

# CFNS 24

Lectures at the  
2024 CFNS Summer School  
on the physics of the EIC



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1. What is the 3D structure?

"What?"

2. Why the 3D structure is interesting?

"Why?"

3. Where can we study the 3D structure?

"Where?"

We have a group of theorists and experimentalists and I will try to give something important and interesting to all.

I will give you arguments and some will be more hand waving, others more rigorous.

Do not be too concerned by them it is physics and the mathematical rigor is not always a guarantee of correctness.

Experimental tests are the key!

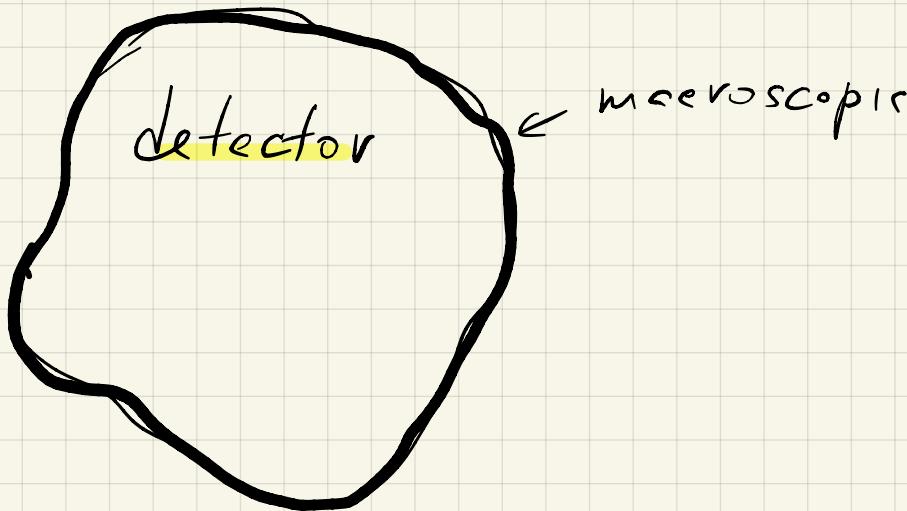
## Questions for the discussion:

1. What is the advantage of the lepton scattering?
2. What are the advantages and disadvantages of a collider with respect to a fixed target experiment?
3. What is the role of "inclusivity" in the 3D structure measurements? Consider pdfs or TMDs.
4. Consider parton model. Could you motivate the physical picture that the scattering happens on almost on-shell particles?
5. Consider parton model. Motivate the usage of coherent  $\sum_a e_a^2$  vs incoherent scattering  $\sum_{q,q'} e_q e_{q'}$ .

Let us start with „What?”

We choose language of Quantum Mechanics (QM) and generically divide the „world” in two categories: system we study, and the observer: detector

system  
microscopic



The system is then described by a wave function

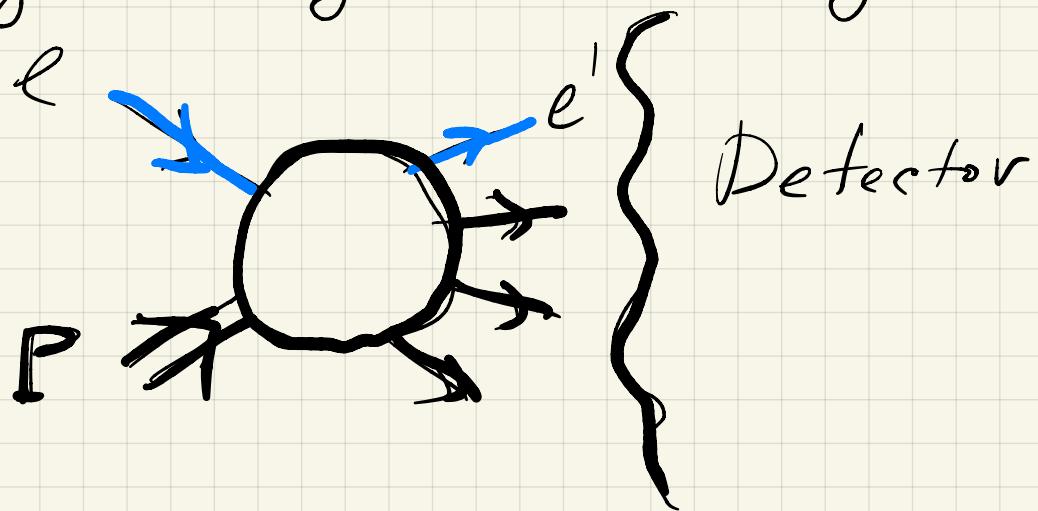
$\Psi$  it evolves according to Schrödinger equation smooth and continuous. Once the measurement is performed, we switch to Born rules and probabilities  $|\Psi|^2$

Compared to classical systems, we can ask approximately a half questions about the system.

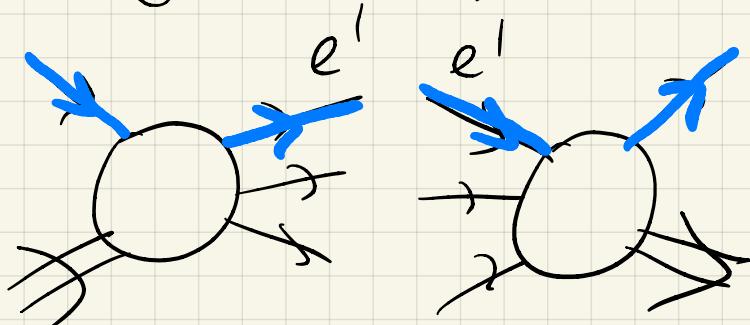
For the proton QM is not enough; we use the language of Quantum Field Theory and use field operators  $\psi$  acting on the state vectors  $|P\rangle$ . (Dirac, Feynman, etc)

The number of constituents is not constant, it is changing.

The measurement is performed by a detector, say we study  $eP$  scattering



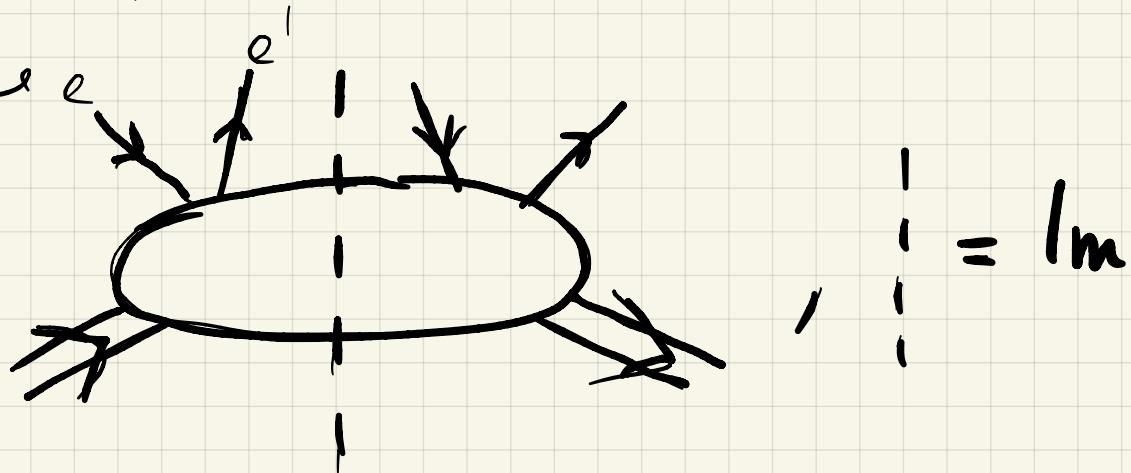
The same process in the mind of the theorist, as you see in George Sterman's lecture, looks differently  $\rightarrow$  we square it " $|\psi|^2$ "



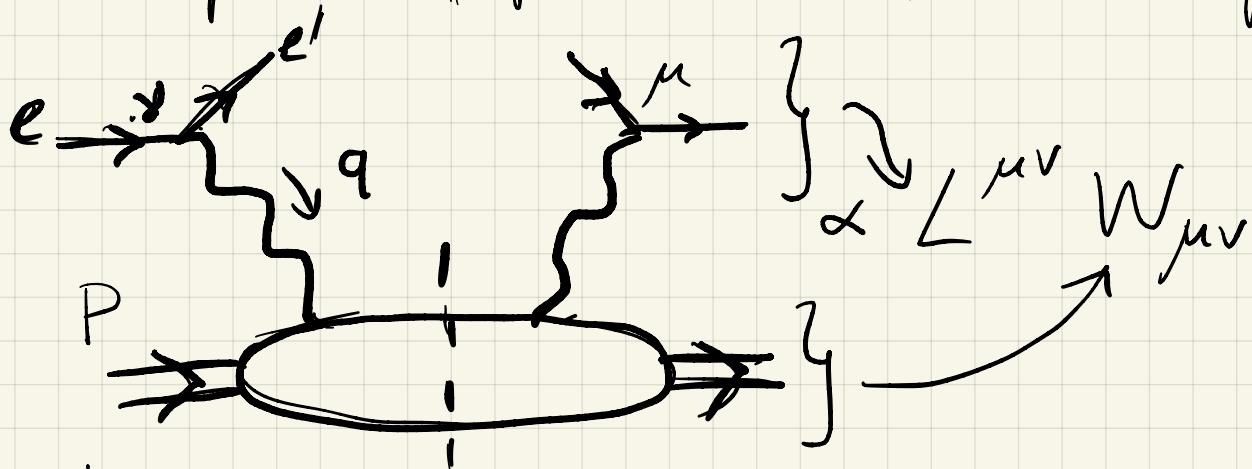
When we perform manipulations, insert a "unity" operator

$$\int dP_S |x\rangle \langle x| = 1$$

apply the optical theorem and derive an amplitude  $e$



One separates "leptonic" and "hadronic" part



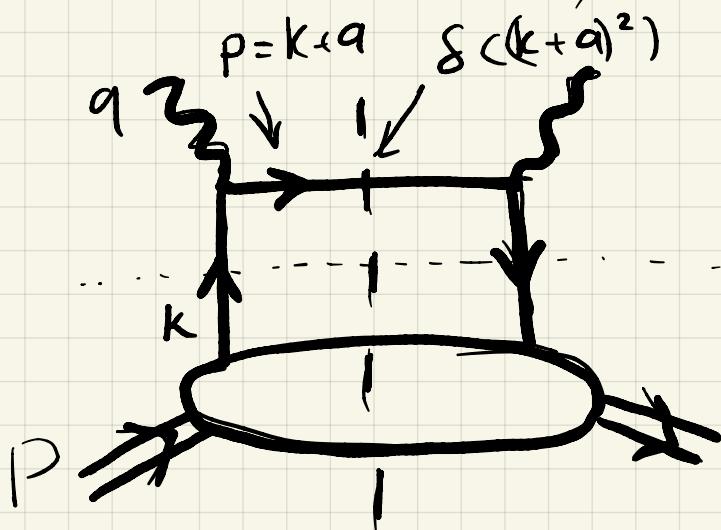
Then one performs an approximation, separation of time scales, known as factorisation

$q^2 = (e - e')^2$  plays an important role

$$Q^2 = -q^2, Q \sim \frac{1}{\tau T}$$

~~(part)~~<sup>2</sup> resolution  
→ parton nature

Such a separation is possible in the Infitite Momentum Frames, an example is the Breit frame



$$p = k + q \quad \delta((k+q)^2)$$

$\propto 6^\circ$ ,  
↑ factorisation

$$p \propto f_i(x) \delta\left(x - \frac{Q^2}{2P \cdot q}\right)$$

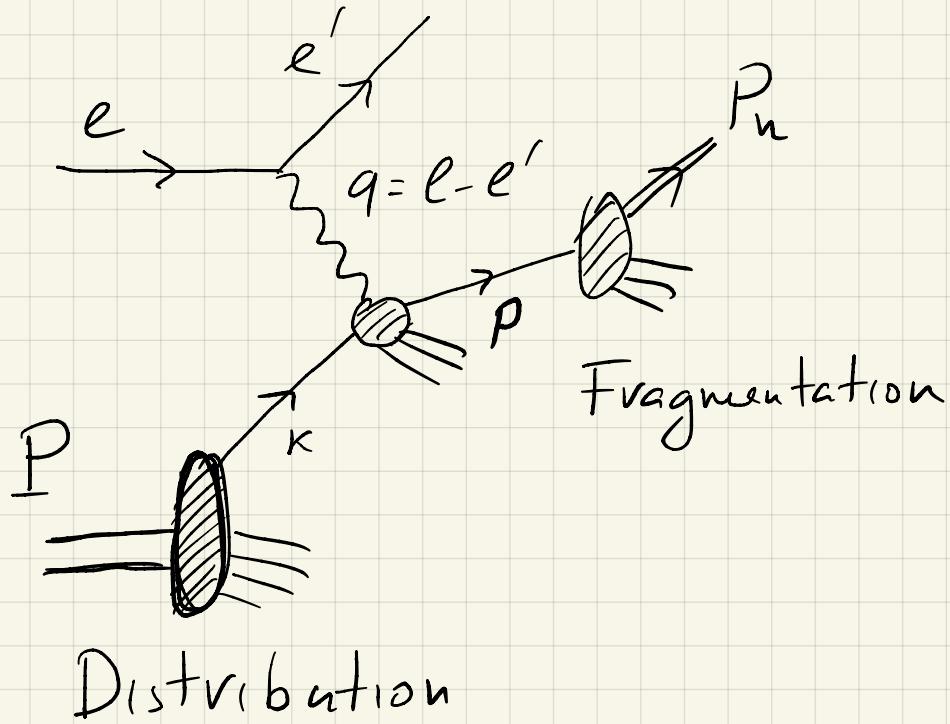
This example here is only the parton model,  
QCD radiation must be dealt with and  
the consequence will be a scale dependence of

$$f_i(x, \mu), \text{ the variable } x = k^+ / P^+, \quad k^+ = \frac{k^0 + k^3}{\sqrt{2}}$$

1 variable  $\Rightarrow$  1 dimensional structure 1D

How can 3D structure be studied?

Consider Semi-Inclusive Deep Inelastic Scattering (SIDIS)



$$x = \frac{Q^2}{2P \cdot q}$$

Bjorken variable  
(parton momentum fraction)

$$y = \frac{P \cdot q}{P \cdot l}$$

Inelasticity  
(energy transfer)

$$z = \frac{P \cdot P_h}{P \cdot q}$$

fraction of the parton momentum carried by the observed hadron

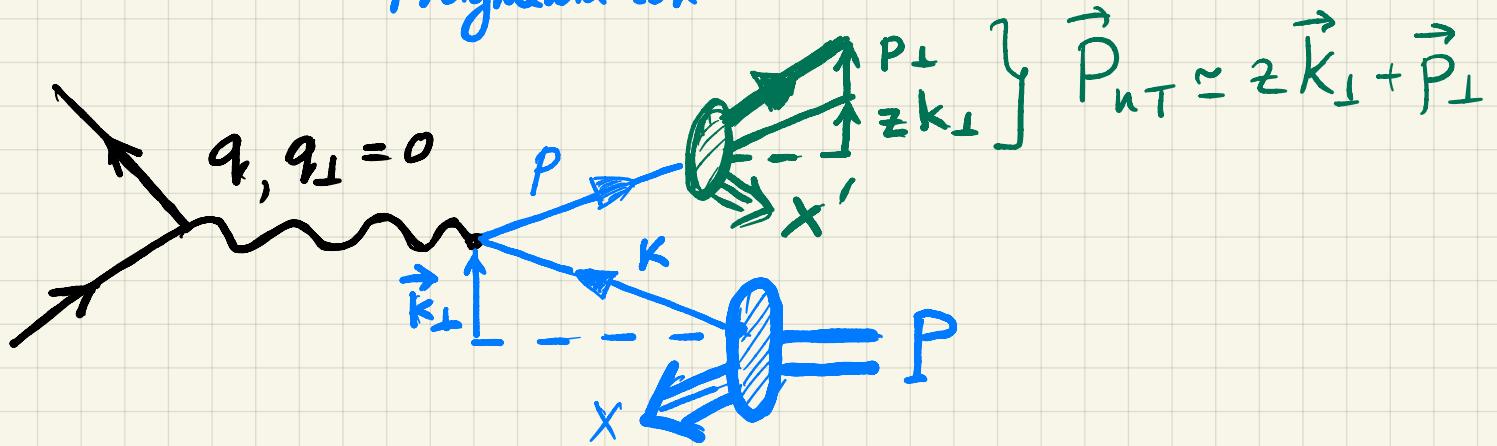
$$\underbrace{Q^2 = -q^2 \approx sxy}$$

an important constraint  
for the experiment

(Home work : derive it!)

$$s = (l + P)^2 \approx 2P \cdot l$$

$P_n \approx 2P$   
Fragmentation



Recoil off allow energy is interesting as it is sensitive to the intrinsic transverse motion,  $\vec{k}_\perp, \vec{P}_\perp$

QCD radiation is important, see the next lecture for a refined treatment.

Now our distributions will depend on  $x$  and  $k_T$

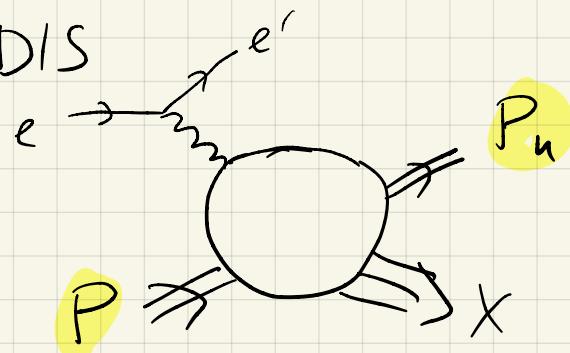
$f_i(x) \rightarrow f_i(x, \vec{k}_\perp)$  - Transverse Momentum Dependent distributions (TMDs)

$\vec{k}_T$  is a 2 vector and thus 3D structure

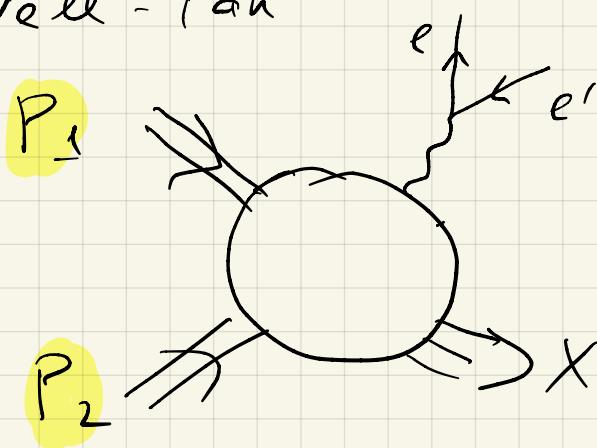
$x, \vec{k}_T$   
↓  
1 2

## Factorisation is proven for

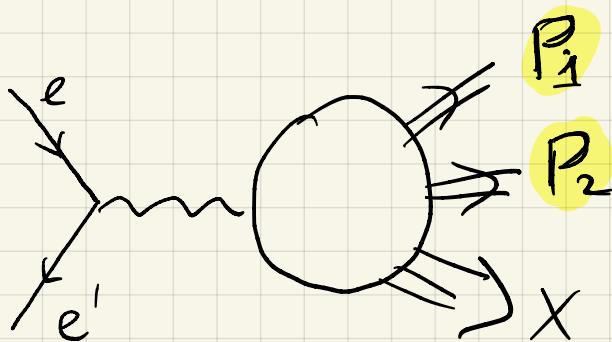
SIDIS



Drell-Yan



$e^+ e^-$  into hadron pairs



Common features

- \* 2 hadrons
- \* recoil with a low momentum,  $q_T$
- \* sufficiently inclusive
- \* 1 electromagnetic system with large virtuality  $Q$
- \*  $q_T \ll Q$

Why?

Experiment  $\rightarrow$  more things to measure,  
(un) polarised cross sections, asymmetries

Theory  $\rightarrow$  more vectors to construct distributions, allows to study correlations of, for instance, the intrinsic momentum and the spin. These should have the imprint of the confinement mechanism.

$$\phi_{ij} = \langle P, S | \bar{\psi}_{(b)} \psi_{(0)} | P, S \rangle$$

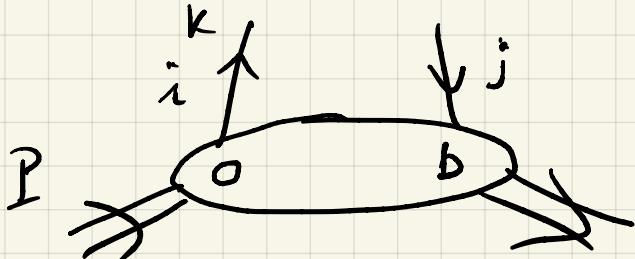
a matrix to be projected on unpolarized

quarks  $\gamma^+$ , longitudinally polarized

quarks  $\gamma^+ \gamma_5$ , transversely polarized

quarks  $i \gamma^d \gamma_5$  Spin projectors

$$N.B. \quad G^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$$



We can define the following projections of the correlator  $\underline{\Phi}(p, P)$  in the case when transverse motion is not ignored.

Parity, time-reversal, and charge conjugation lead to:

$$\underline{\Phi}(x, k_T)_{ij} = \int \frac{db^- d^2 b_T}{(2\pi)^3} e^{-ix^P b^- + i k_T \cdot b_T}$$

$$\langle P | \bar{\psi}_j(b) \psi_i(0) | P \rangle \Big|_{b^+ = 0}$$

$$\underline{\Phi}^{[\Gamma]} \stackrel{\text{def}}{=} \frac{1}{2} \text{Tr} [\Gamma \phi]$$

$$\frac{1}{2} \text{Tr} (\gamma^+ \phi) = f_1 - \frac{\epsilon^{jk} k_T^j S_T^k}{M} f_{1T}^\perp$$

$$\frac{1}{2} \text{Tr} (\gamma^+ \gamma_5 \phi) = S_L g_1 + \frac{\vec{k}_T \cdot \vec{S}_T}{M} g_{1T}^\perp$$

$$\frac{1}{2} \text{Tr} (\gamma^j \gamma^+ \gamma_5 \phi) = S_T^j h_1 + S_L \frac{k_T^j}{M} h_{1L}^\perp + \frac{\kappa^{jk} S_T^k}{M} h_{1T}^\perp$$

$$+ \frac{\epsilon^{jk} k_T^k}{M} h_2 \quad \text{, where } \kappa^{jk} = (k_T^j k_T^k - \frac{1}{2} k_T^2 \delta^{jk})$$

$$\epsilon^{ij} = \epsilon^{-+ij}, \epsilon^{0123} = +1$$

# Leading Quark TMDPDFs

Nucleon Spin

Quark Spin

		Quark Polarization		
		Un-Polarized (U)	Longitudinally Polarized (L)	Transversely Polarized (T)
Nucleon Polarization	U	$f_1 = \bullet$ Unpolarized		$h_1^\perp = \bullet - \bullet$ Boer-Mulders
	L		$g_1 = \bullet \rightarrow - \bullet \rightarrow$ Helicity	$h_{1L}^\perp = \bullet \rightarrow - \bullet \rightarrow$ Worm-gear
	T	$f_{1T}^\perp = \bullet \uparrow - \bullet \downarrow$ Sivers	$g_{1T}^\perp = \bullet \uparrow - \bullet \uparrow$ Worm-gear	$h_1 = \bullet \uparrow - \bullet \uparrow$ Transversity $h_{1T}^\perp = \bullet \uparrow - \bullet \uparrow$ Pretzelosity

Figure 2.5: Leading power quark parton distribution functions for the proton or a spin-1/2 hadron.

$$f_{i/p_S}^{[\gamma^+]}(x, \mathbf{k}_T, \mu, \zeta) = f_1(x, k_T) - \frac{\epsilon_T^{\rho\sigma} k_{T\rho} S_{T\sigma}}{M} \kappa f_{1T}^\perp(x, k_T), \quad (2.123)$$

$$f_{i/p_S}^{[\gamma^+ \gamma_5]}(x, \mathbf{k}_T, \mu, \zeta) = S_L g_1(x, k_T) - \frac{k_T \cdot S_T}{M} g_{1T}^\perp(x, k_T),$$

$$\begin{aligned} f_{i/p_S}^{[i\sigma^{\alpha+}\gamma_5]}(x, \mathbf{k}_T, \mu, \zeta) &= S_T^\alpha h_1(x, k_T) + \frac{S_L k_T^\alpha}{M} h_{1L}^\perp(x, k_T) \\ &\quad - \frac{\mathbf{k}_T^2}{M^2} \left( \frac{1}{2} g_T^{\alpha\rho} + \frac{k_T^\alpha k_T^\rho}{\mathbf{k}_T^2} \right) S_{T\rho} h_{1T}^\perp(x, k_T) - \frac{\epsilon_T^{\alpha\rho} k_{T\rho}}{M} \kappa h_1^\perp(x, k_T). \end{aligned}$$

$$\kappa = \begin{cases} +1 & (\text{DY}) \\ -1 & (\text{SIDIS}) \end{cases}$$

$$\begin{aligned}
\tilde{f}_{i/p_S}^{[\gamma^+]}(x, \mathbf{b}_T, \mu, \zeta) &= \tilde{f}_1(x, b_T) + i\epsilon_{\rho\sigma} b_T^\rho S_T^\sigma M \tilde{f}_{1T}^\perp(x, b_T), \\
\tilde{f}_{i/p_S}^{[\gamma^+ \gamma_5]}(x, \mathbf{b}_T, \mu, \zeta) &= S_L \tilde{g}_1(x, b_T) + i b_T \cdot S_T M \tilde{g}_{1T}^\perp(x, b_T), \\
\tilde{f}_{i/p_S}^{[i\sigma^\alpha \gamma_5]}(x, \mathbf{b}_T, \mu, \zeta) &= S_T^\alpha \tilde{h}_1(x, b_T) - i S_L b_T^\alpha M \tilde{h}_{1L}^\perp(x, b_T) + i \epsilon^{\alpha\rho} b_{\perp\rho} M \tilde{h}_1^\perp(x, b_T) \\
&\quad + \frac{1}{2} \mathbf{b}_T^2 M^2 \left( \frac{1}{2} g_T^{\alpha\rho} + \frac{b_T^\alpha b_T^\rho}{\mathbf{b}_T^2} \right) S_{\perp\rho} \tilde{h}_{1T}^\perp(x, b_T). \tag{2.126}
\end{aligned}$$

All measurements are performed in the momentum space, however it is more convenient to study functions in the configuration space.

How do we perform F.T.?

$$\begin{aligned}
\tilde{f}(b_T) &= \int d^2 k_T e^{-i \vec{b}_T \cdot \vec{k}_T} f(k_T) \\
&= \int_0^\infty k_T dk_T \int_0^{2\pi} d\varphi e^{-i b_T k_T \cos\varphi} f(k_T) \\
&= 2\pi \int_0^\infty k_T dk_T J_0(k_T b_T) f(k_T) \\
f(k_T) &= \int \frac{d^2 b_T}{(2\pi)^2} e^{+i \vec{b}_T \cdot \vec{k}_T} \tilde{f}(b_T) \\
&= \frac{1}{2\pi} \int_0^\infty b_T db_T J_0(b_T k_T) \tilde{f}(b_T)
\end{aligned}$$

We will also need  $b_T$  derivatives:

$$\tilde{f}^{(n)}(b_T) \equiv n! \left( -\frac{1}{M^2 b_T} \frac{\partial}{\partial b_T} \right)^n \tilde{f}(b_T)$$

$$= \frac{2\pi n!}{(M^2)^n} \int k_T dk_T \left( \frac{k_T}{b_T} \right)^n J_n(b_T k_T) f(k_T)$$

Why so? Using  $\lim_{z \rightarrow 0} J_m(z) = \underbrace{\frac{1}{\Gamma(m+1)}}_{m! \text{ for integer } m} \left(\frac{z}{2}\right)^m$

$$\lim_{b_T \rightarrow 0} \tilde{f}^{(n)}(b_T) = \underbrace{2\pi \int k_T dk_T \left( \frac{k_T^2}{2M^2} \right)^n f(k_T)}_{\text{These are called moments}}$$

of TMDs  $\underbrace{i, j \in [1, 2]}$

We need F.T. of the form  $k_T^i \dots k_T^j f(k_T)$

$$\int d^2 k_T k_T^i \dots k_T^j f(k_T) e^{-i \bar{k}_T \bar{b}_T} =$$

$$= \left( +i \frac{\bar{b}_T^i}{b_T} \frac{\partial}{\partial \bar{b}_T} \right) \dots \left( +i \frac{\bar{b}_T^j}{b_T} \frac{\partial}{\partial \bar{b}_T} \right)$$

$$2\pi \int_0^\infty k_T dk_T J_0(k_T \bar{b}_T) f(k_T)$$

An example:

$$\int d^2 k_T k_T^i e^{-i \bar{b}_T \bar{k}_T} f(k_T) =$$

$$= i \frac{b_T^i}{b_T} \frac{\partial}{\partial b_T} 2\pi \int k_T dk_T J_0(k_T b_T) f(k_T) =$$

$$= i \frac{b_T^i}{b_T} 2\pi \int k_T dk_T (-J_1(k_T b_T)) k_T f(k_T)$$

$$= (-i) b_T^i 2\pi \int k_T dk_T \left(\frac{k_T}{b_T}\right) J_1(k_T b_T) f(k_T)$$

$$= (-i) b_T^i M^2 \tilde{f}^{(1)}(b_T)$$

and thus

$$\hat{D}^{[\gamma^+]} = f_{1\perp}(x, k_{\perp}) - \frac{i \epsilon^{ij} k_T^i S_T^j}{M} f_{1T}^{\perp}(x, k_{\perp})$$

↓

$$\tilde{D}^{[\gamma^+]} = \tilde{f}_{1\perp}^{(0)}(x, b_T) + i \epsilon^{ij} b_T^i S_T^j M \tilde{f}_{1T}^{\perp(1)}(x, b_T)$$

Home work → perform the other cases

# Physics

$$f_{a/p^\uparrow}(x, k_T, S_T) = \underbrace{f_{a/p}(x, k_T)}_{\text{unpolarised}} + \frac{\vec{S}_T \cdot (\vec{k}_T \times \hat{P})}{M} \underbrace{f_{1T}^\perp(x, k_T)}_{\text{Sivers function}}$$

If one applies  $P$  and  $T$  invariance

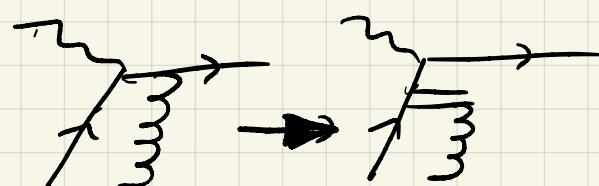
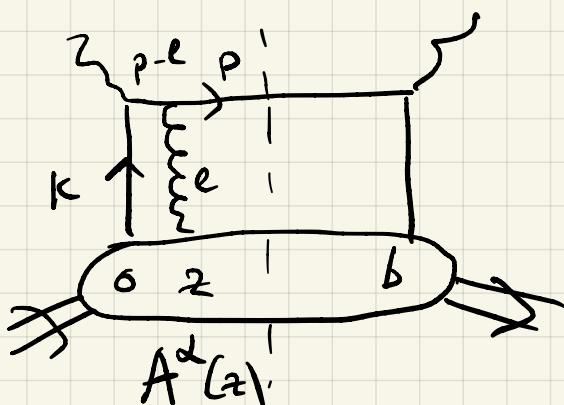
$$f_{a/p^\uparrow}(x, k_T, S_T) = f_{a/p^\uparrow}(x, k_T, -S_T)$$

and

$$f_{a/p}(x, k_T) \text{ remains } f_{1T}^\perp(x, k_T) \text{ vanishes}$$

A famous "mistake" that took 10 years to resolve

One needs to account for initial or final state radiation and introduce the gauge link



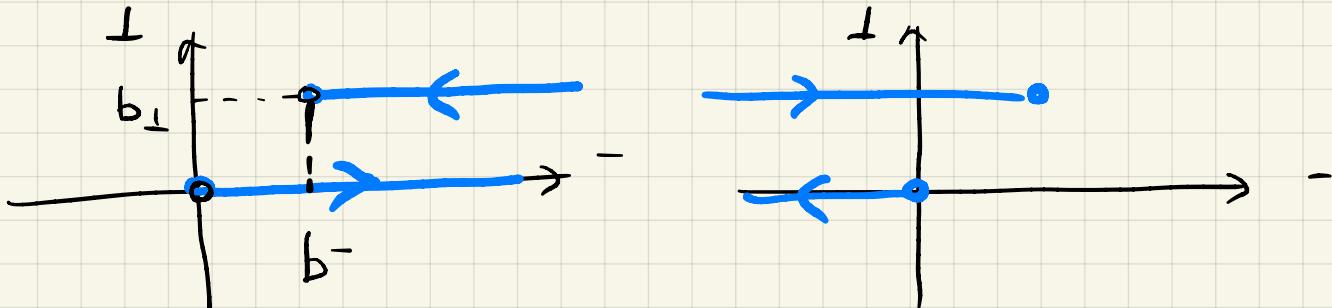
$$\begin{aligned} (p - e)^2 &\approx -2p \cdot l + e^2 \approx -2p \cdot e \\ &= -2p^- e^+ \end{aligned}$$

the eikonal approximation

$$i \frac{p - e}{(p - e)^2 + i\epsilon} \simeq i \frac{p^- \gamma^+}{-2p^- e^+ + i\epsilon} = \frac{i}{2} \frac{\gamma^+}{-e^+ + i\epsilon}$$

It was found instead that

$$f_{a/p}^{\text{SIDIS}}(x, k_T, s_T) = f_{a/p}^{\text{DY}}(x, k_T, -s_T)$$



and thus

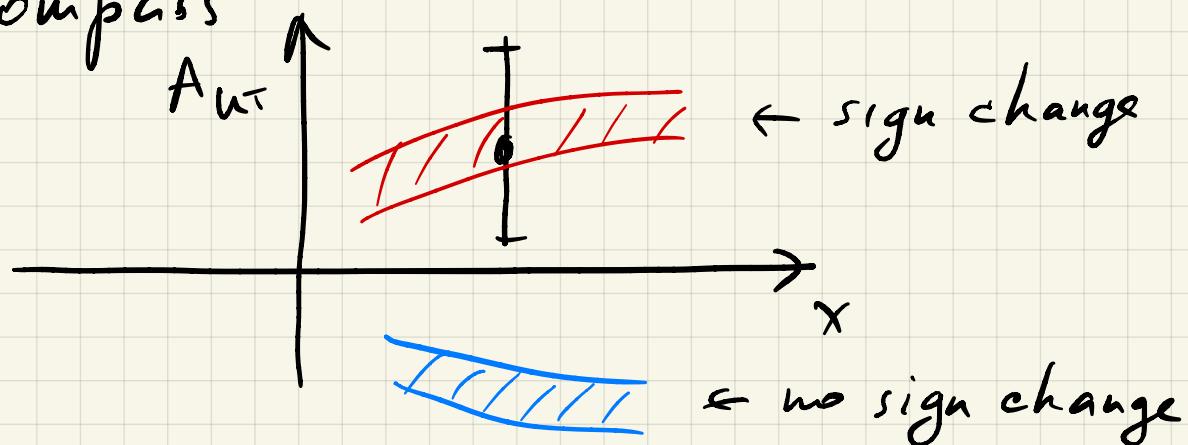
$$f_{q/p}^{\text{SIDIS}}(x, k_T) = \underbrace{f_{q/p}^{\text{DY}}(x, k_T)}$$

Unpolarised are the same

$$f_{1T}^{\perp \text{ SIDIS}}(x, k_T) = - \underbrace{f_{1T}^{\perp \text{ DY}}(x, k_T)}$$

Sivers function changes sign!

Compass



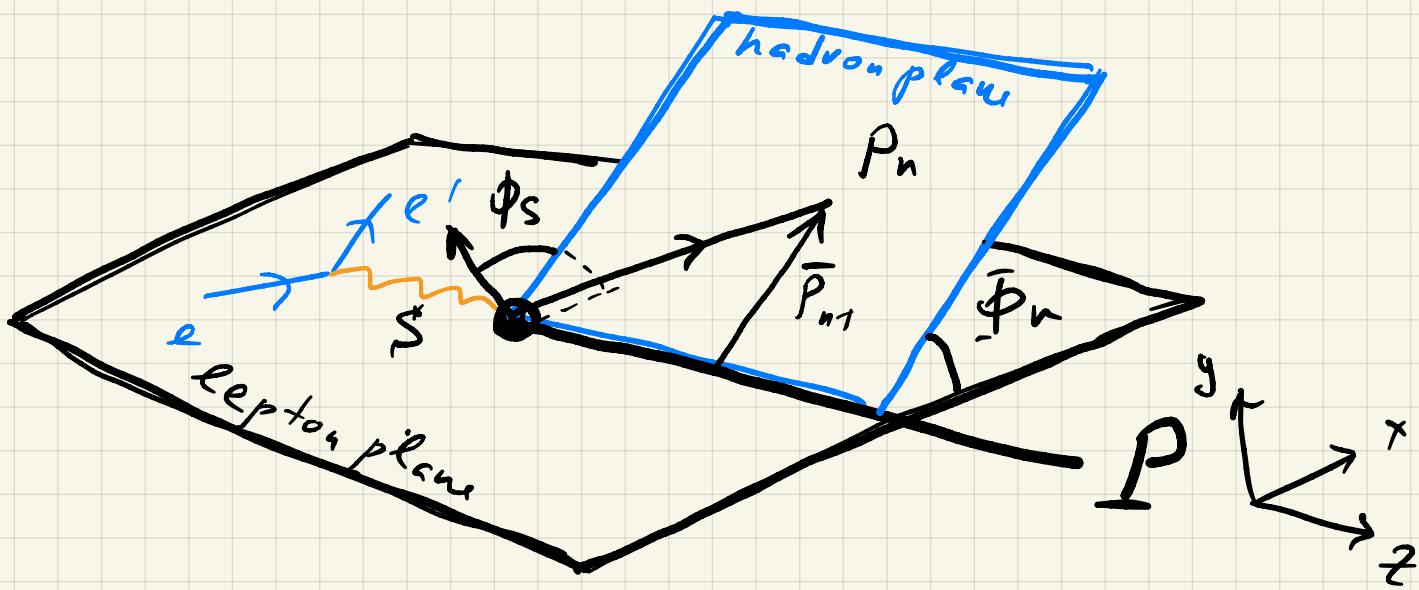
A project: Estimate if the EIC will prove the sign change.

SIDIS cross section has a very rich structure if the polarisation is available in the experiment.

Structure functions:

$$C[\omega f D] = \int d^2 k_T d^2 p_T S^{(2)} (\bar{P}_{uT} - z \bar{k}_T \cdot \bar{P}_T)$$

$$\omega(k, p_T) f(x, k_T) D(z, p_T)$$



Experimentally,  $S_L$ ,  $S_T$ ,  $\phi_h$ ,  $\phi_s$  are the tools to disentangle structure functions.



Now convolution in the momentum space  $\rightarrow b_T$  space

$$C[\omega f D] = \sum_a e_a^2 \int d^2 k_T d^2 p_T \delta^{(2)}(\bar{P}_{aT} - z \bar{k}_T - \bar{p}_T)$$

$$\omega(k_T b_T) f(x, k_T) D(z, p_T), \quad \bar{P}_{aT} = -z \bar{q}_T$$

$$\text{We rewrite } \delta^{(2)}(\bar{P}_{aT} - z \bar{k}_T - \bar{p}_T) =$$

$$= \delta^{(2)}(-z \bar{q}_T - z \bar{k}_T - \bar{p}_T) = \frac{1}{z^2} \delta^{(2)}(\bar{q}_T + \bar{k}_T + \frac{\bar{p}_T}{z})$$

$$= \frac{1}{z^2} \int \frac{d^2 b_T}{(2\pi)^2} e^{-i(\bar{q}_T + \bar{k}_T + \bar{p}_T/z) \cdot \bar{b}_T}$$

$$F_{aT} = C[1f_1D_1] = \sum_a e_a^2 \int \frac{d^2 b_T}{(2\pi)^2} e^{-i \bar{b}_T \cdot \bar{q}_T}$$

$$* \int d^2 k_T e^{-i \bar{b}_T \cdot \bar{k}_T} f_1(x, k_T)$$

$$\frac{1}{z^2} \int d^2 p_T e^{-i \bar{b}_T \cdot \bar{p}_T/z} D_1(z, p_T) =$$

$$= \sum_a e_a^2 \int \frac{d^2 b_T}{(2\pi)^2} e^{-i \bar{b}_T \cdot \bar{q}_T} \tilde{f}_1(x, b_T) \tilde{D}_1(z, b_T)$$

$$= \sum_a e_a^2 \int \frac{b_T db_T}{2\pi} g_0(b_T q_T) \tilde{f}_1(x, b_T) \tilde{D}_1(z, b_T)$$

$$\text{Let us call } B(\tilde{f}_1, \tilde{D}_1) = \sum_a e_a^2 \int \frac{b_T db_T}{2\pi} g_0(b_T q_T) \tilde{f}_1 \tilde{D}_1$$

A more complicated example, Sivers asymmetry

$$F_{u\bar{u}} \stackrel{\sin(\phi_u - \phi_s)}{=} C \left[ - \frac{\hat{h} \cdot \vec{k}_T}{M} f_{1T}^\perp D_1 \right]$$

$$= \sum_q e_q^2 \int d^2 k_T d^2 p_T \delta^{(2)}(\vec{p}_{u\bar{u}} - z \vec{k}_T - \vec{p}_T) \underbrace{\left( - \frac{\hat{h} \cdot \vec{k}_T}{M} \right)}_{-\frac{k_T}{M} \cos(\varphi - \phi_h)}$$

$$\cdot f_{1T}^\perp(x, k_T^2) D_1(z, p_T^2)$$

$$= \sum_q e_q^2 \int \frac{dz b_T}{(2\pi)^2} e^{-i \bar{q}_T \bar{b}_T} \int d^2 k_T \left( - \frac{k_T}{M} \right) \cos(\varphi - \phi_u)$$

$$e^{-i \bar{k}_T \bar{b}_T} f_{1T}^\perp(x, k_T^2) \underbrace{\int d^2 p_T \frac{1}{z^2} e^{-i \bar{p}_T \bar{b}_T/z} D_1(z, p_T^2)}_{\tilde{D}_1(z, b_T^2)}$$

we have  $\bar{k}_T \bar{b}_T = b_T k_T \cos(\varphi - \varphi_b)$

$$\int d\varphi e^{-i b_T k_T \cos(\varphi - \varphi_b)} \cos(\varphi - \phi_u) = \begin{cases} \text{Use} \\ \text{Mathematica} \end{cases}$$

$$= -2\bar{n} i \Im_1(b_T k_T) \cos(\varphi_b - \phi_u)$$

$$\int d\varphi_b \cos(\varphi_b - \phi_u) e^{-i b_T q_T \cos(\varphi_b - \phi_u)} = \gamma$$

$$= 2\bar{n} i \Im_1(b_T q_T)$$

So that

$$F_{UT}^{\sin(\phi_u - \phi_s)} = - \sum_q e_q^2 \int \frac{db_T b_T}{2\pi} J_1(b_T q_T)$$

$$\int dk_T \frac{k_T^2}{M} J_1(b_T k_T) f_{1T}^{\perp}(x, k_T^2) \tilde{D}_1(z, b_T^2)$$

$$\tilde{f}_{1T}^{\perp(1)}(x, b_T^2) = \frac{2\pi}{M^2} \int k_T dk_T \frac{k_T}{b_T} J_1(k_T b_T) f_{1T}^{\perp}(x, k_T^2)$$

Thus

$$F_{UT}^{\sin(\phi_u - \phi_s)} = (-M) \sum_q e_q^2 \int \frac{b_T db_T}{2\pi} \underbrace{b_T J_1(b_T q_T)}_{\text{①}} \\ \cdot \tilde{f}_{1T}^{\perp(1)}(x, b_T^2) \tilde{D}_1(z, b_T^2) \quad \text{②}$$

$$F_{UT}^{\sin(\phi_u - \phi_s)} = -M B [\tilde{f}_{1T}^{\perp(1)} \tilde{D}_1]$$

General definition

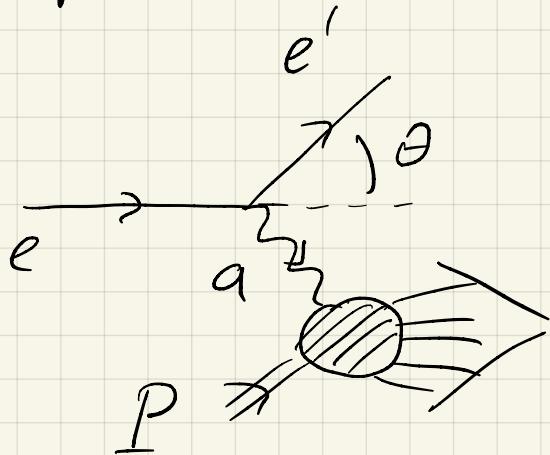
$$B[\tilde{f}^{(n)} \tilde{D}^{(m)}] = \int \frac{b_T db_T}{2\pi} b_T^{n+m} J_{n+m}(b_T q_T)$$

$$\tilde{f}^{(n)}(x, b_T^2) \tilde{D}^{(m)}(z, b_T^2)$$



During lecture 1 we speak of a certain process. Let us look into its kinematics

## Deep Inelastic Scattering (DIS)



1) Show that  $q^2 < 0$

$$\ell = (E, 0, 0, E)$$

$$\ell' = (E', E' \sin\theta, 0, E' \cos\theta)$$

$$\begin{aligned} \text{Thus } q^2 &= (\ell - \ell')^2 \approx -2\ell\ell' = -2(EE' - EE'\cos\theta) \\ &= -2EE'(1 - \cos\theta) \leq 0. \end{aligned}$$

It is customary to introduce  $q^2 = -Q^2$   
where  $Q^2 \geq 0$

Now let us explore other kinematical variables

$$x = \frac{Q^2}{2P \cdot q} - \text{Bjorken } x, y = \frac{P \cdot q}{P \cdot e} - \text{inelasticity}$$

These variables are constructed off scalar products which are Lorentz invariants and therefore we can use any frame to estimate them. Let us choose target rest frame

~~mass~~  $P = (M, \vec{0})$

$q = (\gamma, \vec{q}), \gamma = E - E' > 0$

$$2P \cdot q = 2M\gamma \rightarrow \infty \text{ Bjorken limit}$$

$$q^2 = -Q^2, Q^2 \geq 0, \gamma \geq 0 \Rightarrow$$

$$x = \frac{Q^2}{2P \cdot q} \geq 0$$

$W$  is the energy of  $q$

$$W^2 = (P+q)^2 = P^2 + 2P \cdot q - Q^2 =$$

$$P^2 = M^2 + 2P \cdot q - Q^2 \geq M^2$$

in case the proton is intact = elastic scattering

$$\Rightarrow 2P \cdot q \geq Q^2 \Rightarrow x = \frac{Q^2}{2P \cdot q} \leq 1$$

$$\text{inelasticity } y = \frac{P \cdot q}{P \cdot \ell} = \frac{M(E - E')}{ME} = 1 - \frac{E'}{E}$$

$E' \in [0, E]$  thus  $y \in [0, 1]$

$x \in [0, 1]$       } we can estimate the reach  
 $y \in [0, 1]$       } of experiments

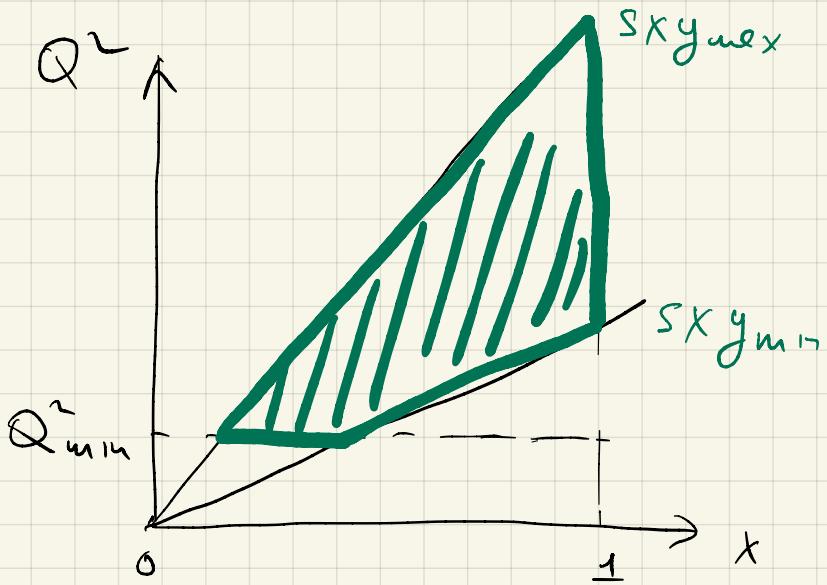
$$s = (P + Q)^2 \approx M^2 + 2P \cdot \ell \approx 2P \cdot \ell$$

=>

$$x = \frac{Q^2}{2P \cdot q} = \frac{Q^2}{2P \cdot q} \cdot \frac{P \cdot \ell}{P \cdot \ell} = \frac{Q^2}{y s}$$

Therefore

$$Q^2 \approx y s x$$

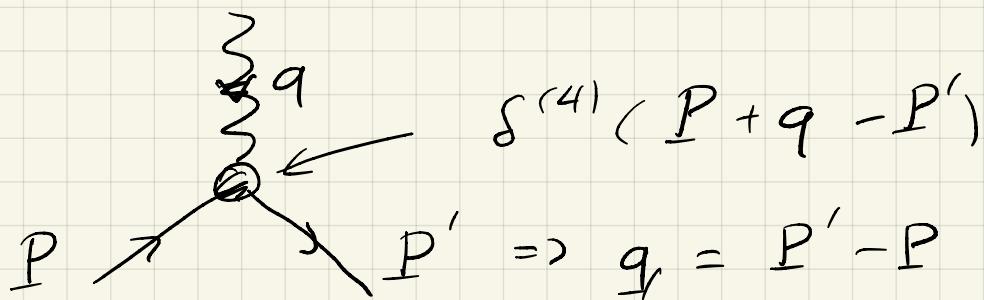


$Q^2_{\min} \approx 1 \text{ (GeV}^2\text{)}$  to ensure DIS regime

$y \in [y_{\min}, y_{\max}]$   
experimental resolution

Why do we need  $x_{Bj}$ ?

Let us consider form factors  $\rightarrow$  elastic scattering



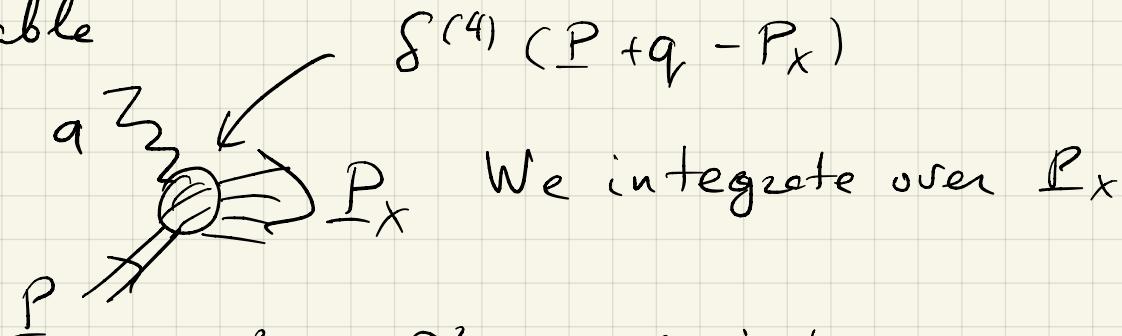
$$\begin{aligned} q^2 &= -Q^2 = (\underline{P}' - \underline{P})^2 = \underline{P}'^2 + \underline{P}^2 - 2 \underline{P}' \cdot \underline{P} = \\ &= 2M^2 - 2\underline{P}' \cdot \underline{P} \rightarrow -2\underline{P}' \cdot \underline{P} \end{aligned}$$

$$\underline{P} \cdot \underline{q} = \underline{P} (\underline{P}' - \underline{P}) = \underline{P} \cdot \underline{P}' - M^2 \rightarrow \underline{P} \cdot \underline{P}'$$

therefore

$$x = \frac{Q^2}{2\underline{P} \cdot \underline{q}} \rightarrow 1 \text{ not an independent variable}$$

variable



$$q^2 = -Q^2 \rightarrow \infty \text{ independently}$$

$$2\underline{P} \cdot \underline{q} \rightarrow \infty$$

$$x_{Bj} = \frac{Q^2}{2\underline{P} \cdot \underline{q}} \in [0, 1]$$

Experiments : fixed target vs collider

$$s = (P + \ell)^2 \text{ cm energy}$$

Fixed target

$$\ell = (P_{\text{lab}}, 0, 0, -P_{\text{lab}})$$

$$P = (M_P, 0, 0, 0)$$

$$s = (P + \ell)^2 = P^2 + 2P \cdot \ell + \ell^2 \simeq 2 M_P P_{\text{lab}}$$

Collider

$$\ell = (E_e, 0, 0, -E_e)$$

$$P \simeq (E_P, 0, 0, E_P) \text{ (neglect the mass)}$$

$$s = (P + \ell)^2 \simeq (E_e + E_p)^2 - (E_p - E_e)^2 = 4 E_p E_e$$

The energy increased easily in collider

# The elements of Quantum Field Theory

The wave function  $\Psi(x)$  is a coordinate projection of the state vector in the Hilbert space  $|\Psi\rangle$

$$\int d^3x |\Psi(x)|^2 < \infty$$

The scalar product

$$\langle \Psi | \phi \rangle = \int d^3x \Psi^*(x) \phi(x)$$

We will use  $|P; S\rangle$  to denote the proton with momentum  $P$  and spin vector  $S$ .

Unitarity of  $S$  matrix and optical theorem

(Taylor, "Scattering theory")

The probability of one state going to the other is described by  $S$  matrix

$$w(X \leftarrow \phi) = |\langle X | S | \phi \rangle|^2$$

One can write

$$S_{ab} = S_{ab} + i (2\pi)^4 \delta^{(4)}(p_a - p_b) T_{ab}$$

$\uparrow$                        $\uparrow$   
no interaction              momentum conservation

interaction  
 $\downarrow$

Probability is conserved

$$SS^+ = S^+ S = 1 \quad \text{Unitarity of } S \text{ matrix}$$

One can use it to write  $\underbrace{\dots}_{\text{all possible states}}$

$$\text{Im } \langle X | T | \phi \rangle = \frac{1}{2} \sum_X \langle X | T | X \rangle \langle X | T^+ | \phi \rangle (2\pi)^4 \delta^{(4)}(p_\phi - p_X)$$

Diagrammatically

$$2 \text{Im} \rightarrow \text{---} = \sum_X - \text{---} \times \text{---} \times \text{---} .$$

↑  
Im part

$$SS^+ = (1 + i(2\pi)^4 \delta^{(4)}(P_\phi - P_\chi) T) (1 - i(2\pi)^4 \delta^{(4)}(P_\phi - P_\chi) T^+) = 1$$

$$\cancel{1 - i(2\pi)^4 (T^+ - T) \delta^{(4)}(P_\phi - P_\chi) + (2\pi)^8 T T^+ (\delta^{(4)}(P_\phi - P_\chi))^2} \neq 1$$

$$T = \text{Re } T + i \text{Im } T, \quad T^+ = \text{Re } T - i \text{Im } T$$

$$T^+ - T = -2i \text{Im } T$$

$$2(2\pi)^4 \text{Im } T = (2\pi)^8 T T^+ \delta^{(4)}(P_\phi - P_\chi)$$

Let us insert  $\square = \sum_x |x\rangle \langle x|$

and sandwich this expression with  $\langle X| \dots |\phi\rangle$

$$\text{Im } \langle X | T | \phi \rangle = \frac{1}{2} \sum_x \langle X | T | x \rangle \langle x | T^+ | \phi \rangle (2\pi)^4 \delta^{(4)}(P_\phi - P_\chi)$$

∴

$$2 \cdot \phi \left\{ \begin{array}{c} | \\ \text{---} \\ | \end{array} \right\} x = \sum_x \phi \cancel{\text{---}} x \cancel{\text{---}} x$$

$\text{Im}$

If  $\phi = \chi$  then, for instance  $pp \rightarrow pp$

$$\begin{array}{ccc} p & | & p \\ & \diagup \quad \diagdown & \\ & \text{---} & \\ & \diagdown \quad \diagup & \\ p & | & p \end{array} = \frac{1}{2} \sum_x \left( \begin{array}{c} p \\ p \\ \diagup \quad \diagdown \\ \text{---} \\ \diagdown \quad \diagup \\ p \\ p \end{array} \right)^2$$

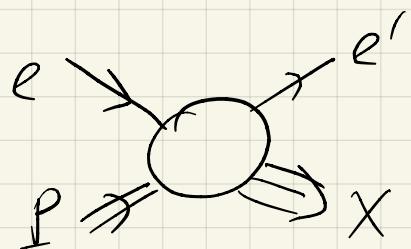
We are interested in photon-proton interactions

$$2 \text{Im} \frac{1}{P} = \sum_x |E_x|^2$$

Experimentally one measures cross-sections

$$\sigma_{\phi \rightarrow \psi} = \underbrace{\frac{1}{F_\phi}}_{\text{Flux of } \phi} |A_{\phi \rightarrow \psi}|^2 \underbrace{\frac{d^3 P_\psi}{(2\pi)^3 2 E_\psi}}_{\text{Phase space of } \psi}$$

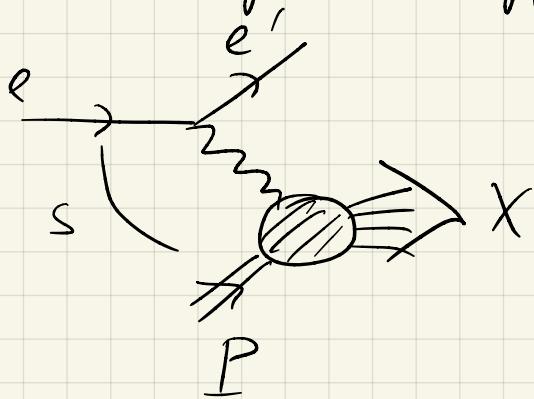
We want to calculate cross-section  
of this process



$$e + P \rightarrow e' + X$$

$$q^2 = (e - e')^2 \quad \xrightarrow{\text{Deep Inelastic Scattering}} \quad P \rightarrow X$$

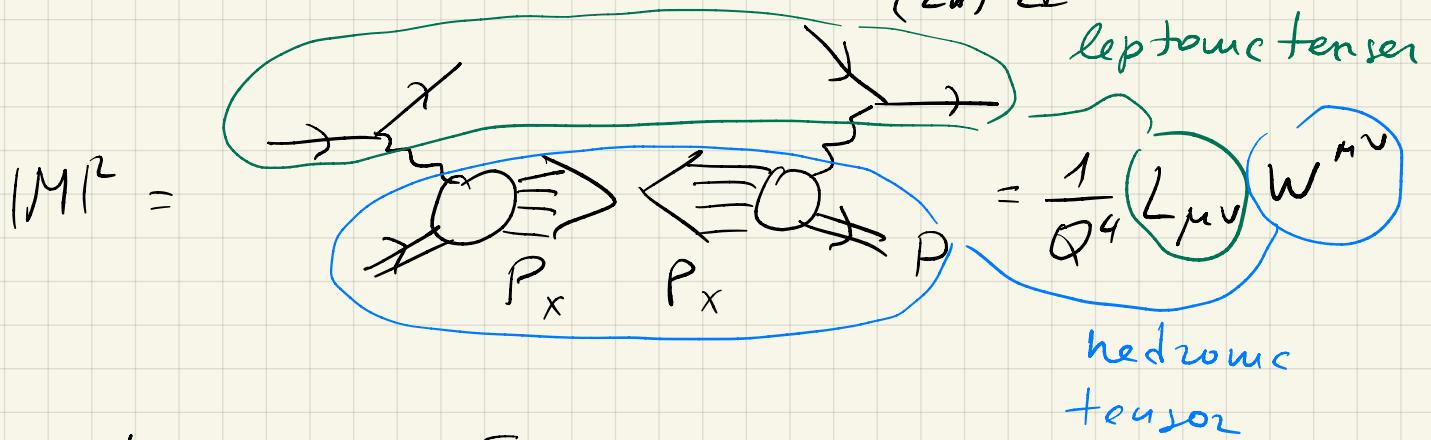
We use one photon approximation



$$\sigma = \frac{1}{\pi} |M|^2 d\mathcal{PS}$$

$$\mathcal{F} \approx 2s = 2(e + P)^2 \text{ flux}$$

$$d\mathcal{PS} = \frac{d^3 e'}{(2\pi)^3 2E'}$$



$$\frac{1}{Q^4} \sim \text{product of photon propagators}$$

Before we calculate  $L_{\mu\nu}$  and  $W^{\mu\nu}$  let us recapitulate some basics of QFT.

Consider the Lagrangian of a spin- $\frac{1}{2}$  particle with the mass  $m$ :

$$\mathcal{L} = \bar{\psi} (i \gamma^\mu \partial_\mu - m) \psi$$

$\gamma^\mu$ - gamma matrices

$$\partial_\mu = \frac{\partial}{\partial x^\mu}, \quad \psi(x), \quad \bar{\psi}(x) = \psi^+(x) \delta^0 \text{ fields}$$

Global gauge transformations

$$\psi'(x) = e^{i\alpha} \psi(x)$$

$$\bar{\psi}'(x) = \bar{\psi}(x) e^{-i\alpha}$$

$$\mathcal{L} \rightarrow \mathcal{L}', \text{ current } j^\mu(x) = \bar{\psi}(x) \gamma^\mu \psi(x)$$

Local gauge transformations

$$\begin{cases} \psi'(x) = e^{i\alpha(x)} \psi(x) \\ \bar{\psi}'(x) = \bar{\psi}(x) e^{-i\alpha(x)} \end{cases}$$

$$\partial_\mu \psi(x) = e^{-i\alpha(x)} (\partial_\mu - i \partial_\mu \alpha(x)) \psi'(x)$$

Thus

$$\mathcal{L} = \bar{\psi}(x) (i \gamma^\mu (\partial_\mu - i \partial_\mu \alpha(x)) - m) \psi'(x)$$

We can restore gauge invariance if we use

$$(\partial_\mu + ie A_\mu(x)) \psi(x)$$

$$(\partial_\mu + ie A_\mu(x)) \psi(x) = e^{-i\alpha(x)} (\partial_\mu + ie A'_\mu(x)) \psi'(x)$$

where

$$A'_\mu(x) = A_\mu(x) - \frac{1}{e} \partial_\mu \alpha(x)$$

$\partial_\mu + ie A_\mu(x) \rightarrow$  covariant derivative  $D_\mu$

$$\mathcal{L} = \bar{\psi}(x) (i \gamma^\mu (D_\mu + ie A_\mu(x)) - m) \psi(x)$$

or

$$\mathcal{L} = \bar{\psi}(x) (i \gamma^\mu D_\mu - m) \psi(x)$$

is invariant also under the local gauge transform.

We also have introduced interactions!

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I, \text{ where}$$

$$\mathcal{L}_I = -e j^\mu A_\mu \text{ where } j^\mu = \bar{\psi}(x) \gamma^\mu \psi(x)$$

Remember

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = \{ \gamma^\mu, \gamma^\nu \} = 2\gamma^{\mu\nu}$$

$$\gamma^0 (\gamma^\mu)^+ \gamma^0 = \gamma^\mu, \Rightarrow (\gamma^0)^+ = \gamma^0, (\gamma^\mu)^+ = -\gamma^\mu$$

Independent fields  $\psi(x)$  &  $\bar{\psi}(x) = \psi^+(x) \gamma^0$

Euler-Lagrange equations (for  $\mathcal{L} = \mathcal{L}_0$ )

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi} = 0 \\ \frac{\partial \mathcal{L}}{\partial \bar{\psi}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \bar{\psi}} = 0 \end{cases}$$

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = (i \gamma^\mu \partial_\mu - m) \psi, \quad \frac{\partial \mathcal{L}}{\partial \partial_\mu \bar{\psi}} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \psi} = -m \bar{\psi}, \quad \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi} = i \bar{\psi} \gamma^\mu$$

$$\Rightarrow \begin{cases} (i \gamma^\mu \partial_\mu - m) \psi(x) = 0 \\ i \partial_\mu \bar{\psi}(x) \gamma^\mu + m \bar{\psi}(x) = 0 \end{cases}$$

4 solutions, 2 with  $p_0 > 0$ , 2 with  $p_0 < 0$

Let us consider only positive energy

$$\Psi(x) = u(p, s) e^{-i p \cdot x}, \quad p^2 = m^2, \quad p_0 > 0$$

$$(i \gamma^\mu \partial_\mu - m) \Psi(x) = 0$$

$$\Rightarrow (\gamma^\mu p_\mu - m) u(p) = 0, \quad \gamma^\mu p_\mu = p$$

$(p - m) u(p) = 0$ ,  $u(p)$  is called spinor

$$\bar{u}(p)(p - m) = 0$$

Feynman diagram illustrating the annihilation of an electron ( $e$ ) and the creation of a virtual electron-positron pair ( $e' \bar{e}'$ ). The incoming electron ( $e$ ) and outgoing virtual electron ( $e'$ ) are shown with arrows pointing from left to right. The virtual positron ( $\bar{e}'$ ) is shown with an arrow pointing from right to left.

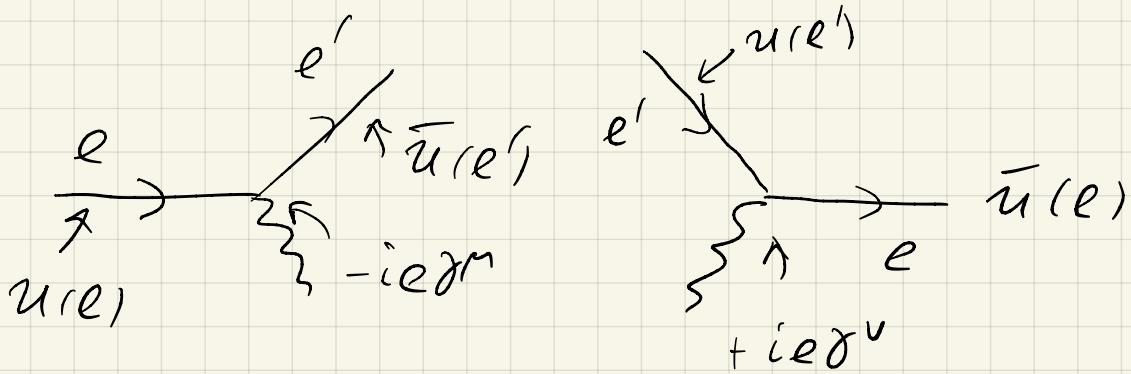
$$q = e - e'$$

Current conservation  $\partial_\mu j^\mu = 0$

$$j^\mu = \bar{u}(e') \gamma^\mu u(e) e^{-i(e-e') \cdot x}$$

$$\partial_\mu j^\mu(x) = -i q_\mu j^\mu = 0 \Rightarrow \boxed{q_\mu j^\mu(x) = 0}$$

Let us calculate  $L^{\mu\nu}$ :



$$L^{\mu\nu} = \frac{1}{2s+1} \sum_{s'} \bar{u}_\alpha(\ell, s) (-ie\delta^\nu)_{\alpha\beta} u(\ell', s') \bar{u}_\beta(\ell', s') (+ie\delta^\mu)_{\beta\alpha} u(\ell)$$

Spin products  $\sum_{s'} u_\beta(\ell', s') u_\alpha(\ell', s') = (\ell' + m)_{\beta\alpha}$

$$u_b(\ell, s) \bar{u}_\alpha(\ell, s) = \left[ \frac{(\ell + m)(1 + \delta_{sS})}{2} \right]_{b\alpha}$$

where

$$\delta_S = +i\delta^0\delta^1\delta^2\delta^3, \quad \delta^{5+} = \delta_S, \quad (\delta_S)^2 = 1, \quad \{ \delta_S, \delta^{\mu\nu} \} = 0$$

$$L^{\mu\nu} = \frac{e^2}{2} \underbrace{(\ell + m)_{b\alpha} \delta^\nu_{\alpha\beta} (\ell' + m)_{\beta\alpha} (\delta^{\mu})_{ab}}_{\text{trace}}$$

trace

neglect  $m$  and we have

$$L^{\mu\nu} = \frac{e^2}{2} \text{Tr} (\ell \delta^\mu \ell' \delta^\nu)$$

Traces

$$\text{Tr}(\text{odd } \# \gamma) = 0$$

$$\text{Tr}(\not{a}\not{b}) = 4a \cdot b$$

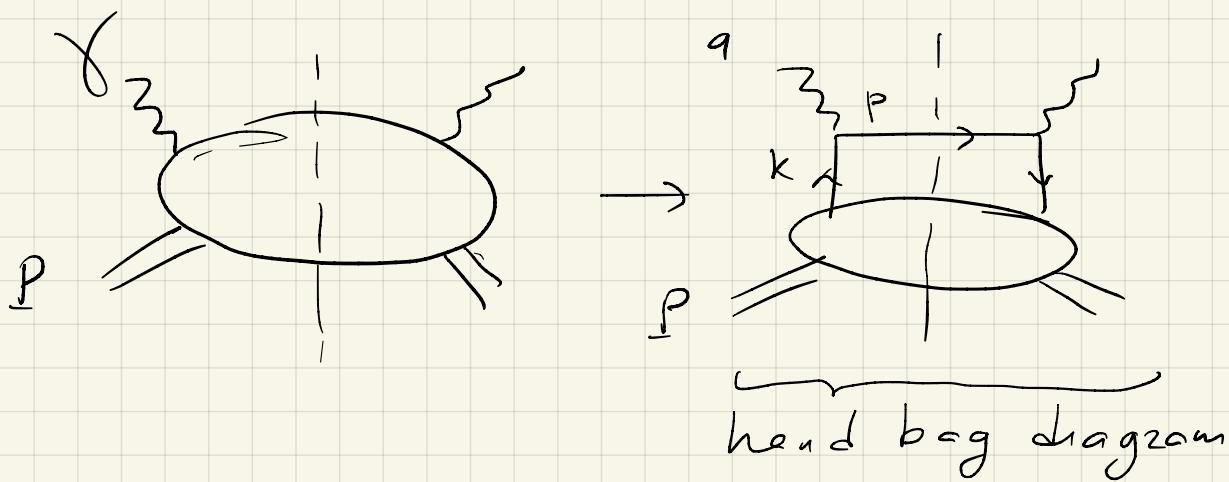
$$\text{Tr}(\not{a}\not{b}\not{c}\not{d}) = 4[(a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(c \cdot b)]$$

$$\text{Tr}(\gamma^a \gamma^b \gamma^c \gamma^d) = 4(g^{ab}g^{cd} - g^{ac}g^{bd} + g^{ad}g^{cb})$$

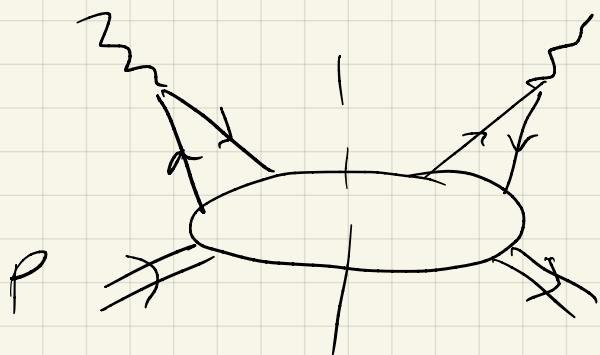
Thus,

$$L^{\mu\nu} = 2e^2(e^\mu e'^\nu + e^\nu e'^\mu - g^{\mu\nu}(e \cdot e'))$$

Now let us consider the hadronic tensor



why do we not consider?

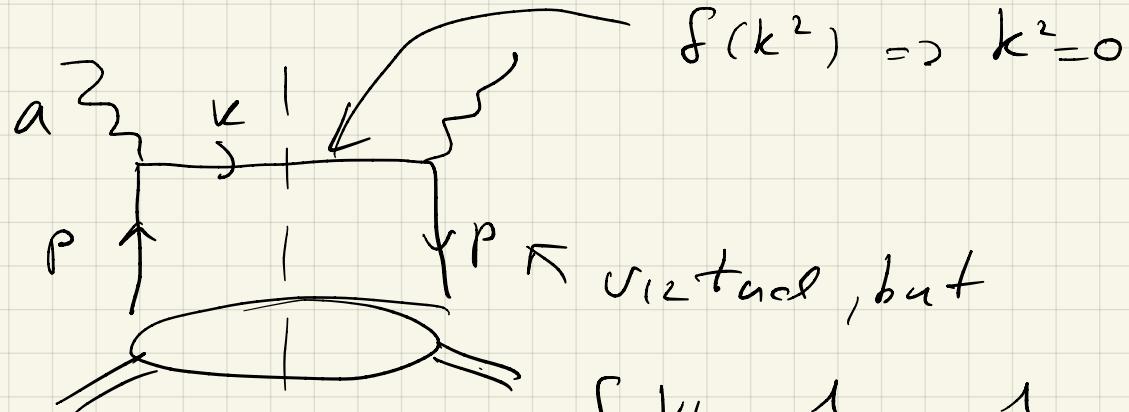


$\propto e_{q_1} e_{q_2}$   
suppressed by  $(\frac{1}{Q^2})^2$   
as at least one  
of the propagators is hard

$$\frac{1}{p^2 + i\epsilon} = \text{Im} \frac{1}{p^2 + i\epsilon} = \pi \delta(p^2)$$

$$q \quad p \\ \swarrow \quad \searrow \\ k \quad \rightarrow \quad \delta^{(4)}(q + k - p) \Rightarrow p = k + q$$

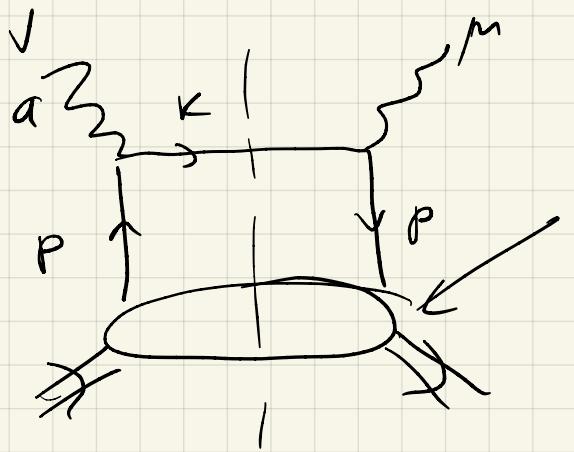
Why partons are almost on mass shell?



$$\int d^4 p \frac{1}{(p^2 + i\epsilon)} \frac{1}{(p^2 - i\epsilon)} \rightarrow p^2 \approx 0$$

The contribution from  
this integral is much then

$p^2 \approx 0$  as well!



let us call this matrix  $\phi(p, P)$

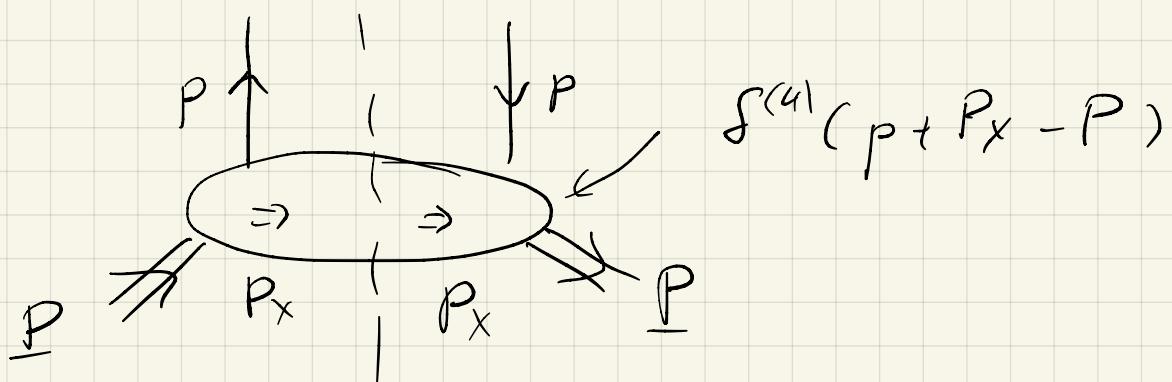
$$W^{\mu\nu} = \sum_q e_q^2 \int \frac{d^4 p}{(2\pi)^4} \text{Tr} (\delta^\mu(p+q) \delta^\nu(p, P) \delta((p+q)^2))$$

Let us parametrize  $p = x \frac{P}{x + P}$ ,  $x \in (-\infty, \infty)$

$$\delta((p+q)^2) = \delta(-Q^2 + 2xP \cdot q) = \frac{1}{2P \cdot q} \delta(x_{B_J} - x)$$

quarks are probed at  $x = x_{B_J}$  !

What is  $\phi$ ?



$$\delta^4(p) = \int \frac{d^4 \vec{z}}{(2\pi)^4} e^{-i p \cdot \vec{z}}$$

$$\oint_x = \int \frac{d^3 P_x}{2E_x (2\pi)^3} = \int \frac{d^4 P_x}{(2\pi)^4} \partial(E_x)$$

$$\bar{\Phi} = \oint_x \delta^{(4)}(p + P_x - P) \langle p | \bar{\psi}(0) | x \rangle \langle x | \psi(0) | p \rangle$$

$$= \oint_x \int \frac{d^4 \vec{z}}{(2\pi)^4} e^{-i \vec{z} \cdot (p + P_x - P)} \langle p | \bar{\psi}(0) | x \rangle \langle x | \psi(0) | p \rangle$$

$$= \oint_x \int \frac{d^4 \vec{z}}{(2\pi)^4} e^{-i \vec{z} \cdot P} \underbrace{\langle p | e^{i \vec{z} \cdot P} \bar{\psi}(0) e^{-i \vec{z} \cdot P_x} | x \rangle}_{\langle p | e^{i \vec{z} \cdot P} \bar{\psi}(0) e^{-i \vec{z} \cdot P} | x \rangle} \underbrace{\langle x | \psi(0) | p \rangle}_{\bar{\psi}(z) - \text{shift of the field}}$$

Thus

$$\Phi(p, P) = \int_X \frac{d^4 z}{(2\pi)^4} e^{-i p \cdot z} \langle P | \bar{\psi}(z) | X \rangle \langle X | \psi(0) | P \rangle$$

now we use

$$\int_X |X\rangle \langle X| = \mathbb{I} \text{ completeness}$$

of states and obtain

$$\Phi(p, P) = \int \frac{d^4 z}{(2\pi)^4} e^{-i p \cdot z} \langle P | \bar{\psi}(z) \psi(0) | P \rangle$$

Let us introduce light cone variables  $A^\pm = \frac{A^0 \pm A^3}{\sqrt{2}}$

$$A \cdot B = A^+ B^- + A^- B^+ - \vec{A}_T \cdot \vec{B}_T, \quad A_T = (A^1, A^2)$$

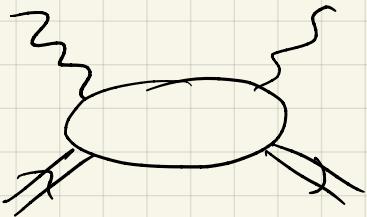
$$P \approx (P^+, \frac{M^2}{2P^+}, 0) \approx (P^+, 0, 0)$$

$$p = x P \approx (P^+, 0, p_T) \quad \begin{matrix} \nwarrow \\ \text{important for TMDs} \end{matrix}$$

$$p \cdot z \rightarrow p^+ z^- - \underbrace{\vec{p}_T \cdot \vec{z}_T}_{\text{small}} \approx p^+ z^-$$

$$\Rightarrow z \approx (0, z^-, 0)$$

Let us see how distributions are introduced  
in DIS.



$$W^{\mu\nu} = -\left(g^{\mu\nu} + \frac{q^\mu q^\nu}{Q^2}\right)W_1 + \left(P^\mu + \frac{q^\mu}{2x}\right)\left(P^\nu + \frac{q^\nu}{2x}\right)W_2$$

$$(\text{only } W^{\mu\nu} = W^{\nu\mu} \text{ & } q_\mu W^{\mu\nu} = 0)$$

Remember  $P \cdot q = \gamma$

One usually uses  $\begin{cases} F_1(x, Q^2) = W_1(x, \theta) \\ F_2(x, Q) = \gamma W_2(x, \theta) \end{cases}$

$$F_L = F_2 - 2x F_1 \approx 0$$

$$P^\mu = (P, 0, 0, P)$$

$$\underline{P}^2 = h^2 = 0$$

$$h^\mu = \left(\frac{1}{2P}, 0, 0, -\frac{1}{2P}\right)$$

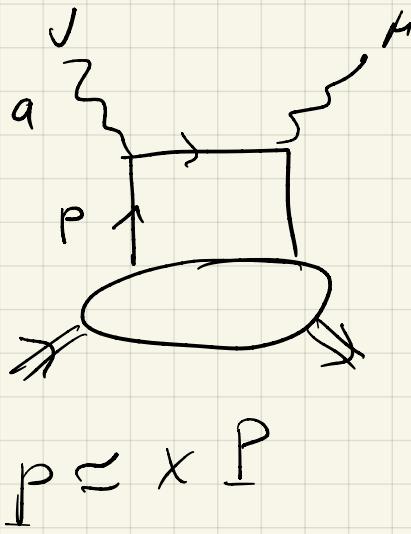
$$P \cdot h = 1$$

$$q^\mu = q_{\perp}^\mu + \gamma n^\mu$$

$$q^2 = -\vec{q}_{\perp}^2 = -Q^2$$

Then

$$n^\mu n^\nu W_{\mu\nu} = W_L = \frac{1}{v} F_2$$



$$W^{\mu\nu} = e_a^4 \int \frac{d^4 p}{(2\pi)^4} \text{Tr} (\gamma^\mu (\not{q} + \not{p}) \gamma^\nu \phi) \cdot \underbrace{\delta((p+q)^\nu)}_{\frac{1}{2\sqrt{v}} \delta(x - x_B)}$$

$$F_2 = v n^\mu n^\nu W_{\mu\nu} = \frac{1}{2} e_a^2 \int \frac{d^4 p}{(2\pi)^4} \text{Tr} (\not{p} \not{q} \not{\phi}) \delta(x - x_B) \underbrace{- \not{p} + 2n \cdot p}_{2x \text{Tr}(\not{p} \not{\phi})}$$

we can define

$$f(x_B) = \int \frac{d^4 p}{(2\pi)^4} \text{Tr} (\not{p} \not{\phi}(p, P)) \delta(x - x_B)$$

Parton distributions

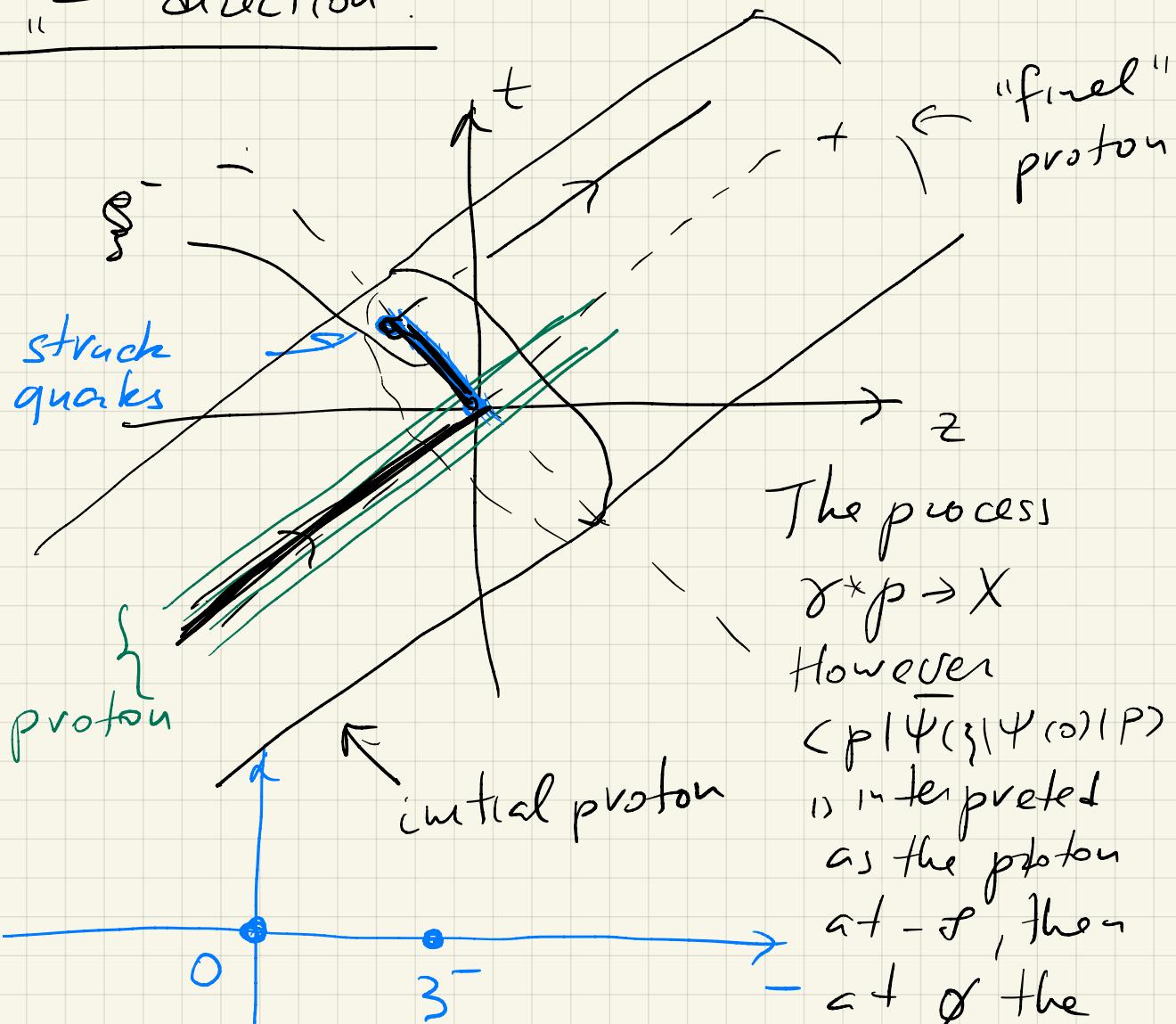
$$\Rightarrow F_2(x, Q^2) = \sum_q e_q^2 \times f(x)$$

Bjorken scaling!

$$\not{p} = \not{\gamma}^+$$

The fields are separated by some distance  $\beta$

In " - " direction!



The process

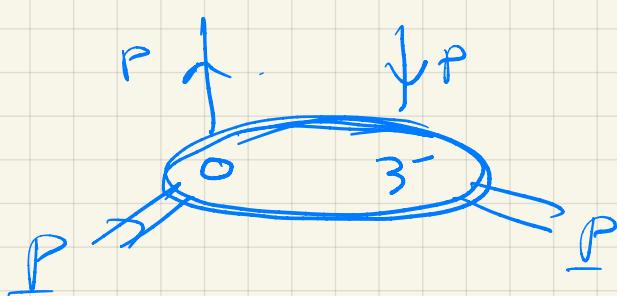
$$\gamma^* p \rightarrow X$$

However

$$\langle p | \bar{\psi}(3) \psi(0) | P \rangle$$

is interpreted  
as the proton  
at  $-s$ , then

at  $t$  of the  
quark field is  
shifted to  $3$   
and then  
the proton  
continues to  $+s$



Analogously to DIS we can define the following projections of the correlator  $\underline{\Phi}(p, \underline{P})$  in the case when transverse motion is not ignored.

It is customary to call the parton's momentum  $k$   
so

$$\underline{\Phi}(x, k_T)_{ij} = \int \frac{d\zeta - d\zeta_T^2}{(2\pi)^3} e^{-ix^+ \zeta^- + ik_T \cdot \zeta_T}$$

$$\langle P | \bar{\psi}_j(\zeta) \psi_i(0) | P \rangle \Big|_{\zeta^+ = 0}$$

$$\frac{1}{2} T_2(\gamma^+ \phi) = f_1 - \frac{\epsilon^{jk} k_T^j S_T^k}{M} f_{1T}^\perp$$

$$\frac{1}{2} T_2(\gamma^+ \gamma_S \phi) = S_L g_1 + \frac{\bar{k}_T \cdot \bar{S}_T}{M} g_{1T}^\perp$$

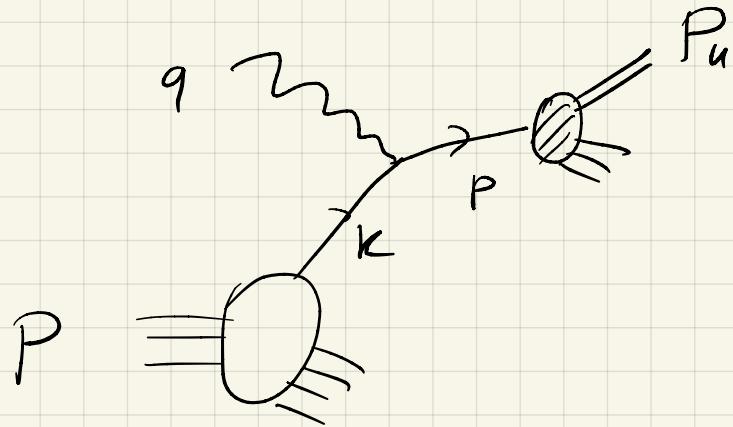
$$\frac{1}{2} T_2(i \gamma^j \gamma_S \phi) = S_T^j h_1 + S_L \frac{k_T^j}{M} h_{1L}^\perp + \frac{\kappa^{jk} S_T^k}{M} h_{1T}^\perp$$

$$+ \frac{\epsilon^{jk} k_T^k}{M} h_2 \quad \text{, where } \kappa^{jk} = (k_T^j k_T^k - \frac{1}{2} k_T^2 \delta^{jk})$$

$$\epsilon^{ij} = \epsilon^{-+ij}, \epsilon^{0123} = +1$$

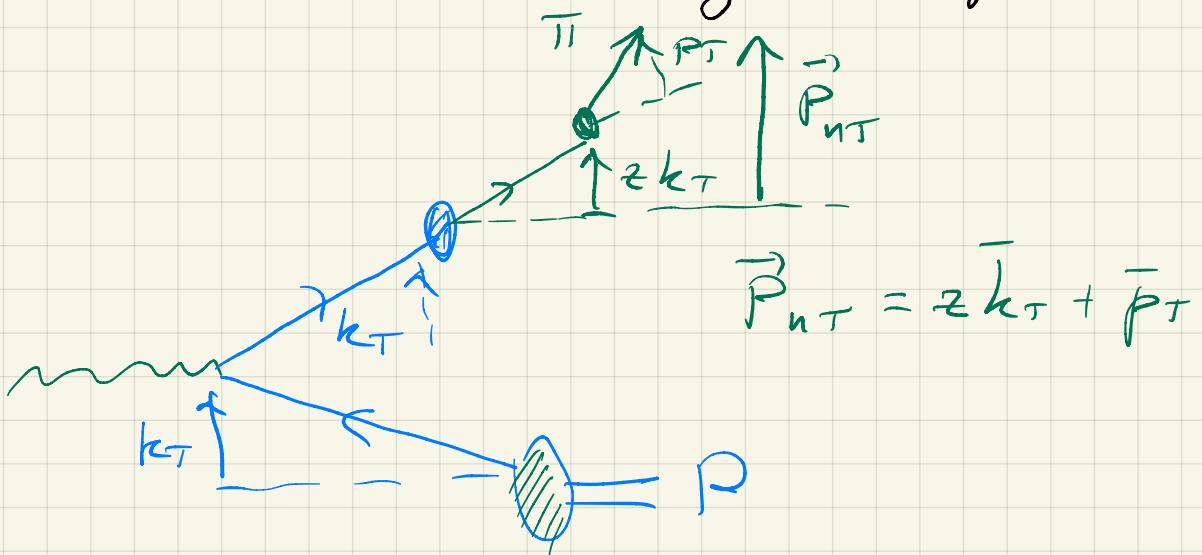
Let us work out convolutions for SIDIS

$$C[\omega fD] = \sum_q e_q^2 \int d^2 k_T d^2 p_T \delta^{(2)}(\vec{P}_{qT} - \vec{k}_T - \vec{p}_T) \omega(k_T, p_T) f(x, k_T) D(z, p_T)$$

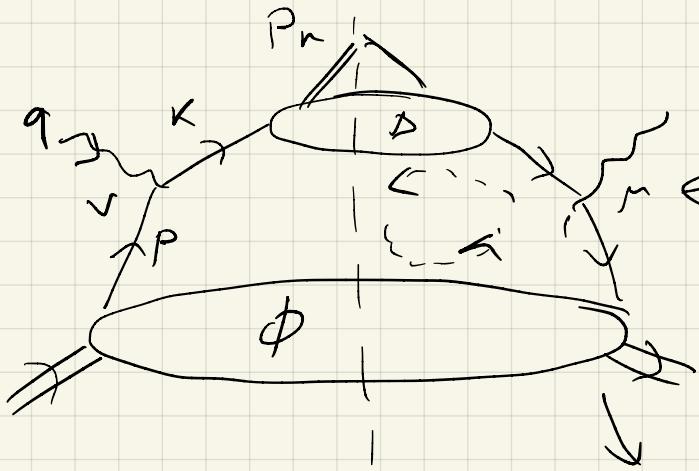


In order to study evolution we need to rewrite the convolution in the configuration space.

We will see that TMD evolution equations are to be solved in configuration space



Let us write the cut amplitude:



Starting from here we "read" the diagram similar to others in clockwise

$$W^{\mu\nu} = \sum_q e_q^2 \int \frac{d^4 p}{(2\pi)^4} \text{Tr} (\gamma^\mu D \gamma^\nu \phi) S^{(u)}(k - p - q)$$

$\phi$  and  $D$  contain TMDs

$$\frac{1}{2} \text{Tr} [\gamma^+ \phi] = f_1 - \frac{\epsilon^{ijk} P_T^j S_T^k}{m} f_{1T}$$

$$\frac{1}{2} \text{Tr} [\gamma^- D] = D_1$$

$$G \sim \frac{1}{Q^4} \langle \mu_\nu W^{\mu\nu} \rangle \sim G_0 (F_{\mu\nu} + \dots)$$

structure functions:  
 $F_{\mu\nu} = C [1 f_1 D_1]$  etc

Asymmetries ratios of polarised over unpolarised structure functions

$$A_{UT} = \frac{S_{1T} (f_{1U} - f_{1T})}{F_{1U}} = \frac{F_{1T}}{F_{1U}} \text{ etc}$$

Definitions: of Fourier-Bessel transform

$$f(x, k_T^2) = \int \frac{d^2 b_T}{(2\pi)^2} e^{i \bar{k}_T \bar{b}_T} \tilde{f}(x, b_T^2) =$$

$$= \int \frac{b_T db_T}{2\pi} J_0(k_T b_T) \tilde{f}(x, b_T^2)$$

NB  $\int d\varphi e^{i k_T b_T \cos \varphi} = 2\pi J_0(k_T b_T)$

$$\tilde{f}(x, b_T^2) = \int d^2 k_T e^{-i \bar{k}_T \bar{b}} f(x, k_T^2) =$$

$$= 2\pi \int k_T dk_T J_0(b_T k_T) f(x, k_T^2)$$

Using Mathematica prove these relations, and

prove that if

$$f(x, k_T^2) = f_1(x) \frac{1}{\pi \langle k_T^2 \rangle} e^{-k_T^2 / \langle k_T^2 \rangle}$$

then

$$\tilde{f}(x, b_T^2) = f_1(x) e^{-\langle k_T^2 \rangle b_T^2 / 4}$$

Notice that

$$\int d^2 k_T f(x, k_T^2) = f_1(x) \leftarrow \text{The basis}$$

of the Generalized Parton Model (Feynman 78')

In TMD phenomenology the following moments are used

$$\int d^2 k_T \frac{k_T^2}{2M^2} f(x, k_T^2) = f^{(1)}(x)$$

the first moment

$$\int d^2 k_T \left( \frac{k_T^2}{2M^2} \right)^2 f(x, k_T^2) = f^{(2)}(x)$$

the second moment

In configuration space we have

$$\tilde{f}^{(1)}(x, b_T) = \frac{2\pi}{M^2} \int k_T dk_T \frac{k_T}{b_T} J_1(k_T b_T) f(x, k_T^2)$$

$$\tilde{f}^{(2)}(x, b_T) = \frac{4\pi}{M^4} \int k_T dk_T \left( \frac{k_T}{b_T} \right)^2 J_2(k_T b_T) f(x, k_T^2)$$

Prove that

$$\tilde{f}^{(1)}(x, 0) = f^{(1)}(x)$$

$$\tilde{f}^{(2)}(x, 0) = f^{(2)}(x)$$

For fragmentation functions

$$D(z, p_T^2) = \int \frac{d^2 b_T}{(2\pi)^2} e^{-i \bar{p}_T \bar{b}_T / z} \tilde{D}(z, b_T^2)$$

$$= \int \frac{b_T db_T}{2\pi} J_0\left(\frac{p_T b_T}{z}\right) \tilde{D}(z, b_T^2)$$

and

$$\tilde{D}(z, b_T^2) = \int \frac{d^2 p_T}{2\pi} e^{-i \bar{b}_T \bar{p}_T / z} D(z, p_T^2)$$

$$= 2\pi \int \frac{p_T dp_T}{2\pi} J_0\left(\frac{p_T b_T}{z}\right) D(z, p_T^2)$$

Test functions (check with Mathematica)

$$D(z, p_T^2) = D_1(z) \frac{1}{\pi \langle p_T^2 \rangle} e^{-p_T^2 / \langle p_T^2 \rangle}$$

$$\tilde{D}(z, b_T^2) = \frac{1}{z^2} D_1(z) e^{-\frac{\langle p_T^2 \rangle b_T^2}{4z^2}}$$

Moments are also important for FFs

$$\tilde{D}^{(n)}(z, b_T^2) = \frac{2\pi n!}{(M_n^2)^n} \int \frac{p_T dp_T}{z^2} J_n\left(\frac{p_T + b_T}{z}\right) \left(\frac{p_T}{zb_T}\right)^n D(z, p_T^2)$$

$$D(z, p_T^2) = \frac{(M_n^2)^n}{2\pi n!} \int b_T dh_T \left(\frac{zb_T}{p_T}\right)^n J_n\left(\frac{p_T + b_T}{z}\right) \tilde{D}^{(n)}(z, b_T)$$

$$\lim_{b_T \rightarrow 0} \tilde{D}^{(n)}(z, b_T) = \frac{1}{z^2} D^{(n)}(z) \quad \text{prove it!}$$

where

$$D^{(n)}(z) = \int d^2 p_T \left(\frac{p_T^2}{2z^2 M_n}\right)^n D(z, p_T^2)$$

Test with Mathematica

$$D(z, p_T) = D_1(z) \frac{1}{\bar{n}(p_T^2)} e^{-p_T^2/\langle p_T^2 \rangle} \quad \text{prove}$$

$$\rightarrow \tilde{D}(z, b_T^2) = \frac{1}{z^2} D_1(z) e^{-\frac{b_T^2 \langle p_T^2 \rangle}{4z^2}}$$

$$D(z, b_T^2) = H_1^{(1)}(z) \frac{z^2 M_n^2}{\bar{n}(p_T^2)^2} e^{-p_T^2/\langle p_T^2 \rangle} \quad \text{prove}$$

$$\rightarrow \tilde{D}^{(1)}(z, b_T^2) = H_1^{(1)}(z) \frac{1}{z^2} e^{-\frac{b_T^2 \langle p_T^2 \rangle}{4z^2}}$$

Now convolution in the momentum space  $\rightarrow b_T$  space

$$C[\omega f D] = \sum_a e_a^2 \int d^2 k_T d^2 p_T \delta^{(2)}(\bar{P}_{aT} - z \bar{k}_T - \bar{p}_T)$$

$$\omega(k_T b_T) f(x, k_T) D(z, p_T), \quad \bar{P}_{aT} = -z \bar{q}_T$$

$$\text{We rewrite } \delta^{(2)}(\bar{P}_{aT} - z \bar{k}_T - \bar{p}_T) =$$

$$= \delta^{(2)}(-z \bar{q}_T - z \bar{k}_T - \bar{p}_T) = \frac{1}{z^2} \delta^{(2)}(\bar{q}_T + \bar{k}_T + \frac{\bar{p}_T}{z})$$

$$= \frac{1}{z^2} \int \frac{d^2 b_T}{(2\pi)^2} e^{-i(\bar{q}_T + \bar{k}_T + \bar{p}_T/z) \cdot \bar{b}_T}$$

$$F_{aT} = C[1f_1 D_1] = \sum_a e_a^2 \int \frac{d^2 b_T}{(2\pi)^2} e^{-i \bar{b}_T \cdot \bar{q}_T}$$

$$* \int d^2 k_T e^{-i \bar{b}_T \cdot \bar{k}_T} f_1(x, k_T)$$

$$\frac{1}{z^2} \int d^2 p_T e^{-i \bar{b}_T \cdot \bar{p}_T/z} D_1(z, p_T) =$$

$$= \sum_a e_a^2 \int \frac{d^2 b_T}{(2\pi)^2} e^{-i \bar{b}_T \cdot \bar{q}_T} \tilde{f}_1(x, b_T) \tilde{D}_1(z, b_T)$$

$$= \sum_a e_a^2 \int \frac{b_T db_T}{2\pi} g_0(b_T q_T) \tilde{f}_1(x, b_T) \tilde{D}_1(z, b_T)$$

$$\text{Let us call } B(\tilde{f}_1, \tilde{D}_1) = \sum_a e_a^2 \int \frac{b_T db_T}{2\pi} g_0(b_T q_T) \tilde{f}_1 \tilde{D}_1$$

A more complicated example, Sivers asymmetry

$$F_{u\bar{u}} \stackrel{\sin(\phi_u - \phi_s)}{=} C \left[ - \frac{\hat{h} \cdot \vec{k}_T}{M} f_{1T}^\perp D_1 \right]$$

$$= \sum_q e_q^2 \int d^2 k_T d^2 p_T \delta^{(2)}(\vec{p}_{u\bar{u}} - z \vec{k}_T - \vec{p}_T) \underbrace{\left( - \frac{\hat{h} \cdot \vec{k}_T}{M} \right)}_{-\frac{k_T}{M} \cos(\varphi - \phi_h)}$$

- $f_{1T}^\perp(x, k_T^2) D_1(z, p_T^2)$

$$= \sum_q e_q^2 \int \frac{dz b_T}{(2\pi)^2} e^{-i \vec{q}_T \vec{b}_T} \int d^2 k_T \left( - \frac{k_T}{M} \right) \cos(\varphi - \phi_u)$$

$$e^{-i \vec{k}_T \vec{b}_T} f_{1T}^\perp(x, k_T^2) \underbrace{\int d^2 p_T \frac{1}{z^2} e^{-i \vec{p}_T \vec{b}_T/z} D_1(z, p_T^2)}_{\tilde{D}_1(z, b_T^2)}$$

we have  $\vec{k}_T \vec{b}_T = b_T k_T \cos(\varphi - \varphi_b)$

$$\int d\varphi e^{-i b_T k_T \cos(\varphi - \varphi_b)} \cos(\varphi - \phi_u) = \underbrace{\text{Use Mathematica}}$$

$$= -2\pi i \Im_1(b_T k_T) \cos(\varphi_b - \phi_u)$$

$$\int d\varphi_b \cos(\varphi_b - \phi_u) e^{-i b_T q_T \cos(\varphi_b - \phi_u)} = \underbrace{\gamma}$$

$$= 2\pi i \Im_1(b_T q_T)$$

So that

$$F_{UT}^{\sin(\phi_u - \phi_s)} = - \sum_q e_q^2 \int \frac{db_T b_T}{2\pi} J_1(b_T q_T)$$

$$\int dk_T \frac{k_T^2}{M} J_1(b_T k_T) f_{1T}^{\perp}(x, k_T^2) \tilde{D}_1(z, b_T^2)$$

$$\tilde{f}_{1T}^{\perp(1)}(x, b_T^2) = \frac{2\pi}{M^2} \int k_T dk_T \frac{k_T}{b_T} J_1(k_T b_T) f_{1T}^{\perp}(x, k_T^2)$$

Thus

$$F_{UT}^{\sin(\phi_u - \phi_s)} = (-M) \sum_q e_q^2 \int \frac{b_T db_T}{2\pi} \underbrace{b_T J_1(b_T q_T)}_{\text{①}} \\ \cdot \tilde{f}_{1T}^{\perp(1)}(x, b_T^2) \tilde{D}_1(z, b_T^2) \quad \text{②}$$

$$F_{UT}^{\sin(\phi_u - \phi_s)} = -M B [\tilde{f}_{1T}^{\perp(1)} \tilde{D}_1]$$

General definition

$$B[\tilde{f}^{(n)} \tilde{D}^{(m)}] = \int \frac{b_T db_T}{2\pi} b_T^{n+m} J_{n+m}(b_T q_T)$$

$$\tilde{f}^{(n)}(x, b_T^2) \tilde{D}^{(m)}(z, b_T^2)$$

Collins asymmetry

$$F_{u\bar{u}}^{\sin(\phi_u + \phi_s)} = C [\omega_A^{(1)} h_1 + h_1^\perp]$$

$$\omega_A^{(1)} = \frac{\hat{h} \cdot \vec{p}_+}{2M_u}$$

$$F_{u\bar{u}}^{\sin(\phi_u + \phi_s)} = M_u B \left[ \tilde{h}_1^{(0)} \tilde{h}_1^\perp \right]$$

Let us use (0) moment  
as well for completeness

Prove it!

## Evolution of TMDs

The evolution is automatically studied for TMDs, in the coordinate space by

$$G \propto \int \frac{d^2 b_T}{(2\pi)^2} e^{-i q_T b_T} \tilde{f}(x, b_T, Q, \zeta) \tilde{D}(z, b_T, Q, \zeta)$$

$\zeta$  corresponds to the UV divergence, the same as for collinear densities

$\zeta$  is a new scale that correspond to a new type of divergences "rapidity" divergence of TMDs

Let us define momentum regions

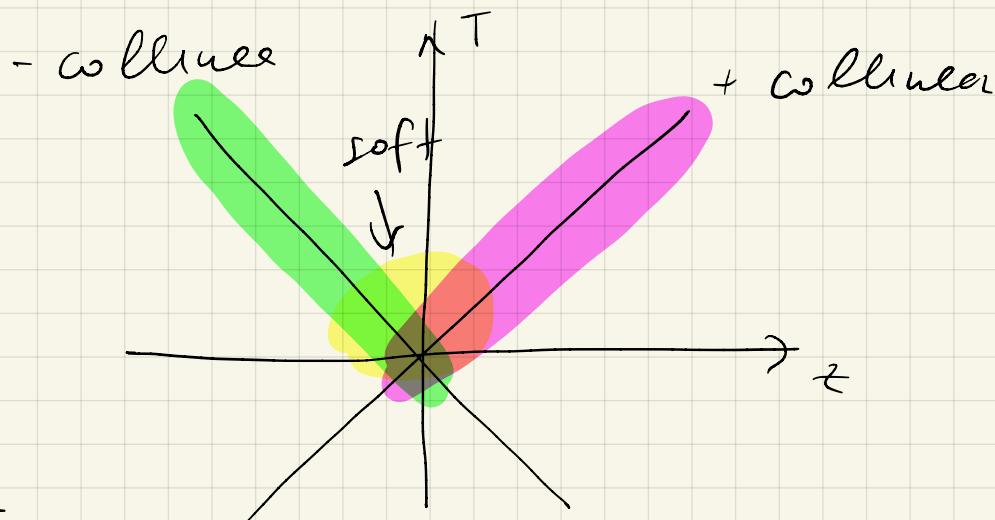
1) Hard region - momentum with large virtualities

$$\sim Q, \quad k \sim Q(1, 1, 1)$$

$\begin{matrix} \nearrow & \nearrow & \nearrow \\ + & - & T \end{matrix}$

2) Collinear region - momentum close to some beam/jet directions  $k \sim Q(1, \lambda, \lambda)$  for example

3) Central (soft) region  $k \sim Q(\lambda^u, \lambda^u, \lambda^u)$   $u > 0$   
 $x \ll 1$



For each region approximations are applied  
and then double counting is subtracted

Result



Some work is still needed to fully factorise. Ward identities are used to strip off collinear polarized gluons from hard part and organise them into Wilson lines

Simple

$$p - k \xrightarrow{\quad} k \xrightarrow{\quad} (p - k)^2 = -2p \cdot k + k^2 \approx -2p \cdot k$$

eikonal approximation

TMD factorization describes processes differential  
in transverse momentum  $\frac{dG}{dq_T^2}$

$q_T \sim \Lambda_{\text{QCD}} \ll Q$  : small  $k_T$  of partons

play an important role at small  $q_T \Rightarrow$  TMD factorization  
with  $\text{TMDsf}(x, k_T)$

Generalized Parton Model

$F(x, k_T)$

$\downarrow \text{QCD}$

$F(x, k_T, \mu, \xi) \rightarrow$  uniquely defined  
deal with all divergencies  
obey evolution equations

QCD evolution is governed by the so-called Collins-Soper equation two Renormalization Group equations

CS equation ↗ to space

$$(1) \quad \frac{\partial \ln \tilde{F}(x, b_T, \mu, \tau)}{\partial \ln \sqrt{s}} = \tilde{K}(b_T, \mu)$$

↑  
CS kernel

RG equations

$$(2) \quad \frac{d \tilde{K}(b_T, \mu)}{d \ln \mu} = -\gamma_K(g(\mu)) \leftarrow \begin{array}{l} \text{Cusp anomalous} \\ \text{dimension of } K \end{array}$$

Very universal in QCD.,  $\gamma_K$  depend only on  $\mu$

$$(3) \quad \frac{d \ln \tilde{F}(x, b_T, \mu, \tau)}{d \ln \mu} = \gamma_F(g(\mu), \tau/\mu) \leftarrow \begin{array}{l} \text{anomalous dimension} \\ \text{of } F. \end{array}$$

$$\frac{d}{d \ln \mu} \left( \frac{\partial \ln F(x, b_T, \mu, \varsigma)}{\partial \ln \sqrt{\varsigma}} \right) = \frac{d}{d \ln \mu} \tilde{K}(b_T; \mu) = -\gamma_K(\mu)$$

$$\frac{\partial}{\partial \ln \sqrt{\varsigma}} \left( \frac{\partial \ln F(x, b_T, \mu, \varsigma)}{\partial \ln \mu} \right) = -\gamma_F(\mu)$$

$$\gamma_F(\mu, \varsigma/\mu^2)$$

$$\gamma_F(\mu, 3/\mu^2) - \gamma_F(\mu, 3/\mu^1) = -\gamma_K(\mu) \ln \sqrt{\varsigma_0} + \gamma_K \ln \sqrt{\varsigma}$$

if  $\varsigma_0 = \mu^2$  then

$$\underline{\gamma_F(\mu, 3/\mu^2) = \gamma_F(\mu, 1) - \frac{1}{2} \gamma_K(\mu) \ln 3/\mu^2}$$

Solutions

$$1) \quad \frac{d \tilde{K}(b_T, \mu)}{d \ln \mu} = -\gamma_K(\mu) \Rightarrow$$

$$\int_{\mu_0}^{\mu} d \tilde{K}(b_T, \mu') = - \int_{\mu_0}^{\mu} \gamma_K(\mu') \frac{d \mu'}{\mu'}$$

$$\tilde{K}(b_T, \mu) = - \int_{\mu_0}^{\mu} \frac{d \mu'}{\mu'} \gamma_K(\mu') + \tilde{K}(b_T, \mu_0)$$

$$2) \quad \tilde{F}(x, b_T, \mu, \zeta) = \tilde{F}(x, b_T, \mu, \zeta_0) \exp \left[ \tilde{K}(b_T, \mu) \ln \sqrt{\frac{\zeta}{\zeta_0}} \right]$$

$$3) \quad \tilde{F}(x, b_T, \mu_0, \zeta) = \tilde{F}(x, b_T, \mu_0, \zeta) \exp \left[ \int_{\mu_0}^{\mu} \frac{d \mu'}{\mu'} \gamma_F(\mu') \frac{\zeta - \zeta_0}{\mu'} \right]$$

## Implementing evolution

We start with low  $b_T$

$$\tilde{F}_f(x, b_T, \mu, \bar{s}) = \sum_j \int_x^{\hat{x}} \frac{dx}{\hat{x}} \underbrace{\tilde{C}_{j/f}(\frac{x}{\hat{x}}, b_T, \mu, \bar{s})}_{\text{coefficient functions}} \underbrace{f_j(\hat{x}, \mu)}_{\text{collinear PDFs}}$$

at the lowest order

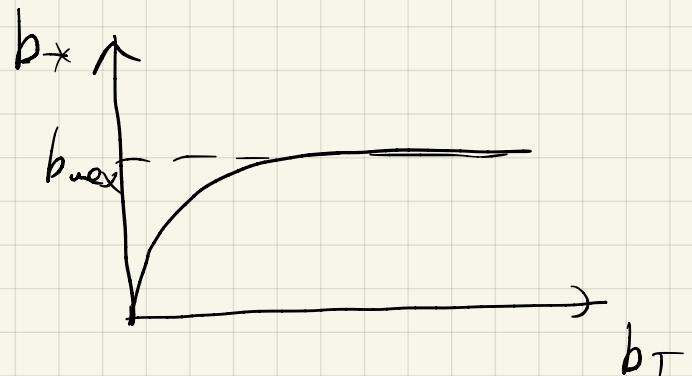
$$\tilde{C}_{j/f} = \delta_{jf} \delta\left(\frac{x}{\hat{x}} - 1\right)$$

Next step: combine perturbative & non perturbative  $\rightarrow b_*$

Problem:  $\tilde{K}(b_T)$ ,  $\tilde{F}(b_T)$  are non perturbative at large  $b_T$

We want: write functions such that they are perturbatively calculable with non perturb. corrections

$$b_*(b_T) = \frac{b_T}{\sqrt{1 + b_T^2/b_{\max}^2}}$$



$$\tilde{K}(b_T, \mu) = \tilde{K}(b_*, \mu) + \underbrace{[\tilde{K}(b_T, \mu) - \tilde{K}(b_*, \mu)]}_{g_K(b_T)}$$

non pert. function

$$\tilde{K}(b_T, \mu) = \tilde{K}(b_*, \mu_0) - \underbrace{\int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \delta_K(\mu') - g_K(b_T)}$$

study in Mathematics

$$g_K = \frac{1}{2} g_2 b_T, \quad g_K = g_0 b_T b_*, \quad g_K = g_2 \ln \frac{b_T}{b_*}$$

different groups

Euler gamma  
↑  
 $-\gamma_E$

We can choose  $\mu_0 \sim 1/b_T$ ,  $\mu_0 = \frac{2e^{-\gamma_E}}{b_*}$  is  
the standard choice

$$\tilde{F}(x, b_T, \mu, \beta) = \tilde{F}(x, b_*, \mu, \beta) \left[ \frac{\tilde{F}(x, b_T, \mu, \beta)}{\tilde{F}(x, b_*, \mu, \beta)} \right] =$$

$$= \tilde{F}(x, b_*, \mu, \beta_0) \exp \left[ \tilde{K}(b_*, \mu) \ln \sqrt{\frac{\beta}{\beta_0}} \right] \underbrace{\left[ \frac{\tilde{F}(x, b_T, \mu, \beta_0)}{\tilde{F}(x, b_*, \mu, \beta_0)} \right]}_{\exp[-g(x, b_T)]}$$

$$\times \exp \left[ \ln \sqrt{\frac{\beta}{\beta_0}} (\tilde{K}(b_T, \mu) - \tilde{K}(b_*, \mu)) \right]$$

$$= \tilde{F}(x, b_*, \mu, \beta_0) \exp \left[ \ln \sqrt{\frac{\beta}{\beta_0}} \tilde{K}(b_*, \mu) \exp[-g(x, b_T) - \ln \sqrt{\frac{\beta}{\beta_0}} g_K(b_T)] \right]$$

$$= \tilde{F}(x, b_*, \mu_0, \gamma_0) \exp \left[ \sum_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \left( \gamma_F(\mu') - \ln \sqrt{\frac{\beta_0}{\beta}} \gamma_K(\mu') \right) \right]$$

$$\exp \left[ \ln \sqrt{\frac{\beta}{\beta_0}} \tilde{K}(b_*, \mu_0) - \sum_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \ln \sqrt{\frac{\beta}{\beta_0}} \gamma_K(\mu') \right]$$

$$\exp \left[ -g(x, b_T) - \ln \sqrt{\frac{\beta}{\beta_0}} g_K(b_T) \right]$$

$$\begin{aligned}
&= \tilde{F}(x, b_*, \mu_0, \beta_0) \exp \left\{ \ln \sqrt{\frac{3}{\beta_0}} \tilde{K}(b_*, \mu_0) \right. \\
&+ \left. \sum_{\mu_0} \frac{d\mu}{\mu} \left[ \gamma_F(\mu, 1) - \ln \sqrt{\frac{3}{\mu^2}} \gamma_K(\mu) \right] \right\} \\
&\times \exp \left\{ -g(x, b_T) - \ln \sqrt{\frac{3}{\beta_0}} g_K(b_T) \right\}
\end{aligned}$$

at small  $b_T$  and large  $\mu$ :

$$\begin{aligned}
\tilde{K}(b_T, \mu) &= - \frac{g^2 C_F}{4\pi^2} \left( \ln(\mu^2 b_T) - \underbrace{\ln 4 + 2\gamma_E}_{\text{the reason why}} \right) \\
&\mu_b \sim \frac{2e^{-\gamma_E}}{b_*} \text{ is chosen}
\end{aligned}$$

$$\tilde{K}(b_*, \mu_b) = 0$$

$$\beta_0 \text{ is a scale } \sim 1-2 \text{ (GeV}^2\text{)}$$

Generalized parton model weasley is.

$$\tilde{F}(x_1, b_T) \sim F(x_1) \exp[-g(x_1, b_T)].$$

## Lecture 3

### Elements of evolution of TMDs

We have studied so far how structure function can be written in terms of TMDs.

For instance

$$F_{\text{uu}} = C [1 f_1 D_1]$$

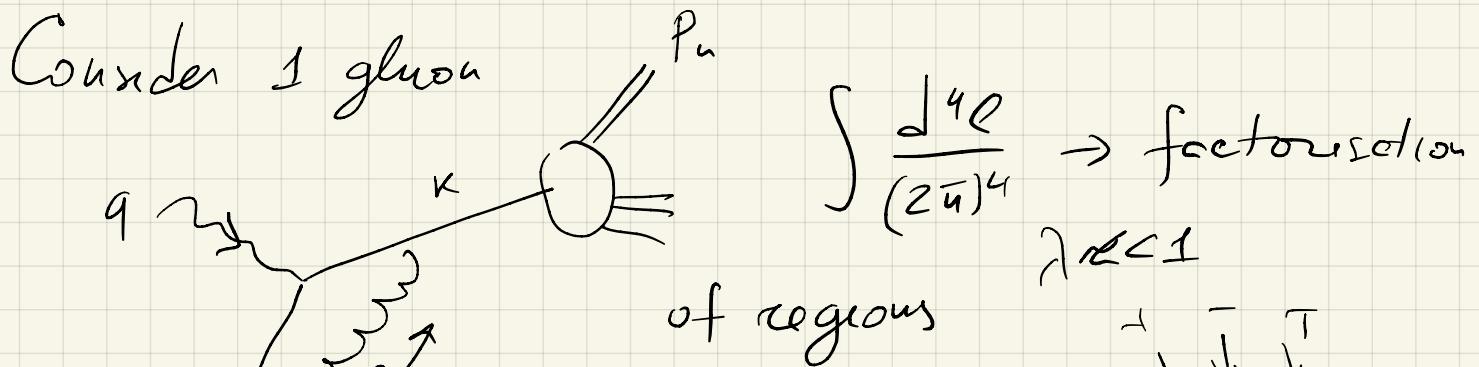
In the Generalized Parton Model one often uses

$$f_1(x, k_T) = f_1(x) \frac{1}{\pi \langle k_T^2 \rangle} e^{-k_T^2 / \langle k_T^2 \rangle}$$

$$D_1(z, p_T) = D_1(z) \frac{1}{\pi \langle p_T^2 \rangle} e^{-p_T^2 / \langle p_T^2 \rangle}$$

Of course this Gaussian dependence can be a good approximation of jetwise  $k_T$  dependence but what happens if we take into account gluon radiation?

My turn?  
Your turn?



$$P = \text{---}$$

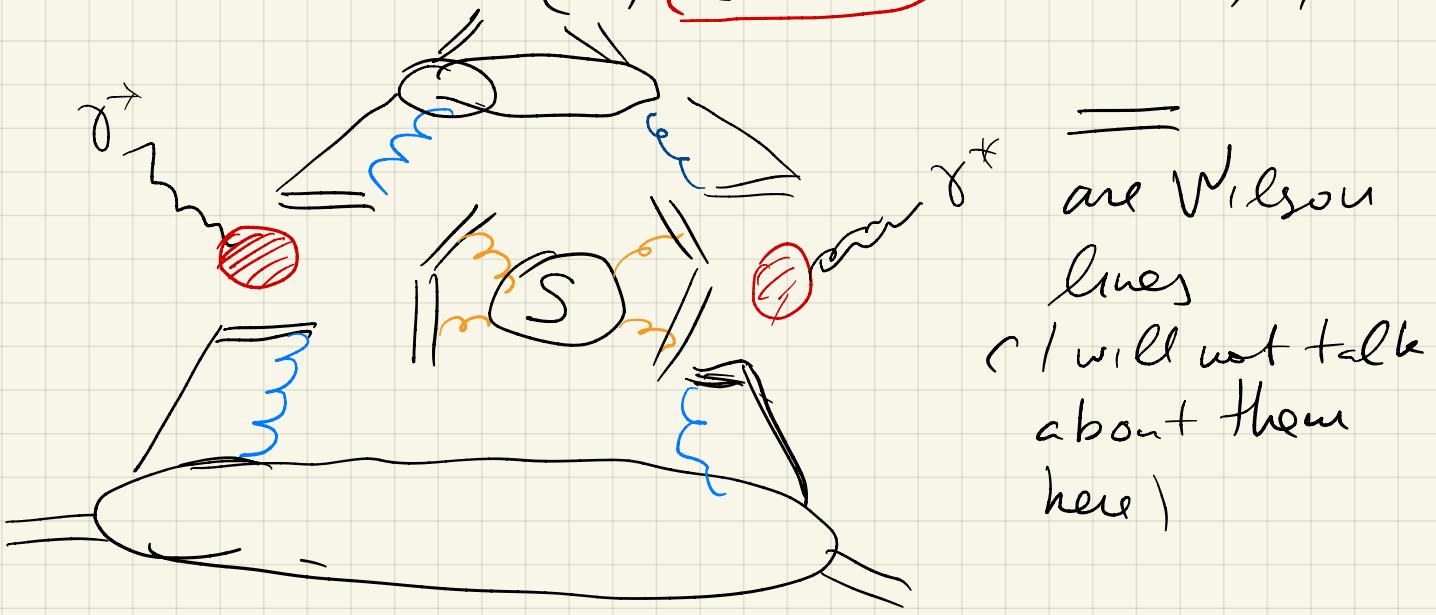
of regions

(1)  $\ell \parallel P$   $e \sim Q(1, \lambda^2, \lambda)$

(2)  $\ell \parallel k$   $e \sim Q(\lambda^2, 1, \lambda)$

(3)  $\ell$  soft  $e \sim Q(\lambda, \lambda, \lambda)$

(4)  $\ell$  hard  $e \sim Q(1, 1, 1)$



$$G \sim \int d^2 k_T d^2 p_T d^2 \ell_T H(Q) f(\alpha, k_T) D(z, p_T)$$

$\cdot S(e_T) \delta^{(2)}(\vec{P}_{nT} - z \vec{k}_T - \vec{p}_T - \vec{\ell}_T)$

gluon radiation

$$G \sim \int \frac{d^2 b_T}{(2\pi)^2} e^{i \bar{P}_{bT} \bar{b}_T / z} \underbrace{H(Q) \tilde{f}(x, b_T) \tilde{D}(z, b_T) S(b_T)}$$

See next lecture for the proof

The additional factors  $S(b)$  is absorbed into  $\tilde{f}$  &  $\tilde{D}$ . It leads to cancellation of divergencies and self consistent definition

$$\tilde{f}(x, b_T, Q, S) \rightarrow \tilde{f}(x, b_T) \sqrt{S(b_T)}$$

$$\tilde{D}(z, b_T, Q, S) \rightarrow \tilde{D}(z, b_T) \underbrace{\sqrt{S(b_T)}}_{\text{UV scale } \nearrow \text{rapidity scale } \nearrow \text{effect of radiation}}$$

$$G \sim H(Q) \int \frac{d^2 b_T}{(2\pi)^2} e^{i \bar{P}_{bT} \bar{b}_T / z} \underbrace{\tilde{f}(x, b_T, Q, S) \tilde{D}(z, b_T, Q, S)}_{\text{exactly like in generalized parton model!}}$$

We would like to write

$\tilde{f}(x, b_T, Q, S)$  starting from some initial scales  $Q_0, S_0$

QCD evolution of TMDs is governed by 3 equations

① Collins-Soper equation (CS)

$$\frac{\partial \ln \tilde{F}(x, b_T, \mu, \xi)}{\partial \ln \sqrt{s}} = \tilde{K}(b_T, \mu)$$

$\tilde{K}$  is the so-called Collins-Soper kernel

it can be calculated perturbatively for small  $b_T$  & large  $\mu$  (so that  $d_s(\mu)$  is small)

$$\tilde{K}(b_T, \mu) = -8 \cdot C_F \frac{d_s(\mu)}{4\pi} \ln \left( \frac{b_T \mu}{2e^{-\gamma_E}} \right) + O(d_s^2)$$

$\gamma_E \approx 0.57$  Euler constant

The problem:

$$\text{We need to } \int \frac{dz_{b_T}}{(2\pi)^2} \rightarrow \int_0^\infty b_T db_T$$

but  $\ln \left( \frac{b_T \mu}{2e^{-\gamma_E}} \right)$  will become large for  $b_T \rightarrow \infty$

if corresponds to non-perturbative regime  
of  $k_T \rightarrow 0$ .

Solution later

② Renormalisation group equation

$$\frac{d K(b_T, \mu)}{d \ln \mu} = - \gamma_K(\alpha_s(\mu))$$

$\gamma_K$  is Casp anomalous dimension. It is present in many areas of physics

$$\gamma_K(\alpha_s) = \sum_{i=1}^{\infty} \gamma_K^i \left( \frac{\alpha_s}{4\pi} \right)^i = 8 C_F \left( \frac{\alpha_s}{4\pi} \right) + \mathcal{O}(\alpha_s^2)$$

③

$$\frac{d \ln \tilde{f}(x, b_T, \mu_1, S)}{d \ln \mu} = \gamma_F(\alpha_s(\mu), S/\mu^2)$$

$\gamma_F$  is the anomalous dimension of  $f$

$$\gamma_F(\alpha_s(\mu), 1) = \sum_{i=1}^{\infty} \gamma_F^i \left( \frac{\alpha_s}{4\pi} \right)^i = 6 C_F \left( \frac{\alpha_s}{4\pi} \right) + \mathcal{O}(\alpha_s^2)$$

Let's write the solutions:

$$\frac{d \ln \tilde{f}(x, b_T, \mu, \varsigma)}{d \ln \mu} = \gamma_F(\mu, \varsigma/\mu^2)$$

$$\int_{\mu_0}^{\mu} d \ln \tilde{f}(x, b_T, \mu, \varsigma) = \int_{\mu_0}^{\mu} \gamma_F(\mu', \varsigma/\mu'^2) \frac{d \mu'}{\mu'}$$

$$\frac{\tilde{f}(x, b_T, \mu, \varsigma)}{\tilde{f}(x, b_T, \mu_0, \varsigma)} = \exp \left[ \int_{\mu_0}^{\mu} \gamma_F(\mu', \varsigma/\mu'^2) \frac{d \mu'}{\mu'} \right]$$

$$\tilde{f}(x, b_T, \mu, \varsigma) = \tilde{f}(x, b_T, \mu_0, \varsigma) \exp \left[ \int_{\mu_0}^{\mu} \gamma_F(\mu', \varsigma/\mu'^2) \frac{d \mu'}{\mu'} \right]$$

$$\textcircled{2} \quad \frac{\partial \ln \tilde{f}(x, b_T, \mu, \varsigma)}{\partial \ln \sqrt{s}} = \tilde{K}(b_T, \mu)$$

$$\Rightarrow \tilde{f}(x, b_T, \mu, \varsigma) = f(x, b_T, \mu, \varsigma_0) \exp \left[ \tilde{K}(b_T, \mu) \ln \sqrt{\frac{s}{s_0}} \right]$$

$$\textcircled{3} \quad \frac{d \tilde{K}(b_T, \mu)}{d \ln \mu} = -\gamma_K(\mu)$$

$$\Rightarrow \tilde{K}(b_T, \mu) = \tilde{K}(b_T, \mu_0) - \sum_{\mu_0}^{\mu} \frac{d \mu'}{\mu'} \gamma_K(\mu')$$

Let us also combine 2 equations

$$\frac{d}{d \ln \mu} \left( \frac{\partial \ln \tilde{f}(x, b_T, \mu, \bar{s})}{\partial \ln \bar{s}} \right) = \frac{d}{d \ln \mu} \tilde{K}(b_T, f) = -\gamma_K(f)$$

those commute!

$$\frac{\partial}{\partial \ln \bar{s}} \left( \underbrace{\frac{d \ln \tilde{f}(x, b_T, \mu, \bar{s})}{d \ln \mu}}_{\gamma_F(\mu, \bar{s}/\mu^2)} \right) = -\gamma_K(f)$$

$$\Rightarrow \gamma_F(\mu, \bar{s}/\mu^2) - \gamma_F(\mu, \bar{s}_0/\mu^2) = -\gamma_K(f) \ln \sqrt{\bar{s}/\bar{s}_0}$$

If we use  $\bar{s}_0 = \mu^2$  then

$$\gamma_F(\mu, \bar{s}/\mu^2) = \gamma_F(\mu, 1) - \frac{1}{2} \gamma_K(f) \ln \left( \frac{\bar{s}}{\mu^2} \right)$$

# Implementing the evolution

## 1) Operator Product Expansion (OPE)

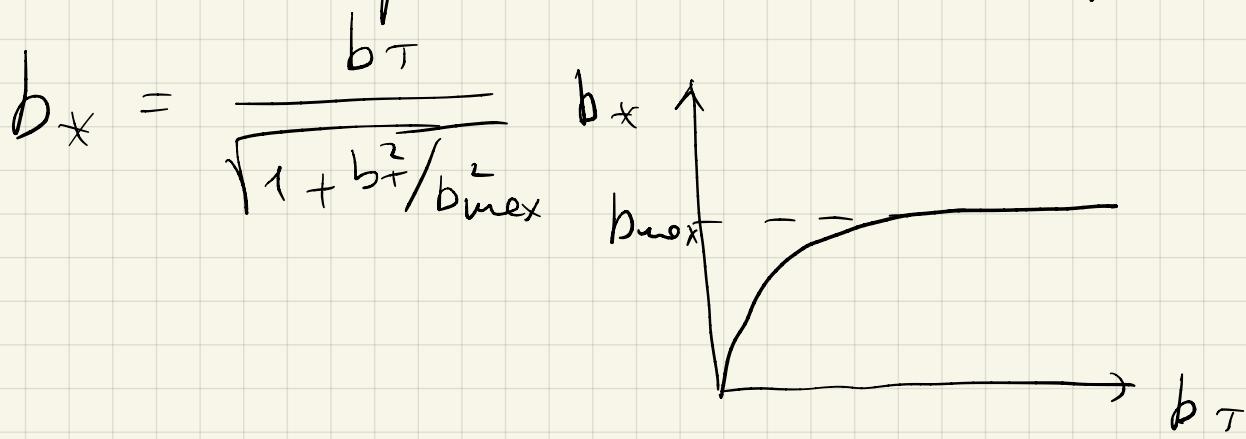
at low  $b_T$ :

$$\tilde{f}_f(x, b_T, \mu, \tau) = \sum_j \int_x^1 \frac{dx}{x} \underbrace{\tilde{C}_{j/f}\left(\frac{x}{\hat{x}}, b_T, \mu, \tau\right)}_{\substack{\text{coefficient} \\ \text{functions}}} f_j(\hat{x}, \mu) + \mathcal{O}(b_T^3)$$

collinear  
 PDFs  
 for upol. f

$$\tilde{C}_{j/f} = \delta_{jf} \delta\left(\frac{x}{\hat{x}} - 1\right) + \mathcal{O}(d_s^2)$$

2) Combine perturbative & non perturbative  
 (solution to problem of  $\tilde{K}(b_T)$  non perturbative at large  $b_T$ )



If  $b_{\max}$  is small ( $\sim 1 \text{ GeV}^{-1}$ ) then

$b_*$  is always perturbative for  $\forall b_T$

Start from CS kernel

$$\tilde{K}(b_T, \mu) = \tilde{K}(b_*, \mu) + \underbrace{\tilde{K}(b_T, \mu) - \tilde{K}(b_*, \mu)}_{g_K(b_T)}$$

universal non pert. function!

$g_K$  does not depend on  $\mu$ , in fact

$$\frac{d}{d \ln \mu} [\tilde{K}(b_T, \mu) - \tilde{K}(b_*, \mu)] = \gamma_K(\mu) - \gamma_K(\mu) = 0$$

next

$$\tilde{K}(b_T, \mu) = \tilde{K}(b_*, \mu_0) - \int_{\mu_0}^{\mu} \frac{dt}{t} \gamma_F(t) - g_K(b_T)$$

For convergence, remember,

$$\tilde{K} = -8C_F \frac{\alpha_s(\mu)}{4\pi} \ln \left( \frac{b_T \mu}{2e^{-\gamma_E}} \right)$$

$$\mu_0 = \frac{2e^{-\gamma_E}}{b_T} \equiv \mu_b \quad \text{but at large } b_T \rightarrow \infty$$

$$\mu_b \rightarrow 0 \Rightarrow \alpha_s(0) \rightarrow \infty \quad \text{Landau pole}$$

Because the function is non perturbative.

To avoid it

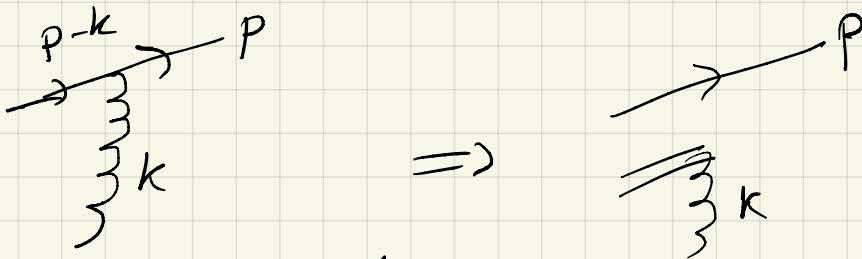
$$\mu_b = \frac{2e^{-\gamma_E}}{b_*} \rightarrow \frac{2e^{-\gamma_E}}{b_{\max}} \gg \Lambda_{QCD}$$

## Study in Mathematics

$$\tilde{K}(b, f) = \tilde{K}(b_*, f_b) - \int_{f_b}^f \frac{df'}{f'} \delta_K(f') - g_K(b)$$

for a realistic  $g_K(b) = g_0 \ln(b/b_*)$

# The origin of Wilson lines:



Eikonal approximation

$$(P - k)^2 \approx -2P \cdot k + k^2 \approx -2P \cdot k \approx -2P^- k^+$$

$$i \frac{(P - k)}{(P - k)^2 + i\epsilon} \approx i \frac{P^- \gamma^+}{-2P^- k^+ + i\epsilon} = \frac{i}{2} \frac{\gamma^+}{-k^+ + i\epsilon}$$

$P^-$  should be  $\geq 0$ , this is true as it goes through the final state cut

$$\langle P | \bar{\psi}(z) \psi(0) | P \rangle \text{ gauge tree if:}$$

$$\bar{\psi}(z) \rightarrow \bar{\psi}(z) u^+(z), \psi(0) \rightarrow \psi(0) u(0)$$

$$\Rightarrow \langle P | \bar{\psi}(z) u^+(z) u(0) | P \rangle$$

Wilson line

$$W(z, 0) = P \exp \left[ +ig \int_0^z dz \cdot A(z) \right]$$

$$W(z, 0) \rightarrow u(z) W(1, 0) u^\dagger(0)$$

$$\Rightarrow \bar{\psi}(z) W(z, 0) \psi(0) \rightarrow \underbrace{\bar{\psi}}_1 \underbrace{u^\dagger(z) u(0)}_1 \underbrace{W(1, 0) u^\dagger(0) u(0)}_1 \underbrace{\psi(0)}_1$$

Now let us write  $f(x, b_T, \mu, \Sigma)$  in terms of  
 $f(x, b_T, \mu_0, \Sigma_0)$ :

$$\tilde{f}(x, b_T, \mu, \Sigma) = \tilde{f}(x, b_*, \mu, \Sigma) \left[ \frac{\tilde{f}(x, b_T, \mu, \Sigma)}{\tilde{f}(x, b_*, \mu, \Sigma)} \right] =$$

$$= \tilde{f}(x, b_*, \mu, \Sigma_0) \exp \left[ \tilde{K}(b_*, \mu) \ln \sqrt{\frac{\Sigma}{\Sigma_0}} \right] \left[ \frac{\tilde{f}(x, b_T, \mu, \Sigma_0)}{\tilde{f}(x, b_*, \mu, \Sigma_0)} \right]$$

- $\exp \left[ \ln \sqrt{\frac{\Sigma}{\Sigma_0}} \underbrace{(\tilde{K}(b_T, \mu) - \tilde{K}(b_*, \mu))}_{g_K(b_T)} \right] \underbrace{\exp [-g(x, b_T)]}_{\text{up behav. of TnD}}$

$$= \tilde{f}(x, b_*, \mu_0, \Sigma_0) \exp \left[ \sum_{\mu_1}^{\mu} \frac{d\mu'}{\mu'} (\delta_F(\mu', 1) - \ln \sqrt{\frac{\Sigma_0}{\mu'^2}} \delta_K(\mu')) \right]$$

- $\exp \left[ \ln \sqrt{\frac{\Sigma}{\Sigma_0}} \tilde{K}(b_*, \mu_0) - \sum_{\mu_1}^{\mu} \frac{d\mu'}{\mu'} \ln \sqrt{\frac{\Sigma}{\mu'^2}} \delta_K(\mu') \right]$

- $\exp \left[ -g(x, b_T) - \ln \sqrt{\frac{\Sigma}{\Sigma_0}} g_K(b_T) \right]$

$$= \tilde{f}(x, b_*, \mu_0, \Sigma_0) \exp \left[ \ln \sqrt{\frac{\Sigma}{\Sigma_0}} \tilde{K}(b_*, \mu_0) + \sum_{\mu_1}^{\mu} \frac{d\mu'}{\mu'} [\delta_F(\mu', 1) - \ln \sqrt{\frac{\Sigma}{\mu'^2}} \delta_K(\mu')] \right]$$

- $\exp \left[ -g(x, b_T) - \ln \sqrt{\frac{\Sigma}{\Sigma_0}} g_K(b_T) \right]$

$$\text{Let us use } \mu_0 = \mu_b = \frac{2e^{-\delta_F}}{b^*}$$

$$\zeta_0 = Q_0^2 \sim 1-2 \text{ (GeV}^2)$$

Then:  $\zeta = Q^2$  (the scale)

$$\tilde{f}(x, b_T, Q, Q^2) = \tilde{f}(x, b^*, \mu_b, Q_0^2) \left( \frac{Q}{Q_0} \right)^{\tilde{K}(b^*, \mu_b)} - g_K(b)$$

$$\exp \left[ \int_{\mu_b}^{\mu} \frac{d\mu'}{\mu'} \left( \delta_F(\mu', 1) - \ln \frac{Q}{\mu'} \delta_K(\mu') \right) \right]$$

$$\cdot \exp [-g(x, b_T)]$$

$\downarrow$  Sudakov form factor

$$\exp [S]$$

$\uparrow$   
Contains result of  
gluon radiation

$$\tilde{f}(x, b_T, Q, Q^2) = \tilde{f}(x, b^*, \mu_b, Q_0^2) e^{-g(x, b_T)} e^S$$

almost like GPM!

$$\approx \tilde{f}(x, \mu_b) e^{-g(x, b_T)} e^S$$

if  $\begin{cases} g(x, b_T) \approx \frac{b_T^2}{4} \\ S \approx 0 \end{cases}$

study in Mathematics.

We will use Mathematics to study how it  
differs from GPM at higher scales.