# Extension

In this section we are going to consider the Cheryl's birthday puzzle, as before, but assume that we do not know which 10 dates she has mentioned at the start of the puzzle. If Albert and Bernard make the same public announcements as before, what can we say about the 10 possible dates?

## 1.1 The Model

Let $M = \langle S, \sim, V \rangle$ be the model.

Firstly let the set of states be the 10 dates that Cheryl gives, we can number them $s_1 = (m_1, d_1), s_2 = (m_2, d_2), ..., s_{10} = (m_{10}, d_{10})$ where $m_i \in \{\text{January, ..., December}\}$ and $d_i \in \{1, ..., 31\}$ such that the pair is a permissible date (for example, we do not allow the pair (February, 31)).

Let us define the set of propositional atoms $\mathbf{P} = \{m_i | i \in \{1, 2, ..., 10\}\} \cup \{d_i | i \in \{1, 2, ..., 10\}\}$. Something to note is that we allow $m_i = m_k$ for $i \neq k$ as well as $d_i = d_k$ for $i \neq k$.

Next, because Albert knows the month, we have the following relations for Albert: $(m_i, d_i) \sim_A (m_k, d_k)$ if and only if $m_i = d_k$. Similarly, because Bernard knows the day we have the following relations for Bernard: $(m_i, d_i) \sim_B (m_k, d_k)$ if and only if $d_i = d_k$. Notice that this means we have reflexive, symmetric, and transitive relations.

For $i \in \{1, ..., 10\}$, we have the following valuations: $V(m_i) = \{s_k \in S | m_i = m_k\}$ and $V(d_i) = \{s_k \in S | d_i = d_k\}$.

We can notice that this is a distributed system.

Finally let us denote $s = (m, d) \in S$ as Cheryl's true birthday.

### 1.1.1 Example

Let us consider an example, using the dates from Cheryl's birthday puzzle: May 15, May 16, May 19, June 17, June 18, July 14, July 16, August 14, August 15, and August 17.

Then let us number these states in the order that they appear. Then the set of states consists of $s_1 = (m_1, d_1) = (\text{May}, 15), ..., s_{10} = (m_{10}, d_{10}) = (\text{August}, 17)$.

In this case our propositional atoms $\mathbf{P} = \{\text{May, June, July, August}\} \cup \{14, 15, 16, 17, 18, 19\}$. We note that $m_1 = m_2 = m_3 = \text{May}$ but it only appears once in the set of propositional atoms.

Again we have relations between states with the same month for Albert, and relations between states with the same day for Bernard. For example $(m_1, d_1) = (\text{May}, 15) \sim_A (\text{May}, 16) = (m_2, d_2)$ because $m_1 = m_2$. $(m_1, d_1) = (\text{May}, 15) \sim_B (\text{August}, 15) = (m_9, d_9)$ because $d_1 = d_9$.

An example for the valuations is $V(\text{May}) = \{(\text{May}, 15), (\text{May}, 16), (\text{May}, 19)\} = \{s_i \in S | m_i = \text{May}\}$ and $V(15) = \{(\text{May}, 15), (\text{August}, 15)\} = \{s_i \in S | d_i = 15\}$.

## 1.2 Notes

First let us establish some properties of the system.

### 1.2.1 Albert knows Cheryl's birth month, and Bernard knows Cheryl's birth day

By construction, in a state $s_i$ Albert knows Cheryls birth month, and Bernard knows Cheryl's birth day:

$(M, s_i) \models K_A m_i$ Definition of $K_A$

$\iff$

$(M, s_j) \models m_i$ for all $s_j$ such that $s_i \sim_A s_j$ Definition of $(M, s_j) \models m_i$

$\iff$

$s_j \in V(m_i)$ for all $s_j$ such that $s_i \sim_A s_j$ Definition of $V(m_i)$

$\iff$

$m_j = m_i$ for all $s_j$ such that $s_i \sim_A s_j$

We can see that this is true, because by definition $s_i \sim_A s_j$ if and only if $m_j = m_i$.

Similarly $(M, s_i) \models K_B d_i$.

### 1.2.2 What Albert or Bernard knowing a month or day implies about the month or day

If $(M, s_i) \models K_A m_k$ then $m_k = m_i$. This can be shown as follows:

$(M, s_i) \models K_A m_k$ Definition of $K_A$

$\iff$

$(M, s_j) \models m_k$ for all $s_j$ such that $s_i \sim_A s_j$ Because $s_i \sim_A s_i$

$\Rightarrow$

$(M, s_i) \models m_k$ Definition of $(M, s_i) \models m_k$

$\iff$

$s_i \in V(m_k) = \{s_l | m_k = m_l\}$

$\iff$

$m_i = m_k$

Similarly $(M, s_i) \models K_A d_k$ implies $d_k = d_i$, $(M, s_i) \models K_B m_k$ implies $m_k = m_i$, and $(M, s_i) \models K_B d_k$ implies $d_k = d_i$.

### 1.2.3 When do Albert and Bernard know Cheryl's birthday

We say that Albert knows Cheryl's birthday if he knows both her birth month and her birth day: $K_A(m_i \wedge d_i)$ for some $i \in \{1, 2, ..., 10\}$. Because he already knows Cheryl's birth month, we claim that it is sufficient to know her birth day:

$(M, s_i) \models K_A(m_1 \wedge d_1) \vee \cdots \vee K_A(m_{10} \wedge d_{10})$ Definition of $\vee$

$\iff$

$(M, s_i) \models K_A(m_k \wedge d_k)$ for some $k \in \{1, 2, ..., 10\}$ Definition of $K_A$

$\iff$

For some k, $(M, s_j) \vDash (m_k \wedge d_k)$ for all $s_j$ such that $s_i \sim_A s_j$

$\iff$

For some k, $(M, s_j) \vDash m_k$ and $(M, s_j) \vDash d_k$ for all $s_j$ such that $s_i \sim_A s_j$

$\iff$

For some k, $(M, s_i) \vDash K_A m_k$ and $(M, s_i) \vDash K_A d_k$

$\Rightarrow$

$(M, s_i) \vDash K_A m_i$ and $(M, s_i) \vDash K_A d_i$

$\Rightarrow$

$(M, s_i) \vDash K_A d_i$

$\Rightarrow$

$(M, s_i) \vDash K_A d_1 \vee \cdots \vee K_A d_{10}$

Now to prove the converse:

$(M, s_i) \vDash K_A d_1 \vee \cdots \vee K_A d_{10}$

$\iff$

$(M, s_i) \vDash K_A d_k$ for some $k \in \{1, 2, ..., 10\}$ $\quad$ $d_k = d_i$ by subsubsection 1.2.2

$\Rightarrow$

$(M, s_i) \vDash K_A d_i$ $\quad$ Because $(M, s_i) \vDash K_A m_i$ holds in all states

$\iff$

$(M, s_i) \vDash K_A m_i$ and $(M, s_i) \vDash K_A d_i$ $\quad$ Definition of $K_A$

$\Rightarrow$

$(M, s_j) \vDash m_i$ for all $s_j$ such that $s_i \sim_A s_j$, and $(M, s_j) \vDash d_i$ for all $s_j$ such that $s_i \sim_A s_j$

$\iff$

$(M, s_j) \vDash m_i$ and $(M, s_j) \vDash d_i$ for all $s_j$ such that $s_i \sim_A s_j$ $\quad$ Definition of $\wedge$

$\iff$

$(M, s_j) \vDash m_i \wedge d_i$ for all $s_j$ such that $s_i \sim_A s_j$ $\quad$ Definition of $K_A$

$\iff$

$(M, s_i) \vDash K_A(m_i \wedge d_i)$

$\Rightarrow$

$(M, s_i) \vDash K_A(m_k \wedge d_k)$ for some $k \in \{1, 2, ..., 10\}$

$\iff$

$(M, s_i) \vDash K_A(m_1 \wedge d_1) \vee \cdots \vee K_A(m_{10} \wedge d_{10})$

So we can conclude that in our model $M \vDash [K_A(m_1 \wedge d_1) \vee \cdots \vee K_A(m_{10} \wedge d_{10})] \leftrightarrow [K_A d_1 \vee \cdots \vee K_A d_{10}]$

Similarly we can say that Bernard knows Cheryl's birthday, $(K_B(m_1 \wedge d_1) \vee \cdots \vee K_B(m_{10} \wedge d_{10}))$ if and only if he knows her birth month $(K_B m_1 \vee \cdots \vee K_B m_{10})$.

We thus write $K_A d_1 \vee \cdots \vee K_A d_{10}$ for the statement "Albert knows Cheryl's birthday" and $K_B m_1 \vee \cdots \vee K_B m_{10}$ for the statement "Bernard knows Cheryl's birthday"

### 1.2.4 Unique months or days

We say a month, $m'$, is unique if there is exactly one state $s_i$ such that $m_i = m'$. Similarly we say a day, $d'$, is unique if there is exactly one state $s_i$ such that $d_i = d'$.

Albert knows Cheryl's birthday in a state if and only if the month is unique (i.e. only reflexive relations for Albert) .

$(M, s_i) \vDash K_A d_1 \vee \cdots \vee K_A d_{10}$
$\Longleftrightarrow$
$(M, s_i) \vDash K_A d_k$ for some $k \in \{1, ..., 10\}$
$\Longleftrightarrow$
$(M, s_i) \vDash K_A d_i$
$\Longleftrightarrow$
$(M, s_j) \vDash d_i$ for all $s_j$ such that $s_i \sim_A s_j$
$\Longleftrightarrow$
$s_j \in V(d_i) = \{s' | d_i = d'\}$ for all $s_j$ such that $s_i \sim_A s_j$
$\Longleftrightarrow$
$d_j = d_i$ for all $s_j$ such that $s_i \sim_A s_j$
$\Longleftrightarrow$
$d_j = d_i$ for all $s_j$ such that $m_i = m_j$
$\Longleftrightarrow$
For all $s_j$ such that $m_i = m_j$, $s_i = s_j$

This means that Albert knows Cheryl's birthday in a state $s_i$ if and only the month $m_i$ is unique.

Similarly Bernard knows Cheryl's birthday in a state $s_i$ if and only if the day $d_i$ is unique (i.e. only reflexive relations for Bernard).

## 1.3 The public announcements

The public announcements made in the puzzle are as follows:

Albert: "I don't know when Cheryl's birthday is, but I know that you (Bernard) don't know either."
Bernard: "At first I didn't know when Cheryl's birthday is, but now I know."
Albert: "Now I also know when Cheryl's birthday is."

Using the previously derived meanings of when Albert and Bernard know Cheryl's birthday, We can write the first public announcement as $\varphi_1 = \neg(K_A d_1 \vee \cdots \vee K_A d_{10}) \wedge K_A \neg(K_B m_1 \vee \cdots K_B m_{10})$.

For the second announcement, the first part does not add any additional information so we can reduce the announcement to $\varphi_2 = K_B m_1 \vee \cdots \vee K_B m_{10}$.

The last public announcement is then $\varphi_3 = K_A d_1 \vee \cdots \vee K_A d_{10}$.

It follows that in the true state $s$, $(M, s) \vDash \langle\varphi_1\rangle\langle\varphi_2\rangle\langle\varphi_3\rangle\top$.

Let us define models $M_1 = M|\varphi_1 = \langle S_1, \sim^1, V_1\rangle$, $M_2 = M_1|\varphi_2 = S_2, \sim^2, V_2$ and $M_3 = M_2|\varphi_3 = S_3, \sim^3$ $, V_3$ then this means the following:

$(M, s) \vDash \langle\varphi_1\rangle\langle\varphi_2\rangle\langle\varphi_3\rangle\top$
$\iff$
$(M, s) \vDash \varphi_1$ and $(M_1, s) \vDash \langle\varphi_2\rangle\langle\varphi_3\rangle\top$
$\iff$
$(M, s) \vDash \varphi_1$ and $(M_1, s) \vDash \varphi_2$ and $(M_2, s) \vDash \langle\varphi_3\rangle\top$
$\iff$
$(M, s) \vDash \varphi_1$ and $(M_1, s) \vDash \varphi_2$ and $(M_2, s) \vDash \varphi_3$ and $(M_3, s) \vDash \top$

The condition $(M_3, s) \vDash \top$ holds automatically if the previous conditions hold so we do not need to check this.

For the solution of the puzzle to be unique, we however need that $M \vDash [\varphi_1][\varphi_2][\varphi_3](m \wedge d)$. Let us consider an arbitrary state $s'$:

$(M, s') \vDash [\varphi_1][\varphi_2][\varphi_3](m \wedge d)$
$\iff$
if $(M, s') \vDash \varphi_1$ then $(M_1, s') \vDash [\varphi_2][\varphi_3](m \wedge d)$
$\iff$
if $(M, s') \vDash \varphi_1$, then if $(M_1, s') \vDash \varphi_2$ then $(M_2, s') \vDash [\varphi_3](m \wedge d)$
$\iff$
if $(M, s') \vDash \varphi_1$, then if $(M_1, s') \vDash \varphi_2$, then if $(M_2, s') \vDash \varphi_3$ then $(M_3, s') \vDash (m \wedge d)$

If for a state $s'$ every subsequent public announcement is truthfully made then $s'$ has to be the true state.

### 1.4 The first public announcement

Albert makes the first announcement "I don't know when Cheryl's birthday is, but I know that you (Bernard) don't know either."

By the equivalences determined in subsubsection 1.2.3 we can write this public announcement as $\varphi_1 = \neg(K_Ad_1 \vee \cdots \vee K_Ad_{10}) \wedge K_A\neg(K_Bm_1 \vee \cdots K_Bm_{10})$

$(M, s) \vDash \varphi_1$
$\iff$
$(M, s) \vDash \neg(K_Ad_1 \vee \cdots \vee K_Ad_{10}) \wedge K_A\neg(K_Bm_1 \vee \cdots K_Bm_{10})$
$\iff$
$(M, s) \vDash \neg(K_Ad_1 \vee \cdots \vee K_Ad_{10})$ and $(M, s) \vDash K_A\neg(K_Bm_1 \vee \cdots K_Bm_{10})$
Firstly let us consider $(M, s) \vDash \neg(K_Ad_1 \vee \cdots \vee K_Ad_{10})$.

$(M, s) \vDash \neg(K_A d_1 \vee \cdots \vee K_A d_{10})$

$\iff$

Not $(M, s) \vDash K_A d_1 \vee \cdots \vee K_A d_{10}$

$\iff$

Not $(M, s) \vDash K_A d_i$ for some $i \in \{1, ..., 10\}$

$\iff$

Not for some $i \in \{1, ..., 10\}$, $(M, s_k) \vDash d_i$ for all $s_k$ such that $s \sim_A s_k$

$\iff$

For all $i \in \{1, ..., 10\}$, not $((M, s_k) \vDash d_i$ for all $s_k$ such that $s \sim_A s_k)$

$\iff$

For all $i \in \{1, ..., 10\}$, there exists an $s_k$ such that $s \sim_A s_k$ and not $(M, s_k) \vDash d_i$

$\iff$

For all $i \in \{1, ..., 10\}$, there exists an $s_k$ such that $s \sim_A s_k$ and $(M, s_k) \nvDash d_i$

$\iff$

For all $i \in \{1, ..., 10\}$, there exists an $s_k$ such that $s \sim_A s_k$ and $s_k \notin V(d_i)$

We can consider this by a case distinction: either $d = d_i$ or not.
Suppose $d \neq d_i$, then the statement follows by considering $s_k = s$ which we are allowed to do because of the reflexive relations. If $d \neq d_i$ then $s \notin V(d_i)$.
Now suppose that $d = d_i$, then it follows that $s \in V(d_i)$ so there must exist another state $s'$ such that $s \sim_A s'$ and $s' \in V(d_i)$. As a reminder $s \sim_A s'$ if and only if $m = m'$, i.e. the states agree on the month.

We can thus conclude that there have to be at least two states with the true month.

It follows that if there is a state with a unique month $m_i$ we know that this is not Cheryl's birthday.

Now let us consider $(M, s) \vDash K_A \neg(K_B m_1 \vee \cdots K_B m_{10})$.

Because of the reflexive relations this implies that Bernard does not know Cheryl's birthday: $(M, s) \vDash \neg(K_B m_1 \vee \cdots K_B m_{10})$.

By similar reasoning as before it follows that there are at least two states that have true day $d$, and if there is a state with a unique day $d_i$ we know that this is not Cheryl's birthday.

Furthermore, the fact that Albert knows that Bernard does not know Cheryl's birthday implies that *if* there are states with unique days, the true month is not one of them.

### 1.4.1 Implications of the Transitivity of $\sim_A$

**Claim:** Something to notice is that if for some state $s_i$, $(M, s_i) \vDash \varphi_1$, then every state $s_j$ such that $s_j \sim_A s_i$ (and by symmetry $s_i \sim_A s_j$) is also such that $(M, s_j) \vDash \varphi_1$.

Let $s_i \sim_A s_j$, then: $(M, s_i) \vDash \neg K_A \varphi$

$\iff$

not $(M, s_i) \models K_A\varphi$

$\iff$

not $(M, s_k) \models \varphi$ for all $s_k$ such that $s_i \sim_A s_k$

$\iff$

for some $s_k$ such that $s_i \sim_A s_k$, not $(M, s_k) \models \varphi$

$\iff$

for some $s_k$ such that $s_j \sim_A s_k$, not $(M, s_k) \models \varphi$ by transitivity of $\sim_A$

$\iff$

not $(M, s_k) \models \varphi$ for all $s_k$ such that $s_j \sim_A s_k$

$\iff$

not $(M, s_j) \models K_A\varphi$

$\iff$

$(M, s_j) \models \neg K_A\varphi$

also: $(M, s_i) \models K_A\varphi$

$\iff$

$(M, s_k) \models \varphi$ for all $s_k$ such that $s_i \sim_A s_k$

$\iff$

$(M, s_k) \models \varphi$ for all $s_k$ such that $s_j \sim_A s_k$ by transitivity of $\sim_A$

$\iff$

$(M, s_j) \models K_A\varphi$

Now we can apply this to $\varphi_1$:

$(M, s_i) \models \varphi_1$

$\iff$

$(M, s_i) \models \neg(K_Ad_1 \vee \cdots \vee K_Ad_{10}) \wedge K_A\neg(K_Bm_1 \vee \cdots K_Bm_{10})$

$\iff$

$(M, s_i) \models \neg(K_Ad_1 \vee \cdots \vee K_Ad_{10})$ and $(M, s_i) \models K_A\neg(K_Bm_1 \vee \cdots K_Bm_{10})$

$\iff$

$(M, s_j) \models \neg(K_Ad_1 \vee \cdots \vee K_Ad_{10})$ and $(M, s_j) \models K_A\neg(K_Bm_1 \vee \cdots K_Bm_{10})$

$\iff$

$(M, s_j) \models \neg(K_Ad_1 \vee \cdots \vee K_Ad_{10}) \wedge K_A\neg(K_Bm_1 \vee \cdots K_Bm_{10})$

$\iff$

$(M, s_j) \models \varphi_1$

### 1.4.2 Summary

$\varphi_1$ is true in a state $s_i$ of model $M$ if and only if the following hold:

- There exists a state $s_j \neq s_i$ such that $m_i = m_j$, and
- For every state $s_j$ such that $m_i = m_j$, there exists a state $s_k \neq s_j$ such that $d_k = d_j$.

### 1.5 The second public announcement

The next public announcement is made by Bernard where he states “At first I didn’t know when Cheryl’s birthday is, but now I know.”

We can model this as follows:
If Bernard didn’t know Cheryl’s birthday at first, that means he didn’t know it in the model $M$. We can model the first part of the statement as follows: $(M, s) \vDash \neg(K_B m_1 \vee \cdots K_B m_{10})$.

This however does not add new information as it is implied $(M, s) \vDash \varphi_1$. We can thus reduce the second public announcement to the statement "I know now", and we let $\varphi_2 = (K_B m_1 \vee \cdots K_B m_{10})$.

Let $M_1 = M|\varphi_1$ be the model after the first public announcement, then we let $\varphi_2 = K_B m_1 \vee \cdots K_B m_{10}$ and $(M_1, s) \vDash \varphi_2$.

If Bernard knows Cheryl’s birthday in $M_1$ it follows that her birth day $d$ is unique in the updated model. However it was shown that her birth day $d$ is not unique in the original model $M$.

If $M_1 = \langle S_1, \sim^1, V_1 \rangle$, then for all states such that $s \sim_B s'$, $s' \notin S_1$. This means $(M, s') \vDash \neg\varphi_1$.

$(M, s') \vDash \neg\varphi_1$
$\iff$
not $(M, s') \vDash \varphi_1$
$\iff$
not $(M, s') \vDash \neg(K_A d_1 \vee \cdots \vee K_A d_{10}) \wedge K_A \neg(K_B m_1 \vee \cdots K_B m_{10})$
$\iff$
not $((M, s') \vDash \neg(K_A d_1 \vee \cdots \vee K_A d_{10})$ and $(M, s) \vDash K_A \neg(K_B m_1 \vee \cdots K_B m_{10}))$
$\iff$
not $(M, s') \vDash \neg(K_A d_1 \vee \cdots \vee K_A d_{10})$ or not $(M, s') \vDash K_A \neg(K_B m_1 \vee \cdots K_B m_{10})$ $\iff$
$(M, s') \vDash (K_A d_1 \vee \cdots \vee K_A d_{10})$ or $(M, s') \vDash \neg K_A \neg(K_B m_1 \vee \cdots K_B m_{10})$
$\iff$
$(M, s') \vDash (K_A d_1 \vee \cdots \vee K_A d_{10})$ or $(M, s') \vDash \hat{K}_A(K_B m_1 \vee \cdots K_B m_{10})$

So either $(M, s') \vDash (K_A d_1 \vee \cdots \vee K_A d_{10})$, and Albert knows Cheryl’s birthday in the state $s'$ which happens if and only if the month is unique, or $(M, s') \vDash \hat{K}_A(K_B m_1 \vee \cdots K_B m_{10})$, which means that there is a state with the same month as $s'$ where Bernard knows Cheryl’s birthday which is true if and only if there is a state $s''$ such that $m' = m''$ and $s''$ has a unique day.

#### 1.5.1 Summary

$\varphi_2$ is true in a state $s_i$ of model $M_1$ if and only if for all states $s_j \neq s_i$ such that $s_i \sim_B s_j$ (in model $M$), $(M, s_j) \vDash \neg\varphi_1$ which occurs if and only if for all states $s_j$ such that $s_i \sim_B s_j$ either:

- For all states $s_k$ such that $s_j \sim_A s_k$, $s_j = s_k$ (the month is unique), or
- There exists a state $s_k$ such that $m_j = m_k$ and for all states $s'$ such that $s_k \sim_B s'$, $s_k = s'$ (the day is unique).

### 1.6 The third public announcement

The final public announcement is by Albert who then states “Now I also know when Cheryl’s birthday is.”

$(M_2, s) \vDash K_A d_1 \vee \cdots \vee K_A d_{10}$.

This means that the true month $m$ is unique. This means that Albert only has reflexive relations from the true state $s$.

Something to note is that the true month $m$ has to be non-unique in the original model $M$ (by the first public announcement), and then also non-unique in the model $M_1$ because if $(M, s) \vDash \varphi_1$ then $(M, s') \vDash \varphi_1$ for all worlds $s'$ such that $s \sim_A s'$ (as shown in subsubsection 1.4.1).

Thus for all worlds $s'$ such that $s' \neq s$ and $s \sim_A s'$ in model M, it follows that $s \sim_A^1 s'$ in model $M_1$ and furthermore $(M_1, s') \vDash \neg\varphi_2$.

$(M_1, s') \vDash \neg\varphi_2$
$\iff$
$(M_1, s') \vDash \neg(K_B m_1 \vee \cdots K_B m_{10})$

This holds if and only if the day $d'$ is not unique in the model $M_1$, i.e. there exists another state $s'' \in S_1$ such that $m \neq m''$ and $d' = d''$.

This furthermore implies that because the state $s'' \in S_1$, $(M, s'') \vDash \varphi_1$.

As a reminder $m'' \neq m$, so it follows that for the states $s$ and $s''$ their months are non-unique in $M$, and for all states with the month $m$ or $m''$ there exists another state with the same day. Furthermore there is a state with the month $m$ that has the same day as a state with the month $m''$.

Something to notice is that Albert knows Cheryl’s birthday in a state $s_i$ in $M_2$ if the month $m_i$ is unique, so for Albert to know Cheryl’s birthday there needs to be at least one unique month in $M_2$. However for the puzzle to be solveable to an outsider, there needs to be exactly one unique month in $M_2$.

#### 1.6.1 Summary

$\varphi_3$ is true in the state $s_i$ of model $M_2$ if and only if for all states $s_j$ such that $s_i \sim_A s_j$, there exists another state $s_k \neq s_j$ such that $s_j \sim_B s_k$ and $(M, s_k) \vDash \varphi_1$ (which holds if and only if $m_k \neq m$ is a unique month and the month has no unique days).