Равенство в Теории Типов

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Intentional Type Theory
Martin-Löf 1975

Definitional equality $=_{\text{def}}$

Definition, αβ-equivalence, $\dfrac{a =_{\text{def}} b}{x[a] =_{\text{def}} x[b]}$,

$$\dfrac{a : A \qquad A =_{\text{def}} B}{a : B} .$$

Propositional equality (a.k.a. Identity) $I(A, a, b)$

$$r(a) : I(A, a, a)$$

$$\dfrac{x : A \qquad y : A \qquad z : I(A, x, y)}{C[x, y, z]} \qquad c[x] : C[x, x, r(x)]$$
$$\overline{f(x, x, r(x)) =_{\text{def}} c[x]}$$

Extentional Type Theory
Martin-Löf 1983

Definitional equality

Definition, Equivalence, Substituiting equals for equals, α. Checking that in $\dfrac{A \quad B}{A \& B}$ A, B on top are the same (i.e. definitionally equal) to the ones on the bottom.

Judgemental equality $A = B$, $a = b : A$

Defined for every type, βη,

$$\xi \, \dfrac{\overset{[x:A]}{f(x) = g(x) : B(x)}}{\lambda x.\ f(x) = \lambda x.\ g(x) : \Pi(x:A)B(x)} \ ,$$

$$\dfrac{a \in A \quad A = B}{a \in B} \ , \quad \dfrac{a = b \in A \quad A = B}{a = b \in B} \ .$$

Propositional equality

$$\text{ER} \ \dfrac{c : I(A, a, b)}{a = b : A} \ , \quad \text{UIP} \ \dfrac{c : I(A, a, b)}{c = r : I(A, a, b)} \ .$$

Observational Type Theory

Definitional equality: αβη.

Propositional equality is heterogeneous equality:

$(x : A) = (y : B), \quad refl : (x : A) \to (x : A) = (x : A)$. Coercion:

$$\frac{p : S_0 = S_1 \qquad s_0 : S_0}{s_0[p\rangle : S_1}$$ Coherence:

$$\frac{p : S_0 = S_1 \qquad s_0 : S_0}{s_0[[p|\rangle : (s_0 : S_0) = (s_0[p\rangle : S_1)}$$

Coercion, coherence rules are defined for each and every type.

Result: propositional equality is truly extensional.

Unified Type Theory

ITT + impredicative Prop. Equality is Leibniz:
$\forall P : A \to Prop. \; P(a) \implies P(b)$.

Computational Type Theory

Based on ETT. Define the meaning of $x = y : A$ for the types you want. $x : A$ is then a shortcut for $x = x : A$. And propositional equality is truly identified with judgemental equality.

Set types: $\{x : A | p(x)\}$ Quotient types: $(x, y) : A//E$. Change an equivalence relation to E.

Homotopy Type Theory

Definitionally like ITT but with η.
Proof relevance for equality. Think βreduction chain. $a = b : A$
when there is a path from a to b. As such no UIP (and K). OTOH
have function extensionality.

ZOMBIE

Forget automatic β. Get equality relations from the context and use them.