

Равенство в Теории Типов

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Propositions as Types

Proposition \leftrightarrow Type

Proof \leftrightarrow Program

Proof simplification \leftrightarrow Computation

► Syntactic =. $\frac{A \quad B}{A \wedge B}$

► = on propositions.

► = as proposition.

Intentional Type Theory

Martin-Löf 1975

Definitional equality $=_{\text{def}}$

$$\text{Definition, } \alpha\beta\text{-equivalence, SUB } \frac{a =_{\text{def}} b}{x[a] =_{\text{def}} x[b]},$$
$$\frac{a : A \quad A =_{\text{def}} B}{a : B}.$$

Propositional equality (a.k.a. Identity) $I(A, a, b)$

$$r(a) : I(A, a, a)$$

$$\frac{c[x] : C[x, x, r(x)]}{f(x, x, r(x)) =_{\text{def}} c[x]}$$

$$\begin{aligned} J : & ((x : A) \rightarrow C(x, x, r(x))) \\ & \rightarrow (xy : A)(p : I(A, x, y)) \\ & \rightarrow C(x, y, p) \end{aligned}$$

Extentional Type Theory

Martin-Löf 1983

Definitional (syntactic, intentional) equality

Definition, α -equivalence, SUB.

Judgemental equality $A = B$, $a = b : A$

Defined for every type, $\beta\eta$,

$[x : A]$

$$\xi \frac{f(x) = g(x) : B(x)}{\lambda x. f(x) = \lambda x. g(x) : \Pi(x : A)B(x)},$$
$$\frac{a : A \quad A = B}{a : B}, \quad \frac{a = b : A \quad A = B}{a = b : B}.$$

Propositional equality

$$\text{ER} \frac{c : I(A, a, b)}{a = b : A}, \quad \text{UIP} \frac{c : I(A, a, b)}{c = r : I(A, a, b)}.$$

Observational Type Theory

Definitional equality $\alpha\beta\eta$.

Propositional equality

$$(x : A) = (y : B)$$

$$\text{refl} : (x : A) \rightarrow (x : A) = (x : A)$$

$$\text{Coercion} \frac{p : S_0 = S_1 \quad s_0 : S_0}{s_0[p] : S_1}$$

$$\text{Coherence} \frac{p : S_0 = S_1 \quad s_0 : S_0}{s_0[p] : (s_0 : S_0) = (s_0[p] : S_1)}$$

Result: propositional equality is extensional, definitional equality is decidable.

Unified Type Theory

ITT + impredicative Prop. Equality is Leibniz:

$$\forall P : A \rightarrow Prop. P(a) \implies P(b).$$

Computational Type Theory

Based on ETT. Define the meaning of $x = y : A$ for the types you want. $x : A$ is then a shortcut for $x = x : A$. And propositional equality is truly identified with judgemental equality.

Set types: $\{x : A \mid p(x)\}$ Quotient types: $(x, y) : A // E$. Change an equivalence relation to E .

Homotopy Type Theory

Definitionally like ITT but with η .

Proof relevance for equality. Think β reduction chain. $a = b : A$ when there is a path from a to b . As such no UIP (and K). OTOH have function extensionality.

ZOMBIE

Forget automatic β . Get equality relations from the context and use them.