Cheryl's Birthday Puzzle - worked out example

Definitions

Consider the following three public announcements made by Albert and Bernard.
Albert: "I don't know when Cheryl's birthday is, but I know that you (Bernard) don't know either."
$\varphi_1 = \neg(K_A(m_5 \wedge d_{15}) \vee ... \vee K_A(m_8 \wedge d_{17})) \wedge K_A\neg(K_B(m_5 \wedge d_{15}) \vee ... \vee K_B(m_8 \wedge d_{17}))$
(For all possible dates given by Cheryl.)

Bernard: "At first I didn't know when Cheryl's birthday is, but now I know."
The first part of the statement does not give new information, so we only consider "I know now"
$\varphi_2 = K_B(m_5 \wedge d_{15}) \vee ... \vee K_B(m_8 \wedge d_{17})$ (For all possible dates given by Cheryl.)

Albert: "Now I also know when Cheryl's birthday is."
$\varphi_3 = K_A(m_5 \wedge d_{15}) \vee ... \vee K_A(m_8 \wedge d_{17})$ (For all possible dates given by Cheryl.)

(With this definition, φ_1 can also be written as $\varphi_1 = \neg\varphi_3 \wedge K_A\neg\varphi_2$.)

We define model M as the model after Cheryl's gave all the possible dates and told Albert the month and told Bernard the day, see Figure 1.
We define the reduced model $M_1 = M|\varphi_1$, see Figure 2.
Reduced model $M_2 = M_1|\varphi_2$, see Figure 3.
Reduced model $M_3 = M_2|\varphi_3$, see Figure 4.

The true state of Cheryl's Birthday is July 16, which Albert and Bernard don't know. Call this state $(16, July)$.

$$(M, (16, July)) \models m_7 \wedge d_{16} \tag{1}$$

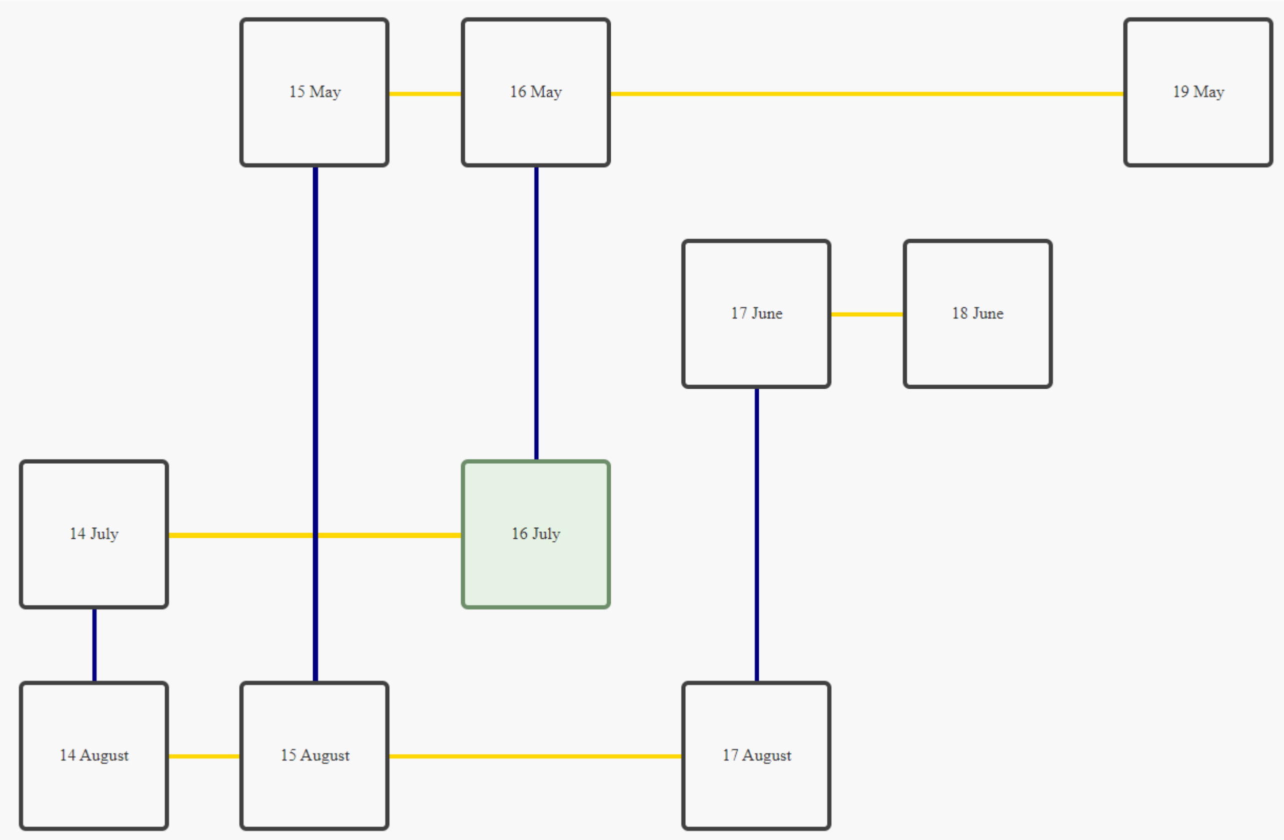

Figure 1: Model M. Reflexive and further transitive relations not shown. Albert's relations are shown in orange, Bernard's relations are shown in blue.

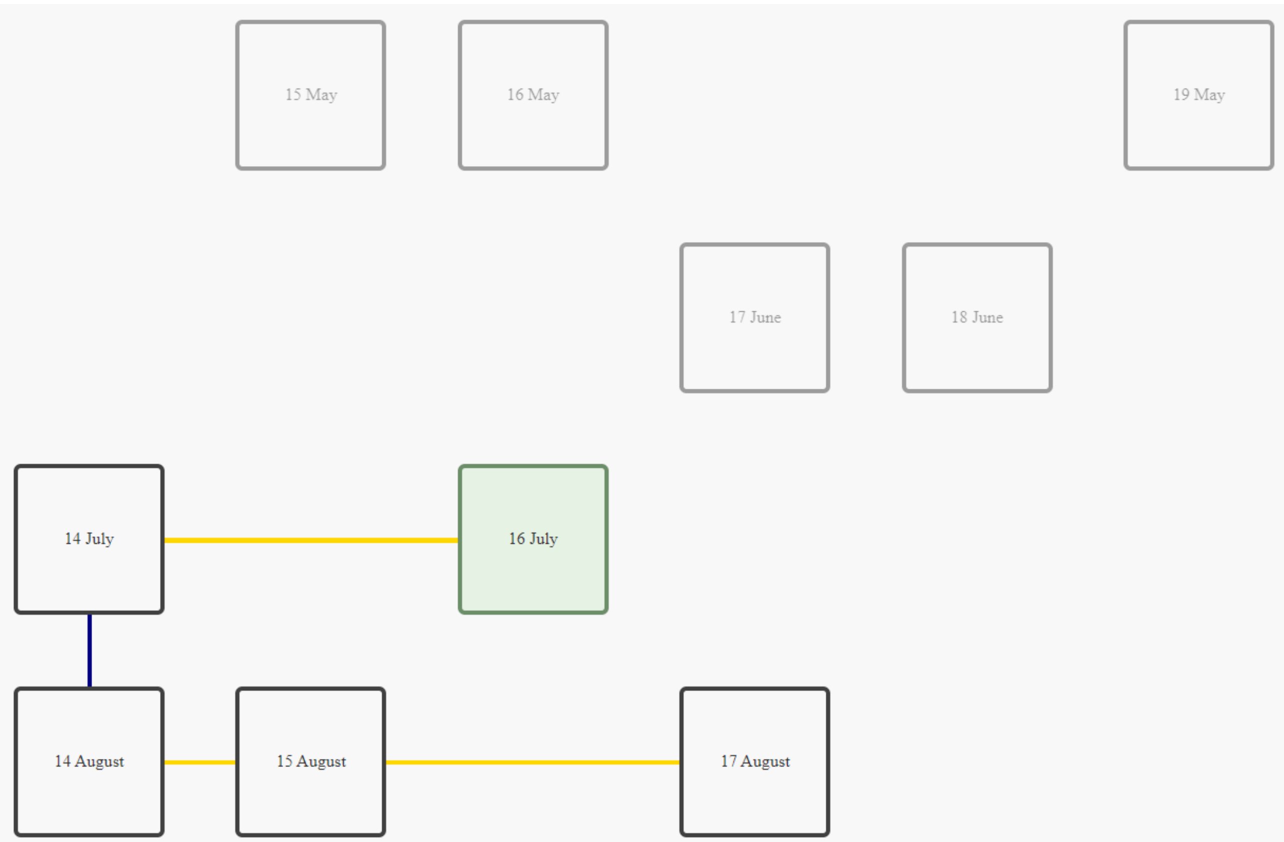

Figure 2: Reduced model M_1. Grayed-out states are not included.

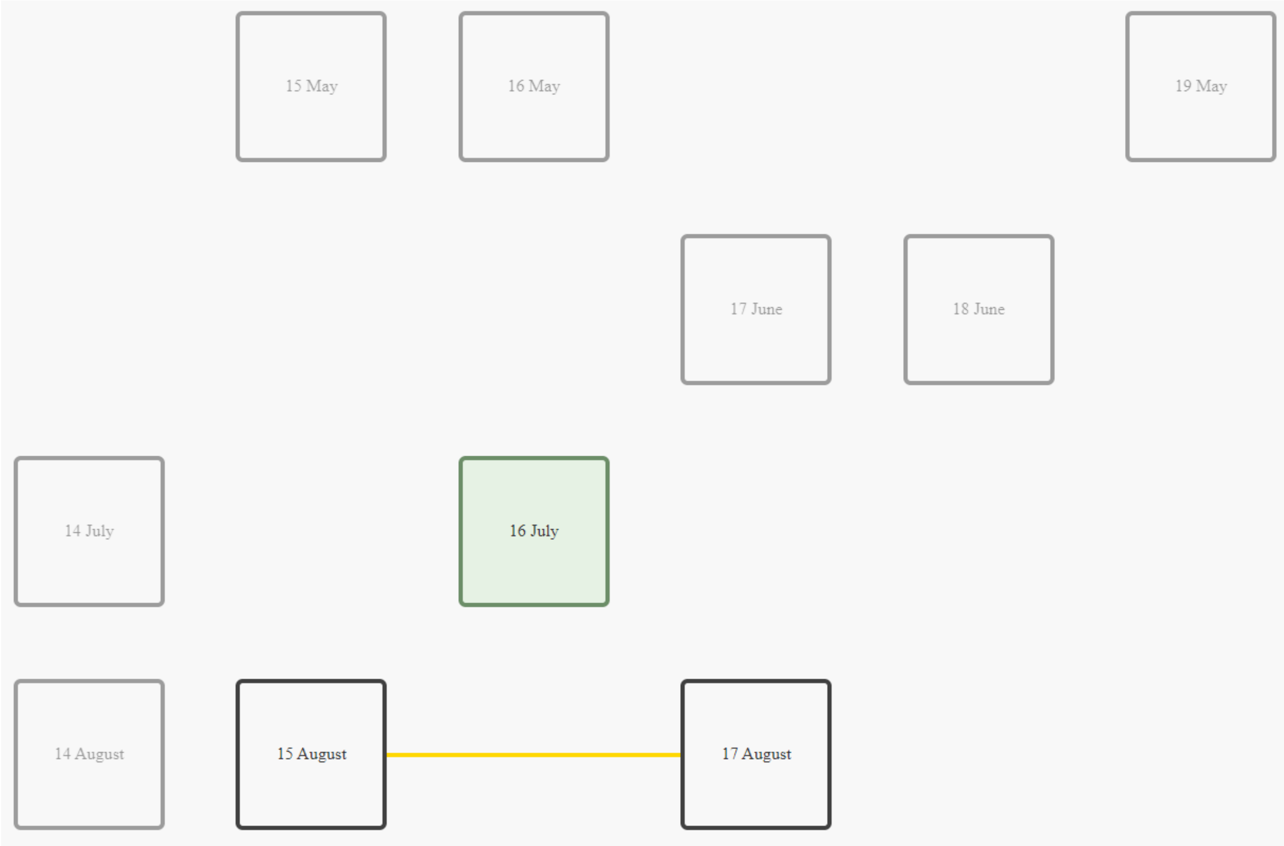

Figure 3: Reduced model M_2.

Figure 4: Reduced model M_3. Only a single state remains.

Derivation

We want to show that after the public announcements, $\varphi_1, \varphi_2, \varphi_3$, the exact date of Cheryl's birthday is known to Albert and Bernard.

Show that $(M, (16, July)) \models \langle\varphi_1\rangle\langle\varphi_2\rangle\langle\varphi_3\rangle\top$ is true.

$(M, (16, July)) \models \langle\varphi_1\rangle\langle\varphi_2\rangle\langle\varphi_3\rangle\top$ $\Leftrightarrow$ (Definition of $\langle\varphi_1\rangle$)

$(M, (16, July)) \models \varphi_1$ and $(M_1, (16, July)) \models \langle\varphi_2\rangle\langle\varphi_3\rangle\top$

Let's first show whether $(M, (16, July)) \models \varphi_1$ is true.

- $(M, (16, July)) \models \varphi_1$ $\Leftrightarrow$ (Definition of φ_1)
  $(M, (16, July)) \models \neg(K_A(m_5 \wedge d_{15}) \vee ... \vee K_A(m_8 \wedge d_{17})) \wedge K_A\neg(K_B(m_5 \wedge d_{15}) \vee ... \vee K_B(m_8 \wedge d_{17}))$ $\Leftrightarrow$ (Def of $\wedge$)
  $(M, (16, July)) \models \neg(K_A(m_5 \wedge d_{15}) \vee ... \vee K_A(m_8 \wedge d_{17}))$
  and $(M, (16, July)) \models K_A\neg(K_B(m_5 \wedge d_{15}) \vee ... \vee K_B(m_8 \wedge d_{17}))$
  First conjunct:

  - $(M, (16, July)) \models \neg(K_A(m_5 \wedge d_{15}) \vee ... \vee K_A(m_8 \wedge d_{17}))$ $\Leftrightarrow$ (Definition of $\neg$)
    $(M, (16, July)) \not\models K_A(m_5 \wedge d_{15}) \vee ... \vee K_A(m_8 \wedge d_{17})$ $\Leftrightarrow$ (Definition of $\vee$)
    $(M, (16, July)) \not\models K_A(m_5 \wedge d_{15})$ or ... or $(M, (16, July)) \not\models K_A(m_8 \wedge d_{17})$ $\Leftrightarrow$ (Def of K_A)
    For all t with $(16, July \sim_A t, (M, t) \not\models (m_5 \wedge d_{15})$ or ...
    or for all u with $(16, July) \sim_A u, (M, u) \not\models (m_8 \wedge d_{17})$
    $t = u \in \{(14, July), (16, July)\}$. In the first state only the propositional atoms m_7 and d_{14} are true. In the second state only the propositional atoms m_7 and d_{16} are true. Any choice of d_i is not true in both states at the same time. So this statement is true.

  Continuing with the other conjunct:
  $(M, (16, July)) \models K_A\neg(K_B(m_5 \wedge d_{15}) \vee ... \vee K_B(m_8 \wedge d_{17}))$ $\Leftrightarrow$ (Definition of K_A)
  For all t with $(16, July) \sim_A t, (M, t) \models \neg(K_B(m_5 \wedge d_{15}) \vee ... \vee K_B(m_8 \wedge d_{17}))$ $\Leftrightarrow$ (De Morgan law)
  For all t with $(16, July) \sim_A t, (M, t) \models \neg K_B(m_5 \wedge d_{15}) \wedge ... \wedge \neg K_B(m_8 \wedge d_{17})$
  $t \in \{(14, July), (16, July)\}$, specifying both states:
  $(M, (14, July)) \models \neg K_B(m_5 \wedge d_{15}) \wedge ... \wedge \neg K_B(m_8 \wedge d_{17})$
  and $(M, (16, July)) \models \neg K_B(m_5 \wedge d_{15}) \wedge ... \wedge \neg K_B(m_8 \wedge d_{17})$ $\Leftrightarrow$ (Definition of $\wedge$)
  $(M, (14, July)) \models \neg K_B(m_5 \wedge d_{15})$ and ... and $(M, (14, July)) \models \neg K_B(m_8 \wedge d_{17})$
  and $(M, (16, July)) \models \neg K_B(m_5 \wedge d_{15})$ and ... and $(M, (16, July)) \models \neg K_B(m_8 \wedge d_{17})$ $\Leftrightarrow$ (Def $\neg$)
  $(M, (14, July)) \not\models K_B(m_5 \wedge d_{15})$ and ... and $(M, (14, July)) \not\models K_B(m_8 \wedge d_{17})$
  and $(M, (16, July)) \not\models K_B(m_5 \wedge d_{15})$ and ... and $(M, (16, July)) \not\models K_B(m_8 \wedge d_{17})$ $\Leftrightarrow$ (Def K_B)
  For all t with $(14, July) \sim_B t, (M, t) \not\models m_5 \wedge d_{15}$ and ...
  and for all u with $(14, July) \sim_B u, (M, u) \not\models m_8 \wedge d_{17}$ and ...
  and for all v with $(16, July) \sim_B v, (M, v) \not\models m_5 \wedge d_{15}$ and ...
  and for all w with $(16, July) \sim_B w, (M, w) \not\models m_8 \wedge d_{17}$.
  $t = u \in \{(14, July), (14, August)\}$, $v = w \in \{(16, May), (16, July)\}$. With similar reasoning as before, any choice of propositional atom m_i is not true in two different states at the same time. So this statement is true.

Continuing with the other conjunct:

$(M_1, (16, July)) \models \langle\varphi_2\rangle\langle\varphi_3\rangle\top$ $\Leftrightarrow$ (Definition of $\langle\varphi_2\rangle$)

$(M_1, (16, July)) \models \varphi_2$ and $(M_2, (16, July)) \models \langle\varphi_3\rangle\top$

Let's show whether $(M_1, (16, July)) \models \varphi_2$ is true.

- $(M_1, (16, July)) \models \varphi_2$ $\Leftrightarrow$ (Definition of φ_2)
  $(M_1, (16, July)) \models K_B(m_5 \wedge d_{15}) \vee ... \vee K_B(m_8 \wedge d_{17})$ $\Leftrightarrow$ (Definition of $\vee$)
  $(M_1, (16, July)) \models K_B(m_5 \wedge d_{15})$ or ... or $(M_1, (16, July)) \models K_B(m_8 \wedge d_{17})$ $\Leftrightarrow$ (Def of K_B)
  For all t with $(16, July) \sim_B t, (M_1, t) \models (m_5 \wedge d_{15})$ or ...
  or for all u with $(16, July) \sim_B u, (M_1, u) \models (m_7 \wedge d_{16})$ or ...
  or for all v with $(16, July) \sim_B v, (M_1, v) \models (m_8 \wedge d_{17})$
  $t = u = v \in \{(16, July)\}$. $(M_1, (16, July)) \models (m_7 \wedge d_{16})$ is true by Equation 1.
  Therefore $(M_1, (16, July)) \models \varphi_2$ is true.

Continuing with the other conjunct:
$(M_2, (16, July)) \models \langle\varphi_3\rangle\top$ $\Leftrightarrow$ (Definition of $\langle\varphi_3\rangle$)
$(M_2, (16, July)) \models \varphi_3$ and $(M_3, (16, July)) \models \top$
$(M_3, (16, July)) \models \top$ is true by tautology.
Last part:
$(M_2, (16, July)) \models \varphi_3$ $\Leftrightarrow$ (Definition of φ_3)
$(M_2, (16, July)) \models K_A(m_5 \wedge d_{15}) \vee ... \vee K_A(m_8 \wedge d_{17})$ $\Leftrightarrow$ (Definition of $\vee$)
$(M_2, (16, July)) \models K_A(m_5 \wedge d_{15})$ or ... or $(M_2, (16, July)) \models K_A(m_8 \wedge d_{17})$ $\Leftrightarrow$ (Definition of K_A)
For all t with $(16, July) \sim_A t, (M_2, t) \models (m_5 \wedge d_{15})$ or ...
or for all u with $(16, July) \sim_A u, (M_2, u) \models (m_7 \wedge d_{16})$ or ...
or for all v with $(16, July) \sim_A v, (M_2, v) \models (m_8 \wedge d_{17})$
$t = u = v \in \{(16, July)\}$. $(M_2, (16, July)) \models (m_7 \wedge d_{16})$ is true by Equation 1.
Therefore $(M, (16, July)) \models \langle\varphi_1\rangle\langle\varphi_2\rangle\langle\varphi_3\rangle\top$ is true.