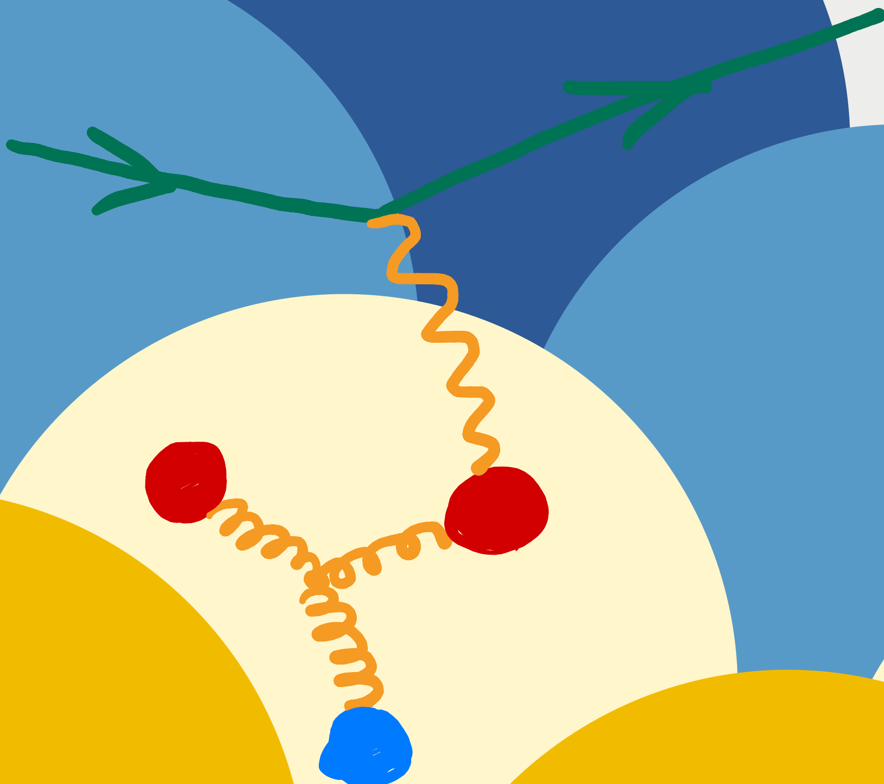
CFNS24

Lectures at the 2024 CFNS Summer School on the physics of the EIC



by: Alexei Prokudin

prokudin@jlab.org

1. What is the 3D structure?

„What?"

2. Why the 3D structure is interesting?

„Why?"

3. Where can we study the 3D structure?

„Where?"

We have a group of theorists and experimentalists and I will try to give something important and interesting to all.

I will give you arguments and some will be more handwaiving, others more rigorous.
Do not be too convinced by them
it is physics and the mathematical rigor is not always a guaratee of correctness.
Experimental tests are the key!

Questions for the discussion:

1. What is the advantage of the lepton scattering?

2. What are the advantages and disadvantages of a collider with respect to a fixed target experiment?

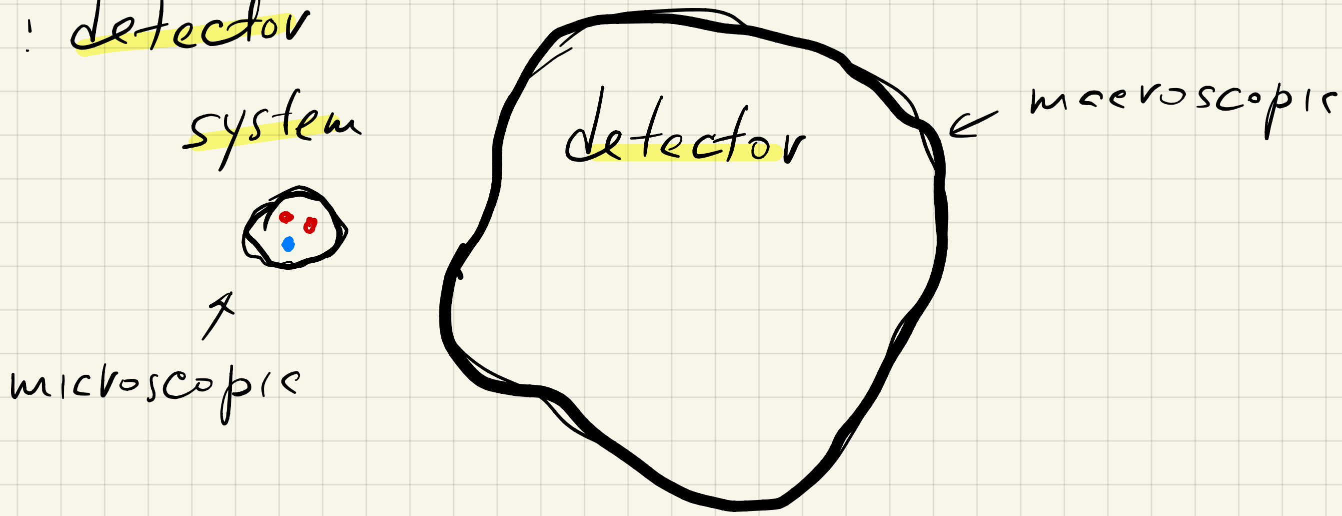
3. What is the role of „inclusivity" in the 3D structure measurements? Consider pdfs or TMDs.

4. Consider parton model. Could you motivate the physical picture that the scattering happens on almost on-shell particles?

5. Consider parton model. Motivate the usage of coherent $\sum_a e_a^2$ vs incoherent scattering $\sum_{a,a'} e_a e_{a'}$.

Let us start with „What?"

We choose language of Quantum Mechanics (QM) and generically divide the „world" in two categories: system we study, and the observer : detector



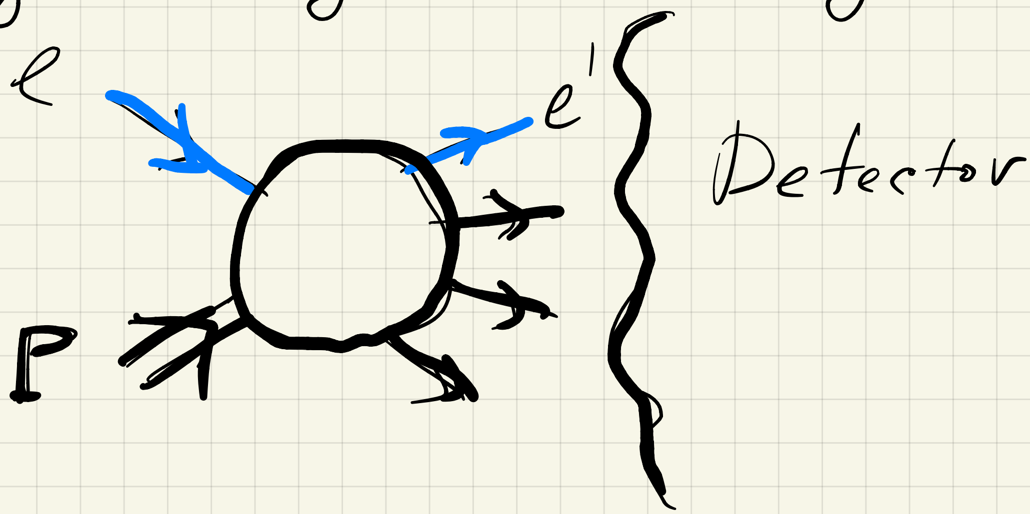
The system is then described by a wave function Ψ it evolves according to Shrödinger equation smooth and continuous. Once the measurement is performed, we switch to Born rules and probabilities $|\Psi|^2$

Compared to classical systems we can ask approximetely a half questions about the system.

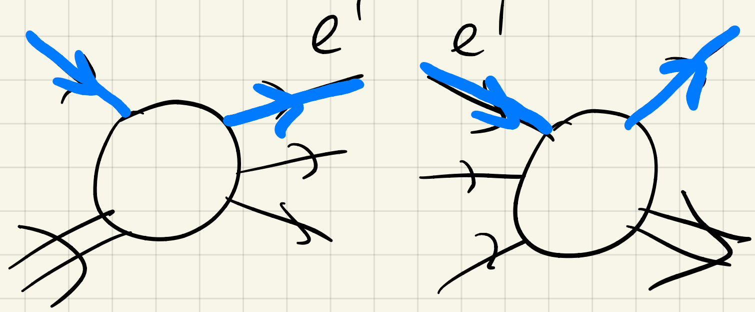
For the proton QM is not enough; we use the language of Quantum Field Theory and use field operators ψ acting on the state vectors $|P\rangle$. (Dirac, Feynman, etc)

The number of constituents is not constant, it is changing.

The measurement is performed by a detector, say we study eP scattering



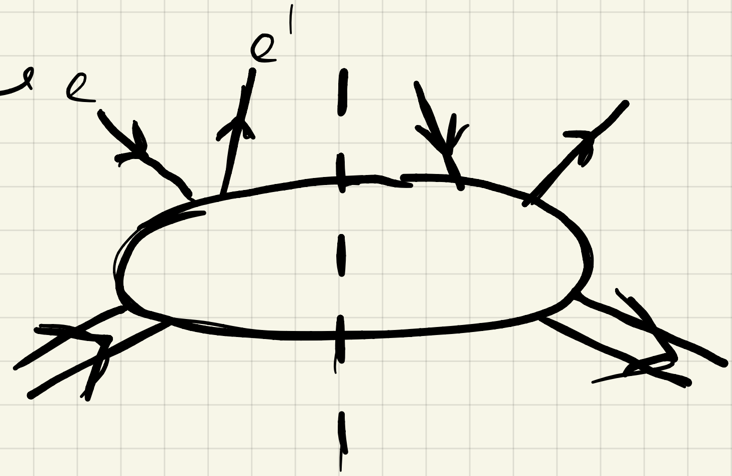
The same process in the mind of the theorist, as you see in George Sterman's lecture, looks differently $\rightarrow$ we square it „$|\psi|^2$"



When we perform manipulations, insert a „unity" operator

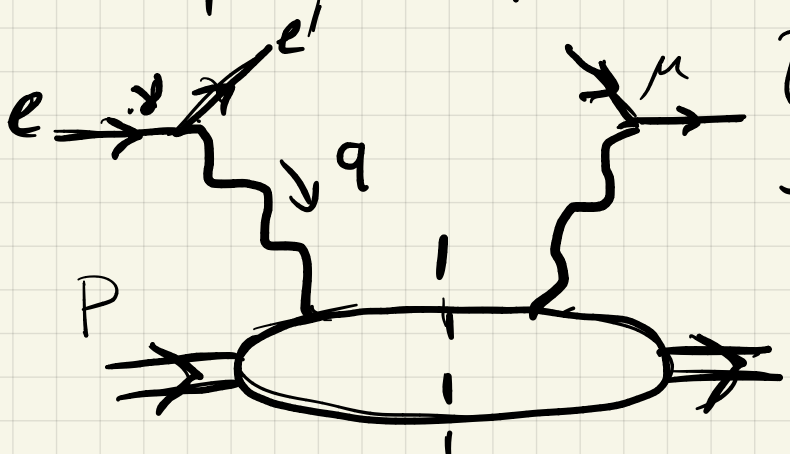
$$\int d\mathrm{PS}\, |X\rangle\langle X| = \mathbb{1}$$

apply the optical theorem and derive an amplitude



, $= \mathrm{Im}$

One separates „leptonic" and „hadronic" part

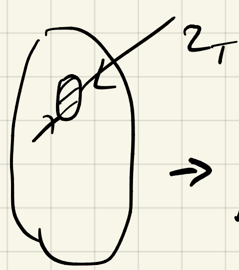


$\propto L^{\mu\nu} W_{\mu\nu}$

Then one performs an approximation, separation of time scales, known as factorisation

$q^2 = (e - e')^2$ plays and imortant role

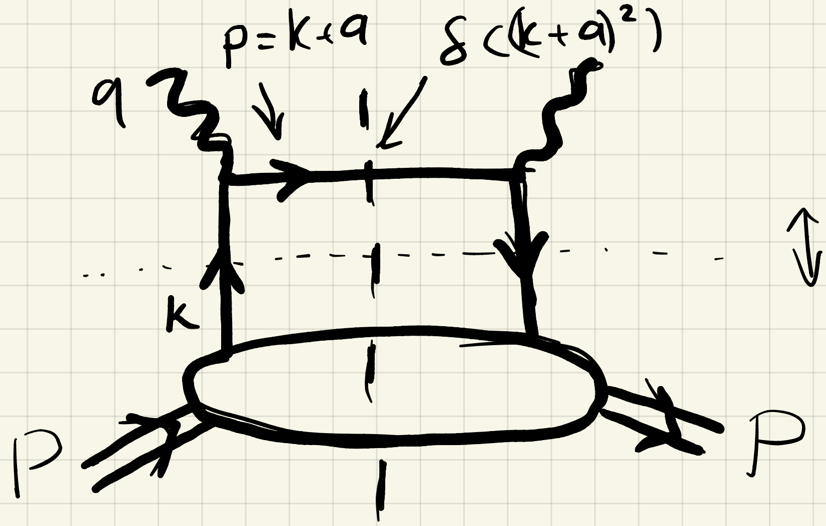
$Q^2 = -q^2$, $Q \sim \frac{1}{r_T}$



resolution

$\rightarrow$ parton nature

Such a separation is possible in the Infitite Momentum Frames, an example is the Breit frame



$\propto \sigma_0$,

$\Updownarrow$ factorisation

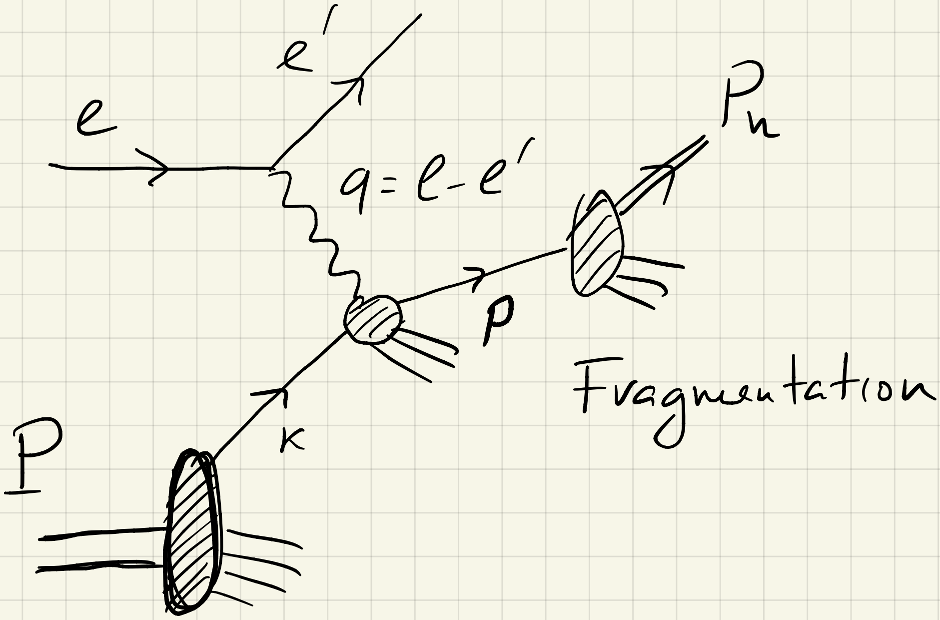
$\propto f_1(x)\, \delta\Big(x - \underbrace{\frac{Q^2}{2P\cdot q}}_{x_{Bj}}\Big)$

This example here is only the parton model, QCD radiation must be dealt with and the consequence will be a scale dependence of $f_1(x,\mu)$, the variable $x = k^+/P^+$, $k^+ = \frac{k^0 + k^3}{\sqrt{2}}$

1 variable => 1 dimensional structure 1D

How can 3D structure be studied?

Consider Semi Inclusive Deep Inelastic Scattering (SIDIS)



Distribution

$$\underbrace{Q^2 = -q^2 \simeq s x y}$$

an important constraint for the experiment

(Homework: derive it!)

$$s = (l + P)^2 \simeq 2 P \cdot l$$

$$x = \frac{Q^2}{2 P \cdot q}$$

Bjorken variable (parton momentum fraction)

$$y = \frac{P \cdot q}{P \cdot l}$$

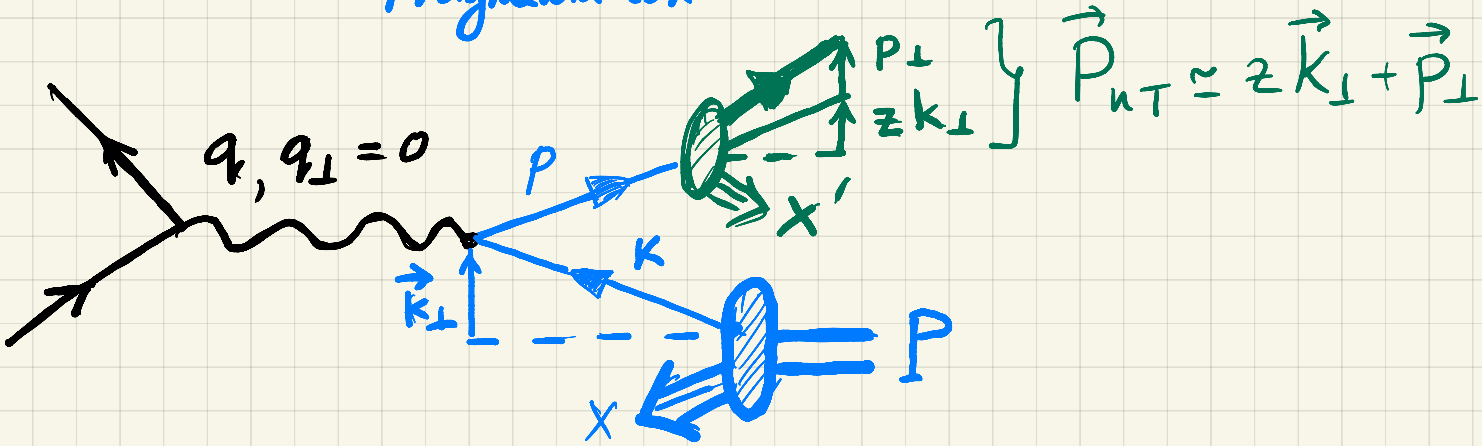
Inelasticity (energy transfer)

$$z = \frac{P \cdot P_h}{P \cdot q}$$

fraction of the parton momentum carried by the observed hadron

$P_h \simeq z p$

Fragmentation



Recoil offalow energy is interesting as it is sensitive to the intrinsic transverse motion, $\vec{k}_\perp$, $\vec{p}_\perp$

QCD radiation is important, see the next lecture for a refined treatment.

Now our distributions will depend on x and k_T

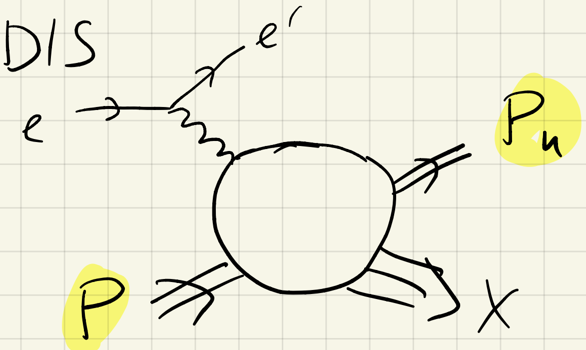
$f_1(x) \to f_1(x, \vec{k}_\perp)$ – Transverse Momentum Dependent distributions (TMDs)

$\vec{k}_T$ is a 2 vector and thus 3D structure

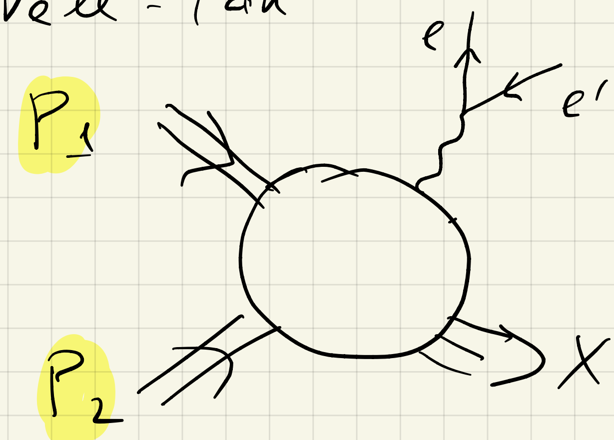
$$\begin{array}{ccc} x & , & \vec{k}_T \\ \downarrow & & \downarrow \\ 1 & & 2 \end{array}$$

Factorisation is proven for

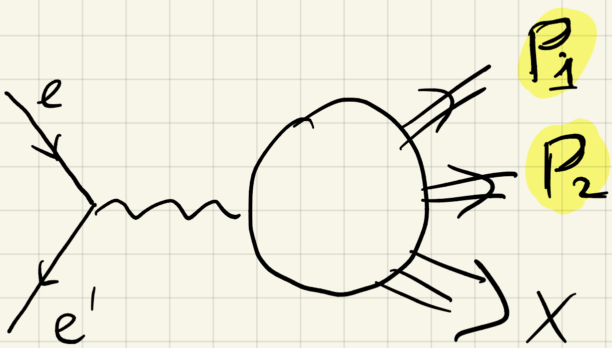
SIDIS



Drell-Yan



e^+e^- into hadron pairs



Common features

* 2 hadrons
* recoil with a low momentum, q_T
* sufficiently inclusive
* 1 electromagnetic system with large virtuality Q
* $q_T \ll Q$

Why?

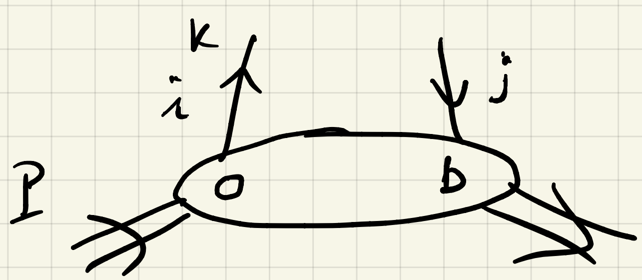
Experiment → more things to measure, (un)polarised cross sections, asymmetries

Theory → more vectors to construct distributions, allows to study correlations of, for instance, the intrinsic momentum and the spin. These should have the imprint of the confinement mechanism.

$$\phi_{ij} = \langle P, S \,|\, \bar{\psi}_j(b)\, \psi_i(0) \,|\, P, S \rangle$$

a matrix to be projected on unpolarised quarks γ^+, longitudinally polarised quarks $\gamma^+\gamma_5$, transversely polarised quarks $i\sigma^{\alpha +}\gamma_5$ Spin projectors

N.B. $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$



We can define the following projections of the correlator $\Phi(p, P)$ in the case when transverse motion is not ignored.

Parity, time-reversal, and charge conjugation lead to:

$$\Phi(x, k_T)_{ij} = \int \frac{db^- d^2 b_T}{(2\pi)^3} e^{-ixP^+ b^- + i k_T \cdot b_T}$$

$$\langle P | \bar{\psi}_j(b) \psi_i(0) | P \rangle \Big|_{b^+ = 0}$$

$$\Phi^{[\Gamma]} \stackrel{\text{def}}{\equiv} \frac{1}{2} \mathrm{Tr}[\Gamma \phi]$$

$$\frac{1}{2} \mathrm{Tr}(\gamma^+ \phi) = f_1 - \frac{\epsilon^{jk} k_T^j S_T^k}{M} f_{1T}^\perp$$

$$\frac{1}{2} \mathrm{Tr}(\gamma^+ \gamma_5 \phi) = S_L g_1 + \frac{\vec{k}_T \cdot \vec{S}_T}{M} g_{1T}^\perp$$

$$\frac{1}{2} \mathrm{Tr}(i \sigma^{j+} \gamma_5 \phi) = S_T^j h_1 + S_L \frac{k_T^j}{M} h_{1L}^\perp + \frac{\kappa^{jk} S_T^k}{M} h_{1T}^\perp$$

$$+ \frac{\epsilon^{jk} k_T^k}{M} h_1^\perp, \quad \text{where } \kappa^{jk} \equiv \left(k_T^j k_T^i - \frac{1}{2} k_T^2 \delta^{jk}\right)$$

$$\epsilon^{ij} \equiv \epsilon^{-+ij}, \quad \epsilon^{0123} = +1$$

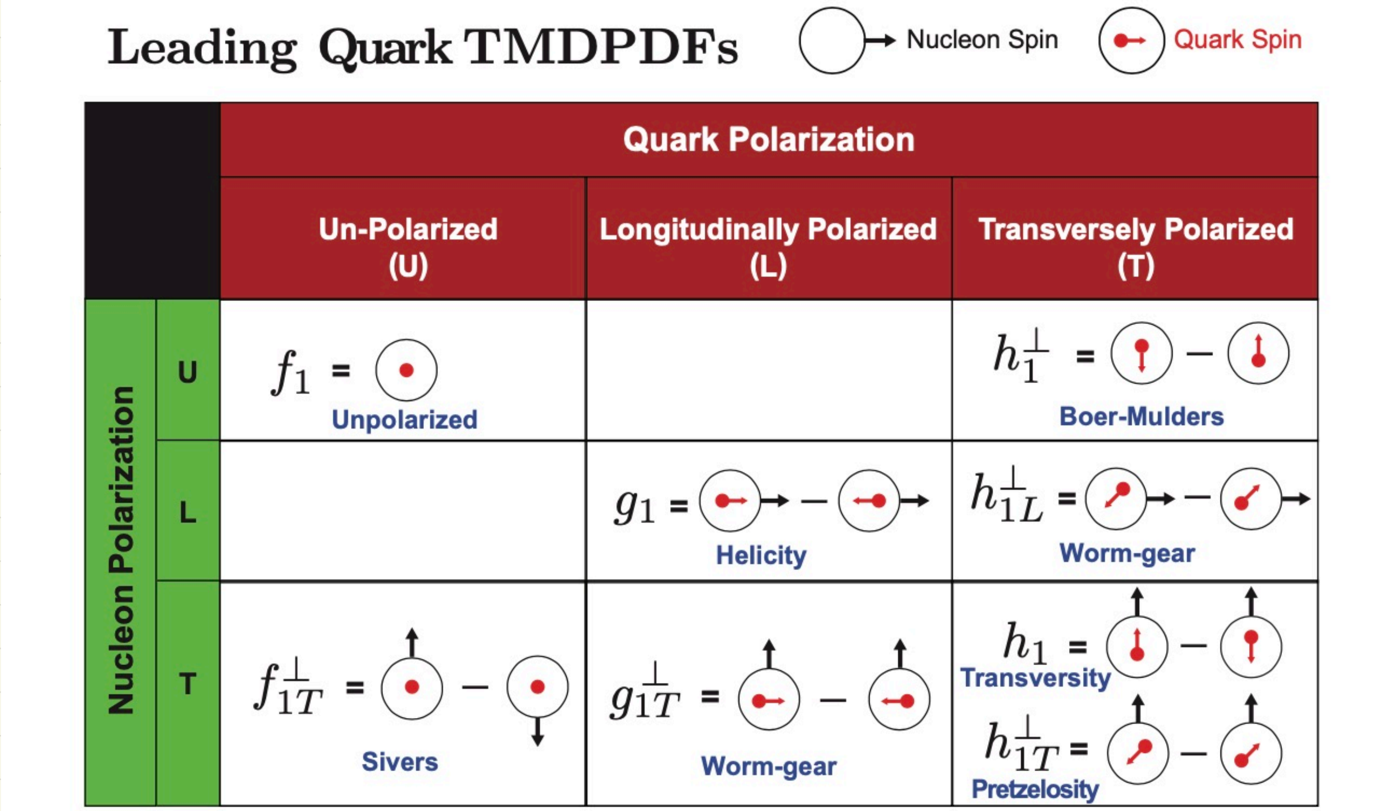


Figure 2.5: Leading power quark parton distribution functions for the proton or a spin-1/2 hadron.

$$
\begin{aligned}
f_{i/p_S}^{[\gamma^+]}(x,\mathbf{k}_T,\mu,\zeta) &= f_1(x,k_T) - \frac{\epsilon_T^{\rho\sigma}k_{T\rho}S_{T\sigma}}{M}\kappa\, f_{1T}^{\perp}(x,k_T)\,, \\
f_{i/p_S}^{[\gamma^+\gamma_5]}(x,\mathbf{k}_T,\mu,\zeta) &= S_L\, g_1(x,k_T) - \frac{k_T\cdot S_T}{M} g_{1T}^{\perp}(x,k_T)\,, \\
f_{i/p_S}^{[i\sigma^{\alpha+}\gamma_5]}(x,\mathbf{k}_T,\mu,\zeta) &= S_T^\alpha h_1(x,k_T) + \frac{S_L k_T^\alpha}{M} h_{1L}^{\perp}(x,k_T) \\
&\quad - \frac{\mathbf{k}_T^2}{M^2}\left(\frac{1}{2}g_T^{\alpha\rho} + \frac{k_T^\alpha k_T^\rho}{\mathbf{k}_T^2}\right) S_{T\rho} h_{1T}^{\perp}(x,k_T) - \frac{\epsilon_T^{\alpha\rho}k_{T\rho}}{M}\kappa\, h_1^{\perp}(x,k_T)\,.
\end{aligned}
\tag{2.123}
$$

$$
\kappa = \begin{cases} +1 & (\mathrm{DY}) \\ -1 & (\mathrm{SIDIS}) \end{cases}
$$

$$\tilde{f}_{i/p_S}^{[\gamma^+]}(x, \mathbf{b}_T, \mu, \zeta) = \tilde{f}_1(x, b_T) + i\epsilon_{\rho\sigma} b_T^\rho S_T^\sigma M \tilde{f}_{1T}^\perp(x, b_T)\,,$$
$$\tilde{f}_{i/p_S}^{[\gamma^+\gamma_5]}(x, \mathbf{b}_T, \mu, \zeta) = S_L \tilde{g}_1(x, b_T) + i b_T \cdot S_T M \tilde{g}_{1T}^\perp(x, b_T)\,,$$
$$\tilde{f}_{i/p_S}^{[i\sigma^{\alpha+}\gamma_5]}(x, \mathbf{b}_T, \mu, \zeta) = S_T^\alpha \tilde{h}_1(x, b_T) - i S_L b_T^\alpha M \tilde{h}_{1L}^\perp(x, b_T) + i\epsilon^{\alpha\rho} b_{\perp\rho} M \tilde{h}_1^\perp(x, b_T)$$
$$+ \frac{1}{2}\mathbf{b}_T^2 M^2 \left(\frac{1}{2} g_T^{\alpha\rho} + \frac{b_T^\alpha b_T^\rho}{\mathbf{b}_T^2}\right) S_{\perp\rho} \tilde{h}_{1T}^\perp(x, b_T)\,. \qquad (2.126)$$

All measurements are performed in the momentum space, however it is more convenient to study functions in the configuration space.

How do we perform F.T. ?

$$\tilde{f}(b_T) = \int d^2k_T\, e^{-i\vec{b}_T\cdot\vec{k}_T} f(k_T)$$

$$= \int_0^\infty k_T\, dk_T \int_0^{2\pi} d\varphi\, e^{-i b_T k_T \cos\varphi} f(k_T)$$

$$= 2\pi \int_0^\infty k_T\, dk_T\, J_0(k_T b_T)\, f(k_T)$$

$$f(k_T) = \int \frac{d^2 b_T}{(2\pi)^2} e^{+i\bar{b}_T\cdot\bar{k}_T} \tilde{f}(b_T)$$

$$= \frac{1}{2\pi} \int_0^\infty b_T\, db_T\, J_0(b_T k_T)\, \tilde{f}(b_T)$$

We will also need b_T derivatives:

$$\tilde{f}^{(n)}(b_T) \equiv n!\left(-\frac{1}{M^2 b_T}\frac{\partial}{\partial b_T}\right)^n \tilde{f}(b_T)$$

$$= \frac{2\pi\, n!}{(M^2)^n}\int k_T\, dk_T \left(\frac{k_T}{b_T}\right)^n J_n(b_T k_T)\, f(k_T)$$

Why so?. Using $\lim_{z\to 0} J_m(z) = \underbrace{\frac{1}{\Gamma(m+1)}}_{m! \text{ for integer } m}\left(\frac{z}{2}\right)^m$

$$\lim_{b_T\to 0} \tilde{f}^{(n)}(b_T) = \underbrace{2\pi\int k_T\, dk_T \left(\frac{k_T^2}{2M^2}\right)^n f(k_T)}_{\text{These are called moments of TMDs}}$$

We need F.T. of the form $\overbrace{k_T^i \ldots k_T^j}^{i,j \in [1,2]} f(k_T)$

$$\int d^2k_T\, k_T^i \ldots k_T^j f(k_T)\, e^{-i\bar{k}_T \bar{b}_T} =$$

$$= \left(+i\frac{b_T^i}{b_T}\frac{\partial}{\partial b_T}\right)\ldots\left(+i\frac{b_T^j}{b_T}\frac{\partial}{\partial b_T}\right)$$

$$2\pi\int_0^\infty k_T\, dk_T\, J_0(k_T b_T)\, f(k_T)$$

An example:

$$\int d^2k_T\, k_T^i\, e^{-i\bar{b}_T \bar{k}_T} f(k_T) =$$

$$= i\,\frac{b_T^i}{b_T}\frac{\partial}{\partial b_T}\, 2\pi \int k_T\, dk_T\, J_0(k_T b_T)\, f(k_T) =$$

$$= i\,\frac{b_T^i}{b_T}\, 2\pi \int k_T\, dk_T \left(-J_1(k_T b_T)\right) k_T\, f(k_T)$$

$$= (-i)\, b_T^i\, 2\pi \int k_T\, dk_T \left(\frac{k_T}{b_T}\right) J_1(k_T b_T)\, f(k_T)$$

$$= (-i)\, b_T^i\, M^2\, \tilde{f}^{(1)}(b_T)$$

and thus

$$\Phi^{[\gamma^+]} = f_1(x,k_T) - \frac{\epsilon^{ij} k_T^i S_T^j}{M} f_{1T}^\perp(x,k_T)$$

$$\downarrow$$

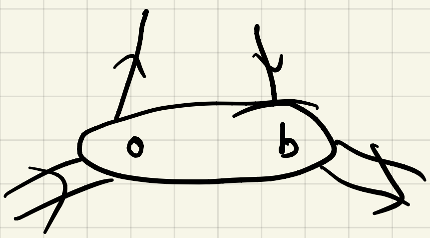
$$\tilde{\Phi}^{[\gamma^+]} = \tilde{f}_1^{(0)}(x,b_T) + i\,\epsilon^{ij}\, b_T^i\, S_T^j\, M\, \tilde{f}_{1T}^{\perp(1)}(x,b_T)$$

Homework → perform the other cases

Physics of the phase $\bar{k}_T \cdot \bar{b}_T$

$$f(x, k_T, S_T) = f_1(x, k_T) - \underbrace{\frac{\epsilon^{ij} k_T^i S_T^j}{M}}_{\text{correlation of } \vec{k}_T \text{ and } \vec{S}_T} f_{1T}^{\perp}(x, k_T)$$

OAM $\propto \vec{r} \times \vec{p}$, $\vec{r}$ is missing



Assume $\tilde{\bar{\psi}}(\bar{x}_\perp, \bar{\ell}_\perp)$, $\tilde{\psi}(\bar{y}_\perp, \bar{m}_\perp)$

$$\psi(y_\perp) = \int d^2 m_\perp \, e^{-i y_\perp m_\perp} \tilde{\psi}(\bar{y}_\perp, \bar{m}_\perp)$$

then

$$\bar{\psi}(x_\perp)\psi(y_\perp) = \int d^2 \ell_\perp \, d^2 m_\perp \, e^{+i \bar{x}_\perp \bar{\ell}_\perp - i \bar{y}_\perp \bar{m}_\perp} \tilde{\bar{\psi}}(\bar{x}_\perp, \bar{\ell}_\perp) \tilde{\psi}(\bar{y}_\perp, \bar{m}_\perp)$$

$$\bar{x}_\perp \bar{\ell}_\perp - \bar{y}_\perp \bar{m}_\perp = \frac{1}{2}(\bar{x}_\perp + \bar{y}_\perp)(\bar{\ell}_\perp - \bar{m}_\perp) + \frac{1}{2}(\bar{x}_\perp - \bar{y}_\perp)(\bar{\ell}_\perp + \bar{m}_\perp)$$

For TMDs we have a forward amplitude $\bar{\ell}_\perp = \bar{m}_\perp \equiv \vec{k}_T$

$$\Rightarrow \quad \frac{1}{2}(\bar{x}_\perp - \bar{y}_\perp)(\bar{\ell}_\perp + \bar{m}_\perp) \overset{\text{def}}{=} \bar{b}_T \cdot \bar{k}_T$$

$\bar{b}_T$ is the difference of quark field positions

$\bar{k}_T$ is the transverse momentum

If the amplitude is off forward $\frac{1}{2}(x_\perp + y_\perp) \equiv \vec{b}$ impact parameter w.r.t. the center of mass of the hadron

$\bar{\ell}_\perp - \bar{m}_\perp = \bar{\Delta}$ momentum transfer $\Rightarrow$ „GPDs"

Thus $\vec{k}_T$ is not Fourier conjugate to $\vec{b}$ and therefore there are no model indep. relations of Generalized Parton Distributions and TMDs

This is the reason I use „configuration" space $\vec{b}_T$, I do not call it „impact" parameter.

The impact parameter is Fourier conjugate to the momentum transfer.

Physics

Lecture 2

$$f_{a/p^\uparrow}(x, k_T, S_T) = \underbrace{f_{a/p}(x, k_T)}_{\text{unpolarised}} + \frac{\vec{S}_T \cdot (\vec{k}_T \times \hat{P})}{M} \underbrace{f_{1T}^{\perp}(x, k_T)}_{\text{Sivers function}}$$

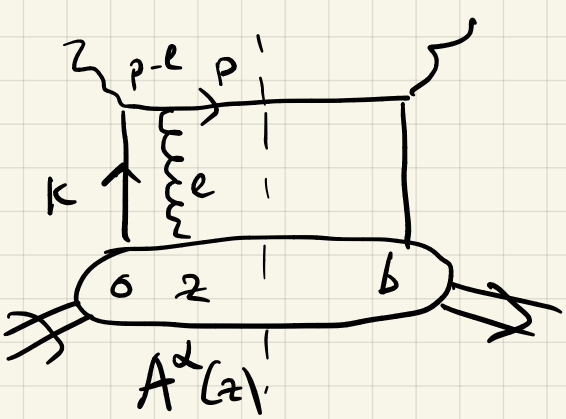
If one applies P and T invariance

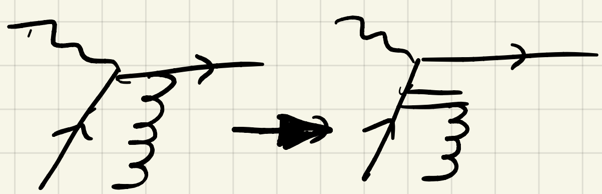
$$f_{a/p^\uparrow}(x, k_T, S_T) = f_{a/p^\uparrow}(x, k_T, -S_T)$$

and $f_{a/p}(x, k_T)$ remains $\underline{f_{1T}^{\perp}(x, k_T) \text{ vanishes}}$

A famous „mistake" that took 10 years to resolve

One needs to account for initial or final state radiation and introduce the gauge link





$$(p-\ell)^2 \simeq -2p\cdot\ell + \ell^2 \simeq -2p\cdot\ell$$

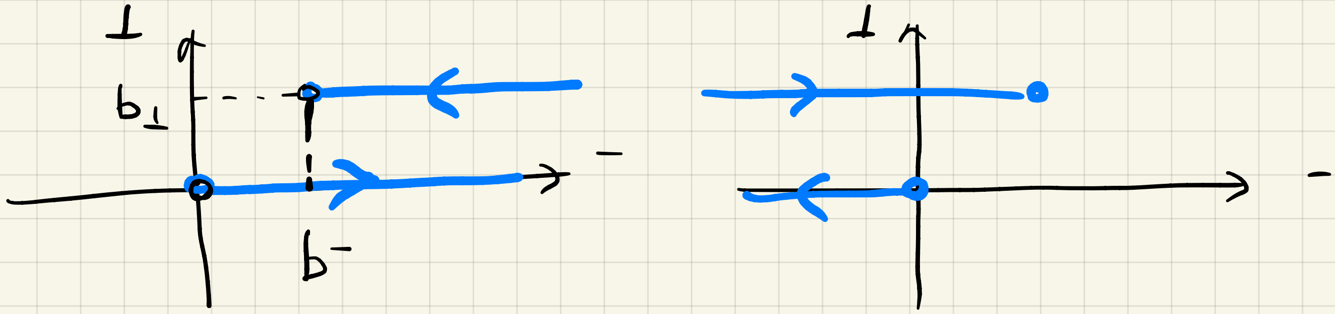
$$\simeq -2p^- \ell^+$$

the eikonal approximation

$$i\frac{\not{p} - \not{\ell}}{(p-\ell)^2 + i\epsilon} \simeq i\frac{p^-\gamma^+}{-2p^-\ell^+ + i\epsilon} = \frac{i}{2}\frac{\gamma^+}{-\ell^+ + i\epsilon}$$

It was found instead that

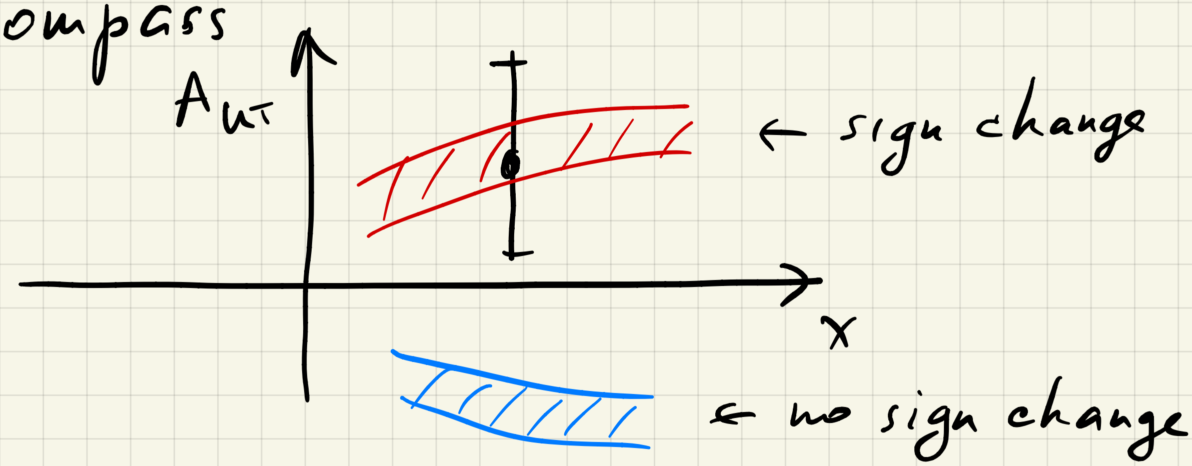
$$f^{SIDIS}_{a/p^\uparrow}(x, k_T, S_T) = f^{DY}_{a/p^\uparrow}(x, k_T, -S_T)$$



and thus
$$\underbrace{f^{SIDIS}_{a/p}(x, k_T) = f^{DY}_{a/p}(x, k_T)}_{\text{Unpolarised are the same}}$$

$$\underbrace{f^{\perp\, SIDIS}_{1T}(x, k_T) = - f^{\perp\, DY}_{1T}(x, k_T)}_{\text{Sivers function changes sign!}}$$

Compass



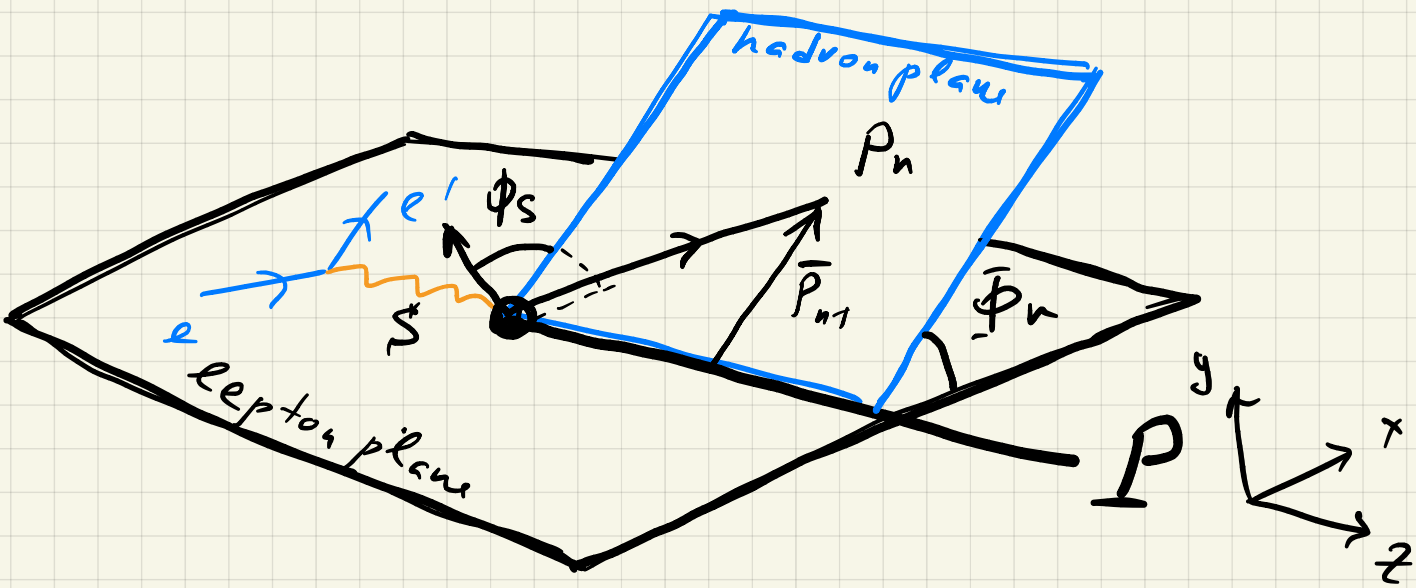
A project: Estimate if the EIC will proove the sign change.

SIDIS cross section has a very rich structure if the polarisation is availabe in the experiment.

Structure functions:

$$\mathcal{C}[w f D] = \int d^2 k_T \, d^2 p_T \, \delta^{(2)}(\bar{P}_{hT} - z\bar{k}_T - \bar{p}_T)$$

$$w(k_i, p_i)\, f(x, k_T)\, D(z, p_T)$$



Experimentally, S_L, S_T, ϕ_h, ϕ_S are the tools to disentangle structure functions.

Now convolutions in the momentum space $\to$ b_T space

$$\mathcal{C}[w f D] \equiv \sum_a e_a^2 \int d^2k_T\, d^2p_T\, \delta^{(2)}(\bar{P}_{hT} - z\bar{k}_T - \bar{p}_T)$$

$$w(k_T b_T)\, f(x, k_T)\, D(z, p_T), \qquad \bar{P}_{hT} \equiv -z\bar{q}_T$$

We rewrite $\delta^{(2)}(\bar{P}_{hT} - z\bar{k}_T - \bar{p}_T) =$

$$= \delta^{(2)}(-z\bar{q}_T - z\bar{k}_T - \bar{p}_T) = \frac{1}{z^2}\delta^{(2)}\left(\bar{q}_T + \bar{k}_T + \frac{\bar{p}_T}{z}\right)$$

$$= \frac{1}{z^2}\int \frac{d^2b_T}{(2\pi)^2} e^{-i(\bar{q}_T + \bar{k}_T + \bar{p}_T/z)\bar{b}_T}$$

$$F_{UU} = \mathcal{C}[1 f_1 D_1] = \sum_a e_a^2 \int \frac{d^2b_T}{(2\pi)^2} e^{-i\bar{b}_T\bar{q}_T}$$

$$\cdot \int d^2k_T\, e^{-i\bar{b}_T\bar{k}_T} f_1(x, k_T^2)$$

$$\cdot \frac{1}{z^2}\int d^2p_T\, e^{-i\bar{b}_T\bar{p}_T/z} D_1(z, p_T^2) =$$

$$= \sum_a e_a^2 \int \frac{d^2b_T}{(2\pi)^2} e^{-i\bar{b}_T\bar{q}_T} \tilde{f}_1(x, b_T^2)\, \tilde{D}_1(z, b_T^2)$$

$$= \sum_a e_a^2 \int \frac{b_T\, db_T}{2\pi} J_0(b_T q_T)\, \tilde{f}_1(x, b_T^2)\, \tilde{D}_1(z, b_T^2)$$

Let us call $\mathcal{B}\left[\tilde{f}_1^{(0)} \tilde{D}_1^{(0)}\right] = \sum_a e_a^2 \int \frac{b_T\, db_T}{2\pi} J_0(b_T q_T)\, \tilde{f}\, \tilde{D}$

A more complicated example, Sivers asymmetry

$$F_{UT}^{\sin(\phi_h-\phi_S)} = \mathcal{C}\left[-\frac{\hat{h}\cdot\bar{k}_T}{M} f_{1T}^{\perp} D_1\right] \qquad \hat{h} = \frac{\bar{P}_{hT}}{|P_{hT}|}$$

$$= \sum_a e_a^2 \int d^2k_T\, d^2p_T\, \delta^{(2)}(\bar{P}_{hT} - z\bar{k}_T - \bar{p}_T) \underbrace{\left(-\frac{\hat{h}\cdot\bar{k}_T}{M}\right)}_{-\frac{k_T}{M}\cos(\varphi-\phi_h)}$$

$$\cdot\, f_{1T}^{\perp}(x, k_T^2)\, D_1(z, p_T^2)$$

$$= \sum_a e_a^2 \int \frac{d^2 b_T}{(2\pi)^2} e^{-i\bar{q}_T\bar{b}_T} \int d^2k_T \left(-\frac{k_T}{M}\right)\cos(\varphi-\phi_h)$$

$$e^{-i\bar{k}_T\bar{b}_T} f_{1T}^{\perp}(x, k_T^2) \underbrace{\int d^2p_T \frac{1}{z^2} e^{-i\bar{p}_T\bar{b}_T/z} D_1(z, p_T^2)}_{\tilde{D}_1(z, b_T^2)}$$

we have $\bar{k}_T\bar{b}_T = b_T k_T \cos(\varphi-\varphi_b)$

$$\int d\varphi\, e^{-ib_Tk_T\cos(\varphi-\varphi_b)} \cos(\varphi-\phi_h) =$$

$$= -2\pi i\, J_1(b_Tk_T)\cos(\varphi_b-\phi_h)$$

Use Mathematica

$$\int d\varphi_b \cos(\varphi_b-\phi_h)\, e^{-ib_Tq_T\cos(\varphi_b-\phi_h)} =$$

$$= 2\pi i\, J_1(b_Tq_T)$$

So that

$$F_{UT}^{\sin(\phi_h - \phi_S)} = - \sum_q e_a^2 \int \frac{db_T\, b_T}{2\pi} J_1(b_T q_T)$$

$$\int dk_T \, \frac{k_T^2}{M} J_1(b_T k_T) f_{1T}^{\perp}(x, k_T^2) \, \tilde{D}_1(z, b_T^2)$$

$$\tilde{f}_{1T}^{\perp(1)}(x, b_T^2) = \frac{2\pi}{M^2} \int k_T\, dk_T \, \frac{k_T}{b_T} J_1(k_T b_T) f_{1T}^{\perp}(x, k_T^2)$$

Thus

$$F_{UT}^{\sin(\phi_h - \phi_S)} = (-M) \sum_q e_a^2 \int \frac{b_T\, db_T}{2\pi} \underbrace{b_T J_1(b_T q_T)}_{b_T^{1} J_{1}(b_T q_T)}$$

$$\cdot \tilde{f}_{1T}^{\perp(1)}(x, b_T^2) \, \tilde{D}_1(z, b_T^2)$$

$$F_{UT}^{\sin(\phi_h - \phi_S)} = - M \, \mathcal{B}\left[\tilde{f}_{1T}^{\perp(1)} \tilde{D}_1\right]$$

General definition

$$\mathcal{B}\left[\tilde{f}^{(n)} \tilde{D}^{(m)}\right] = \int \frac{b_T\, db_T}{2\pi} \, b_T^{n+m} J_{n+m}(b_T q_T)$$

$$\tilde{f}^{(n)}(x, b_T^2) \, \tilde{D}^{(m)}(z, b_T^2)$$

See TMD book for other examples.

The main advantage of b_T space: TMDs always come in product, not a complicated convolution as in k_T space

The consequence of factorisation theorems are evolution equations. For TMDs these are Collins-Soper equation and two renorm group equations

$$\tilde{F}(x, b_T) \rightarrow \tilde{F}(x, b_T; \mu, \zeta)$$

μ - Ultraviolet (UV) scale, UV divergence

ζ - Collins-Soper parameter, rapidity divergence

<u>CS equation</u>

$$(1) \quad \frac{\partial \ln \tilde{F}(x, b_T; \mu, \zeta)}{\partial \ln \sqrt{\zeta}} = \underbrace{\tilde{K}(b_T, \mu)}_{\text{CS kernel}}$$

<u>RG equations</u>

$$(2) \quad \frac{d \ln \tilde{K}(b_T, \mu)}{d \ln \mu} = - \underbrace{\gamma_K(\mu)}_{\text{Cusp anomalous dimension}}$$

$$(3) \quad \frac{d \ln \tilde{F}(x, b_T; \mu, \zeta)}{d \ln \mu} = \underbrace{\gamma_F(\mu, \zeta/\mu^2)}_{\substack{\text{anomalous dimension} \\ \text{of TMD}}}$$

γ_F, γ_K, and $\tilde{K}(b_T)$ at small b_T can be expanded in α_s.

Differentiate (1)

$$\frac{d}{d\ln\mu}\left(\frac{\partial \ln \tilde{F}(x,b_T,\mu,\zeta)}{\partial \ln\sqrt{\zeta}}\right) = \frac{d}{d\ln\mu}\tilde{K}(b_T,\mu) = -\gamma_K(\mu)$$

thus

$$\frac{\partial}{\partial \ln\sqrt{\zeta}}\Bigg(\underbrace{\frac{d\ln \tilde{F}(x,b_T,\mu,\zeta)}{d\ln\mu}}_{\gamma_F(\mu,\,\zeta/\mu^2)}\Bigg) = -\gamma_K(\mu)$$

Thus

$$\gamma_F(\mu, \zeta_0/\mu^2) - \gamma_F(\mu, \zeta/\mu^2) = -\gamma_K \ln\sqrt{\zeta_0} + \gamma_K \ln\sqrt{\zeta}$$

if $\zeta_0 = \mu^2$, then

$$\boxed{\gamma_F(\mu, \zeta/\mu^2) = \gamma_F(\mu; 1) - \frac{1}{2}\gamma_K(\mu)\ln \zeta/\mu^2}$$

Let us solve (2)

$$\frac{d\tilde{K}(b_T,\mu)}{d\ln\mu} = -\gamma_K(\mu)$$

$$\int_{\mu_0}^{\mu} d\,\tilde{K}(b_T,\mu) = -\int_{\mu_0}^{\mu} \gamma_K(\mu')\frac{d\mu'}{\mu'}$$

$$\tilde{K}(b_T,\mu) = -\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\gamma_K(\mu') + \tilde{K}(b_T,\mu_0) \tag{2.1}$$

Let us solve (1)

$$\tilde{F}(x,b_T,\mu,\zeta) = \tilde{F}(x,b_T,\mu,\zeta_0)\exp\left[\tilde{K}(b_T,\mu)\ln\sqrt{\frac{\zeta}{\zeta_0}}\right] \tag{1.1}$$

Let us solve (3)

$$\tilde{F}(x,b_T,\mu,\zeta) = \tilde{F}(x,b_T,\mu_0,\zeta)\exp\left[\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\gamma_F(\mu',\zeta/\mu'^2)\right] \tag{3.1}$$

CSS organisation

We start from low b_T, and expand our operator in terms of collinear operators

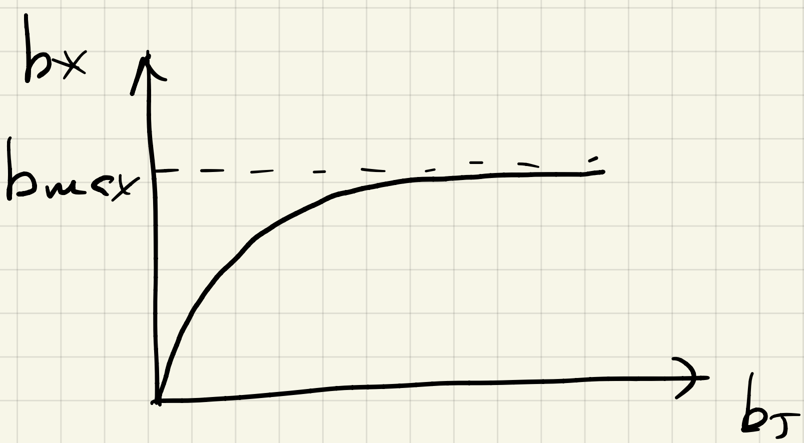
$$\tilde{F}_f(x, b_T, \mu, \zeta) = \sum_j \int_x^1 \frac{d\hat{x}}{\hat{x}} \underbrace{\tilde{C}_{j/f}\left(\frac{x}{\hat{x}}, b_T\right)}_{\text{Coeff. function}} \underbrace{f_j(\hat{x}, \mu)}_{\text{collinear pdf}}$$

at the lowest order

$$\tilde{C}_{j/f} = \delta_{jf}\, \delta\left(\frac{x}{\hat{x}} - 1\right)$$

Next step → combine perturbative and non perturbative. $\tilde{K}(b_T)$, $\tilde{F}(b_T)$ are non perturbative at large b_T

Introduce $b_* = \dfrac{b_T}{\sqrt{1 + b_T^2/b_{max}^2}}$



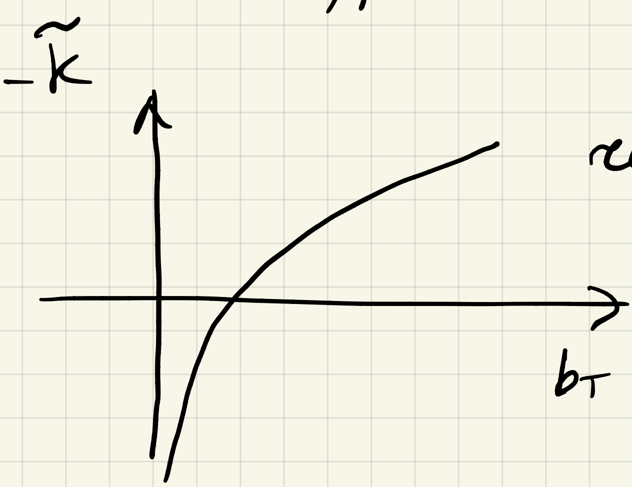
Rewrite

$$\tilde{K}(b_T,\mu) = \tilde{K}(b_*,\mu) + \underbrace{\left[\tilde{K}(b_T,\mu) - \tilde{K}(b_*,\mu)\right]}_{-g_K(b_T)}$$

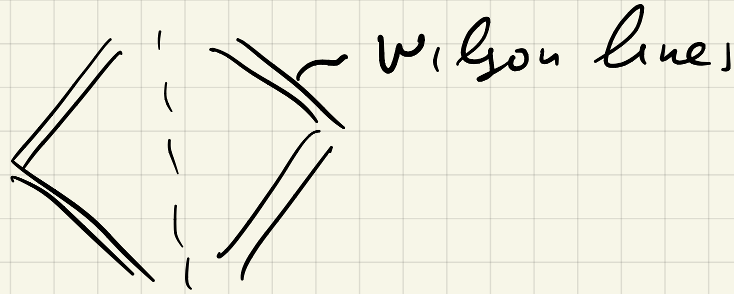
μ independent!
prove it using ev. eq.

Thus

$$\tilde{K}(b_T,\mu) = \tilde{K}(b_*,\mu_0) - \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'}\,\gamma_K(\mu') - g_K(b_T)$$



related to properties of QCD vacuum



$g_K(b_T)$ is a non perturbative function

$\sim g_2 b^2,\ g_2 \ln(b_T/b_*),\ g_2 b_T b_*\ ?$

Usually one uses $\mu_0 \sim 1/b_*$, namely

$$\mu_0 = \underbrace{\frac{2e^{-\gamma_E}}{b_*}}_{\text{also known as } \mu_{b_*}}, \quad \gamma_E \text{ Euler gamma}$$

$2e^{-\gamma_E} = C_1 \simeq 1.12$

Now let us write the following

$$\tilde{F}(x,b_T,\mu,\zeta) = \tilde{F}(x,b_*,\mu,\zeta)\,\frac{\tilde{F}(x,b_T,\mu,\zeta)}{\tilde{F}(x,b_*,\mu,\zeta)}$$

$$= \tilde{F}(x,b_T,\mu,\zeta_0)\exp\left[\tilde{K}(b_*,\mu)\ln\sqrt{\zeta/\zeta_0}\right]\underbrace{\frac{\tilde{F}(x,b_T,\mu,Q_0^2)}{\tilde{F}(x,b_*,\mu,Q_0^2)}}_{\overset{\text{def}}{\equiv}\ \exp\left[-g(x,b_T)\right]}$$

Q_0 here is some reference scale ~ 1-2 GeV

does not depend on μ!
n.p. intrinsic function

$$* \exp\left[\ln\sqrt{\zeta/Q_0^2}\,\underbrace{\left(\tilde{K}(b_T,\mu)-\tilde{K}(b_*,\mu)\right)}_{-g_K(b_T)}\right]$$

$$= \tilde{F}(x,b_*,\mu,\zeta_0)\exp\left[\ln\sqrt{\zeta/\zeta_0}\,\tilde{K}(b_*,\mu)\right]$$

$$\times \exp\left[-g(x,b_T) - \ln\sqrt{\zeta/Q_0^2}\,g_K(b_T)\right]$$

$$= \tilde{F}(x,b_*,\mu_0,\zeta_0)\exp\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\left(\gamma_F(\mu',1) - \ln\sqrt{\frac{\zeta_0}{\mu'^2}}\,\gamma_K(\mu')\right)$$

$$\exp\left[\ln\sqrt{\zeta/\zeta_0}\,\tilde{K}(b_*,\mu_0) - \int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\ln\sqrt{\zeta/\zeta_0}\,\gamma_K(\mu')\right]$$

$$\exp\left[-g(x,b_T) - \ln\sqrt{\zeta/Q_0}\,g_K(b_T)\right]$$

Combine alltogether:

$$\tilde{F}(x, b_T, \mu, \zeta) = \tilde{F}(x, b_*, \mu_0, \zeta_0) \exp\left(\ln\sqrt{\zeta/\zeta_0}\, \tilde{K}(b_*, \mu_0)\right)$$

$$\exp \underbrace{\int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \left[\gamma_F(\mu, 1) - \ln\sqrt{\zeta/\mu'^2}\, \gamma_K(\mu') \right]}_{\text{Sudakov F.F.}}$$

$$\exp\left[-g(x, b_T) - \ln\sqrt{\zeta/Q_0^2}\, g_K(b_T) \right] \Big\} \text{ n.p. structure}$$

Now we will use $\mu_0 = \mu_{b_*} = \dfrac{2e^{-\gamma_E}}{b_*}$,

$\zeta_0 = \mu_{b_*}^2$, $\zeta = Q^2$, $\mu = \mu_Q = Q$

and rewrite

$$\tilde{F}(x, b_T, \mu, \zeta) = \tilde{F}(x, b_*, \mu_{b_*}, \mu_{b_*}^2)$$

$$\left(\frac{Q^2}{\mu_{b_*}^2}\right)^{\frac{1}{2}\tilde{K}(b_*, \mu_{b_*})} \left(\frac{Q^2}{Q_0^2}\right)^{-\frac{1}{2} g_K(b_T)}$$

$$e^{-g(x, b_T)} \exp\left[\int_{\mu_{b_*}}^{\mu} \frac{d\mu'}{\mu'} \left(\gamma_F(\mu, 1) - \ln\frac{Q}{\mu'}\, \gamma_K(\mu') \right) \right]$$

$\Rightarrow$ This is CSS solution of TMD evolution equations

Some remarks about physics:

$$\left.\begin{array}{l} g(x, b_T) \\ g_K(b_T) \end{array}\right\}$$ to be extracted from the data

$g_K(b_T)$ – a universal function related to the properties of the vacuum of QCD

$g(x, b_T)$ – depends on the hadron, but parametrises the nonperturbative intrinsic structure

$$\int_{\mu_{b_*}}^{\mu_Q} \underbrace{\frac{d\mu'}{\mu'}}_{d\ln\mu'} \ln\frac{Q}{\mu'}\,\gamma_K \sim \underbrace{\alpha_s \ln^2 \mu'}\Big|_{\mu_{b_*}}^{\mu_Q}$$

famous double logs in the Sudakov form factor

In the momentum space $\sim \alpha_s \ln^2 q_T/Q$

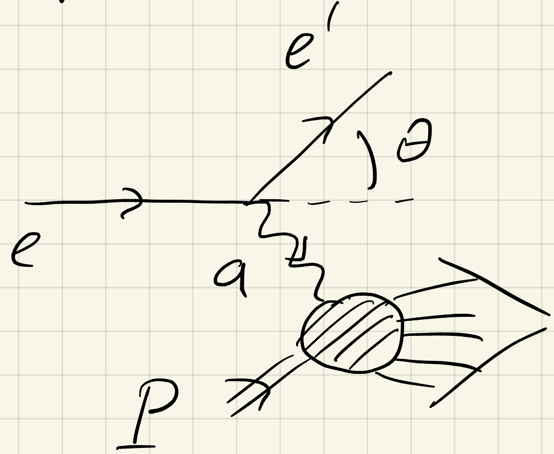
$q_T \to 0$, $\ln^2 q_T/Q$ can be very large and spoil perturbative convergence, thus must be „resummed" (exponentiated) to all orders

that is why CSS formalism is also called resummation

Backup notes

During lecture 1 we speak of a certain process. Let us look into its kinematics

Deep Inelastic Scattering (DIS)



1) Show that $q^2 < 0$

$$\ell = (E, 0, 0, E)$$

$$e' = (E', E' \sin\theta, 0, E' \cos\theta)$$

Thus $q^2 = (\ell - e')^2 \simeq -2\ell e' = -2(EE' - EE'\cos\theta)$

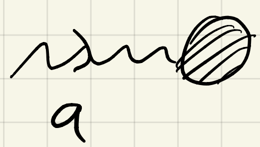
$= -2EE'(1-\cos\theta) \leq 0$.

It is customary to introduce $q^2 \equiv -Q^2$ where $Q^2 \geq 0$

Now let us explore other kinematical variables

$$x = \frac{Q^2}{2P\cdot q} \text{ - Bjorken-}x, \quad y = \frac{P\cdot q}{P\cdot e} \text{ - inelasticity}$$

These variables are constructed off scaler products which are Lorentz invariants and wherefore we can use any frame to estimate them. Let us choose target rest frame



$$P = (M, \vec{0})$$

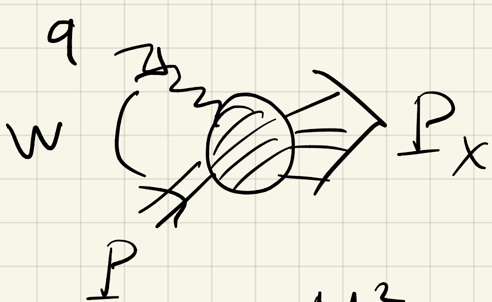
$$q = (\nu, \vec{q}), \quad \nu = E - E' > 0$$

$$2P\cdot q = 2M\nu \to \infty \quad \text{Bjorken limit}$$

$$q^2 = -Q^2, \quad Q^2 \geq 0, \quad \nu \geq 0 \Rightarrow$$

$$x = \frac{Q^2}{2P\cdot q} \geq 0$$

W is the energy of qP



$$W^2 = (P+q)^2 = P^2 + 2P\cdot q - Q^2 =$$

$$= M^2 + 2P\cdot q - Q^2 \geq M^2$$

in case the proton is intact = elastic scattering

$$\Rightarrow \quad 2P\cdot q \geq Q^2 \Rightarrow \quad x = \frac{Q^2}{2P\cdot q} \leq 1$$

Inelasticity $y = \frac{P \cdot q}{P \cdot e} = \frac{M(E-E')}{ME} = 1 - E'/E$

$E' \in [0, E]$ thus $y \in [0,1]$

$$\left.\begin{array}{l} x \in [0,1] \\ y \in [0,1] \end{array}\right\}$$ we can estimate the reach of experiments

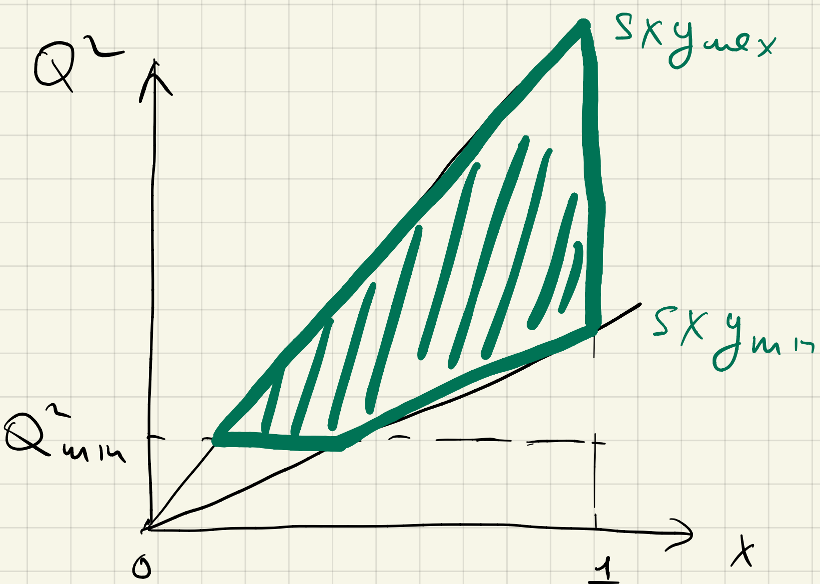
$$s = (P+e)^2 \simeq M^2 + 2P\cdot e \simeq 2P\cdot e$$

$\Rightarrow$

$$x = \frac{Q^2}{2P\cdot q} = \frac{Q^2}{2P\cdot q} \cdot \frac{P\cdot e}{P\cdot e} = \frac{Q^2}{ys}$$

Therefore

$$Q^2 \simeq syx$$



$Q^2_{min} \simeq 1\ (\mathrm{GeV}^2)$ to ensure DIS regime

$y \in \underbrace{[y_{min}, y_{max}]}_{\text{experimental resolution}}$

Why do we need x_{Bj}?

Let us consider form factors → elastic scattering

$\delta^{(4)}(\underline{P}+q-\underline{P}')$

$\Rightarrow \underline{q} = \underline{P}' - \underline{P}$

$$q^2 = -Q^2 = (\underline{P}'-\underline{P})^2 = \underline{P}'^2 + \underline{P}^2 - 2\,\underline{P}'\cdot\underline{P} =$$

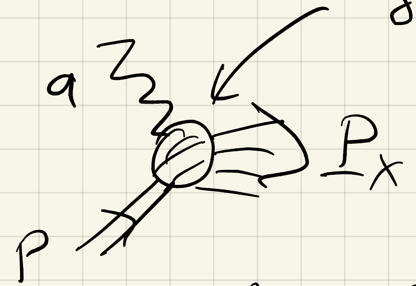
$$= 2M^2 - 2\,\underline{P}'\cdot\underline{P} \rightarrow -2\,\underline{P}'\cdot\underline{P}$$

$$\underline{P}\cdot q = \underline{P}\,(P' - P) = \underline{P}\cdot\underline{P}' - M^2 \rightarrow P\cdot P'$$

therefore

$$x = \frac{Q^2}{2P\cdot q} \rightarrow 1$$ not an independent variable

$\delta^{(4)}(\underline{P}+q-P_X)$



We integrate over $\underline{P}_X$

$$\left.\begin{array}{l} q^2 = -Q^2 \rightarrow \infty \\ 2P\cdot q \rightarrow \infty \end{array}\right\}$$ independently

$$x_{Bj} \equiv \frac{Q^2}{2\underline{P}\cdot q} \in [0,1].$$

Experiments : fixed target vs collider

$$s = (\underline{P}+\ell)^2 \quad \text{cm energy}$$

Fixed target

$$\ell = (P_{lab}, 0, 0, -P_{lab})$$

$$\underline{P} = (M_p, 0, 0, 0)$$

$$s = (\underline{P}+\ell)^2 = \underline{P}^2 + 2\underline{P}\cdot\ell + \ell^2 \simeq 2 M_p P_{lab}$$

Collider

$$\ell = (E_e, 0, 0, -E_e)$$

$$\underline{P} \simeq (E_p, 0, 0, E_p) \quad \text{(neglect the mass)}$$

$$s = (\underline{P}+\ell)^2 \simeq (E_e+E_p)^2 - (E_p - E_e)^2 = 4 E_p E_e$$

The energy increased easly in collider

The elements of Quantum Field Theory

The wave function $\psi(x)$ is a coordinate projection of the state vector in the Hilbert space $|\psi\rangle$

$$\int d^3x \, |\psi(x)|^2 < \infty$$

The scalar product

$$\langle \psi | \phi \rangle = \int d^3x \, \psi^*(x) \, \phi(x)$$

We will use $|P; S\rangle$ to denote the proton with momentum P and spin vector S.

Unitarity of S matrix and optical theorem

(Taylor „Scattering theory")

The probability of one state going to the other is described by „S" matrix

$$w(\chi \leftarrow \phi) = |\langle \chi | S | \phi \rangle|^2$$

One can write

$$S_{ab} = \underbrace{\delta_{ab}}_{\text{no interaction}} + i(2\pi)^4 \underbrace{\delta^{(4)}(p_a - p_b)}_{\text{momentum conservation}} \overset{\text{interaction}}{T_{ab}}$$

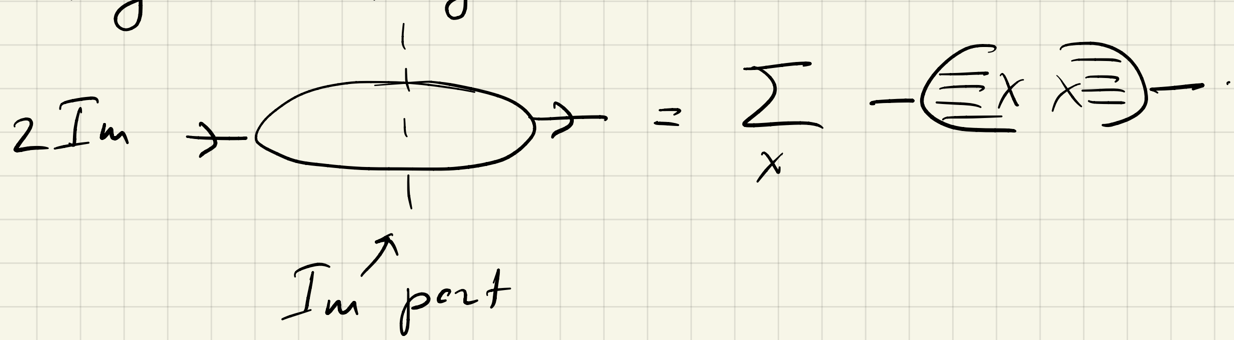
Probability is conserved

$$SS^\dagger = S^\dagger S = 1 \quad \text{Unitarity of S matrix}$$

One can use it to write

$$\mathrm{Im}\,\langle \chi | T | \phi \rangle = \frac{1}{2} \sum_X \langle \chi | T \overbrace{| X \rangle \langle X |}^{\text{all possible states}} T^\dagger | \phi \rangle (2\pi)^4 \delta^{(4)}(p_\phi - p_X)$$

Diagrammatically



$$SS^{\dagger} \simeq \left(1 + i(2\pi)^4 \delta^{(4)}(P_\phi - P_\chi)T\right)\left(1 - i(2\pi)^4 \delta^{4}(P_\phi - P_\chi)T^{\dagger}\right) = 1$$

$$\cancel{1} - i(2\pi)^4\left(T^{\dagger} - T\right)\delta^{(4)}(P_\phi - P_\chi) + (2\pi)^8\, T T^{\dagger}\left(\delta^{4}(P_\phi - P_\chi)\right)^2 = \cancel{1}$$

$$T = \mathrm{Re}\,T + i\,\mathrm{Im}\,T, \quad T^{\dagger} = \mathrm{Re}\,T - i\,\mathrm{Im}\,T$$

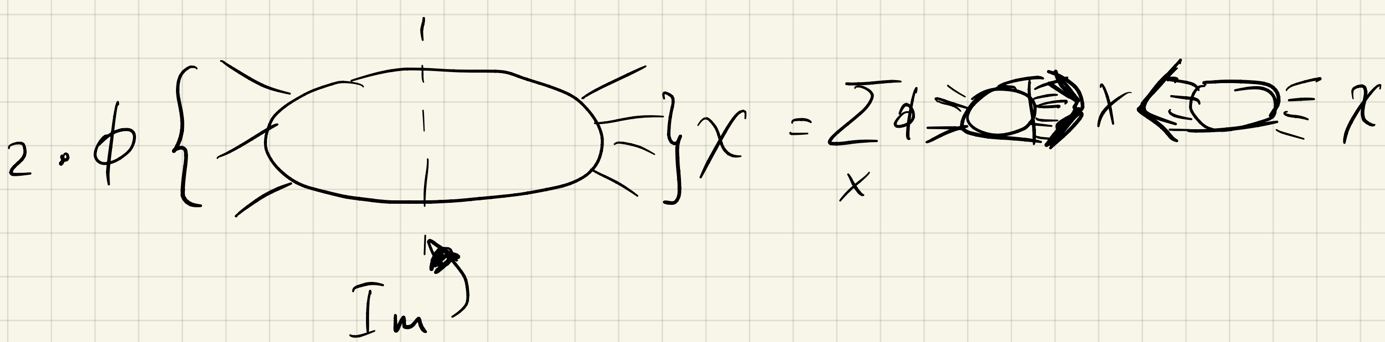
$$T^{\dagger} - T = -2i\,\mathrm{Im}\,T$$

$$2(2\pi)^4\,\mathrm{Im}\,T = (2\pi)^8\, T T^{\dagger}\,\delta^{(4)}(P_\phi - P_\chi)$$

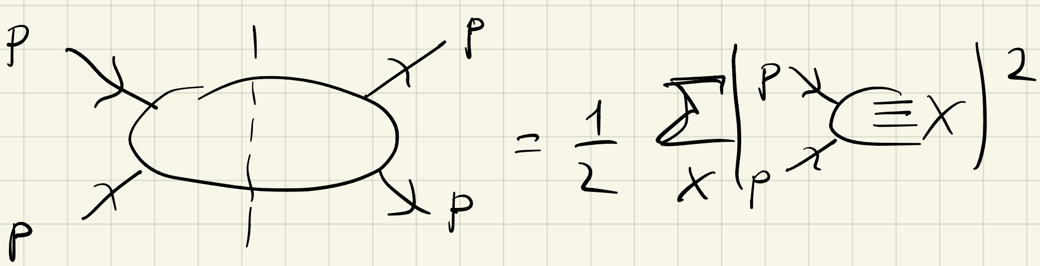
Let us insert $1 = \sum_X |X\rangle\langle X|$

and sandwich this expression with $\langle\chi| \dots |\phi\rangle$

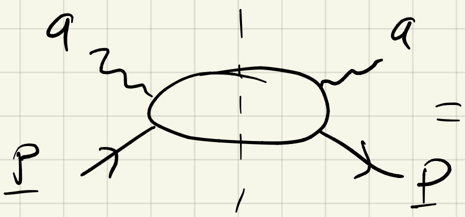
$$\mathrm{Im}\,\langle\chi|T|\phi\rangle = \frac{1}{2}\sum_X \langle\chi|T|X\rangle\langle X|T^{\dagger}|\phi\rangle(2\pi)^4\delta(P_\phi - P_\chi)$$



If $\phi = \chi$ then, for instance $pp \to pp$



We are interested in photon-proton interactions

$2\,\mathrm{Im}$ 

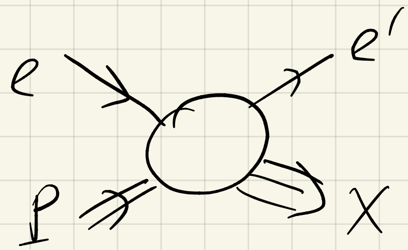
$= \sum_X \Big|$ 

$\Big|^2$

Experimentally one measures cross-sections

$$\sigma_{\phi\to\psi} = \underbrace{\frac{1}{F_\phi}}_{\text{Flux of } \phi} |A_{\phi\to\psi}|^2 \underbrace{\frac{d^3P_\psi}{(2\pi)^3\, 2E_\psi}}_{\text{Phase space of } \psi}$$

We want to calculate cross-section of this process

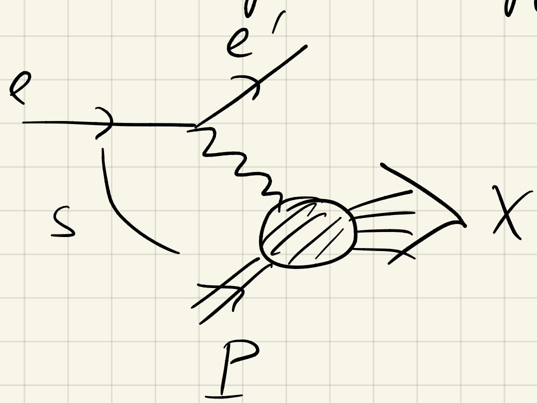


$e + P \to e' + X$

$q^2 = (e-e')^2$ $\qquad P \to X$

Deep Inelastic Scattering

We use one photon approximation

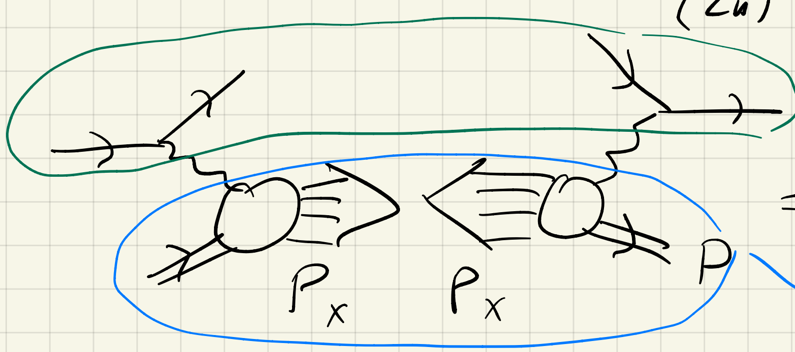


$\sigma = \frac{1}{F} |M|^2 \, dPS$

$F \simeq 2s = 2(e+P)^2$ flux

$dPS = \frac{d^3 e'}{(2\pi)^3 \, 2E'}$

leptonic tensor

$|M|^2 =$ 

$= \frac{1}{Q^4} L_{\mu\nu} W^{\mu\nu}$

hadronic tensor

$\frac{1}{Q^4} \sim \xi \cdot \xi$ product of photon propagators

Before we calculate $L_{\mu\nu}$ and $W^{\mu\nu}$ let us recapitulate some basics of QFT.

Consider the Lagrangean of a spin-½ particle with the mass m:

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi$$

γ^μ – gamma matrices

$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$, $\Psi(x)$, $\bar{\Psi}(x) \equiv \Psi^+(x)\gamma^0$ fields

Global gauge transformations

$$\Psi'(x) = e^{i\alpha}\Psi(x)$$

$$\bar{\Psi}'(x) = \bar{\Psi}(x)e^{-i\alpha}$$

$\mathcal{L} \to \mathcal{L}'$, current $j^\mu(x) = \bar{\Psi}(x)\gamma^\mu\Psi(x)$

Local gauge transformations

$$\begin{cases} \Psi'(x) = e^{i\alpha(x)}\Psi(x) \\ \bar{\Psi}'(x) = \bar{\Psi}(x)\, e^{-\alpha(x)} \end{cases}$$

$$\partial_\mu \Psi(x) = e^{-i\alpha(x)}(\partial_\mu - i\partial_\mu \alpha(x))\Psi'(x)$$

Thus

$$\mathcal{L} = \bar{\Psi}'(x)(i\gamma^\mu(\partial_\mu - i\partial_\mu \alpha(x)) - m)\Psi'(x)$$

We can restore gauge invariance if we use

$$(\partial_\mu + ieA_\mu(x))\psi(x)$$

$$(\partial_\mu + ieA_\mu(x))\psi(x) = e^{-i\alpha(x)}(\partial_\mu + ieA'_\mu(x))\psi'(x)$$

where

$$A'_\mu(x) = A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x)$$

$\partial_\mu + ieA_\mu(x) \rightarrow$ covariant derivative D_μ

$$\mathcal{L} = \bar{\psi}(x)(i\gamma^\mu(\partial_\mu + ieA_\mu(x)) - m)\psi(x)$$

or

$$\mathcal{L} = \bar{\psi}(x)(i\gamma^\mu D_\mu - m)\psi(x)$$

is invariant also under the local gauge transform.

We also have introduced interactions!

$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$, where

$\mathcal{L}_I = -e j^\mu A_\mu$ where $j^\mu = \bar{\psi}(x)\gamma^\mu\psi(x)$

Remember

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \equiv \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

$$\gamma^0 (\gamma^\mu)^\dagger \gamma^0 = \gamma^\mu \quad , \Rightarrow (\gamma^0)^\dagger = \gamma^0 , (\gamma^k)^\dagger = -\gamma^k$$

Independent fields $\psi(x)$ & $\bar\psi(x) = \psi^\dagger(x)\gamma^0$

Euler-Lagrange equations (for $\mathcal{L} \equiv \mathcal{L}_0$)

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \dfrac{\partial \mathcal{L}}{\partial \partial_\mu \psi} = 0 \\[2ex] \dfrac{\partial \mathcal{L}}{\partial \bar\psi} - \partial_\mu \dfrac{\partial \mathcal{L}}{\partial \partial_\mu \bar\psi} = 0 \end{cases}$$

$$\frac{\partial \mathcal{L}}{\partial \bar\psi} = (i\gamma^\mu\partial_\mu - m)\psi \quad , \quad \frac{\partial \mathcal{L}}{\partial \partial_\mu \bar\psi} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \psi} = -m\bar\psi \quad , \quad \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi} = i\bar\psi\gamma^\mu$$

$$\Rightarrow \begin{cases} (i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \\ i\partial_\mu \bar\psi(x)\gamma^\mu + m\bar\psi(x) = 0 \end{cases}$$

4 solutions, 2 with $p_0 > 0$, 2 with $p_0 < 0$

Let us consider only positive energy

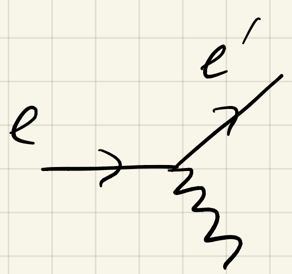
$$\psi(x) = u(p,s)\, e^{-ip\cdot x} \quad , \; p^2 = m^2 , \; p_0 \geq 0$$

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$$

$$\Rightarrow \; (\gamma^\mu p_\mu - m)\, u(p) = 0 \quad , \quad \gamma^\mu p_\mu \equiv \not{p}$$

$(\not{p} - m)\, u(p) = 0$, $u(p)$ is called spinor

$$\bar{u}(p)(\not{p} - m) = 0$$



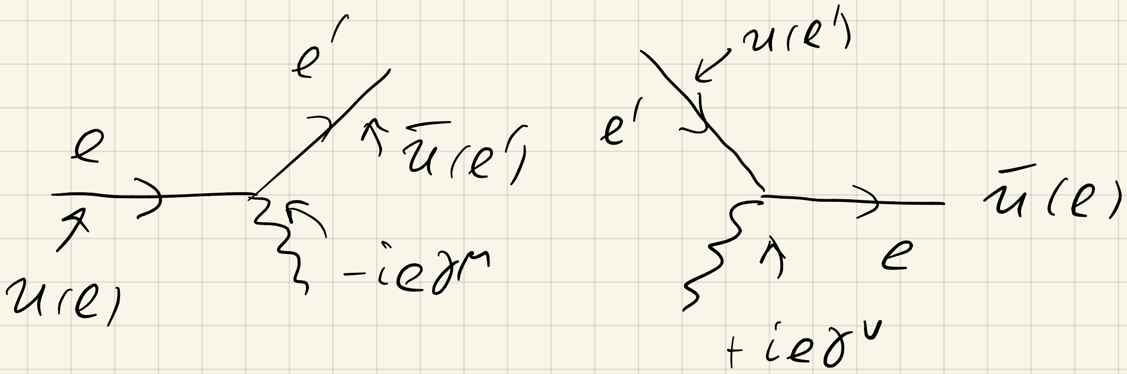
$$q = e - e'$$

Current conservation $\partial_\mu J^\mu(x) = 0$

$$J^\mu(x) = \bar{u}(e')\gamma^\mu u(e)\, e^{-i(e-e')\cdot x}$$

$$\partial_\mu J^\mu(x) = -i q_\mu J^\mu = 0 \Rightarrow q_\mu J^\mu(x) = 0$$

Let us calculate $L^{\mu\nu}$:



$$L^{\mu\nu} = \frac{1}{2s+1}\sum_{s'} \bar{u}_\alpha(e,s)(-ie\gamma^\nu)_{\alpha\beta}\, u(e',s')_\beta\, \bar{u}_a(e',s')(+ie\gamma^\mu)_{ab}\, u(e)_b$$

Spin products
$$\sum_{s'} u_\beta(e',s')\, u_a(e',s') = (\not{e}' + m)_{\beta a}$$

$$u_b(e,s)\,\bar{u}_\alpha(e,s) = \left[\frac{(\not{e}+m)(1+\gamma_5\not{s})}{2}\right]_{b\alpha}$$

where $\gamma_5 = +i\gamma^0\gamma^1\gamma^2\gamma^3$, $\gamma_5^{\dagger} = \gamma_5$, $(\gamma_5)^2 = 1$, $\{\gamma_5, \gamma^\mu\} = 0$

$$L^{\mu\nu} = \frac{e^2}{2}\underbrace{(\not{e}+m)_{b\alpha}\,\gamma^\nu_{\alpha\beta}(\not{e}'+m)_{\beta a}(\gamma^\mu)_{ab}}_{\text{trace}}$$

neglect m and we have

$$L^{\mu\nu} = \frac{e^2}{2}\,\mathrm{Tr}(\not{e}\,\gamma^\mu\,\not{e}'\,\gamma^\nu)$$

Traces

$$\mathrm{Tr}(\text{odd } \# \gamma) = 0$$

$$\mathrm{Tr}(\not{a}\not{b}) = 4\, a \cdot b$$

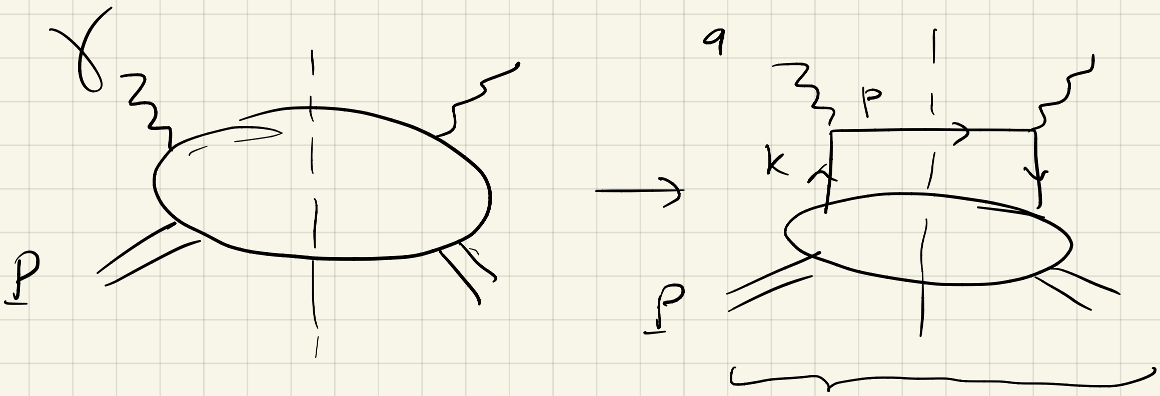
$$\mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) = 4\left[(a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(c \cdot b)\right]$$

$$\mathrm{Tr}(\gamma^a \gamma^b \gamma^c \gamma^d) = 4(g^{ab} g^{cd} - g^{ac} g^{bd} + g^{ad} g^{cb})$$

Thus

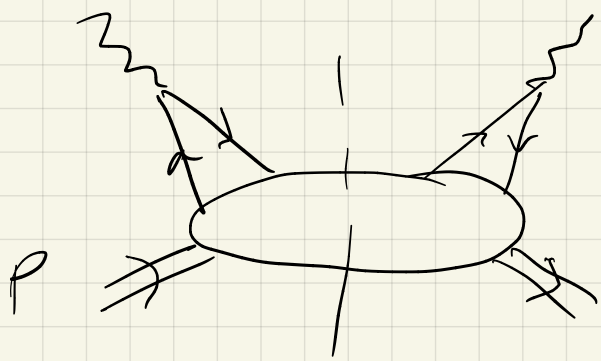
$$L^{\mu\nu} = 2e^2(e^\mu e'^\nu + e^\nu e'^\mu - g^{\mu\nu}(e \cdot e'))$$

Now let us consider the hadronic tensor



hand bag diagram

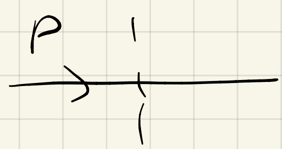
why do we not consider?



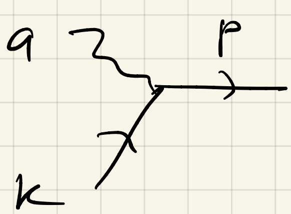
$\propto e_{q_1} e_{q_2}$

suppressed by $\left(\frac{1}{Q^2}\right)^2$

as at least one of the propagators is hard

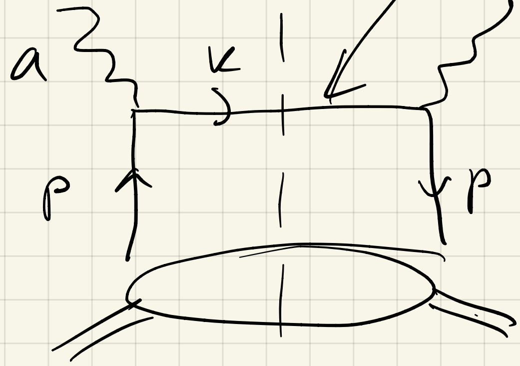


$$= \mathrm{Im}\, \frac{1}{p^2 + i\epsilon} = \pi\, \delta(p^2)$$



$$\rightarrow \delta^{(4)}(q + k - p) \Rightarrow p = k + q$$

Why partons are almost on mass shell?

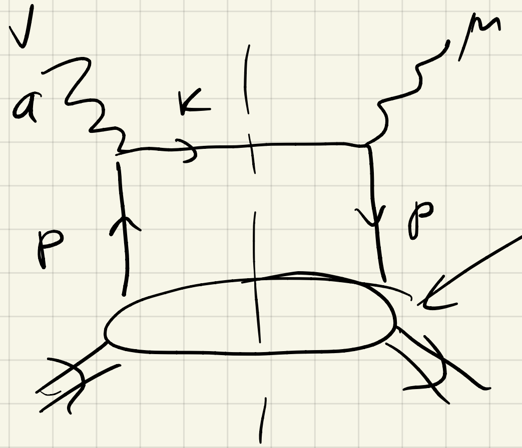


$\delta(k^2) \Rightarrow k^2 = 0$

virtual, but

$$\underbrace{\int d^4p \frac{1}{(p^2+i\epsilon)} \frac{1}{(p^2-i\epsilon)}} \to p^2 \approx 0$$

The contribution from this integral is max then $p^2 \simeq 0$ as well!



let us call this matrix $\Phi(p,P)$

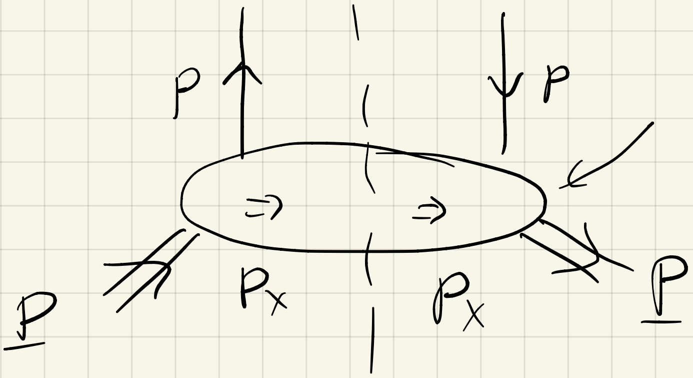
$$W^{\mu\nu} = \sum_q e_q^2 \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left(\gamma^\mu (\not p + \not q)\gamma^\nu \Phi(p,P)\right) \delta\left((p+q)^2\right)$$

Let us parametrize $p = xP$, $x \in (-\infty, \infty)$

$$\delta((p+q)^2) \simeq \delta(-Q^2 + 2xP\cdot q) = \frac{1}{2P\cdot q}\,\delta(x_{BJ} - x)$$

quarks are probed at $x = x_{BJ}$!

What is ϕ?



$\delta^{(4)}(p+P_X-P)$

$$\delta^4(p) = \int \frac{d^4\xi}{(2\pi)^4} e^{-ip\cdot\xi}$$

$$\sum\!\!\!\!\!\!\int_X \equiv \int \frac{d^3P_X}{2E_X(2\pi)^3} = \int \frac{d^4P_X}{(2\pi)^4}\,\theta(E_X)$$

$$\underline{\Phi} = \sum\!\!\!\!\!\!\int_X \delta^{(4)}(p+P_X-P)\,\langle p|\bar{\Psi}(0)|X\rangle\langle X|\Psi(0)|P\rangle$$

$$= \sum\!\!\!\!\!\!\int_X \int \frac{d^4\xi}{(2\pi)^4} e^{-i\xi(p+P_X-P)} \langle \underline{P}|\bar{\Psi}(0)|X\rangle\langle X|\Psi(0)|P\rangle$$

$$= \sum\!\!\!\!\!\!\int_X \int \frac{d^4\xi}{(2\pi)^4} e^{-i\xi\cdot p}\, \underbrace{\langle P|e^{i\xi \underline{P}}}\,\bar{\Psi}(0)\underbrace{e^{-i\xi P_X}|X\rangle}\langle X|\Psi(0)|P\rangle$$

$$\langle P|\underbrace{e^{i\xi\cdot\hat{\underline{P}}}\,\bar{\Psi}(0)\,e^{-i\xi\cdot\hat{P}}}_{\bar{\Psi}(\xi)}|\cancel{X}\rangle$$

$\bar{\Psi}(\xi)$ - <u>shift of the field</u>

Thus

$$\underline{\Phi}(p,\underline{P}) = \sum\!\!\!\!\!\!\int_X \int \frac{d^4 z}{(2\pi)^4} e^{-ip\cdot z} \langle P|\bar{\psi}(z)|X\rangle\langle X|\psi(0)|P\rangle$$

now we use

$$\sum\!\!\!\!\!\!\int_X |X\rangle\langle X| = \mathbb{1} \text{ completeness}$$

of states and obtain

$$\underline{\Phi}(p,\underline{P}) = \int \frac{d^4 z}{(2\pi)^4} e^{-ip\cdot z} \langle P|\bar{\psi}(z)\psi(0)|P\rangle$$

Let us introduce light cone variables $A^{\pm} = \frac{A^0 \pm A^3}{\sqrt{2}}$

$$A\cdot B = A^+B^- + A^-B^+ - \vec{A}_T\cdot\vec{B}_T \quad , \quad A_T \equiv (A^1, A^2)$$

$$\underline{P} \simeq \left(P^+, \frac{M^2}{2P^+}, 0\right) \simeq (P^+, 0, 0)$$

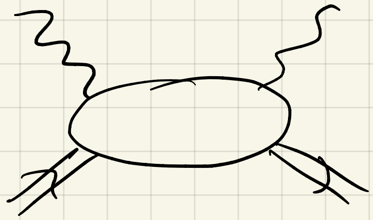
$$p = x\underline{P} \simeq (p^+, 0, p_\perp)$$

$\nwarrow$ important for TMDs

$$p\cdot z \to p^+z^- - \underbrace{\vec{p}_T\cdot\vec{z}_T}_{\text{small}} \simeq p^+z^-$$

$$\Rightarrow z \simeq (0, z^-, 0)$$

Let us see how distributions are introduced in DIS.



$$W^{\mu\nu} = -\left(g^{\mu\nu} + \frac{q^\mu q^\nu}{Q^2}\right) W_1 + \left(P^\mu + \frac{q^\mu}{2x}\right)\left(P^\nu + \frac{q^\nu}{2x}\right) W_2$$

(only $W^{\mu\nu} = W^{\nu\mu}$ & $q_\mu W^{\mu\nu} = 0$)

Remember $P \cdot q = \nu$

One usually uses $\begin{cases} F_1(x,Q^2) = W_1(x,Q) \\ F_2(x,Q) = \nu\, W_2(x,Q) \end{cases}$

$$F_L = F_2 - 2x F_1 \simeq 0$$

$$p^\mu = (P, 0, 0\, P) \qquad p^2 = n^2 = 0$$

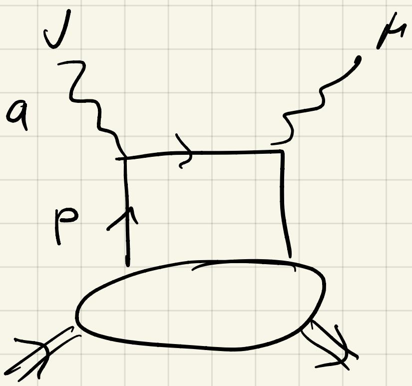
$$n^\mu = \left(\frac{1}{2P}, 0, 0, -\frac{1}{2P}\right) \qquad p \cdot n = 1$$

$$q^\mu = q^\mu_\perp + \nu n^\mu$$

$$q^2 = -\vec{q}_\perp^{\,2} = -Q^2$$

Then

$$n^\mu n^\nu W_{\mu\nu} = W_L = \frac{1}{\nu} F_2$$



$$W^{\mu\nu} = e_q^4 \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left(\gamma^\mu (\not q + \not p)\gamma^\nu \phi\right) \cdot \underbrace{\delta\left((p+q)^2\right)}_{\frac{1}{2\nu}\delta(x - x_{Bj})}$$

$$\underline{p} \simeq x \underline{P}$$

$$F_2 = \nu n^\mu n^\nu W_{\mu\nu} = \frac{1}{2} e_q^2 \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}(\underbrace{\not n \underbrace{\not p \not n}_{-\not n \not p + 2n\cdot p} \phi}_{2x \mathrm{Tr}(\not n \phi)})\delta(x - x_{Bj})$$

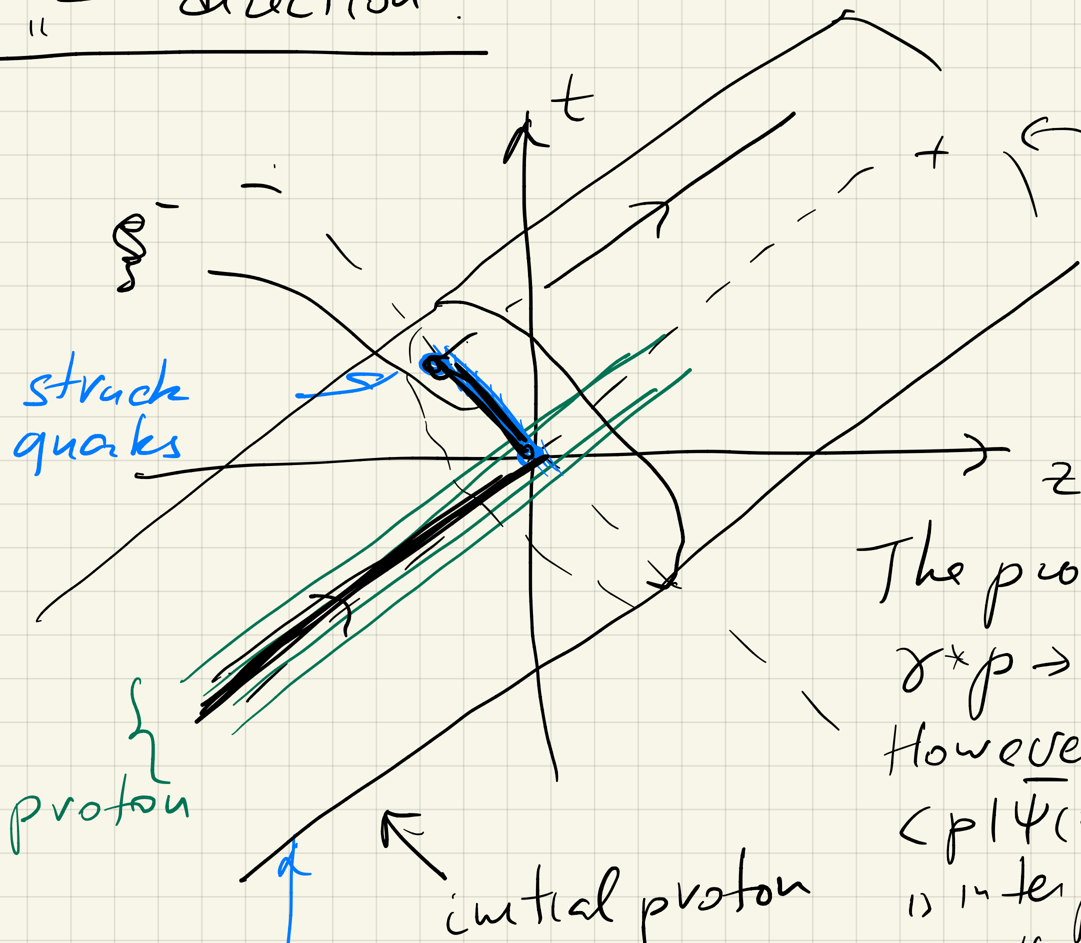
we can define

$$\underbrace{f(x_{Bj}) \equiv \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}(\not n \phi(p, P))\delta(x - x_{Bj})}_{\text{Parton distributions}}$$

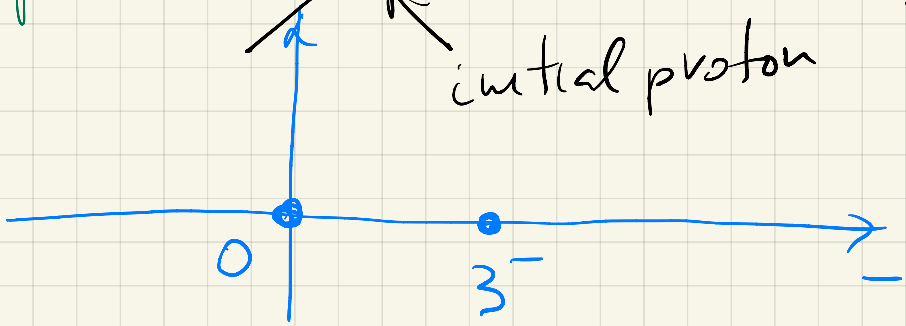
$$\Rightarrow \underbrace{F_2(x, Q^2) = \sum_q e_q^2 \, x f(x)}_{\text{Bjorken scaling!}}$$

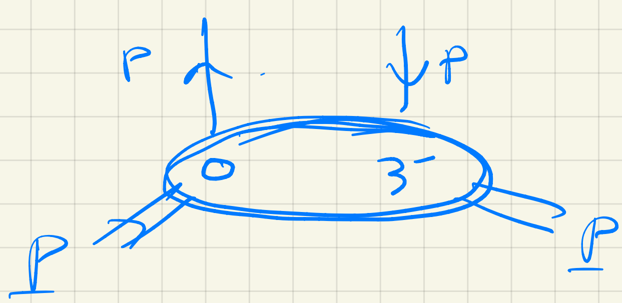
$$\underline{\not n = \gamma^+}$$

The fields are separated by some distance z in "$-$" direction!



The process
$\gamma^* p \to X$
However
$\langle p|\bar{\psi}(z)\psi(0)|p\rangle$
is interpreted
as the proton
at $-\infty$, then
at 0 the
quark field is
shifted to z
and then
the proton
continues to $+\infty$





Analogously to DIS we can define the following projections of the correlator $\Phi(p, P)$ in the case when transverse motion is not ignored.

It is customary to call the parton's momentum k

so

$$\Phi(x, k_T)_{ij} = \int \frac{d\xi^- d^2\xi_T}{(2\pi)^3} e^{-ixP^+\xi^- + ik_T\cdot\xi_T}$$

$$\langle P| \bar{\psi}_j(\xi)\, \psi_i(0)|P\rangle \Big|_{\xi^+=0}$$

$$\frac{1}{2}\mathrm{Tr}(\gamma^+\phi) = f_1 - \frac{\epsilon^{jk} k_T^j S_T^k}{M} f_{1T}^{\perp}$$

$$\frac{1}{2}\mathrm{Tr}(\gamma^+\gamma_5\phi) = S_L g_1 + \frac{\bar{k}_T\cdot\bar{S}_T}{M} g_{1T}^{\perp}$$

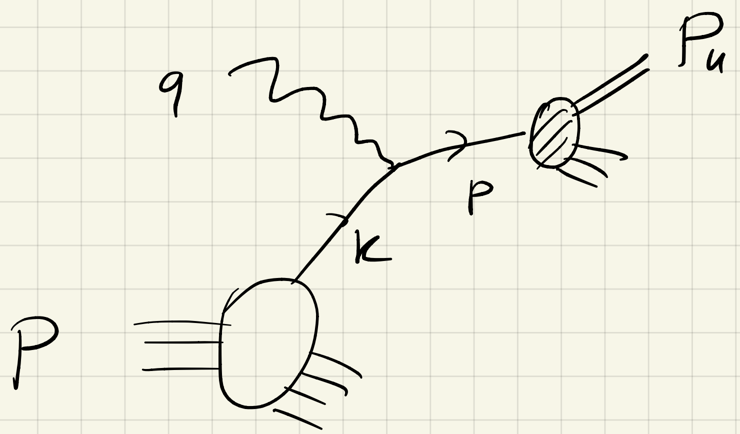
$$\frac{1}{2}\mathrm{Tr}(i\sigma^{j+}\gamma_5\phi) = S_T^j h_1 + S_L\frac{k_T^j}{M} h_{1L}^{\perp} + \frac{\kappa^{jk} S_T^k}{M} h_{1T}^{\perp}$$

$$+ \frac{\epsilon^{jk} k_T^k}{M} h_1 \quad , \text{where } \kappa^{jk} \equiv \left(k_T^j k_T^i - \frac{1}{2} k_T^2 \delta^{jk}\right)$$

$$\epsilon^{ij} \equiv \epsilon^{-+ij} \ , \ \epsilon^{0123} = +1$$

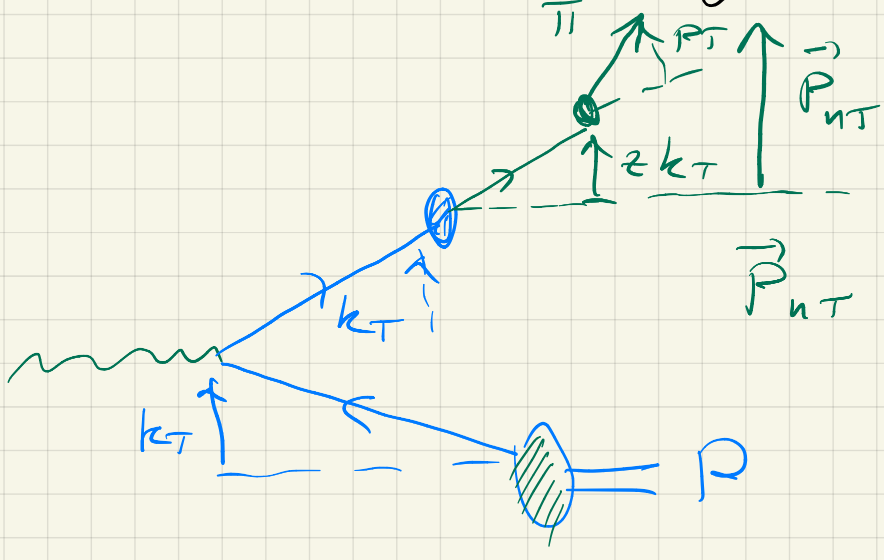
Let us work out convolutions for SIDIS

$$C[wfD] \equiv \sum_q e_a^2 \int d^2k_T \, d^2p_T \, \delta^{(2)}(\vec{P}_{hT} - z\vec{k}_T - \vec{p}_T)$$
$$w(k_T, p_T)\, f(x, k_T)\, D(z, p_T)$$



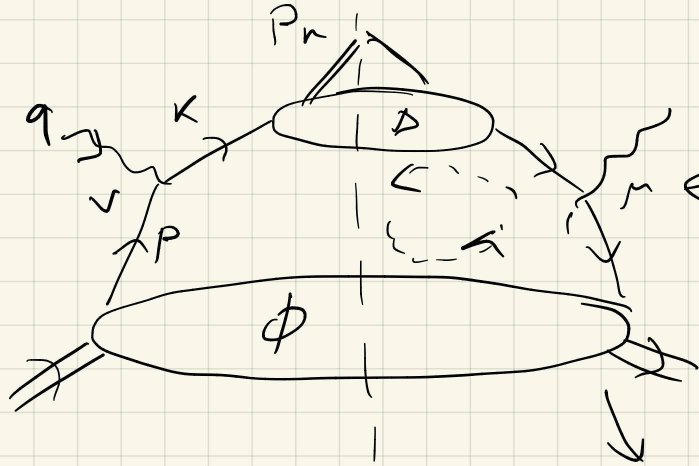
In order to study evolution we need to rewrite the convolutions in the configuration space.

We will see that TMD evolution equations are to be solved in configuration space



$$\vec{P}_{hT} = z\vec{k}_T + \vec{p}_T$$

Let us write the cut amplitude:



Starting from here we „read" the diagram similar to others in counterclockwise

$$W^{\mu\nu} = \sum_q e_q^2 \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}(\gamma^\mu \Delta \gamma^\nu \phi)\, \delta^{(4)}(k-p-q)$$

ϕ and Δ contain TMDs

$$\frac{1}{2}\mathrm{Tr}[\gamma^+\phi] = f_1 - \frac{\epsilon^{jk} p_T^j S_T^k}{M} f_{1T}^\perp$$

$$\frac{1}{2}\mathrm{Tr}[\gamma^-\Delta] = D_1$$

$$\sigma \sim \frac{1}{Q^4} L_{\mu\nu} W^{\mu\nu} \sim \sigma_0 (F_{UU} + \dots)$$

structure functions:

$F_{UU} = C[1\, f_1 D_1]$ etc

Asymmetries a ratios of polarised over unpolarised structure functions

$$A_{UT}^{\sin(\phi_h - \phi_S)} = \frac{F_{UT}^{\sin(\phi_h-\phi_S)}}{F_{UU}} \quad \text{etc}$$

Definitions: of Fourier - Bessel transform

$$f(x,k_T^2) = \int \frac{d^2 b_T}{(2\pi)^2} e^{i\bar{k}_T \bar{b}_T} \tilde{f}(x,b_T^2) =$$

$$= \int \frac{b_T\, db_T}{2\pi} J_0(k_T b_T) \tilde{f}(x,b_T^2)$$

NB $\quad \int d\varphi\, e^{i k_T b_T \cos\varphi} = 2\pi J_0(k_T b_T)$

$$\tilde{f}(x,b_T^2) = \int d^2 k_T\, e^{-i\bar{k}_T \bar{b}} f(x,k_T^2) =$$

$$= 2\pi \int k_T\, dk_T\, J_0(b_T k_T) f(x,k_T^2)$$

Using Mathematica prove these relations, and prove that if

$$f(x,k_T^2) = f_1(x) \frac{1}{\pi \langle k_T^2 \rangle} e^{-k_T^2/\langle k_T^2 \rangle}$$

then

$$\tilde{f}(x,b_T^2) = f_1(x)\, e^{-\langle k_T^2 \rangle b_T^2/4}$$

Notice that

$$\int d^2 k_T\, f(x,k_T^2) \equiv f_1(x) \leftarrow \text{The basis}$$

of the Generalized Parton Model (Feynman 78')

In TMD phenomenology the following <u>moments</u> are used

$$\int d^2k_T \, \frac{k_T^2}{2M^2} f(x, k_T^2) = f^{(1)}(x)$$

the first moment

$$\int d^2k_T \left(\frac{k_T^2}{2M^2}\right)^2 f(x, k_T^2) = f^{(2)}(x)$$

the second moment

In configuration space we have

$$\tilde{f}^{(1)}(x, b_T^2) = \frac{2\pi}{M^2} \int k_T \, dk_T \, \frac{k_T}{b_T} J_1(k_T b_T) f(x, k_T^2)$$

$$\tilde{f}^{(2)}(x, b_T^2) = \frac{4\pi}{M^4} \int k_T \, dk_T \left(\frac{k_T}{b_T}\right)^2 J_2(k_T b_T) f(x, k_T^2)$$

<u>Prove</u> that

$$\tilde{f}^{(1)}(x, 0) \equiv f^{(1)}(x)$$

$$\tilde{f}^{(2)}(x, 0) = f^{(2)}(x)$$

For fragmentation functions

$$D(z, p_T^2) = \int \frac{d^2 b_T}{(2\pi)^2} e^{i \vec{p}_T \vec{b}_T / z} \tilde{D}(z, b_T^2)$$

$$= \int \frac{b_T \, db_T}{2\pi} J_0\left(\frac{p_T b_T}{z}\right) \tilde{D}(z, b_T^2)$$

and

$$\tilde{D}(z, b_T^2) = \int \frac{d^2 p_T}{z^2} e^{-i \vec{b}_T \vec{p}_T / z} D(z, p_T^2)$$

$$= 2\pi \int \frac{p_T \, dp_T}{z^2} J_0\left(\frac{p_T b_T}{z}\right) D(z, p_T^2)$$

Test functions (check with Mathematica)

$$D(z, p_T^2) = D_1(z) \frac{1}{\pi \langle p_T^2 \rangle} e^{-p_T^2 / \langle p_T^2 \rangle}$$

$$\tilde{D}(z, b_T^2) = \frac{1}{z^2} D_1(z) \, e^{-\frac{\langle p_T^2 \rangle b_T^2}{4 z^2}}$$

Moments are also important for FFs

$$\tilde{D}^{(n)}(z, b_T^2) = \frac{2\pi n!}{(M_h^2)^n} \int \frac{p_T\, dp_T}{z^2} J_n\left(\frac{p_T b_T}{z}\right) \left(\frac{p_T}{z b_T}\right)^n D(z, p_T^2)$$

$$D(z, p_T^2) = \frac{(M_h^2)^n}{2\pi n!} \int b_T\, db_T \left(\frac{z b_T}{p_T}\right)^n J_n\left(\frac{p_T b_T}{z}\right) \tilde{D}^{(n)}(z, b_T^2)$$

$$\lim_{b_T \to 0} \tilde{D}^{(n)}(z, b_T^2) = \frac{1}{z^2} D^{(n)}(z)$$

prove it!

where

$$D^{(n)}(z) = \int d^2 p_T \left(\frac{p_T^2}{2 z^2 M_h^2}\right)^n D(z, p_T^2)$$

Test with Mathematica

$$D(z, p_T) = D_1(z) \frac{1}{\pi \langle p_T^2 \rangle} e^{-p_T^2 / \langle p_T^2 \rangle}$$

prove

$$\rightarrow \quad \tilde{D}(z, b_T^2) = \frac{1}{z^2} D_1(z)\, e^{-\frac{b_T^2 \langle p_T^2 \rangle}{4 z^2}}$$

$$D(z, b_T^2) = H_1^{\perp(1)}(z) \frac{2 z^2 M_h^2}{\pi \langle p_T^2 \rangle^2} e^{-p_T^2 / \langle p_T^2 \rangle}$$

prove

$$\rightarrow \quad \tilde{D}^{(1)}(z, b_T^2) = H_1^{\perp(1)}(z) \frac{1}{z^2} e^{-\frac{b_T^2 \langle p_T^2 \rangle}{4 z^2}}$$

Now convolutions in the momentum space $\to$ b_T space

$$C[\omega f D] \equiv \sum_a e_a^2 \int d^2k_T d^2p_T \, \delta^{(2)}(\bar{P}_{hT} - z\bar{k}_T - \bar{p}_T)$$

$$\omega(k_T b_T) f(x, k_T) D(z, p_T) \,, \qquad \bar{P}_{hT} \equiv -z\bar{q}_T$$

We rewrite $\delta^{(2)}(\bar{P}_{hT} - z\bar{k}_T - \bar{p}_T) =$

$$= \delta^{(2)}(-z\bar{q}_T - z\bar{k}_T - \bar{p}_T) = \frac{1}{z^2}\delta^{(2)}\left(\bar{q}_T + \bar{k}_T + \frac{\bar{p}_T}{z}\right)$$

$$= \frac{1}{z^2}\int \frac{d^2b_T}{(2\pi)^2} e^{-i(\bar{q}_T + \bar{k}_T + \bar{p}_T/z)\bar{b}_T}$$

$$F_{UU} = C[1 f_1 D_1] = \sum_a e_a^2 \int \frac{d^2b_T}{(2\pi)^2} e^{-i\bar{b}_T\bar{q}_T}$$

$$\cdot \int d^2k_T \, e^{-i\bar{b}_T\bar{k}_T} f_1(x, k_T^2)$$

$$\cdot \frac{1}{z^2}\int d^2p_T \, e^{-i\bar{b}_T\bar{p}_T/z} D_1(z, p_T^2) =$$

$$= \sum_a e_a^2 \int \frac{d^2b_T}{(2\pi)^2} e^{-i\bar{b}_T\bar{q}_T} \tilde{f}_1(x, b_T^2)\tilde{D}_1(z, b_T^2)$$

$$= \sum_a e_a^2 \int \frac{b_T db_T}{2\pi} J_0(b_T q_T) \tilde{f}_1(x, b_T^2)\tilde{D}_1(z, b_T^2)$$

Let us call $\mathcal{B}(\tilde{f}_1 \tilde{D}_1) \equiv \sum_a e_a^2 \int \frac{b_T db_T}{2\pi} J_0(b_T q_T)\tilde{f}\tilde{D}$

A more complicated example, Sivers asymmetry

$$F_{UT}^{\sin(\phi_h-\phi_S)} = \mathcal{C}\left[-\frac{\hat{h}\cdot\bar{k}_T}{M} f_{1T}^{\perp} D_1\right] \qquad \hat{h} = \frac{\bar{P}_{hT}}{|P_{hT}|}$$

$$= \sum_q e_q^2 \int d^2k_T\, d^2p_T\, \delta^{(2)}(\bar{P}_{hT} - z\bar{k}_T - \bar{p}_T) \underbrace{\left(-\frac{\hat{h}\cdot\bar{k}_T}{M}\right)}_{-\frac{k_T}{M}\cos(\varphi-\phi_h)}$$

$$\cdot f_{1T}^{\perp}(x, k_T^2)\, D_1(z, p_T^2)$$

$$= \sum_q e_q^2 \int \frac{d^2b_T}{(2\pi)^2} e^{-i\bar{q}_T\bar{b}_T} \int d^2k_T \left(-\frac{k_T}{M}\right)\cos(\varphi-\phi_h)$$

$$e^{-i\bar{k}_T\bar{b}_T} f_{1T}^{\perp}(x, k_T^2) \underbrace{\int d^2p_T \frac{1}{z^2} e^{-i\bar{p}_T\bar{b}_T/z} D_1(z, p_T^2)}_{\tilde{D}_1(z, b_T^2)}$$

we have $\bar{k}_T\bar{b}_T = b_T k_T \cos(\varphi - \varphi_b)$

$$\int d\varphi\, e^{-ib_Tk_T\cos(\varphi-\varphi_b)} \cos(\varphi-\phi_h) = \Big\}$$

$$= -2\pi i\, J_1(b_Tk_T)\cos(\varphi_b - \phi_h)$$

Use Mathematica

$$\int d\varphi_b \cos(\varphi_b-\phi_h)\, e^{-ib_Tq_T\cos(\varphi_b-\phi_h)} = \Big\}$$

$$= 2\pi i\, J_1(b_Tq_T)$$

So that

$$F_{UT}^{\sin(\phi_h-\phi_S)} = -\sum_q e_a^2 \int \frac{db_T\, b_T}{2\pi} J_1(b_T q_T)$$

$$\int dk_T \frac{k_T^2}{M} J_1(b_T k_T) f_{1T}^{\perp}(x,k_T^2)\ \tilde{D}_1(z,b_T^2)$$

$$\tilde{f}_{1T}^{\perp(1)}(x,b_T^2) = \frac{2\pi}{M^2}\int k_T\, dk_T \frac{k_T}{b_T} J_1(k_T b_T) f_{1T}^{\perp}(x,k_T^2)$$

Thus

$$F_{UT}^{\sin(\phi_h-\phi_S)} = (-M)\sum_q e_a^2 \int \frac{b_T\, db_T}{2\pi} \underbrace{b_T J_1(b_T q_T)}_{b_T^{1} J_{1}(b_T q_T)}$$

$$\cdot\ \tilde{f}_{1T}^{\perp(1)}(x,b_T^2)\ \tilde{D}_1(z,b_T^2)$$

$$F_{UT}^{\sin(\phi_h-\phi_S)} = -M\ \mathcal{B}\left[\tilde{f}_{1T}^{\perp(1)}\tilde{D}_1\right]$$

General definition

$$\mathcal{B}\left[\tilde{f}^{(n)}\tilde{D}^{(m)}\right] = \int \frac{b_T\, db_T}{2\pi} b_T^{n+m} J_{n+m}(b_T q_T)$$

$$\tilde{f}^{(n)}(x,b_T^2)\ \tilde{D}^{(m)}(z,b_T^2)$$

Collins asymmetry

$$F_{UT}^{\sin(\phi_h+\phi_S)} \equiv \mathcal{C}\left[w_A^{\{1\}} h_1 H_1^{\perp}\right]$$

$$w_A^{\{1\}} \equiv \frac{\hat{h}\cdot \vec{p}_T}{z M_h}$$

$$F_{UT}^{\sin(\phi_h+\phi_S)} = M_h\, \mathcal{B}\left[\tilde{h}_1^{(0)} \tilde{H}_1^{\perp(1)}\right]$$

let us use (0) moment as well for completeness

Prove it!

Evolution of TMDs

The evolution is customarily studied for TMDs in the coordinate space b_T

$$\sigma \propto \int \frac{d^2 b_T}{(2\pi)^2} e^{-iq_T b_T} \tilde{f}(x, b_T, Q, \zeta)\, \tilde{D}(z, b_T, Q, \zeta)$$

Q corresponds to the UV divergence, the same as for collinear densities

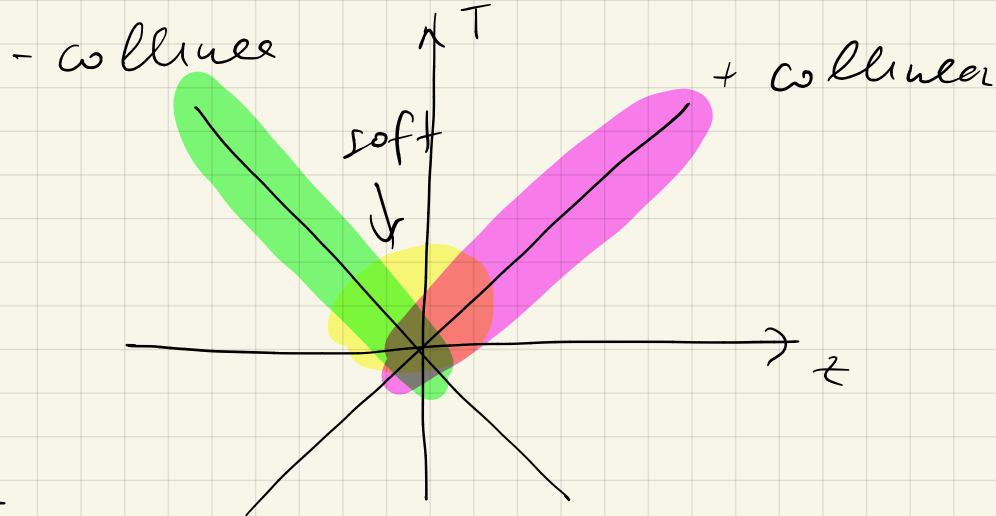
ζ is a new scale that correspond to a new type of divergens „rapidity" divergence of TMDs

Let us define momentum regions

1) Hard region - momentum with large virtualities $\sim Q$, $k \sim Q(\underset{+}{1}, \underset{-}{1}, \underset{T}{1})$

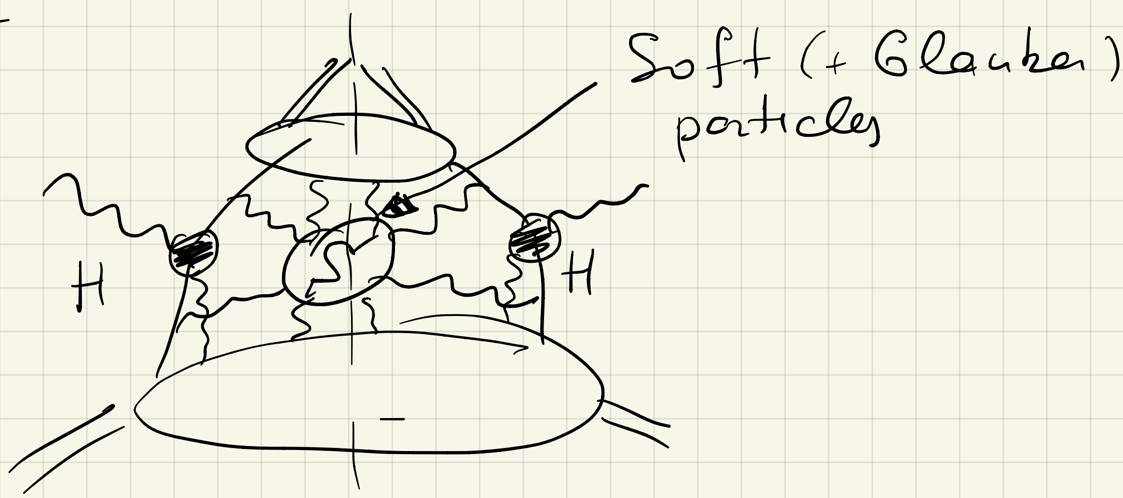
2) Collinear region - momentum close to some beam/jet directions $k \sim Q(1, \lambda^2, \lambda)$ for example

3) Central (soft) region $k \sim Q(\lambda^n, \lambda^n, \lambda^n)$ $n > 0$
$\lambda \ll 1$



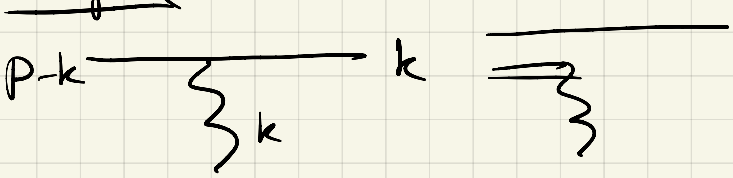
For each region approximations are applied and then double counting is subtracted

Result



Some work is still needed to fully factorise. Ward identities are used to stripe collinear polarized gluons from hard part and organise them into Wilson lines

Simple



$$(p-k)^2 = -2p\cdot k + k^2 \simeq \underbrace{-2p\cdot k}_{\text{eikonal approximation}}$$

TMD factorization describes processes differential in transverse momentum $\frac{d\sigma}{dq_T^2}$

$q_T \sim \Lambda_{QCD} << Q$: small k_T of partons play an important role at small q_T => TMD factorization with TMDs $F(x, k_T)$

Generalized Parton Model

$F(x, k_T)$

$\downarrow$ QCD

$F(x, k_T, \mu, \zeta) \rightarrow$ uniquely defined
deal with all divergencies
obey evolution equations

QCD evolution is governed by the so-called Collins-Soper equation two Renormalization Group equations

<u>CS equation</u>

b_T space

$$(1) \quad \frac{\partial \ln \tilde{F}(x, b_T, \mu, \zeta)}{\partial \ln \sqrt{\zeta}} = \tilde{K}(b_T, \mu)$$

CS kernel

<u>RG equations</u>

$$(2) \quad \frac{d \tilde{K}(b_T, \mu)}{d \ln \mu} = -\gamma_K(g(\mu))$$

← Cusp anomalous dimention of K

Very universal in QCD., γ_K depend only on μ

$$(3) \quad \frac{d \ln \tilde{F}(x, b_T, \mu, \zeta)}{d \ln \mu} = \gamma_F(g(\mu), \zeta/\mu^2)$$

anomalous dimention of F.

$$\frac{d}{d\ln\mu}\left(\frac{\partial \ln F(x, b_T, \mu, \zeta)}{\partial \ln\sqrt{\zeta}}\right) = \frac{d}{d\ln\mu}\tilde{K}(b_T; \mu) = -\gamma_K(\mu)$$

$$\frac{\partial}{\partial\ln\sqrt{\zeta}}\left(\frac{d\ln F(x, b_T, \mu, \zeta)}{d\ln\mu}\right) = -\gamma_K(\mu)$$

$$\gamma_F(\mu, \zeta/\mu^2)$$

$$\gamma_F(\mu, \zeta_0/\mu^2) - \gamma_F(\mu, \zeta/\mu^2) = -\gamma_K(\mu)\ln\sqrt{\zeta_0} + \gamma_K \ln\sqrt{\zeta}$$

if $\zeta_0 = \mu^2$ then

$$\underline{\gamma_F(\mu, \zeta/\mu^2) = \gamma_F(\mu, 1) - \frac{1}{2}\gamma_K(\mu)\ln \zeta/\mu^2}$$

Solutions

1) $$\frac{d\tilde{K}(b_T,\mu)}{d\ln\mu} = -\gamma_K(\mu) \Rightarrow$$

$$\int_{\mu_0}^{\mu} d\tilde{K}(b_T,\mu) = -\int_{\mu_0}^{\mu}\gamma_K(\mu')\frac{d\mu'}{\mu'}$$

$$\tilde{K}(b_T,\mu) = -\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\gamma_K(\mu') + \tilde{K}(b_T,\mu_0)$$

2) $$\tilde{F}(x,b_T,\mu,\zeta) = \tilde{F}(x,b_T,\mu,\zeta_0)\exp\left[\tilde{K}(b_T,\mu)\ln\sqrt{\frac{\zeta}{\zeta_0}}\right]$$

3) $$\tilde{F}(x,b_T,\mu_0,\zeta) = \tilde{F}(x,b_T,\mu_0,\zeta)\exp\left[\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\gamma_F(\mu';\zeta/\mu'^2)\right]$$

Implementing evolution

We start with low b_T

$$\tilde{F}_f(x, b_T, \mu, \zeta) = \sum_j \int_x \frac{d\hat{x}}{\hat{x}} \underbrace{\tilde{C}_{j/f}\left(\frac{x}{\hat{x}}, b_T, \mu, \zeta\right)}_{\text{coefficient functions}} \underbrace{f_j(\hat{x}, \mu)}_{\text{collinear pdfs}}$$

at the lowest order

$$\tilde{C}_{j/f} = \delta_{jf}\, \delta\left(\frac{x}{\hat{x}} - 1\right)$$

Next step: combine perturbative & non perturbative $\to b_*$

Problem: $\tilde{K}(b_T)$, $\tilde{F}(b_T)$ are non perturbative at large b_T

We want: write functions such that they are perturbatively calculable with non perturb. corrections

$$b_*(b_T) = \frac{b_T}{\sqrt{1 + b_T^2/b_{max}^2}}$$

$$\tilde{K}(b_T,\mu) = \tilde{K}(b_*,\mu) + \underbrace{\left[\tilde{K}(b_T,\mu) - \tilde{K}(b_*,\mu)\right]}_{g_K(b_T)}$$

non pert. function

$$\underbrace{\tilde{K}(b_T,\mu) = \tilde{K}(b_*,\mu_0) - \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'}\gamma_K(\mu') - g_K(b_T)}_{\text{study in Mathematica}}$$

$$\underbrace{g_K = \tfrac{1}{2} g_2 b_T^2, \quad g_K = g_0 b_T b_*, \quad g_K = g_2 \ln b_T/b_*}_{\text{different groups}}$$

We can choose $\mu_0 \sim 1/b_T$, $\mu_0 = \dfrac{2e^{-\gamma_E}}{b_*}$ is the standard choice

Euler gamma

$$\tilde{F}(x,b_T,\mu,\zeta) = \tilde{F}(x,b_*,\mu,\zeta)\left[\frac{\tilde{F}(x,b_T,\mu,\zeta)}{\tilde{F}(x,b_*,\mu,\zeta)}\right] =$$

$$= \tilde{F}(x,b_*,\mu,\zeta_0)\exp\left[\tilde{K}(b_*,\mu)\ln\sqrt{\frac{\zeta}{\zeta_0}}\right]\underbrace{\left[\frac{\tilde{F}(x,b_T,\mu,\zeta_0)}{\tilde{F}(x,b_*,\mu,\zeta_0)}\right]}_{\exp[-g(x,b_T)]}$$

$$\times \exp\left[\ln\sqrt{\frac{\zeta}{\zeta_0}}\left(\tilde{K}(b_T,\mu)-\tilde{K}(b_*,\mu)\right)\right]$$

$$= \tilde{F}(x,b_*,\mu,\zeta_0)\exp\left[\ln\sqrt{\frac{\zeta}{\zeta_0}}\,\tilde{K}(b_*,\mu)\exp\left[-g(x,b_T)-\ln\sqrt{\frac{\zeta}{\zeta_0}}\,g_K(b_T)\right]\right.$$

$$= \tilde{F}(x,b_*,\mu_0,\zeta_0)\exp\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\left(\gamma_F(\mu';1)-\ln\sqrt{\frac{\zeta_0}{\mu'^2}}\,\gamma_K(\mu')\right)$$

$$\exp\left[\ln\sqrt{\frac{\zeta}{\zeta_0}}\,\tilde{K}(b_*,\mu_0)-\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\ln\sqrt{\frac{\zeta}{\zeta_0}}\,\gamma_K(\mu')\right]$$

$$\exp\left[-g(x,b_T)-\ln\sqrt{\frac{\zeta}{\zeta_0}}\,g_K(b_T)\right]$$

$$= \tilde{F}(x, b_*, \mu_0, \zeta_0) \exp\left\{ \ln\sqrt{\frac{\zeta}{\zeta_0}}\, \tilde{K}(b_*, \mu_0) + \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \left[\gamma_F(\mu', 1) - \ln\sqrt{\frac{\zeta}{\mu'^2}}\, \gamma_K(\mu') \right] \right\}$$

$$\times \exp\left[-g(x, b_T) - \ln\sqrt{\frac{\zeta}{\zeta_0}}\, g_K(b_T) \right]$$

at small b_T and large μ:

$$\tilde{K}(b_T, \mu) = -\frac{g^2 C_F}{4\pi^2} \left(\ln(\mu^2 b_T) - \underbrace{\ln 4 + 2\gamma_E}_{\text{the reason why}} \right)$$

$$\mu_b \sim \frac{2e^{-\gamma_E}}{b_*} \text{ is chosen}$$

$$\tilde{K}(b_*, \mu_b) = 0$$

ζ_0 is a scale $\sim 1 - 2\ (GeV^2)$

Generalized parton model usually is.

$$\tilde{F}(x, b_T) \sim F(x) \exp\left[-g(x, b_T)\right].$$

<u>Lecture 3</u>

Elements of evolution of TMDs

We have studied so far how structure function can be written in terms of TMDs.

For instance

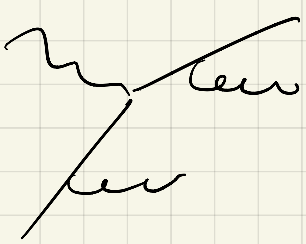
$$F_{UU} = \mathcal{C}\left[1\, f_1\, D_1\right]$$

In the Generalized Parton Model one often uses

$$f_1(x, k_T) = f_1(x)\, \frac{1}{\pi \langle k_T^2 \rangle}\, e^{-k_T^2/\langle k_T^2 \rangle}$$

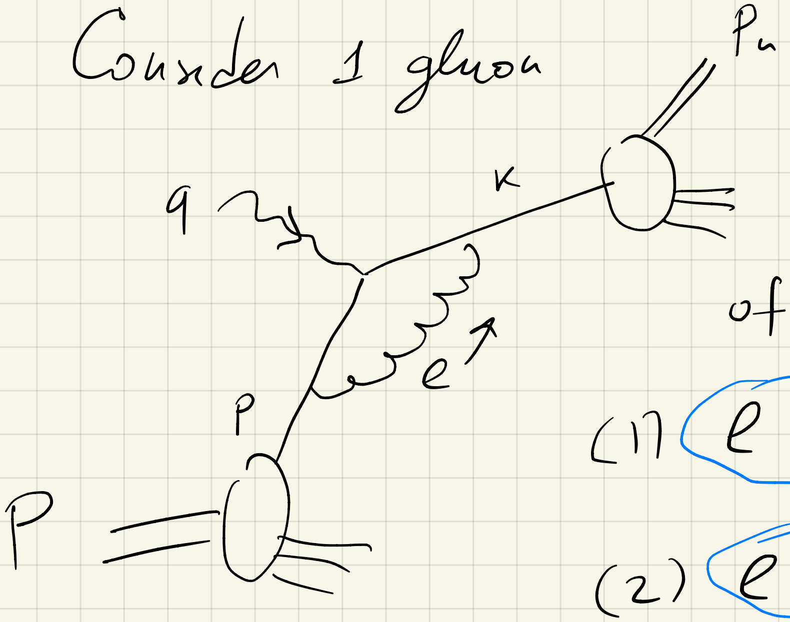
$$D_1(z, p_T) = D_1(z)\, \frac{1}{\pi \langle p_T^2 \rangle}\, e^{-p_T^2/\langle p_T^2 \rangle}$$

Of course this Gaussian dependence can be a good approximation of intrinsic k_T dependence but what happens if we take into account gluon radiation?



?

Consider 1 gluon



$$\int \frac{d^4 l}{(2\pi)^4} \to \text{factorisation}$$

$\lambda \ll 1$

of regions

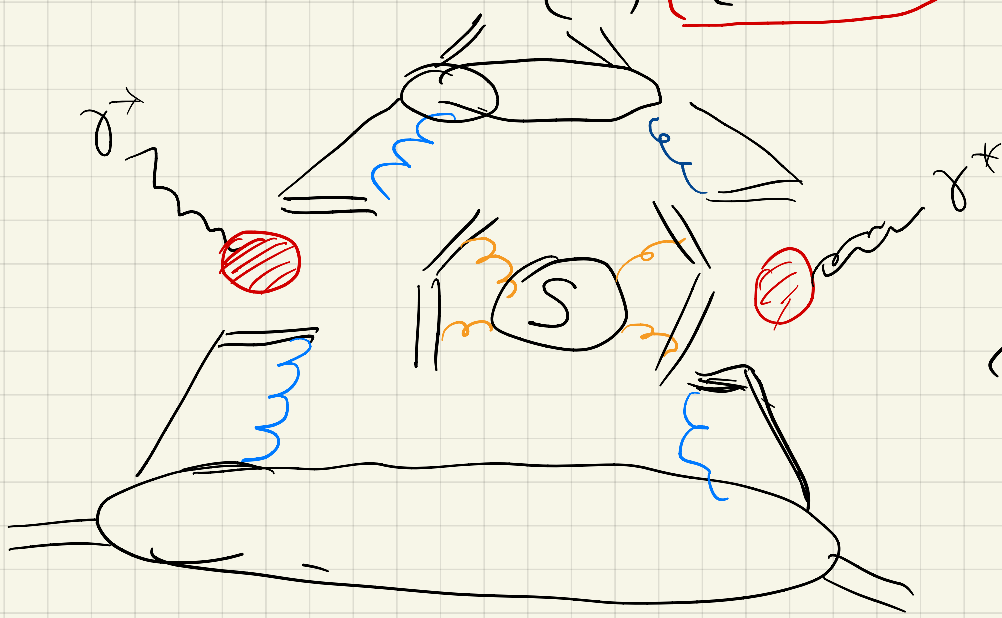
$\qquad\qquad\qquad\qquad\qquad\quad + \quad - \quad T$

(1) $l \parallel p$ $\quad l \sim Q(1, \lambda^2, \lambda)$

(2) $l \parallel k$ $\quad l \sim Q(\lambda^2, 1, \lambda)$

(3) l soft $\quad l \sim Q(\lambda, \lambda, \lambda)$

(4) l hard $\quad l \sim Q(1, 1, 1)$



$=$ are Wilson lines (I will not talk about them here)

$$\sigma \sim \int d^2k_T\, d^2p_T\, d^2l_T\, H(Q)\, f(x, k_T)\, D(z, p_T)$$

$$\cdot\, S(l_T)\, \delta^{(2)}(\vec{P}_{hT} - z\vec{k}_T - \vec{p}_T - \underbrace{\vec{l}_T}_{\text{gluon radiation}})$$

$$G \sim \underbrace{\int \frac{d^2 b_T}{(2\pi)^2} e^{i \vec{P}_{hT} \vec{b}_T / z} H(Q) \tilde{f}(x, b_T) \tilde{D}(z, b_T) S(b_T)}_{}$$

See next lecture for the proof

The additional factor $S(b_T)$ is absorbed into $\tilde{f}$ & $\tilde{D}$. It leads to cancelation of divergencies and self consistent definition

$$\tilde{f}(x, b_T, Q, \zeta) \to \tilde{f}(x, b_T) \sqrt{S(b_T)}$$

$$\tilde{D}(z, b_T, Q, \zeta) \to \tilde{D}(z, b_T) \underbrace{\sqrt{S(b_T)}}_{}$$

UV scale $\quad$ rapidity scale $\quad$ effect of radiation

$$G \sim H(Q) \int \frac{d^2 b_T}{(2\pi)^2} e^{i \vec{P}_{hT} \vec{b}_T / z} \underbrace{\tilde{f}(x, b_T, Q, \zeta) \tilde{D}(z, b_T, Q, \zeta)}_{}$$

exactly like in Generalized parton model!

We would like to write

$\tilde{f}(x, b_T, Q, \zeta)$ starting from some initial scales Q_0, ζ_0

QCD evolution of TMDs is governed by 3 equations

1) Collins-Soper equation (CS)

$$\frac{\partial \ln \tilde{f}(x, b_T, \mu, \zeta)}{\partial \ln \sqrt{\zeta}} = \tilde{K}(b_T, \mu)$$

$\tilde{K}$ is the so-called Collins-Soper kernel

it can be calculated perturbatively for small b_T & large μ (so that $\alpha_s(\mu)$ is small)

$$\tilde{K}(b_T, \mu) = -8 \cdot C_F \frac{\alpha_s(\mu)}{4\pi} \ln\left(\frac{b_T \mu}{2e^{-\gamma_E}}\right) + O(\alpha_s^2)$$

$\gamma_E \simeq 0.57$ Euler constant

The problem:

We need to $\int \frac{d^2 b_T}{(2\pi)^2} \to \int_0^\infty b_T \, db_T$

but $\ln\left(\frac{b_T \mu}{2e^{-\gamma_E}}\right)$ will become large for $b_T \to \infty$

it corresponds to non-perturbative regime of $k_T \to 0$.

Solution later

② Renormalisation group equation

$$\frac{d\tilde{K}(b_T,\mu)}{d\ln\mu} = -\gamma_K(\alpha_s(\mu))$$

γ_K is Cusp anomalous dimention. It is present in many areas of physics

$$\gamma_K(\alpha_s) = \sum_{i=1}^{\infty} \gamma_K^i \left(\frac{\alpha_s}{4\pi}\right)^i = 8C_F\left(\frac{\alpha_s}{4\pi}\right) + \mathcal{O}(\alpha_s^2)$$

③ $$\frac{d\ln\tilde{f}(x,b_T,\mu,\zeta)}{d\ln\mu} = \gamma_F(\alpha_s(\mu), \zeta/\mu^2)$$

γ_F is the anomalous dimention of f

$$\gamma_F(\alpha_s(\mu),1) = \sum_{i=1}^{\infty} \gamma_F^i \left(\frac{\alpha_s}{4\pi}\right)^i = 6C_F\left(\frac{\alpha_s(\mu)}{4\pi}\right) + \mathcal{O}(\alpha_s^2)$$

Let us write the solutions:

$$\frac{d \ln \tilde{f}(x, b_T, \mu, \zeta)}{d \ln \mu} = \gamma_F(\mu, \zeta/\mu^2)$$

$$\int_{\mu_0}^{\mu} d \ln \tilde{f}(x, b_T, \mu, \zeta) = \int_{\mu_0}^{\mu} \gamma_F(\mu', \zeta/\mu'^2) \frac{d\mu'}{\mu'}$$

$$\frac{\tilde{f}(x, b_T, \mu, \zeta)}{\tilde{f}(x, b_T, \mu_0, \zeta)} = \exp\left[\int_{\mu_0}^{\mu} \gamma_F(\mu', \zeta/\mu'^2) \frac{d\mu'}{\mu'} \right]$$

$$\tilde{f}(x, b_T, \mu, \zeta) = \tilde{f}(x, b_T, \mu_0, \zeta) \exp\left[\int_{\mu_0}^{\mu} \gamma_F(\mu', \zeta/\mu'^2) \frac{d\mu'}{\mu'} \right]$$

(2) $$\frac{\partial \ln \tilde{f}(x, b_T, \mu, \zeta)}{\partial \ln \sqrt{\zeta}} = \tilde{K}(b_T, \mu)$$

$$\Rightarrow \quad \tilde{f}(x, b_T, \mu, \zeta) = f(x, b_T, \mu, \zeta_0) \exp\left[\tilde{K}(b_T, \mu) \ln \sqrt{\frac{\zeta}{\zeta_0}}\right]$$

(3) $$\frac{d \tilde{K}(b_T, \mu)}{d \ln \mu} = -\gamma_K(\mu)$$

$$\Rightarrow \quad \tilde{K}(b_T, \mu) = \tilde{K}(b_T, \mu_0) - \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \gamma_K(\mu')$$

Let us also combine 2 equations

$$\frac{d}{d\ln\mu}\left(\frac{\partial \ln \tilde{f}(x,b_T,\mu,\zeta)}{\partial \ln\sqrt{\zeta}}\right) = \frac{d}{d\ln\mu}\tilde{K}(b_T,\mu) = -\gamma_K(\mu)$$

those commute!

$$\frac{\partial}{\partial \ln\sqrt{\zeta}}\Bigg(\underbrace{\frac{d\ln\tilde{f}(x,b_T,\mu,\zeta)}{d\ln\mu}}_{\gamma_F(\mu,\zeta/\mu^2)}\Bigg) = -\gamma_K(\mu)$$

$$\Rightarrow \quad \gamma_F(\mu,\zeta/\mu^2) - \gamma_F(\mu,\zeta_0/\mu^2) = -\gamma_K(\mu)\ln\sqrt{\zeta/\zeta_0}$$

If we use $\zeta_0=\mu^2$ then

$$\gamma_F(\mu,\zeta/\mu^2) = \gamma_F(\mu,1) - \tfrac{1}{2}\gamma_K(\mu)\ln(\zeta/\mu^2)$$

Implementing the evolution

1) Operator Product Expantion (OPE)
at low b_T:

$$\tilde{f}_f(x, b_T, \mu, \zeta) = \sum_j \int_x^1 \frac{d\hat{x}}{\hat{x}} \underbrace{\tilde{C}_{j/f}\left(\frac{x}{\hat{x}}, b_T, \mu, \zeta\right)}_{\text{coefficient functions}} \underbrace{f_j(\hat{x}, \mu)}_{\text{collinear pdfs for unpol. f}} + O(b_T^2)$$

$$\tilde{C}_{j/f} = \delta_{jf}\, \delta\left(\frac{x}{\hat{x}} - 1\right) + O(\alpha_s^2)$$

2) Combine perturbative & non perturbative
(solution to problem of $\tilde{K}(b_T)$ nonpert @ large b_T)

$$b_* = \frac{b_T}{\sqrt{1 + b_T^2/b_{max}^2}}$$

if b_{max} is small (~ 1 GeV^{-1}) then
b_* is always perturbative for $\forall\, b_T$

Start from CS kernel

$$\tilde{K}(b_T,\mu) = \tilde{K}(b_*,\mu) + \underbrace{\tilde{K}(b_T,\mu) - \tilde{K}(b_*,\mu)}_{g_K(b_T)}$$

universal non pert. function!

g_K does not depend on μ, in fact

$$\frac{d}{d\ln\mu}\left[\tilde{K}(b_T,\mu) - \tilde{K}(b_*,\mu)\right] = \gamma_K(\mu) - \gamma_K(\mu) = \emptyset$$

next

$$\tilde{K}(b_T,\mu) = \tilde{K}(b_*,\mu_0) - \int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\gamma_F(\mu') - g_K(b_T)$$

For convergence, remember,

$$\tilde{K} = -8 C_F \frac{\alpha_s(\mu)}{4\pi}\ln\left(\frac{b_T\mu}{2e^{-\gamma_E}}\right)$$

$$\mu_0 = \frac{2e^{-\gamma_E}}{b_T} \equiv \mu_b \quad \text{but at large } b_T \to \infty$$

$$\mu_b \to 0 \Rightarrow \alpha_s(0) \to \infty \quad \underline{\text{Landau pole}}$$

Because the function is non perturbative.

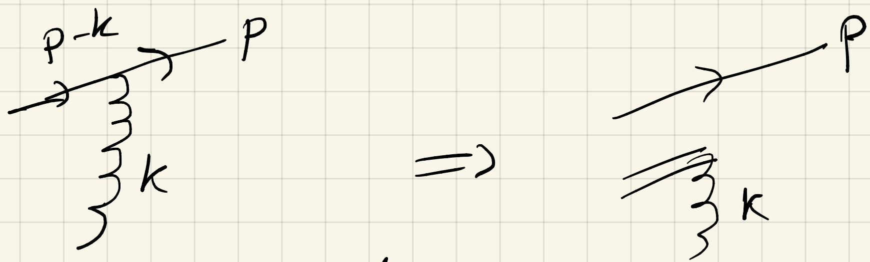
To avoid it $\mu_b = \frac{2e^{-\gamma_E}}{b_*} \to \frac{2e^{-\gamma_E}}{b_{max}} \gg \Lambda_{QCD}$

Study in Mathematica

$$\tilde{K}(b_T, \mu) = \tilde{K}(b_*, \mu_b) - \int_{\mu_b}^{\mu} \frac{d\mu'}{\mu'} \gamma_K(\mu') - g_K(b)$$

for a realistic $g_K(b) = g_0 . \ln(b/b_*)$

The origin of Wilson lines:



Eikonal approximation

$$(p-k)^2 \simeq -2p\cdot k + k^2 \simeq -2p\cdot k \simeq -2p^- k^+$$

$$i\,\frac{(\not p - \not k)}{(p-k)^2+i\epsilon} \simeq i\,\frac{p^-\gamma^+}{-2p^-k^+ + i\epsilon} = \frac{i}{2}\,\frac{\gamma^+}{-k^+ + i\epsilon}$$

p^- should be ≥ 0, this is true as it goes through the final state cut

$\langle P|\bar\psi(\xi)\,\psi(0)\,|P\rangle$ gauge transf:

$\bar\psi(\xi) \to \bar\psi(\xi)\,\mathcal{U}^+(\xi)\,,\ \psi(0) \to \psi(0)\,\mathcal{U}(0)$

$\Rightarrow \langle P|\,\bar\psi(\xi)\,\mathcal{U}^+(\xi)\,\mathcal{U}(0)\,|P\rangle$

Wilson line

$$W(\xi,0) = \mathcal{P}\exp\left[+ig\int_0^\xi dz\cdot A(z)\right]$$

$W(\xi,0) \to \mathcal{U}(\xi)\,W(\xi,0)\,\mathcal{U}^+(0)$

$\Rightarrow \bar\psi(\xi)\,W(\xi,0)\,\psi(0) \to \bar\psi\,\underbrace{\mathcal{U}^+(\xi)\,\mathcal{U}(\xi)}_{1}\,W(\xi,0)\,\underbrace{\mathcal{U}^+(0)\,\mathcal{U}(0)}_{1}\psi(0)$

gauge inv.!

Now let us write $f(x, b_T, \mu, \zeta)$ in terms of $f(x, b_T, \mu_0, \zeta_0)$:

$$\tilde{f}(x, b_T, \mu, \zeta) = \tilde{f}(x, b_*, \mu, \zeta)\left[\frac{\tilde{f}(x, b_T, \mu, \zeta)}{\tilde{f}(x, b_*, \mu, \zeta}\right] =$$

$$= \tilde{f}(x, b_*, \mu, \zeta_0) \exp\left[\tilde{K}(b_*, \mu) \ln\sqrt{\frac{\zeta}{\zeta_0}}\right] \underbrace{\left[\frac{\tilde{f}(x, b_T, \mu, \zeta_0)}{\tilde{f}(x, b_*, \mu, \zeta_0)}\right]}_{\exp[-g(x, b_T)]}$$

$$\cdot \exp\left[\ln\sqrt{\frac{\zeta}{\zeta_0}} \underbrace{(\tilde{K}(b_T, \mu) - \tilde{K}(b_*, \mu))}_{g_K(b_T)}\right]$$

up behav. of TMD

$$= \tilde{f}(x, b_*, \mu_0, \zeta_0) \exp\left[\int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \left(\gamma_F(\mu'; 1) - \ln\sqrt{\frac{\zeta_0}{\mu'^2}}\, \gamma_K(\mu')\right)\right]$$

$$\cdot \exp\left[\ln\sqrt{\frac{\zeta}{\zeta_0}}\, \tilde{K}(b_*, \mu_0) - \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \ln\sqrt{\frac{\zeta}{\zeta_0}}\, \gamma_K(\mu')\right]$$

$$\cdot \exp\left[-g(x, b_T) - \ln\sqrt{\frac{\zeta}{\zeta_0}}\, g_K(b_T)\right]$$

$$= \tilde{f}(x, b_*, \mu_0, \zeta_0) \exp\left[\ln\sqrt{\frac{\zeta}{\zeta_0}}\, \tilde{K}(b_*, \mu_0) + \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \left[\gamma_F(\mu'; 1) - \ln\sqrt{\frac{\zeta}{\mu'^2}}\, \gamma_K(\mu')\right]\right]$$

$$\cdot \exp\left[-g(x, b_T) - \ln\sqrt{\frac{\zeta}{\zeta_0}}\, g_K(b_T)\right]$$

Let us use $\mu_0 = \mu_b = \dfrac{2e^{-\gamma_E}}{b_*}$

$$\zeta_0 = Q_0^2 \sim 1-2 \, (\text{GeV}^2)$$

Then: , $\zeta = Q^2$ (the scale)

$$\tilde{f}(x, b_T, Q, Q^2) = \tilde{f}(x, b_*, \mu_b, Q_0^2) \left(\frac{Q}{Q_0}\right)^{\tilde{K}(b_*, \mu_b) - g_K(b)} \cdot \exp\left[\int_{\mu_b}^{\mu} \frac{d\mu'}{\mu'} \left(\gamma_F(\mu', 1) - \ln \frac{Q}{\mu'} \gamma_K(\mu')\right)\right] \cdot \exp\left[-g(x, b_T)\right]$$

$\downarrow$

Sadakov form factor

$\exp[S]$

Contains result of gluon radiation

$$\underbrace{\tilde{f}(x, b_T, Q, Q^2) = \tilde{f}(x, b_*, \mu_b, Q_0^2)\, e^{-g(x, b_T)} e^{S}}_{\text{almost like GPM!}}$$

$$\simeq \tilde{f}(x, \mu_b)\, e^{-g(x, b_T)} e^{S} \quad \text{if} \begin{cases} g(x, b_T) \simeq \dfrac{b_T^2 \langle k_T^2 \rangle}{4} \\ S \simeq 0 \end{cases}$$

study in Mathematica.

We will use Mathematica to study how it differs from GPM at higher scales.