Co-rotational Formulation in Chrono

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Abstract

This white paper describes the basis of the nonlinear finite element co-rotational formulation, with special emphasis on its implementation in the multi-physics software **Chrono**::Engine .

Keywords: Co-Rotational Formulation, Nonlinear Finite Elements, Flexible Multibody Systems, **Chrono**::Engine

Contents

1. Introduction

The co-rotational formulation (CRF) is a nonlinear finite element method that is used in flexible multibody system applications. The motion of the flexible bodies is split into base and co-rotated. Strains and stresses are measured from the co-rotated frame to the current configuration, whereas the base frame is used to account for rigid body dynamics [4]. The co-rotational formulation describes large rotations and small deformation for linear materials, but it can be extended to nonlinear material under some circumstances: "Displacements and rotations may be arbitrarily large, but deformations must be small" [3]. A reference paper on CRF is authored by Belytschko and Hsieh [1]. For efficiency, the implementation in Chrono follows the projection approach described in [6].

2. Co-Rotated Beam Elements

2.1 Kinematics of the CRF

This section describes the kinematics of the CRF, using beam geometry as an example, as done in [7]. The co-rotational approach, due to the linearization of the strains, allows the use of classical, linear finite elements. Each element i has a floating frame associated with it, $\langle \mathbf{F} \rangle_j$; for brevity, the subscript j will be dropped. The overall displacement of the element, in deformed current configuration $\mathcal{C}_D$, can be obtained in the CRF by superposing the configuration of the element floating frame, $\mathcal{C}_S$ from the reference $\mathcal{C}_0$, and the local small deformation, $\mathcal{C}_D$ from $\mathcal{C}_S$. Further:

- $\mathcal{C}_0$ is the reference configuration, which is described by the initial coordinates of the flexible body. Note that in the derivations made in these notes, the reference configuration involves an unstressed configuration.
- $\mathcal{C}_S$ refers to a floating configuration that captures rigid body motion of the finite element. This configuration is defined by the position and rotation coordinates of the orthogonal frame $\langle \mathbf{F} \rangle$, which is obtained via a Gram-Schmidt orthogonalization algorithm.
- Configuration $\mathcal{C}_D$ is obtained by superposing small deformation to $\mathcal{C}_S$.

The motion of a node i of a beam element is defined by the position vector $\boldsymbol{x}_i$ and a set of quaternions $\boldsymbol{\rho}_i$ of a reference frame. The vector $\boldsymbol{\rho}_i$ captures the rigid body rotation and deformation rotation of the beam cross section at node i. The state of a system with n nodes is, therefore, $\boldsymbol{s} = \left[\boldsymbol{q}, \boldsymbol{v}\right]$, where $\boldsymbol{q} = [\boldsymbol{x}_1,\ \boldsymbol{\rho}_1, \boldsymbol{x}_2,\ \boldsymbol{\rho}_2, ..., \boldsymbol{x}_3,\ \boldsymbol{\rho}_3] \in \mathbb{R}^{(3+4)n}$ and $v = [\boldsymbol{v}_1,\ \bar{\boldsymbol{\omega}}_1, \boldsymbol{v}_2,\ \bar{\boldsymbol{\omega}}_2, ..., \boldsymbol{v}_3,\ \bar{\boldsymbol{\omega}}_3] \in \mathbb{R}^{(3+3)n}$. Angular velocities $\bar{\boldsymbol{\omega}}_i$ are defined with respect to the local (nodal) frame $\langle \mathbf{A} \rangle$ or $\langle \mathbf{B} \rangle$ (see Fig. 1), whereas positions $\boldsymbol{x}_i$, quaternions $\boldsymbol{\rho}_i$, and velocities $\boldsymbol{v}_i$ are referred to the global frame.

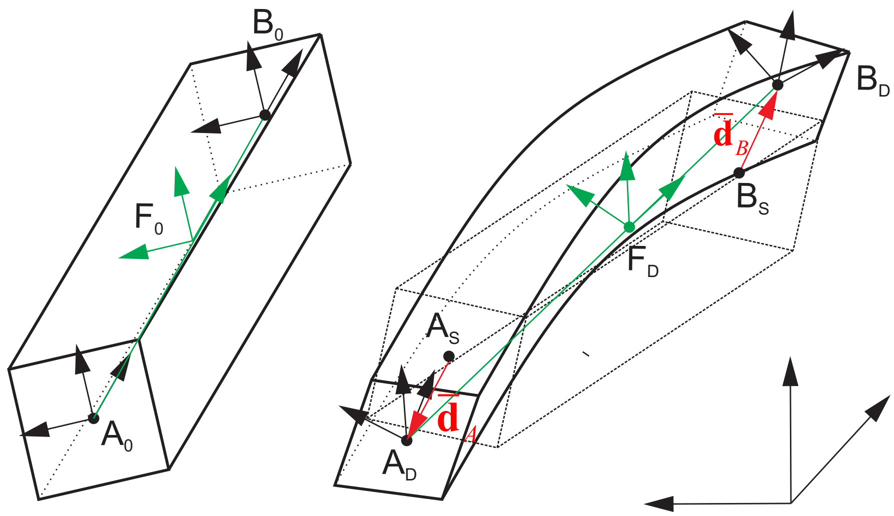

Figure 1: Co-rotational kinematics of a beam element: Reference (0), Co-rotated (S), and Deformed (D) configurations. Transformations between the 3 configurations are used to obtain the strain/stress state of the element and its projection onto global coordinates

When an element j moves, the position and rotation of the floating frame, $\langle \mathbf{F} \rangle$, are updated. In Chrono, the origin of $\langle \mathbf{F} \rangle$ is placed at the element's midpoint $\boldsymbol{x}_F = 1/2\,(\boldsymbol{x}_B - \boldsymbol{x}_A)$. The floating frame's longitudinal axis $\boldsymbol{X}$ is aligned with the vector $\boldsymbol{x}_B - \boldsymbol{x}_A$, whereas the $\boldsymbol{Y}$ and $\boldsymbol{Z}$ axes are obtained via a Gram-Schmidt orthogonalization. The rotation matrix and unit quaternion of $\langle \mathbf{F} \rangle$ will be denoted as $\boldsymbol{R_F}$ and $\boldsymbol{\rho_F}$, respectively. One can compute the local displacements of a node i as $\bar{\boldsymbol{d}}_i = \bar{\boldsymbol{x}}_i - \bar{\boldsymbol{x}}_{i_0} = \boldsymbol{R}_{\boldsymbol{F}}^{\mathrm{T}}\,(\boldsymbol{x}_{i_0} - \boldsymbol{x}_{\boldsymbol{F}_0}) - \boldsymbol{R}_{\boldsymbol{F}}^{\mathrm{T}}\,(\boldsymbol{x}_i - \boldsymbol{x}_{\boldsymbol{F}})$, where the bar over displacement quantities describes locality, and the subscript 0 refers to the initial configuration.

Nodal quaternions can be used to describe linearized angles for the definition of local strains. A unit quaternion $(e_0, \boldsymbol{e})$, where $\boldsymbol{e} = (e_1, e_2, e_3)$, may be written as

$$\boldsymbol{\rho} = \left[\cos\left(\theta/2\ \right), \quad \boldsymbol{n}\sin\left(\theta/2\ \right)\right], \tag{1}$$

with the rotation angle θ being $\theta = \arccos\left(2e_0^2 - 1\right)$, and the vector defining the axis of rotation as

$$\boldsymbol{n} = 2\boldsymbol{e}e_0/\sin\theta. \tag{2}$$

Note that Equations (1)-(2) link Euler parameters to Euler's Rotation Theorem, which expresses any three-dimensional rotation as a finite rotation about a single axis $\boldsymbol{n}$. At this point, we can compute the linearized angles as $\bar{\boldsymbol{\theta}}_i = \theta_i \boldsymbol{n}_i$. For nodes A and B of a beam element, the vector of local deformations is $\bar{\boldsymbol{d}}_{12\times 1} = \left[\bar{\boldsymbol{d}}_A, \bar{\boldsymbol{\theta}}_A, \bar{\boldsymbol{d}}_B, \bar{\boldsymbol{\theta}}_B\right]$. The local stiffness matrix, $\bar{\boldsymbol{K}}_{12\times 12}(\bar{\boldsymbol{d}})$, and the local internal force vector, $\bar{\boldsymbol{f}}_{\text{in}} = \bar{\boldsymbol{K}}(\bar{\boldsymbol{d}})$, are then mapped onto the global frame of reference by introducing new matrices and building projectors, see [3]. The dimensions of the stiffness matrix and internal force vector are dependent upon the number of coordinates used to describe deformation and, therefore, they are dependent on the element. For the beam element described in these notes, $\bar{\boldsymbol{K}}$ is a 12-by-12 matrix, and $\bar{\boldsymbol{d}}$ is a 12-by-1 vectors.

2.2 Mapping of Local Forces and Stiffness Matrix

Several CRF matrices used to transform vector quantities are defined below. Figure 2 summarizes the transformations.

The $\boldsymbol{\Lambda}_{3\times 3}\left(\bar{\boldsymbol{\theta}}\right) = \partial\bar{\boldsymbol{\theta}}_i/\partial\bar{\boldsymbol{\omega}}_i$ matrix. Its analytical expression is given by $\boldsymbol{\Lambda}\left(\bar{\boldsymbol{\theta}}\right) = \boldsymbol{I}_{3\times 3} - \frac{1}{2}\text{skew}\left(\bar{\boldsymbol{\theta}}\right) + \zeta\text{skew}\left(\bar{\boldsymbol{\theta}}\right)^2$, where $\zeta = 1 - \frac{1}{2}\bar{\theta}\cot\left(\frac{1}{2}\bar{\theta}\right)/\bar{\theta}^2$. This matrix denotes the Jacobian derivative of the local rotational axial vector with respect to the local spin axial vector evaluated at a node. Note that a spin vector is a non-unique vector that describes the direction of rotation in space [3]. The local spin vector is obtained by performing an axial operation to the transpose of the rotation matrix that goes from the base reference configuration to the co-rotated (current) frame, that is, $\bar{\boldsymbol{\omega}} = \text{axial}(\boldsymbol{R}_0^{\mathrm{T}})$, where the rotation matrix $\boldsymbol{R}_0$ describes the change in orientation of a node with respect to the reference configuration. This transformation is needed to extract the rotational deformation from the overall change in rotational coordinates.

The $\bar{\boldsymbol{H}}$ transformation matrix. This matrix may be expressed as

$$\bar{\boldsymbol{H}}_{12\times 12} = \begin{pmatrix} \bar{\boldsymbol{H}}_{\boldsymbol{n}}\left(\bar{\boldsymbol{\theta}}_A\right) & \mathbf{0}_{6\times 6} \\ \mathbf{0}_{6\times 6} & \bar{\boldsymbol{H}}_{\boldsymbol{n}}\left(\bar{\boldsymbol{\theta}}_B\right) \end{pmatrix}, \tag{3}$$

where $\bar{\boldsymbol{H}}_{\boldsymbol{n}}\left(\bar{\boldsymbol{\theta}}\right) = \begin{pmatrix} \boldsymbol{I}_{3\times 3} & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & \boldsymbol{\Lambda}\left(\bar{\boldsymbol{\theta}}\right) \end{pmatrix}$. Matrix $\bar{\boldsymbol{H}}$ transforms local corotated deformational spins to local corotated deformational rotations. $\bar{\boldsymbol{H}}$ tends to a unit matrix $\boldsymbol{I}_{12\times 12}$ for $\theta \to 0$.

The projector matrix $\bar{\boldsymbol{P}}$. The projector matrix may be written as $\bar{\boldsymbol{P}}_{12\times 12} = \boldsymbol{I}_{12\times 12} - \bar{\boldsymbol{S}}^D\bar{\boldsymbol{G}}$, where $\bar{\boldsymbol{S}}^D$ is the moment-arm matrix, obtained as follows:

$$\bar{\boldsymbol{S}}^D_{12\times 3} = \begin{pmatrix} -\text{skew}\left(\bar{\boldsymbol{x}}_A\right) \\ \boldsymbol{I}_{3\times 3} \\ -\text{skew}\left(\bar{\boldsymbol{x}}_B\right) \\ \boldsymbol{I}_{3\times 3} \end{pmatrix} \tag{4}$$

where $\bar{\boldsymbol{x}}_A$ and $\bar{\boldsymbol{x}}_B$ denote the position of the end nodes with respect to the element floating frame; $\bar{\boldsymbol{G}}_{3\times 12}$ is the so-called spin fitter matrix, which takes into account the change of orientation of $\langle\mathbf{F}\rangle$ as the end nodes change position or rotation. For the two-node beam, this matrix is defined as

$\bar{\boldsymbol{G}} = [\partial \bar{\boldsymbol{\omega}}_{\boldsymbol{F}} / \partial \bar{\boldsymbol{x}}_A, \partial \bar{\boldsymbol{\omega}}_{\boldsymbol{F}} / \partial \bar{\boldsymbol{\omega}}_A, \ldots\ ]$. Particularizing the previous expression to the location of $\langle \mathbf{F} \rangle$, placed at the middle of the element, one gets

$$\bar{\boldsymbol{G}} = \begin{bmatrix} 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/L & 0 & 0 & 0 & 0 & 0 & -1/L & 0 & 0 & 0 \\ 0 & -1/L & 0 & 0 & 0 & 0 & 0 & 1/L & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (5)$$

The projector matrix $\bar{\boldsymbol{P}}$ extracts the deformation part of the nodal coordinates. It can be expressed as a sum of translational part $\bar{\boldsymbol{P}}_u$ and rotational part $\bar{\boldsymbol{P}}_\omega$.

The element rotation matrix $\boldsymbol{R}_\alpha$. This matrix is given as follows

$$\boldsymbol{R}_{\alpha 12\times 12} = \begin{pmatrix} \boldsymbol{R}_{\boldsymbol{F}} & & & \\ & \boldsymbol{R}_A^{\mathrm{T}} \boldsymbol{R}_{\boldsymbol{F}} & & \\ & & \boldsymbol{R}_{\boldsymbol{F}} & \\ & & & \boldsymbol{R}_B^{\mathrm{T}} \boldsymbol{R}_{\boldsymbol{F}} \end{pmatrix}, \quad (6)$$

where transformation matrices $\boldsymbol{R}_A$ and $\boldsymbol{R}_B$ are included so as to account for local angular velocities of the finite element's state vector.

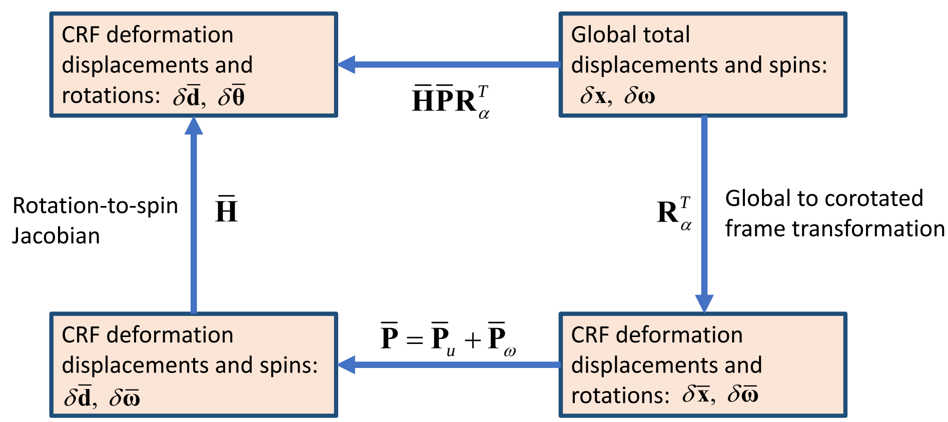

Figure 2: Transformation sequence in the context of CRF. Diagram inspired by Fig. 14.1 of notes by C. Felippa [4]

The expressions above provide all the ingredients for the computation of the global internal force vector, $\boldsymbol{f}_{\mathrm{in}}$, from the vector of local internal forces, $\bar{\boldsymbol{f}}_{\mathrm{in}}$, as follows:

$$\boldsymbol{f}_{in} = \boldsymbol{R}_\alpha \bar{\boldsymbol{P}}^{\mathrm{T}} \bar{\boldsymbol{H}}^{\mathrm{T}} \bar{\boldsymbol{f}}_{in}. \quad (7)$$

$\boldsymbol{f}_{in}$ can be further projected to obtain the generalized forces associated with rotation parameters, e.g. Euler parameters. We can split a subset of Eq. (7) as

$\bar{\boldsymbol{P}}^{\mathrm{T}}\bar{\boldsymbol{H}}^{\mathrm{T}}\bar{\boldsymbol{f}}_{in} = [\bar{\boldsymbol{n}}_A, \bar{\boldsymbol{m}}_A, \bar{\boldsymbol{n}}_B, \bar{\boldsymbol{m}}_B]$. This new definition of vectors helps us define the tangent stiffness matrix by using the following matrices

$$\bar{\boldsymbol{F}}_{nm} = \begin{pmatrix} \text{skew}\left(\bar{\boldsymbol{n}}_A\right) \\ \text{skew}\left(\bar{\boldsymbol{m}}_A\right) \\ \text{skew}\left(\bar{\boldsymbol{n}}_B\right) \\ \text{skew}\left(\bar{\boldsymbol{m}}_B\right) \end{pmatrix} ; \ \bar{\boldsymbol{F}}_{n} = \begin{pmatrix} \text{skew}\left(\bar{\boldsymbol{n}}_A\right) \\ \boldsymbol{0}_{3\times 3} \\ \text{skew}\left(\bar{\boldsymbol{n}}_B\right) \\ \boldsymbol{0}_{3\times 3} \end{pmatrix} . \tag{8}$$

Finally, we can compute the global tangent stiffness matrix as follows

$$\boldsymbol{K} = \boldsymbol{R}_\alpha \left(\underbrace{\bar{\boldsymbol{P}}^{\mathrm{T}}\bar{\boldsymbol{H}}^{\mathrm{T}}\bar{\boldsymbol{K}}\bar{\boldsymbol{H}}\bar{\boldsymbol{P}}}_{\boldsymbol{K}_{GM}} - \underbrace{\bar{\boldsymbol{F}}_{nm}\bar{\boldsymbol{G}}}_{\boldsymbol{K}_{GR}} - \underbrace{\bar{\boldsymbol{G}}^{\mathrm{T}}\boldsymbol{F}_n^{\mathrm{T}}\bar{\boldsymbol{P}}}_{\boldsymbol{K}_{GP}} + \underbrace{\bar{\boldsymbol{P}}^{\mathrm{T}}\boldsymbol{L}_H\bar{\boldsymbol{P}}}_{\boldsymbol{K}_{GH}} \right) \boldsymbol{R}_\alpha^{\mathrm{T}} \tag{9}$$

where the terms $\boldsymbol{K}_{GR}$ (rotational), $\boldsymbol{K}_{GP}$ (equilibrium projection), and $\boldsymbol{K}_{GH}$ (moment-correction) capture geometric stiffness; $\boldsymbol{K}_{GM}$ is the material stiffness.

The following notes apply to Chrono's implementation:

- The term $\boldsymbol{K}_{GH}$ is not computed in the implementation because its influence was found to be negligible (see [3])
- $\boldsymbol{K}_{GM}$ is a symmetric matrix for Euler-Bernoulli beams; however, the overall global stiffness matrix $\boldsymbol{K}$ is non-symmetric due to the influence of geometric stiffness terms
- Under reasonable assumptions, the global stiffness matrix can be symmetrized without affecting convergence of the Newton-Raphson method:

  $$\boldsymbol{K} = \boldsymbol{R}_\alpha \left(\bar{\boldsymbol{P}}^{\mathrm{T}}\bar{\boldsymbol{H}}^{\mathrm{T}}\bar{\boldsymbol{K}}\bar{\boldsymbol{H}}\bar{\boldsymbol{P}} - \bar{\boldsymbol{F}}_{sy}\bar{\boldsymbol{G}} - \bar{\boldsymbol{G}}^{\mathrm{T}}\bar{\boldsymbol{F}}_{sy}^{\mathrm{T}}\bar{\boldsymbol{P}}\right) \boldsymbol{R}_\alpha^{\mathrm{T}} ,$$

  where $\bar{\boldsymbol{F}}_{sy} = \frac{1}{2}\left(\bar{\boldsymbol{F}}_{nm} + \bar{\boldsymbol{F}}_n\right)$.

3. CRF in Chrono

3.1 Euler-Bernoulli beam

This subsection describes the basics of the Euler-Bernoulli beam formulation within the context of the CRF. For more details, see [2]. We describe hereafter the way elastic forces are obtained element-wise.

3.1.1 Axial Strain

The local strains are formulated in terms of local displacements and local linearized angles, which describe infinitesimal rotations about local axes needed for the computation of bending (in two directions) and torsional strains. A frame of reference is attached to each of the nodes of the beam, whose initial configuration is stored. The vector of displacements of each node i is obtained as $\bar{\boldsymbol{u}}_i = \boldsymbol{R}_{c,j}^{\mathrm{T}}\boldsymbol{x}_i - \boldsymbol{R}_{r,j}^{\mathrm{T}}\boldsymbol{x}_{r,i}$, where $\boldsymbol{R}_{c,j}$ and $\boldsymbol{R}_{r,j}$ are the orientation matrices of the element j in the current and reference (initial) configuration, and $\boldsymbol{x}_i$ and $\boldsymbol{x}_{r,i}$ are the global position vectors of such a node in the current and reference (initial) configuration, respectively. Note that the reference configuration, $\mathcal{C}_0$, is considered here unstressed (see Fig. 1).

The axial force in the beam can be computed as $N = EA/l_0(u_l^A - u_l^B)$, where E is the modulus of elasticity, A is the cross section area, l_0 the beam element's reference length, and u_l^A and u_l^B are the displacement along the element frame $\langle \mathbf{F} \rangle$ of nodes A and B, respectively.

3.1.2 Bending Strain

Local linearized angles are obtained by composing rotation matrices. The rotation matrix describing this infinitesimal rotation is given by the expression $\boldsymbol{R_\theta} = \boldsymbol{R}_{c,j}^{\mathrm{T}} \boldsymbol{R}_{c,i} \boldsymbol{R}_{r,i}^{\mathrm{T}}$, which yields a rotation matrix from which three-dimensional infinitesimal angles, which represent local, linear strains, can be extracted. Note that these computations can also be carried out using quaternion algebra, as done in Chrono. The torque produced by local deformations can be expressed as $\boldsymbol{M} = \boldsymbol{D}(\boldsymbol{\theta}_l - \boldsymbol{\theta}_{l_0})$, where

$$\boldsymbol{D} = \frac{1}{l_0} \begin{pmatrix} GJ & 0 & 0 & -GJ & 0 & 0 \\ 0 & 4EI_2 & 0 & 0 & 2EI_2 & 0 \\ 0 & 0 & 4EI_3 & 0 & 0 & 2EI_3 \\ -GJ & 0 & 0 & GJ & 0 & 0 \\ 0 & 2EI_2 & 0 & 0 & 4EI_2 & 0 \\ 0 & 0 & 2EI_3 & 0 & 0 & 4EI_3 \end{pmatrix}, \tag{10}$$

where G is the modulus of rigidity, J is the polar moment of inertia, E is the modulus of elasticity, I_2 is the area moment of inertia about the y axis, I_3 is the area moment of inertia about the z axis, $\boldsymbol{\theta}_l$ is the current local linear rotation pseudo-vector, and $\boldsymbol{\theta}_{l_0}$ is the reference local, linear rotation pseudo-vector. Matrix $\boldsymbol{D}$ is expanded to create a 12-by-12 local stiffness matrix, $\bar{\boldsymbol{K}}$, including axial and shear – in two directions – deformations.

3.1.3 Local Internal Forces

The displacements necessary to compute the local internal forces are grouped together in the vector $\bar{\boldsymbol{d}} = \begin{bmatrix} u_l^A & u_y^A & u_z^A & \theta_x^A & \theta_y^A & \theta_z^A & u_l^B & u_y^B & u_z^B & \theta_x^B & \theta_y^B & \theta_z^B \end{bmatrix}$, where u refers to displacements, θ denotes linearized angles, and superscripts A and B refer to the first and second node of the beam element, respectively. The internal force vector may be simply written as $\bar{\boldsymbol{f}}_{in} = \bar{\boldsymbol{K}} \bar{\boldsymbol{d}}$. Vector $\bar{\boldsymbol{f}}_{in}$ can be mapped into global coordinates as described in Section 2.2. Then, vector $\boldsymbol{f}_{in}$ can be projected into whatever generalized coordinates are used to describe finite rotations in a specific software implementation, e.g., quaternions.

3.2 Other Co-Rotated Finite Elements

Chrono implements other finite elements within the CRF:

- Hexahedron – 8 nodes: `ChElementHexa_8.cpp`
- Hexahedron – 20 nodes: `ChElementHexa_20.cpp`
- Tetrahedron – 4 nodes: `ChElementTetra_4.cpp`
- Tetrahedron – 10 nodes: `ChElementTetra_10.cpp`

These finite elements have linear displacement fields and use Polar Decomposition to find a local frame that best represents the orientation of the element, see [5].

4. General Considerations

This section briefly summarizes some of the general features of the co-rotational formulation:

- The CRF divides the reference motion of the bodies into base and co-rotated. Strains and stresses are computed as the configuration change between co-rotated and current.
- CRF captures well large translations, large rotations, and small elastic strains. Under certain conditions, i.e. strains must remain small, the CRF can be extended to nonlinear materials – e.g. material constants depend upon current strain state. This extension is usually limited to structural approaches (as opposed to continuum mechanics-based), that is, trusses, beams, shells, and plates.
- The formulation is Lagrangian-based since the CRF tracks the motion of material points. For instance, the motion of the A and B ends of a beam.
- The state of each finite element in the CRF defines both rigid body and deformation motion. That is, changes in coordinate values may or may not cause strains. This is in contrast with the *floating frame of reference* formulation, where the motion of finite elements is split into reference (rigid body) and flexible coordinates; thereby allowing the use of model order reduction techniques on the linear, flexible coordinates.

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