Равенство в Теории Типов

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Propositions as Types

$$
\begin{array}{rcl}
\text{Proposition} & \leftrightarrow & \text{Type} \\
\text{Proof} & \leftrightarrow & \text{Program} \\
\text{Proof simplification} & \leftrightarrow & \text{Computation}
\end{array}
$$

- Syntactic $=$. $\dfrac{A \qquad B}{A \wedge B}$
- $=$ on propositions.
- $=$ as proposition.

Intentional Type Theory

Martin-Löf 1975

Definitional equality $=_{def}$

Definition, αβ-equivalence, SUB $\dfrac{a =_{def} b}{x[a] =_{def} x[b]}$,

$\dfrac{a : A \qquad A =_{def} B}{a : B}$.

Propositional equality (a.k.a. Identity) $I(A, a, b)$

$$r(a) : I(A, a, a)$$

$$\dfrac{c[x] : C[x, x, r(x)]}{f(x, x, r(x)) =_{def} c[x]}$$

$$J : ((x : A) \to C(x, x, r(x)))$$
$$\to (xy : A)(p : I(A, x, y))$$
$$\to C(x, y, p)$$

Extentional Type Theory

Martin-Löf 1983

Definitional (syntactic, intentional) equality
> Definition, α-equivalence, SUB.

Judgemental equality $A = B$, $a = b : A$
> Defined for every type, βη,

$$\xi \, \frac{[x : A]}{f(x) = g(x) : B(x)}{\lambda x.\, f(x) = \lambda x.\, g(x) : \Pi(x : A)B(x)} \, ,$$

$$\frac{a : A \quad A = B}{a : B} \, , \quad \frac{a = b : A \quad A = B}{a = b : B} \, .$$

Propositional equality

$$\text{ER} \, \frac{c : I(A, a, b)}{a = b : A} \, , \quad \text{UIP} \, \frac{c : I(A, a, b)}{c = r : I(A, a, b)} \, .$$

Observational Type Theory

Definitional equality $\alpha\beta\eta$.

Propositional equality

$$(x : A) = (y : B)$$

$$refl : (x : A) \to (x : A) = (x : A)$$

$$\text{Coercion} \quad \frac{p : S_0 = S_1 \qquad s_0 : S_0}{s_0[p\rangle : S_1}$$

$$\text{Coherence} \quad \frac{p : S_0 = S_1 \qquad s_0 : S_0}{s_0[\![p\rangle\!] : (s_0 : S_0) = (s_0[p\rangle : S_1)}$$

Result: propositional equality is extensional, definitional equality is decidable.

Computational Type Theory

Based on ETT. But ETT is formal type theory.

Definitional equality

  Define the meaning of $x = y : A$ for the types you
  want. $x : A$ is then a shortcut for $x = x : A$.

Propositional equality

  The same as judgemental.

Quotient types

  $(x, y) : A//E$. Change an equivalence relation to E.

Homotopy Type Theory

Definitionally like ITT but with η.
Proof relevance for equality. Think βreduction chain. $a = b : A$
when there is a path from a to b. As such no UIP (and K). OTOH
have function extensionality.

Unified Type Theory

ITT + impredicative Prop. Equality is Leibniz:
$\forall P : A \to Prop.\ P(a) \implies P(b)$.

ZOMBIE

Forget automatic β. Get equality relations from the context and use them.