Data Structures and Algorithms

Lecture 25: Balanced Binary Search Trees (AVL Trees)

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Outlines

- Tree Sort (BST sort) and its complexity
- Balanced and unbalanced binary search trees
- Height-balance property
- AVL trees
- Insertion on AVL trees

Phase 1: construct BST ⇒ $T(n) = ?$
Phase 2: traverse BST ⇒ $O(n)$

Trees Sort (BST Sort)

"out-of-place"

- *Tree Sort* is a not-in-place sorting algorithm. The algorithm is based on constructing a BST and perform an in-order traversal on the constructed BST to list a sorted output sequence.

BST has n nodes

- Tree Sort requires $\boxed{\Theta(n)}$ extra space to store the BST.
- The time complexity of Tree Sort is determined by the time taken to constructed the BST given an input sequnece.

Case analysis

  - At worst, the complexity of the construction can be $O(n^2)$
  - At best, the complexity of the construction can be $O(n \log n)$

Tree Sort:
1. Construct BST
2. Run inorder ⇒ $\Theta(n)$

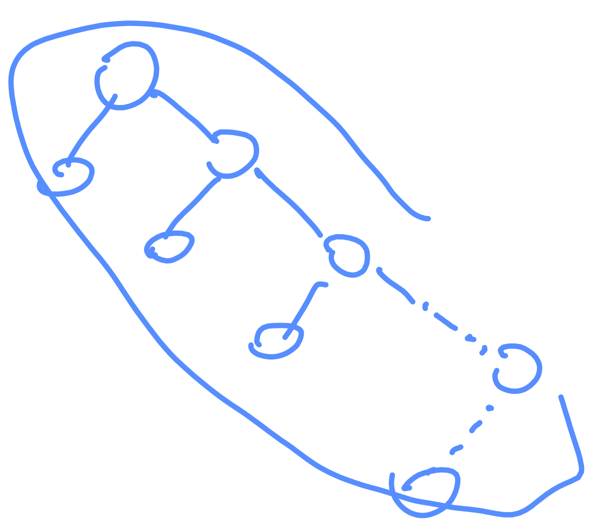
W.C. Complexity of Tree Sort

- **Proposition 1**: The worst case time complexity of Tree Sort is $O(n^2)$

W.C. After each insertion, the height of the BST increases by one.

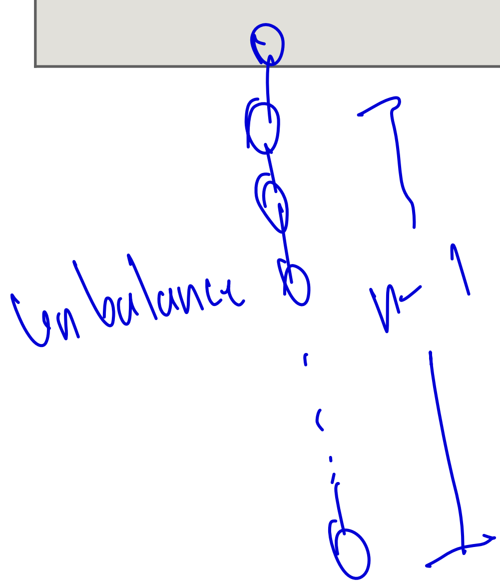
Therefore, the complexity of constructing the BST is

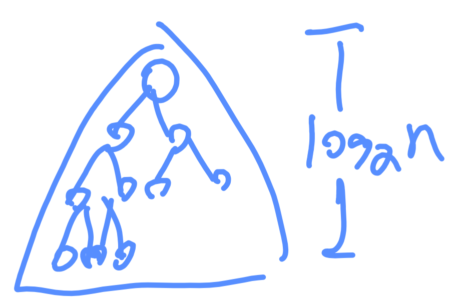
$$\sum_{h=0}^{n-1} \Theta(h) = \Theta\left(\sum_{h=0}^{n-1} h\right) = \Theta(\underline{\underline{n^2}})$$



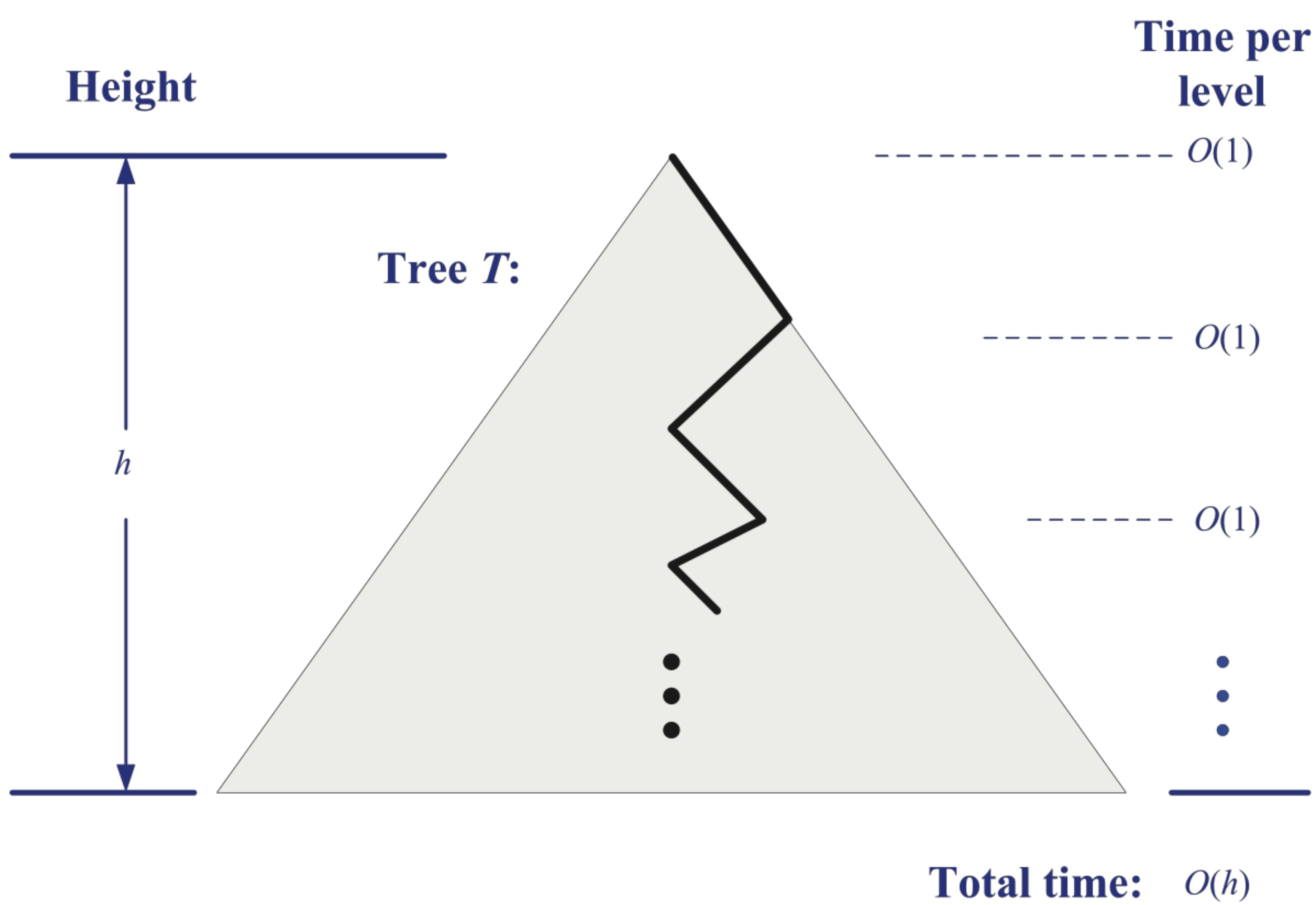
Recall: Complexity of Operations on BST (1)

Operations	Complexity
Search	$O(h)$
Minimum	$O(h)$
Maximum	$O(h)$
Successor	$O(h)$
Predecessor	$O(h)$
Tree-Insert	$O(h)$
Tree-Delete	$O(h)$
	<u>Remark:</u> h is the height of a binary tree. - At worst, h can be n-1. W.C. - At best, h can be $\log_2(n+1)-1$. B.C.

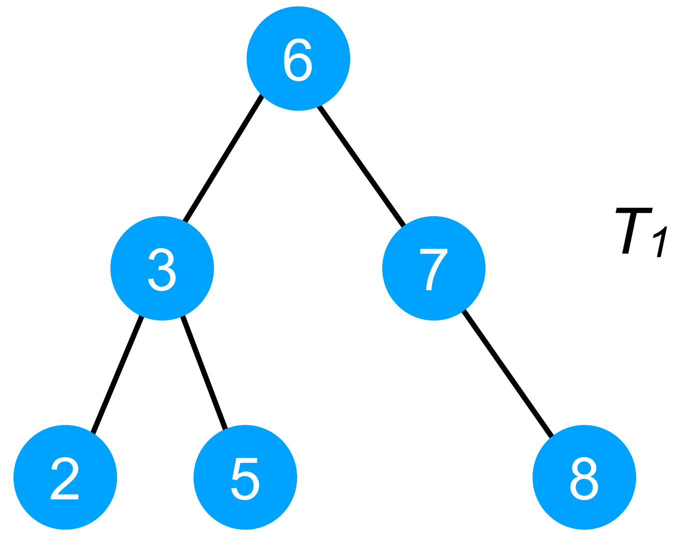


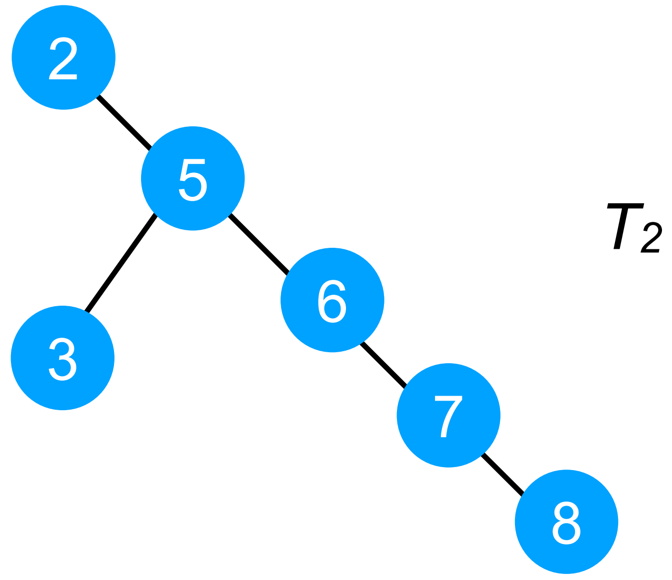


Recall: Complexity of Operations on BST(2)



Binary Search Trees of Different Heights





unbalanced BST $\approx$ BST of height $\Theta(n)$

Unbalanced Binary Search Trees

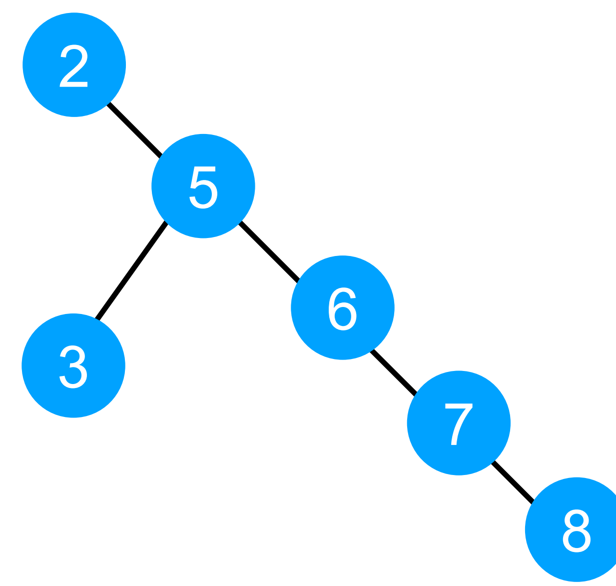
$h = \Theta(\log n)$ in B.C.

- **Unbalanced binary search tree**: The height of the search tree is approximately linear in the number of nodes

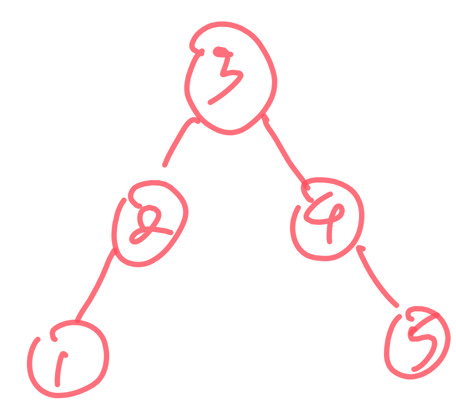
$h = \Theta(n)$

- If a binary search tree is unbalanced, the performance it achieves is no better than the linear data structures

1 2 3 4 5

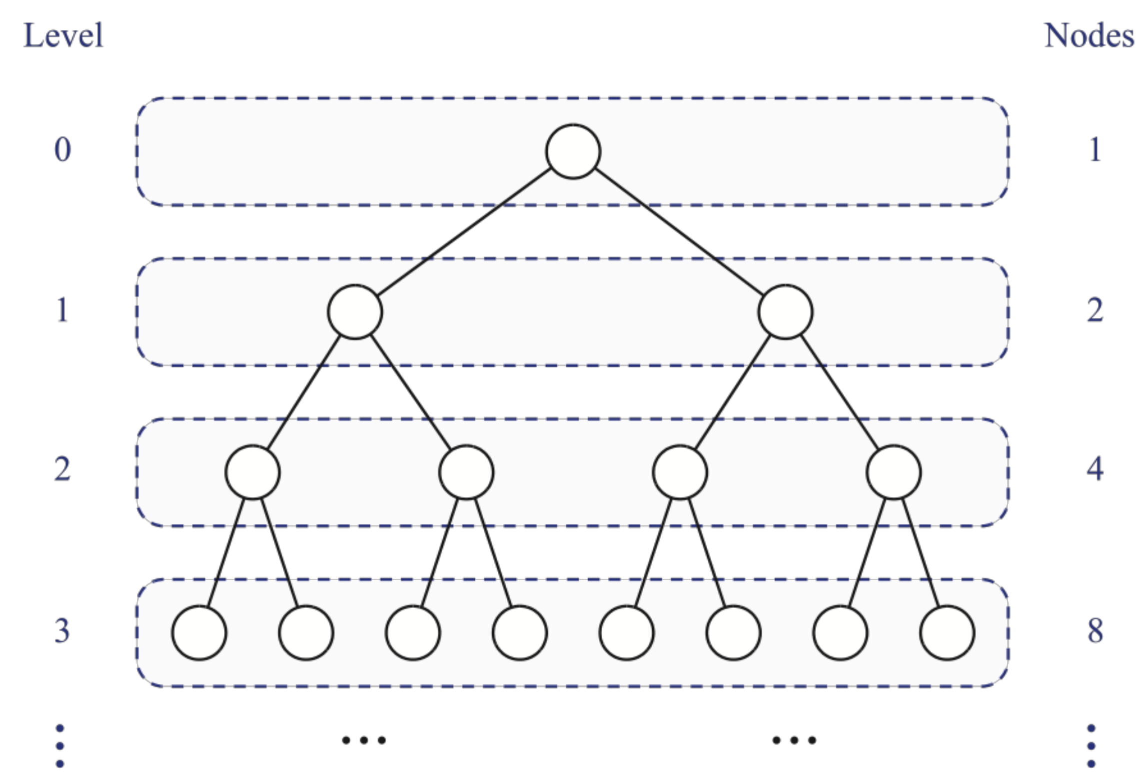


T_2



Complete Binary Tree

- Ideally, we would like to have a binary search tree to be **complete**, that is, every level is filled
    - The height of a complete binary tree is always $\boxed{\log_2(n+1)-1}$
- However, it is almost impossible to always maintain the search tree as a complete binary tree



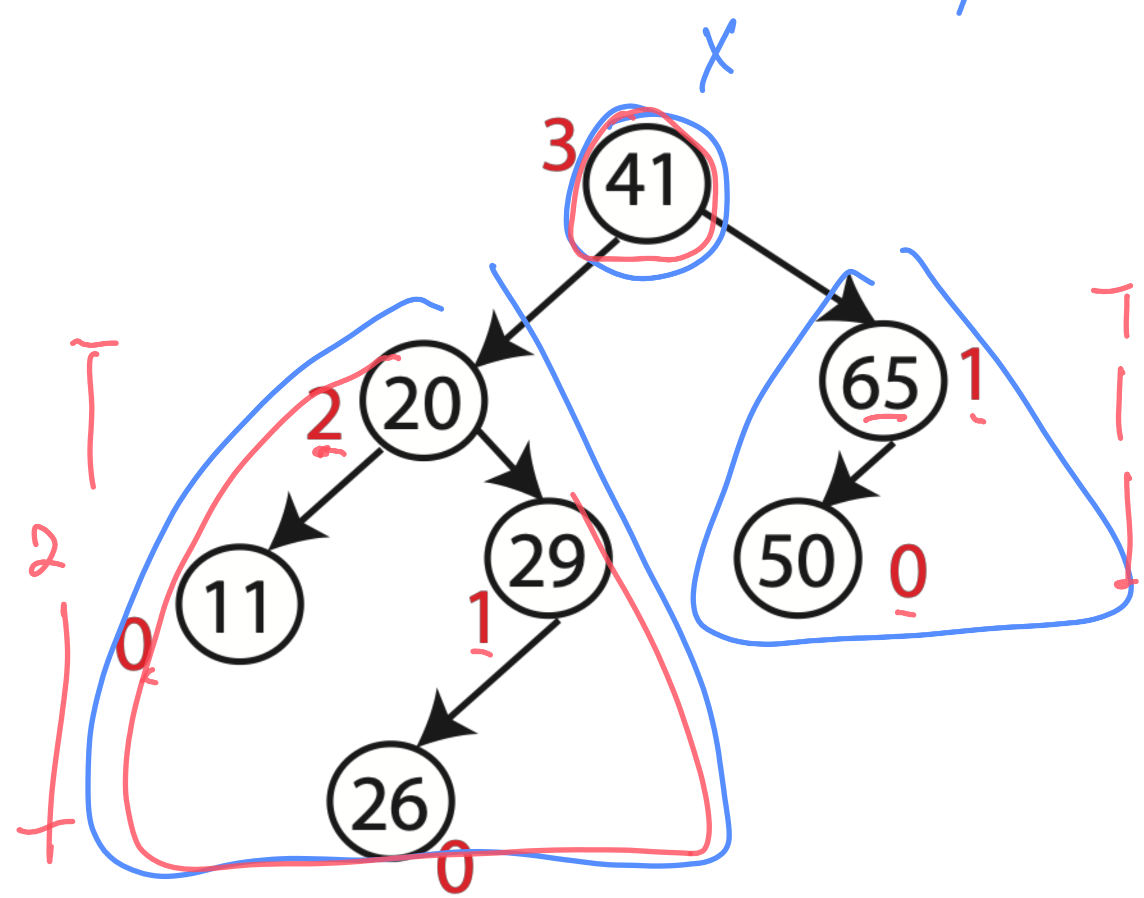
AVL Property

ผลต่างกันไม่เกิน 1

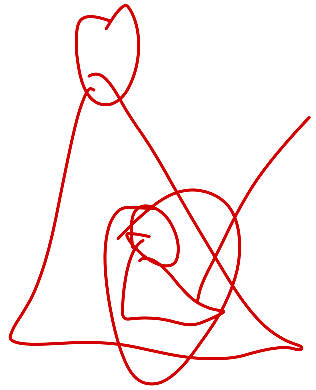
$|\text{height}(L) - \text{hieght}(R)| \le 1 \Rightarrow$ height of BST always $O(\log n)$

- **AVL property:** For ทุกๆ every internal node *v*, the height of the subtrees rooted at children of *v* differs by at most 1

- **Node's height (subtree's height):** The height of a node *u* in a tree is recursively defined by:
  - If *u* is external, then the height of *u* is zero
  - Otherwise, the height of *u* is one plus the maximum height of a child of *u*



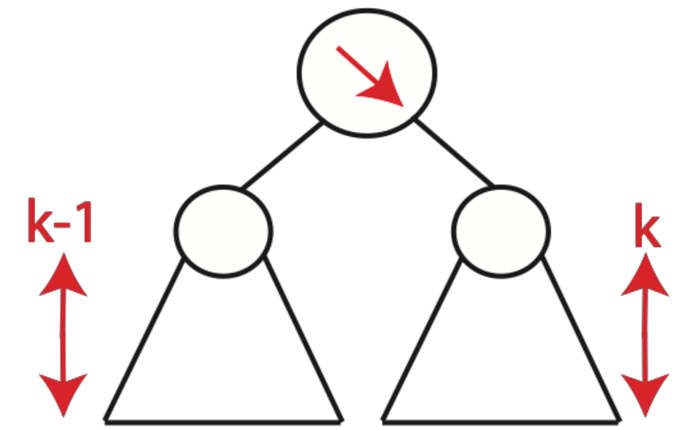
If u is NULL height of u is -1.



AVL Tree ตัดมาต้องเป็นด้วย

AVL Trees

- Any binary search tree that satisfies the AVL property is said to be an **AVL tree**
  - An immediate consequence of the AVL property is that a subtree of an AVL tree itself is an AVL tree
  - In other words, the difference between the heights of left and right subtrees cannot be more than one for all nodes
- AVL Trees are named after the initials of its inventors (Adelson, Velskii, and Landis 1962)



AVL Tree's Height Property

- **Proposition 2**: Let *T* be a binary tree that exhibits the AVL property. The height of the binary tree *T* is O(log *n*).

$\leq 2 \cdot \log_2 n$

Pf. Let n_h be the minimum number of Nodes in an AVL Tree of height h

$$n_h = \underbrace{1}_{\text{internal node}} + \underbrace{n_{h-2}}_{L} + \underbrace{n_{h-1}}_{R}$$

Boundary condition

$n_0 = 1$

$n_1 = 2$

$$n_h > n_{h-2} + n_{h-1} > n_{h-2} + n_{h-2} = 2n_{h-2}$$

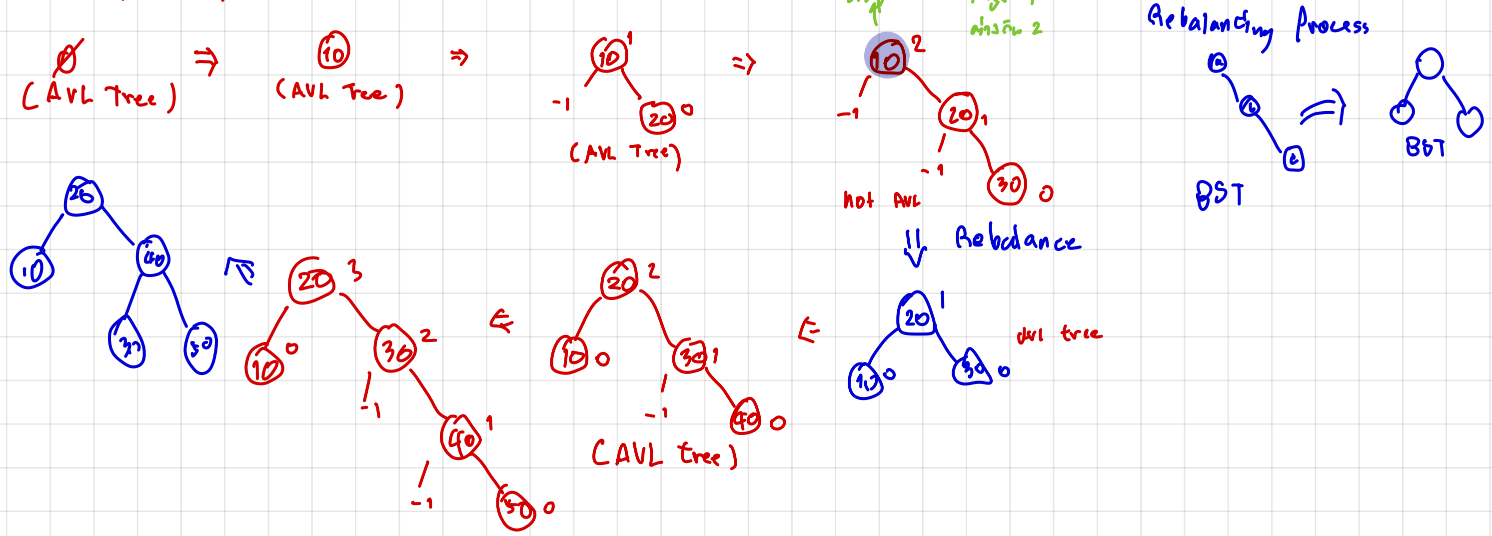
Complexity of Search Operations on AVL Trees

Operations	Complexity
Search	$O(\log n)$
Minimum	$O(\log n)$
Maximum	$O(\log n)$
Successor	$O(\log n)$
Predecessor	$O(\log n)$

Insertion on AVL Trees

- The insert operation on an AVL tree *T* is the same as that on a regular binary search tree, but we must perform additional steps to make sure that *T* still has the AVL property
  - **Step 1:** Perform insertion as in a regular binary search tree
  - **Step 2:** Restructure *T* to restore the AVL property by performing rebalancing process through rotations

Insert 10, 20, 30, 40, 50



Subroutine: Left-Rotate

left rotate x ลง y ขึ้นไปแทน

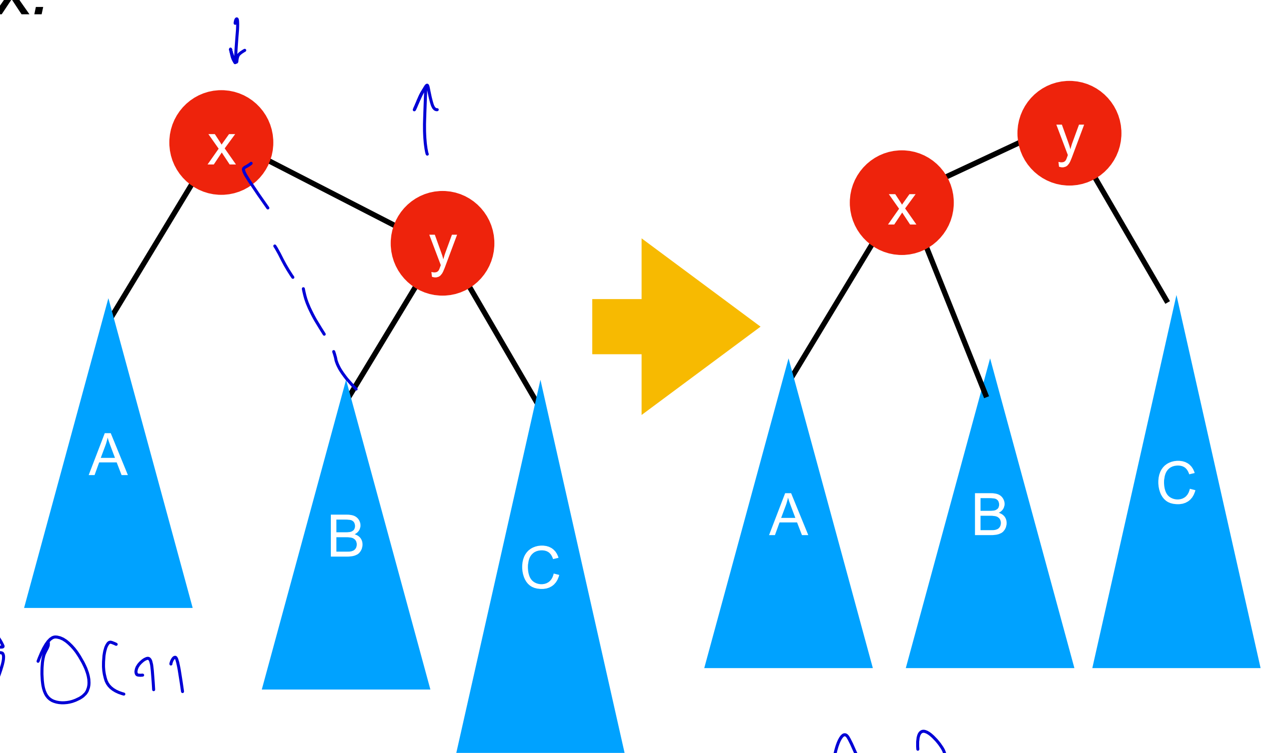
- Left-Rotate(x, *T*): Left rotate subtree rooted at *x* so that *x* becomes the left child of *y* (the current right child of *x*) and the left child of *y* becomes the new right child of x.

```
Left-Rotate(x, T):
        y = x.right
        B = y.left
        y.left = x
        x.parent = y
        x.right = B
        B.parent = x
        // Update heights of x & y
        x.height = 1 + max(height(x.left), height(x.right))
        y.height = 1 + max(height(y.left), height(y.right))
```

$\Theta(1)$

$\Theta(1)$

```
height(u):
  if (u == NULL):
    return -1
  return u.height
```



A x B y C

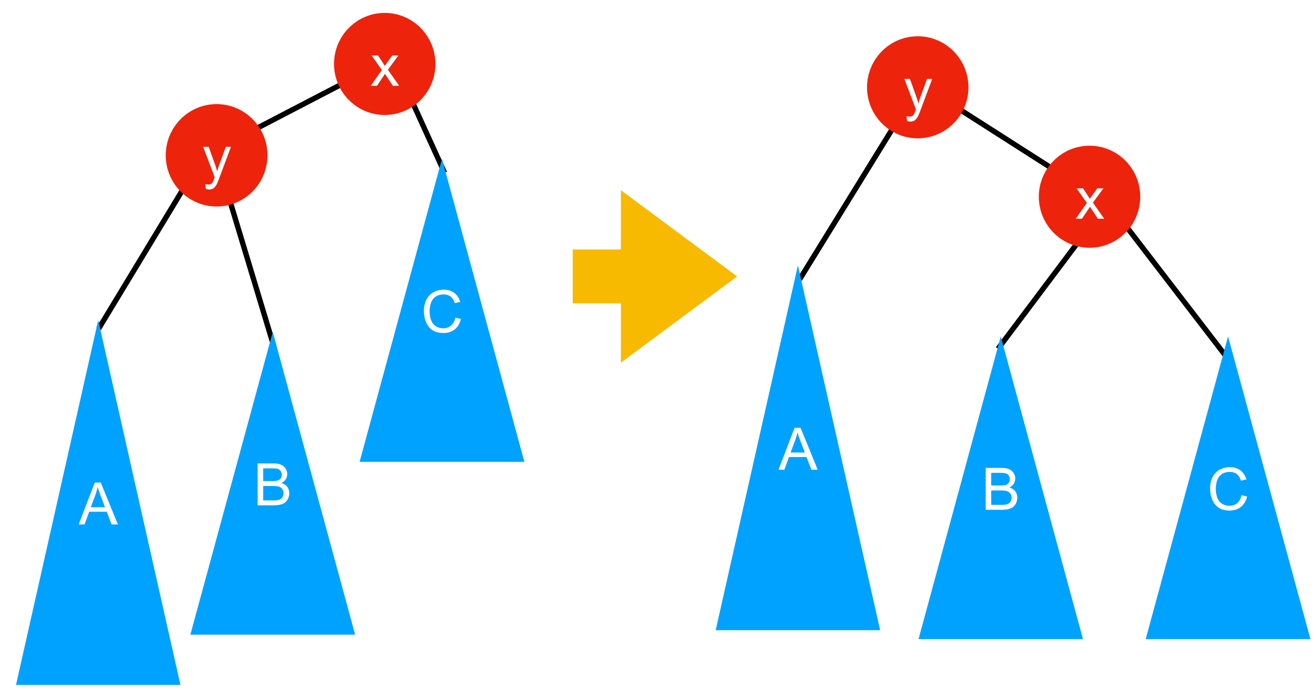
inorder

A x B y C

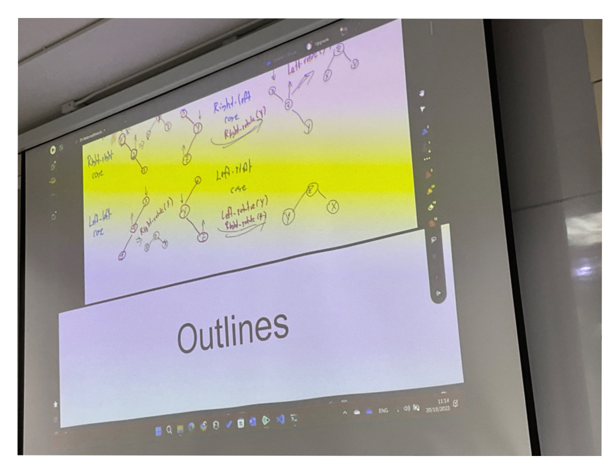
Subroutine: Right-Rotate

- Right-Rotate(x, *T*): Right rotate subtree rooted at *x* so that *x* becomes the right child of *y* (the current left child of *x*) and the right child of *y* becomes the new left child of x.

```
Right-Rotate(x, T):
        y = x.left
        B = y.right
        y.right = x
        x.parent = y
        x.left = B
        B.parent = x
        // Update heights of x & y
        x.height = 1 + max(height(x.left), height(x.right))
        y.height = 1 + max(height(y.left), height(y.right))
```



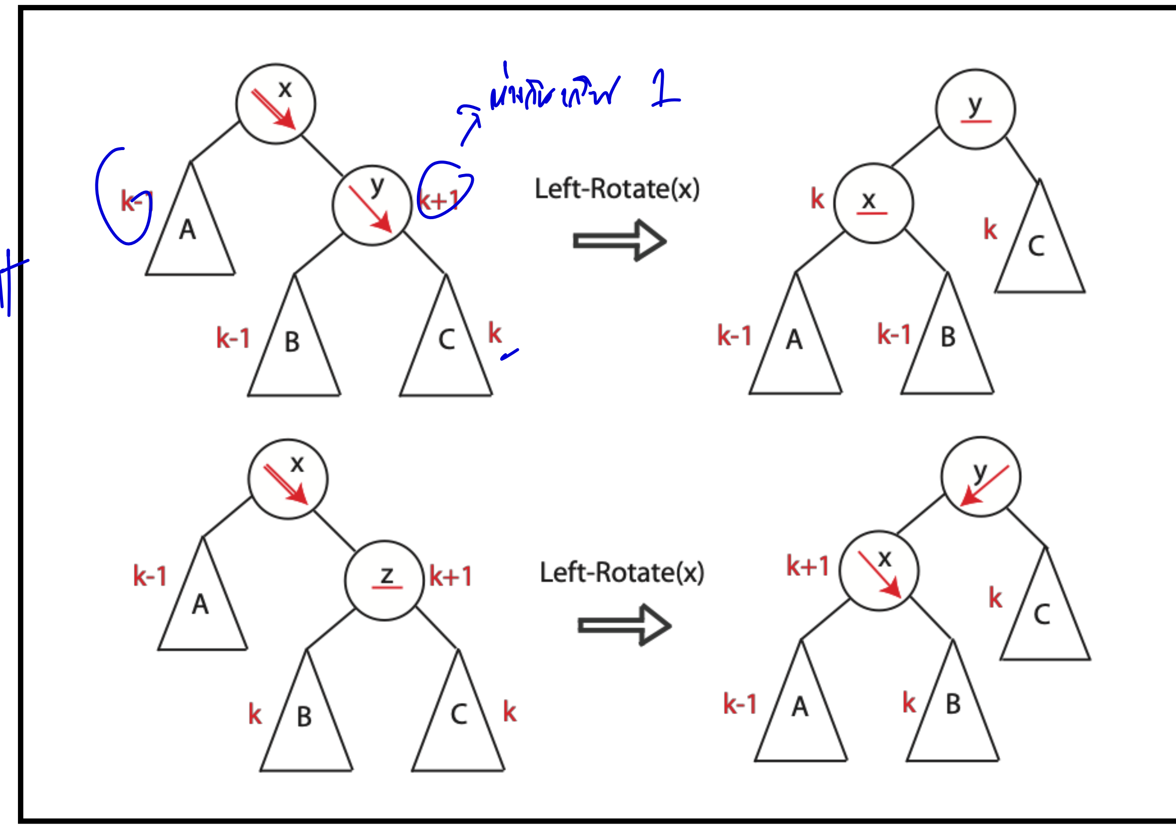

```
height(u):
  if (u == NULL):
    return -1
  return u.height
```

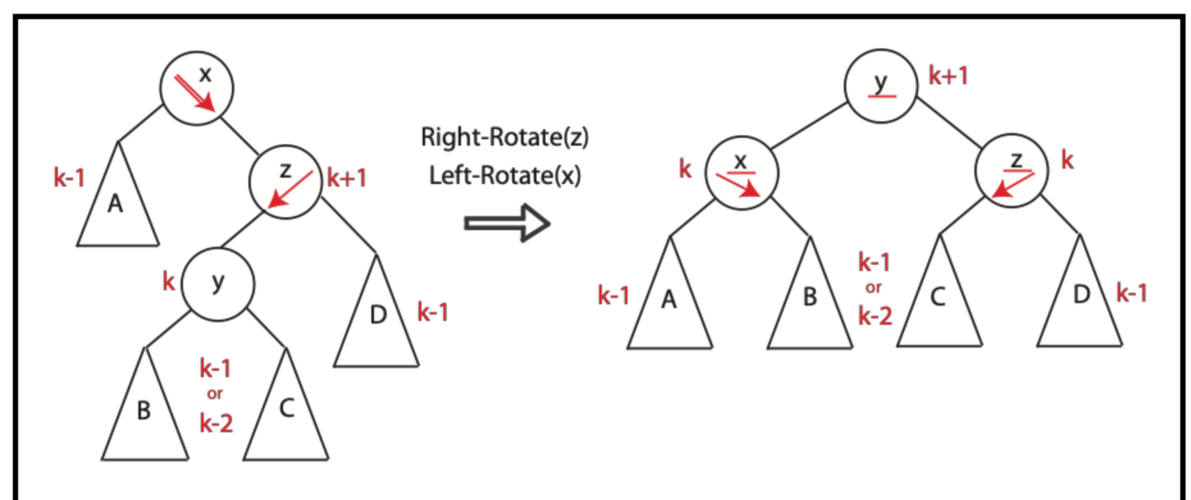


Rebalancing Process

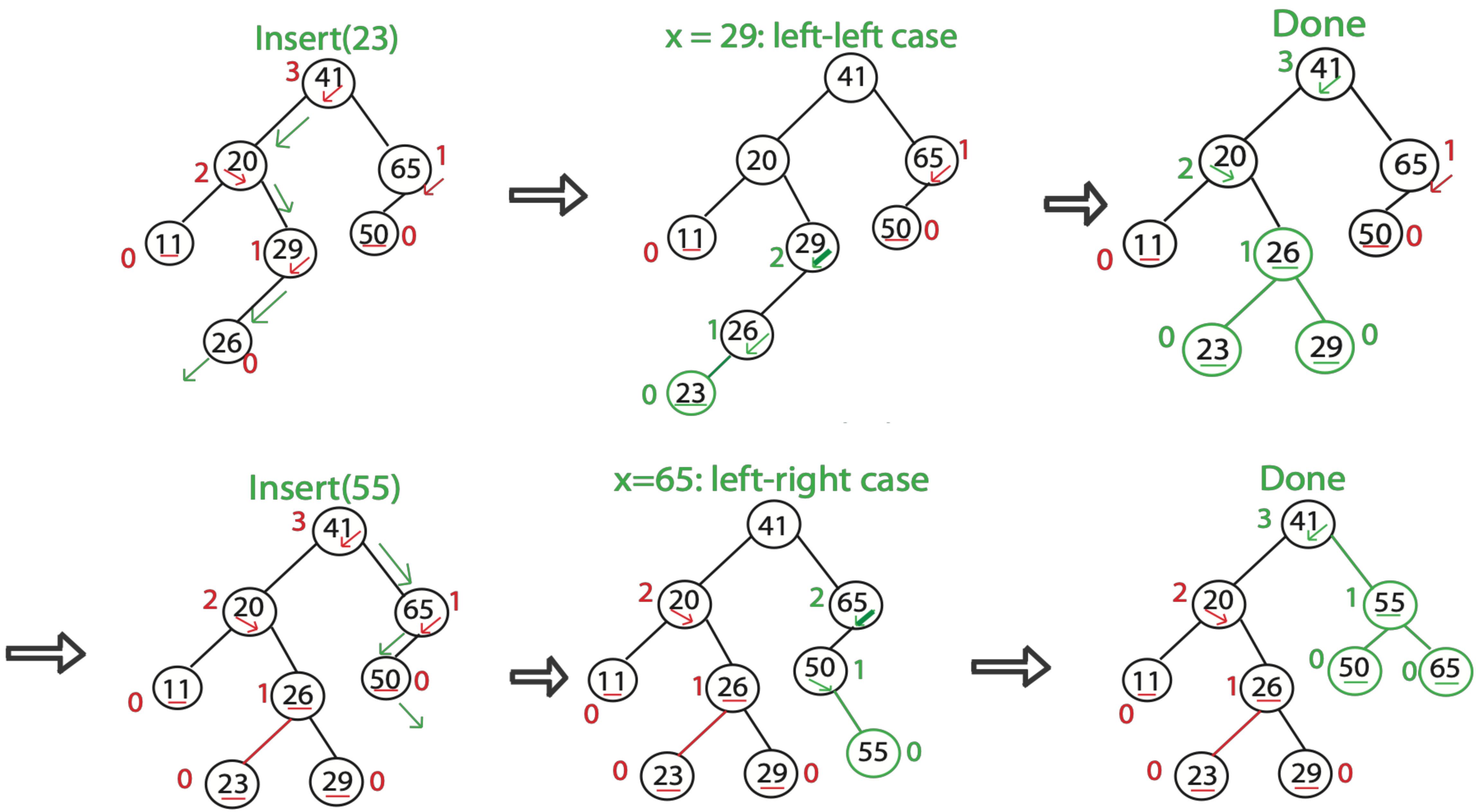
- Suppose *x* is lowest node violating the height-balance property.
- Assume *x* is right-heavy (the left case is symmetric).
  - **Right-right case:** If x's right child is right-heavy or balanced, perform left rotation.
  - **Right-left case:** Else, perform two rotations (right rotation, and left rotation).
- Then, continue up to *x*'s grandparent, great-grandparent, and so on.

right-right





Examples of Insertions on AVL Trees



Operation: AVL-Insert

- AVL-Insert(z, *u*, *T*): insert a new node *z* for which *z.key* = *v*, z.*parent* = NULL, *z.left* = NULL, and *z.right* = NULL into an appropriate position in the AVL subtree of *T*

```
AVL-Insert(z, u, T):
  // Step 1: Perform an insertion as in regular BST
  if (u == NULL):
    return z
  if (z.key < u.key):
    u.left = AVL-Insert(z, u.left, T)
  else:
    u.right = AVL-Insert(z, u.right, T)
  // Step 2: Update height of node u
  u.height = 1 + max(height(u.left), height(u.right))      } O(1)
  b = checkBalanced(u)
  // Step 3: If u becomes unbalanced, then there are 4 cases
  // Right-right case
  if (b < -1 and z.key > u.right.key):
     Left-Rotate(u, T)
  …                                                         } O(1)
  // Right-left case
  if (b < -1 and z.key > u.left.key):
     Right-Rotate(u.right, T)
     Left-Rotate(u, T)
  return u
```

```
height(u):
  if (u == NULL):
    return -1
  return u.height
```

```
node:
  node left
  node right
  node parent
  int key
  int height
```

```
checkBalanced(u):
  if (u == NULL):
    return -1
  return height(u.left)-height(u.right)
```

Complexity of Operations on AVL Trees

Operations	Complexity
Search	$O(\log n)$
Minimum	$O(\log n)$
Maximum	$O(\log n)$
Successor	$O(\log n)$
Predecessor	$O(\log n)$
AVL-insert	$O(\log n)$

W.C. Complexity of Tree Sort (based on AVL Tree)

- **Proposition 3**: The worst case time complexity of Tree Sort based on using AVL Tree is $O(n \log n)$

construct BST + inorder

$\underbrace{O(n\log n)} + O(n) = O(n \log n)$

Implementation of AVL-Insert in C (1)

```c
// C program to insert a node in AVL tree
#include<stdio.h>
#include<stdlib.h>

// An AVL tree node
struct node
{
   int key;
   struct node *left;
   struct node *right;
   struct node *parent;
   int height;
};

// A utility function to get maximum of two integers
int max(int a, int b)
{
   return (a > b)? a : b;
}

// A utility function to get the height of the tree
int height(struct node* node)
{
   if (node == NULL)
      return -1;
   return node->height;
}
```

Implementation of AVL-Insert in C (2)

```
// Helper function that allocates a new node with the given key
struct node* createNode(int key)
{
   struct node* node = (struct node*)
   malloc(sizeof(struct node));
   node->key = key;
   node->left = NULL;
   node->right = NULL;
   node->parent = NULL;
   node->height = 0; // new node is initially added at leaf
   return(node);
}

// A utility function to left rotate subtree rooted with x
struct node* leftRotate(struct node* x)
{
   struct node* y = x->right;
   struct node* B = y->left;
   // Perform rotation
   y->left = x;
   x->parent = y;
   x->right = B;
   if(B != NULL)
     B->parent = x;
   // Update heights
   x->height = max(height(x->left), height(x->right))+1;
   y->height = max(height(y->left), height(y->right))+1;
   // Return new root
   return y;
}
```

Implementation of AVL-Insert in C (3)

```
// A utility function to right rotate subtree rooted at x
struct node* rightRotate(struct node* x)
{
   struct node* y = x->left;
   struct node* B = y->right;
   // Perform rotation
   y->right = x;
   x->parent = y;
   x->left = B;
   if(B != NULL)
      B->parent = x;
   // Update heights
   x->height = max(height(x->left), height(x->right))+1;
   y->height = max(height(y->left), height(y->right))+1;
   // Return new root
   return y;
}

// Get Balance factor of a node
int getBalance(struct node* node)
{
   if (node == NULL)
      return -1;
   return height(node->left)-height(node->right);
}
```

Implementation of AVL-Insert in C (4)

```
// Recursive function to insert a key in the subtree rooted with node and returns the new root of the subtree.
struct node* insert(struct node* node, int key)
{
   /* 1. Perform the normal BST insertion */
   if (node == NULL)
      return(createNode(key));
   if (key < node->key) {
      node->left = insert(node->left, key);
      node->left->parent = node;
   }
   else if (key > node->key) {
      node->right = insert(node->right, key);
      node->right->parent = node;
   }
   else // Equal keys are not allowed in BST
      return node;
   /* 2. Update height of this ancestor node */
   node->height = 1 + max(height(node->left), height(node->right));
   /* 3. Get the balance factor of this ancestor node to check whether this node became unbalanced */
   int balance = getBalance(node);
   // If this node becomes unbalanced, then there are 4 cases
   // Left Left Case
   if (balance > 1 && key < node->left->key)
      return rightRotate(node);
   // Right Right Case
   if (balance < -1 && key > node->right->key)
      return leftRotate(node);
   // Left Right Case
   if (balance > 1 && key > node->left->key)
   {
      node->left = leftRotate(node->left);
      return rightRotate(node);
   }
   // Right Left Case
   if (balance < -1 && key < node->right->key)
   {
      node->right = rightRotate(node->right);
      return leftRotate(node);
   }
   /* return the (unchanged) node pointer */
   return node;
}
```

Implementation of AVL-Insert in C (5)

```c
// A utility function to print preorder traversal of the tree. The function also prints height of every node
void inorder(struct node* node)
{
   if(node != NULL)
   {
      inorder(node->left);
      printf("key: %d, height: %d\n", node->key, node->height);
      inorder(node->right);
   }
}

/* Drier program to test above function*/
int main()
{
   struct node *root = NULL;

   root = insert(root, 10);
   root = insert(root, 20);
   root = insert(root, 30);
   root = insert(root, 40);
   root = insert(root, 50);
   root = insert(root, 25);

   printf("Preorder traversal of the constructed AVL"
         " tree is \n");
   inorder(root);

   return 0;
}
```

```
nj@Nopadons-MacBook-Pro codes % ./AVL_tree
Inorder traversal of the constructed AVL tree is
key: 10, height: 0
key: 20, height: 1
key: 25, height: 0
key: 30, height: 2
key: 40, height: 1
key: 50, height: 0
nj@Nopadons-MacBook-Pro codes %
```