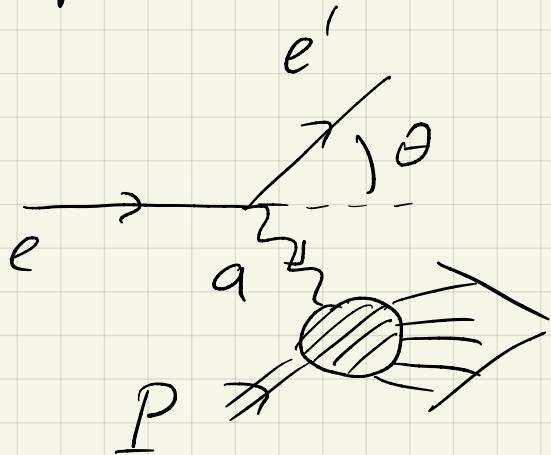




During lecture 1 we speak of a certain process. Let us look into its kinematics

Deep Inelastic Scattering (DIS)



1) Show that $q^2 < 0$

$$\ell = (E, 0, 0, E)$$

$$\ell' = (E', E' \sin\theta, 0, E' \cos\theta)$$

$$\begin{aligned} \text{Thus } q^2 &= (\ell - \ell')^2 \approx -2\ell\ell' = -2(EE' - EE'\cos\theta) \\ &= -2EE'(1 - \cos\theta) \leq 0. \end{aligned}$$

It is customary to introduce $q^2 = -Q^2$
where $Q^2 \geq 0$

Now let us explore other kinematical variables

$$x = \frac{Q^2}{2P \cdot q} - \text{Bjorken } x, y = \frac{P \cdot q}{P \cdot e} - \text{inelasticity}$$

These variables are constructed off scalar products which are Lorentz invariants and therefore we can use any frame to estimate them. Let us choose target rest frame

~~mass~~ $P = (M, \vec{0})$

$q = (\gamma, \vec{q}), \gamma = E - E' > 0$

$$2P \cdot q = 2M\gamma \rightarrow \infty \text{ Bjorken limit}$$

$$q^2 = -Q^2, Q^2 \geq 0, \gamma \geq 0 \Rightarrow$$

$$x = \frac{Q^2}{2P \cdot q} \geq 0$$

W is the energy of q

$$W^2 = (P+q)^2 = P^2 + 2P \cdot q - Q^2 =$$

$$P^2 = M^2 + 2P \cdot q - Q^2 \geq M^2$$

in case the proton is intact = elastic scattering

$$\Rightarrow 2P \cdot q \geq Q^2 \Rightarrow x = \frac{Q^2}{2P \cdot q} \leq 1$$

$$\text{inelasticity } y = \frac{P \cdot q}{P \cdot \ell} = \frac{M(E - E')}{ME} = 1 - \frac{E'}{E}$$

$E' \in [0, E]$ thus $y \in [0, 1]$

$x \in [0, 1]$ } we can estimate the reach
 $y \in [0, 1]$ } of experiments

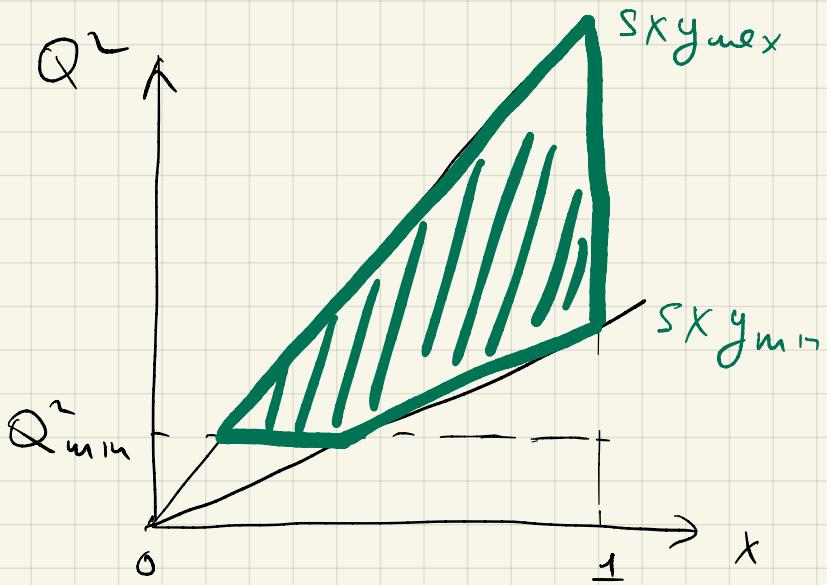
$$s = (P + Q)^2 \approx M^2 + 2P \cdot \ell \approx 2P \cdot \ell$$

=>

$$x = \frac{Q^2}{2P \cdot q} = \frac{Q^2}{2P \cdot q} \cdot \frac{P \cdot \ell}{P \cdot \ell} = \frac{Q^2}{y s}$$

Therefore

$$Q^2 \approx y s x$$

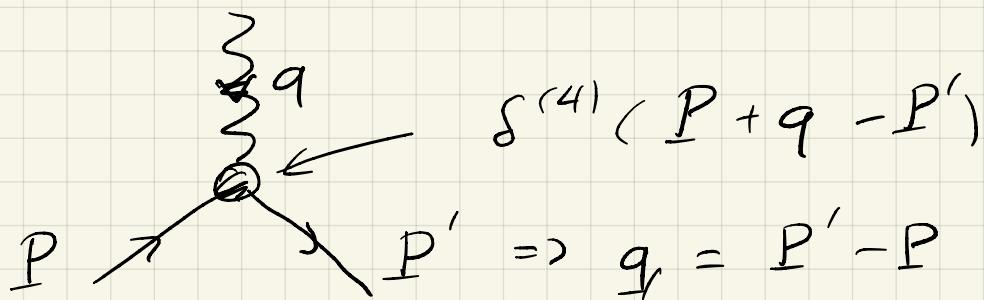


$Q^2_{\min} \approx 1 \text{ (GeV}^2\text{)}$ to ensure DIS regime

$y \in [y_{\min}, y_{\max}]$
experimental resolution

Why do we need x_{Bj} ?

Let us consider form factors \rightarrow elastic scattering



$$q^2 = -Q^2 = (\underline{P}' - \underline{P})^2 = \underline{P}'^2 + \underline{P}^2 - 2 \underline{P}' \cdot \underline{P} =$$

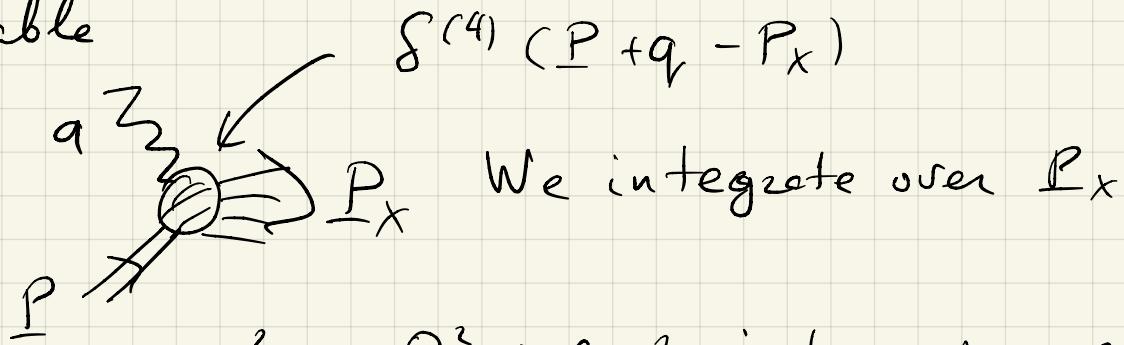
$$= 2M^2 - 2\underline{P}' \cdot \underline{P} \rightarrow -2\underline{P}' \cdot \underline{P}$$

$$\underline{P} \cdot \underline{q} = \underline{P} (\underline{P}' - \underline{P}) = \underline{P} \cdot \underline{P}' - M^2 \rightarrow \underline{P} \cdot \underline{P}'$$

therefore

$$x = \frac{Q^2}{2\underline{P} \cdot \underline{q}} \rightarrow 1 \text{ not an independent variable}$$

variable



$$q^2 = -Q^2 \rightarrow \infty \text{ independently}$$

$$2\underline{P} \cdot \underline{q} \rightarrow \infty$$

$$x_{Bj} = \frac{Q^2}{2\underline{P} \cdot \underline{q}} \in [0, 1]$$

Experiments : fixed target vs collider

$$s = (P + \ell)^2 \text{ cm energy}$$

Fixed target

$$\ell = (P_{\text{lab}}, 0, 0, -P_{\text{lab}})$$

$$P = (M_P, 0, 0, 0)$$

$$s = (P + \ell)^2 = P^2 + 2P \cdot \ell + \ell^2 \simeq 2 M_P P_{\text{lab}}$$

Collider

$$\ell = (E_e, 0, 0, -E_e)$$

$$P \simeq (E_P, 0, 0, E_P) \text{ (neglect the mass)}$$

$$s = (P + \ell)^2 \simeq (E_e + E_p)^2 - (E_p - E_e)^2 = 4 E_p E_e$$

The energy increased easily in collider

The elements of Quantum Field Theory

The wave function $\Psi(x)$ is a coordinate projection of the state vector in the Hilbert space $|\Psi\rangle$

$$\int d^3x |\Psi(x)|^2 < \infty$$

The scalar product

$$\langle \Psi | \phi \rangle = \int d^3x \Psi^*(x) \phi(x)$$

We will use $|P; S\rangle$ to denote the proton with momentum P and spin vector S .

Unitarity of S matrix and optical theorem

(Taylor, "Scattering theory")

The probability of one state going to the other is described by S matrix

$$w(X \leftarrow \phi) = |\langle X | S | \phi \rangle|^2$$

One can write

interaction
↓

$$S_{ab} = S_{ab} + i (2\pi)^4 \delta^{(4)}(p_a - p_b) T_{ab}$$

\nearrow \nearrow
no interaction momentum conservation

Probability is conserved

$$SS^+ = S^+ S = 1 \quad \text{Unitarity of } S \text{ matrix}$$

One can use it to write $\underbrace{\dots}_{\text{all possible states}}$

$$\text{Im } \langle X | T | \phi \rangle = \frac{1}{2} \sum_X \langle X | T | X \rangle \langle X | T^+ | \phi \rangle (2\pi)^4 \delta^{(4)}(p_\phi - p_X)$$

Diagrammatically

$$2 \text{Im} \rightarrow \text{---} = \sum_X - \text{---} \times \text{---} \times \text{---} .$$

\uparrow
 Im part

$$SS^+ = (1 + i(2\pi)^4 \delta^{(4)}(P_\phi - P_\chi) T) (1 - i(2\pi)^4 \delta^{(4)}(P_\phi - P_\chi) T^+) = 1$$

$$\cancel{1 - i(2\pi)^4 (T^+ - T) \delta^{(4)}(P_\phi - P_\chi) + (2\pi)^8 T T^+ (\delta^{(4)}(P_\phi - P_\chi))^2} \neq 1$$

$$T = \text{Re } T + i \text{Im } T, \quad T^+ = \text{Re } T - i \text{Im } T$$

$$T^+ - T = -2i \text{Im } T$$

$$2(2\pi)^4 \text{Im } T = (2\pi)^8 T T^+ \delta^{(4)}(P_\phi - P_\chi)$$

Let us insert $\square = \sum_x |x\rangle \langle x|$

and sandwich this expression with $\langle X| \dots |\phi\rangle$

$$\text{Im } \langle X | T | \phi \rangle = \frac{1}{2} \sum_x \langle X | T | x \rangle \langle x | T^+ | \phi \rangle (2\pi)^4 \delta^{(4)}(P_\phi - P_\chi)$$

∴

$$2 \cdot \phi \left\{ \begin{array}{c} | \\ \text{---} \\ | \end{array} \right\} x = \sum_x \phi \left\{ \begin{array}{c} | \\ \text{---} \\ | \end{array} \right\} x \left\{ \begin{array}{c} | \\ \text{---} \\ | \end{array} \right\} x$$

Im

If $\phi = \chi$ then, for instance $pp \rightarrow pp$

$$\begin{array}{ccc} p & & p \\ & \text{---} & \\ p & & p \end{array} = \frac{1}{2} \sum_x \left(\begin{array}{c} p \\ \chi \\ p \end{array} \right) \left(\begin{array}{c} p \\ \chi \\ p \end{array} \right)^2$$

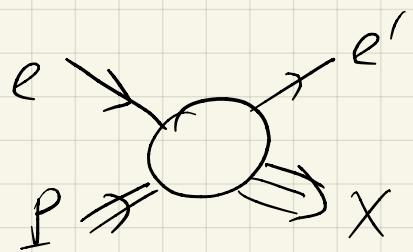
We are interested in photon-proton interactions

$$2 \text{Im} \frac{1}{P} = \sum_x |E_x|^2$$

Experimentally one measures cross-sections

$$\sigma_{\phi \rightarrow \psi} = \underbrace{\frac{1}{F_\phi}}_{\text{Flux of } \phi} |A_{\phi \rightarrow \psi}|^2 \underbrace{\frac{d^3 P_\psi}{(2\pi)^3 2 E_\psi}}_{\text{Phase space of } \psi}$$

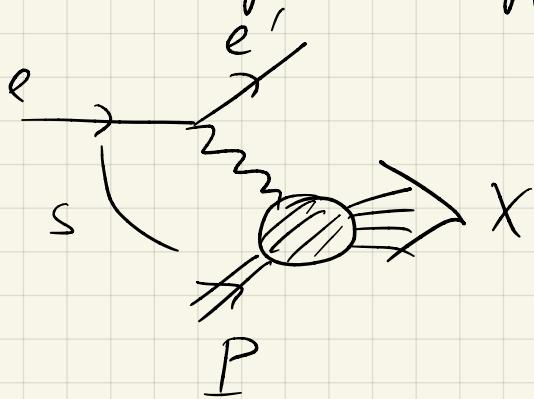
We want to calculate cross-section
of this process



$$e + P \rightarrow e' + X$$

$$q^2 = (e - e')^2 \quad \xrightarrow{\text{Deep Inelastic Scattering}} \quad P \rightarrow X$$

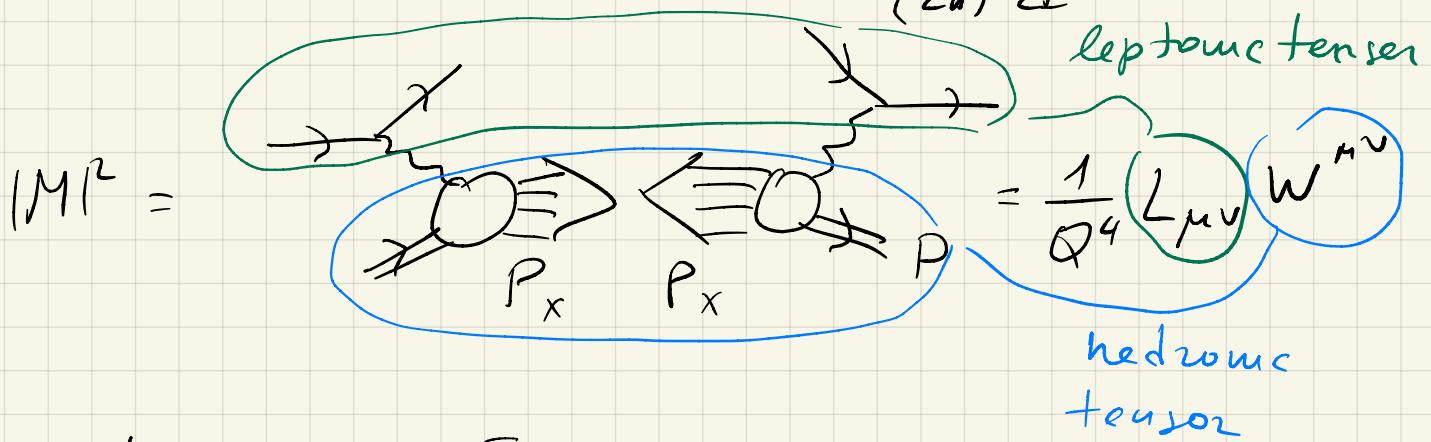
We use one photon approximation



$$\sigma = \frac{1}{\pi} |M|^2 d\mathcal{PS}$$

$$\mathcal{F} \approx 2s = 2(e + P)^2 \text{ flux}$$

$$d\mathcal{PS} = \frac{d^3 e'}{(2\pi)^3 2E'}$$



$$\frac{1}{Q^4} \sim \text{product of photon propagators}$$

Before we calculate $L_{\mu\nu}$ and $W^{\mu\nu}$ let us recapitulate some basics of QFT.

Consider the Lagrangian of a spin- $\frac{1}{2}$ particle with the mass m :

$$\mathcal{L} = \bar{\psi} (i \gamma^\mu \partial_\mu - m) \psi$$

γ^μ - gamma matrices

$$\partial_\mu = \frac{\partial}{\partial x^\mu}, \quad \psi(x), \quad \bar{\psi}(x) = \psi^+(x) \delta^0 \text{ fields}$$

Global gauge transformations

$$\psi'(x) = e^{i\alpha} \psi(x)$$

$$\bar{\psi}'(x) = \bar{\psi}(x) e^{-i\alpha}$$

$$\mathcal{L} \rightarrow \mathcal{L}', \text{ current } j^\mu(x) = \bar{\psi}(x) \gamma^\mu \psi(x)$$

Local gauge transformations

$$\begin{cases} \psi'(x) = e^{i\alpha(x)} \psi(x) \\ \bar{\psi}'(x) = \bar{\psi}(x) e^{-i\alpha(x)} \end{cases}$$

$$\partial_\mu \psi(x) = e^{-i\alpha(x)} (\partial_\mu - i \partial_\mu \alpha(x)) \psi'(x)$$

Thus

$$\mathcal{L} = \bar{\psi}(x) (i \gamma^\mu (\partial_\mu - i \partial_\mu \alpha(x)) - m) \psi'(x)$$

We can restore gauge invariance if we use

$$(\partial_\mu + ie A_\mu(x)) \psi(x)$$

$$(\partial_\mu + ie A_\mu(x)) \psi(x) = e^{-i\alpha(x)} (\partial_\mu + ie A'_\mu(x)) \psi'(x)$$

where

$$A'_\mu(x) = A_\mu(x) - \frac{1}{e} \partial_\mu \alpha(x)$$

$\partial_\mu + ie A_\mu(x) \rightarrow$ covariant derivative D_μ

$$\mathcal{L} = \bar{\psi}(x) (i \gamma^\mu (D_\mu + ie A_\mu(x)) - m) \psi(x)$$

or

$$\mathcal{L} = \bar{\psi}(x) (i \gamma^\mu D_\mu - m) \psi(x)$$

is invariant also under the local gauge transform.

We also have introduced interactions!

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I, \text{ where}$$

$$\mathcal{L}_I = -e j^\mu A_\mu \text{ where } j^\mu = \bar{\psi}(x) \gamma^\mu \psi(x)$$

Remember

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = \{ \gamma^\mu, \gamma^\nu \} = 2\gamma^{\mu\nu}$$

$$\gamma^0 (\gamma^\mu)^+ \gamma^0 = \gamma^\mu, \Rightarrow (\gamma^0)^+ = \gamma^0, (\gamma^\mu)^+ = -\gamma^\mu$$

Independent fields $\psi(x)$ & $\bar{\psi}(x) = \psi^+(x) \gamma^0$

Euler-Lagrange equations (for $\mathcal{L} = \mathcal{L}_0$)

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi} = 0 \\ \frac{\partial \mathcal{L}}{\partial \bar{\psi}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \bar{\psi}} = 0 \end{cases}$$

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = (i \gamma^\mu \partial_\mu - m) \psi, \quad \frac{\partial \mathcal{L}}{\partial \partial_\mu \bar{\psi}} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \psi} = -m \bar{\psi}, \quad \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi} = i \bar{\psi} \gamma^\mu$$

$$\Rightarrow \begin{cases} (i \gamma^\mu \partial_\mu - m) \psi(x) = 0 \\ i \partial_\mu \bar{\psi}(x) \gamma^\mu + m \bar{\psi}(x) = 0 \end{cases}$$

4 solutions, 2 with $p_0 > 0$, 2 with $p_0 < 0$

Let us consider only positive energy

$$\Psi(x) = u(p, s) e^{-i p \cdot x}, \quad p^2 = m^2, \quad p_0 > 0$$

$$(i \gamma^\mu \partial_\mu - m) \Psi(x) = 0$$

$$\Rightarrow (\gamma^\mu p_\mu - m) u(p) = 0, \quad \gamma^\mu p_\mu = p$$

$(p - m) u(p) = 0$, $u(p)$ is called spinor

$$\bar{u}(p)(p - m) = 0$$

Feynman diagram illustrating the annihilation of an electron (e) and the creation of a virtual electron-positron pair ($e' \bar{e}'$). The incoming electron (e) and outgoing virtual electron (e') are shown with arrows pointing to the right. The virtual positron (\bar{e}') is shown with an arrow pointing to the left.

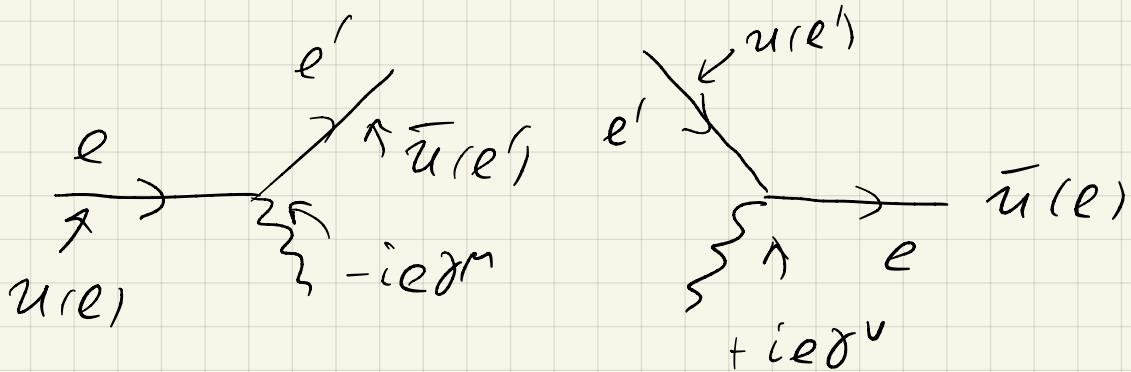
$$q = e - e'$$

Current conservation $\partial_\mu j^\mu = 0$

$$j^\mu = \bar{u}(e') \gamma^\mu u(e) e^{-i(e-e') \cdot x}$$

$$\partial_\mu j^\mu(x) = -i q_\mu j^\mu = 0 \Rightarrow \boxed{q_\mu j^\mu(x) = 0}$$

Let us calculate $L^{\mu\nu}$:



$$L^{\mu\nu} = \frac{1}{2s+1} \sum_{s'} \bar{u}_\alpha(\ell, s) (-ie\delta^\nu)_{\alpha\beta} u_\beta(\ell', s') \bar{u}_\beta(\ell', s') (+ie\delta^\mu)_{\beta\alpha} u_\alpha(\ell)$$

Spin products $\sum_{s'} u_\beta(\ell', s') u_\alpha(\ell', s') = (\ell' + m)_{\beta\alpha}$

$$u_b(\ell, s) \bar{u}_\alpha(\ell, s) = \left[\frac{(\ell + m)(1 + \delta_{sS})}{2} \right]_{b\alpha}$$

where

$$\delta_S = +i\delta^0\delta^1\delta^2\delta^3, \quad \delta^{5+} = \delta_S, \quad (\delta_S)^2 = 1, \quad \{ \delta_S, \delta^{\mu\nu} \} = 0$$

$$L^{\mu\nu} = \frac{e^2}{2} \underbrace{(\ell + m)_{b\alpha} \delta^\nu_{\alpha\beta} (\ell' + m)_{\beta\alpha} (\delta^\mu)_{ab}}_{\text{trace}}$$

trace

neglect m and we have

$$L^{\mu\nu} = \frac{e^2}{2} \text{Tr} (\ell \delta^\mu \ell' \delta^\nu)$$

Traces

$$\text{Tr}(\text{odd } \# \gamma) = 0$$

$$\text{Tr}(\not{a}\not{b}) = 4a \cdot b$$

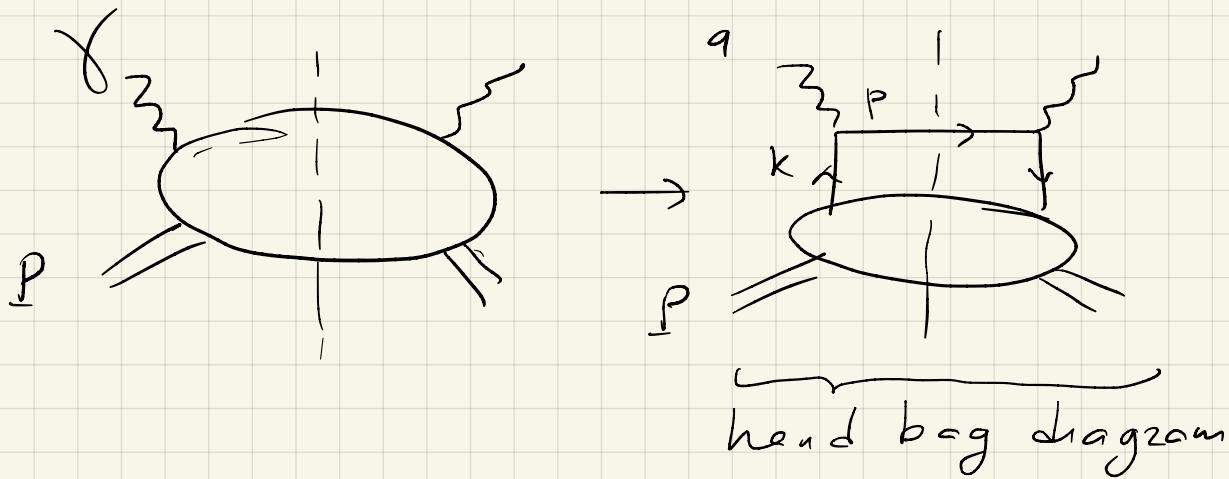
$$\text{Tr}(\not{a}\not{b}\not{c}\not{d}) = 4[(a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(c \cdot b)]$$

$$\text{Tr}(\gamma^a \gamma^b \gamma^c \gamma^d) = 4(g^{ab}g^{cd} - g^{ac}g^{bd} + g^{ad}g^{cb})$$

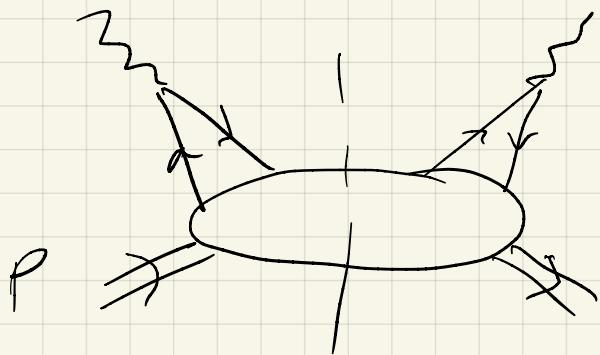
Thus,

$$L^{\mu\nu} = 2e^2(e^\mu e'^\nu + e^\nu e'^\mu - g^{\mu\nu}(e \cdot e'))$$

Now let us consider the hadronic tensor



why do we not consider?

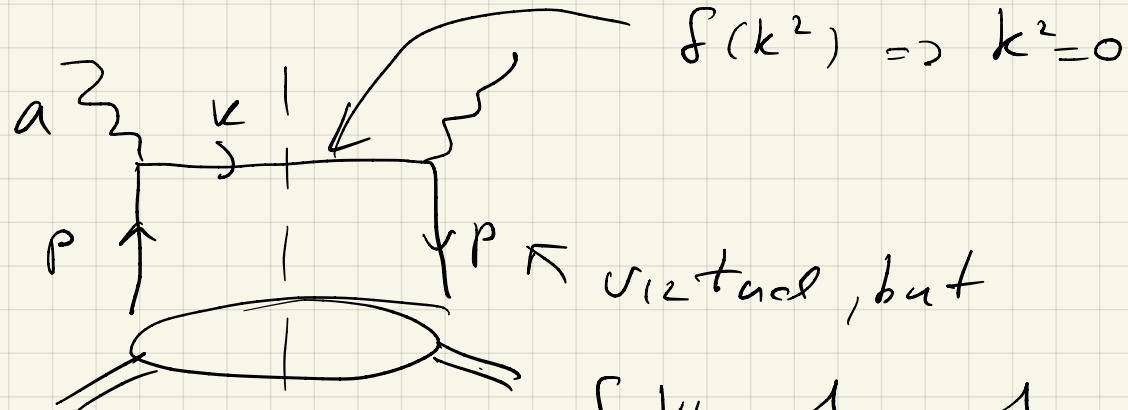


$\propto e_{q_1} e_{q_2}$
suppressed by $(\frac{1}{Q^2})^2$
as at least one
of the propagators is hard

$$\frac{1}{p^2 + i\epsilon} = \text{Im} \frac{1}{p^2 + i\epsilon} = \pi \delta(p^2)$$

$$q \quad p \\ \swarrow \quad \searrow \\ k \quad \rightarrow \quad \delta^{(4)}(q + k - p) \Rightarrow p = k + q$$

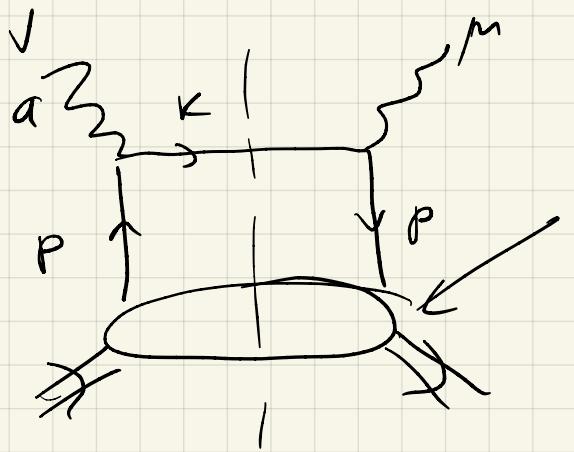
Why partons are almost on mass shell?



$$\int d^4 p \frac{1}{(p^2 + i\epsilon)} \frac{1}{(p^2 - i\epsilon)} \rightarrow p^2 \approx 0$$

The contribution from
this integral is much then

$p^2 \approx 0$ as well!



let us call this matrix $\phi(p, P)$

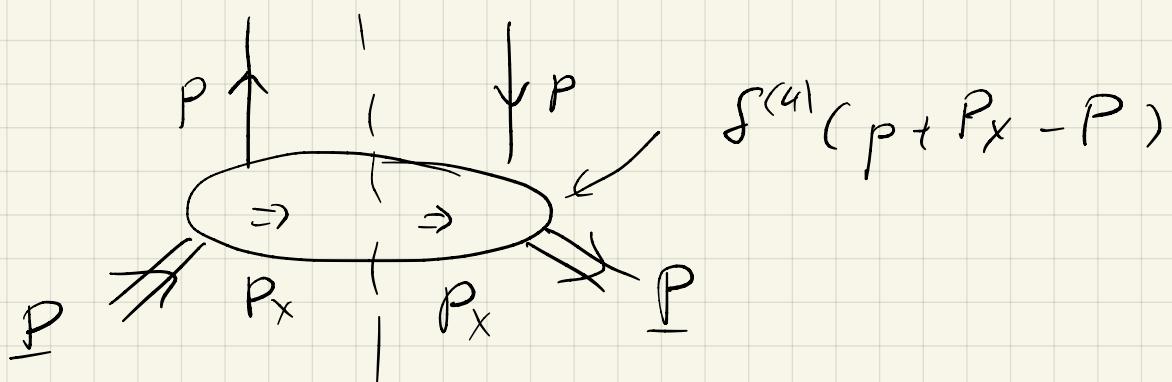
$$W^{\mu\nu} = \sum_q e_q^2 \int \frac{d^4 p}{(2\pi)^4} \text{Tr} (\delta^\mu(p+q) \delta^\nu(p, P) \delta((p+q)^2))$$

Let us parametrize $p = x \frac{P}{x + P}$, $x \in (-\infty, \infty)$

$$\delta((p+q)^2) = \delta(-Q^2 + 2xP \cdot q) = \frac{1}{2P \cdot q} \delta(x_{B_J} - x)$$

quarks are probed at $x = x_{B_J}$!

What is ϕ ?



$$\delta^4(p) = \int \frac{d^4 \zeta}{(2\pi)^4} e^{-i p \cdot \zeta}$$

$$\oint_x = \int \frac{d^3 P_x}{2E_x (2\pi)^3} = \int \frac{d^4 P_x}{(2\pi)^4} \partial(E_x)$$

$$\bar{\Phi} = \oint_x \delta^{(4)}(p + P_x - P) \langle p | \bar{\psi}(0) | x \rangle \langle x | \psi(0) | p \rangle$$

$$= \oint_x \int \frac{d^4 \zeta}{(2\pi)^4} e^{-i \zeta \cdot (p + P_x - P)} \langle p | \bar{\psi}(0) | x \rangle \langle x | \psi(0) | p \rangle$$

$$= \oint_x \int \frac{d^4 \zeta}{(2\pi)^4} e^{-i \zeta \cdot P} \underbrace{\langle p | e^{i \zeta \cdot P} \bar{\psi}(0) e^{-i \zeta \cdot P_x} | x \rangle}_{\langle p | e^{i \zeta \cdot P} \bar{\psi}(0) e^{-i \zeta \cdot P_x} | x \rangle} \underbrace{\langle x | \psi(0) | p \rangle}_{\bar{\psi}(\zeta) - \text{shift of the field}}$$

Thus

$$\Phi(p, P) = \int_X \frac{d^4 z}{(2\pi)^4} e^{-i p \cdot z} \langle P | \bar{\psi}(z) | X \rangle \langle X | \psi(0) | P \rangle$$

now we use

$$\int_X |X\rangle \langle X| = \mathbb{I} \text{ completeness}$$

of states and obtain

$$\Phi(p, P) = \int \frac{d^4 z}{(2\pi)^4} e^{-i p \cdot z} \langle P | \bar{\psi}(z) \psi(0) | P \rangle$$

Let us introduce light cone variables $A^\pm = \frac{A^0 \pm A^3}{\sqrt{2}}$

$$A \cdot B = A^+ B^- + A^- B^+ - \vec{A}_T \cdot \vec{B}_T, \quad A_T = (A^1, A^2)$$

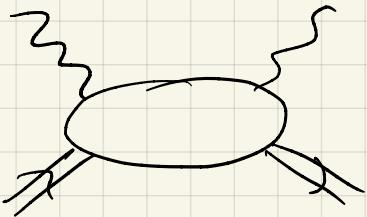
$$P \approx (P^+, \frac{M^2}{2P^+}, 0) \approx (P^+, 0, 0)$$

$$p = \gamma P \approx (P^+, 0, p_T) \quad \begin{matrix} \nwarrow \\ \text{important for TMDs} \end{matrix}$$

$$p \cdot z \rightarrow p^+ z^- - \underbrace{\vec{p}_T \cdot \vec{z}_T}_{\text{small}} \approx p^+ z^-$$

$$\Rightarrow z \approx (0, z^-, 0)$$

Let us see how distributions are introduced
in DIS.



$$W^{\mu\nu} = -\left(g^{\mu\nu} + \frac{q^\mu q^\nu}{Q^2}\right)W_1 + \left(P^\mu + \frac{q^\mu}{2x}\right)\left(P^\nu + \frac{q^\nu}{2x}\right)W_2$$

(only $W^{\mu\nu} = W^{\nu\mu}$ & $q_\mu W^{\mu\nu} = 0$)

Remember $P \cdot q = \gamma$

One usually uses $\begin{cases} F_1(x, Q^2) = W_1(x, \theta) \\ F_2(x, Q) = \gamma W_2(x, \theta) \end{cases}$

$$F_L = F_2 - 2x F_1 \approx 0$$

$$P^\mu = (P, 0, 0, P)$$

$$\underline{P}^2 = h^2 = 0$$

$$h^\mu = \left(\frac{1}{2P}, 0, 0, -\frac{1}{2P}\right)$$

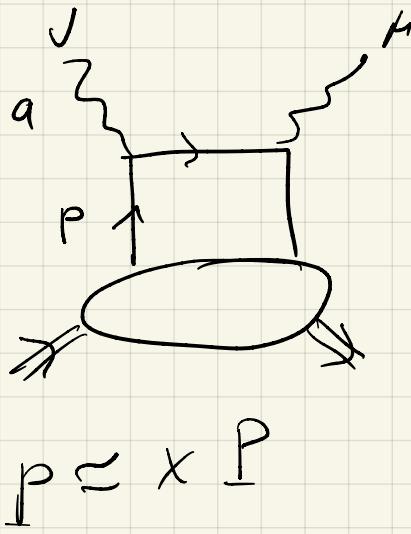
$$P \cdot h = 1$$

$$q^\mu = q_{\perp}^\mu + \gamma n^\mu$$

$$q^2 = -\vec{q}_{\perp}^2 = -Q^2$$

Then

$$n^\mu n^\nu W_{\mu\nu} = W_L = \frac{1}{v} F_2$$



$$W^{\mu\nu} = e_a^4 \int \frac{d^4 p}{(2\pi)^4} \text{Tr} (\gamma^\mu (\not{q} + \not{p}) \gamma^\nu \phi) \cdot \underbrace{\delta((p+q)^\nu)}_{\frac{1}{2\sqrt{v}} \delta(x - x_B)}$$

$$F_2 = v n^\mu n^\nu W_{\mu\nu} = \frac{1}{2} e_a^2 \int \frac{d^4 p}{(2\pi)^4} \text{Tr} (\not{p} \not{q} \not{p} \not{\phi}) \delta(x - x_B) \underbrace{- \not{p} + 2n \cdot p}_{2x \text{Tr}(\not{p} \not{\phi})}$$

we can define

$$f(x_B) = \int \frac{d^4 p}{(2\pi)^4} \text{Tr} (\not{p} \phi(p, P)) \delta(x - x_B)$$

Parton distributions

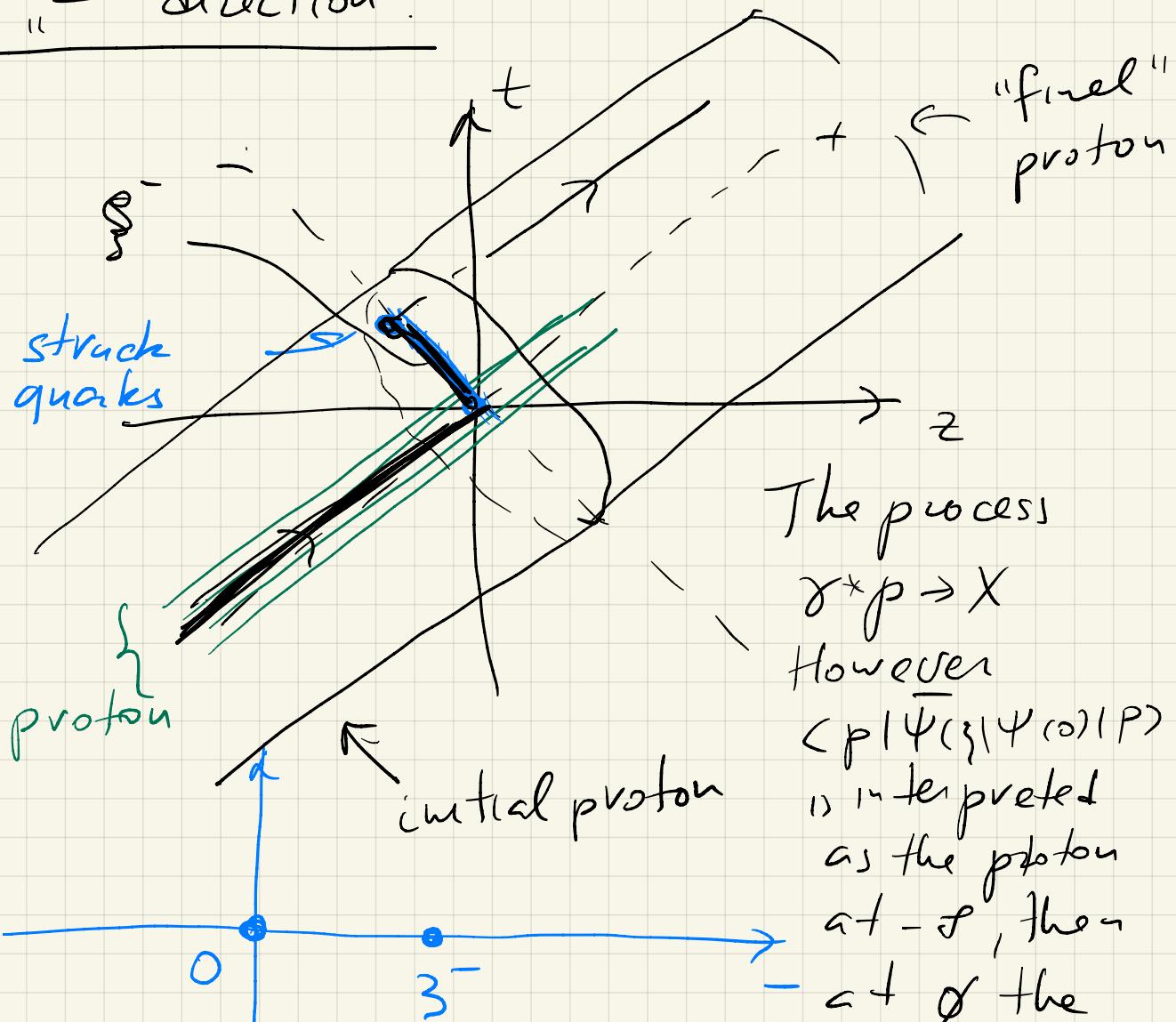
$$\Rightarrow F_2(x, Q^2) = \sum_q e_q^2 \times f(x)$$

Bjorken scaling!

$$\underline{\kappa = \gamma^+}$$

The fields are separated by some distance β

In " - " direction!



The process

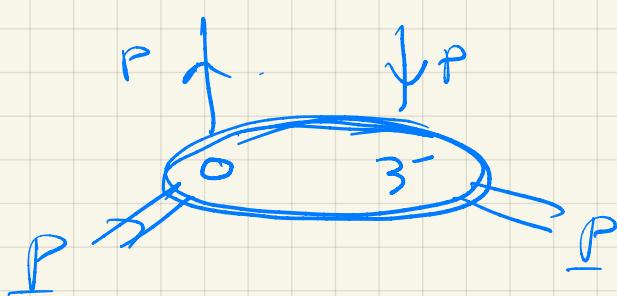
$$\gamma^* p \rightarrow X$$

However

$$\langle p | \bar{\psi}(z) \psi(0) | P \rangle$$

is interpreted
as the proton
at $-\delta$, then

at δ of the
quark field is
shifted to β
and then
the proton
continues to $+\infty$



Analogously to DIS we can define the following projections of the correlator $\underline{\Phi}(p, \underline{P})$ in the case when transverse motion is not ignored.

It is customary to call the parton's momentum k
so

$$\underline{\Phi}(x, k_T)_{ij} = \int \frac{d\zeta - d\zeta_T^2}{(2\pi)^3} e^{-ix^+ \zeta^- + ik_T \cdot \zeta_T}$$

$$\langle P | \bar{\psi}_j(\zeta) \psi_i(0) | P \rangle \Big|_{\zeta^+ = 0}$$

$$\frac{1}{2} T_2(\gamma^+ \phi) = f_1 - \frac{\epsilon^{jk} k_T^j S_T^k}{M} f_{1T}^\perp$$

$$\frac{1}{2} T_2(\gamma^+ \gamma_S \phi) = S_L g_1 + \frac{\bar{k}_T \cdot \bar{S}_T}{M} g_{1T}^\perp$$

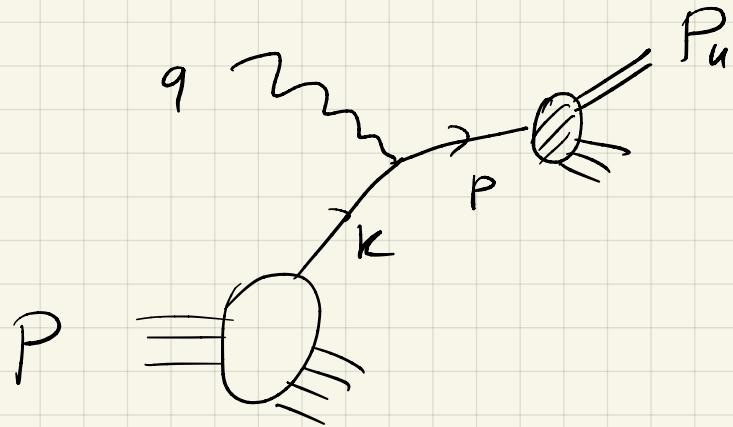
$$\frac{1}{2} T_2(i \gamma^j \gamma_S \phi) = S_T^j h_1 + S_L \frac{k_T^j}{M} h_{1L}^\perp + \frac{\kappa^{jk} S_T^k}{M} h_{1T}^\perp$$

$$+ \frac{\epsilon^{jk} k_T^k}{M} h_2 \quad \text{, where } \kappa^{jk} = (k_T^j k_T^k - \frac{1}{2} k_T^2 \delta^{jk})$$

$$\epsilon^{ij} = \epsilon^{-+ij}, \epsilon^{0123} = +1$$

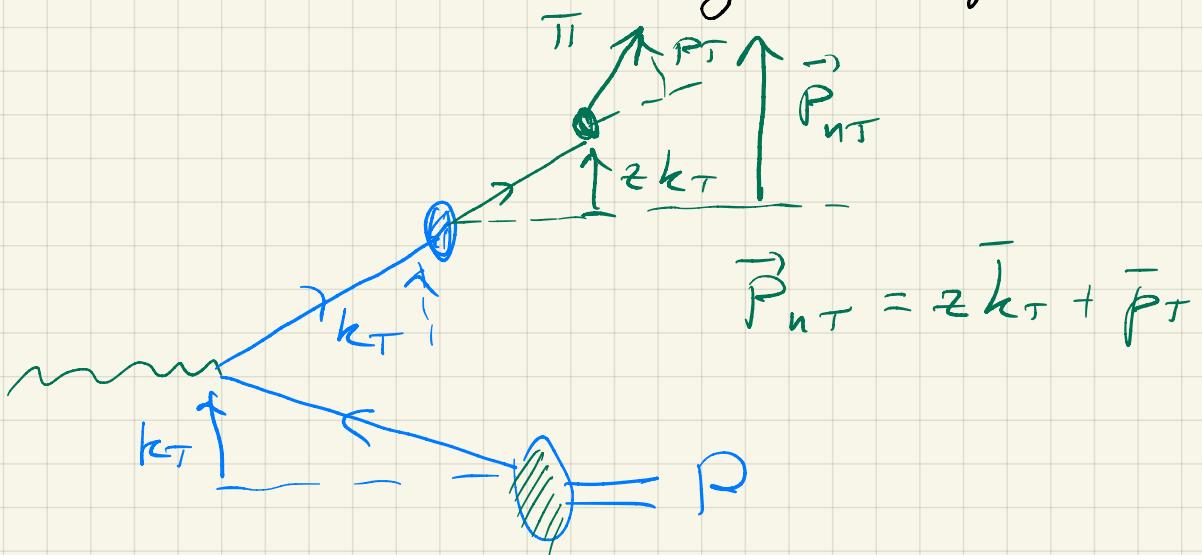
Let us work out convolutions for SIDIS

$$C[\omega fD] = \sum_q e_q^2 \int d^2 k_T d^2 p_T \delta^{(2)}(\vec{P}_{qT} - \vec{k}_T - \vec{p}_T) \omega(k_T, p_T) f(x, k_T) D(z, p_T)$$

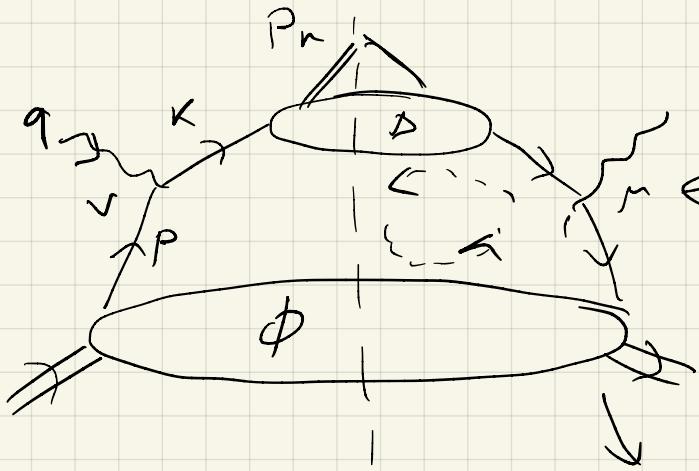


In order to study evolution we need to rewrite the convolution in the configuration space.

We will see that TMD evolution equations are to be solved in configuration space



Let us write the cut amplitude:



Starting from here we "read" the diagram similar to others in clockwise

$$W^{\mu\nu} = \sum_q e_q^2 \int \frac{d^4 p}{(2\pi)^4} \text{Tr} (\gamma^\mu D \gamma^\nu \phi) S^{(u)}(k - p - q)$$

ϕ and D contain TMDs

$$\frac{1}{2} \text{Tr} [\gamma^+ \phi] = f_1 - \frac{\epsilon^{ijk} p_T^j S_T^k}{m} f_{1T}$$

$$\frac{1}{2} \text{Tr} [\gamma^- D] = D_1$$

$$G \sim \frac{1}{Q^4} \langle \mu_\nu W^{\mu\nu} \rangle \sim G_0 (F_{\mu\nu} + \dots)$$

structure functions:
 $F_{\mu\nu} = C [1 f_1 D_1]$ etc

Asymmetries ratios of polarised over unpolarised structure functions

$$A_{UT} = \frac{S_{1T} (f_{1U} - f_{1S})}{F_{1U}} = \frac{F_{1T}}{F_{1U}} \text{ etc}$$

Definitions: of Fourier-Bessel transform

$$f(x, k_T^2) = \int \frac{d^2 b_T}{(2\pi)^2} e^{i \bar{k}_T \bar{b}_T} \tilde{f}(x, b_T^2) =$$

$$= \int \frac{b_T db_T}{2\pi} J_0(k_T b_T) \tilde{f}(x, b_T^2)$$

NB $\int d\varphi e^{i k_T b_T \cos \varphi} = 2\pi J_0(k_T b_T)$

$$\tilde{f}(x, b_T^2) = \int d^2 k_T e^{-i \bar{k}_T \bar{b}} f(x, k_T^2) =$$

$$= 2\pi \int k_T dk_T J_0(b_T k_T) f(x, k_T^2)$$

Using Mathematica prove these relations, and

prove that if

$$f(x, k_T^2) = f_1(x) \frac{1}{\pi \langle k_T^2 \rangle} e^{-k_T^2 / \langle k_T^2 \rangle}$$

then

$$\tilde{f}(x, b_T^2) = f_1(x) e^{-\langle k_T^2 \rangle b_T^2 / 4}$$

Notice that

$$\int d^2 k_T f(x, k_T^2) = f_1(x) \leftarrow \text{The basis}$$

of the Generalized Parton Model (Feynman 78')

In TMD phenomenology the following moments are used

$$\int d^2 k_T \frac{k_T^2}{2M^2} f(x, k_T^2) = f^{(1)}(x)$$

the first moment

$$\int d^2 k_T \left(\frac{k_T^2}{2M^2} \right)^2 f(x, k_T^2) = f^{(2)}(x)$$

the second moment

In configuration space we have

$$\tilde{f}^{(1)}(x, b_T) = \frac{2\pi}{M^2} \int k_T dk_T \frac{k_T}{b_T} J_1(k_T b_T) f(x, k_T^2)$$

$$\tilde{f}^{(2)}(x, b_T) = \frac{4\pi}{M^4} \int k_T dk_T \left(\frac{k_T}{b_T} \right)^2 J_2(k_T b_T) f(x, k_T^2)$$

Prove that

$$\tilde{f}^{(1)}(x, 0) = f^{(1)}(x)$$

$$\tilde{f}^{(2)}(x, 0) = f^{(2)}(x)$$

For fragmentation functions

$$D(z, p_T^2) = \int \frac{d^2 b_T}{(2\pi)^2} e^{-i \bar{p}_T \bar{b}_T / z} \tilde{D}(z, b_T^2)$$

$$= \int \frac{b_T db_T}{2\pi} J_0\left(\frac{p_T b_T}{z}\right) \tilde{D}(z, b_T^2)$$

and

$$\tilde{D}(z, b_T^2) = \int \frac{d^2 p_T}{2\pi} e^{-i \bar{b}_T \bar{p}_T / z} D(z, p_T^2)$$

$$= 2\pi \int \frac{p_T dp_T}{2\pi} J_0\left(\frac{p_T b_T}{z}\right) D(z, p_T^2)$$

Test functions (check with Mathematica)

$$D(z, p_T^2) = D_1(z) \frac{1}{\pi \langle p_T^2 \rangle} e^{-p_T^2 / \langle p_T^2 \rangle}$$

$$\tilde{D}(z, b_T^2) = \frac{1}{z^2} D_1(z) e^{-\frac{\langle p_T^2 \rangle b_T^2}{4z^2}}$$

Moments are also important for FFs

$$\tilde{D}^{(n)}(z, b_T^2) = \frac{2\pi n!}{(M_n^2)^n} \int \frac{p_T dp_T}{z^2} J_n\left(\frac{p_T + b_T}{z}\right) \left(\frac{p_T}{zb_T}\right)^n D(z, p_T^2)$$

$$D(z, p_T^2) = \frac{(M_n^2)^n}{2\pi n!} \int b_T dh_T \left(\frac{zb_T}{p_T}\right)^n J_n\left(\frac{p_T + b_T}{z}\right) \tilde{D}^{(n)}(z, b_T)$$

$$\lim_{b_T \rightarrow 0} \tilde{D}^{(n)}(z, b_T) = \frac{1}{z^2} D^{(n)}(z) \quad \text{prove it!}$$

where

$$D^{(n)}(z) = \int d^2 p_T \left(\frac{p_T^2}{2z^2 M_n}\right)^n D(z, p_T^2)$$

Test with Mathematica

$$D(z, p_T) = D_1(z) \frac{1}{\bar{n}(p_T^2)} e^{-p_T^2/\langle p_T^2 \rangle} \quad \text{prove}$$

$$\rightarrow \tilde{D}(z, b_T^2) = \frac{1}{z^2} D_1(z) e^{-\frac{b_T^2 \langle p_T^2 \rangle}{4z^2}}$$

$$D(z, b_T^2) = H_1^{(1)}(z) \frac{z^2 M_n^2}{\bar{n}(p_T^2)^2} e^{-p_T^2/\langle p_T^2 \rangle} \quad \text{prove}$$

$$\rightarrow \tilde{D}^{(1)}(z, b_T^2) = H_1^{(1)}(z) \frac{1}{z^2} e^{-\frac{b_T^2 \langle p_T^2 \rangle}{4z^2}}$$

Now convolution in the momentum space $\rightarrow b_T$ space

$$C[\omega f D] = \sum_a e_a^2 \int d^2 k_T d^2 p_T \delta^{(2)}(\bar{P}_{aT} - z \bar{k}_T - \bar{p}_T)$$

$$\omega(k_T b_T) f(x, k_T) D(z, p_T), \quad \bar{P}_{aT} = -z \bar{q}_T$$

$$\text{We rewrite } \delta^{(2)}(\bar{P}_{aT} - z \bar{k}_T - \bar{p}_T) =$$

$$= \delta^{(2)}(-z \bar{q}_T - z \bar{k}_T - \bar{p}_T) = \frac{1}{z^2} \delta^{(2)}(\bar{q}_T + \bar{k}_T + \frac{\bar{p}_T}{z})$$

$$= \frac{1}{z^2} \int \frac{d^2 b_T}{(2\pi)^2} e^{-i(\bar{q}_T + \bar{k}_T + \bar{p}_T/z) \cdot \bar{b}_T}$$

$$F_{aT} = C[1f_1 D_1] = \sum_a e_a^2 \int \frac{d^2 b_T}{(2\pi)^2} e^{-i \bar{b}_T \cdot \bar{q}_T}$$

$$* \int d^2 k_T e^{-i \bar{b}_T \cdot \bar{k}_T} f_1(x, k_T)$$

$$\frac{1}{z^2} \int d^2 p_T e^{-i \bar{b}_T \cdot \bar{p}_T/z} D_1(z, p_T) =$$

$$= \sum_a e_a^2 \int \frac{d^2 b_T}{(2\pi)^2} e^{-i \bar{b}_T \cdot \bar{q}_T} \tilde{f}_1(x, b_T) \tilde{D}_1(z, b_T)$$

$$= \sum_a e_a^2 \int \frac{b_T db_T}{2\pi} g_0(b_T q_T) \tilde{f}_1(x, b_T) \tilde{D}_1(z, b_T)$$

$$\text{Let us call } B(\tilde{f}_1, \tilde{D}_1) = \sum_a e_a^2 \int \frac{b_T db_T}{2\pi} g_0(b_T q_T) \tilde{f}_1 \tilde{D}_1$$

A more complicated example, Sivers asymmetry

$$F_{u\bar{u}} \stackrel{\sin(\phi_u - \phi_s)}{=} C \left[- \frac{\hat{h} \cdot \vec{k}_T}{M} f_{1T}^\perp D_1 \right]$$

$$= \sum_q e_q^2 \int d^2 k_T d^2 p_T \delta^{(2)}(\vec{p}_{u\bar{u}} - z \vec{k}_T - \vec{p}_T) \underbrace{\left(- \frac{\hat{h} \cdot \vec{k}_T}{M} \right)}_{-\frac{k_T}{M} \cos(\varphi - \phi_h)}$$

$$\cdot f_{1T}^\perp(x, k_T^2) D_1(z, p_T^2)$$

$$= \sum_q e_q^2 \int \frac{dz b_T}{(2\pi)^2} e^{-i \vec{q}_T \vec{b}_T} \int d^2 k_T \left(- \frac{k_T}{M} \right) \cos(\varphi - \phi_u)$$

$$e^{-i \vec{k}_T \vec{b}_T} f_{1T}^\perp(x, k_T^2) \underbrace{\int d^2 p_T \frac{1}{z^2} e^{-i \vec{p}_T \vec{b}_T/z} D_1(z, p_T^2)}_{\tilde{D}_1(z, b_T^2)}$$

we have $\vec{k}_T \vec{b}_T = b_T k_T \cos(\varphi - \varphi_b)$

$$\int d\varphi e^{-i b_T k_T \cos(\varphi - \varphi_b)} \cos(\varphi - \phi_u) = \begin{cases} \text{Use} \\ \text{Mathematica} \end{cases}$$

$$= -2\pi i \Im_1(b_T k_T) \cos(\varphi_b - \phi_u)$$

$$\int d\varphi_b \cos(\varphi_b - \phi_u) e^{-i b_T q_T \cos(\varphi_b - \phi_u)} = \gamma$$

$$= 2\pi i \Im_1(b_T q_T)$$

So that

$$F_{UT}^{\sin(\phi_u - \phi_s)} = - \sum_q e_q^2 \int \frac{db_T b_T}{2\pi} J_1(b_T q_T)$$

$$\int dk_T \frac{k_T^2}{M} J_1(b_T k_T) f_{1T}^{\perp}(x, k_T^2) \tilde{D}_1(z, b_T^2)$$

$$\tilde{f}_{1T}^{\perp(1)}(x, b_T^2) = \frac{2\pi}{M^2} \int k_T dk_T \frac{k_T}{b_T} J_1(k_T b_T) f_{1T}^{\perp}(x, k_T^2)$$

Thus

$$F_{UT}^{\sin(\phi_u - \phi_s)} = (-M) \sum_q e_q^2 \int \frac{b_T db_T}{2\pi} \underbrace{b_T J_1(b_T q_T)}_{\text{①}} \\ \cdot \tilde{f}_{1T}^{\perp(1)}(x, b_T^2) \tilde{D}_1(z, b_T^2) \quad \text{②}$$

$$F_{UT}^{\sin(\phi_u - \phi_s)} = -M B [\tilde{f}_{1T}^{\perp(1)} \tilde{D}_1]$$

General definition

$$B[\tilde{f}^{(n)} \tilde{D}^{(m)}] = \int \frac{b_T db_T}{2\pi} b_T^{n+m} J_{n+m}(b_T q_T)$$

$$\tilde{f}^{(n)}(x, b_T^2) \tilde{D}^{(m)}(z, b_T^2)$$

Collins asymmetry

$$F_{u\bar{u}}^{\sin(\phi_u + \phi_s)} = C [\omega_A^{(1)} h_1 + h_1^\perp]$$

$$\omega_A^{(1)} = \frac{\hat{h} \cdot \vec{p}_+}{2M_u}$$

$$F_{u\bar{u}}^{\sin(\phi_u + \phi_s)} = M_u B \left[\tilde{h}_1^{(0)} \tilde{h}_1^\perp \right]$$

Let us use (0) moment
as well for completeness

Prove it!

Evolution of TMDs

The evolution is automatically studied for TMDs, in the coordinate space by

$$G \propto \int \frac{d^2 b_T}{(2\pi)^2} e^{-i q_T b_T} \tilde{f}(x, b_T, Q, \zeta) \tilde{D}(z, b_T, Q, \zeta)$$

ζ corresponds to the UV divergence, the same as for collinear densities

ζ is a new scale that correspond to a new type of divergences "rapidity" divergence of TMDs

Let us define momentum regions

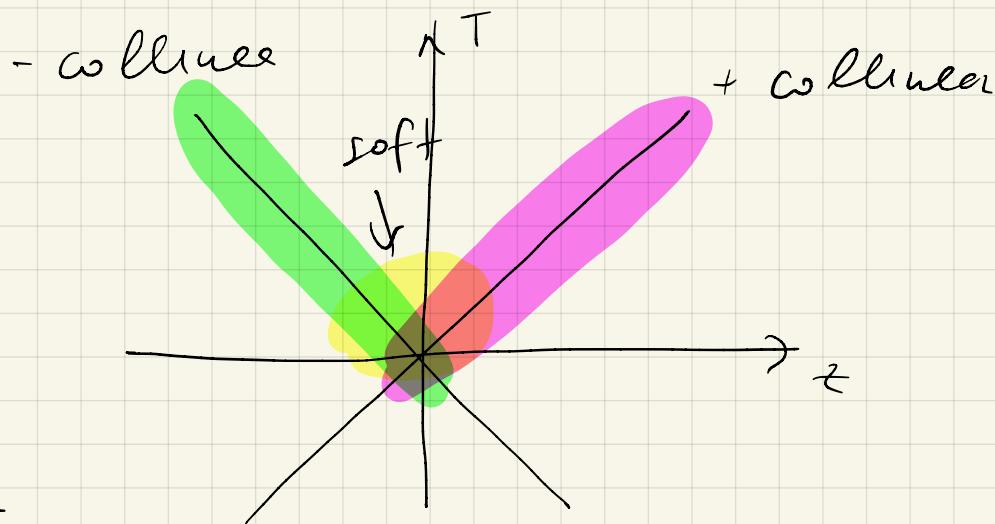
1) Hard region - momentum with large virtualities

$$\sim Q, \quad k \sim Q(1, 1, 1)$$

$\begin{matrix} \nearrow & \nearrow & \nearrow \\ + & - & T \end{matrix}$

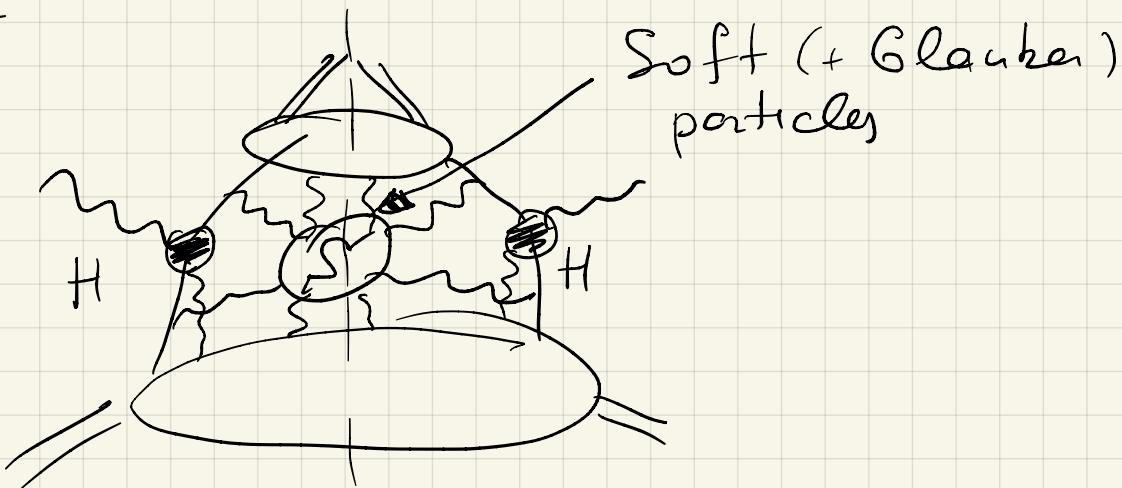
2) Collinear region - momentum close to some beam/jet directions $k \sim Q(1, \lambda, \lambda)$ for example

3) Central (soft) region $k \sim Q(\lambda^u, \lambda^u, \lambda^u)$ $u > 0$
 $x \ll 1$



For each region approximations are applied
and then double counting is subtracted

Result



Some work is still needed to fully factorise. Ward identities are used to strip off collinear polarized gluons from hard part and organise them into Wilson lines

Simple

$$p-k \xrightarrow{\quad} k \xrightarrow{\quad} (p-k)^2 = -2p \cdot k + k^2 \approx -2p \cdot k$$

eikonal approximation

TMD factorization describes processes differential
in transverse momentum $\frac{dG}{dq_T^2}$

$q_T \sim \Lambda_{\text{QCD}} \ll Q$: small k_T of partons

play an important role at small $q_T \Rightarrow$ TMD factorization
with $\text{TMDsf}(x, k_T)$

Generalized Parton Model

$F(x, k_T)$

$\downarrow \text{QCD}$

$F(x, k_T, \mu, \xi) \rightarrow$ uniquely defined
deal with all divergencies
obey evolution equations

QCD evolution is governed by the so-called Collins-Soper equation two Renormalization Group equations

CS equation ↗ to space

$$(1) \quad \frac{\partial \ln \tilde{F}(x, b_T, \mu, \tau)}{\partial \ln \sqrt{s}} = \tilde{K}(b_T, \mu)$$

↑
CS kernel

RG equations

$$(2) \quad \frac{d \tilde{K}(b_T, \mu)}{d \ln \mu} = -\gamma_K(g(\mu)) \leftarrow \begin{array}{l} \text{Cusp anomalous} \\ \text{dimension of } K \end{array}$$

Very universal in QCD., γ_K depend only on μ

$$(3) \quad \frac{d \ln \tilde{F}(x, b_T, \mu, \tau)}{d \ln \mu} = \gamma_F(g(\mu), \tau/\mu^2)$$

↑
anomalous dimension
of F .

$$\frac{d}{d \ln \mu} \left(\frac{\partial \ln F(x, b_T, \mu, \bar{s})}{\partial \ln \bar{s}} \right) = \frac{d}{d \ln \mu} \tilde{K}(b_T; \mu) = -\gamma_K(\mu)$$

$$\frac{\partial}{\partial \ln \bar{s}} \left(\frac{\partial \ln F(x, b_T, \mu, \bar{s})}{\partial \ln \mu} \right) = -\gamma_F(\mu)$$

$$\gamma_F(\mu, \bar{s}/\mu^2)$$

$$\gamma_F(\mu, 3/\mu^2) - \gamma_F(\mu, 3/\mu^1) = -\gamma_K(\mu) \ln \bar{s}_0 + \gamma_K \ln \bar{s}$$

if $\bar{s}_0 = \mu^2$ then

$$\underbrace{\gamma_F(\mu, 3/\mu^2) = \gamma_F(\mu, 1) - \frac{1}{2} \gamma_K(\mu) \ln 3/\mu^2}_{}$$

Solutions

$$1) \quad \frac{d \tilde{K}(b_T, \mu)}{d \ln \mu} = -\gamma_K(\mu) \Rightarrow$$

$$\int_{\mu_0}^{\mu} d \tilde{K}(b_T, \mu') = - \int_{\mu_0}^{\mu} \gamma_K(\mu') \frac{d \mu'}{\mu'}$$

$$\tilde{K}(b_T, \mu) = - \int_{\mu_0}^{\mu} \frac{d \mu'}{\mu'} \gamma_K(\mu') + \tilde{K}(b_T, \mu_0)$$

$$2) \quad \tilde{F}(x, b_T, \mu, \zeta) = \tilde{F}(x, b_T, \mu, \zeta_0) \exp \left[\tilde{K}(b_T, \mu) \ln \sqrt{\frac{\zeta}{\zeta_0}} \right]$$

$$3) \quad \tilde{F}(x, b_T, \mu_0, \zeta) = \tilde{F}(x, b_T, \mu_0, \zeta) \exp \left[\int_{\mu_0}^{\mu} \frac{d \mu'}{\mu'} \gamma_F(\mu') \frac{\zeta - \zeta_0}{\mu'} \right]$$

Implementing evolution

We start with low b_T

$$\tilde{F}_f(x, b_T, \mu, \bar{s}) = \sum_j \int_x^{\hat{x}} \frac{dx}{\hat{x}} \underbrace{\tilde{C}_{j/f}(\frac{x}{\hat{x}}, b_T, \mu, \bar{s})}_{\text{coefficient functions}} \underbrace{f_j(\hat{x}, \mu)}_{\text{collinear PDFs}}$$

at the lowest order

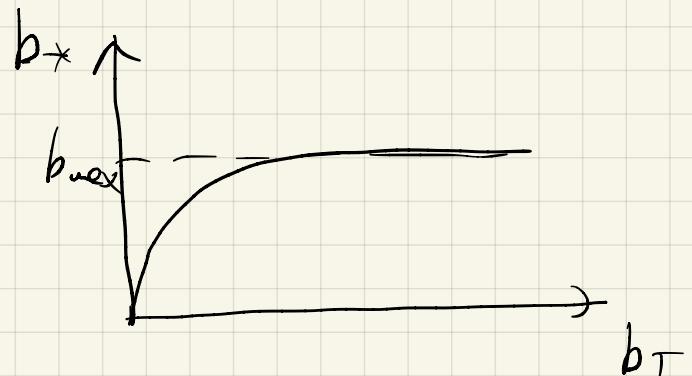
$$\tilde{C}_{j/f} = \delta_{jf} \delta\left(\frac{x}{\hat{x}} - 1\right)$$

Next step: combine perturbative & non perturbative $\rightarrow b_*$

Problem: $\tilde{K}(b_T)$, $\tilde{F}(b_T)$ are non perturbative at large b_T

We want: write functions such that they are perturbatively calculable with non perturb. corrections

$$b_*(b_T) = \frac{b_T}{\sqrt{1 + b_T^2/b_{\max}^2}}$$



$$\tilde{K}(b_T, \mu) = \tilde{K}(b_*, \mu) + \underbrace{[\tilde{K}(b_T, \mu) - \tilde{K}(b_*, \mu)]}_{g_K(b_T)}$$

non pert. function

$$\tilde{K}(b_T, \mu) = \tilde{K}(b_*, \mu_0) - \underbrace{\int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \delta_K(\mu') - g_K(b_T)}$$

study in Mathematics

$$g_K = \frac{1}{2} g_2 b_T, \quad g_K = g_0 b_T b_*, \quad g_K = g_2 \ln \frac{b_T}{b_*}$$

different groups

Euler gamma
↑
 $-\gamma_E$

We can choose $\mu_0 \sim 1/b_T$, $\mu_0 = \frac{2e^{-\gamma_E}}{b_*}$ is
the standard choice

$$\tilde{F}(x, b_T, \mu, \beta) = \tilde{F}(x, b_*, \mu, \beta) \left[\frac{\tilde{F}(x, b_T, \mu, \beta)}{\tilde{F}(x, b_*, \mu, \beta)} \right] =$$

$$= \tilde{F}(x, b_*, \mu, \beta_0) \exp \left[\tilde{K}(b_*, \mu) \ln \sqrt{\frac{\beta}{\beta_0}} \right] \underbrace{\left[\frac{\tilde{F}(x, b_T, \mu, \beta_0)}{\tilde{F}(x, b_*, \mu, \beta_0)} \right]}_{\exp[-g(x, b_T)]}$$

$$\times \exp \left[\ln \sqrt{\frac{\beta}{\beta_0}} (\tilde{K}(b_T, \mu) - \tilde{K}(b_*, \mu)) \right]$$

$$= \tilde{F}(x, b_*, \mu, \beta_0) \exp \left[\ln \sqrt{\frac{\beta}{\beta_0}} \tilde{K}(b_*, \mu) \exp[-g(x, b_T) - \ln \sqrt{\frac{\beta}{\beta_0}} g_K(b_T)] \right]$$

$$= \tilde{F}(x, b_*, \mu_0, \gamma_0) \exp \left[\sum_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \left(\gamma_F(\mu') - \ln \sqrt{\frac{\beta_0}{\beta}} \gamma_K(\mu') \right) \right]$$

$$\exp \left[\ln \sqrt{\frac{\beta}{\beta_0}} \tilde{K}(b_*, \mu_0) - \sum_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \ln \sqrt{\frac{\beta}{\beta_0}} \gamma_K(\mu') \right]$$

$$\exp \left[-g(x, b_T) - \ln \sqrt{\frac{\beta}{\beta_0}} g_K(b_T) \right]$$

$$\begin{aligned}
&= \tilde{F}(x, b_*, \mu_0, \beta_0) \exp \left\{ \ln \sqrt{\frac{3}{\beta_0}} \tilde{K}(b_*, \mu_0) \right. \\
&+ \left. \sum_{\mu_0} \frac{d\mu}{\mu} \left[\gamma_F(\mu, 1) - \ln \sqrt{\frac{3}{\mu^2}} \gamma_K(\mu) \right] \right\} \\
&\times \exp \left\{ -g(x, b_T) - \ln \sqrt{\frac{3}{\beta_0}} g_K(b_T) \right\}
\end{aligned}$$

at small b_T and large μ :

$$\begin{aligned}
\tilde{K}(b_T, \mu) &= - \frac{g^2 C_F}{4\pi^2} \left(\ln(\mu^2 b_T) - \underbrace{\ln 4 + 2\gamma_E}_{\text{the reason why}} \right) \\
&\mu_b \sim \frac{2e^{-\gamma_E}}{b_*} \text{ is chosen}
\end{aligned}$$

$$\tilde{K}(b_*, \mu_b) = 0$$

$$\beta_0 \text{ is a scale } \sim 1-2 \text{ (GeV}^2\text{)}$$

Generalized parton model weasley is.

$$\tilde{F}(x_1, b_T) \sim F(x_1) \exp[-g(x_1, b_T)].$$

Lecture 3

Elements of evolution of TMDs

We have studied so far how structure function can be written in terms of TMDs.

For instance

$$F_{\text{uu}} = C [1 f_1 D_1]$$

In the Generalized Parton Model one often uses

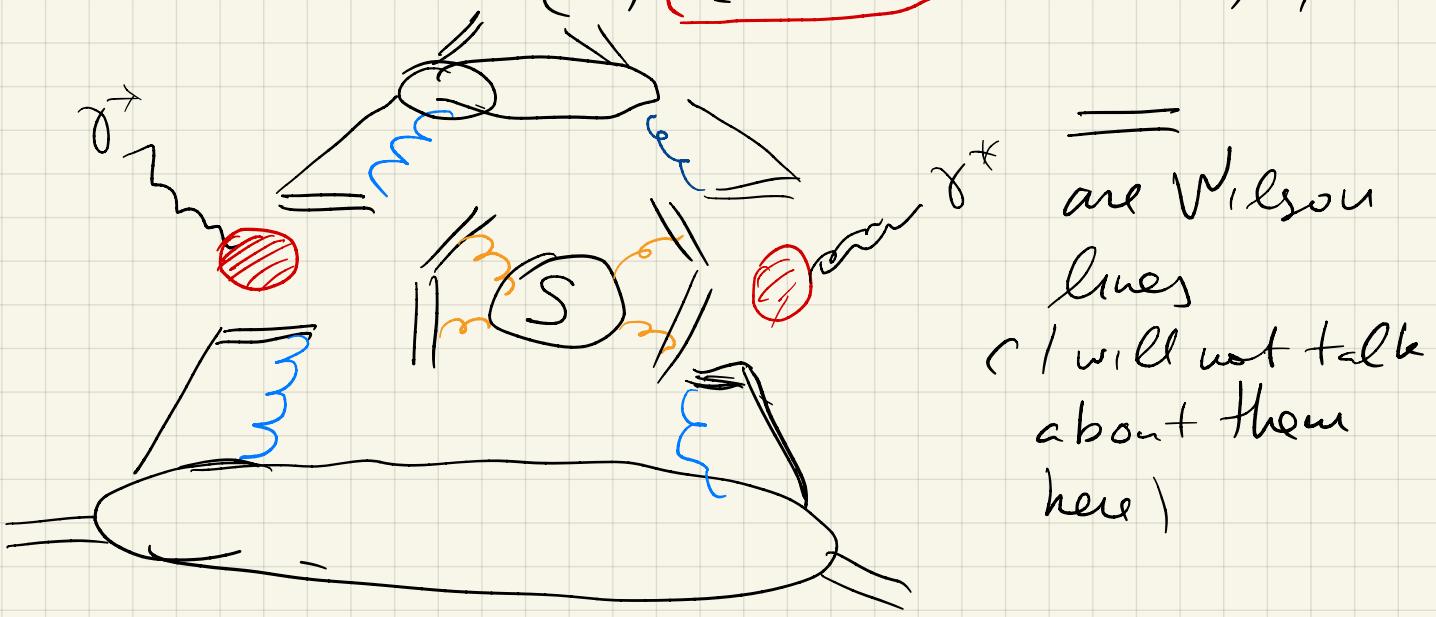
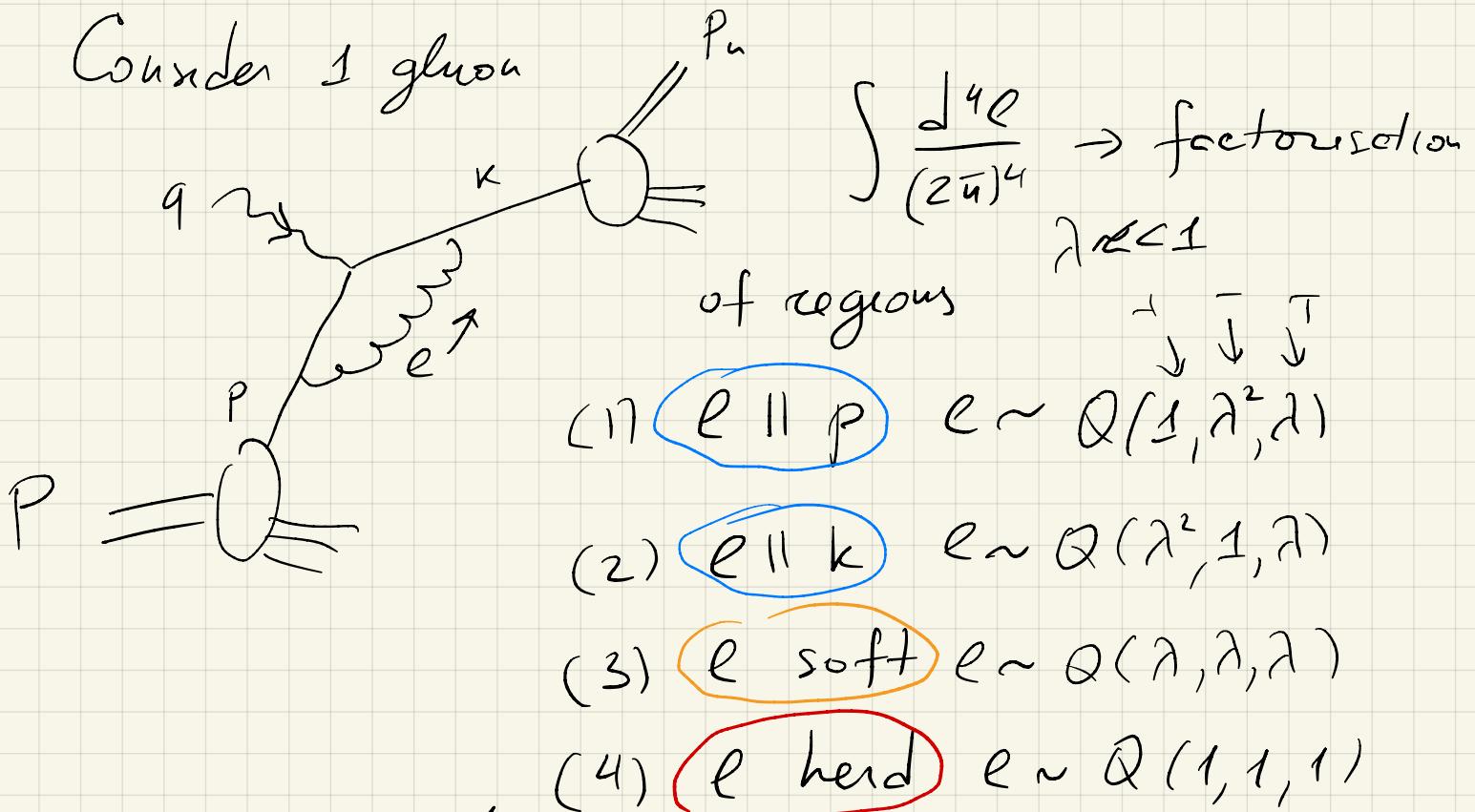
$$f_1(x, k_T) = f_1(x) \frac{1}{\pi \langle k_T^2 \rangle} e^{-k_T^2 / \langle k_T^2 \rangle}$$

$$D_1(z, p_T) = D_1(z) \frac{1}{\pi \langle p_T^2 \rangle} e^{-p_T^2 / \langle p_T^2 \rangle}$$

Of course this Gaussian dependence can be a good approximation of jetwise k_T dependence but what happens if we take into account gluon radiation?

My turn?
Your turn?

Consider 1 gluon



$$G \sim \int d^2 k_T d^2 p_T d^2 Q_T H(Q) f(\alpha, k_T) D(z, p_T)$$

$$\cdot S(e_T) \delta^{(2)}(\vec{P}_{uT} - z \vec{k}_T - \vec{p}_T - \vec{Q}_T)$$

gluon radiation

$$G \sim \int \frac{d^2 b_T}{(2\pi)^2} e^{i \bar{P}_{bT} \bar{b}_T / z} \underbrace{H(Q) \tilde{f}(x, b_T) \tilde{D}(z, b_T) S(b_T)}$$

See next lecture for the proof

The additional factors $S(b)$ is absorbed into \tilde{f} & \tilde{D} . It leads to cancellation of divergencies and self consistent definition

$$\tilde{f}(x, b_T, Q, S) \rightarrow \tilde{f}(x, b_T) \sqrt{S(b_T)}$$

$$\tilde{D}(z, b_T, Q, S) \rightarrow \tilde{D}(z, b_T) \underbrace{\sqrt{S(b_T)}}_{\text{UV scale } \nearrow \text{rapidity scale } \nearrow \text{effect of radiation}}$$

$$G \sim H(Q) \int \frac{d^2 b_T}{(2\pi)^2} e^{i \bar{P}_{bT} \bar{b}_T / z} \underbrace{\tilde{f}(x, b_T, Q, S) \tilde{D}(z, b_T, Q, S)}_{\text{exactly like in generalized parton model!}}$$

We would like to write

$\tilde{f}(x, b_T, Q, S)$ starting from some initial scales Q_0, S_0

QCD evolution of TMDs is governed by 3 equations

① Collins-Soper equation (CS)

$$\frac{\partial \ln \tilde{F}(x, b_T, \mu, \xi)}{\partial \ln \sqrt{s}} = \tilde{K}(b_T, \mu)$$

\tilde{K} is the so-called Collins-Soper kernel

it can be calculated perturbatively for small b_T & large μ (so that $\alpha_s(\mu)$ is small)

$$\tilde{K}(b_T, \mu) = -8 \cdot C_F \frac{\alpha_s(\mu)}{4\pi} \ln \left(\frac{b_T \mu}{2e^{-\gamma_E}} \right) + O(\alpha_s^2)$$

$\gamma_E \approx 0.57$ Euler constant

The problem:

$$\text{We need to } \int \frac{dz_{b_T}}{(2\pi)^2} \rightarrow \int_0^\infty b_T db_T$$

but $\ln \left(\frac{b_T \mu}{2e^{-\gamma_E}} \right)$ will become large for $b_T \rightarrow \infty$

if corresponds to non-perturbative regime
of $k_T \rightarrow 0$.

Solution later

② Renormalisation group equation

$$\frac{d K(b_T, \mu)}{d \ln \mu} = -\gamma_K(\alpha_s(\mu))$$

γ_K is Casp anomalous dimension. It is present in many areas of physics

$$\gamma_K(\alpha_s) = \sum_{i=1}^{\infty} \gamma_K^i \left(\frac{\alpha_s}{4\pi} \right)^i = 8 C_F \left(\frac{\alpha_s}{4\pi} \right) + \mathcal{O}(\alpha_s^2)$$

③

$$\frac{d \ln \tilde{f}(x, b_T, \mu_1, S)}{d \ln \mu} = \gamma_F(\alpha_s(\mu), S/\mu^2)$$

γ_F is the anomalous dimension of f

$$\gamma_F(\alpha_s(\mu), 1) = \sum_{i=1}^{\infty} \gamma_F^i \left(\frac{\alpha_s}{4\pi} \right)^i = 6 C_F \left(\frac{\alpha_s}{4\pi} \right) + \mathcal{O}(\alpha_s^2)$$

Let's write the solutions:

$$\frac{d \ln \tilde{f}(x, b_T, \mu, \varsigma)}{d \ln \mu} = \gamma_F(\mu, \varsigma/\mu^2)$$

$$\int_{\mu_0}^{\mu} d \ln \tilde{f}(x, b_T, \mu, \varsigma) = \int_{\mu_0}^{\mu} \gamma_F(\mu', \varsigma/\mu'^2) \frac{d \mu'}{\mu'}$$

$$\frac{\tilde{f}(x, b_T, \mu, \varsigma)}{\tilde{f}(x, b_T, \mu_0, \varsigma)} = \exp \left[\int_{\mu_0}^{\mu} \gamma_F(\mu', \varsigma/\mu'^2) \frac{d \mu'}{\mu'} \right]$$

$$\tilde{f}(x, b_T, \mu, \varsigma) = \tilde{f}(x, b_T, \mu_0, \varsigma) \exp \left[\int_{\mu_0}^{\mu} \gamma_F(\mu', \varsigma/\mu'^2) \frac{d \mu'}{\mu'} \right]$$

$$\textcircled{2} \quad \frac{\partial \ln \tilde{f}(x, b_T, \mu, \varsigma)}{\partial \ln \sqrt{s}} = \tilde{K}(b_T, \mu)$$

$$\Rightarrow \tilde{f}(x, b_T, \mu, \varsigma) = f(x, b_T, \mu, \varsigma_0) \exp \left[\tilde{K}(b_T, \mu) \ln \sqrt{\frac{s}{s_0}} \right]$$

$$\textcircled{3} \quad \frac{d \tilde{K}(b_T, \mu)}{d \ln \mu} = -\gamma_K(\mu)$$

$$\Rightarrow \tilde{K}(b_T, \mu) = \tilde{K}(b_T, \mu_0) - \sum_{\mu_0}^{\mu} \frac{d \mu'}{\mu'} \gamma_K(\mu')$$

Let us also combine 2 equations

$$\frac{d}{d \ln \mu} \left(\frac{\partial \ln \tilde{f}(x, b_T, \mu, \bar{s})}{\partial \ln \bar{s}} \right) = \frac{d}{d \ln \mu} \tilde{K}(b_T, f) = -\gamma_K(f)$$

those commute!

$$\frac{\partial}{\partial \ln \bar{s}} \left(\underbrace{\frac{d \ln \tilde{f}(x, b_T, \mu, \bar{s})}{d \ln \mu}}_{\gamma_F(\mu, \bar{s}/\mu^2)} \right) = -\gamma_K(f)$$

$$\Rightarrow \gamma_F(\mu, \bar{s}/\mu^2) - \gamma_F(\mu, \bar{s}_0/\mu^2) = -\gamma_K(f) \ln \sqrt{\bar{s}/\bar{s}_0}$$

If we use $\bar{s}_0 = \mu^2$ then

$$\gamma_F(\mu, \bar{s}/\mu^2) = \gamma_F(\mu, 1) - \frac{1}{2} \gamma_K(f) \ln \left(\frac{\bar{s}}{\mu^2} \right)$$

Implementing the evolution

1) Operator Product Expansion (OPE)

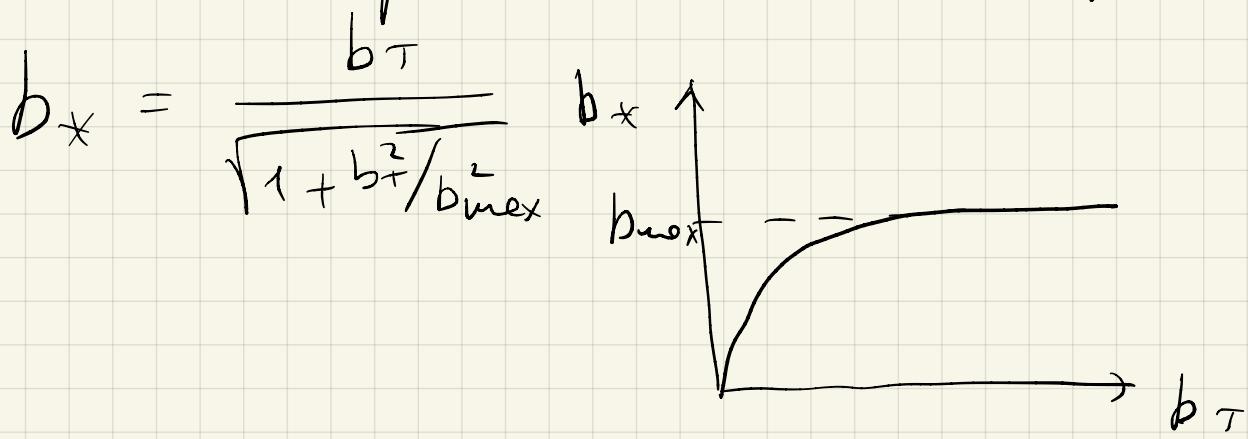
at low b_T :

$$\tilde{f}_f(x, b_T, \mu, \tau) = \sum_j \int_x^1 \frac{dx}{x} \underbrace{\tilde{C}_{j/f}\left(\frac{x}{\hat{x}}, b_T, \mu, \tau\right)}_{\substack{\text{coefficient} \\ \text{functions}}} f_j(\hat{x}, \mu) + \mathcal{O}(b_T^3)$$

collinear
 PDFs
 for upol. f

$$\tilde{C}_{j/f} = \delta_{jf} \delta\left(\frac{x}{\hat{x}} - 1\right) + \mathcal{O}(d_s^2)$$

2) Combine perturbative & non-perturbative
 (solution to problem of $\tilde{K}(b_T)$ nonpert @ large b_T)



if b_{\max} is small ($\sim 1 \text{ GeV}^{-1}$) then

b_* is always perturbative for $\forall b_T$

Start from CS kernel

$$\tilde{K}(b_T, \mu) = \tilde{K}(b_*, \mu) + \underbrace{\tilde{K}(b_T, \mu) - \tilde{K}(b_*, \mu)}_{g_K(b_T)}$$

universal non pert. function!

g_K does not depend on μ , in fact

$$\frac{d}{d \ln \mu} [\tilde{K}(b_T, \mu) - \tilde{K}(b_*, \mu)] = \gamma_K(\mu) - \gamma_K(\mu) = 0$$

next

$$\tilde{K}(b_T, \mu) = \tilde{K}(b_*, \mu_0) - \int_{\mu_0}^{\mu} \frac{dt}{\mu'} \gamma_F(t') - g_K(b_T)$$

For convergence, remember,

$$\tilde{K} = -8 \cdot C_F \frac{\alpha_s(\mu)}{4\pi} \ln \left(\frac{b_T \mu}{2e^{-\gamma_E}} \right)$$

$$\mu_0 = \frac{2e^{-\gamma_E}}{b_T} \equiv \mu_b \quad \text{but at large } b_T \rightarrow \infty$$

$$\mu_b \rightarrow 0 \Rightarrow \alpha_s(0) \rightarrow \infty \quad \text{Landau pole}$$

Because the function is non perturbative.

To avoid it

$$\mu_b = \frac{2e^{-\gamma_E}}{b_*} \rightarrow \frac{2e^{-\gamma_E}}{b_{\max}} \gg \Lambda_{QCD}$$

Study in Mathematics

$$\tilde{K}(b, f) = \tilde{K}(b_*, f_b) - \int_{f_b}^f \frac{df'}{f'} \delta_K(f') - g_K(b)$$

for a realistic $g_K(b) = g_0 \ln(b/b_*)$

Now let us write $f(x, b_T, \mu, \Sigma)$ in terms of
 $f(x, b_T, \mu_0, \Sigma_0)$:

$$\tilde{f}(x, b_T, \mu, \Sigma) = \tilde{f}(x, b_*, \mu, \Sigma) \left[\frac{\tilde{f}(x, b_T, \mu, \Sigma)}{\tilde{f}(x, b_*, \mu, \Sigma)} \right] =$$

$$= \tilde{f}(x, b_*, \mu, \Sigma_0) \exp \left[\tilde{K}(b_*, \mu) \ln \sqrt{\frac{\Sigma}{\Sigma_0}} \right] \left[\frac{\tilde{f}(x, b_T, \mu, \Sigma_0)}{\tilde{f}(x, b_*, \mu, \Sigma_0)} \right]$$

- $\exp \left[\ln \sqrt{\frac{\Sigma}{\Sigma_0}} \underbrace{(\tilde{K}(b_T, \mu) - \tilde{K}(b_*, \mu))}_{g_K(b_T)} \right] \underbrace{\exp [-g(x, b_T)]}_{\text{up behav. of TnD}}$

$$= \tilde{f}(x, b_*, \mu_0, \Sigma_0) \exp \left[\sum_{\mu_1}^{\mu} \frac{d\mu'}{\mu'} (\delta_F(\mu', 1) - \ln \sqrt{\frac{\Sigma_0}{\mu'^2}} \gamma_K(\mu')) \right]$$

- $\exp \left[\ln \sqrt{\frac{\Sigma}{\Sigma_0}} \tilde{K}(b_*, \mu_0) - \sum_{\mu_1}^{\mu} \frac{d\mu'}{\mu'} \ln \sqrt{\frac{\Sigma}{\mu'^2}} \gamma_K(\mu') \right]$

- $\exp \left[-g(x, b_T) - \ln \sqrt{\frac{\Sigma}{\Sigma_0}} g_K(b_T) \right]$

$$= \tilde{f}(x, b_*, \mu_0, \Sigma_0) \exp \left[\ln \sqrt{\frac{\Sigma}{\Sigma_0}} \tilde{K}(b_*, \mu_0) + \sum_{\mu_1}^{\mu} \frac{d\mu'}{\mu'} [\delta_F(\mu', 1) - \ln \sqrt{\frac{\Sigma}{\mu'^2}} \gamma_K(\mu')] \right]$$

- $\exp \left[-g(x, b_T) - \ln \sqrt{\frac{\Sigma}{\Sigma_0}} g_K(b_T) \right]$

$$\text{Let us use } \mu_0 = \mu_b = \frac{2e^{-\delta_F}}{b^*}$$

$$\zeta_0 = Q_0^2 \sim 1-2 \text{ (GeV}^2)$$

Then: $\zeta = Q^2$ (the scale)

$$\tilde{f}(x, b_T, Q, Q^2) = \tilde{f}(x, b^*, \mu_b, Q_0^2) \left(\frac{Q}{Q_0} \right)^{\tilde{K}(b^*, \mu_b)} - g_K(b)$$

$$\exp \left[\int_{\mu_b}^{\mu} \frac{d\mu'}{\mu'} \left(\delta_F(\mu', 1) - \ln \frac{Q}{\mu'} \delta_K(\mu') \right) \right]$$

$$\cdot \exp [-g(x, b_T)]$$

\downarrow Sudakov form factor

$$\exp [S]$$

\uparrow
Contains result of
gluon radiation

$$\tilde{f}(x, b_T, Q, Q^2) = \tilde{f}(x, b^*, \mu_b, Q_0^2) e^{-g(x, b_T)} e^S$$

almost like GPM!

$$\approx \tilde{f}(x, \mu_b) e^{-g(x, b_T)} e^S$$

if $\begin{cases} g(x, b_T) \approx \frac{b_T^2}{4} \\ S \approx 0 \end{cases}$

study in Mathematics.

We will use Mathematics to study how it
differs from GPM at higher scales.