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TMD evolution: implementation guide for SIDIS, DY, and e^+e^-

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Abstract: Details of TMD evolution implementation are highlighted for all polarized and unpolarized structure functions and asymmetries in SIDIS, Drell-Yan, and e^+e^-.

Contents

1 Disclaimer

The following text is my vision of implementation of TMD evolution for all polarized and unpolarized structure functions and asymmetries in SIDIS, Drell-Yan, and e^+e^-. It contains parts of articles from me and other authors. The formulas should be updated as needed in future.

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Alexei Prokudin, 2020

2 Definition of TMDs

TMDs are defined in terms of light-front correlators

$$\Phi(x,\boldsymbol{k}_T)_{ij} = \int \frac{d\xi^- d^2\boldsymbol{\xi}_\perp}{(2\pi)^3}\, e^{-ik\xi}\, \langle N(P,S)|\bar\psi_j(\xi/2)\,\mathcal{W}_{(\xi/2,\,\infty)}\mathcal{W}_{(\infty,\,-\xi/2)}\,\psi_i(-\xi/2)|N(P,S)\rangle\Big|_{\substack{\xi^+=0\\ k^+=xP^+}}\,, \tag{2.1}$$

where the Wilson lines $\mathcal{W}_{(\xi/2,\,\infty)}\mathcal{W}_{(\infty,\,-\xi/2)}$ refer to the SIDIS process [1]. For a generic four-vector a^μ we define the light-cone coordinates $a^\mu = (a^+, a^-, a_\perp)$ with $a^\pm = (a^0 \pm a^3)/\sqrt{2}$. The light-cone direction is singled out by the virtual-photon momentum and transverse vectors like $\boldsymbol{k}_T$ are perpendicular to it. In the virtual-photon–nucleon center-of-mass frame, the nucleon and the partons inside it move in the $(+)$–lightcone direction, while the struck quark and the produced hadron move in the $(-)$–light-cone direction. In the nucleon rest frame the polarization vector is given by $S = (0, \boldsymbol{S}_T, S_L)$ with $\boldsymbol{S}_T^2 + S_L^2 = 1$.

The 8 leading-twist TMDs [2] are projected out from the correlator (2.1) as follows (blue: T-even TMDs, red: T-odd TMDs; all TMDs depend on x, $k_\perp$, renormalization scale and carry a flavor index which we do not indicate for brevity):

$$\frac{1}{2}\,\mathrm{Tr}\Big[\gamma^+\,\Phi(x,\boldsymbol{k}_T)\Big] = f_1 - \frac{\varepsilon^{jk}k_T^j S_T^k}{M_N}\, f_{1T}^\perp\,, \tag{2.2a}$$

$$\frac{1}{2}\,\mathrm{Tr}\Big[\gamma^+\gamma_5\,\Phi(x,\boldsymbol{k}_T)\Big] = S_L\, g_1 + \frac{\boldsymbol{k}_T\cdot\boldsymbol{S}_T}{M_N}\, g_{1T}^\perp\,, \tag{2.2b}$$

$$\frac{1}{2}\,\mathrm{Tr}\Big[i\sigma^{j+}\gamma_5\,\Phi(x,\boldsymbol{k}_T)\Big] = S_T^j\, h_1 + S_L\,\frac{k_T^j}{M_N}\, h_{1L}^\perp + \frac{\kappa^{jk}S_T^k}{M_N^2}\, h_{1T}^\perp + \frac{\varepsilon^{jk}k_T^k}{M_N}\, h_1^\perp\,, \tag{2.2c}$$

and the 16 subleading-twist TMDs [3, 4] are given by

$$\frac{1}{2}\mathrm{Tr}\Big[1\,\Phi(x,\boldsymbol{k}_T)\Big] = \frac{M_N}{P^+}\Big[e - \frac{\varepsilon^{jk}k_T^j S_T^k}{M_N}\, e_T^\perp\Big], \tag{2.2d}$$

$$\frac{1}{2}\mathrm{Tr}\Big[i\gamma_5\Phi(x,\boldsymbol{k}_T)\Big] = \frac{M_N}{P^+}\Big[S_L e_L + \frac{\boldsymbol{k}_T\cdot\boldsymbol{S}_T}{M_N}\, e_T\Big], \tag{2.2e}$$

$$\frac{1}{2}\mathrm{Tr}\Big[\gamma^j\,\Phi(x,\boldsymbol{k}_T)\Big] = \frac{M_N}{P^+}\Big[\frac{k_T^j}{M_N} f^\perp + \varepsilon^{jk}S_T^k f_T + S_L\frac{\varepsilon^{jk}k_T^k}{M_N} f_L^\perp - \frac{\kappa^{jk}\varepsilon^{kl}S_T^l}{M_N^2} f_T^\perp\Big], \tag{2.2f}$$

$$\frac{1}{2}\mathrm{Tr}\Big[\gamma^j\gamma_5\Phi(x,\boldsymbol{k}_T)\Big] = \frac{M_N}{P^+}\Big[S_T^j\, g_T + S_L\,\frac{k_T^j}{M_N} g_L^\perp + \frac{\kappa^{jk}S_T^k}{M_N^2}\, g_T^\perp + \frac{\varepsilon^{jk}k_T^k}{M_N}\, g^\perp\Big], \tag{2.2g}$$

$$\frac{1}{2}\mathrm{Tr}\Big[i\,\sigma^{jk}\gamma_5\Phi(x,\boldsymbol{k}_T)\Big] = \frac{M_N}{P^+}\Big[\frac{S_T^j k_T^k - S_T^k k_T^j}{M_N}\, h_T^\perp - \varepsilon^{jk}\, h\Big], \tag{2.2h}$$

$$\frac{1}{2}\mathrm{Tr}\Big[i\,\sigma^{+-}\,\gamma_5\,\Phi(x,\boldsymbol{k}_T)\Big] = \frac{M_N}{P^+}\Big[S_L\, h_L + \frac{\boldsymbol{k}_T\cdot\boldsymbol{S}_T}{M_N}\, h_T\Big], \tag{2.2i}$$

where $\kappa^{jk} \equiv (k_T^j k_T^k - \frac{1}{2}\,\boldsymbol{k}_T^2\delta^{jk})$. The indices j,k,l refer to the plane transverse with respect to the light cone, $\epsilon^{ij} \equiv \epsilon^{-+ij}$ and $\epsilon^{0123} = +1$. Dirac structures not listed in (2.2a–2.2i) are twist-4 [5]. Integrating out transverse momenta in the correlator (2.1) leads to the "usual" PDFs known from collinear kinematics [6, 7], namely at twist-2 level

$$\frac{1}{2}\,\mathrm{Tr}\Big[\gamma^+\,\Phi(x)\Big] = \quad f_1\,, \tag{2.3a}$$

$$\frac{1}{2}\,\mathrm{Tr}\Big[\gamma^+\gamma_5\,\Phi(x)\Big] = S_L\,g_1\,, \tag{2.3b}$$

$$\frac{1}{2}\,\mathrm{Tr}\Big[i\sigma^{j+}\gamma_5\,\Phi(x)\Big] = S_T^j\,h_1\,, \tag{2.3c}$$

and at twist-3 level

$$\frac{1}{2}\,\mathrm{Tr}\Big[1\,\Phi(x)\Big] = \frac{M_N}{P^+}\,e\,, \tag{2.3d}$$

$$\frac{1}{2}\,\mathrm{Tr}\Big[\gamma^j\gamma_5\,\Phi(x)\Big] = \frac{M_N}{P^+}\,S_T^j\,g_T\,, \tag{2.3e}$$

$$\frac{1}{2}\,\mathrm{Tr}\Big[i\,\sigma^{+-}\gamma_5\,\Phi(x)\Big] = \frac{M_N}{P^+}\,S_L\,h_L\,. \tag{2.3f}$$

Other structures drop out either due to explicit k_T–dependence, or due to the sum rules [4]

$$\int d^2\boldsymbol{k}_T\;f_T^a(x,k_T^2) = \int d^2\boldsymbol{k}_T\;e_L^a(x,k_T^2) = \int d^2\boldsymbol{k}_T\;h^a(x,k_T^2) = 0 \tag{2.4}$$

imposed by time reversal constraints.

Following Ref. [8], fragmentation functions are defined through the following correlator [9] (where $\boldsymbol{p}_T$ denotes the transverse momentum of the produced hadron h acquired during the fragmentation process with respect to the quark q):

$$\Delta(z,\boldsymbol{p}_T)_{ij} = \sum_X \int \frac{d\xi^+ d^2\boldsymbol{\xi}_\perp}{2z(2\pi)^3}\,e^{ip\cdot\xi}\,\langle 0|\mathcal{W}_{(\infty,\xi)}\psi_i(\xi)\,|h,X\rangle\,\langle h,X|\bar{\psi}_j(0)\mathcal{W}_{(0,\infty)}|0\rangle\Big|_{\substack{\xi^-=0\\ p^-=P_h^-/z\\ \boldsymbol{p}_\perp=-\boldsymbol{p}_T/z}}\,. \tag{2.5}$$

If the produced hadron moves fast in the $(-)$ light-cone direction, the TMD correlator that result can then be written in terms of twist-2 Dirac structures:

$$\Delta(z,\boldsymbol{p}_T) = \Delta^{[\gamma^-]}(z,\boldsymbol{p}_T)\,\gamma^+ - \Delta^{[\gamma^-\gamma_5]}(z,\boldsymbol{p}_T)\,\gamma^+\gamma_5 + \Delta^{[i\sigma^{i-}\gamma_5]}(z,\boldsymbol{p}_T)\,i\sigma^{i+}\gamma_5, \tag{2.6}$$

where

$$\Delta^{[\Gamma]}(z,\boldsymbol{p}_T) = \frac{1}{2}\,\mathrm{Tr}\,[\Delta(z,\boldsymbol{p}_T)\,\Gamma]\,, \tag{2.7}$$

The correlator (2.7) gives eight twist-2 TMD FFs [10]:

$$\Delta^{[\gamma^-]}(z,\boldsymbol{p}_T) = D_1(z,\boldsymbol{p}_T) - \frac{\epsilon_\perp^{ij}p_T^i S_{hT}^j}{zM_h}\,D_{1T}(z,\boldsymbol{p}_T)\,, \tag{2.8}$$

$$\Delta^{[\gamma^-\gamma_5]}(z,\boldsymbol{p}_T) = S_{hL}\,G_{1L}(z,\boldsymbol{p}_T) - \frac{\vec{p}_T\cdot\vec{S}_{hT}}{zM_h}\,G_{1T}(z,\boldsymbol{p}_T)\,, \tag{2.9}$$

$$\begin{aligned}\Delta^{[i\sigma^{i-}\gamma_5]}(z,\boldsymbol{p}_T) =\,& S_{hT}^i\,H_{1T}(z,\boldsymbol{p}_T) + \frac{\epsilon_\perp^{ij}p_T^j}{zM_h}\,H_1^\perp(z,\boldsymbol{p}_T)\\ &- \frac{p_T^i}{zM_h}\left[S_{hL}\,H_{1L}^\perp(z,\boldsymbol{p}_T) - \frac{\vec{p}_T\cdot\vec{S}_{hT}}{zM_h}\,H_{1T}^\perp(z,\boldsymbol{p}_T)\right],\end{aligned} \tag{2.10}$$

where $\epsilon_\perp^{ij} \equiv \epsilon^{-+ij}$ and $\epsilon^{0123} = 1$. Similar expressions hold for $\bar{\Delta}^{[\Gamma]}$ for a quark q fragmenting into a hadron h moving along (+) light-cone direction, so that h in this situation has a large plus- (rather than a large minus-) component of momentum, with $\Gamma = \gamma^+$, $\gamma^+\gamma_5$, $i\sigma^{i+}\gamma_5$ if one keeps in mind the relation [11]

$$\bar{\Delta}^{h/q[\Gamma]} = \begin{cases} +\Delta^{h/\bar{q}[\Gamma]} \text{ for } \gamma^\mu,\, i\sigma^{\mu\nu}\gamma_5 \\ -\Delta^{h/\bar{q}[\Gamma]} \text{ for } \mathbb{1},\, i\gamma_5,\, \gamma^\mu\gamma_5\,, \end{cases} \tag{2.11}$$

For a production of an unpolarized hadron h one has

$$\frac{1}{2}\mathrm{Tr}[\gamma^-\Delta(z,\boldsymbol{p}_T)] = D_1\,, \tag{2.12a}$$

$$\frac{1}{2}\mathrm{Tr}[i\sigma^{j-}\gamma_5\Delta(z,\boldsymbol{p}_T)] = \epsilon^{jk}\,\frac{p_T^k}{zM_h} H_1^\perp\,, \tag{2.12b}$$

and at twist-3 level

$$\frac{1}{2}\,\mathrm{Tr}\Big[1\,\Delta(z,\boldsymbol{p}_T)\Big] = \frac{M_h}{P_h^-}\,E\,, \tag{2.12c}$$

$$\frac{1}{2}\,\mathrm{Tr}\Big[\gamma^j\,\Delta(z,\boldsymbol{p}_T)\Big] = -\frac{p_T^j}{zP_h^-}\,D^\perp\,, \tag{2.12d}$$

$$\frac{1}{2}\,\mathrm{Tr}\Big[\gamma^j\gamma_5\,\Delta(z,\boldsymbol{p}_T)\Big] = \varepsilon^{jk}\,\frac{p_T^k}{zP_h^-}\,G^\perp\,, \tag{2.12e}$$

$$\frac{1}{2}\,\mathrm{Tr}\Big[i\,\sigma^{jk}\gamma_5\,\Delta(z,\boldsymbol{p}_T)\Big] = -\varepsilon^{jk}\,\frac{M_h}{P_h^-}\,H\,. \tag{2.12f}$$

The FFs depend on z, $\boldsymbol{p}_T$, renormalization scale, quark flavor and type of hadron which we do not indicate for brevity. Integration over transverse hadron momenta leaves us with $D_1(z)$, $E(z)$, $H(z)$ while the other structures drop out due to their p_T dependence.

2.1 Coordinate b_T space and Fourier Transform

The transverse component of position ξ in Eqs. (2.1),(2.5) is usually called b_T and Eqs. (2.1),(2.5) lead to the following definitions of Fourier Transforms for TMD PDF:

$$f(x,k_T^2) \equiv \int \frac{d^2\boldsymbol{b}_T}{(2\pi)^2} e^{i\boldsymbol{k}_T\boldsymbol{b}_T}\tilde{f}^{(0)}(x,b_T^2) = \int_0^\infty \frac{b_T db_T}{2\pi} J_0(k_Tb_T)\tilde{f}^{(0)}(x,b_T^2) \tag{2.13}$$

$$\tilde{f}^{(0)}(x,b_T^2) \equiv \int d^2\boldsymbol{k}_T e^{-i\boldsymbol{k}_T\boldsymbol{b}_T} f(x,p_T^2) = 2\pi\int_0^\infty k_T dk_T J_0(k_Tb_T) f(x,p_T^2) \tag{2.14}$$

and TMD FF:

$$D(z,p_T^2) \equiv \int \frac{d^2\boldsymbol{b}_T}{(2\pi)^2} e^{i\boldsymbol{p}_T\boldsymbol{b}_T/z}\tilde{D}^{(0)}(x,b_T^2) = \int_0^\infty \frac{b_T db_T}{2\pi} J_0\left(\frac{p_Tb_T}{z}\right)\tilde{D}^{(0)}(z,b_T^2) \tag{2.15}$$

$$\tilde{D}^{(0)}(z,b_T^2) \equiv \int \frac{d^2\boldsymbol{p}_T}{z^2} e^{-i\boldsymbol{p}_T\boldsymbol{b}_T/z} D(x,p_T^2) = 2\pi\int_0^\infty \frac{p_T dp_T}{z^2} J_0\left(\frac{p_Tb_T}{z}\right) D(z,p_T^2) \tag{2.16}$$

We will see in following sections, that often moments of TMDs are important for description of structure functions. Those moments are naively defined [3] in momentum space as

$$f^{(n)}(x_h) = \int \mathrm{d}^2\boldsymbol{k}_{Th}\left(\frac{\boldsymbol{k}_{Th}^2}{2M_N^2}\right)^n f(x_h,\boldsymbol{k}_{Th}^2)\,, \tag{2.17}$$

We will use b_T-space TMD moments [12]

$$\tilde{f}^{(n)}(x_h, b_T^2) = n! \left(\frac{-1}{M_N^2} \frac{\partial}{b_T \partial b_T} \right)^n \tilde{f}(x_h, b_T^2) \,. \tag{2.18}$$

These moments have the important feature,

$$\lim_{b_T \to 0} \tilde{f}^{(n)}(x_h, b_T^2) = f^{(n)}(x_h) \,, \tag{2.19}$$

where $f^{(n)}$ are conventional transverse moments of TMDs In these expressions, $x_h = x_p$ and $M_N = M_p$ will correspond to the proton TMDs, while $x_h = x_\pi$ and $M_N = M_\pi$ will correspond to the pion TMDs.

We obtain for TMD PDFs:

$$\tilde{f}^{(n)}(x_h, b_T^2) = \frac{2\pi n!}{(M_N^2)^n} \int k_T dk_T \left(\frac{k_T}{b_T} \right)^n J_n(k_T b_T) f(x_h, k_T^2) \,, \tag{2.20}$$

$$f(x_h, k_T^2) = \frac{(M_N^2)^n}{2\pi n!} \int b_T db_T \left(\frac{b_T}{p_T} \right)^n J_n(k_T b_T) \tilde{f}^{(n)}(x_h, b_T^2) \tag{2.21}$$

and for TMD FFs we will use b_T-space TMD moments(compare to Ref. [12])

$$\tilde{D}^{(n)}(z, b_T^2) = n! \left(\frac{-1}{M_h^2} \frac{\partial}{b_T \partial b_T} \right)^n \tilde{D}(z, b_T^2) \,. \tag{2.22}$$

such that

$$\tilde{D}^{(n)}(z, b_T^2) = \frac{2\pi n!}{(M_h^2)^n} \int \frac{p_T dp_T}{z^2} \left(\frac{p_T}{z b_T} \right)^n J_n\left(\frac{p_T b_T}{z} \right) D(z, p_T^2) \,, \tag{2.23}$$

$$D(z, p_T^2) = \frac{(M_h^2)^n}{2\pi n!} \int b_T db_T \left(\frac{z b_T}{p_T} \right)^n J_n\left(\frac{p_T b_T}{z} \right) \tilde{D}^{(n)}(z, b_T^2) \,, \tag{2.24}$$

where the functions $D^{(n)}(z, b_T^2)$ in Eq. (2.24) have the following property:

$$\lim_{b_T \to 0} \tilde{D}^{(n)}(z, b_T^2) = \frac{1}{z^2} D^{(n)}(z) \,, \tag{2.25}$$

$$D^{(n)}(z) = \int d^2\boldsymbol{p}_T \left(\frac{p_T^2}{2z^2 M_h^2} \right)^n D(z, p_T^2) \,. \tag{2.26}$$

2.2 The model for transverse momentum dependence of TMDs at the initial scale

We will utilize the following parametrizations [13] for TMDs at the initial scale Q_0:

$$\begin{aligned}
f_h^a(x_h, \boldsymbol{k}_{Th}, Q_0, Q_0^2) &= f_h^{(0)a}(x_h, Q_0) \, \frac{e^{-\boldsymbol{k}_{Th}^2/\langle k_{Th}^2 \rangle_{f_h}}}{\pi \, \langle k_{Th}^2 \rangle_{f_h}} \,, && f_h^{(0)a} = f_{1,p}^a,\ f_{1,\pi}^a,\ h_{1,p}^a, \\
f_h^a(x_h, \boldsymbol{k}_{Th}, Q_0, Q_0^2) &= f_h^{(1)a}(x_h, Q_0) \, \frac{2M_N^2}{\pi \langle k_{Th}^2 \rangle_{f_h}^2} e^{-\boldsymbol{k}_{Th}^2/\langle k_{Th}^2 \rangle_{f_h}} , && f_h^a = f_{1T,p}^{\perp a},\ h_{1,p}^{\perp a},\ h_{1,\pi}^{\perp a},\ h_{1L,p}^{\perp a}, \\
f_h^a(x_h, \boldsymbol{k}_{Th}, Q_0, Q_0^2) &= f_h^{(2)a}(x_h, Q_0) \, \frac{2M_N^4}{\pi \langle k_{Th}^2 \rangle_{f_h}^3} e^{-\boldsymbol{k}_{Th}^2/\langle k_{Th}^2 \rangle_{f_h}} , && f_h^a = h_{1T,p}^{\perp a},
\end{aligned} \tag{2.27}$$

where transverse moments of TMDs are defined in Eq. (2.17). We added an index h in order to distinguish two different hadrons in Drell-Yan process that we will consider later. Parametrizations from Eqs. (2.27) are often used in phenomenology to describe polarized SIDIS and DY data. These parametrizations correspond to the the following b_T-space expressions

$$\begin{aligned}
\tilde{f}_h^{(0)a}(x_h, b_T, Q_0, Q_0^2) &= f_h^{(0)a}(x_h, Q_0)\; e^{-\frac{1}{4}\, b_T^2 \langle k_{Th}^2\rangle_{f_h}}, \quad f_h^{(0)a} = f_{1,p}^a,\; f_{1,\pi}^a,\; h_{1,p}^a, \\
\tilde{f}_h^{(1)a}(x_h, b_T, Q_0, Q_0^2) &= f_h^{(1)a}(x_h, Q_0)\, e^{-\frac{1}{4}\, b_T^2 \langle k_{Th}^2\rangle_{f_h}}, \quad f_h^{(1)a} = f_{1T,p}^{\perp(1)a},\; h_{1,p}^{\perp(1)a},\; h_{1,\pi}^{\perp(1)a},\; h_{1L,p}^{\perp(1)a}, \\
\tilde{f}_h^{(2)a}(x_h, b_T, Q_0, Q_0^2) &= f_h^{(2)a}(x_h, Q_0)\, e^{-\frac{1}{4}\, b_T^2 \langle k_{Th}^2\rangle_{f_h}}, \quad f_h^{(2)a} = h_{1T,p}^{\perp(2)a}, \qquad (2.28)
\end{aligned}$$

where the collinear functions are the same as in Eqs. (2.27).

We will utilize the following parametrizations [13] for TMD FFs at the initial scale Q_0:

$$\begin{aligned}
D^a(z, p_T, Q_0, Q_0^2) &= D^{(0)a}(z, Q_0)\, \frac{e^{-p_T^2/\langle p_T^2\rangle_D}}{\pi\, \langle p_T^2\rangle_D}\,, \qquad D^{(0)a} = D_1^a \\
D^a(z, p_T, Q_0, Q_0^2) &= D^{(1)a}(z, Q_0)\, \frac{2z^2 M_h^2}{\pi \langle p_T^2\rangle_D^2} e^{-p_T^2/\langle p_T^2\rangle_D^2}, \quad D^a = H_1^{\perp a}, \\
D^a(z, p_T, Q_0, Q_0^2) &= D^{(2)a}(z, Q_0)\, \frac{2z^4 M_h^4}{\pi \langle p_T^2\rangle_D^3} e^{-p_T^2/\langle p_T^2\rangle_D^3}, \quad D^a = ?\,, \qquad (2.29)
\end{aligned}$$

where transverse moments of TMDs are defined in Eq. (2.26).

These parametrizations correspond to the the following b_T-space expressions (for the tree level coefficient functions), see Eqs. (2.24,2.25)

$$\begin{aligned}
\tilde{D}^{(0)a}(z, b_T, Q_0, Q_0^2) &= \frac{1}{z^2} D^{(0)a}(z, Q_0)\; e^{-\, b_T^2 \langle p_T^2\rangle_D/(4z^2)}, \quad D^{(0)a} = D_1^a, \\
\tilde{D}^{(1)a}(z, b_T, Q_0, Q_0^2) &= \frac{1}{z^2} D^{(1)a}(z, Q_0)\, e^{-\, b_T^2 \langle p_T^2\rangle_D/(4z^2)}, \quad D^{(1)a} = H_1^{\perp(1)a}, \\
\tilde{D}^{(2)a}(z, b_T, Q_0, Q_0^2) &= \frac{1}{z^2} D^{(2)a}(z, Q_0)\, e^{-\, b_T^2 \langle p_T^2\rangle_D/(4z^2)}, \quad D^{(2)a} = ?\,. \qquad (2.30)
\end{aligned}$$

One can see that our parametrizations corresponds to an assumption that $g_f^q(x, b_T; b_{\max}) = \frac{1}{4}\, b_T^2 \langle k_{Th}^2\rangle_{f_h}$ in Eqs. (3.21), and $g_D^q(z, b_T; b_{\max}) = \frac{1}{4}\, (b_T/z)^2 \langle p_T^2\rangle_D$ as b_T dependence of g_f^q, g_D^q and can and will be more involved, however generically the form of equations will still be the same, compare to the second and the third lines of Eq. (3.22).

Notice that in the two ways of writing solutions to the evolution equations, that we will introduce in the next chapters, the functional form of b_T shape may and most certainly will be different for those solutions.

3 TMD evolution equations and their solutions

The transversely differential unpolarized cross section for SIDIS is, Ref. [14]:

$$\frac{d^6\sigma}{dx\,dy\,dz\,d\psi_l\,d\phi_h\,dP_{hT}^2} = \frac{\alpha_{em}^2}{x\,y\,Q^2}\left(1 - y + \frac{1}{2}y^2\right) F_{UU}(x, z, Q^2, P_{hT}^2)\,, \tag{3.1}$$

where the unpolarized structure function can be interpreted in the appropriate region of validity of TMD factorization as:

$$\begin{aligned} F_{UU}(x, z, Q^2, P_{hT}^2) = \; & x \sum_q e_q^2 \mathcal{H}^{(SIDIS)}(\alpha_s(Q)) \int \frac{d^2\boldsymbol{b}_T}{(2\pi)^2} e^{-i\boldsymbol{b}_T\cdot\boldsymbol{P}_{hT}/z}\, \tilde{f}_1^q(x, b_T; Q, Q^2)\, \tilde{D}_1^q(z, b_T; Q, Q^2) \\ & + Y_{\rm SIDIS}\,. \end{aligned} \tag{3.2}$$

The two-dimensional Fourier transform is needed to convert the transverse coordinate space expression into momentum space. We will mainly be working with the coordinate space integrand in the first term in Eq. (3.2). The scales in Eq. (3.2) are chosen to be Q so that the hard part, $\mathcal{H}(\alpha_s(Q))$, has good convergence properties. The kinematical variables x, z and Q for SIDIS are defined in the usual way, see Section 4.1.

3.1 TMD evolution equations

The TMD functions in Eq. (3.2) obey a set of evolution equations [15]. They are the Collins-Soper (CS) equations for each TMD:

$$\frac{\partial \ln \tilde{f}_1(x, b_T; \mu, \zeta_1)}{\partial \ln \sqrt{\zeta_1}} = \frac{\partial \ln \tilde{D}_1(z, b_T; \mu, \zeta_2)}{\partial \ln \sqrt{\zeta_2}} = \tilde{K}(b_T; \mu)\,, \tag{3.3}$$

and the renormalization group (RG) equations

$$\frac{d\tilde{K}(b_T;\mu)}{d\ln\mu} = -\,\gamma_K(\alpha_s(\mu))\,, \qquad \text{RG CS kernel} \tag{3.4}$$

$$\frac{d\ln \tilde{f}_1(x, b_T; \mu, \zeta_1)}{d\ln\mu} = \gamma_j(\alpha_s(\mu); \zeta_1/\mu^2)\,, \qquad \text{RG TMD PDF} \tag{3.5}$$

$$\frac{d\ln \tilde{D}_1(z, b_T; \mu, \zeta_2)}{d\ln\mu} = \gamma_{jFF}(\alpha_s(\mu); \zeta_2/\mu^2)\,. \qquad \text{RG TMD FF} \tag{3.6}$$

3.2 TMD anomalous dimensions

The anomalous dimensions $\gamma_K(\alpha_s(\mu))$ and $\gamma_j(\alpha_s(\mu); \zeta_F/\mu^2)$ are perturbatively calculable, and we will keep up to order α_s^3 terms. The CS kernel, $\tilde{K}(b_T;\mu)$, is also perturbatively calculable as long as $b_T \ll\sim 1/\Lambda_{\rm QCD}$. See [15] for one-loop calculations of γ_K from its definition. Three loop result is reported in Appendix C Eq. (C.1).

3.3 Perturbative Collins-Soper kernel

Over short transverse distance scales, $1/b_T$ becomes a legitimate hard scale, and the transverse coordinate dependence in the TMD PDFs can itself be calculated in perturbation theory. With the choice of renormalization scale $\mu \sim 1/b_T$, $\alpha_s(\sim 1/b_T)$ approaches zero for small sizes due to asymptotic freedom, thus ensuring that the small size transverse coordinate dependence is optimally calculable in perturbation theory. We report three loop result for $\tilde{K}$ in Appendix D Eq. (D.1).

3.4 Operator expansion for TMDs

Predictions are obtained with the aid of evolution equations and the small-b_T OPE of the TMD parton densities:

$$\tilde{f}_1^q(x, b_T; \mu, \zeta) = \sum_k \int_{x-}^{1+} \frac{d\hat{x}}{\hat{x}} \tilde{C}_{q/k}^{\text{PDF}} \left(\frac{x}{\hat{x}}, b_T; \zeta, \mu, \alpha_s(\mu)\right) f_1^k(\hat{x}; \mu) \ + \ O\left[(m b_T)^p\right] \ , \tag{3.7}$$

$$\tilde{D}_1^q(z, b_T; \mu, \zeta) = \sum_k \int_{z^-}^{1^+} \frac{d\hat{z}}{\hat{z}^3} \tilde{C}_{q/k}^{FF} \left(\frac{z}{\hat{z}}, b_T; \mu^2, \mu, \alpha_s(\mu)\right) D_1^k(\hat{z}; \mu) \ + \ O\left[(m b_T)^p\right] \ . \tag{3.8}$$

(For an explanation of the notations $x-$ and $1+$ for the integration limits, see [15, pp. 248 & 249].) We report results for $\tilde{C}$ in Appendix F Eq. (**??**).

3.5 Hard factors

Process dependent hard factors $\mathcal{H}$ are reported in Appendix E.

3.6 Combining perturbative and nonperturbative regions

For very large b_T, the transverse coordinate dependence corresponds to intrinsic nonperturbative behavior associated with the hadron wave function. (In momentum space, this corresponds to the onset of effects from intrinsic bound state transverse momentum in the hadron wavefunction). There, a prescription is needed to tame the growth of $\alpha_s(1/b_T)$ and match to a nonperturbative, large distance description of the b_T-dependence. The renormalization group scale is therefore chosen to be

$$\mu_b \equiv C_1/|\boldsymbol{b}_*(b_T)| \, , \tag{3.9}$$

where $\boldsymbol{b}_*(b)$ is a function of b_T that equals b_T at small b_T, but freezes in the limit where b_T becomes nonperturbatively large, i.e., when b_T is larger than some fixed $b_{\max}$. This function must obey

$$\boldsymbol{b}_*(\boldsymbol{b}_T) = \begin{cases} \boldsymbol{b}_T & b_T \ll b_{\max} \\ \boldsymbol{b}_{\max} & b_T \gg b_{\max} \, . \end{cases} \tag{3.10}$$

The most common taming prescription is

$$\boldsymbol{b}_*(b_T) \equiv \frac{b_T}{\sqrt{1 + b_T^2/b_{\max}^2}}. \tag{3.11}$$

Although any function obeying Eq. (3.10) is consistent with both TMD factorization and the standard CSS formalism, Eq. (3.11) is one of the simplest choices and is the one that we will adopt in this paper. The factor C_1 is an arbitrary numerical constant that can be chosen to minimize higher order corrections. It is typically fixed at $C_1 = 2e^{-\gamma_{\rm E}}$.

3.7 Solutions of TMD evolution equations

Ref. [16] proposes a few ways to write solutions of TMD evolution equations (3.3),(3.4),(3.5),(3.6). We will review two of them that are important for phenomenological studies.

3.7.1 Fixed Q_0-scale solution

The solution with the parton densities and the CSS kernel at fixed scales is:

$$
\begin{aligned}
F_{UU}(x,z,Q^2,P_{hT}^2) = x \sum_q e_q^2 \, \mathcal{H}_q^{(SIDIS)}(\mu_Q,\alpha_s(Q)) \int \frac{d^2\boldsymbol{b}_T}{(2\pi)^2} e^{-i\boldsymbol{b}_T\cdot \boldsymbol{P}_{hT}/z} \\
\times \tilde{f}_1^{(0)q}(x,b_T;\mu_0,Q_0^2)\, \tilde{D}_1^{(0)q}(z,b_T;\mu_0,Q_0^2) \\
\times \left(\frac{Q^2}{Q_0^2}\right)^{\tilde{K}(b_T;\mu_0)} \exp\left\{ \int_{\mu_0}^{\mu_Q} \frac{d\mu'}{\mu'} \left[2\gamma_j(\alpha_s(\mu');1) - \ln\frac{Q^2}{(\mu')^2}\gamma_K(\alpha_s(\mu')) \right] \right\} \\
+ \text{polarized terms} + \text{large } P_{hT}/z \text{ correction, } Y + \text{p.s.c.}
\end{aligned}
\tag{3.12}
$$

Here Q_0 and μ_0 are chosen fixed reference scales. They have exactly the same status as a similar parameter that is used in DGLAP evolution of ordinary parton densities (e.g., [17]), and that is often denoted by Q_0. In both cases, there is a functional form of parton densities at the fixed reference scale (or scales), and evolution has been used to obtain Q-dependent parton densities. It will generally be convenient to set $\mu_0 = \mu_{Q_0} = C_2 Q_0$. In Eq. (3.12) (and in similar later equations) "p.s.c." is a short-hand for "power suppressed corrections".

Although the reference scale μ_0 is in principle arbitrary, it should in practice be chosen large enough to be treated as being in the perturbative region. This allows finite-order perturbative calculations of the anomalous dimensions, γ_j and γ_K to be appropriate for all the values in the integral over μ'. Given the choice $\mu_0 = C_2 Q_0$, it will generally be sensible to choose Q_0 to be near the lower end of the range of Q for the data to which one applies factorization. This is exactly the same as with typical implementations of DGLAP evolution.

In the hard scattering, we have preserved $\mu_Q = C_2 Q, C_2 = 1$, so that it has no large logarithms in its perturbative coefficients, and can therefore be effectively calculated by low order perturbation theory in powers of $\alpha_s(\mu_Q)$.

It is convenient to notate the evolution factor on the third line of (3.12) as $e^{-S(b_T,Q,Q_0,\mu_0)}$, where

$$
S(b_T,Q,\mu_Q,Q_0,\mu_0) = -\tilde{K}(b_T;\mu_0)\ln\frac{Q^2}{Q_0^2} + \int_{\mu_0}^{\mu_Q}\frac{d\mu'}{\mu'}\left[-2\gamma_j(\alpha_s(\mu');1) + \ln\frac{Q^2}{(\mu')^2}\gamma_K(\alpha_s(\mu'))\right]. \tag{3.13}
$$

The Eq. (3.12) exhibits a very close relationship to a TMD parton model formula, except that: (a) the hard part has perturbative higher order corrections, (b) the TMD parton densities are scale dependent, and (c) there is a Y-term. But when the cross section is expressed in terms of TMD parton densities at the reference scales, we find in Eq. (3.12) a factor e^{-S} that gives the important effects of gluon radiation. Notice that in the literature one uses A and B coefficients in in Eq. (3.13), A is $\gamma_K/2$ and B is $-\gamma_j$ in Eq. (3.13). They can be expanded as perturbative series, i.e., $A = \sum_{n=1}^{\infty} A^{(n)} \left(\alpha_s/\pi\right)^n$, $B = \sum_{n=1}^{\infty} B^{(n)} \left(\alpha_s/\pi\right)^n$ [1]. The evolution factor $S(b_T,Q,Q_0,\mu_0)$ in Eq. (3.13), which results from solving the CSS evolution equation and the renormalization group equations for the rapidity dependence of the TMD parton densities and for the soft factor [15, 21], can be rewritten as

---

[1] The so-called Next-to-Leading Logarithmic (NLL) logarithmic precision mean taking $A^{(1)}$, $A^{(2)}$ and $B^{(1)}$ for the (NLL) accuracy.

TODO add other definitions of NNLL, N^{3}LL

This part is spin-independent and the coefficients are given by [16, 20–27]:

$$
A^{(1)} = C_F,\; A^{(2)} = \frac{C_F}{2}\left[C_A\left(\frac{67}{18}-\frac{\pi^2}{6}\right) - \frac{10}{9}T_R\, n_f\right],\; B^{(1)} = -\frac{3}{2}C_F\,, \tag{3.14}
$$

where $C_F = 4/3$, $C_A = 3$, $T_R = 1/2$ and n_f is the number of active flavors.

$$S(b_T, Q, \mu_Q, Q_0, Q_0) = -\tilde{K}(b_T, Q_0) \ln \frac{Q^2}{Q_0^2} + \int_{Q_0^2}^{\mu_Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \left[A(\alpha_s(\bar{\mu})) \ln \frac{Q^2}{\bar{\mu}^2} + B(\alpha_s(\bar{\mu})) \right] , \tag{3.15}$$

Furthermore, the solution (3.12) exhibits universality properties of the TMD densities that are the same as in the parton model. That is, under all circumstances where a TMD factorization theorem holds, the same TMD densities, functions of x and b_T, are used, up to possible factors of -1 for T-odd functions. Both the perturbatively calculable hard scattering and the factor involving perturbatively calculable anomalous dimensions only affect the normalization of the cross section, but not its shape as a function of $q_T \simeq P_{hT}/z$. The remaining factor $(Q^2/Q_0^2)^{\tilde{K}(b_T;\mu_0)} = \exp\left[\tilde{K}(b_T;\mu_0)\ln(Q^2/Q_0^2)\right]$ gives a Q-dependent change in shape of the distribution, a very characteristic effect of gluonic emission in a gauge theory.

As is well known, a minor modification to universality arises because the appropriate TMD parton densities differ between processes of the Drell-Yan type and those of the SIDIS type. The operators in the definition of the TMD densities use oppositely directed Wilson lines in the two cases. Most TMD densities are numerically unchanged, but T-odd densities, like the Sivers function, change sign [1].

It is important to recall that the derivation of Eq. (3.12) depends on the TMD factorization and evolution equations, and that these in turn depend on properties of the particular definitions used for the TMD parton densities; see Ref. [15, Eq. (13.106, 13.108)] and the discussions leading up to this definition. (These remarks apply equally to the original CSS derivations [18–20].)

Since the integral in Eq. (3.12) extends over all b_T, one cannot avoid using $\tilde{K}$ in the CS evolution factor Eq. (3.13) in the non-perturbative large b_T region. In order to combine the perturbative and non-perturbative regions, we use the b_* prescription, namely, from Eq. (3.11) which introduces a smooth upper cutoff $b_{\max}$ in the transverse distance.

Then, the perturbative part of $\tilde{K}$ is defined by replacing b_T by b_* and the non-perturbative part is defined by the difference $\tilde{K}(b_*, \mu) - \tilde{K}(b_T, \mu) = g_K(b_T; b_{\max})$. Furthermore to combine the perturbative and non-perturbative regions using the fixed scale evolution, it is optimal to use the renormalization group running scheme for $\tilde{K}$ in Eq. (3.13), evolved from the fixed scale Q_0, i.e.

$$\tilde{K}(b_T, Q_0) = \tilde{K}(b_*, \mu_{b_*}) - \int_{\mu_{b_*}}^{Q_0} \frac{d\bar{\mu}}{\bar{\mu}} \gamma_K(\alpha_s(\bar{\mu})) - g_K(b_T, b_{\max}) , \tag{3.16}$$

where μ_{b_*} is chosen to be

$$\mu_{b_*} \equiv \frac{2e^{-\gamma_E}}{b_*} , \tag{3.17}$$

such that the two-loop result [28] for $\tilde{K}$, valid at small values of b_T, is free of large logarithmic corrections and reads Eq. (D.1)

$$\tilde{K}(b_*; \mu_{b_*}) = \frac{C_F}{2} \left(\frac{\alpha_s(\mu_{b_*})}{\pi} \right)^2 \left[\left(\frac{7}{2}\zeta_3 - \frac{101}{27} \right) C_A + \frac{14}{27} n_f \right] + \mathcal{O}(\alpha_s(\mu_{b_*})^3) . \tag{3.18}$$

Furthermore, we adopt the functional form of $g_K(b_T; b_{\max})$ given by Collins and Rogers [16],

$$g_K(b_T, b_{\max}) = g_0(b_{\max}) \left(1 - \exp\left[-\frac{C_F \alpha_s(\mu_{b_*}) b_T^2}{\pi g_0(b_{\max}) b_{\max}^2} \right] \right) , \tag{3.19}$$

which interpolates smoothly between the small and large-b_T regions, where at small b_T it approximates a power series in b_T^2, while at large b_T the resulting value of $\tilde{K}$ goes to a constant [16]. We choose $g_0(b_{\max}) = 0.84$ and $b_{max} = 1$ (GeV^{-1}) to match the non perturbative behavior of g_K used in Refs. [29, 30] to describe the polarized SIDIS data and in Ref. [31] to describe unpolarized SIDIS, Drell-Yan and weak boson production data.

3.7.2 Perturbatively optimized μ_b-scale solution

From Ref. [16]: Note, however, that perturbatively calculated functions may appear in an exponent—as for $\tilde{K}$ and the anomalous dimensions. Thus any errors in a perturbative calculation of such a function can be magnified by a large logarithm.

In any case, perturbative calculations are not applicable at large enough values of b_T, which, given our knowledge of QCD, is a nonperturbative region. An indication of where the nonperturbative region is likely to be quantitatively important is given by an analysis by Schweitzer, Strikman, and Weiss [32]. Using a chiral effective theory, they found that there are two relevant nonperturbative distance scales: a chiral scale $0.3\,\mathrm{fm} = 1.5\,\mathrm{GeV}^{-1}$ and a confinement scale $1\,\mathrm{fm} = 5\,\mathrm{GeV}^{-1}$. At large b_T, they find that a TMD density behaves like an exponential $e^{-b_T/l}$ times a power of b_T, with l being a characteristic scale. The chiral and confinement scales manifest themselves in the large-b_T dependence of the sea and valence quark densities.

To get maximum predictive information, one should therefore combine the use of perturbative calculations at small b_T with fits to data to measure $\tilde{K}$ and the TMD densities at large b_T. Undoubtedly, fits to data will eventually be supplemented by further constraints from nonperturbative calculations like those of Ref. [32] from chiral models, and those of Ref. [33] from lattice gauge theory.

Since the integral in Eq. (3.2) extends from $b_T = 0$ to $b_T = \infty$, one cannot avoid using parton densities and $\tilde{K}$ in the nonperturbative large-b_T region[2]. Therefore it is necessary to combine nonperturbative information with perturbative calculations. CSS [20] provided a prescription[3] for doing this; we will call their method the "b_* method", after the name of a variable defined by CSS.

What are called the perturbative parts of the TMD densities and of $\tilde{K}$ were defined by replacing b_T by b_*. Then the nonperturbative[4] parts were defined as whatever is left over. This idea is implemented with the aid of functions $g_{j/H}(x, b_T; b_{\max})$ and $g_K(b_T; b_{\max})$ defined by

$$g_K(b_T; b_{\max}) = -\tilde{K}(b_T, \mu) + \tilde{K}(b_*, \mu) \tag{3.20}$$

and

$$e^{-g_f^q(x,b_T;b_{\max})} = \frac{\tilde{f}_1^q(x, b_T; \mu, \zeta)}{\tilde{f}_1^q(x, b_*; \mu, \zeta)} e^{g_K(b_T;b_{\max}) \ln(\sqrt{\zeta}/Q_0)}. \tag{3.21}$$

Here Q_0 is a chosen reference scale, that simply determines how much of the TMD density is in $e^{-g_f^q}$ and how much is put into the exponential of g_K times a logarithm that appears in Eq. (3.21) and in (3.22) below. As indicated by our notation, we will choose Q_0 here to have the same value as in Eq. (3.12). We treat g_f^q and g_K as needing to be fit to data.

[2] An important, but separate, practical question is whether or not the integrand in Eq. (3.2) is large enough in the nonperturbative region for the details of the nonperturbative parametrization to matter in the context of particular calculations.

[3] The CSS prescription is not the only possibility. See Refs. [22, 34] for one alternative. In addition Bozzi et al. [35, Eq. (17)], motivated by [36], proposed a modification to improve the behavior of the formalism at small b_T. This involves the replacement of b_T^2 by $b_T^2 + 4e^{-2\gamma_E}/Q^2$ in the parton densities and evolution factors in Eq. (3.12), and our other solutions, together with a consequent change in Y, as computed in [35, App. B]. This is probably a generally useful prescription.

[4] "nonperturbative" is somewhat of a misnomer. If $b_{\max}$ were chosen to be excessively small, the values of the "nonperturbative" parts near $b_T = b_{\max}$ could be reliably estimated perturbatively.

Both of g_f^q and g_K vanish approximately[5] like b_T^2 at small b_T, from their definition, and become significant when b_T approaches $b_{\max}$ and beyond.

Both functions are independent of both μ and ζ. This is because there is an exact cancellation in the terms obtained by applying the CSS and RG equations to the quantities on the right of Eqs. (3.20) and (3.21). The functions do depend, however, on the choice of the value of $b_{\max}$ and on the particular CSS prescription for segregating nonperturbative information. It is the full TMD parton densities and the function $\tilde{K}$ that are independent of $b_{\max}$ and of the use of the b_* prescription of CSS.

As regards the possible flavor and x dependence of g_K and $g_{j/H}$, this follows from that of the corresponding parent functions, i.e., $\tilde{K}$ and the TMD parton densities. Since $\tilde{K}$ is independent of quark flavor, hadron flavor, and parton x, so is g_K. But the TMD parton densities can depend on quark and hadron flavor and on x, so the same is true of the $g_{j/H}$ functions.

Given these definitions, the evolution equations and the small-b_T expansion can be used to write the factorization formula as

$$
\begin{aligned}
F_{UU}(x,z,Q^2,P_{hT}^2) = x \sum_q e_q^2 \sum_{j,i} &\mathcal{H}^{(SIDIS)}_{q,i,j}(\mu_Q,\alpha_s(Q)) \int \frac{d^2\boldsymbol{b}_T}{(2\pi)^2} e^{-i\boldsymbol{b}_T\cdot \boldsymbol{P}_{hT}/z} \\
&\times e^{-g_f^q(x,b_T;b_{\max})} \int_x^1 \frac{d\hat{x}}{\hat{x}} f_1^j(\hat{x};\mu_{b_*})\, \tilde{C}_{q/j}\left(\frac{x}{\hat{x}}, b_*;\mu_{b_*}^2,\mu_{b_*},\alpha_s(\mu_{b_*})\right) \\
&\times e^{-g_D^q(z,b_T;b_{\max})} \int_z^1 \frac{d\hat{z}}{\hat{z}^3} D_1^i(\hat{z};\mu_{b_*})\, \tilde{C}_{q/i}\left(\frac{z}{\hat{z}}, b_*;\mu_{b_*}^2,\mu_{b_*},\alpha_s(\mu_{b_*})\right) \\
&\times \left(\frac{Q^2}{Q_0^2}\right)^{-g_K(b_T;b_{\max})} \left(\frac{Q^2}{\mu_{b_*}^2}\right)^{\tilde{K}(b_*;\mu_{b_*})} \\
&\times \exp\left\{ \int_{\mu_{b_*}}^{\mu_Q} \frac{d\mu'}{\mu'} \left[2\gamma_j(\alpha_s(\mu');1) - \ln\frac{Q^2}{(\mu')^2}\gamma_K(\alpha_s(\mu'))\right]\right\} \\
&+ \text{polarized terms} + \text{large } P_{hT}/z \text{ correction, } Y + \text{p.s.c.}
\end{aligned} \tag{3.22}
$$

Here μ_{b_*} is chosen to allow perturbative calculations of b_*-dependent quantities without large logarithms:

$$
\mu_{b_*} = C_1/b_*, \tag{3.23}
$$

where C_1 is a numerical constant typically chosen to be $C_1 = 2e^{-\gamma_E}$.

It is important to note that, although $g_K(b_T;b_{\max})$ is totally universal, $g_f^q(x,b_T;b_{\max})$ and $g_D^q(z,b_T;b_{\max})$ depend in general on the species of the incoming and outgoing hadrons respectively, as well as on the fact that one TMD is a PDF while the other is an FF, just as in the case of collinear PDFs and FFs, see examples of parametrizations in Eq. (2.28),(2.30).

As in the previos section, it is convenient to combine the fourth and the fifth line of Eq. (3.22) into the evolution factor as $e^{-S(b_T,Q,\mu_Q,Q_0,\mu_{b_*})}$, where

$$
\begin{aligned}
S(b_T,Q,\mu_Q,Q_0,\mu_{b_*}) = &-\tilde{K}(b_*;\mu_{b_*})\ln\frac{Q^2}{\mu_{b_*}^2} + \int_{\mu_{b_*}}^{\mu_Q}\frac{d\mu'}{\mu'}\left[-2\gamma_j(\alpha_s(\mu');1) + \ln\frac{Q^2}{(\mu')^2}\gamma_K(\alpha_s(\mu'))\right] \\
&+ g_K(b_T;b_{\max})\ln\frac{Q^2}{Q_0^2}\,,
\end{aligned} \tag{3.24}
$$

[5]The existence of perturbatively controlled logarithmic singular behavior of $\tilde{K}$ and $\tilde{f}$ at small b_T implies that g_f^q and g_K are not exactly quadratic at small b_T.

which we can also rewrite in terms of A and B as

$$\boxed{\begin{aligned} S(b_T, Q, \mu_Q, Q_0, \mu_{b_*}) = &- \tilde{K}(b_*; \mu_{b_*}) \ln \frac{Q^2}{\mu_{b_*}^2} + \int_{\mu_{b_*}^2}^{\mu_Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \left[A(\alpha_s(\bar{\mu})) \ln \frac{Q^2}{\bar{\mu}^2} + B(\alpha_s(\bar{\mu})) \right] \\ &+ g_K(b_T; b_{\max}) \ln \frac{Q^2}{Q_0^2} \,, \qquad (3.25) \end{aligned}}$$

Notice that now one can use perturbative expression to calculate $\tilde{K}(b_*; \mu_{b_*})$ and the obvious advantage of the solution spelled in Eq. (3.22) is the usage of knowledge of collinear PDFs and FFs.

4 The SIDIS process in terms of TMDs and FFs

In this section we review the description of the SIDIS process, define structure functions, PDFs, TMDs, FFs and recall how they describe the SIDIS structure functions.

4.1 The SIDIS process

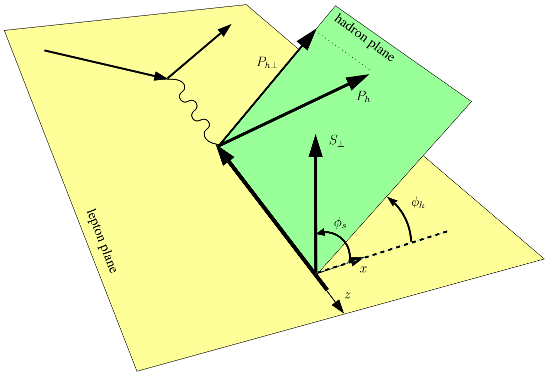

Figure 1. Kinematics of the SIDIS process $lN \to l'hX$ in the 1-photon exchange approximation.

The SIDIS process $lN \to l'hX$ is sketched in Fig. 1. Here, l and P are the momenta of the incoming lepton and nucleon, l' and P_h are the momenta of the outgoing lepton and produced hadron. The virtual-photon momentum $q = l - l'$ defines the z-axis, and l' points in the direction of the x-axis from which azimuthal angles are counted. The relevant kinematic invariants are

$$x = \frac{Q^2}{2\,P\cdot q},\quad y = \frac{P\cdot q}{P\cdot l},\quad z = \frac{P\cdot P_h}{P\cdot q},\quad Q^2 = -q^2. \tag{4.1}$$

Note that we consider the production of unpolarized hadrons in DIS off charged leptons (electrons, positrons, muons) at $Q^2 \ll M_Z^2$ in the single-photon exchange approximation, where M_Z denotes the mass of the Z^0 electroweak gauge boson. In addition to x, y, and z, the cross section is also differential in the azimuthal angle ϕ_h of the produced hadron and in the square of the hadron's momentum component P_{hT} perpendicular with respect to the virtual-photon momentum. The cross section is also differential with respect to the azimuthal angle ψ_l characterizing the overall orientation of the lepton scattering plane around the incoming lepton direction. The angle is calculated with respect to an arbitrary reference axis, which in case of transversely polarized targets is chosen to be the transverse component S_T of the target-spin direction. In the DIS limit, $\psi_l \approx \phi_S$, where the latter is the azimuthal angle of the spin-vector defined as in Fig. 1.

To leading order in $1/Q$ the SIDIS cross-section is given by

$$\begin{aligned}\frac{d^6\sigma_{\rm leading}}{dx\,dy\,dz\,d\psi_l\,d\phi_h\,dP_{hT}^2} &= \frac{\alpha_{em}^2}{x\,y\,Q^2}\left(1-y+\frac{1}{2}y^2\right)F_{UU}(x,z,P_{hT}^2)\\ &\times\Bigg\{1+\cos(2\phi_h)\,p_1\,A_{UU}^{\cos(2\phi_h)}+S_L\sin(2\phi_h)\,p_1\,A_{UL}^{\sin(2\phi_h)}+\lambda\,S_L\,p_2\,A_{LL}\\ &\qquad+S_T\sin(\phi_h-\phi_S)\,A_{UT}^{\sin(\phi_h-\phi_S)}+S_T\sin(\phi_h+\phi_S)\,p_1\,A_{UT}^{\sin(\phi_h+\phi_S)}\\ &\qquad+S_T\sin(3\phi_h-\phi_S)\,p_1\,A_{UT}^{\sin(3\phi_h-\phi_S)}+\lambda\,S_T\cos(\phi_h-\phi_S)\,p_2\,A_{LT}^{\cos(\phi_h-\phi_S)}\Bigg\}.\end{aligned} \tag{4.2a}$$

Here F_{UU} is the structure function due to transverse polarization of the virtual photon (sometimes denoted as $F_{UU,T}$), and we neglect $1/Q^2$ corrections in kinematic factors and a structure function (sometimes denoted as $F_{UU,L}$) arising from longitudinal polarization of the virtual photon (and another structure function $\propto S_T\ \sin(\phi_h - \phi_S)$, see below).

At subleading order in the $1/Q$ expansion one has

$$\frac{d^6\sigma_{\text{subleading}}}{dx\, dy\, dz\, d\psi_l\, d\phi_h\, dP_{hT}^2} = \frac{\alpha_{em}^2}{x\, y\, Q^2}\left(1 - y + \frac{1}{2}y^2\right) F_{UU}(x, z, P_{hT}^2)\Bigg\{ \cos(\phi_h)\, p_3\, A_{UU}^{\cos(\phi_h)}$$
$$+\lambda \sin(\phi_h)\, p_4\, A_{LU}^{\sin(\phi_h)} + S_L \sin(\phi_h)\, p_3\, A_{UL}^{\sin(\phi_h)} + \lambda\, S_L \cos(\phi_h)\, p_4\, A_{LL}^{\cos(\phi_h)}$$
$$+S_T \sin(2\phi_h - \phi_S)\, p_3\, A_{UT}^{\sin(2\phi_h-\phi_S)} + S_T \sin(\phi_S)\, p_3\, A_{UT}^{\sin(\phi_S)}$$
$$+\lambda\, S_T \cos(\phi_S)\, p_4\, A_{LT}^{\cos(\phi_S)} + \lambda\, S_T \cos(2\phi_h - \phi_S)\, p_4\, A_{LT}^{\cos(2\phi_h-\phi_S)} \Bigg\}. \tag{4.2b}$$

Neglecting $1/Q^2$ corrections, the kinematic prefactors p_i are given by

$$p_1 = \frac{1-y}{1-y+\frac{1}{2}\,y^2}\,, \quad p_2 = \frac{y(1-\frac{1}{2}\,y)}{1-y+\frac{1}{2}\,y^2}\,, \quad p_3 = \frac{(2-y)\sqrt{1-y}}{1-y+\frac{1}{2}\,y^2}\,, \quad p_4 = \frac{y\sqrt{1-y}}{1-y+\frac{1}{2}\,y^2}\,, \tag{4.3}$$

and the asymmetries A_{XY}^{weight}, are defined in terms of structure functions F_{XY}^{weight}, as follows

$$A_{XY}^{\text{weight}} \equiv A_{XY}^{\text{weight}}(x, z, P_{hT}) = \frac{F_{XY}^{\text{weight}}(x, z, P_{hT})}{F_{UU}(x, z, P_{hT})}. \tag{4.4}$$

Here, the first subscript $X = U(L)$ denotes the unpolarized beam (longitudinally polarized beam with helicity λ). The second subscript $Y = U(L \text{ or } T)$ refers to the target, which can be unpolarized (longitudinally or transversely polarized with respect to the virtual photon). The superscript "weight" indicates the azimuthal dependence with no index indicating a ϕ_h-independent asymmetry or structure function.

In the partonic description the structure functions in (4.2a) are "twist-2." Those in (4.2b) are "twist-3" and contain a factor M_N/Q in their definitions, see below, where M_N is the nucleon mass. In our treatment to $1/Q^2$ accuracy we neglect two structure functions due to longitudinal virtual-photon polarization, which contribute at order $\mathcal{O}(M_N^2/Q^2)$ in the partonic description of the process, one being $F_{UU,L}$ and the other contributing to the $\sin(\phi_h - \phi_S)$ angular distribution [4].

Experimental collaborations often define asymmetries in terms of counts $N(\phi_h)$. This means the kinematic prefactors p_i and $1/(x\, y\, Q^2)$ are included in the numerators or denominators of the asymmetries which are averaged over y within experimental kinematics. We will call the corresponding asymmetries $A_{XY,\langle y\rangle}^{\text{weight}}$. For instance, in the unpolarized case one has

$$N(x, \ldots, \phi_h) = \frac{N_0(x, \ldots)}{2\pi}\left(1 + \cos\phi_h\ A_{UU,\langle y\rangle}^{\cos\phi_h}(x, \ldots) + \cos 2\phi_h\ A_{UU,\langle y\rangle}^{\cos 2\phi_h}(x, \ldots)\right) \tag{4.5}$$

where N_0 denotes the total (ϕ_h–averaged) number of counts and the dots indicate further kinematic variables in the kinematic bin of interest (which may also be averaged over). It would be preferable if asymmetries were analyzed with known kinematic prefactors divided out on event-by-event basis. One could then directly compare asymmetries A_{XY}^{weight} measured in different experiments and kinematics, and focus on effects of evolution or power suppression for twist-3. In practice, often the kinematic factors were included. We will define and comment on the explicit expressions as needed.

For completeness we remark that after integrating the cross section over transverse hadron momenta one obtains

$$
\frac{d^4\sigma_{\rm leading}}{dx\,dy\,dz\,d\psi_l} = \frac{1}{2\pi}\,\frac{4\pi\alpha_{em}^2}{xyQ^2}\left(1-y+\frac{1}{2}y^2\right)\,F_{UU}(x,z)\Bigg\{\,1+\lambda\,S_L\,p_2\,A_{LL}\Bigg\} \tag{4.6a}
$$

$$
\frac{d^4\sigma_{\rm subleading}}{dx\,dy\,dz\,d\psi_l} = \frac{1}{2\pi}\,\frac{4\pi\alpha_{em}^2}{xyQ^2}\left(1-y+\frac{1}{2}y^2\right)\,F_{UU}(x,z)\Bigg\{S_T\sin(\phi_S)\,p_3\,A_{UT}^{\sin(\phi_S)}+
$$

$$
\lambda\,S_T\cos(\phi_S)\,p_4\,A_{LT}^{\cos(\phi_S)}\Bigg\}\,, \tag{4.6b}
$$

where (and analogous for the other structure functions)

$$
F_{UU}(x,z)=\int d^2P_{hT}\;F_{UU}(x,z,P_{hT}) \tag{4.7}
$$

and the asymmetries are defined as

$$
A_{XY}^{\rm weight}(x,z)=\frac{F_{XY}^{\rm weight}(x,z)}{F_{UU}(x,z)}\,. \tag{4.8}
$$

The connection of "collinear" SIDIS structure functions in (4.6a, 4.6b) to those known from inclusive DIS is established by integrating over z and summing over hadrons as

$$
\sum_h\int dz\;z\;F_{UU}(x,z) \quad\equiv\quad 2\,x\,F_1(x)\,, \tag{4.9a}
$$

$$
\sum_h\int dz\;z\;F_{LL}(x,z) \quad\equiv\quad 2\,x\,g_1(x)\,, \tag{4.9b}
$$

$$
\sum_h\int dz\;z\;F_{LT}^{\cos\phi_S}(x,z) \;\equiv\; -\gamma\,2\,x\Big(g_1(x)+g_2(x)\Big)\,, \tag{4.9c}
$$

$$
\sum_h\int dz\;z\;F_{UT}^{\sin\phi_S}(x,z) \;=\; 0\,, \tag{4.9d}
$$

where $\gamma=2M_Nx/Q$ signals the twist-3 character of $F_{LT}^{\cos\phi_S}(x,z)$. Notice that $F_{UT}^{\sin\phi_S}(x,z)$ has no DIS counterpart due to time-reversal symmetry of strong interactions, and terms suppressed by $1/Q^2$ are consequently neglected throughout this work including the twist-4 DIS structure function $F_L(x)$.

4.2 SIDIS structure functions and their interpretation as convolutions of TMDs

The structure functions in Eqs. (4.2a, 4.2b) are described in the Bjorken limit at tree level in terms of convolutions of TMDs and FFs. We define the unit vector $\hat{\boldsymbol{h}}=\boldsymbol{P}_{hT}/P_{hT}$ and use the following convolution integrals (see Appendix A for details)

$$
\mathcal{C}\Big[\omega\;f\;D\Big]=x\sum_a e_a^2\mathcal{H}_a^{(SIDIS)}\int d^2\boldsymbol{k}_Td^2\boldsymbol{p}_T\;\delta^{(2)}(z\boldsymbol{k}_T+\boldsymbol{p}_T-\boldsymbol{P}_{hT})\;\omega\;f^a(x,\boldsymbol{k}_T^2)\;D^a(z,\boldsymbol{p}_T^2)\,, \tag{4.10}
$$

where ω is a weight function, which in general depends on $\boldsymbol{k}_T$ and $\boldsymbol{p}_T$. The 8 leading-twist structure functions are

$$F_{UU} = \mathcal{C}\Big[\,\omega^{\{0\}}\, f_1 D_1\,\Big]\,, \tag{4.11a}$$

$$F_{UU}^{\cos 2\phi_h} = \mathcal{C}\Big[\,\omega_{\rm AB}^{\{2\}}\, h_1^{\perp}\, H_1^{\perp}\,\Big]\,, \tag{4.11b}$$

$$F_{UU}^{\cos 2\phi_h} = \frac{4M_N^2}{Q^2}\,\mathcal{C}\Big[\,\omega_{\rm C}^{\{2\}}\, f_1\, D_1\,\Big] \ \text{twist-4 contribution,} \tag{4.11c}$$

$$F_{UL}^{\sin 2\phi_h} = \mathcal{C}\Big[\,\omega_{\rm AB}^{\{2\}}\, h_{1L}^{\perp} H_1^{\perp}\,\Big]\,, \tag{4.11d}$$

$$F_{LL} = \mathcal{C}\Big[\,\omega^{\{0\}}\, g_1 D_1\,\Big]\,, \tag{4.11e}$$

$$F_{LT}^{\cos(\phi_h-\phi_S)} = \mathcal{C}\Big[\omega_{\rm B}^{\{1\}}\, g_{1T}^{\perp} D_1\Big]\,, \tag{4.11f}$$

$$F_{UT}^{\sin(\phi_h+\phi_S)} = \mathcal{C}\Big[\,\omega_{\rm A}^{\{1\}}\, h_1 H_1^{\perp}\,\Big]\,, \tag{4.11g}$$

$$F_{UT}^{\sin(\phi_h-\phi_S)} = \mathcal{C}\Big[-\omega_{\rm B}^{\{1\}}\, f_{1T}^{\perp} D_1\Big]\,, \tag{4.11h}$$

$$F_{UT}^{\sin(3\phi_h-\phi_S)} = \mathcal{C}\Big[\,\omega^{\{3\}}\, h_{1T}^{\perp} H_1^{\perp}\,\Big]\,. \tag{4.11i}$$

At subleading-twist we have the structure functions

$$F_{UU}^{\cos\phi_h} = \frac{2M_N}{Q}\,\mathcal{C}\Big[\ \omega_{\rm A}^{\{1\}}\Big(xh\, H_1^{\perp} + r_h\, f_1 \frac{\tilde{D}^{\perp}}{z}\Big) - \omega_{\rm B}^{\{1\}}\Big(xf^{\perp} D_1 + r_h\, h_1^{\perp}\frac{\tilde{H}}{z}\Big)\Big], \tag{4.12a}$$

$$F_{LU}^{\sin\phi_h} = \frac{2M_N}{Q}\,\mathcal{C}\Big[\ \omega_{\rm A}^{\{1\}}\Big(x\, e\, H_1^{\perp} + r_h\, f_1 \frac{\tilde{G}^{\perp}}{z}\Big) + \omega_{\rm B}^{\{1\}}\Big(xg^{\perp} D_1 + r_h\, h_1^{\perp}\frac{\tilde{E}}{z}\Big)\Big], \tag{4.12b}$$

$$F_{UL}^{\sin\phi_h} = \frac{2M_N}{Q}\,\mathcal{C}\Big[\ \omega_{\rm A}^{\{1\}}\Big(xh_L H_1^{\perp} + r_h\, g_1 \frac{\tilde{G}^{\perp}}{z}\Big) + \omega_{\rm B}^{\{1\}}\Big(xf_L^{\perp} D_1 - r_h\, h_{1L}^{\perp}\frac{\tilde{H}}{z}\Big)\Big], \tag{4.12c}$$

$$F_{LL}^{\cos\phi_h} = \frac{2M_N}{Q}\,\mathcal{C}\Big[-\omega_{\rm A}^{\{1\}}\Big(xe_L H_1^{\perp} - r_h\, g_1 \frac{\tilde{D}^{\perp}}{z}\Big) - \omega_{\rm B}^{\{1\}}\Big(xg_L^{\perp} D_1 + r_h\, h_{1L}^{\perp}\frac{\tilde{E}}{z}\Big)\Big], \tag{4.12d}$$

$$\begin{aligned} F_{UT}^{\sin\phi_S} = \frac{2M_N}{Q}\,\mathcal{C}\Big[\ &\omega^{\{0\}}\Big(xf_T D_1 - r_h\, h_1 \frac{\tilde{H}}{z}\Big) \\ &-\frac{\omega_{\rm B}^{\{2\}}}{2}\Big(xh_T H_1^{\perp} + r_h\, g_{1T}^{\perp}\frac{\tilde{G}^{\perp}}{z} - xh_T^{\perp} H_1^{\perp} + r_h\, f_{1T}^{\perp}\frac{\tilde{D}^{\perp}}{z}\Big)\Big]\,, \end{aligned} \tag{4.12e}$$

$$
\begin{aligned}
F_{LT}^{\cos\phi_S} &= \frac{2M_N}{Q}\,\mathcal{C}\Big[-\omega^{\{0\}}\left(xg_T D_1 + r_h\; h_1\frac{\tilde{E}}{z}\right) \\
&\qquad +\frac{\omega_{\rm B}^{\{2\}}}{2}\left(xe_T H_1^\perp - r_h\, g_{1T}^\perp\,\frac{\tilde{D}^\perp}{z} + xe_T^\perp H_1^\perp + r_h\, f_{1T}^\perp\,\frac{\tilde{G}^\perp}{z}\right)\Big]\,, \qquad (4.12f) \\
F_{UT}^{\sin(2\phi_h-\phi_S)} &= \frac{2M_N}{Q}\,\mathcal{C}\Big[\;\frac{\omega_{\rm AB}^{\{2\}}}{2}\left(xh_T H_1^\perp + r_h\, g_{1T}^\perp\,\frac{\tilde{G}^\perp}{z} + xh_T^\perp H_1^\perp - r_h\, f_{1T}^\perp\,\frac{\tilde{D}^\perp}{z}\right) \\
&\qquad +\omega_{\rm C}^{\{2\}}\left(xf_T^\perp D_1 - r_h\; h_{1T}^\perp\frac{\tilde{H}}{z}\right)\Big]\,, \qquad (4.12g) \\
F_{LT}^{\cos(2\phi_h-\phi_S)} &= \frac{2M_N}{Q}\,\mathcal{C}\Big[-\frac{\omega_{\rm AB}^{\{2\}}}{2}\left(xe_T H_1^\perp - r_h\, g_{1T}^\perp\,\frac{\tilde{D}^\perp}{z} - xe_T^\perp H_1^\perp - r_h\, f_{1T}^\perp\,\frac{\tilde{G}^\perp}{z}\right) \\
&\qquad -\omega_{\rm C}^{\{2\}}\left(xg_T^\perp D_1 + r_h\; h_{1T}^\perp\frac{\tilde{E}}{z}\right)\Big]\,, \qquad (4.12h)
\end{aligned}
$$

where $r_h = M_h/M_N$ and $F_{XY}^{\rm weight} \equiv F_{XY}^{\rm weight}(x,z,P_{hT})$. The tilde-functions $\tilde{D}^\perp$, $\tilde{G}^\perp$, $\tilde{H}$, $\tilde{E}$ are defined in terms of $\bar{q}gq$-correlators. The weight functions are defined as

$$
\begin{aligned}
&\omega^{\{0\}} = 1\,, \\
&\omega_{\rm A}^{\{1\}} = \frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_T}{zM_h}\,,\quad \omega_{\rm B}^{\{1\}} = \frac{\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_T}{M_N}\,, \\
&\omega_{\rm A}^{\{2\}} = \frac{2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_T)\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_T)}{zM_NM_h}\,,\quad \omega_{\rm B}^{\{2\}} = -\frac{\boldsymbol{p}_T\cdot\boldsymbol{k}_T}{zM_NM_h}\,,\quad \omega_{\rm C}^{\{2\}} = \frac{2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_T)^2-\boldsymbol{k}_T^2}{2M_N^2}\,, \\
&\omega^{\{3\}} = \frac{4\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_T)\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_T)^2 - 2\,(\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_T)\,(\boldsymbol{k}_T\cdot\boldsymbol{p}_T) - (\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_T)\,\boldsymbol{k}_T^2}{2zM_N^2M_h}\,, \qquad (4.13)
\end{aligned}
$$

and $\omega_{\rm AB}^{\{2\}} = \omega_{\rm A}^{\{2\}} + \omega_{\rm B}^{\{2\}}$. In $\omega_i^{\{n\}}$ the index $n = 0, 1, 2, 3$ indicates the (maximal) power $(P_{hT})^n$ with which the corresponding contribution scales, and index i (if any) distinguishes different types of contributions at the given order n. Notice that twist-3 structure functions in Eqs. (4.12a–4.12h) contain an explicit factor M_N/Q. We also recall that we neglect two structure functions (denoted in [4] as $F_{UU,L}$ and $F_{UT,L}^{\sin(\phi_h-\phi_S)}$) due to longitudinal virtual-photon polarization, which are of order $\mathcal{O}(M^2/Q^2)$ in the TMD partonic description.

The structure functions that survive P_{hT}–integration of the SIDIS cross section in (4.6a, 4.6b) are associated with the trivial weights $\omega^{\{0\}}$ and expressed in terms of collinear PDFs and FFs as follows (here the sum rules (2.4) are used):

$$
\begin{aligned}
F_{UU}(x,z) &= x\sum_a e_a^2\, f_1^a(x)\, D_1^a(z)\,, \qquad (4.14a) \\
F_{LL}(x,z) &= x\sum_a e_a^2\, g_1^a(x)\, D_1^a(z)\,, \qquad (4.14b) \\
F_{LT}^{\cos\phi_S}(x,z) &= -\,\frac{2M_N}{Q}\;x\sum_a e_a^2\left(x\, g_T^q(x)\, D_1^a(z) + r_h\, h_1^a(x)\frac{\tilde{E}^a(z)}{z}\right), \qquad (4.14c) \\
F_{UT}^{\sin\phi_S}(x,z) &= -\,\frac{2\,M_h}{Q}\;x\sum_a e_a^2\, h_1^a(x)\frac{\tilde{H}^a(z)}{z}\,. \qquad (4.14d)
\end{aligned}
$$

Finally, integrating over z, summing over hadrons, and using the sum rules for the T-odd FFs, $\sum_h\int dz\;\tilde{E}^a(z) = 0$ and $\sum_h\int dz\;\tilde{H}^a(z) = 0$, we recover Eqs. (4.9a–4.9d) and obtain for the DIS structure functions

$$F_1(x) = \frac{1}{2}\sum_a e_a^2\, f_1^a(x)\,, \tag{4.15a}$$

$$g_1(x) = \frac{1}{2}\sum_a e_a^2\, g_1^a(x)\,, \tag{4.15b}$$

$$g_2(x) = \frac{1}{2}\sum_a e_a^2\, g_T^a(x) \;-\; g_1(x)\,. \tag{4.15c}$$

4.3 SIDIS structure functions and their interpretation as convolutions of TMDs in b_T space

$$\mathcal{B}[\tilde{f}^{(n)}\ \tilde{D}^{(m)}] \equiv x\sum_a e_a^2 \mathcal{H}_a^{(SIDIS)}(Q,\mu_Q)\int_0^\infty \frac{db_T\, b_T}{2\pi}\ b_T^{n+m}\ J_{n+m}(q_T b_T) \\ \times\, \tilde{f}^{(n)a}(x,b_T,\mu_Q,\zeta_1)\ \tilde{D}^{(m)a}(z,b_T,\mu_Q,\zeta_2)\,, \tag{4.16}$$

where $\zeta_1\zeta_2 = Q^4$ and here $q_T = P_{hT}/z_n \simeq P_{hT}/z$, so that

$$\mathcal{B}[\tilde{f}^{(n)}\ \tilde{D}^{(m)}] \equiv x\sum_a e_a^2 \mathcal{H}_a^{(SIDIS)}(Q,\mu_Q)\int_0^\infty \frac{db_T\, b_T}{2\pi}\ b_T^{n+m}\ J_{n+m}(q_T b_T) \\ \times\, \tilde{f}^{(n)a}(x,b_T,Q_0,Q_0^2)\ \tilde{D}^{(m)a}(z,b_T,Q_0,Q_0^2)\ e^{-S(b_T,Q,\mu_Q,Q_0)}\,, \tag{4.17}$$

if solution from Eq. (3.12) is used. For functions $\tilde{f}^{(n)a}$ and $\tilde{D}^{(m)a}$ we obtain

$$\tilde{f}^{(n)a}(x,b_T,\mu_Q,\zeta_1) \equiv \tilde{f}^{(n)a}(x,b_T,Q_0,Q_0^2)\ e^{-\frac{1}{2}S(b_T,Q,\mu_Q,Q_0)}\,, \\ \tilde{D}^{(m)a}(z,b_T,\mu_Q,\zeta_2) \equiv \tilde{D}^{(m)a}(z,b_T,Q_0,Q_0^2)\ e^{-\frac{1}{2}S(b_T,Q,\mu_Q,Q_0)}\,, \tag{4.18}$$

where functions at the initial scale $\tilde{f}^{(n)a}(x,b_T,Q_0,Q_0^2)$ and $\tilde{D}^{(m)a}(z,b_T,Q_0,Q_0^2)$ are understood as nonperturbative functions to be parametrized phenomenologically and the evolution exponent $S(b_T,Q,\mu_Q,Q_0)$ is from Eqs. (3.13,3.15).

If the solution from Eq. (3.22), optimized for the usage of Operator Product Relation to collinear function, is used them one has

$$\mathcal{B}[\tilde{f}^{(n)}\ \tilde{D}^{(m)}] \equiv x\sum_a e_a^2 \sum_{i,j}\mathcal{H}_{a,i,j}^{(SIDIS)}(Q,\mu_Q)\int_0^\infty \frac{db_T\, b_T}{2\pi}\ b_T^{n+m}\ J_{n+m}(q_T b_T) \\ \times\, e^{-g_f^a(x,b_T;b_{\max})}\int_x^1 \frac{d\hat{x}}{\hat{x}}\tilde{C}_{f/i}(x/\hat{x},\mu_{b_*})\otimes f^{(n)i}(\hat{x},\mu_{b_*}) \\ \times\, e^{-g_D^a(z,b_T;b_{\max})}\int_z^1 \frac{d\hat{z}}{\hat{z}^3}\tilde{C}_{D/j}(z/\hat{z},\mu_{b_*})\otimes D^{(m)j}(\hat{z},\mu_{b_*}) \\ \times\, e^{-S(b_T,Q,\mu_Q,\mu_{b_*})}\,, \tag{4.19}$$

For functions $\tilde{f}^{(n)a}$ and $\tilde{D}^{(m)a}$ we obtain

$$\tilde{f}^{(n)a}(x,b_T,\mu_Q,\zeta_1) \equiv e^{-g_f^a(x,b_T;b_{\max})}\ e^{-\frac{1}{2}S(b_T,Q,\mu_Q,\mu_{b_*})}\int_x^1 \frac{d\hat{x}}{\hat{x}}\tilde{C}_{f/i}(x/\hat{x},\mu_{b_*})\otimes f^{(n)i}(\hat{x},\mu_{b_*})\,, \\ \tilde{D}^{(m)a}(z,b_T,\mu_Q,\zeta_2) \equiv e^{-g_D^a(z,b_T;b_{\max})}\ e^{-\frac{1}{2}S(b_T,Q,\mu_Q,\mu_{b_*})}\int_z^1 \frac{d\hat{z}}{\hat{z}^3}\tilde{C}_{D/j}(z/\hat{z},\mu_{b_*})\otimes D^{(m)j}(\hat{z},\mu_{b_*})\,, \tag{4.20}$$

where now only the nonperturbative shape functions $g_f^a(x, b_T; b_{\max})$ and $g_D^a(z, b_T; b_{\max})$ and the universal function $g_K(b_T; b_{\max})$ are to be parametrized phenomenologically. The evolution exponent S is from Eq. (3.24).

This leads to the following expressions for the twist-2 structure functions,

$$F_{UU}(x,z,P_{hT},Q^2) = \mathcal{B}[\tilde{f}_1^{(0)}\,\tilde{D}_1^{(0)}]\,, \tag{4.21}$$

$$F_{UU}^{\cos 2\phi_h}(x,z,P_{hT},Q^2) = M_N\,M_h\,\mathcal{B}[\tilde{h}_1^{\perp(1)}\,\tilde{H}_1^{\perp(1)}]\,, \tag{4.22}$$

$$F_{UU}^{\cos 2\phi_h}(x,z,P_{hT},Q^2) = \frac{M_N^4}{Q^2}\,\mathcal{B}[\tilde{f}_1^{(2)}\,\tilde{D}_1^{(0)}]\,,\ \text{twist-4 contribution,} \tag{4.23}$$

$$F_{UL}^{\sin 2\phi_h}(x,z,P_{hT},Q^2) = M_N\,M_h\,\mathcal{B}[\tilde{h}_{1L}^{\perp(1)}\,\tilde{H}_1^{\perp(1)}]\,, \tag{4.24}$$

$$F_{LL}(x,z,P_{hT},Q^2) = \mathcal{B}[\tilde{g}_1^{(0)}\,\tilde{D}_1^{(0)}]\,, \tag{4.25}$$

$$F_{LT}^{\cos(\phi_h-\phi_S)}(x,z,P_{hT},Q^2) = M_N\,\mathcal{B}[\tilde{g}_{1T}^{\perp(1)}\,\tilde{D}_1^{(0)}]\,, \tag{4.26}$$

$$F_{UT}^{\sin(\phi_h+\phi_S)}(x,z,P_{hT},Q^2) = M_h\,\mathcal{B}[\tilde{h}_1^{(0)}\,\tilde{H}_1^{\perp(1)}]\,, \tag{4.27}$$

$$F_{UT}^{\sin(\phi_h-\phi_S)}(x,z,P_{hT},Q^2) = -M_N\,\mathcal{B}[\tilde{f}_{1T}^{\perp(1)}\,\tilde{D}_1^{(0)}]\,, \tag{4.28}$$

$$F_{UT}^{\sin(3\phi_h-\phi_S)}(x,z,P_{hT},Q^2) = \frac{M_N^2\,M_h}{4}\,\mathcal{B}[\tilde{h}_{1T}^{\perp(2)}\,\tilde{H}_1^{\perp(1)}]\,. \tag{4.29}$$

5 Drell-Yan process with pions and polarized protons

In this section we briefly review the DY formalism, and provide the description of the DY structure functions in our approach.

5.1 Structure functions

In the tree-level description a dilepton l, l' is produced from the annihilation of a quark and anti-quark carrying the fractions x_π, x_p of the longitudinal momenta of respectively the pion and the proton. The process is shown in the Collins-Soper frame in Fig. 2. In the case of pions colliding with polarized protons the DY cross section is described in terms of 6 structure functions [37]

$$
\begin{aligned}
F_{UU}^{1} &= \mathcal{C}\Big[f_{1,\pi}^{\bar a}\, f_{1,p}^{a} \Big], \\
F_{UU}^{\cos 2\phi} &= \mathcal{C}\Big[\frac{2(\hat{\boldsymbol h}\cdot\vec{\boldsymbol k}_{T\pi})(\hat{\boldsymbol h}\cdot\vec{\boldsymbol k}_{Tp}) - \vec{\boldsymbol k}_{T\pi}\cdot\vec{\boldsymbol k}_{Tp}}{M_\pi\, M_p}\, h_{1,\pi}^{\perp\bar a}\, h_{1,p}^{\perp a} \Big], \\
F_{UL}^{\sin 2\phi} &= -\mathcal{C}\Big[\frac{2(\hat{\boldsymbol h}\cdot\vec{\boldsymbol k}_{T\pi})(\hat{\boldsymbol h}\cdot\vec{\boldsymbol k}_{Tp}) - \vec{\boldsymbol k}_{T\pi}\cdot\vec{\boldsymbol k}_{Tp}}{M_\pi\, M_p}\, h_{1,\pi}^{\perp\bar a}\, h_{1L,p}^{\perp a} \Big], \\
F_{UT}^{\sin \phi_S} &= \mathcal{C}\Big[\frac{\hat{\boldsymbol h}\cdot\vec{\boldsymbol k}_{Tp}}{M_p}\, f_{1,\pi}^{\bar a}\, f_{1T,p}^{\perp a} \Big], \\
F_{UT}^{\sin(2\phi-\phi_S)} &= -\mathcal{C}\Big[\frac{\hat{\boldsymbol h}\cdot\vec{\boldsymbol k}_{T\pi}}{M_\pi}\, h_{1,\pi}^{\perp\bar a}\, h_{1,p}^{a} \Big], \\
F_{UT}^{\sin(2\phi+\phi_S)} &= -\mathcal{C}\Big[\frac{2(\hat{\boldsymbol h}\cdot\vec{\boldsymbol k}_{Tp})[2(\hat{\boldsymbol h}\cdot\vec{\boldsymbol k}_{T\pi})(\hat{\boldsymbol h}\cdot\vec{\boldsymbol k}_{Tp}) - \vec{\boldsymbol k}_{T\pi}\cdot\vec{\boldsymbol k}_{Tp}] - \vec{\boldsymbol k}_{Tp}^2(\hat{\boldsymbol h}\cdot\vec{\boldsymbol k}_{T\pi})}{2\, M_\pi\, M_p^2}\, h_{1,\pi}^{\perp\bar a}\, h_{1T,p}^{\perp a} \Big].
\end{aligned} \tag{5.1}
$$

The subscripts indicate the hadron polarization which can be unpolarized U (pions, protons), longitudinally L, or transversely T polarized (protons). The azimuthal angles ϕ, ϕ_S are defined in Fig. 2, where the unit vector $\hat{\boldsymbol h} = \boldsymbol q_T/q_T$ points along x-axis. Notice that in the Collins-Soper frame the dilepton is at rest, and each incoming hadron carries the transverse momentum $\boldsymbol q_T/2$, see Fig. 2. The convolution integrals in Eq. (5.1) are defined [37] as

$$
\mathcal{C}[\omega\, f_\pi^{\bar a}\, f_p^{a}] = \frac{1}{N_c}\sum_a e_a^2 \int d^2\boldsymbol k_{T\pi}\, d^2\boldsymbol k_{Tp}\, \delta^{(2)}(\boldsymbol q_T - \boldsymbol k_{T\pi} - \boldsymbol k_{Tp})\, \omega\, f_\pi^{\bar a}(x_\pi, \boldsymbol k_{T\pi}^2) f_p^{a}(x_p, \boldsymbol k_{Tp}^2)\,, \tag{5.2}
$$

where ω, a function of the transverse momenta $\boldsymbol k_{T\pi}$, $\boldsymbol k_{Tp}$ and $\boldsymbol q_T$, projects out the corresponding azimuthal angular dependence. The sum over $a = u,\, \bar u,\, d,\, \bar d,\, \dots$ includes the active flavors.

This partonic interpretation of DY is based on a TMD factorization [15, 20] and valid at $q_T \ll Q$.

In our work we will focus on the description of asymmetries of the kind

$$
A_{XY}^{\text{weight}}(x_\pi, x_p, q_T, Q^2) = \frac{F_{XY}^{\text{weight}}(x_\pi, x_p, q_T, Q^2)}{F_{UU}^{1}(x_\pi, x_p, q_T, Q^2)}. \tag{5.3}
$$

In such asymmetries there is tendency for various types of higher order corrections to largely cancel out [38–44].

The Q^2 dependence of the structure functions and asymmetries will often not be explicitly indicated for brevity. In the following we will display results for the asymmetries as functions of one of the variables x_π, x_p, q_T. It is then understood that the structure functions are integrated over other variables within the acceptance of the experiment, keeping in mind that x_π, x_p are connected to each other by $x_\pi\, x_p = Q^2/s$.

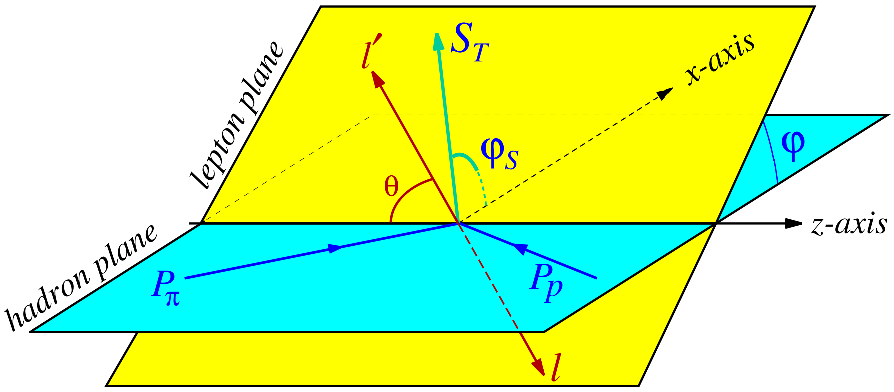

Figure 2. The DY process in the Collins-Soper frame where the pion and the proton come in with different momenta P_π, P_p, but each carries the same transverse momentum $\frac{1}{2}\boldsymbol{q}_T$, and the produced lepton pair is at rest. The angle ϕ describes the inclination of the leptonic frame with respect to the hadronic plane, and ϕ_S is the azimuthal angle of the transverse-spin vector of the proton.

5.2 TMD evolution of Drell-Yan structure functions

The basis for TMD evolution are QCD factorization theorems [15, 16, 18, 20, 21, 45–51], they constrain the operator definition of distribution and fragmentation TMDs and define QCD evolution of TMDs. Here we will adopt the Collins-Soper-Sterman (CSS) framework. We will perform a study of pion induced Drell-Yan including TMD evolution [12, 21, 30]. Here we will use the TMD evolution formalism starting from a fixed scale Q_0 [16] in the structure functions from Eqs. (5.1).

The evolution of TMDs is a double-scale problem, and conveniently addressed in the impact-parameter space with $\boldsymbol{b}_T$ the Fourier-conjugate variable to $\boldsymbol{k}_{Th}$. The TMDs in the impact-parameter space are generically given by $\tilde{f}(x_h, b_T, \mu, \zeta)$ where $\mu \sim Q$ is the "standard" renormalization scale for ultraviolet logarithms, and $\zeta \sim Q^2$ is the rapidity renormalization scale. In principle one can solve TMD evolution equations starting from some initial scale Q_0 without employing operator product expansion at low $\boldsymbol{b}_T$, Ref. [16]. The TMD at this initial scale is then $f(x_h, b_T, Q_0, Q_0^2)$. The unpolarized structure function is then very similar to parton model result, see Ref. [16]. For instance for the unpolarized structure function one obtains [16]

$$F^1_{UU}(x_\pi, x_p, q_T, Q^2) = \frac{1}{N_c}\sum_a e_a^2 \mathcal{H}^{(DY)}(Q,\mu_Q) \int \frac{b_T db_T}{2\pi} J_0(q_T b_T) \tilde{f}^{\bar{a}}_{1,\pi}(x_\pi, b_T, Q_0, Q_0^2) \tilde{f}^{a}_{1,p}(x_p, b_T, Q_0, Q_0^2) \times e^{-S(b_T,Q,\mu_Q,Q_0)}\,, \tag{5.4}$$

where the factor $S(b_T, Q, \mu_Q, Q_0)$ Eq. (3.13) contains important effects of gluon radiation with $S(b_T, Q_0, Q_0, Q_0) = 0$ by construction [16]. One can parametrize TMDs at initial scale Q_0 as

$$\tilde{f}^a_{1,p}(x_p, b_T, Q_0, Q_0^2) = f^a_{1,p}(x_p, Q_0)\, e^{-\frac{1}{4}b_T^2 \langle k^2_{Tp}\rangle_{f_{1,p}}}\,, \tag{5.5}$$

$$\tilde{f}^a_{1,\pi}(x_\pi, b_T, Q_0, Q_0^2) = f^a_{1,\pi}(x_\pi, Q_0)\, e^{-\frac{1}{4}b_T^2 \langle k^2_{T\pi}\rangle_{f_{1,\pi}}}\,, \tag{5.6}$$

where x-dependent functions correspond to collinear distributions and the exponential factors are primordial shapes of TMDs at the initial scale. This particular dependence is often used in phenomenology and corresponds to the Gaussian anzatz. The average widths of TMDs may be flavour dependent and may depend on x, we will take the values from the known phenomenological parametrizations at Q_0^2.

Based on the b_T space formalism given in Ref. [12] we write down the rest of the twist-2 structure functions. We use the convenient notation from Ref. [52],

$$\mathcal{B}[\tilde{f}_{1,\pi}^{(n)}\ \tilde{f}_{2,p}^{(m)}] \equiv \frac{1}{N_c}\sum_a e_a^2 \mathcal{H}^{(DY)}(Q,\mu_Q)\int_0^\infty \frac{db_T\, b_T}{2\pi}\ b_T^{n+m}\ J_{n+m}(q_T b_T) \\ \times \tilde{f}_{1,\pi}^{(n)\bar{a}}(x_\pi, b_T, \mu_Q, \zeta_a)\ \tilde{f}_{2,p}^{(m)a}(x_p, b_T, \mu_Q, \zeta_b), \tag{5.7}$$

where $\zeta_a\zeta_b = Q^4$ and

$$\mathcal{B}[\tilde{f}_{1,\pi}^{(n)}\ \tilde{f}_{2,p}^{(m)}] \equiv \frac{1}{N_c}\sum_a e_a^2 \mathcal{H}^{(DY)}(Q,\mu_Q)\int_0^\infty \frac{db_T\, b_T}{2\pi}\ b_T^{n+m}\ J_{n+m}(q_T b_T) \\ \times \tilde{f}_{1,\pi}^{(n)\bar{a}}(x_\pi, b_T, Q_0, Q_0^2)\ \tilde{f}_{2,p}^{(m)a}(x_p, b_T, Q_0, Q_0^2)\ e^{-S(b_T,Q,\mu_Q,Q_0)}\,, \tag{5.8}$$

if solution from Eq. (3.12) is used and

$$\mathcal{B}[\tilde{f}_{1,\pi}^{(n)}\ \tilde{f}_{2,p}^{(m)}] \equiv \frac{1}{N_c}\sum_a e_a^2 \mathcal{H}^{(DY)}(Q,\mu_Q)\int_0^\infty \frac{db_T\, b_T}{2\pi}\ b_T^{n+m}\ J_{n+m}(q_T b_T) \\ \times \tilde{C}_{\bar{f}} \otimes f_{1,\pi}^{(n)\bar{a}}(x_\pi, \mu_{b_*})\ e^{-g_{f_1}^{\bar{a}}(x_\pi, b_T; b_{\max})} \\ \times \tilde{C}_{f} \otimes f_{2,p}^{(m)a}(x_p, \mu_{b_*})\ e^{-g_{f_2}^{a}(x_p, b_T; b_{\max})} \\ \times e^{-S(b_T,Q,\mu_Q,\mu_{b_*})}\,, \tag{5.9}$$

if solution from Eq. (3.22) is used. This leads to the following expressions for the twist-2 structure functions,

$$F_{UU}^{1}(x_\pi, x_p, q_T, Q^2) = \mathcal{B}[\tilde{f}_{1,\pi}^{(0)}\ \tilde{f}_{1,p}^{(0)}]\,, \tag{5.10}$$

$$F_{UU}^{\cos 2\phi}(x_\pi, x_p, q_T, Q^2) = M_\pi\, M_p\ \mathcal{B}[\tilde{h}_{1,\pi}^{\perp(1)}\ \tilde{h}_{1,p}^{\perp(1)}]\,, \tag{5.11}$$

$$F_{UL}^{\sin 2\phi}(x_\pi, x_p, q_T, Q^2) = -M_\pi\, M_p\ \mathcal{B}[\tilde{h}_{1,\pi}^{\perp(1)}\ \tilde{h}_{1L,p}^{\perp(1)}]\,, \tag{5.12}$$

$$F_{UT}^{\sin \phi_S}(x_\pi, x_p, q_T, Q^2) = M_p\ \mathcal{B}[\tilde{f}_{1,\pi}^{(0)}\ \tilde{f}_{1T,p}^{\perp(1)}]\,, \tag{5.13}$$

$$F_{UT}^{\sin(2\phi-\phi_S)}(x_\pi, x_p, q_T, Q^2) = -M_\pi\ \mathcal{B}[\tilde{h}_{1,\pi}^{\perp(1)}\ \tilde{h}_{1,p}^{(0)}]\,, \tag{5.14}$$

$$F_{UT}^{\sin(2\phi+\phi_S)}(x_\pi, x_p, q_T, Q^2) = -\frac{M_\pi\, M_p^2}{4}\ \mathcal{B}[\tilde{h}_{1,\pi}^{\perp(1)}\ \tilde{h}_{1T,p}^{\perp(2)}]\,, \tag{5.15}$$

We will numerically calculate the integral in Eq. (**??**) using the two-loop result for the strong coupling constant, tuned to the world average [53] $\alpha_s(M_Z) = 0.118$ as in the CTEQ analysis [54].

5.3 Structure functions at the initial scale

Using Eqs. (2.27) or Eqs. (2.28) one obtains for the convolution integrals in Eqs. (5.1) or Eqs. (5.10)-(5.15) the following results

$$
\begin{aligned}
F_{UU}^{1}(x_\pi, x_p, q_T, Q_0^2) &= \frac{1}{N_c}\sum_a e_a^2 \; f_{1,\pi}^{\bar a}(x_\pi, Q_0)\; f_{1,p}^{a}(x_p, Q_0)\; \frac{e^{-q_T^2/\langle q_T^2\rangle}}{\pi\,\langle q_T^2\rangle}, \\
F_{UT}^{\sin\phi_S}(x_\pi, x_p, q_T) &= \frac{1}{N_c}\sum_a e_a^2 \; f_{1,\pi}^{\bar a}(x_\pi, Q_0)\; f_{1T,p}^{\perp(1)a}(x_p, Q_0)\, 2M_p\, \frac{q_T}{\langle q_T^2\rangle}\, \frac{e^{-q_T^2/\langle q_T^2\rangle}}{\pi\,\langle q_T^2\rangle}, \\
F_{UT}^{\sin(2\phi-\phi_S)}(x_\pi, x_p, q_T, Q_0^2) &= -\frac{1}{N_c}\sum_a e_a^2 \; h_{1,\pi}^{\perp(1)\bar a}(x_\pi, Q_0)\; h_{1,p}^{a}(x_p, Q_0)\, 2M_\pi\, \frac{q_T}{\langle q_T^2\rangle}\, \frac{e^{-q_T^2/\langle q_T^2\rangle}}{\pi\,\langle q_T^2\rangle}, \\
F_{UU}^{\cos 2\phi}(x_\pi, x_p, q_T, Q_0^2) &= \frac{1}{N_c}\sum_a e_a^2 \; h_{1,\pi}^{\perp(1)\bar a}(x_\pi, Q_0)\; h_{1,p}^{\perp(1)a}(x_p, Q_0)\, 4M_\pi M_p\, \frac{q_T^2}{\langle q_T^2\rangle^2}\, \frac{e^{-q_T^2/\langle q_T^2\rangle}}{\pi\,\langle q_T^2\rangle}, \\
F_{UL}^{\sin 2\phi}(x_\pi, x_p, q_T, Q_0^2) &= -\frac{1}{N_c}\sum_a e_a^2 \; h_{1,\pi}^{\perp(1)\bar a}(x_\pi, Q_0)\; h_{1L,p}^{\perp(1)a}(x_p, Q_0)\, 4M_\pi M_p\, \frac{q_T^2}{\langle q_T^2\rangle^2}\, \frac{e^{-q_T^2/\langle q_T^2\rangle}}{\pi\,\langle q_T^2\rangle}, \\
F_{UT}^{\sin(2\phi+\phi_S)}(x_\pi, x_p, q_T, Q_0^2) &= -\frac{1}{N_c}\sum_a e_a^2 \; h_{1,\pi}^{\perp(1)\bar a}(x_\pi, Q_0)\; h_{1T,p}^{\perp(2)a}(x_p, Q_0)\, 2M_\pi M_p^2\, \frac{q_T^3}{\langle q_T^2\rangle^3}\, \frac{e^{-q_T^2/\langle q_T^2\rangle}}{\pi\,\langle q_T^2\rangle},
\end{aligned}
\tag{5.16}
$$

where the index $a = u,\, \bar u,\, d,\, \bar d,\, \ldots$ and the mean square transverse momenta $\langle q_T^2\rangle$ are defined in each case as the sum of the mean square transverse momenta of the corresponding TMDs, e.g. in (5.16) in first equation $\langle q_T^2\rangle = \langle k_{T\pi}^2\rangle_{f_{1,\pi}} + \langle k_{Tp}^2\rangle_{f_{1,p}}$, in second equation $\langle q_T^2\rangle = \langle k_{T\pi}^2\rangle_{f_{1,\pi}} + \langle k_{Tp}^2\rangle_{f_{1T,p}^{\perp}}$, etc. In these expressions it is evident that the asymmetries decrease if the transverse parton momentum distributions are broadened, which is qualitatively expected at higher energies and is the feature of TMD evolution spelled in Eqs. (5.10)- (5.15), as we shall discuss in more detail below.

6 e^+e^- annihilation process

The first process where TMD factorization has been proven is back-to-back hadron production in e^+e^- annihilation [18],

$$e^+(P_{e^+}) + e^-(P_{e^-}) \to h_1(P_{h_1}) + h_2(P_{h2}) + X\,, \tag{6.1}$$

where $h_{1,2}$ are the identified hadrons with momenta $P_{h1,2}$, and one is inclusive over additional hadronic final states X. Similar to the detected outgoing hadron in SIDIS, see Sec. **??**, these hadrons arise from the fragmentation of quarks in the underlying partonic process, and are described by fragmentation functions characterized by the longitudinal momentum fractions

$$z_{h1} = \frac{2|\mathbf{P}_{h1}|}{Q}\,, \quad z_{h2} = \frac{2|\mathbf{P}_{h2}|}{Q}\,, \tag{6.2}$$

where the center-of-mass energy $S = Q^2 = (P_{e^+} + P_{e^-})^2$ defines the hard scale of the process.

At leading order, the hadrons are produced exactly back to back, which is spoiled at higher orders due to the additional radiation X, which thus gives rise to an imbalance between the hadron momenta. The near back-to-back region is characterized by a small transverse momentum of the dihadron system compared to Q, which is the realm of TMD factorization.

As before, measuring angular distributions of the final-state hadrons can give access to spin correlations in the fragmenting hadrons, most famously in the form of the Collins effect that gives rise to an azimuthal asymmetry of the form $\cos(2\phi)$ [55]. To define the azimuthal angle ϕ, two different reference frames have been proposed in the literature [10]:

1. One defines the thrust axis of the e^+e^- annihilation and measures the relative azimuthal angular correlation between the two hadrons in the two back-to-back jets. In this case, one has to measure two azimuthal angles ϕ_1 and ϕ_2, and the Collins effect manifests itself as a $\cos(\phi_1 + \phi_2)$ asymmetry, and is referred to as the A_{12} asymmetry.

2. One aligns the z axis along one of the detected hadrons, and measures the azimuthal angle ϕ_0 of the other hadron with respect to this axis and the lepton plane, as illustrated in Fig. 3. The Collins effect then appears as a $\cos(2\phi_0)$ asymmetry, and is referred to as the A_0 asymmetry.

The A_{12} asymmetry can not be directly described within TMD factorization, as one needs to define the jet directions, which goes beyond standard TMD factorization. Hence, we will only consider the second approach of the A_0 asymmetry, which is described within TMD factorization in terms of the Collins function.

In the limit of small transverse momentum $P_{h\perp}$, the cross section as predicted by TMD factorization reads [8**?**]

$$\frac{\mathrm{d}^5\sigma^{e^+e^-\to h_1h_2+X}}{\mathrm{d}z_{h1}\mathrm{d}z_{h2}\mathrm{d}^2\mathbf{q}_T\mathrm{d}\cos\theta} = \frac{N_c\pi\alpha_{\mathrm{em}}^2}{2Q^2} z_{h1}^2 z_{h2}^2 \left[\left(1+\cos^2\theta\right)F_{UU} + \sin^2\theta\cos(2\phi_0)F_{UU}^{\cos 2\phi_0}\right]\,. \tag{6.3}$$

As illustrated in Fig. 3, the transverse momentum $\mathbf{P_{h\perp}} = -\mathbf{q}_T z_{h1}$ is defined as the component of P_1 transverse to P_2, its azimuthal angle ϕ_0 is measured relative to the lepton plane, and θ is the polar angle between the hadron h_2 and the e^+e^- beam.

Other structure functions including electro-weak contributions can be found in Ref. [8].

6.1 Structure functions

The structure function F_{UU} is a convolution of unpolarized TMD fragmentation functions for a quark and an anti-quark. The polarized structure function $F_{UU}^{\cos 2\phi_0}$ is a convolution of Collins

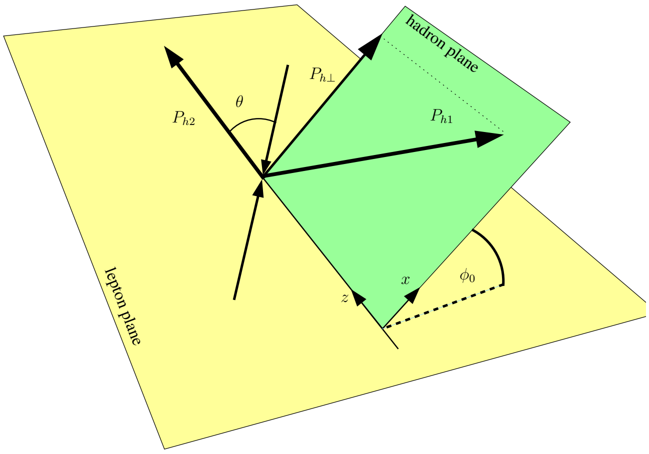

Figure 3. $e^+ + e^- \to h_1 + h_2 + X$ process in the frame of method (2). The figure is from [30].

fragmentation functions for a quark and an anti-quark.

$$\begin{aligned} F_{UU} &= \ \mathcal{C}_{ee}[D_1 \bar{D}_1], \\ F_{UU}^{\cos 2\phi} &= -\mathcal{C}_{ee}\left[\frac{2(\hat{h}\cdot \mathbf{k}_{T1})(\hat{h}\cdot \mathbf{k}_{T2}) - \mathbf{k}_{T1}\cdot \mathbf{k}_{T2}}{z_{h1} z_{h2} M_{h1} M_{h2}} H_1^\perp \bar{H}_1^\perp\right] \end{aligned} \tag{6.4}$$

Here

$$\mathcal{C}_{ee}\left[w(k_{T1}, k_{T2}) D_1\, \bar{D}_2\right] = \sum_q e_q^2 \mathcal{H}^{(e^+e^-)}(Q^2,\mu) \int \frac{d^2\mathbf{k}_{T1}}{z_{h1}^2}\frac{d^2\mathbf{k}_{T2}}{z_{h2}^2}\, \delta^{(2)}\left(-\frac{\mathbf{k}_{T1}}{z_{h1}} - \frac{\mathbf{k}_{T2}}{z_{h2}} - \boldsymbol{q}_T\right) w(k_{T1}, k_{T2}) \\ \times \left[D_1^{h_1/q}(z_{h1}, k_{T1}, \mu, \zeta_1) D_2^{h_2/\bar{q}}(z_{h2}, k_{T2}, \mu, \zeta_2) + D_1^{h_1/\bar{q}}(z_{h1}, k_{T1}, \mu, \zeta_1) D_2^{h_2/q}(z_{h2}, k_{T2}, \mu, \zeta_2)\right]. \tag{6.5}$$

6.2 Structure functions on b_T space

Defining

$$\mathcal{B}[\tilde{D}_1^{(n)}\ \tilde{\bar{D}}_2^{(m)}] \equiv \sum_a e_a^2 \mathcal{H}^{(e^+e^-)}(Q,\mu_Q) \int_0^\infty \frac{db_T\, b_T}{2\pi}\, b_T^{n+m}\, J_{n+m}(q_T b_T) \\ \times \left(\tilde{D}_1^{(n)\bar{a}}(z_{h1}, b_T, \mu, \zeta_1)\ \tilde{D}_2^{(m)a}(z_{h2}, b_T, \mu, \zeta_2) + \tilde{D}_1^{(n)a}(z_{h1}, b_T, \mu, \zeta_1)\ \tilde{D}_2^{(m)\bar{a}}(z_{h2}, b_T, \mu, \zeta_2)\right), \tag{6.6}$$

so that

$$\mathcal{B}[\tilde{D}_1^{(n)}\ \tilde{\bar{D}}_2^{(m)}] \equiv \sum_a e_a^2 \mathcal{H}^{(e^+e^-)}(Q,\mu_Q) \int_0^\infty \frac{db_T\, b_T}{2\pi}\, b_T^{n+m}\, J_{n+m}(q_T b_T) \\ \times \Big(\tilde{D}_1^{(n)\bar{a}}(z_{h1}, b_T, Q_0, Q_0^2)\ \tilde{D}_2^{(m)a}(z_{h2}, b_T, Q_0, Q_0^2) \\ + \tilde{D}_1^{(n)a}(z_{h1}, b_T, Q_0, Q_0^2)\ \tilde{D}_2^{(m)\bar{a}}(z_{h2}, b_T, Q_0, Q_0^2)\Big) e^{-S(b_T, Q, \mu_Q, Q_0)}, \tag{6.7}$$

if the solution from Eq. (3.12) is used and

$$
\begin{aligned}
\mathcal{B}[\tilde{D}_1^{(n)}\ \tilde{\bar{D}}_2^{(m)}] \equiv \sum_a e_a^2 \mathcal{H}^{(e^+e^-)}(Q,\mu_Q) \int_0^\infty \frac{db_T\, b_T}{2\pi}\ b_T^{n+m}\ J_{n+m}(q_T b_T) \\
\times \Big(e^{-g_{D_1}^{\bar{a}}(z_{h1},b_T;b_{\max})} \int_{z_{h1}}^1 \frac{d\hat{z}}{\hat{z}^3} \tilde{C}_{D_1/j}(z_{h1}/\hat{z},\mu_{b_*}) \otimes D_1^{(n)j}(\hat{z},\mu_{b_*}) \\
\times e^{-g_{D_2}^{a}(z_{h1},b_T;b_{\max})} \int_{z_{h2}}^1 \frac{d\hat{z}}{\hat{z}^3} \tilde{C}_{D_2/j}(z_{h2}/\hat{z},\mu_{b_*}) \otimes D_2^{(m)j}(\hat{z},\mu_{b_*}) \\
+ e^{-g_{D_1}^{a}(z_{h1},b_T;b_{\max})} \int_{z_{h1}}^1 \frac{d\hat{z}}{\hat{z}^3} \tilde{C}_{D_1/j}(z_{h1}/\hat{z},\mu_{b_*}) \otimes D_1^{(n)j}(\hat{z},\mu_{b_*}) \\
\times e^{-g_{D_2}^{\bar{a}}(z_{h1},b_T;b_{\max})} \int_{z_{h2}}^1 \frac{d\hat{z}}{\hat{z}^3} \tilde{C}_{D_2/j}(z_{h2}/\hat{z},\mu_{b_*}) \otimes D_2^{(m)j}(\hat{z},\mu_{b_*}) \Big) e^{-S(b_T,Q,\mu_Q,\mu_{b_*})} ,
\end{aligned}
$$

if the solution from Eq. (3.22) is used. One obtains the following compact formulas

$$
F_{UU}(z_{h1},z_{h2},q_T,Q^2) = \quad \mathcal{B}[\tilde{D}_1^{(0)}\ \tilde{\bar{D}}_1^{(0)}] , \tag{6.8}
$$

$$
F_{UU}^{\cos 2\phi_0}(z_{h1},z_{h2},q_T,Q^2) = -M_{h1}\, M_{h2}\, \mathcal{B}[\tilde{H}_1^{\perp(1)}\ \tilde{\bar{H}}_1^{\perp(1)}] , \tag{6.9}
$$

6.3 Structure functions at the initial scale

Using Eqs. (2.29) or Eqs. (2.30) one obtains for the convolution integrals in Eqs. (6.4) or Eqs. (6.9) the following results

$$
\begin{aligned}
F_{UU}(z_{h1},z_{h2},q_T,Q_0^2) &= \quad \sum_a e_a^2\ \left(D_1^{\bar{a}}(z_{h1},Q_0)\ D_1^{a}(z_{h2},Q_0) + D_1^{a}(z_{h1},Q_0)\ D_1^{\bar{a}}(z_{h2},Q_0)\right)\ \frac{e^{-q_T^2 z_{h1}^2 z_{h2}^2/\langle q_T^2\rangle}}{\pi\ \langle q_T^2\rangle} , \\
F_{UU}^{\cos 2\phi_0}(z_{h1},z_{h2},,q_T,Q_0^2) &= -\sum_a e_a^2\ \left(H_1^{\perp(1)\bar{a}}(z_{h1},Q_0)\ H_1^{\perp(1)a}(z_{h2},Q_0) + H_1^{\perp(1)a}(z_{h1},Q_0)\ H_1^{\perp(1)\bar{a}}(z_{h2},Q_0)\right) \\
&\quad \times 4M_{h1}\, M_{h2}\, z_{h1}^4\, z_{h2}^4\, q_T^2\ \frac{e^{-q_T^2 z_{h1}^2 z_{h2}^2/\langle q_T^2\rangle}}{\pi\ \langle q_T^2\rangle^3}
\end{aligned} \tag{6.10}
$$

where the index $a = u,\, \bar{u},\, d,\, \bar{d},\, \ldots$ and the mean square transverse momenta $\langle q_T^2\rangle$ are defined in each case as the sum of the mean square transverse momenta of the corresponding TMDs, in (6.10): in the first equation $\langle q_T^2\rangle = \langle p_T^2\rangle_{D_1(z_{h1})} z_{h2}^2 + \langle p_T^2\rangle_{D_1(z_{h2})} z_{h1}^2$ and in the second equation $\langle q_T^2\rangle = \langle p_T^2\rangle_{H_1^\perp(z_{h1})} z_{h2}^2 + \langle p_T^2\rangle_{H_1^\perp(z_{h2})} z_{h1}^2$.

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A Notation for convolution integrals

Structure functions are expressed as convolutions of TMDs and FFs in the Bjorken limit at tree level. For reference we quote the convolution integrals in "Amsterdam notation" [4]

$$\mathcal{C}[w\, f\, D] = x \sum_a e_a^2 \int d^2\boldsymbol{p}_T\, d^2\boldsymbol{k}_T\, \delta^{(2)}(\boldsymbol{p}_T - \boldsymbol{k}_T - \boldsymbol{P}_{h\perp}/z)\, w(\boldsymbol{p}_T, \boldsymbol{k}_T)\, f^a(x, p_T^2)\, D^a(z, z^2 k_T^2), \quad \text{(A.1)}$$

where all transverse momenta refer to the virtual photon-proton center-of-mass frame and $\hat{\boldsymbol{h}} = \boldsymbol{P}_{h\perp}/P_{h\perp}$. Hereby $\boldsymbol{p}_T$ is the transverse momentum of quark with respect to nucleon, $\boldsymbol{k}_T$ is the transverse momentum of the fragmenting quark with respect to produced hadron. The notation is not unique. The one chosen in this work, in comparison to other works, is

$$\begin{aligned}
\text{transverse momentum in TMD:}\quad & [\boldsymbol{k}_T]_{\text{our}} = [\boldsymbol{k}_\perp]_{\text{Ref.[56]}} = \quad [\boldsymbol{p}_T]_{\text{Ref.[4]}}\,, && \text{(A.2)}\\
\text{transverse momentum in FF:}\quad & [\boldsymbol{p}_T]_{\text{our}} = [\boldsymbol{p}_\perp]_{\text{Ref.[56]}} = -z\,[\boldsymbol{k}_T]_{\text{Ref.[4]}}\,, && \text{(A.3)}\\
\text{transverse hadron momenta:}\quad & [\boldsymbol{P}_{hT}]_{\text{our}} = [\boldsymbol{P}_T]_{\text{Ref.[56]}} = \quad [\boldsymbol{P}_{h\perp}]_{\text{Ref.[4]}}\,. && \text{(A.4)}
\end{aligned}$$

Notice that $[\boldsymbol{p}_T]_{\text{our}} = -z\,[\boldsymbol{k}_T]_{\text{Ref.[4]}}$ is the transverse momentum the hadron acquires in the fragmentation process. The normalization for unpolarized fragmentation functions is

$$D_1^a(z) = \left[\int d^2\boldsymbol{p}_T D_1^a(z, \boldsymbol{p}_T^2)\right]_{\text{our}} = \left[z^2 \int d^2\boldsymbol{k}_T D_1^a(z, z^2 k_T^2)\right]_{\text{Ref. [4]}}\,. \quad \text{(A.5)}$$

The "Amsterdam" convolution integral (A.1) reads in our notation

$$\mathcal{C}[w\, f\, D] = x \sum_a e_a^2 \int d^2\boldsymbol{k}_T\, d^2\boldsymbol{p}_T\, \delta^{(2)}(z\boldsymbol{k}_T + \boldsymbol{p}_T - \boldsymbol{P}_{hT})\, w\left(\boldsymbol{k}_T, -\frac{\boldsymbol{p}_T}{z}\right) f^a(x, k_T^2)\, D^a(z, p_T^2). \quad \text{(A.6)}$$

B Fourier transforms WORK ON IT!

Here, we collect some useful definitions and identities for Fourier transforms in transverse space. Our convention for the Fourier transform and its inverse are

$$\tilde{f}(\mathbf{b}_T) = \int \mathrm{d}^2\mathbf{k}_T\, e^{-i\mathbf{b}_T\cdot\mathbf{k}_T} f(\mathbf{k}_T)\,, \qquad f(\mathbf{k}_T) = \int \frac{\mathrm{d}^2\mathbf{b}_T}{(2\pi)^2} e^{+i\mathbf{b}_T\cdot\mathbf{k}_T} \tilde{f}(\mathbf{b}_T)\,, \quad \text{(B.1)}$$

where $\tilde{f}(\mathbf{b}_T)$ is the function in Fourier or position space, and $f(\mathbf{k}_T)$ is the function in momentum space. If $f(\mathbf{k}_T)$ is independent of the azimuthal angle, i.e. $f(\mathbf{k}_T) \equiv f(|\mathbf{k}_T|)$, then one can use the identity

$$\tilde{f}(b_T) = \int_0^\infty \mathrm{d}b_T\, b_T \int_0^{2\pi} \mathrm{d}\phi\, e^{ib_T k_T \cos\phi} f(p_T) = 2\pi \int_0^\infty \mathrm{d}b_T\, b_T J_0(b_T k_T) f(k_T)\,, \quad \text{(B.2)}$$

where $J_0(x)$ is the 0-th order Bessel function of the first kind. In this case, $\tilde{f}(\mathbf{b}_T) \equiv \tilde{f}(b_T)$ is independent of the azimuthal angle as well, which yields the corresponding identity for the inverse transform

$$f(k_T) = \frac{1}{(2\pi)^2} \int_0^\infty \mathrm{d}b_T\, b_T \int_0^{2\pi} \mathrm{d}\phi\, e^{-ib_T k_T \cos\phi} \tilde{f}(b_T) = \frac{1}{2\pi} \int_0^\infty \mathrm{d}b_T\, b_T J_0(b_T k_T) \tilde{f}(b_T)\,. \quad \text{(B.3)}$$

From Eqs. (B.2) and (B.3), it is clear that the Fourier transform $\tilde{f}(b_T)$ of a real function $f(k_T)$ is real, and likewise for the inverse Fourier transform.

A key feature of the Fourier transform is that it turns convolutions in momentum space into simple products,

$$\int \mathrm{d}^2\mathbf{k}_1 \mathrm{d}^2\mathbf{k}_2\, \delta^{(2)}(\mathbf{k}_T - \mathbf{k}_1 - \mathbf{k}_2) f(\mathbf{k}_1) g(\mathbf{k}_2) = \int \frac{\mathrm{d}^2\mathbf{b}_T}{(2\pi)^2} e^{i\mathbf{b}_T\cdot\mathbf{k}_T} \tilde{f}(\mathbf{b}_T)\tilde{g}(\mathbf{b}_T)\,, \tag{B.4}$$

which can be easily seen by inserting Eq. (B.1) together with the distributional identity

$$\delta^{(2)}(\mathbf{k}_T - \mathbf{k}_1 - \mathbf{k}_2) = \int \frac{\mathrm{d}^2\mathbf{b}_T}{(2\pi)^2} e^{i\mathbf{b}_T\cdot(\mathbf{k}_T - \mathbf{k}_1 - \mathbf{k}_2)}\,. \tag{B.5}$$

In Sec. **??**, we also need Fourier transforms of functions of the form $p_T^\mu f(p_T)$, which can be obtained as

$$\int \mathrm{d}^2\mathbf{k}_T e^{-i\mathbf{k}_T\cdot\mathbf{b}_T}\, (k_T^\mu \cdots k_T^\nu) f(p_T) = \Big(-i\frac{\partial}{\partial b_{T\mu}}\Big)\cdots\Big(-i\frac{\partial}{\partial b_{T\nu}}\Big) \int \mathrm{d}^2\mathbf{k}_T e^{-i\mathbf{k}_T\cdot\mathbf{b}_T} f(k_T) \tag{B.6}$$

$$= (-i\partial^\mu)\cdots(-i\partial^\nu)\tilde{f}(b_T) \tag{B.7}$$

$$= (-i\partial^\mu)\cdots(-i\partial^\nu)\, 2\pi \int_0^\infty \mathrm{d}k_T\, k_T J_0(b_T k_T) f(k_T)\,. \tag{B.8}$$

By acting with $\partial^\mu \equiv \partial/\partial b_{T\mu}$ on the exponential phase, one induces the desired tensor structure $k_T^\mu \ldots k_T^\nu$ in the Fourier integral. (Recall that $\mathbf{k}_T\cdot\mathbf{b}_T = -k_T^\mu b_{T\mu}$, which fixes the sign of the derivative factors.) Thus, we can conveniently express this Fourier transform as derivatives acting on the Fourier transform $\tilde{f}(b_T)$, which in the last line was expressed using Eq. (B.2). Using Eq. (B.6) together with the Bessel function identity

$$\frac{\mathrm{d}}{\mathrm{d}z} z^{-m} J_m(z) = -z^{-m} J_{m+1}(z)\,, \tag{B.9}$$

we easily obtain the explicit results

$$\int \mathrm{d}^2\mathbf{p}_T\, e^{i\mathbf{k}_T\cdot\mathbf{b}_T}\, \frac{k_T^\mu}{k_T} f(k_T) = (-i)\frac{b_T^\mu}{b_T} \times 2\pi \int_0^\infty \mathrm{d}k_T\, k_T\, J_1(b_T k_T) f(k_T)\,, \tag{B.10}$$

$$\int \mathrm{d}^2\mathbf{k}_T\, e^{i\mathbf{k}_T\cdot\mathbf{b}_T} \left(\frac{g_T^{\mu\nu}}{2} + \frac{k_T^\mu k_T^\nu}{\mathbf{k}_T^2}\right) f(k_T) = (-i)^2 \left(\frac{g_T^{\mu\nu}}{2} + \frac{b_T^\mu b_T^\nu}{\mathbf{b}_T^2}\right) \times 2\pi \int_0^\infty \mathrm{d}k_T\, p_T\, J_2(b_T k_T)\, f(k_T)\,. \tag{B.11}$$

The integrals over k_T have the same structure as in Eq. (B.2), up to exchanging $J_0(x)$ by $J_1(x)$ and $J_2(x)$, respectively. From Eq. (B.10), we easily obtain the relations

$$\int \mathrm{d}^2\mathbf{k}_T\, e^{i\mathbf{k}_T\cdot\mathbf{b}_T}\, \frac{k_T^\mu}{M} f(k_T) = (-i) b_T^\mu M \tilde{f}^{(1)}(b_T)\,, \tag{B.12}$$

$$\int \mathrm{d}^2\mathbf{k}_T\, e^{i\mathbf{k}_T\cdot\mathbf{b}_T}\, \frac{\mathbf{k}_T^2}{M^2} \left(\frac{g_T^{\mu\nu}}{2} + \frac{k_T^\mu k_T^\nu}{\mathbf{p}_T^2}\right) f(k_T) = \frac{(-i)^2}{2} b_T^2 M^2 \left(\frac{g_T^{\mu\nu}}{2} + \frac{b_T^\mu b_T^\nu}{\mathbf{b}_T^2}\right) \tilde{f}^{(2)}(b_T)\,, \tag{B.13}$$

where the $\tilde{f}^{(n)}$ denote derivatives with respect to b_T as defined in Eq. (**??**),

$$\tilde{f}^{(n)}(b_T) \equiv n! \left(\frac{-1}{M^2 b_T}\partial_{b_T}\right)^n \tilde{f}(b_T) = \frac{2\pi\, n!}{(b_T M)^n} \int_0^\infty \mathrm{d}k_T\, k_T \left(\frac{k_T}{M}\right)^n J_n(b_T k_T)\, f(k_T)\,. \tag{B.14}$$

The equality in the second step follows directly from Eq. (B.9). The factor of $n!$ arises from following the convention of [12]. Also note that the Eq. (B.14) is manifestly real if $f(p_T)$ is real, and hence the explicit factors of i have been extracted in Eq. (B.12).

Eq. (B.14) can be inverted using the the orthogonality relation of Bessel functions,

$$\int_0^\infty \mathrm{d}b_T\, b_T\, J_n(k_T b_T)\, J_n(p'_T b_T) = \frac{1}{k_T}\delta(k_T - k'_T)\,, \tag{B.15}$$

from which one easily finds that

$$f(k_T) = \frac{M^{2n}}{2\pi\, n!}\int_0^\infty \mathrm{d}b_T\, b_T \left(\frac{b_T}{p_T}\right)^n J_n(b_T k_T)\, \tilde{f}^{(n)}(b_T)\,. \tag{B.16}$$

C Anomalous dimensions

The value to three-loop order was found by Moch, Vermaseren, and Vogt [24]; they compute a quantity they call A, which is our $\gamma_K/2$ — see their Eq. (2.4). Their value was recently confirmed by Grozin et al. [27]. Using $a_s(\mu) \equiv \alpha_s(\mu)/(4\pi)$ Ref. [28] has Eqs.(58,59) , we rewrite it using α_s:

$$\begin{aligned}
\gamma_j = 6C_F\left(\frac{\alpha_s(\mu)}{4\pi}\right) &+ \left(\frac{\alpha_s(\mu)}{4\pi}\right)^2\left[C_F^2\left(3 - 4\pi^2 + 48\zeta_3\right) + C_F C_A\left(\frac{961}{27} + \frac{11\pi^2}{3} - 52\zeta_3\right) + C_F n_f\left(-\frac{130}{27} - \frac{2\pi^2}{3}\right)\right] \\
&+ \left(\frac{\alpha_s(\mu)}{4\pi}\right)^3\left[C_F^2 n_f\left(-\frac{2953}{27} + \frac{26\pi^2}{9} + \frac{28\pi^4}{27} - \frac{512\zeta_3}{9}\right) + C_F n_f^2\left(-\frac{4834}{729} + \frac{20\pi^2}{27} + \frac{16\zeta_3}{27}\right)\right. \\
&\quad + C_F^3\left(29 + 6\pi^2 + \frac{16\pi^4}{5} + 136\zeta_3 - \frac{32\pi^2\zeta_3}{3} - 480\zeta_5\right) \\
&\quad + C_A^2 C_F\left(\frac{139345}{1458} + \frac{7163\pi^2}{243} + \frac{83\pi^4}{45} - \frac{7052\zeta_3}{9} + \frac{88\pi^2\zeta_3}{9} + 272\zeta_5\right) \\
&\quad + C_A C_F n_f\left(\frac{17318}{729} - \frac{2594\pi^2}{243} - \frac{22\pi^4}{45} + \frac{1928\zeta_3}{27}\right) \\
&\quad \left. + C_A C_F^2\left(\frac{151}{2} - \frac{410\pi^2}{9} - \frac{494\pi^4}{135} + \frac{1688\zeta_3}{3} + \frac{16\pi^2\zeta_3}{3} + 240\zeta_5\right)\right] \\
&+ \mathcal{O}\left(\left(\frac{\alpha_s(\mu)}{4\pi}\right)^4\right),
\end{aligned} \tag{C.1}$$

$$\begin{aligned}
\gamma_K = 8C_F\left(\frac{\alpha_s(\mu)}{4\pi}\right) &+ \left(\frac{\alpha_s(\mu)}{4\pi}\right)^2\left[C_A C_F\left(\frac{536}{9} - \frac{8\pi^2}{3}\right) - \frac{80}{9}C_F n_f\right] \\
&+ \left(\frac{\alpha_s(\mu)}{4\pi}\right)^3\left[-\frac{32}{27}C_F n_f^2 + C_A C_F n_f\left(-\frac{1672}{27} + \frac{160\pi^2}{27} - \frac{224\zeta_3}{3}\right) + C_A^2 C_F\left(\frac{980}{3} - \frac{1072\pi^2}{27} + \frac{88\pi^4}{45} + \frac{176\zeta_3}{3}\right)\right. \\
&\quad \left. + C_F^2 n_f\left(-\frac{220}{3} + 64\zeta_3\right)\right] \\
&+ \mathcal{O}\left(\left(\frac{\alpha_s(\mu)}{4\pi}\right)^4\right).
\end{aligned} \tag{C.2}$$

D Collins-Soper kernel $\tilde{K}$

Ref. [28] has Eq. (69) which is derived utilizing the results up to order a_s^2 for $B_{\mathrm{CSS1,\,DY}}(a_s; 2e^{-\gamma_E}, 1)$ from Ref. [57], and reads

$$\begin{aligned}\tilde{K}(b_T;\mu) = &-8C_F\left(\frac{\alpha_s(\mu)}{4\pi}\right)\ln\left(\frac{b_T\mu}{2e^{-\gamma_E}}\right)\\ &+8C_F\left(\frac{\alpha_s(\mu)}{4\pi}\right)^2\left[\left(\frac{2}{3}n_f-\frac{11}{3}C_A\right)\ln^2\left(\frac{b_T\mu}{2e^{-\gamma_E}}\right)\right.\\ &\qquad\left.+\left(-\frac{67}{9}C_A+\frac{\pi^2}{3}C_A+\frac{10}{9}n_f\right)\ln\left(\frac{b_T\mu}{2e^{-\gamma_E}}\right)+\left(\frac{7}{2}\zeta_3-\frac{101}{27}\right)C_A+\frac{14}{27}n_f\right]\\ &+\mathcal{O}\left(\left(\frac{\alpha_s(\mu)}{4\pi}\right)^3\right). \end{aligned} \tag{D.1}$$

E Hard factors

From Ref. [28]: For DY we find:

$$
\begin{aligned}
\mathcal{H}^{\mathrm{DY}}_{j\bar{j}}(Q,\mu;a_s(\mu)) = 1 &+ C_F a_s\left(-16+\frac{7\pi^2}{3}+6T-2T^2\right)\\
&+ a_s^2\Bigg\{C_F^2\left[\frac{511}{4}-\frac{83\pi^2}{3}+\frac{67\pi^4}{30}-60\zeta_3+T\left(-93+10\pi^2+48\zeta_3\right)+T^2\left(-\frac{14\pi^2}{3}+50\right)-12T^3+2T^4\right]\\
&\quad + C_F C_A\left[-\frac{51157}{324}+\frac{1061\pi^2}{54}-\frac{8\pi^4}{45}+\frac{626}{9}\zeta_3+T\left(\frac{2545}{27}-\frac{44\pi^2}{6}-52\zeta_3\right)+T^2\left(\frac{2\pi^2}{3}-\frac{233}{9}\right)+\frac{22}{9}T^3\right]\\
&\quad + n_f C_F\left[\frac{4085}{162}-\frac{91\pi^2}{27}+\frac{4}{9}\zeta_3+T\left(\frac{8\pi^2}{9}-\frac{418}{27}\right)+\frac{38}{9}T^2-\frac{4}{9}T^3\right]\Bigg\}\\
&+ a_s^3\Bigg\{C_F^3\Bigg[32\zeta_3^2-\frac{140\pi^2\zeta_3}{3}-460\zeta_3+1328\zeta_5+\frac{27403\pi^6}{17010}-\frac{346\pi^4}{15}+\frac{4339\pi^2}{36}-\frac{5599}{6}\\
&\qquad + T\left(\frac{304\pi^2\zeta_3}{3}-992\zeta_3-480\zeta_5+\frac{109\pi^4}{15}-89\pi^2+\frac{1495}{2}\right)+T^2\left(408\zeta_3-\frac{67\pi^4}{15}+\frac{220\pi^2}{3}-\frac{1051}{2}\right)\\
&\qquad + T^3\left(-96\zeta_3-20\pi^2+222\right)+T^4\left(\frac{14\pi^2}{3}-68\right)+12T^5-\frac{4T^6}{3}\Bigg]\\
&\quad + C_A C_F^2\Bigg[\frac{592\zeta_3^2}{3}+\frac{1690\pi^2\zeta_3}{9}-\frac{52564\zeta_3}{27}-\frac{5512\zeta_5}{9}-\frac{1478\pi^6}{1701}+\frac{92237\pi^4}{2430}-\frac{406507\pi^2}{972}+\frac{824281}{324}\\
&\qquad\quad + T\left(-116\pi^2\zeta_3+2252\zeta_3+240\zeta_5-\frac{1694\pi^4}{135}+\frac{24268\pi^2}{81}-\frac{14269}{6}\right)\\
&\qquad\quad + T^2\left(-\frac{5644\zeta_3}{9}+\frac{86\pi^4}{45}-\frac{3376\pi^2}{27}+\frac{208099}{162}\right)\\
&\qquad\quad + T^3\left(104\zeta_3+\frac{526\pi^2}{27}-\frac{10340}{27}\right)+T^4\left(\frac{598}{9}-\frac{4\pi^2}{3}\right)-\frac{44T^5}{9}\Bigg]\\
&\quad + C_A^2 C_F\Bigg[-\frac{2272\zeta_3^2}{9}-\frac{1168\pi^2\zeta_3}{9}+\frac{505087\zeta_3}{243}-\frac{868\zeta_5}{9}+\frac{4784\pi^6}{25515}-\frac{4303\pi^4}{4860}+\frac{596513\pi^2}{2187}-\frac{51082685}{26244}\\
&\qquad\quad + T\left(\frac{88\pi^2\zeta_3}{9}-\frac{34928\zeta_3}{27}+272\zeta_5+\frac{85\pi^4}{27}-\frac{34276\pi^2}{243}+\frac{1045955}{729}\right)\\
&\qquad\quad + T^2\left(176\zeta_3-\frac{22\pi^4}{45}+\frac{752\pi^2}{27}-\frac{37364}{81}\right)+T^3\left(\frac{5738}{81}-\frac{44\pi^2}{27}\right)-\frac{121T^4}{27}\Bigg]\\
&\quad C_A C_F n_f\Bigg[\frac{148\pi^2\zeta_3}{9}-\frac{8576\zeta_3}{27}-\frac{8\zeta_5}{3}-\frac{35\pi^4}{243}-\frac{201749\pi^2}{2187}+\frac{3400342}{6561}\\
&\qquad\quad + T\left(\frac{1448\zeta_3}{9}-\frac{98\pi^4}{135}+\frac{11668\pi^2}{243}-\frac{309838}{729}\right)\\
&\qquad\quad + T^2\left(-16\zeta_3-8\pi^2+\frac{11752}{81}\right)+T^3\left(\frac{8\pi^2}{27}-\frac{1948}{81}\right)+\frac{44T^4}{27}\Bigg]\\
&\quad + C_F n_f^2\Bigg[-\frac{832\zeta_3}{243}+\frac{86\pi^4}{1215}+\frac{1612\pi^2}{243}-\frac{190931}{6561}+T\left(\frac{32\zeta_3}{27}-\frac{304\pi^2}{81}+\frac{19676}{729}\right)\\
&\qquad\quad + T^2\left(\frac{16\pi^2}{27}-\frac{812}{81}\right)+\frac{152T^3}{81}-\frac{4T^4}{27}\Bigg]\\
&\quad + C_F^2 n_f\Bigg[-\frac{148}{9}\pi^2\zeta_3+\frac{26080\zeta_3}{81}-\frac{832\zeta_5}{9}-\frac{1463\pi^4}{243}+\frac{13705\pi^2}{243}-\frac{56963}{486}\\
&\qquad\quad + T\left(-\frac{1208\zeta_3}{9}+\frac{332\pi^4}{135}-\frac{4060\pi^2}{81}+\frac{6947}{27}\right)+T^2\left(\frac{136\zeta_3}{9}+\frac{520\pi^2}{27}-\frac{14948}{81}\right)\\
&\qquad\quad + T^3\left(\frac{1676}{27}-\frac{76\pi^2}{27}\right)-\frac{100T^4}{9}+\frac{8T^5}{9}\Bigg]\\
&\qquad\quad \Big[\,28\zeta_3 N \quad 160\zeta_5 N \quad 112\zeta_3 \quad 640\zeta_5 \quad \pi^4 N \quad 10\pi^2 N \quad 4\pi^4 \quad 40\pi^2 \quad 32\,\Big]\Big)
\end{aligned}
$$

For SIDIS, we get

$$
\begin{aligned}
\mathcal{H}^{\mathrm{SIDIS}}_{j\bar{\jmath}}(Q,\mu;a_s(\mu)) &= 1 + C_F a_s\left(-16+\frac{\pi^2}{3}+6T-2T^2\right)\\
&+ a_s^2\Bigg\{ C_F^2\left[\frac{511}{4}+\frac{13\pi^2}{3}-\frac{13\pi^4}{30}-60\zeta_3+T(-93-2\pi^2+48\zeta_3)+T^2\left(-\frac{2\pi^2}{3}+50\right)-12T^3+2T^4\right]+\\
&\quad + C_A C_F\left[-\frac{51157}{324}-\frac{337\pi^2}{54}+\frac{22\pi^4}{45}+\frac{626\zeta_3}{9}+T\left(\frac{2545}{27}+\frac{22\pi^2}{9}-52\zeta_3\right)+T^2\left(\frac{2\pi^2}{3}-\frac{233}{9}\right)+\frac{22T^3}{9}\right]\\
&\quad + C_F n_f\left[\frac{4085}{162}+\frac{23\pi^2}{27}+\frac{4\zeta_3}{9}-T\left(\frac{418}{27}+\frac{4\pi^2}{9}\right)+\frac{38T^2}{9}-\frac{4T^3}{9}\right]\Bigg\}\\
&+ a_s^3\Bigg\{ C_F^3\Bigg[32\zeta_3^2+\frac{220\pi^2\zeta_3}{3}-460\zeta_3+1328\zeta_5+\frac{1625\pi^6}{3402}+\frac{4\pi^4}{15}-\frac{4859\pi^2}{36}-\frac{5599}{6}\\
&\qquad + T\left(\frac{16\pi^2\zeta_3}{3}-992\zeta_3-480\zeta_5-\frac{11\pi^4}{15}+97\pi^2+\frac{1495}{2}\right)+T^2\left(408\zeta_3+\frac{13\pi^4}{15}-\frac{80\pi^2}{3}-\frac{1051}{2}\right)\\
&\qquad + T^3\left(-96\zeta_3+4\pi^2+222\right)+T^4\left(\frac{2\pi^2}{3}-68\right)+12T^5-\frac{4T^6}{3}\Bigg]\\
&\quad + C_A C_F^2\Bigg[\frac{592\zeta_3^2}{3}-\frac{382\pi^2\zeta_3}{3}-\frac{52564\zeta_3}{27}-\frac{5512\zeta_5}{9}-\frac{2476\pi^6}{8505}-\frac{14503\pi^4}{2430}+\frac{292367\pi^2}{972}+\frac{824281}{324}\\
&\qquad + T\left(-12\pi^2\zeta_3+2252\zeta_3+240\zeta_5+\frac{496\pi^4}{135}-\frac{13088\pi^2}{81}-\frac{14269}{6}\right)\\
&\qquad + T^2\left(-\frac{5644\zeta_3}{9}-\frac{34\pi^4}{45}+\frac{608\pi^2}{27}+\frac{208099}{162}\right)+T^3\left(104\zeta_3-\frac{2\pi^2}{27}-\frac{10340}{27}\right)\\
&\qquad + T^4\left(\frac{598}{9}-\frac{4\pi^2}{3}\right)-\frac{44T^5}{9}\Bigg]\\
&\quad + C_A^2 C_F\Bigg[-\frac{2272\zeta_3^2}{9}+\frac{416\pi^2\zeta_3}{9}+\frac{505087\zeta_3}{243}-\frac{868\zeta_5}{9}-\frac{1538\pi^6}{5103}+\frac{22157\pi^4}{4860}-\frac{412315\pi^2}{2187}-\frac{51082685}{26244}\\
&\qquad + T\left(\frac{88\pi^2\zeta_3}{9}-\frac{34928\zeta_3}{27}+272\zeta_5-\frac{47\pi^4}{27}+\frac{17366\pi^2}{243}+\frac{1045955}{729}\right)\\
&\qquad + T^2\left(176\zeta_3-\frac{22\pi^4}{45}+\frac{26\pi^2}{27}-\frac{37364}{81}\right)+T^3\left(\frac{5738}{81}-\frac{44\pi^2}{27}\right)-\frac{121T^4}{27}\Bigg]\\
&\quad C_A C_F n_f\Bigg[\frac{4\pi^2\zeta_3}{9}-\frac{8576\zeta_3}{27}-\frac{8\zeta_5}{3}+\frac{\pi^4}{243}+\frac{115555\pi^2}{2187}+\frac{3400342}{6561}\\
&\qquad + T\left(\frac{1448\zeta_3}{9}+\frac{22\pi^4}{135}-\frac{5864\pi^2}{243}-\frac{309838}{729}\right)+T^2\left(-16\zeta_3+\frac{16\pi^2}{9}+\frac{11752}{81}\right)\\
&\qquad + T^3\left(\frac{8\pi^2}{27}-\frac{1948}{81}\right)+\frac{44T^4}{27}\Bigg]\\
&\quad + C_F n_f^2\Bigg[-\frac{832\zeta_3}{243}-\frac{94\pi^4}{1215}-\frac{824\pi^2}{243}-\frac{190931}{6561}+T\left(\frac{32\zeta_3}{27}+\frac{152\pi^2}{81}+\frac{19676}{729}\right)\\
&\qquad + T^2\left(-\frac{812}{81}-\frac{8\pi^2}{27}\right)+\frac{152T^3}{81}-\frac{4T^4}{27}\Bigg]\\
&\quad + C_F^2 n_f\Bigg[-\frac{4}{3}\pi^2\zeta_3+\frac{26080\zeta_3}{81}-\frac{832\zeta_5}{9}-\frac{131\pi^4}{243}-\frac{8567\pi^2}{243}-\frac{56963}{486}\\
&\qquad + T\left(-\frac{1208\zeta_3}{9}+\frac{32\pi^4}{135}+\frac{1904\pi^2}{81}+\frac{6947}{27}\right)\\
&\qquad + T^2\left(\frac{136\zeta_3}{9}-\frac{152\pi^2}{27}-\frac{14948}{81}\right)+T^3\left(\frac{1676}{27}+\frac{20\pi^2}{27}\right)-\frac{100T^4}{9}+\frac{8T^5}{9}\Bigg]\\
&\quad + C_F N_{j,v}\left[\frac{28\zeta_3 N}{3}-\frac{160\zeta_5 N}{3}-\frac{112\zeta_3}{3N}+\frac{640\zeta_5}{3N}-\frac{\pi^4 N}{45}+\frac{10\pi^2 N}{3}+8N+\frac{4\pi^4}{45N}-\frac{40\pi^2}{3N}-\frac{32}{N}\right]\Bigg\}+O(a_s^4)\,.
\end{aligned}
$$

The quantity $N_{j,v}$ is defined as

$$N_{j,v} \equiv \frac{\sum_q e_q}{e_j} \,. \tag{E.3}$$

and is needed for graphs first encountered at a_s^3 where the quark line at the electromagnetic current is in an internal loop instead of being connected to the external lines.

In both of these equations, $T = \ln(Q^2/\mu^2)$, and $N_{j,v}$ is defined by Eq. (E.3). With $n_f = 3$, the ratio of the Drell-Yan to SIDIS hard factors is

$$\frac{H_{j\bar{j}}^{\rm DY}}{H_{j\bar{j}}^{\rm SIDIS}} = 1 + 2.0944\,\alpha_s(\mu) + 5.96498\,\alpha_s(\mu)^2 + 18.6104\,\alpha_s(\mu)^3 + O(\alpha_s^4)\,, \tag{E.4}$$

and we have verified that we match Eq. (4.4) of Ref. [24] for $n_f = 4$.

In our later calculations, we will need the coefficients of the Drell-Yan hard factor at $T = 0$, i.e., with $\mu = Q$ or $C_2 = 1$. So we write

$$\mathcal{H}_{j\bar{j}}^{\rm DY}(Q, Q; a_s(Q)) = 1 + \sum_{n=1}^{\infty} a_s^n \hat{H}_{j\bar{j}}^{{\rm DY}\,(n)}, \tag{E.5}$$

and we have

$$\hat{H}^{\text{DY}\,(1)}_{j\bar{j}} = C_F\left(-16+\frac{7\pi^2}{3}\right), \tag{E.6a}$$

$$\begin{aligned}\hat{H}^{\text{DY}\,(2)}_{j\bar{j}} &= C_F^2\left[\frac{511}{4}-\frac{83\pi^2}{3}+\frac{67\pi^4}{30}-60\zeta_3\right]+C_FC_A\left[-\frac{51157}{324}+\frac{1061\pi^2}{54}-\frac{8\pi^4}{45}+\frac{626}{9}\zeta_3\right]\\ &\quad+n_fC_F\left[\frac{4085}{162}-\frac{91\pi^2}{27}+\frac{4}{9}\zeta_3\right],\end{aligned} \tag{E.6b}$$

$$\begin{aligned}\hat{H}^{\text{DY}\,(3)}_{j\bar{j}} &= C_F^3\left[32\zeta_3^2+\frac{220\pi^2\zeta_3}{3}-460\zeta_3+1328\zeta_5+\frac{1625\pi^6}{3402}+\frac{4\pi^4}{15}-\frac{4859\pi^2}{36}-\frac{5599}{6}\right]\\
&\quad+C_AC_F^2\left[\frac{592\zeta_3^2}{3}-\frac{382\pi^2\zeta_3}{3}-\frac{52564\zeta_3}{27}-\frac{5512\zeta_5}{9}-\frac{2476\pi^6}{8505}-\frac{14503\pi^4}{2430}\right.\\
&\qquad\left.+\frac{292367\pi^2}{972}+\frac{824281}{324}\right]\\
&\quad+C_A^2C_F\left[-\frac{2272\zeta_3^2}{9}+\frac{416\pi^2\zeta_3}{9}+\frac{505087\zeta_3}{243}-\frac{868\zeta_5}{9}-\frac{1538\pi^6}{5103}+\frac{22157\pi^4}{4860}\right.\\
&\qquad\left.-\frac{412315\pi^2}{2187}-\frac{51082685}{26244}\right]\\
&\quad C_AC_Fn_f\left[\frac{4\pi^2\zeta_3}{9}-\frac{8576\zeta_3}{27}-\frac{8\zeta_5}{3}+\frac{\pi^4}{243}+\frac{115555\pi^2}{2187}+\frac{3400342}{6561}\right]\\
&\quad+C_Fn_f^2\left[-\frac{832\zeta_3}{243}-\frac{94\pi^4}{1215}-\frac{824\pi^2}{243}-\frac{190931}{6561}\right]\\
&\quad+C_F^2n_f\left[-\frac{4}{3}\pi^2\zeta_3+\frac{26080\zeta_3}{81}-\frac{832\zeta_5}{9}-\frac{131\pi^4}{243}-\frac{8567\pi^2}{243}-\frac{56963}{486}\right]\\
&\quad+C_FN_{j,v}\left[\frac{28\zeta_3N}{3}-\frac{160\zeta_5N}{3}-\frac{112\zeta_3}{3N}+\frac{640\zeta_5}{3N}-\frac{\pi^4N}{45}\right.\\
&\qquad\left.+\frac{10\pi^2N}{3}+8N+\frac{4\pi^4}{45N}-\frac{40\pi^2}{3N}-\frac{32}{N}\right].\end{aligned} \tag{E.6c}$$

F Wilson coefficients

From Ref. [28]: **TODO I still need to figure it out...**

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