QCD Evolution for TMDs

J. Collins, Foundations of Perturbative QCD (Cambridge U. Press, 2011)

M.A, T. Rogers, Phys. Rev. D83, 114042 (2011) [1101.5057] ← unpolarized

M.A, J. Collins, J.W. Qiu & T. Rogers, arXiv: 1110.6428 ← Sivers

Outline: L1: - Collinear vs TMD factorization

- TMD factorization & TMD Definitions (*)

L2: - Evolution equations for TMDs

- Implementing evolution

- b_* matching (Perturbative - Non-perturbative matching)

L3: - NLO calculations for evolution kernel & anomalous dimensions

- Sivers (if there is time)

Collinear and TMD Factorization

Factorization theorems $\Longleftrightarrow$ predictive power of QCD

Collinear factorization for inclusive processes $\Rightarrow$ well understood. Proof in full theory by Collins-Soper-Sterman

For the DY process, schematically factorization theorem states that the differential cross section can be written as

$$\frac{d\sigma}{dQ^2} \sim f_a(x_1,\mu) \otimes f_b(x_2,\mu) \otimes \mathcal{H}(Q,\mu,\ldots)$$

convolutions

Essential ingredients of collinear factorization that gives QCD its predictive power are

- Unambiguous prescription for calculating perturbative higher order corrections to hard scattering.
- Universality of pdfs.
- Evolution equations for pdfs to relate different scales and different experiments.

As for the collinear correlation functions themselves, there has been a tremendous ammount of effort put into extracting them and tabulating them over the years: CTEQ, MRST, Neural Network PDF, ...

TMD factorization: Collinear factorization is not adequate to describe processes where transverse momentum of partons start to become important.

When does this happen?

Suppose we are interested in DY differential cross section

$\frac{d\sigma^{DY}}{dq_T^2}$ with q_T : transverse momenta in the final state. for <u>all</u> q_T.

<u>Regions</u> :

- $q_T \sim Q \gg \Lambda_{QCD}$: large q_T transfer could happen as a part of the hard scattering (via a gluon emission) and small k_T of partons in this case is not important
  $\Rightarrow$ <u>collinear factorization</u> with $\mathcal{H}(Q, \mu, q_T, \ldots)$

- $q_T \sim \Lambda_{QCD} \ll Q$: small k_T of partons play an important role for small q_T transfers. $\Rightarrow$ <u>TMD factorization</u> with pdfs as also a function of k_T.

- intermediate region $\Lambda_{QCD} \ll q_T \ll Q$: matching between collinear factorization & collinear factorization.

Common approaches in TMD phenomenology has been

- CSS approach : based on the Collins Soper Sterman TMD factorization formulation ('85)
  - ? look like collinear factorization (explicit soft factors)
  - cumbersome & process dependent fits
  - no direct link with TMD correlation themselves.

- Generalized Parton Model (GPM) approach: extraction of TMDs assuming a literal parton model interpretation of TMDs.
  - fixed scale, no evolution

- Ressummation : Begin with collinear factorization treatment valid at large q_T and by resumming logs of q_T/Q attempt to improve the treatment to lower q_T.
  - will fail at some q_T since collinear factorization is not the appropriate description at small q_T.
- Model calculations : – not clear where (at what scale) the models are valid
  – not clear how to make pQCD calculations using models.
- Lattice calculations : – again not clear how to incorporate this in real pQCD calculations.

Aim of the TMD Project : Extend the collinear factorization methodology to TMD factorization. Requires

- an analogue TMD factorization with ✓ J.C.
- well defined unique TMD correlation functions with ✓ J.C
- QCD evolution incorporated. ✓ J.C. ; MA & TR

TMD Factorization & Definitions of TMDs

J. Collins, starting with the old CSS formulation, gives a new TMD factorization formu. For example for the DY process one can rite (for the hadronic tenso)

$$W^{\mu\nu} = \sum_f H_f^{\mu\nu}(Q,\mu) \int d^2\vec{k}_{1T}\, d^2\vec{k}_{2T}\; F_{f/P_1}(x_1, k_{1T}, \mu, \zeta_1)\, F_{f/P_2}(x_2, k_{2T}, \mu, \zeta_2)$$

$$\times\, \delta^2(\vec{k}_{1T} + \vec{k}_{2T} - \vec{q}_T) \; + \; Y(Q, q_T) + \mathcal{O}(\Lambda_{QCD}/Q)$$

Written in this form the TMD factorization looks very much like the collinear factorization.

GPM $\longrightarrow$ QCD

$F(x, k_T) \longrightarrow F(x, k_T, \mu, \zeta)$

μ: renormalization scale

ζ: rapidity cut off.

and just like for the standard parton model picture we expect to have QCD evolution for the new pa[illegible]

- $F(x, k_T, \mu, \zeta)$ are
  - uniquely defined
  - they deal with <u>all</u> divergences
  - requirements of factorization.
  - generalized universality.

(Definitions are given in terms of limits of operators.)

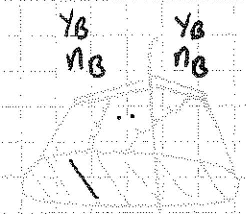
First define $\quad n_A = (1, -e^{-2y_A}, \vec{0}) \qquad n_B = (-e^{2y_B}, 1, \vec{0})$

$$\lim_{y_A \to \infty} n_A = (1, 0, \vec{0}) \quad \& \quad \lim_{y_B \to -\infty} n_B = (0, 1, \vec{0})$$

The definition is $F(x, k_T, \mu, \zeta) = \frac{1}{(2\pi)^2} \int d^2\vec{b}_T \, e^{i\vec{k}_T\cdot\vec{b}_T} \tilde{F}(x, b_T, \mu, \zeta)$

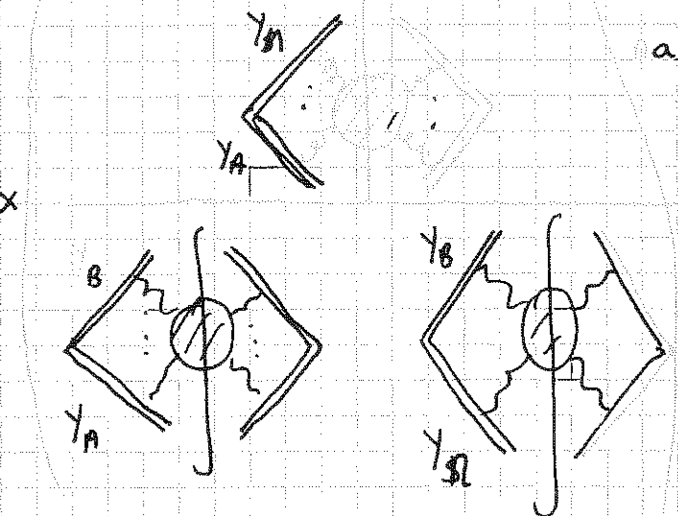
and

$$\tilde{F}(x, b_T, \mu, \zeta) = \lim_{\substack{y_A \to \infty \\ y_B \to -\infty}}$$



½ "Unsubtracted" naive definition

$\times$



implements subtractions/cancellations

$\times Z_F^2$ ← field strength renormalization factor

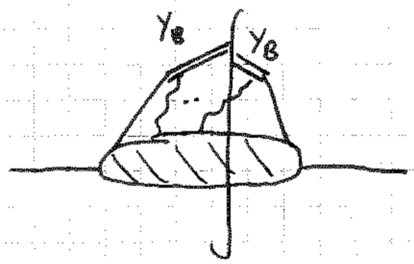
y_s : parameter that separates extreme plus and minus directions

$$\zeta = 2 M_p^2 x^2 e^{2(y_p - y_s)}$$

But we will treat ζ as a free parameter and choose $\zeta = Q^2$.

- in b_T space
- no explicit operator definition (due to soft factors)

Written in another way



$$\tilde{F}(x, b_T, \mu, \zeta) = \lim_{\substack{y_A \to \infty \\ y_B \to -\infty}}$$

$$\times \left(\frac{\tilde{S}_0(b_T; y_A, y_n)}{\tilde{S}_0(b_T; y_A, {}_B) \quad \tilde{S}_0(b_T, y_n, y_B)} \right)^{1/2} Z_F \; Z_2$$

IV. DEFINITIONS OF THE TMDS

As explained in Sect. II, our calculations are based on the formulation of TMD-factorization explained in detail in Ref. [26]. A repeat of the derivation is beyond the scope of this paper. However, in order to put our later calculations into their proper context, we will give an overview of the basic features of the formalism in this and the next section. We refer the reader directly to Ref. [26] for pertinent details.

A. Soft Factor Definition

We have already stressed in Sect. II C that the definitions of the TMDs in Eq. (7) are not the often quoted matrix elements of the form $\sim \langle P|\bar{\psi}\ \text{Wilson Line}\ \psi|P\rangle$ with simple light-like Wilson lines connecting the field operators. Using such definitions in a factorization formula leads to inconsistencies, including unregulated light-cone divergences. Also, soft gluons with rapidity intermediate between the two nearly light-like directions need to be accounted for in the form of soft factors. Therefore, before we can discuss the definitions of the TMDs that will ultimately be used in Eq. (7), we must provide the precise definition of the soft factor. In coordinate space it is an expectation value of a Wilson loop:

$$\tilde{S}_{(0)}(\mathbf{b}_T; y_A, y_B) = \frac{1}{N_c}\langle 0|W(\mathbf{b_T}/2,\infty;n_B)^\dagger_{ca}\, W(\mathbf{b}_T/2,\infty;n_A)_{ad} W(-\mathbf{b_T}/2,\infty;n_B)_{bc} W(-\mathbf{b_T}/2,\infty;n_A)^\dagger_{db}|0\rangle_{\text{No S.I.}}. \tag{8}$$

We have used the vectors in Eq. (6) to define the directions of the Wilson lines so that, as long as y_A and y_B are finite, the Wilson lines in Eq. (8) are non-light-like. The subscripts a, b, c and d are color triplet indices, and repeated indices are summed over. The "(0)" subscript indicates that bare fields are used. The soft factor contains Wilson line self-interaction (S.I.) divergences that are very badly divergent and are unrelated to the original unfactorized graphs. They must therefore be excluded, and we indicate this with a subscript "No S.I.". We emphasize, however, that this is only a temporary requirement because all Wilson line self-energy contributions will cancel in the final definitions. Another potential complication, pointed out in Refs. [19, 20], is that exact gauge invariance requires the Wilson lines to be closed by the insertion of links at light-cone infinity in the transverse direction. However, the transverse segments will not contribute in the final definitions of the TMDs (at least in non-singular gauges), so we do not show them explicitly in Eq. (8). Again, the final arrangement of soft factors will ensure a cancellation.

Rather than appearing as a separate factor in the TMD-factorization formula, soft factors like Eq. (8) will be part of the final definitions of the TMDs. Their role in the definitions will be essential for the internal consistency of the TMDs and their validity in a factorization formula like Eq. (7).

B. TMD PDF and FF Definitions

Now we turn to the definitions of the TMDs themselves, starting with the unpolarized TMD PDF. The most natural first attempt at an operator definition is obtained simply by direct extension of the collinear integrated parton distribution, though with the Wilson line tilted to avoid light-cone singularities. The operator definition is

$$\begin{aligned}&\tilde{F}^{\text{unsub}}_{f/P}(x,\mathbf{b}_T;\mu;y_P-y_B)\\ &\qquad = \mathrm{Tr}_C \int \frac{dw^-}{2\pi} e^{-ixP^+w^-}\langle P|\bar{\psi}_f(w/2)W(w/2,\infty,n_B)^\dagger \frac{\gamma^+}{2} W(-w/2,\infty,n_B)\psi_f(-w/2)|P\rangle_{\text{c,No S.I.}}.\end{aligned} \tag{9}$$

This definition does not account for the overlap of the soft and collinear regions, so we refer to it as the "unsubtracted" TMD PDF. Here $w = (0, w^-, \mathbf{b}_T)$ and y_P is the physical rapidity of the hadron. As usual, the struck

QCD Evolution for TMDs

QCD evolution is governed by a CS equation and two renormalization group (RG) equations.

CS equation:
$$\frac{\partial \tilde{F}(x, b_T, \mu, \zeta)}{\partial \ln\sqrt{\zeta}} = \tilde{K}(b_T, \mu) \quad (1)$$

CS kernel

Note: Derivative wrt. $\ln\sqrt{\zeta}$ is equivalent to a derivative w.r.t. $-Y_n$. The only dependence of $\tilde{F}$ on ζ or Y_n is through the soft factors. So then from the definition of $\tilde{F}$ we get by direct computation

$$\tilde{K}(b_T, \mu) = \frac{\partial}{\partial Y_n}\left[\frac{1}{2}\ln \tilde{S}(b_T, Y_n, -\infty) - \frac{1}{2}\ln \tilde{S}(b_T, +\infty, Y_n)\right]$$
$$= \frac{1}{2\tilde{S}(b_T, Y_n, -\infty)}\frac{\partial \tilde{S}(b_T, Y_n, -\infty)}{\partial Y_n} - \frac{1}{2\tilde{S}(b_T, +\infty, Y_n)}\frac{\partial \tilde{S}(b_T, +\infty, Y_n)}{\partial Y_n}$$

RG equations:

- $\frac{d\tilde{K}(b_T, \mu)}{d\ln\mu} = -\gamma_K(g(\mu)) \quad (2)$
  - $\tilde{K}$ is renormalized by adding a counterterm.
  - This results in an additive anomalous dimension.
  - UV divergence arises from virtual diagrams only and therefore γ_K has no b_T dependence.

- $\dfrac{d \ln \tilde{F}(x, b_T, \mu, \zeta)}{d \ln \mu} = \gamma_F\left(g(\mu), \zeta/\mu^2\right)$ (3) – Doesn't depend on b_T, as for $\gamma_K(g(\mu))$

Using (1) & (3) ~~and~~ inserting (1) in (2) and changing order of differentiation one can show

$$\gamma_F\left(g(\mu), \zeta/\mu^2\right) = \gamma_F\left(g(\mu), 1\right) - \frac{1}{2} \ln \frac{\zeta}{\mu^2} \gamma_K\left(g(\mu)\right) \quad (4)$$

<u>Note</u>: • Evolution equations are in b_T space!

<u>Solutions</u>:

- $\tilde{F}(x, b_T, \mu, \zeta) = \tilde{F}(x, b_T, \mu, \zeta_0) \exp\left[\tilde{K}(b_T, \mu) \ln \sqrt{\dfrac{\zeta}{\zeta_0}}\right]$
- $\tilde{F}(x, b_T, \mu_0, \zeta) = \tilde{F}(x, b_T, \mu_0, \zeta) \exp\left[\displaystyle\int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \gamma_F\left(g(\mu'), \zeta/\mu'^2\right)\right]$
- $\tilde{K}(b_T, \mu) = \tilde{K}(b_T, \mu_0) - \displaystyle\int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \gamma_K\left(g(\mu')\right)$

Implementing Evolution

We start with the low b_T (high k_T collinear) region. In region the TMDs satisfy a factorization formalism so that

$$\tilde{F}_f(x, b_T, \mu, \zeta) = \sum_j \int_x \frac{d\hat{x}}{\hat{x}} \tilde{C}_{j/f}\left(\frac{x}{\hat{x}}, b_T, \mu, \zeta\right) f_j(\hat{x}, \mu) + \mathcal{O}((m b_T)^a)$$

with $f_j(\hat{x}, \mu)$ the collinear pdf.

At lowest order $\tilde{C}_{j/f}\left(\frac{x}{\hat{x}}, b_T, \mu, \zeta\right) = \delta_{jf}\, \delta\left(\frac{x}{\hat{x}} - 1\right) + \mathcal{O}(\alpha_s)$

Next step is to combine the perturbative information at small b_T with non-perturbative information at large b_T which is to be determined through experiment. $\Rightarrow$ b_* matching prescription

* matching prescription

Problem : Functions like $\tilde{K}(b_T, \ldots)$, $\tilde{F}(b_T, \ldots)$ are non-perturbative for large b_T. (therefore not calculable)

Want : to be able to write these functions as a function of b_T so that for <u>all</u> b_T they are perturbatively calculable <u>with</u> non-perturbative corrections.

$$\vec{b}_*(\vec{b}_T) \equiv \frac{\vec{b}_T}{\left(1 + \frac{b_T^2}{b_{max}^2}\right)^{1/2}}$$

, b_{max} : max. distance at which perturbation theory is to be trusted.

$b_T \to$ (collinear limit, perturbation theory is ok) $\Rightarrow b_* \to b_T$

$b_T \to \infty$ (non-perturbative limit) $\Rightarrow b_* \to b_{max}$

$$\tilde{K}(b_T, \mu, g(\mu)) = \tilde{K}(b_*, \mu, g(\mu)) + \underbrace{\left[\tilde{K}(b_T, \mu, g(\mu)) - \tilde{K}(b_*, \mu, g(\mu))\right]}$$

$$= \tilde{K}(b_*, \mu_0, g(\mu_0)) - \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \gamma_K \, g(\mu')) - g_K(b_T)$$

$g_K(b_T)$:
- non-perturbative
- universal (same for all TMDs)
- $g_K(b_T) = \frac{1}{2} g_2 b_T^2$ (Landry et. al)

Dimensionally $\mu \sim \frac{1}{b_T}$. We choose $\mu_0 = \frac{C_1}{b_*}$ so that for <u>each</u> value of b_T, μ_0 is a rturbative scale.

a ly et. (2003), Konychev & Nadolsky (2006 find

$$2 = \begin{cases} 0.68^{+0.01}_{-0.02} \quad V^2 & \text{with} \quad b_{max} = 0.5 \; GeV^{-1} \\ 0.17 \pm 0.02 \quad GeV^2 & \text{with} \quad b_{max} = 1 \; GeV^{-1} \end{cases}$$

imilar analysis for $\tilde{F}(x, b_T, \mu, \zeta)$

$$\tilde{F}(x, b_T, \mu, \zeta) = \tilde{F}(x, b_*, \mu, \zeta) \left[\frac{\tilde{F}(x, b_T, \mu, \zeta)}{\tilde{F}(x, b_*, \mu, \zeta)} \right]$$

$$= \tilde{F}(x, b_*, \mu, \zeta_0) \exp\left[\tilde{K}(b_*, \mu) \ln\sqrt{\frac{\zeta}{\zeta_0}} \right] \cdot \underbrace{\left[\frac{\tilde{F}(x, b_T, \mu, \zeta_0)}{\tilde{F}(x, b_*, \mu, \zeta_0)} \right]}$$

$$\times \exp\left[n\sqrt{\frac{\zeta}{\zeta_0}} \left(\tilde{K}(b_T, \mu) - \tilde{K}(b_*, \mu) \right) \right]$$

$$= \tilde{F}(x, \; *, \mu, \zeta_0) \exp\left[\ln\sqrt{\frac{\zeta}{\zeta_0}} \; \tilde{K}(b_*, \mu) \right] \exp\left[- g_{H/f} \; x, b_T) - \ln\sqrt{\frac{\zeta}{\zeta_0}} \, g_K(b_T) \right]$$

$$= \tilde{F}(x, b_*, \mu_0, \zeta_0) \exp \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \left(\gamma_F \; g(\mu') \cdot 1) - n\sqrt{\frac{\;\;}{\mu'^2}} \gamma_K(g(\mu')) \right)$$

$$\exp\left[\ln\sqrt{\frac{\zeta}{\zeta_0}} \; \tilde{K}(b_*, \mu_0) - \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \ln\sqrt{\frac{\zeta}{\zeta_0}} \; \gamma_K(g \; \mu')) \right.$$

$$\exp\left[-g \quad (x, b_T) - \ln\sqrt{\zeta} \quad g \; (b_T) \right]$$

$$
= \tilde{F}(x, b_*, \mu_0, \zeta_0) \, \exp\left[\ln\sqrt{\frac{\zeta}{\zeta_0}} \, \tilde{K}(b_*, \mu_0) + \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \left(\gamma_F(g(\mu'), 1) - \ln\sqrt{\frac{\zeta}{\mu'^2}} \, \gamma_K(g(\mu')) \right) \right]
$$

$$
\times \exp\left[- g_{H/f}(x, b_T) - \ln\sqrt{\frac{\zeta}{\zeta_0}} \, g_K(b_T) \right]
$$

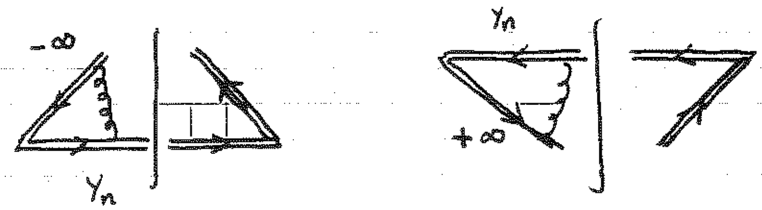
TMD Kernel $\tilde{K}(b_T, \mu)$

Recall the definition

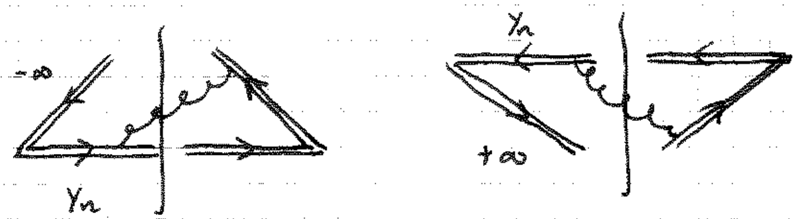
$$\tilde{K}(b_T,\mu) = \frac{\partial}{\partial y_n}\left[\frac{1}{2}\ln \tilde{S}(b_T, y_n, -\infty) - \frac{1}{2}\ln \tilde{S}(b_T, +\infty, y_n)\right]$$

$$= \frac{1}{2\tilde{S}(b_T, y_n, -\infty)}\,\frac{\partial \tilde{S}(b_T, y_n, -\infty)}{\partial y_n} - \frac{1}{2\tilde{S}(b_T, +\infty, y_n)}\,\frac{\partial \tilde{S}(b_T, +\infty, y_n)}{\partial y_n}$$

Diagrams that contribute at $\mathcal{O}(\alpha_s)$ are

(a) 

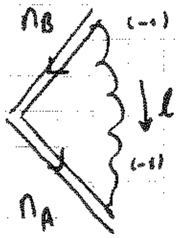
$+$ h.c.

(b) 

$+$ h.c.

(c) U.V. counterterms

Evalua ion of he Virtual Diagrams



$$n_B = \left(-e^{2y_B},\ 1,\ \vec{0}\right)$$

$$n_A = \left(1,\ -e^{-2y_A},\ \vec{0}\right)$$

$$i\int\frac{d^D\ell}{(2\pi)^D}\ \frac{n_A\cdot n_B}{(2\ell^+\ell^- - \ell_T^2 + i\epsilon)\,(\ell\cdot n_B + i\epsilon)\,(-\ell\cdot n_A + i\epsilon)}$$

$$= i\int\frac{d^D\ell}{(2\pi)^D}\ \frac{1+e^{-2(y_A-y_B)}}{\underbrace{(2\ell^+\ell^- - \ell_T^2 + i\epsilon)}_{\text{LHP for } \ell^+>0}\ \underbrace{(\ell^+ - \ell^- e^{2y_B} + i\epsilon)}_{\text{UHP}}\ \underbrace{(-\ell^- + \ell^+ e^{-2y_A} + i\epsilon)}_{\text{UHP}}}$$

Evaluate minus integral closing the contour in LHP

$$= +\int\frac{d^{D-2}\ell_T}{(2\pi)^{D-2}}\int_0^\infty\frac{d\ell^+}{2\pi}\ \frac{1}{2\ell^+}\ \frac{1+e^{-2(y_A-y_B)}}{\left(\ell^+ - e^{2y_B}\dfrac{\ell_T^2}{2\ell^+} + i\epsilon\right)\left(\ell^+ e^{-2y_A} - \dfrac{\ell_T^2}{2\ell^+} + i\epsilon\right)}$$

$$= \int\frac{d^{D-2}\ell_T}{(2\pi)^{D-2}}\int_0^\infty\frac{d\ell^+}{2\pi}\ \frac{1}{4}\ \left(1+e^{-2(y_A-y_B)}\right)\frac{\ell^+ e^{2y_A}}{\left(\ell^{+2} - e^{2y_B}\dfrac{\ell_T^2}{2}\right)\left(\ell^{+2} - e^{2y_A}\dfrac{\ell_T^2}{2}\right)}$$

Evaluating the ℓ^+ integral we get

$$= -\,(y_A - y_B)\ \mathrm{Co\,h}\,(y_A - y_B)\int\frac{d^{D-2}\ell_T}{(2\pi)^{D-1}}\ \frac{1}{\ell_T^2}$$

Evaluation of the Real Diagrams

$$n_B = (0, 1, \vec{0}) \qquad (D = 4-2\epsilon)$$

$$n_A = (e^{y_n}, -e^{-y_n}, \vec{0})$$

from $-g_{\mu\nu}$ of the polarization sum for gluons

$$\frac{\partial}{\partial y_n}\int \frac{d^D k}{(2\pi)^D} \frac{(-1)\, n_A\cdot n_B\, (-1)}{(n_A\cdot k + i\epsilon)(+n_B\cdot k - i\epsilon)}\, 2\pi\,\delta(k^2)\,\theta(k^+)$$

$$= +\frac{\partial}{\partial y_n} \frac{d^{D-2}k_T}{(2\pi)^{D-2}} \int \frac{dk^+ dk^-}{(2\pi)^2} \frac{e^{n}}{(-k^+ e^{-y_n} + k^- e^{y_n} + i\epsilon}\, \frac{1}{(k^+ + i\epsilon)}\, 2\pi\,\delta(2k k^- - k_T^2)\,\theta(k^+)$$

$$= +\frac{\partial}{\partial y_n}\int \frac{d^{D-2}k_T}{(2\pi)^{D-2}} \int \frac{dk^+ dk^-}{(2\pi)^2} \frac{1}{(-k^+ e^{-2y_n} + k^- + i\epsilon)}\, \frac{1}{k^+ + i\epsilon}\, 2\pi\,\delta(2k^+k^- - k_T^2)\,\theta(k^+)$$

$$= -\int \frac{d^{D-2}k_T}{(2\pi)^{D-2}} \int \frac{dk^+ dk^-}{(2\pi)^2} \frac{2k^+ e^{-2y_n}}{(-k^+ e^{-2y_n} + k^- + i\epsilon)^2}\, \frac{1}{k^+ + i\epsilon}\, 2\pi\,\delta(2k^+k - k_T^2)\,\theta(k^+)$$

$$= -\int \frac{d^{D-2}k_T}{(2\pi)^{D-2}} \int \frac{dk^+ dk^-}{(2\pi)^2} \frac{2}{(-{}^{+} e^{-y_n} + k^- e^{y_n} + i\epsilon)^2}\, 2\pi\,\delta(2k k^- - k_T^2)\,\theta(k^+)$$

Evaluate the minus integral using the delta function

$$= -\int \frac{d^{D-}k_T}{2\pi)^{D-}} \int_0^\infty \frac{dk^+}{2\pi}\, \frac{1}{2k^+}\, \frac{2}{\left(-k^+ e^{-y_n} + \frac{k_T^2}{2k^+} e^{y_n}\right)^2}$$

$$= -\int \frac{d^{D-2}k_T}{(2\pi)^{D-2}} \int_0^\infty \frac{dk^+}{2\pi} \frac{2k^+}{(2e^{-y_n}k^{+2} - k_T^2 e^{y_n})^2}$$

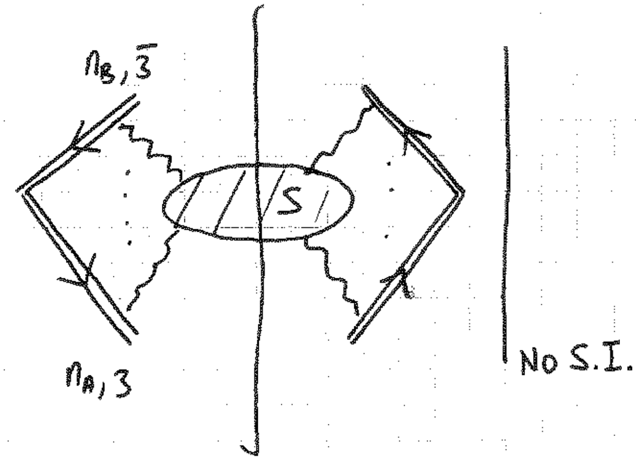
The + integral is elementary of the form $\int_0^\infty dx \, \frac{2x}{(ax^2-b)^2} = \frac{-1}{ab}$

$$= + \int \frac{d^{D-2}k_T}{(2\pi)^{D-1}} \quad \frac{1}{k_T^2}$$

The IR divergence at $k_T = 0$ will cancel against the virtual graphs.

Putting it all together:

Recall: $S_{Bare}(k_T) = \frac{1}{N_C} \int \frac{dk^+ dk^-}{(2\pi)^D}$



Taking a Fourier transform

$$\tilde{K}_R(b_T, \ldots) = \frac{g_s^2 C_F (4\pi^2 \mu^2)^\epsilon}{4\pi^3} \int d^{D-2}\vec{k}_T \, \frac{e^{i\vec{k}_T \cdot \vec{b}_T}}{k_T^2}$$

$$\tilde{K}_V(b_T, \mu) + U.V. = -\frac{g_s^2 C_F (4\pi^2 \mu^2)^\epsilon}{4\pi^3} \int d^{D-2}\vec{l}_T \, \frac{1}{l_T^2} + \frac{g^2 C_F (4\pi \cancel{\mu^2})^\epsilon}{4\pi^2 \Gamma(1-\epsilon)\,\epsilon}$$

$$\tilde{K}(b_T, \mu) = \frac{g^2 C_F (4\pi \mu^2)^\epsilon}{4\pi^3} \int d^{D-2}k_T \, \frac{e^{i\vec{k}_T \cdot \vec{b}_T} - 1}{k_T^2} + \frac{g^2 C_F (4\pi \cancel{\mu})^\epsilon}{4\pi^2 \Gamma(1-\epsilon)\,\epsilon}$$

Perform k_T integral using

$$\int d^{D-2}\vec{k}_T \, \frac{e^{i\vec{k}_T \cdot \vec{b}_T}}{(k_T^2)^\alpha} = \left(\frac{b_T^2}{4\pi}\right)^{\epsilon+\alpha-1} \frac{\pi^\alpha \, \Gamma(1-\epsilon-\alpha)}{\Gamma(\alpha)}.$$

$$\tilde{K}(b_T, \mu) = -\frac{g^2 C_F}{4\pi^2} \left[\ln(\mu^2 b_T) - \ln 4 + 2\gamma_E \right]$$