(self, polynomial_a: List[int], polynomial_b: List[int]) -> List[int
      ]:
        """Multiply two polynomials with the option to normalize then after.

        Complexity per call: Time: O(n log n), T(4(n log n)),
                     Space: O(n),  S(4n) | [n == |a| + |b|]
        Optimizations: listed fft_in_place_fast_fourier_transform.
        """
        a_len, b_len = len(polynomial_a), len(polynomial_b)
        n = 2**((a_len + b_len).bit_length())      # computes 2^(log2(a+b) + 1)
        n = n if (a_len + b_len) != n//2 else n//2  # optimization that fixes n when (a+b) % 2 == 0
        a_vector = [complex(i) for i in polynomial_a] + [complex(0)] * (n - a_len)
        b_vector = [complex(i) for i in polynomial_b] + [complex(0)] * (n - b_len)
        self.fft_prepare_swap_indices(n)     # these three calls are for optimization with
        self.fft_prepare_lengths_list(n)     # multiplying, if calling fft outside multiply
        self.fft_prepare_roots_of_unity(False)  # you will need to call for each size of array.
        self.fft_in_place_fast_fourier_transform(a_vector, False)
        self.fft_in_place_fast_fourier_transform(b_vector, False)
        a_vector = [i * j for i, j in zip(a_vector, b_vector)]
        self.fft_prepare_roots_of_unity(True)
        self.fft_in_place_fast_fourier_transform(a_vector, True)
        a_vector = [int(round(el.real)) for el in a_vector]   # optional
        return self.fft_normalize(a_vector, 10)       # optional turns into base 10 num


##################################################


from math import isclose, dist, sin, cos, acos, sqrt, fsum, pi
from itertools import combinations
# remember to sub stuff out for integer ops when you want only integers
# for ints need to change init, eq and
# hard code these in for performance speedup
CCW = 1  # counterclockwise
CW = -1  # clockwise
CL = 0   # collinear
EPS = 1e-12  # used in some spots


class Pt2d:  # TODO RETEST
    __slots__ = ("x", "y")
    # def __init__(self, x_val, y_val): self.x, self.y = map(float, (x_val, y_val))
    def __init__(self, x_val, y_val): self.x, self.y = x_val, y_val

    def __add__(self, other): return Pt2d(self.x + other.x, self.y + other.y)
    def __sub__(self, other): return Pt2d(self.x - other.x, self.y - other.y)
    def __mul__(self, scale): return Pt2d(self.x * scale, self.y * scale)
    def __truediv__(self, scale): return Pt2d(self.x / scale, self.y / scale)
    def __floordiv__(self, scale): return Pt2d(self.x // scale, self.y // scale)

    # def __eq__(self, other): return isclose(self.x, other.x) and isclose(self.y, other.y)
    def __eq__(self, other): return self.x == other.x and self.y == other.y

    # def __lt__(self, other):
    #    return self.x < other.x if not isclose(self.x, other.x) else self.y < other.y
    def __lt__(self, other): return self.x < other.x if self.x != other.x else self.y < other.y

    def __str__(self): return "{} {}".format(self.x, self.y)
    # def __str__(self): return "(x = {:20}, y = {:20})".format(self.x, self.y)
    def __round__(self, n): return Pt2d(round(self.x, n), round(self.y, n))
    def __hash__(self): return hash((self.x, self.y))


    def get_tup(self): return self.x, self.y


Triangle = Tuple[Pt2d, Pt2d, Pt2d] | None
ClosestPair = Tuple[Numeric, Pt2d, Pt2d]


class Pt3d:  # TODO RETEST
    def __init__(self, x_val, y_val, z_val):
        self.x, self.y, self.z = map(float, (x_val, y_val, z_val))

    def __add__(self, other): return Pt3d(self.x + other.x, self.y + other.y, self.z + other.z)
    def __sub__(self, other): return Pt3d(self.x - other.x, self.y - other.y, self.z - other.z)
    def __mul__(self, scale): return Pt3d(self.x * scale, self.y * scale, self.z * scale)
    def __truediv__(self, scale): return Pt3d(self.x / scale, self.y / scale, self.z / scale)
    def __floordiv__(self, scale): return Pt3d(self.x // scale, self.y // scale, self.z // scale)

    def __eq__(self, other):
        return isclose(self.x, other.x) and isclose(self.y, other.y) and isclose(self.z, other.z)

    def __lt__(self, other):
        return False if self == other else (self.x, self.y, self.z) < (other.x, other.y, other.y)

    def __str__(self): return "{} {} {}".format(self.x, self.y, self.z)
    # def __str__(self): return "(x = {:20}, y = {:20}), z = {:20})".format(self.x, self.y, self.z
      )
    def __round__(self, n): return Pt3d(round(self.x, n), round(self.y, n), round(self.z, n))
    def __hash__(self): return hash((self.x, self.y, self.z))


class QuadEdge:
    __slots__ = ("origin", "rot", "o_next", "used")

    def __init__(self):
        self.origin = None
        self.rot = None
        self.o_next = None
        self.used = False

    def rev(self): return self.rot.rot
    def l_next(self): return self.rot.rot.o_next.rot
    def o_prev(self): return self.rot.o_next.rot
    def dest(self): return self.rot.rot.origin


QuadEdgePair = Tuple[QuadEdge, QuadEdge]


class QuadEdgeDataStructure:
    def __init__(self):
        pass

    def make_edge(self, in_pt, out_pt):
        e1 = QuadEdge()
        e2 = QuadEdge()
        e3 = QuadEdge()
        e4 = QuadEdge()
        e1.origin = in_pt
        e2.origin = out_pt
        e3.origin = None
        e4.origin = None
        e1.rot = e3
        e2.rot = e4
        e3.rot = e2
        e4.rot = e1
        e1.o_next = e1
        e2.o_next = e2
        e3.o_next = e4
        e4.o_next = e3
        return e1

    def splice(self, a, b):
        a.o_next.rot.o_next, b.o_next.rot.o_next = b.o_next.rot.o_next, a.o_next.rot.o_next
        a.o_next, b.o_next = b.o_next, a.o_next

    def delete_edge(self, edge):
        self.splice(edge, edge.o_prev())
        self.splice(edge.rev(), edge.rev().o_prev())
        # del edge.rot.rot.rot
```

```python
        # del edge.rot.rot
        # del edge.rot
        # del edge

    def connect(self, a, b):
        e = self.make_edge(a.dest(), b.origin)
        self.splice(e, a.l_next())
        self.splice(e.rev(), b)
        return e


class GeometryAlgorithms:
    def __init__(self):
        self.quad_edges = QuadEdgeDataStructure()

    def compare_ab(self, a: Numeric, b: Numeric) -> int:
        """Compare a b, for floats and ints. It is useful when you want set values to observe.
        paste directly into code and drop isclose for runtime speedup."""
        return 0 if isclose(a, b) else -1 if a < b else 1

    def dot_product(self, left_vector: Pt2d, right_vector: Pt2d) -> Numeric:
        """Compute the scalar product a.b of a,b equivalent to: a . b"""
        return left_vector.x*right_vector.x + left_vector.y*right_vector.y

    def cross_product(self, left_vector: Pt2d, right_vector: Pt2d) -> Numeric:
        """Computes the scalar value perpendicular to a,b equivalent to: a x b"""
        return left_vector.x*right_vector.y - left_vector.y*right_vector.x

    def distance_normalized(self, left_point: Pt2d, right_point: Pt2d) -> float:
        """Normalized distance between two points a, b equivalent to: sqrt(a^2 + b^2) = distance."""
        return dist(left_point, right_point)

    def distance(self, left_point: Pt2d, right_point: Pt2d) -> Numeric:
        """Squared distance between two points a, b equivalent to: a^2 + b^2 = distance."""
        return self.dot_product(left_point - right_point, left_point - right_point)

    def rotate_cw_90_wrt_origin(self, point: Pt2d) -> Pt2d:
        """Compute a point rotation on pt. Just swap x and y and negate x."""
        return Pt2d(point.y, -point.x)

    def rotate_ccw_90_wrt_origin(self, point: Pt2d) -> Pt2d:
        """Compute a point rotation on pt. Just swap x and y and negate y."""
        return Pt2d(-point.y, point.x)

    def rotate_ccw_rad_wrt_origin(self, point: Pt2d, degree_in_radians: float) -> Pt2d:
        """Compute a counterclockwise point rotation on pt. Accurate only for floating point cords.
        formula: x = (x cos(rad) - y sin(rad)), y = (x sin(rad) + y cos (rad)).

        Complexity per call: Time: O(1), Space: O(1).
        Optimizations: calculate cos and sin outside the return, so you don't double call each.
        """
        return Pt2d(point.x * cos(degree_in_radians) - point.y * sin(degree_in_radians),
                    point.x * sin(degree_in_radians) + point.y * cos(degree_in_radians))

    def point_c_rotation_wrt_line_ab(self, a_point: Pt2d, b_point: Pt2d, c_point: Pt2d) -> int:
        """Determine orientation of c wrt line ab, in terms of collinear clockwise counterclockwise.
        Since 2d cross-product is the area of the parallelogram, we can use this to accomplish this.

        Complexity per call: Time: O(1), Space: O(1).
        Returns collinear(cl): 0, counterclockwise(ccw): 1, clockwise(cw): -1
        Optimizations: if x,y are ints, use 0 instead of 0.0 or just paste the code here directly.
        """
        return self.compare_ab(self.cross_product(b_point - a_point, c_point - a_point), 0.0)

    def angle_point_c_wrt_line_ab(self, a_point: Pt2d, b_point: Pt2d, c_point: Pt2d) -> float:
        """For a line ab and point c, determine the angle of a to b to c in radians.
        formula: arc-cos(dot(vec_ba, vec_bc) / sqrt(dist_sq(vec_ba) * dist_sq(vec_bc))) = angle

        Complexity per call: Time: O(1), Space: O(1).
        Optimizations: for accuracy we sqrt both distances can remove if distances are ints.
        """
        vector_ba, vector_bc = a_point - b_point, c_point - b_point
        dot_ba_bc = self.dot_product(vector_ba, vector_bc)
        dist_sq_ba = self.dot_product(vector_ba, vector_ba)
        dist_sq_bc = self.dot_product(vector_bc, vector_bc)
        return acos(dot_ba_bc / (sqrt(dist_sq_ba) * sqrt(dist_sq_bc)))
        # return acos(dot_ba_bc / sqrt(dist_sq_ba * dist_sq_bc))

    def pt_on_line_segment_ab(self, point_a: Pt2d, point_b: Pt2d, point: Pt2d) -> bool:
```

```python
        """Logic is cross == 0 mean parallel, and dot being <= 0 means different directions."""
        vec_pa, vec_pb = point_a - point, point_b - point
        return (self.compare_ab(self.cross_product(vec_pa, vec_pb), 0) == 0
                and self.compare_ab(self.dot_product(vec_pa, vec_pb), 0) <= 0)

    def project_pt_c_to_line_ab(self, a_point: Pt2d, b_point: Pt2d, c_point: Pt2d) -> Pt2d:
        """Compute the point closest to c on the line ab.
        formula: pt = a + u x vector_ba, where u is the scalar projection of vector_ca onto
        vector_ba via dot-product
        # TODO RETEST
        Complexity per call: Time: O(1), Space: O(1).
        """
        vec_ab, vec_ac = b_point - a_point, c_point - a_point
        translation = vec_ab * (self.dot_product(vec_ac, vec_ab) / self.dot_product(vec_ab, vec_ab))
        return a_point + translation

    def project_pt_c_to_line_seg_ab(self, a_point: Pt2d, b_point: Pt2d, c_point: Pt2d) -> Pt2d:
        """Compute the point closest to c on the line segment ab.
        Rule if a==b, then if c closer to a or b, otherwise we can just use the line version.
        # TODO RETEST
        Complexity per call: Time: O(1), Space: O(1).
        Optimizations: use compare_ab on the last line if needed better accuracy.
        """
        if a_point == b_point:  # base case, closest point is either, avoids division by 0 below
            return a_point
        vec_ab, vec_ac = b_point - a_point, c_point - a_point
        u = self.dot_product(vec_ac, vec_ab) / self.dot_product(vec_ab, vec_ab)
        return (a_point if u < 0.0 else b_point if u > 1.0          # closer to a or b
                else self.project_pt_c_to_line_ab(a_point, b_point, c_point))  # inbetween a and b

    def distance_pt_c_to_line_ab(self, a_point: Pt2d, b_point: Pt2d, c_point: Pt2d) -> float:
        """Just return the distance between c and the projected point :)."""
        closest_point: Pt2d = self.project_pt_c_to_line_ab(a_point, b_point, c_point)
        return self.distance_normalized(c_point, closest_point)

    def distance_pt_c_to_line_seg_ab(self, a_point: Pt2d, b_point: Pt2d, c_point: Pt2d) -> float:
        """Same as above, just return the distance between c and the projected point :)."""
        # TODO RETEST
        closest_point: Pt2d = self.project_pt_c_to_line_seg_ab(a_point, b_point, c_point)
        return self.distance_normalized(c_point, closest_point)

    def is_parallel_lines_ab_and_cd(self, endpoint_a: Pt2d, endpoint_b: Pt2d,
                                    endpoint_c: Pt2d, endpoint_d: Pt2d) -> bool:  # TODO RETEST
        """Two lines are parallel if the cross_product between vec_ab and vec_dc is 0."""
        vec_ab, vec_dc = endpoint_b - endpoint_a, endpoint_c - endpoint_d
        return self.compare_ab(self.cross_product(vec_ab, vec_dc), 0.0) == 0

    def is_collinear_lines_ab_and_cd_1(self, end_pt_a: Pt2d, end_pt_b: Pt2d,
                                       end_pt_c: Pt2d, end_pt_d: Pt2d) -> bool:  # TODO RETEST
        """Old function. a!=b and c!=d and then returns correctly"""
        return (self.is_parallel_lines_ab_and_cd(end_pt_a, end_pt_b, end_pt_c, end_pt_d)
                and self.is_parallel_lines_ab_and_cd(end_pt_b, end_pt_a, end_pt_a, end_pt_c)
                and self.is_parallel_lines_ab_and_cd(end_pt_d, end_pt_c, end_pt_c, end_pt_a))

    def is_collinear_lines_ab_and_cd_2(self, end_point_a: Pt2d, endpoint_b: Pt2d,
                                       endpoint_c: Pt2d, endpoint_d: Pt2d) -> bool:  # TODO RETEST
        """Two lines are collinear iff a!=b and c!=d, and both c and d are collinear to line ab."""
        return (self.point_c_rotation_wrt_line_ab(end_point_a, endpoint_b, endpoint_c) == 0
                and self.point_c_rotation_wrt_line_ab(end_point_a, endpoint_b, endpoint_d) == 0)

    def is_segments_intersect_ab_to_cd(self, end_pt_a: Pt2d, end_pt_b: Pt2d,
                                       end_pt_c: Pt2d, end_pt_d: Pt2d) -> bool:
        """4 distinct points as two lines intersect if they are collinear and at least one of the
        end points c or d are in between a and b otherwise, need to compute cross products."""
        if self.is_collinear_lines_ab_and_cd_2(end_pt_a, end_pt_b, end_pt_c, end_pt_d):
            lo, hi = (end_pt_a, end_pt_b) if end_pt_a < end_pt_b else (end_pt_b, end_pt_a)
            return lo <= end_pt_c <= hi or lo <= end_pt_d <= hi
        vec_ad, vec_ab, vec_ac = end_pt_d - end_pt_a, end_pt_b - end_pt_a, end_pt_c - end_pt_a
        vec_ca, vec_cd, vec_cb = end_pt_a - end_pt_c, end_pt_d - end_pt_c, end_pt_b - end_pt_c
        point_a_value = self.cross_product(vec_ad, vec_ab) * self.cross_product(vec_ac, vec_ab)
        point_c_value = self.cross_product(vec_ca, vec_cd) * self.cross_product(vec_cb, vec_cd)
        return not (point_a_value > 0 or point_c_value > 0)

    def is_lines_intersect_ab_to_cd(self, end_pt_a: Pt2d, end_pt_b: Pt2d,
                                    end_pt_c: Pt2d, end_pt_d: Pt2d) -> bool:
        """Two lines intersect if they aren't parallel or if they collinear."""
        return (not self.is_parallel_lines_ab_and_cd(end_pt_a, end_pt_b, end_pt_c, end_pt_d)
                or self.is_collinear_lines_ab_and_cd_2(end_pt_a, end_pt_b, end_pt_c, end_pt_d))
```

```python
def pt_lines_intersect_ab_to_cd(self, end_pt_a: Pt2d, end_pt_b: Pt2d,
                                end_pt_c: Pt2d, end_pt_d: Pt2d) -> Pt2d:   # TODO RETEST
    """Compute the intersection point between two lines via cross products of the vectors."""
    vec_ab, vec_ac, vec_dc = end_pt_b - end_pt_a, end_pt_c - end_pt_a, end_pt_c - end_pt_d
    vec_t = vec_ab * (self.cross_product(vec_ac, vec_dc) / self.cross_product(vec_ab, vec_dc))
    return end_pt_a + vec_t

def pt_line_seg_intersect_ab_to_cd(self, a: Pt2d, b: Pt2d, c: Pt2d, d: Pt2d) -> Pt2d:
    """Same as for line intersect but this time we need to use a specific formula.
    Formula: # TODO RETEST"""
    x, y, cross_prod = c.x - d.x, d.y - c.y, self.cross_product(d, c)
    u = abs(y * a.x + x * a.y + cross_prod)
    v = abs(y * b.x + x * b.y + cross_prod)
    return Pt2d((a.x * v + b.x * u) / (v + u), (a.y * v + b.y * u) / (v + u))

def is_point_in_radius_of_circle(self, point: Pt2d, center_point: Pt2d, radius: float) -> bool:
    """True if point p in circle False otherwise. Use <= for circumference inclusion."""
    return self.compare_ab(self.distance_normalized(point, center_point), radius) < 0

def pt_circle_center_given_pt_abc(self, a_point: Pt2d, b_point: Pt2d, c_point: Pt2d) -> Pt2d:
    """Find the center of a circle based of 3 distinct points
    TODO add in the formula
    =# TODO RETEST
    """
    ab, ac = (a_point + b_point) / 2, (a_point + c_point) / 2
    ab_rotated: Pt2d = self.rotate_cw_90_wrt_origin(a_point - ab) + ab
    ac_rotated: Pt2d = self.rotate_cw_90_wrt_origin(a_point - ac) + ac
    return self.pt_lines_intersect_ab_to_cd(ab, ab_rotated, ac, ac_rotated)

def pts_line_ab_intersects_circle_cr(self, a, b, c, r):   # TODO RETEST
    """Compute the point(s) that line ab intersects circle c radius r. from stanford 2016
    TODO add in the formula
    """
    vec_ba, vec_ac = b-a, a-c
    dist_sq_ba = self.dot_product(vec_ba, vec_ba)
    dist_sq_ac_ba = self.dot_product(vec_ac, vec_ba)
    dist_sq_ac = self.dot_product(vec_ac, vec_ac) - r * r
    dist_sq = dist_sq_ac_ba*dist_sq_ac_ba - dist_sq_ba*dist_sq_ac
    result = self.compare_ab(dist_sq, 0.0)
    if result >= 0:
        first_intersect = c + vec_ac + vec_ba*(-dist_sq_ac_ba + sqrt(dist_sq + EPS))/dist_sq_ba
        second_intersect = c + vec_ac + vec_ba*(-dist_sq_ac_ba - sqrt(dist_sq))/dist_sq_ba
        return first_intersect if result == 0 else first_intersect, second_intersect
    return None   # no intersect

def pts_two_circles_intersect_cr1_cr2(self, c1: Pt2d, c2: Pt2d, r1, r2):   # TODO RETEST
    """I think this is the points on the circumference but not fully sure. from stanford 2016
    TODO add in teh formula
    """
    center_dist = self.distance_normalized(c1, c2)
    if (self.compare_ab(center_dist, r1 + r2) <= 0 <= self.compare_ab(center_dist + min(r1, r2),
                                                                      max(r1, r2))):
        x = (center_dist*center_dist - r2*r2 + r1*r1)/(2*center_dist)
        y = sqrt(r1*r1 - x*x)
        v = (c2-c1)/center_dist
        pt1, pt2 = c1 + v * x, self.rotate_ccw_90_wrt_origin(v) * y
        return (pt1 + pt2) if self.compare_ab(y, 0.0) <= 0 else (pt1+pt2, pt1-pt2)
    return None   # no overlap

def pt_tangent_to_circle_cr(self, center_point: Pt2d, radius: Numeric, pt: Pt2d) -> List[Pt2d
        ]:
    """Find the two points that create tangent lines from p to the circumference.
    TODO add in teh formula
    # TODO RETEST
    """
    vec_pc = pt - center_point
    x = self.dot_product(vec_pc, vec_pc)
    dist_sq = x - radius * radius
    result = self.compare_ab(dist_sq, 0.0)
    if result >= 0:
        dist_sq = dist_sq if result else 0
        q1 = vec_pc * (radius * radius / x)
        q2 = self.rotate_ccw_90_wrt_origin(vec_pc * (-radius * sqrt(dist_sq) / x))
        return [center_point + q1 - q2, center_point + q1 + q2]
    return []

def tangents_between_2_circles(self, c1, r1, c2, r2):   # TODO RETEST
    """Between two circles there should be at least 4 points that make two tangent lines.
    TODO add in teh formula
    """
    if self.compare_ab(r1, r2) == 0:
        c2c1 = c2 - c1
        multiplier = r1/sqrt(self.dot_product(c2c1, c2c1))
        tangent = self.rotate_ccw_90_wrt_origin(c2c1 * multiplier)   # need better name
        r_tangents = [(c1+tangent, c2+tangent), (c1-tangent, c2-tangent)]
    else:
        ref_pt = ((c1 * -r2) + (c2 * r1)) / (r1 - r2)
        ps = self.pt_tangent_to_circle_cr(c1, r1, ref_pt)
        qs = self.pt_tangent_to_circle_cr(c2, r2, ref_pt)
        r_tangents = [(ps[i], qs[i]) for i in range(min(len(ps), len(qs)))]
    ref_pt = ((c1 * r2) + (c2 * r1)) / (r1 + r2)
    ps = self.pt_tangent_to_circle_cr(c1, r1, ref_pt)
    qs = self.pt_tangent_to_circle_cr(c2, r2, ref_pt)
    for i in range(min(len(ps), len(qs))):
        r_tangents.append((ps[i], qs[i]))
    return r_tangents

def sides_of_triangle_abc(self, a, b, c):   # TODO RETEST
    """Compute the side lengths of a triangle."""
    dist_ab = self.distance_normalized(a, b)
    dist_bc = self.distance_normalized(b, c)
    dist_ca = self.distance_normalized(c, a)
    return dist_ab, dist_bc, dist_ca

def pt_p_in_triangle_abc(self, a, b, c, p):   # TODO RETEST
    """Compute if a point is in or on a triangle. If all edges return the same orientation this
    should return true and the point should be in or on the triangle."""
    return (self.point_c_rotation_wrt_line_ab(a, b, p) >= 0
            and self.point_c_rotation_wrt_line_ab(b, c, p) >= 0
            and self.point_c_rotation_wrt_line_ab(c, a, p) >= 0)

def perimeter_of_triangle_abc(self, side_ab, side_bc, side_ca):   # TODO RETEST
    """Computes the perimeter of triangle given the side lengths."""
    return side_ab + side_bc + side_ca

def triangle_area_bh(self, base, height):   # TODO RETEST
    """Simple triangle area formula: area = b*h/2."""
    return base*height/2

def triangle_area_from_heron_abc(self, side_ab, side_bc, side_ca):   # TODO RETEST
    """Compute heron's formula which gives us the area of a triangle given the side lengths."""
    s = self.perimeter_of_triangle_abc(side_ab, side_bc, side_ca) / 2
    return sqrt(s * (s-side_ab) * (s-side_bc) * (s-side_ca))

def triangle_area_from_cross_product_abc(self, a, b, c):   # TODO RETEST
    """Compute triangle area, via cross-products of the pairwise sides ab, bc, ca."""
    return (self.cross_product(a, b) + self.cross_product(b, c) + self.cross_product(c, a))/2

# def incircle_radis_of_triangle_abc_helper(self, ab, bc, ca):
#     area = self.triangle_area_from_heron_abc(ab, bc, ca)
#     perimeter = self.perimeter_of_triangle_abc(ab, bc, ca) / 2
#     return area/perimeter

def incircle_radius_of_triangle_abc(self, a, b, c):   # TODO RETEST
    """Computes the radius of the incircle, achieved by computing the side lengths then finding
    the area and perimeter to use in this Formula: r = area/(perimeter/2) Author: TODO Author
    """
    side_ab, side_bc, side_ca = self.sides_of_triangle_abc(a, b, c)
    area = self.triangle_area_from_heron_abc(side_ab, side_bc, side_ca)
    perimeter = self.perimeter_of_triangle_abc(side_ab, side_bc, side_ca) / 2
    return area / perimeter

# def circumcircle_radis_of_triangle_abc_helper(self, ab, bc, ca):
#     area = self.triangle_area_from_heron_abc(ab, bc, ca)
#     return (ab*bc*ca) / (4*area)

def circumcircle_radius_of_triangle_abc(self, a, b, c):   # TODO RETEST
    """Computes the radius of the circum-circle, achieved by computing the side lengths then
    gets the area for Formula: r = (ab * bc * ca) / (4 * area) Author: TODO Author
    """
    side_ab, side_bc, side_ca = self.sides_of_triangle_abc(a, b, c)
    area = self.triangle_area_from_heron_abc(side_ab, side_bc, side_ca)
    return (side_ab * side_bc * side_ca) / (4 * area)

def incircle_pt_for_triangle_abc_1(self, a, b, c):   # TODO RETEST
    """Get the circle center of an incircle.
```

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        Complexity per call: Time: lots of ops but still O(1), Space O(1)
        Formula: TODO add in the formula
        Optimization: get sides individually instead of through another call
        """
        radius = self.incircle_radius_of_triangle_abc(a, b, c)
        if self.compare_ab(radius, 0.0) == 0:  # if the radius was 0 we don't have a point
            return False, 0, 0
        side_ab, side_bc, side_ca = self.sides_of_triangle_abc(a, b, c)
        ratio_1 = side_ab/side_ca
        ratio_2 = side_ab/side_bc
        pt_1 = b + (c-b) * (ratio_1/(ratio_1 + 1.0))
        pt_2 = a + (c-a) * (ratio_2/(ratio_2 + 1.0))

        if self.is_lines_intersect_ab_to_cd(a, pt_1, b, pt_2):
            intersection_pt = self.pt_lines_intersect_ab_to_cd(a, pt_1, b, pt_2)
            return True, radius, round(intersection_pt, 12)  # can remove the round function
        return False, 0, 0

    def triangle_circle_center_pt_abcd(self, a, b, c, d):  # TODO RETEST
        """A 2 in one method that can get the middle point of both incircle circumcenter.
        Method: TODO add in the formula

        Complexity per call: Time: lots of ops but still O(1), Space O(1)
        Optimization: paste rotation code instead of function call
        """
        pt_1 = self.rotate_cw_90_wrt_origin(b-a)  # rotation on the vector b-a
        pt_2 = self.rotate_cw_90_wrt_origin(d-c)  # rotation on the vector d-c
        cross_product_1_2 = self.cross_product(pt_1, pt_2)
        # cross_product_2_1 = -cross_product_1_2  # self.cross_product(pt_2, pt_1)
        if self.compare_ab(cross_product_1_2, 0.0) == 0:
            return None
        pt_3 = Pt2d(self.dot_product(a, pt_1), self.dot_product(c, pt_2))
        x = ((pt_3.x * pt_2.y) - (pt_3.y * pt_1.y)) / cross_product_1_2
        y = ((pt_3.x * pt_2.x) - (pt_3.y * pt_1.x)) / -cross_product_1_2  # cross(pt_2, pt_1)
        return round(Pt2d(x, y), 12)

    def angle_bisector_for_triangle_abc(self, a, b, c):  # TODO RETEST
        """Compute the angle bisector point.
        Method: TODO add in the formula
        """
        dist_ba = self.distance_normalized(b, a)
        dist_ca = self.distance_normalized(c, a)
        ref_pt = (b-a) / dist_ba * dist_ca
        return ref_pt + (c-a) + a

    def perpendicular_bisector_for_triangle_ab(self, a, b):  # TODO RETEST
        """Compute the perpendicular bisector point.
        Method: TODO add in the formula
        """
        rotated_vector_ba = self.rotate_ccw_90_wrt_origin(b-a)  # code is a ccw turn. check formula
        return rotated_vector_ba + (a+b)/2

    def incircle_pt_for_triangle_abc_2(self, a, b, c):  # TODO RETEST
        """An alternative way to compute incircle. This one uses bisectors
        Method: TODO add in the formula
        """
        bisector_abc = self.angle_bisector_for_triangle_abc(a, b, c)
        bisector_bca = self.angle_bisector_for_triangle_abc(b, c, a)
        return self.triangle_circle_center_pt_abcd(a, bisector_abc, b, bisector_bca)

    def circumcenter_pt_of_triangle_abc_2(self, a, b, c):  # TODO RETEST
        """An alternative way to compute circumcenter. This one uses bisectors
        Method: TODO add in the formula
        """
        bisector_ab = self.perpendicular_bisector_for_triangle_ab(a, b)
        bisector_bc = self.perpendicular_bisector_for_triangle_ab(b, c)
        ab2, bc2 = (a+b)/2, (b+c)/2
        return self.triangle_circle_center_pt_abcd(ab2, bisector_ab, bc2, bisector_bc)

    def orthocenter_pt_of_triangle_abc_v2(self, a, b, c):  # TODO RETEST
        """Compute the orthogonal center of triangle abc.Z
        Method: TODO add in the formula
        """
        return a + b + c - self.circumcenter_pt_of_triangle_abc_2(a, b, c) * 2

    def perimeter_of_polygon_pts(self, pts: List[Pt2d]) -> float:
        """Compute summed pairwise perimeter of polygon in CCW ordering."""
        return fsum([self.distance_normalized(a, b) for a, b in pairwise_func(pts)])
```

```python
    def signed_area_of_polygon_pts(self, pts: List[Pt2d]) -> float:
        """Compute sum of area of polygon, via shoelace method: half the sum of the pairwise
        cross-products."""  # see start of booklet for pairwise
        return fsum([self.cross_product(a, b) for a, b in pairwise_func(pts)]) / 2

    def area_of_polygon_pts(self, pts: List[Pt2d]) -> float:
        """Positive area of polygon using above method."""
        return abs(self.signed_area_of_polygon_pts(pts))

    def is_polygon_pts_convex(self, pts: List[Pt2d]) -> bool:
        """Determines if polygon is convex with options of allowing or disallowing collinearity.

        Complexity per call: Time: O(n), Space: O(1) ? maybe its O(n)
        Optimizations and Notes: pts[0] != pts[-1], Use iterators with zip instead of costly mod.
        """
        if len(pts) > 3:
            n, func = len(pts), self.point_c_rotation_wrt_line_ab
            rotations = {func(pts[i], pts[(i + 1) % n], pts[(i + 2) % n]) for i in range(n)}
            # return (len(rotations) == 1) and (CL not in rotations)  # use when CL not allowed
            return (CCW in rotations) != (CW in rotations)        # use when CL is allowed
        return False

    def pt_p_in_polygon_pts(self, pts: List[Pt2d], p: Pt2d) -> bool:
        """Determine if a point is in a polygon via, ray casting.

        Complexity per call: Time: O(n), Space: O(1)
        """
        ans, px, py = False, p.x, p.y
        for a, b in pairwise_func(pts):
            xi, yi = a.x, a.y
            xj, yj = b.x, b.y
            if min(yi, yj) <= py < max(yi, yj) and px < (xi + (xj - xi) * (py - yi) / (yj - yi)):
                ans = not ans
        return ans

    def pt_p_in_polygon_pts_alternative(self, pts: List[Pt2d], p: Pt2d) -> bool:
        """Determine if a point is in a polygon based on the sum of the angles.

        Complexity per call: Time: O(n), Space: O(1)
        """
        if len(pts) > 3:
            angle_sum = 0.0
            for a, b in pairwise_func(pts):
                angle = self.angle_point_c_wrt_line_ab(a, p, b)
                angle_sum += angle if self.point_c_rotation_wrt_line_ab(p, a, b) < CL else -angle
            return True if self.compare_ab(abs(angle_sum), pi) > 0 else False
        return False

    def pt_p_on_polygon_perimeter_pts(self, pts: List[Pt2d], p: Pt2d) -> bool:
        """Determine if a point is on the perimeter of a polygon simply via a distance check.

        Complexity per call: Time: O(n), Space: O(1)
        Optimizations: turn pts into a set and check that case first
        """
        return any(self.pt_on_line_segment_ab(a, b, p) for a, b in pairwise_func(pts)) or p in pts

    def pt_position_wrt_polygon_pts(self, pts: List[Pt2d], p: Pt2d) -> int:
        """Will determine if a point is in on or outside a polygon.

        Complexity per call: Time: O(n) Convex(log n), Space: O(1)
        Optimizations: use log n version if you need superfast, and it's a convex polygon
        Return: 0 for on, 1 for in, -1 for out
        """
        return (0 if self.pt_p_on_polygon_perimeter_pts(pts, p)
                else 1 if self.pt_p_in_polygon_pts(pts, p) else -1)

    def remove_collinear_points(self, pts: List[Pt2d]):
        """Removes all collinear points in O(n) time. MUTATES pts."""
        pts[:] = [pts[-1]] + [pt for pt in pts] + [pts[0]]
        pts[:] = [pts[i] for i in range(1, len(pts) - 1)
                  if self.point_c_rotation_wrt_line_ab(pts[i - 1], pts[i], pts[i + 1]) != CL]

    def pt_p_in_convex_polygon_pts(self, pts: List[Pt2d], query_pt: Pt2d) -> bool:
        """For a convex Polygon we are able to search if point is in the polygon fast.
        Must be ordered in CounterClockWise ordering and no collinear points.

        Complexity per call: Time: O(log n), Space: O(1)
        Optimizations: pre-compute pts[mid] - pts[0] for all values then use cross product.
        """
```

```python
    left, right, min_point = 1, len(pts) - 1, pts[0]
    while left < right:
      mid = 1 + ((left + right) // 2)
      if self.point_c_rotation_wrt_line_ab(min_point, pts[mid], query_pt) == 1:
        left = mid
      else:
        right = mid - 1
    if (self.point_c_rotation_wrt_line_ab(min_point, pts[left], query_pt) == -1
        or left == len(pts) - 1):
      return False
    return self.point_c_rotation_wrt_line_ab(pts[left], pts[left + 1], query_pt) >= 0

def centroid_pt_of_convex_polygon(self, pts: List[Pt2d]) -> Pt2d:  # TODO RETEST
    """Compute the centroid of a convex polygon.

    Complexity per call: Time: O(n), Space: O(1)
    Optimizations:
    """
    ans = Pt2d(0, 0)
    for a, b in pairwise_func(pts):  # defined at start of booklet
      ans = ans + (a + b) * self.cross_product(a, b)
    return ans / (6.0 * self.signed_area_of_polygon_pts(pts))

def is_polygon_pts_simple_quadratic(self, pts: List[Pt2d]) -> bool:
    """Brute force method to check if a polygon is simple. check all line pairs

    Complexity per call: Time: O(n^2), Space: O(n)
    """
    lines = [(a, b) for a, b in pairwise_func(pts)]
    return not any(self.is_segments_intersect_ab_to_cd(pt_a, pt_b, pt_c, pt_d)
                   for (pt_a, pt_b), (pt_c, pt_d) in combinations(lines, 2)
                   if pt_a != pt_d and pt_b != pt_c)  # avoids lines that neighbor each other

def polygon_cut_from_line_ab(self, pts: List[Pt2d], left, right) -> List[Pt2d]:  # TODO RETEST
    """Method computes the left side polygon resulting from a cut from the line a-b.
    Method: Walk around the polygon and only take points that return CCW to line ab

    Complexity per call: Time: O(n), Space: O(n)
    Optimizations:
    """
    left_partition = []
    for u, v in pairwise_func(pts):
      rot_1 = self.point_c_rotation_wrt_line_ab(left, right, u)
      rot_2 = self.point_c_rotation_wrt_line_ab(left, right, v)
      if 0 >= rot_1:
        left_partition.append(u)
        if 0 == rot_1:
          continue
      if rot_1 * rot_2 < 0:  # CCW -1, CW 1 so tests if they are opposite ie lines intersect.
        left_partition.append(self.pt_line_seg_intersect_ab_to_cd(u, v, left, right))
    if left_partition and left_partition[0] != left_partition[-1]:
      left_partition.append(left_partition[0])
    return left_partition

def convex_hull_monotone_chain(self, pts: List[Pt2d]) -> List[Pt2d]:
    """Compute convex hull of a list of points via Monotone Chain method. CCW ordering returned.

    Complexity per call: Time: O(nlog n), Space: final O(n), aux O(nlog n)
    Optimizations: can use heapsort for Space: O(n)[for set + heap] or O(1) [if we consume pts],
    Can also optimize out the append and pop with using a stack like index.
    """
    def func(points, cur_hull, min_size):
      for p in points:
        while (len(cur_hull) > min_size
               and self.point_c_rotation_wrt_line_ab(cur_hull[-2], cur_hull[-1], p) == CW):
          cur_hull.pop()
        cur_hull.append(p)
      cur_hull.pop()
    unique_points, convex_hull = sorted(set(pts)), []
    if len(unique_points) > 1:
      func(unique_points, convex_hull, 1)
      func(unique_points[::-1], convex_hull, 1 + len(convex_hull))
      return convex_hull + [convex_hull[0]]  # add the first point because lots of our methods
    return unique_points              # require the first and last point be the same

def rotating_caliper_of_polygon_pts(self, pts: List[Pt2d]) -> float:
    """Computes the max distance of two points in the convex polygon?

    Complexity per call: Time: O(nlog n) unsorted O(n) sorted, Space: O(1)
    Optimizations:
    """
    convex_hull = self.convex_hull_monotone_chain(pts)
    n, t, ans = len(convex_hull) - 1, 1, 0.0
    for i, (a, b) in enumerate(pairwise_func(convex_hull)):
      p = b - a
      while ((t + 1) % n != i
             and self.compare_ab(self.cross_product(p, convex_hull[(t + 1) % n] - a),
                                 self.cross_product(p, convex_hull[t] - a)) >= 0):
        t = (t + 1) % n
      ans = max(ans, self.distance(a, convex_hull[t]))
      ans = max(ans, self.distance(b, convex_hull[t]))
    return sqrt(ans)

def closest_pair_helper(self, lo: int, hi: int, x_ord: List[Pt2d]) -> ClosestPair:
    """brute force function, for small range will brute force find the closet pair. O(n^2)"""
    closest_pair = (self.distance(x_ord[lo], x_ord[lo + 1]), x_ord[lo], x_ord[lo + 1])
    for pt_a, pt_b in combinations(x_ord[lo:hi], 2):  # 2 for every pair of combinations
      distance_ij = self.distance(pt_a, pt_b)
      if distance_ij < closest_pair[0]:
        closest_pair = (distance_ij, pt_a, pt_b)
    return closest_pair

def closest_pair_recursive(self, lo: int, hi: int,
                           x_ord: List[Pt2d], y_ord: List[Pt2d]) -> ClosestPair:
    """Recursive part of computing the closest pair. Divide by y recurse then do a special check

    Complexity per call T(n/2) halves each time, T(n/2) halves each call, O(n) at max tho
    Optimizations and notes: If not working use y_part_left and y_part_right again. I did add in
    the optimization of using y_partition 3 times over rather than having 3 separate lists
    also can remove compare_ab for direct compare
    """
    n = hi - lo
    if n < 32:  # base case, just brute force: powers of 2 between with 32 working the fastest.
      return self.closest_pair_helper(lo, hi, x_ord)
    left_len, right_len = lo + n - n // 2, lo + n // 2
    mid = round((x_ord[left_len].x + x_ord[right_len].x) / 2)
    partition_left, partition_right = [], []
    append_left, append_right = partition_left.append, partition_right.append
    for pt in y_ord:
      append_right(pt) if pt.x > mid else append_left(pt)
    best_left = self.closest_pair_recursive(lo, left_len, x_ord, partition_left)
    best_right = self.closest_pair_recursive(left_len, hi, x_ord, partition_right)
    best_pair = best_left if best_left[0] <= best_right[0] else best_right
    if self.compare_ab(best_pair[0], 0) == 0:
      return best_pair
    y_partition = [pt for pt in y_ord if best_pair[0] > (pt.x - mid) ** 2]
    for i, pt_a in enumerate(y_partition):
      a_y = pt_a.y
      for pt_b in y_partition[i+1:]:  # slicing seemed to run the fastest, range was second.
        dist_ij = a_y - pt_b.y
        if dist_ij ** 2 >= best_pair[0]:
          break
        dist_ij = self.distance(pt_a, pt_b)
        if dist_ij < best_pair[0]:
          best_pair = (dist_ij, pt_a, pt_b)
    return best_pair

def compute_closest_pair(self, pts: List[Pt2d]) -> ClosestPair:
    """Compute the closest pair of points in a set of points. method is divide and conqur

    Complexity per call Time: O(nlog n), Space O(nlog n)
    Optimizations: use c++ if too much memory, haven't found the way to do it without nlog n
    """
    x_ord = sorted(pts, key=lambda pt: pt.x)
    y_ord = sorted(pts, key=lambda pt: pt.y)
    return self.closest_pair_recursive(0, len(pts), x_ord, y_ord)

def delaunay_triangulation_slow(self, pts: List[Pt2d]) -> List[Triangle]:
    """A very slow version of  Delaunay Triangulation. Can beat the faster version when n small.

    Complexity per call Time: O(n^4), Space O(n)
    Optimizations: use c++ if too much memory, haven't found the way to do it without nlog n
    """
    n, ans = len(pts), []
    z_arr = [el.x ** 2 + el.y ** 2 for el in pts]
    x_arr = [el.x for el in pts]
    y_arr = [el.y for el in pts]
    for i in range(n - 2):
```

```python
        for j in range(i + 1, n):
            for k in range(i + 1, n):
                if j == k:
                    continue
                xn = ((y_arr[j] - y_arr[i]) * (z_arr[k] - z_arr[i])
                      - (y_arr[k] - y_arr[i]) * (z_arr[j] - z_arr[i]))
                yn = ((x_arr[k] - x_arr[i]) * (z_arr[j] - z_arr[i])
                      - (x_arr[j] - x_arr[i]) * (z_arr[k] - z_arr[i]))
                zn = ((x_arr[j] - x_arr[i]) * (y_arr[k] - y_arr[i])
                      - (x_arr[k] - x_arr[i]) * (y_arr[j] - y_arr[i]))
                flag = zn < 0.0
                for m in range(n):
                    if flag:
                        flag = flag and (self.compare_ab((x_arr[m] - x_arr[i]) * xn +
                                         (y_arr[m] - y_arr[i]) * yn +
                                         (z_arr[m] - z_arr[i]) * zn, 0.0) <= 0)
                    else:
                        break
                if flag:
                    ans.append(sort_3_elements(pts[i], pts[j], pts[k]))
        return ans

    def is_pt_left_of_edge(self, pt: Pt2d, edge: QuadEdge) -> bool:
        """A helper function with a name to describe the action. Remove for speedup."""
        return CCW == self.point_c_rotation_wrt_line_ab(pt, edge.origin, edge.dest())

    def is_pt_right_of_edge(self, pt: Pt2d, edge: QuadEdge) -> bool:
        """A helper function with a name to describe the action. Remove for speedup."""
        return CW == self.point_c_rotation_wrt_line_ab(pt, edge.origin, edge.dest())

    def det3_helper(self, a1, a2, a3, b1, b2, b3, c1, c2, c3) -> Numeric:
        """A helper function for determining the angle. Remove for speedup."""
        return (a1 * (b2 * c3 - c2 * b3) -
                a2 * (b1 * c3 - c1 * b3) +
                a3 * (b1 * c2 - c1 * b2))

    def is_in_circle(self, a: Pt2d, b: Pt2d, c: Pt2d, d: Pt2d) -> bool:
        """Expensive calculation function that determines if """
        a_dot = self.dot_product(a, a)
        b_dot = self.dot_product(b, b)
        c_dot = self.dot_product(c, c)
        d_dot = self.dot_product(d, d)
        det = -self.det3_helper(b.x, b.y, b_dot, c.x, c.y, c_dot, d.x, d.y, d_dot)
        det += self.det3_helper(a.x, a.y, a_dot, c.x, c.y, c_dot, d.x, d.y, d_dot)
        det -= self.det3_helper(a.x, a.y, a_dot, b.x, b.y, b_dot, d.x, d.y, d_dot)
        det += self.det3_helper(a.x, a.y, a_dot, b.x, b.y, b_dot, c.x, c.y, c_dot)
        return det > 0
        # use this if above doesn't work for what ever reason
        # def angle(l, mid, r):
        #     x = self.dot_product(l-mid, r-mid)
        #     y = self.cross_product(l-mid, r-mid)
        #     return atan2(x, y)
        # kek = angle(a, b, c) + angle(c, d, a) - angle(b, c, d) - angle(d, a, b)
        # return self.compare_ab(kek, 0.0) > 0

    def build_triangulation(self, left: int, right: int, pts: List[Pt2d]) -> QuadEdgePair:
        if right - left + 1 == 2:
            res = self.quad_edges.make_edge(pts[left], pts[right])
            return res, res.rev()
        if right - left + 1 == 3:
            edge_a = self.quad_edges.make_edge(pts[left], pts[left + 1])
            edge_b = self.quad_edges.make_edge(pts[left + 1], pts[right])
            self.quad_edges.splice(edge_a.rev(), edge_b)
            sg = self.point_c_rotation_wrt_line_ab(pts[left], pts[left + 1], pts[right])
            if sg == 0:
                return edge_a, edge_b.rev()
            edge_c = self.quad_edges.connect(edge_b, edge_a)
            return (edge_a, edge_b.rev()) if sg == 1 else (edge_c.rev(), edge_c)
        mid = (left + right) // 2
        ldo, ldi = self.build_triangulation(left, mid, pts)
        rdi, rdo = self.build_triangulation(mid + 1, right, pts)
        while True:
            if self.is_pt_left_of_edge(rdi.origin, ldi):
                ldi = ldi.l_next()
                continue
            if self.is_pt_right_of_edge(ldi.origin, rdi):
                rdi = rdi.rev().o_next
                continue
            break
        base_edge_l = self.quad_edges.connect(rdi.rev(), ldi)
        if ldi.origin == ldo.origin:
            ldo = base_edge_l.rev()
        if rdi.origin == rdo.origin:
            rdo = base_edge_l
        while True:
            l_cand_edge = base_edge_l.rev().o_next
            if self.is_pt_right_of_edge(l_cand_edge.dest(), base_edge_l):
                while self.is_in_circle(base_edge_l.dest(), base_edge_l.origin,
                                        l_cand_edge.dest(), l_cand_edge.o_next.dest()):
                    temp_edge = l_cand_edge.o_next
                    self.quad_edges.delete_edge(l_cand_edge)
                    l_cand_edge = temp_edge
            r_cand_edge = base_edge_l.o_prev()
            if self.is_pt_right_of_edge(r_cand_edge.dest(), base_edge_l):
                while self.is_in_circle(base_edge_l.dest(), base_edge_l.origin,
                                        r_cand_edge.dest(), r_cand_edge.o_prev().dest()):
                    temp_edge = r_cand_edge.o_prev()
                    self.quad_edges.delete_edge(r_cand_edge)
                    r_cand_edge = temp_edge
            l_check = self.is_pt_right_of_edge(l_cand_edge.dest(), base_edge_l)
            r_check = self.is_pt_right_of_edge(r_cand_edge.dest(), base_edge_l)
            if not (l_check or r_check):
                break
            if ((not l_check)
                    or (r_check and self.is_in_circle(l_cand_edge.dest(), l_cand_edge.origin,
                                                      r_cand_edge.origin, r_cand_edge.dest()))):
                base_edge_l = self.quad_edges.connect(r_cand_edge, base_edge_l.rev())
            else:
                base_edge_l = self.quad_edges.connect(base_edge_l.rev(), l_cand_edge.rev())
        return ldo, rdo

    def delaunay_triangulation_fast(self, pts: List[Pt2d]) -> List[Triangle]:
        def add_helper():
            cur = edge
            while True:
                cur.used = True
                pts.append(cur.origin)
                edges.append(cur.rev())
                cur = cur.l_next()
                if cur == edge:
                    return
        pts.sort()
        result = self.build_triangulation(0, len(pts) - 1, pts)
        edge, edges = result[0], [result[0]]
        while self.point_c_rotation_wrt_line_ab(edge.o_next.dest(), edge.dest(), edge.origin) < CL:
            edge = edge.o_next
        add_helper()
        pts, kek = [], 0
        while kek < len(edges):
            edge = edges[kek]
            kek += 1
            if not edge.used:
                add_helper()
        return [sort_3_elements(pts[i], pts[i + 1], pts[i + 2]) for i in range(0, len(pts), 3)]


##################################################


from itertools import takewhile


# constants can paste into code for speedup
GREATER_EQUAL = -1
GREATER_THAN = 0


class StringAlgorithms:
    def __init__(self):
        """Attributes declared here must be passed in or global if not used in class format."""
        self.text_len = self.pattern_len = 0
        self.prime_p = self.mod_m = 0
        self.text, self.pattern = '', ''

        self.back_table = []

        self.suffix_array = []
        self.text_ord, self.pattern_ord = [], []
```

```python
        self.longest_common_prefix = []
        self.owner = []
        self.seperator_list = []

        self.math_algos = MathAlgorithms()
        self.hash_powers = []
        self.hash_h_values = []
        self.left_mod_inverse = []

    def kmp_preprocess(self, new_pattern):
        """Preprocess the pattern for KMP. TODO add a bit more to this description ?

        Complexity per call: Time O(m + m), Space: O(m)
        """
        pattern = new_pattern
        pattern_len = len(pattern)  # m = length of pattern
        back_table = [0] * (pattern_len + 1)
        back_table[0], j = -1, -1
        for i, character in enumerate(pattern):
            while j >= 0 and character != pattern[j]:
                j = back_table[j]
            j += 1
            back_table[i + 1] = j
        self.pattern, self.pattern_len, self.back_table = pattern, pattern_len, back_table

    def kmp_search_find_indices(self, text_to_search):
        """Search the text for the pattern we preprocessed.

        Complexity per call: Time O(n + m), Space: O(n + m)
        """
        ans, j = [], 0
        for i, character in enumerate(text_to_search):
            while j >= 0 and character != self.pattern[j]:
                j = self.back_table[j]
            j += 1
            if j == self.pattern_len:
                ans.append(1 + i - j)
                j = self.back_table[j]
        return ans

    def suffix_array_counting_sort(self, k, s_array, r_array):
        """Basic count sort for the radix sorting part of suffix arrays.

        Complexity per call. Time: O(n), T(6n), Space: O(n), S(2n)
        Switching to non itertools versions and non enumerating version can speed it up.
        """
        n = self.text_len
        maxi, tmp = max(255, n), 0
        suffix_array_temp, frequency_array = [0] * n, [0] * maxi
        frequency_array[0] = n - (n - k)  # allows us to skip k values, handled in the second loop
        for i in range(k, n):           # here we skip those k iterations
            frequency_array[r_array[i]] += 1
        for i in range(maxi):
            frequency_array[i], tmp = tmp, tmp + frequency_array[i]
        for suffix_i in s_array:
            pos = 0 if suffix_i + k >= n else r_array[suffix_i + k]
            suffix_array_temp[frequency_array[pos]] = suffix_i
            frequency_array[pos] += 1
        for i, value in enumerate(suffix_array_temp):
            s_array[i] = value

    def suffix_array_build_array(self, new_texts):
        """Suffix array construction on a list of texts. n = sum lengths of all the texts.

        Complexity per call: Time O(nlog n), T(3n log n), Space: O(n), S(6n)
        Optimizations: remove take while, don't use list(map(ord, text)), remove the pairwise
        """
        num_strings = len(new_texts)
        new_text = ''.join([txt + chr(num_strings - i) for i, txt in enumerate(new_texts)])
        self.text_len = new_len = len(new_text)
        suffix_arr, rank_arr = [i for i in range(new_len)], [ord(c) for c in new_text]
        for power in takewhile(lambda x: 2 ** x < new_len, range(32)):  # iterate powers of 2
            k = 2 ** power
            self.suffix_array_counting_sort(k, suffix_arr, rank_arr)
            self.suffix_array_counting_sort(0, suffix_arr, rank_arr)
            rank_array_temp = [0] * new_len
            rank = 0
            for last, curr in pairwise_func(suffix_arr):  # suffix[i] = curr, suffix[i - 1] last
                rank = rank if (rank_arr[curr] == rank_arr[last]
                                and rank_arr[curr + k] == rank_arr[last + k]) else rank + 1
                rank_array_temp[curr] = rank
            rank_arr = rank_array_temp
            if rank_arr[suffix_arr[-1]] == new_len - 1:  # exit loop early optimization
                break
        self.suffix_array, self.text = suffix_arr, new_text
        self.text_ord = [ord(c) for c in new_text]  # optional used in the binary search
        self.seperator_list = [num_strings - i for i in range(len(new_texts))]  # optional owners

    def compute_longest_common_prefix(self):
        """After generating a suffix array you can use that to find the longest common pattern.

        Complexity per call: Time: O(n), T(4n), Space: O(n), S(3n)
        """
        permuted_lcp, phi = [0] * self.text_len, [0] * self.text_len
        phi[0], left = -1, 0
        for last, curr in pairwise_func(self.suffix_array):
            phi[curr] = last
        for i, phi_i in enumerate(phi):
            if phi_i == -1:
                permuted_lcp[i] = 0
                continue
            while (i + left < self.text_len
                    and phi_i + left < self.text_len
                    and self.text_ord[i + left] == self.text_ord[phi_i + left]):
                left += 1
            permuted_lcp[i] = left
            left = 0 if left < 1 else left - 1  # this replaced max(left - 1, 0)
        self.longest_common_prefix = [permuted_lcp[suffix] for suffix in self.suffix_array]

    def suffix_array_string_matching_multiple(self, patterns: List[str], text: str) -> List[List]:
        """Generates location indices for all patterns.

        Complexity per call: Time: O(k log n), T(2(k log n)), Space: O(k)
        """
        text += '\0'
        text_len, matches = len(text), []
        for pattern in patterns:
            lo, hi, pattern_len = 0, text_len - 1, len(pattern)
            while lo < hi:
                mid = (lo + hi) // 2
                if text[self.suffix_array[mid]: self.suffix_array[mid] + pattern_len] >= pattern:
                    hi = mid
                else:
                    lo = mid + 1
            if text[self.suffix_array[lo]: self.suffix_array[lo] + pattern_len] != pattern:
                matches.append([])
            else:
                start, hi = lo, text_len - 1
                while lo < hi:
                    mid = (lo + hi) // 2
                    if text[self.suffix_array[mid]: self.suffix_array[mid] + pattern_len] > pattern:
                        hi = mid
                    else:
                        lo = mid + 1
                end = hi-(text[self.suffix_array[hi]: self.suffix_array[hi]+pattern_len] != pattern)+1
                matches.append(sorted([self.suffix_array[i] for i in range(start, end)]))
        return matches

    def compute_longest_repeated_substring(self):
        """The longest repeated substring is just the longest common pattern. Require lcp to be
        computed already. Returns the first longest repeat pattern, so for other ones implement a
        forloop.

        Complexity per call: Time: O(n), T(2n), Space: O(1)
        for optimization implement the physical forloop itself, however it's still O(n).
        """
        max_lcp = max(self.longest_common_prefix)
        return max_lcp, self.longest_common_prefix.index(max_lcp)

    def compute_owners(self):  # TODO RETEST
        """Used to compute the owners of each position in the text. O(n) time and space."""
        tmp_owner = [0] * self.text_len
        it = iter(self.seperator_list)
        seperator = next(it)
        for i, ord_value in enumerate(self.text_ord):
            tmp_owner[i] = seperator
            if ord_value == seperator:
                seperator = next(it, None)
```

```python
        self.owner = [tmp_owner[suffix_i] for suffix_i in self.suffix_array]

    def compute_longest_common_substring(self):  # TODO RETEST
        """Computes the longest common substring between two strings. returns index, value pair.

        Complexity per call: Time: O(n), Space: O(1)
        Pre-Requirements: owner, and longest_common_prefix must be built (also suffix array for lcp)
        Variants: LCS pair from k strings, LCS between all k strings.
        """
        it = iter(self.longest_common_prefix)
        max_lcp_index, max_lcp_value = 0, next(it) - 1  # - 1 here since next(it) should return 0
        for i, lcp_value in enumerate(it, 1):
            if lcp_value > max_lcp_value and self.owner[i] != self.owner[i - 1]:
                max_lcp_index, max_lcp_value = i, lcp_value
        return max_lcp_index, max_lcp_value

    def compute_rolling_hash(self, new_text):  # TODO RETEST
        """For a given text compute and store the rolling hash. we use the smallest prime lower than
        2^30 since python gets slower after 2^30, p = 131 is a small prime below 256.

        Complexity per call: Time: O(n), T(4n),  Space O(n), mid-call S(6n), post call S(4n)
        """
        len_text, p, m = len(new_text), 131, 2**30 - 35  # p is prime m is the smallest prime < 2^30
        h_vals, powers, ord_iter = [0] * len_text, [0] * len_text, map(ord, new_text)
        powers[0], h_vals[0] = 1, 0
        for i in range(1, len_text):
            powers[i] = (powers[i - 1] * p) % m
        for i, power in enumerate(powers):
            h_vals[i] = ((h_vals[i-1] if i != 0 else 0) + (next(ord_iter) * power) % m) % m
        self.text = new_text
        self.hash_powers, self.hash_h_values = powers, h_vals
        self.prime_p, self.mod_m, self.math_algos = p, m, MathAlgorithms()
        self.left_mod_inverse = [pow(power, m-2, m) for power in powers]  # optional

    def hash_fast_log_n(self, left, right):  # TODO RETEST
        """Log n time calculation of rolling hash formula: h[right]-h[left] * mod_inverse(left).

        Complexity per call: Time: O(log mod_m), Space: O(1)
        """
        ans = self.hash_h_values[right]
        if left != 0:
            ans = ((ans - self.hash_h_values[left - 1]) * pow(self.hash_powers[left],
                                                              self.mod_m - 2,
                                                              self.mod_m)) % self.mod_m
        return ans

    def hash_fast_constant(self, left, right):  # TODO RETEST
        """Constant time calculation of rolling hash. formula: h[right]-h[left] * mod_inverse[left]

        Complexity per call: Time: O(1), Space: O(1)
        more uses: string matching in n + m or constant if you know that it's a set size
        kattis typo: uses a variant of this were you will expand on the code further.
        """
        ans = self.hash_h_values[right]
        if left != 0:
            ans = ((ans - self.hash_h_values[left - 1]) * self.left_mod_inverse[left]) % self.mod_m
        return ans


###############################################

class Matrix:
    """Optimization notes: prefixed names of form local_NAME are used to avoid expensive load_attr
    calls they can be replaced with self.NAME or passed in as global values,
    REASON: even in pypy it seems N^3 operations can be speed up by doing this.
    """
    def __init__(self, n, m, init_value: Num):
        self.matrix = [[init_value] * m for _ in range(n)]
        self.num_rows = n
        self.num_cols = m
        self.matrix_left = []
        self.matrix_right = []

    def prep_matrix_multiply(self, matrix_a, matrix_b, size_n):  # TODO RETEST
        """loads up the left and right matrices for a matrix multiply. O(n^2) but optimized."""
        self.matrix_left = [[el for el in row] for row in matrix_a]
        self.matrix_right = [[el for el in row] for row in matrix_b]
        self.matrix = [[0] * size_n for _ in range(size_n)]
```

```python
        self.num_rows = self.num_cols = size_n

    def fast_copy(self, other):  # TODO RETEST
        """Quickly copy one matrix to another. MIGHT CHANGE SIZE, O(n^2), is fast copy tho."""
        other_mat = other.matrix
        self.matrix = [[el for el in row] for row in other_mat]

    def fill_matrix_from_row_col(self, new_matrix, row_offset: int, col_offset: int):  # TODO TEST
        """fill our matrix from row and col offset with new_matrix, O(n^2). Doesn't change size."""
        local_matrix = self.matrix  # prefix optimization see class docs for more info
        for i, row in enumerate(new_matrix):
            for j, new_value in enumerate(row):
                local_matrix[i + row_offset][j + col_offset] = new_value

    def matrix_multiply_mod_a_times_b(self, multiplier_1, multiplier_2, mod_m):  # TODO RETEST
        """Performs (A*B)%mod_m on matrix A,B and mod=mod_m, for normal multiply just remove mod_m.

        Complexity per call: Time: O(n^3), T(n^3 + n^2), Space: (1), does require 3n^2 in memory tho
        """
        local_num_rows = self.num_rows  # prefix optimization see class docs for more info.
        self.prep_matrix_multiply(multiplier_1.matrix, multiplier_2.matrix, local_num_rows)
        local_matrix = self.matrix  # prefix optimization see class docs for more info.
        mat_a = self.matrix_left  # prefix optimization see class docs for more info.
        mat_b = self.matrix_right  # prefix optimization see class docs for more info.
        for i in range(local_num_rows):
            for k in range(local_num_rows):
                mat_a_ik = mat_a[i][k]
                if 0 != mat_a_ik:  # skip because we will just be adding 0 to local
                    for j in range(local_num_rows):
                        local_matrix[i][j] = (local_matrix[i][j] + mat_a_ik * mat_b[k][j]) % mod_m
        self.matrix_left, self.matrix_right = [], []   # optional? keep space small

    def set_identity(self):  # TODO RETEST
        """Used for pow and pow mod on matrices. O(n^2) but memset(0) lvl speed for python."""
        local_matrix, local_num_rows = self.matrix, self.num_rows
        local_matrix = [[0] * local_num_rows for _ in range(local_num_rows)]  # memset(0) in python
        for i in range(local_num_rows):
            local_matrix[i][i] = 1

    def get_best_sawp_row(self, row: int, col: int, local_matrix, local_num_rows):  # TODO RETEST
        """Find the best pivot row defined as the row with the highest absolute value.

        Complexity per call: Time: O(n), Space: O(1)
        """
        best, pos = 0.0, -1
        for i in range(row, local_num_rows):
            column_value = abs(local_matrix[i][col])  # compute and lookup the value only once
            if column_value > best:
                best, pos = column_value, i
        return pos

    def swap_rows(self, row_a: int, row_b: int, local_matrix):  # TODO RETEST
        """Swaps two rows a and b, via reference swapping so should be constant time.

        Complexity per call: Time: O(1)[or very fast O(n) like memset], Space: O(1)
        """
        local_matrix[row_a], local_matrix[row_b] = local_matrix[row_b], local_matrix[row_a]

    def row_divide(self, row: int, div: float, local_matrix):  # TODO RETEST
        """Applies a vector divide operation on the whole row, via optimised list comprehension.

        Complexity per call: Time: O(n), Space: during O(n), post O(1)
        """
        local_matrix[row] = [el/div for el in local_matrix[row]]  # use int divide if possible

    def row_reduce_helper(self, i: int, row: int, val: Num, local_matrix):  # TODO RETEST
        """Applies a vector row reduce to a single row, via optimised list comprehension.

        Complexity per call: Time: O(n), Space: during O(n), post O(1)
        """
        local_matrix[i] = [el - val * local_matrix[row][col]
                           for col, el in enumerate(local_matrix[i])]

    def row_reduce(self, row: int, col: int, row_begin: int):  # TODO RETEST
        """Applies the whole row reduction step to the matrix.

        Complexity per call: Time: O(n^2), Space: during O(n), post O(1)
        """
        local_matrix = self.matrix  # prefix optimization see class docs for more info
```

```python
    for i in range(row_begin, self.num_rows):
      if i != row:
        self.row_reduce_helper(i, row, local_matrix[i][col], local_matrix)

  def row_reduce_2(self, row: int, col: int, determinant):  # TODO RETEST
    """Applies the whole row reduction step to both the matrix and its determinant."""
    determinant_matrix = determinant.matrix  # prefix optimization see class docs for more info
    local_matrix = self.matrix           # it is optional
    for i in range(self.num_rows):
      if i != row:
        const_value, local_matrix[i][col] = local_matrix[i][col], 0.0
        self.row_reduce_helper(i, row, const_value, local_matrix)
        determinant.row_reduce_helper(i, row, const_value, determinant_matrix)

  def get_augmented_matrix(self, matrix_b):  # TODO RETEST
    """Given matrix A and B return augmented matrix = A | B.

    Complexity per call: Time: O(n^2), Space: O(n^2)
    """
    augmented = Matrix(self.num_rows + matrix_b.num_rows,
                       self.num_cols + matrix_b.num_cols,
                       self.matrix[0][0])
    augmented.fill_matrix_from_row_col(self.matrix, 0, 0)
    augmented.fill_matrix_from_row_col(matrix_b.matrix, 0, self.num_cols)
    return augmented

  def get_determinant_matrix(self):  # TODO RETEST
    """Compute the determinant of a matrix and return the result.

     Complexity per call: Time: O(n^3), T(n^3 + n^2), Space: (n^2)
    """
    det_num_rows = self.num_rows
    determinant = Matrix(det_num_rows, self.num_cols, self.matrix[0][0])
    determinant.fast_copy(self)
    det_matrix = determinant.matrix
    r = 1.0
    for i in range(det_num_rows):
      for j in range(det_num_rows):
        while det_matrix[j][i] != 0:
          ratio = det_matrix[i][i] / det_matrix[j][i]
          for k in range(i, det_num_rows):
            det_matrix[i][k] = (det_matrix[i][k] - ratio * det_matrix[j][k])
            det_matrix[i][k], det_matrix[j][k] = det_matrix[j][k], det_matrix[i][k]
          r = -r
      r = r * det_matrix[i][i]
    return r


#################################################

class MatrixAlgorithms:
  def __init__(self):
    pass

  def matrix_pow_base_exponent_mod(self, base: Matrix, exponent: int, mod_m: int) -> Matrix:
    """Modular exponentiation applied to square matrices. For normal pow omit mod_m.
    [translated from Competitive Programming 4 part 2 c++ book, in the math section]
    # TODO RETEST
    Complexity per call: Time: [big n]O(n^3 log p), [small n] O(log p), Space: O(n^2)
    Input:
      base: the matrix represent the base, nxn matrix
    Mutations and return:
      base get  mutated to keep up with the calculation.
      returns the resulting matrix from (b^e)%m
    """
    result = Matrix(base.num_rows, base.num_rows, 0)
    result.set_identity()
    while exponent:
      if exponent % 2 == 1:
        result.matrix_multiply_mod_a_times_b(result, base, mod_m)
      base.matrix_multiply_mod_a_times_b(base, base, mod_m)
      exponent //= 2
    return result

  def get_rank_via_reduced_row_echelon(self, aug_ab: Matrix) -> int:  # TODO RETEST
    """Method used is Gauss-jordan elimination, only partial pivoting.
    [translated from standford 2016 c++ acm icpc booklet]
```

```python
    Complexity per call: Time: O(n^3), Space: O(n)
    Input:
      aug_ab: a nxm augmented matrix of A | b
    Mutations and return:
      a -> rref: a mutations into nxm matrix of rref
      returns the rank of the matrix.
    """
    local_num_rows, local_num_cols = aug_ab.num_rows, aug_ab.num_cols
    augmented_matrix = aug_ab.matrix
    rank = 0
    for col in range(local_num_cols):
      if rank == local_num_rows:
        break
      swap_row = aug_ab.get_best_sawp_row(rank, col, augmented_matrix, local_num_rows)
      if swap_row != -1:
        aug_ab.swap_rows(swap_row, rank, augmented_matrix)
        aug_ab.row_divide(rank, augmented_matrix[rank][col], augmented_matrix)
        aug_ab.row_reduce(rank, col, 0)
        rank += 1
    return rank

  def gauss_elimination(self, aug_ab: Matrix) -> int:  # TODO RETEST
    """Computes gauss with constraints, Ax=b | A -> nxn matrix, b -> nx1 matrix aug_ab = A | b.
     [translated from foreverbell 2014 c++ acm icpc cheat sheet repo]

    Complexity per call: Time: O(n^3), Space: O(n)
    More uses: solving systems of linear equations (AX=B), getting the rank

    Input:
      aug_ab = nxm matrix | m = n + 1
    Mutation and return:
      aug_ab -> X
      returns rank
    """
    local_num_rows, local_num_cols = aug_ab.num_rows, aug_ab.num_cols
    n = local_num_rows
    augmented_matrix = aug_ab.matrix
    rank = 0
    for col in range(local_num_cols):
      if rank == local_num_rows:
        break
      swap_row = aug_ab.get_best_sawp_row(rank, col, augmented_matrix, local_num_rows)
      if swap_row != -1:
        aug_ab.swap_rows(swap_row, rank, augmented_matrix)
        aug_ab.row_divide(rank, augmented_matrix[rank][col], augmented_matrix)
        aug_ab.row_reduce(rank, col, rank + 1)
        rank += 1
    for i in range(n - 1, -1, -1):
      for j in range(i):
        augmented_matrix[j][n] -= (augmented_matrix[i][n] * augmented_matrix[j][i])
        augmented_matrix[j][i] = 0
    return rank

  def gauss_jordan_elimination(self, a: Matrix, b: Matrix) -> float:  # TODO RETEST
    """Full pivoting. Mutates a and b [translated from standford 2016 c++ acm icpc booklet]

    Complexity per call: Time: O(n^3), Space: O(n), S(4n)
    More uses: solving systems of linear equations (AX=B), Inverting matrices (AX=I), and
    Computing determinants of square matrices.

    Input:
      a = nxn matrix
      b = nxm matrix
    Mutation:
      b -> X | X = nxm matrix
      a -> A^{-1} | nxn matrix
      returns determinant of a
    """
    n = a.num_rows
    matrix_a = a.matrix
    matrix_b = b.matrix
    det = 1.0
    i_row, i_col, i_piv_j, i_piv_k = [0] * n, [0] * n, set(range(n)), set(range(n))
    for i in range(n):
      pj, pk = -1, -1
      for j in i_piv_j:
        for k in i_piv_k:
          if pj == -1 or abs(matrix_a[j][k]) > abs(matrix_a[pj][pk]):
            pj, pk = j, k
```

```python
            i_piv_j.remove(pk)
            i_piv_k.remove(pk)
            a.swap_rows(pj, pk, matrix_a)
            b.swap_rows(pj, pk, matrix_b)
            if pj != pk:
                det = -det
            i_row[i], i_col[i] = pj, pk
            div = matrix_a[pk][pk]
            det /= div
            matrix_a[pk][pk] = 1.0
            a.row_divide(pk, div, matrix_a)
            b.row_divide(pk, div, matrix_b)
            a.row_reduce_2(pk, pk, b)
        for p in range(n - 1, -1, -1):
            if i_row[p] != i_col[p]:
                i_row_p, i_col_p = i_row[p], i_col[p]
                for k in range(n):
                    matrix_a[k][i_row_p], matrix_a[k][i_col_p] = (matrix_a[k][i_col_p],
                                                                  matrix_a[k][i_row_p])
        return det


class book_v1_forgot_to_add(GraphAlgorithms):
    def prime_count_dp_slower(self, limit):
        root = isqrt(limit) + 1  # after this we mostly refer to root + 1 so just add one here
        end = limit // root
        low_and_high = [i for i in range(1, end)] + [limit // i for i in range(root, 0, -1)]
        dp = [el for el in low_and_high]
        dp_len, prime_index = len(dp), 0
        for prime in range(2, root):
            if dp[prime - 1] != dp[prime - 2]:
                prime_index += 1
                end = bisect_right(low_and_high, prime * prime) - 2
                for i in range(dp_len - 1, end, -1):
                    k = low_and_high[i] // prime
                    dp[i] -= dp[k - 1 if k < root else dp_len - (limit // k)] - prime_index
        return dp[dp_len - 1] - 1

    def prime_count_dp_faster(self, n: int):
        root, number_of_primes, step_multiplier = isqrt(n) + 1, n - 1, 1
        high, low, visited_sieve = [0] * root, [0] * root, [0] * root
        for divisor in range(2, root):
            low[divisor], high[divisor] = divisor - 1, n // divisor - 1
        for prime in range(2, root):
            if low[prime] != low[prime - 1]:  # happens only when n is prime
                t, end = low[prime - 1], min(root - 1, n // (prime * prime))
                number_of_primes -= high[prime] - t
                for i in range(prime + step_multiplier, end + 1, step_multiplier):
                    if not visited_sieve[i]:  # already visited this composite number
                        multiple = i * prime
                        is_above_root = multiple > root - 1
                        high[i] -= (low[n // multiple] if is_above_root else high[multiple]) - t
                for i in range(root - 1, prime * prime - 1, -1):
                    low[i] -= low[i // prime] - t
                for prime_multiple in range(prime * prime, end + 1, prime):
                    visited_sieve[prime_multiple] = 1
                if prime == 2:
                    step_multiplier += 1  # for, odds we can skip even numbers by stepping by 2
        return number_of_primes

    def bellman_ford_shortest_path(self, source: int):
        """shortest path when graph has negative cycle.

        Complexity per call Time: O(k|E|), Space: O(|V|)
        Optimizations: use adj_list and move the INF check to before visiting neighbours.
        """
        num_verts = self.graph.num_nodes
        distances, parents = [INF] * num_verts, [-1] * num_verts
        distances[source] = 0
        for i in range(num_verts - 1):                 # this code can be copied over the negative cycle
            modified = False                           # check it will speed up the check a fair bit with
            for u, v, wt in self.graph.edge_list:      # the only thing to change being the inner line
                if distances[u] != INF and distances[v] > distances[u] + wt:
                    distances[v], parents[v], modified = distances[u] + wt, u, True
            if not modified:
                break
        for i in range(num_verts - 1):                 # this is the negative cycle check, can be used
            for u, v, wt in self.graph.edge_list:      # with spfa algorithm too
                if distances[u] != INF and distances[v] > distances[u] + wt:
                    distances[v] = -INF
        self.dist = distances

    def bellman_ford_moore_spfa(self, source):
        """Same complexity as bellmanford but faster in practice."""
        num_verts = self.graph.num_nodes
        distances, times_seen, in_queue = [INF] * num_verts,  [0] * num_verts, [False] * num_verts
        queue, distances[source], in_queue[source] = deque([source]), 0, True
        while queue:
            u = queue.popleft()
            in_queue[u] = False
            for v, wt in self.graph.adj_list[u]:
                if distances[v] > distances[u] + wt and times_seen[v] <= self.graph.num_nodes:
                    distances[v] = distances[u] + wt
                    if not in_queue[v]:
                        times_seen[v], in_queue[v] = times_seen[v] + 1, True
                        queue.append(v)

    def solve_tower_of_powers(self):
        from math import log10, log
        EPS = 1e-14
        LIM = 309
        n = int(input())
        ori = [[*map(int, input().split('^'))] for _ in range(n)]
        pos = [0] * n
        fmt = lambda x: print(x, '^'.join(map(str, ori[x])).ljust(40), end=' ')
        for i in range(n - 1):
            for j in range(i + 1, n):
                expi, expj = ori[i].copy(), ori[j].copy()
                if 1 in expi: expi = expi[:expi.index(1)]
                if 1 in expj: expj = expj[:expj.index(1)]
                si = sj = cmpi = cmpj = 1
                infi = infj = 0
                for k in range(len(expi) - 1, -1, -1):
                    if si * log10(expi[k]) < LIM:
                        si = expi[k] ** si; cmpi = si
                    else:
                        infi += 1
                        if infi == 1: cmpi = si * log(expi[k])
                        if infi == 2: cmpi = si * log(expi[k + 1]) + log(log(expi[k]))
                for k in range(len(expj) - 1, -1, -1):
                    if sj * log10(expj[k]) < LIM:
                        sj = expj[k] ** sj; cmpj = sj
                    else:
                        infj += 1
                        if infj == 1: cmpj = sj * log(expj[k])
                        if infj == 2: cmpj = sj * log(expj[k + 1]) + log(log(expj[k]))
                if infi > infj:
                    pos[i] += 1
                elif infj > infi:
                    pos[j] += 1
                else:
                    if (cmpi - cmpj) / (cmpi + cmpj) > EPS:
                        pos[i] += 1
                    elif (cmpi - cmpj) / (cmpi + cmpj) < -EPS:
                        pos[j] += 1
                    else:
                        expi, expj = expi[:infi][:-1], expj[:infj][:-1]
                        while expi and expj and expi[-1] == expj[-1]: expi.pop(), expj.pop()
                        if not expi and not expj: continue
                        if (not expi and expj) or expj[-1] > expi[-1]: pos[j] += 1
                        if (expi and not expj) or expi[-1] > expj[-1]: pos[i] += 1
                # fmt(i), print(infi, cmpi), fmt(j), print(infj, cmpj), print('-'*80)
        print('Case 1:')
        for tow in map(lambda x: '^'.join(map(str, ori[x])),
                       sorted(range(n), key=lambda x: pos[x])): print(tow)
```