Равенство в Теории Типов

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Intentional Type Theory

Per Martin-Löf "An Intuitionstic Theory of Types: Predicative Part", 1975

Definitional equality $=_{\text{def}}$

Definition, Equivalence, αβ, $\dfrac{a =_{\text{def}} b}{x[a] =_{\text{def}} x[b]}$,

$$\frac{a : A \qquad A =_{\text{def}} B}{a : B}.$$

Propositional equality (a.k.a. Identity) $I(A, a, b)$

$$r(a) : I(A, a, a)$$

$$\dfrac{\dfrac{x : A \qquad y : A \qquad z : I(A, x, y)}{C[x, y, z]} \qquad c[x] : C[x, x, r(x)]}{f(x, x, r(x)) =_{\text{def}} c[x]}$$

TODO: Also mention J as a modern alternative.

Extentional Type Theory

Per Martin-Löf "Intuitionstic Type Theory", 1980

Definitional equality

Definition, Equivalence, Substituiting equals for equals, α. Checking that in $\dfrac{A \qquad B}{A\&B}$ A, B on top are the same (i.e. definitionally equal) to the ones on the bottom.

Judgemental equality $A = B$, $a = b : A$

Defined for every type, $\beta\eta$,

$$\xi \, \frac{\begin{array}{c}[x : A] \\ f(x) = g(x) : B(x)\end{array}}{\lambda x.\ f(x) = \lambda x.\ g(x) : \Pi(x : A)B(x)} ,$$

$$\frac{a \in A \qquad A = B}{a \in B} , \quad \frac{a = b \in A \qquad A = B}{a = b \in B} .$$

Propositional equality

$$\mathrm{ER}\, \frac{c : I(A, a, b)}{a = b : A} , \quad \mathrm{UIP}\, \frac{c : I(A, a, b)}{c = r : I(A, a, b)} .$$

Observational Type Theory

Definitional equality: αβη.
Propositional equality is heterogeneous equality:
$(x : A) = (y : B)$, $\quad$ $\mathit{refl} : (x : A) \to (x : A) = (x : A)$. Coercion:
$$\frac{p : S_0 = S_1 \qquad s_0 : S_0}{s_0[p\rangle : S_1}$$
Coherence:
$$\frac{p : S_0 = S_1 \qquad s_0 : S_0}{s_0[[p|\rangle : (s_0 : S_0) = (s_0[p\rangle : S_1)}$$
Coercion, coherence rules are defined for each and every type.
Result: propositional equality is truly extensional.

Propositions as Types

Proposition	$\leftrightarrow$	Type
Proof	$\leftrightarrow$	Term
Proof simplification	$\leftrightarrow$	Computation

Propositions as Types

$$
\begin{array}{rcl}
A & \leftrightarrow & A \\
A \wedge B & \leftrightarrow & A \times B \\
A \supset B & \leftrightarrow & A \rightarrow B \\
\forall x B(x) & \leftrightarrow & \Pi(x : A)B \\
x = y & \leftrightarrow & ???
\end{array}
$$

$=$

Logic:

- Reflexivity $\quad x = x$
- Congruence $\quad \dfrac{x = y}{f(x) = f(y)}$
- Substiutivity $\quad \dfrac{x = y \qquad A_x}{A_y}$

Type theory:

- $\mathcal{I}xx$
- $\dfrac{\mathcal{I}xy}{\mathcal{I}(fx)(fy)}$
- $\dfrac{\mathcal{I}xy \qquad A_x}{A_y}$

Intentional Type Theory

aka Martin-Löf theory of types

Definitional $f \equiv g$.

- Intentional
- Up to α
- Substiutive

Checking that in $\dfrac{A \qquad B}{A \& B}$ A, B on top are the same (i.e. definitionally equal) to the ones on the bottom.

Intentional Type Theory

aka Martin-Löf theory of types

Judgemental $A = B, \quad a = b \in A$

- Holds by definition for canonical elements
- Up to β
- Substiutive
- $\dfrac{a \in A \qquad A = B}{a \in B}$
- $\dfrac{a = b \in A \qquad A = B}{a = b \in B}$

Intentional Type Theory

aka Martin-Löf theory of types

Propositional $I(a, b, A)$

- A type
- Canonical element $r(a) : I(a, a, A)$
- Eliminator J :.
- $\dfrac{a = b \in A}{r(a) : I(a, b, A)}$

Extensional Type Theory

$$\frac{x \in I(a, b, A)}{a = b \in A}$$

CiC

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