SIDIS in b_T-space

Definitions of Fourier transform

1) $$f(x,k_T^2) = \int \frac{d^2 b_T}{(2\pi)^2} e^{i \bar{k}_T \bar{b}_T} \tilde{f}(x,b_T^2) =$$

$$= \int \frac{b_T\, db_T}{2\pi} J_0(k_T b_T)\, \tilde{f}(x,b_T^2)$$

$$\tilde{f}(x,b_T^2) = \int d^2 k_T\, e^{-i \bar{b}_T \bar{k}_T} f(x,k_T^2)$$

$$= 2\pi \int k_T\, dk_T\, J_0(b_T k_T)\, f(x,k_T^2)$$

Using a test function

$$f(x,k_T^2) = f_1(x) \frac{1}{\pi \langle k_T^2 \rangle} e^{-k_T^2/\langle k_T^2 \rangle}$$

we obtain

$$\tilde{f}(x,b_T^2) = f_1(x)\, e^{-\frac{\langle k_T^2 \rangle b_T^2}{4}}$$

2) For functions where 1^{st} or 2^{nd} moments are useful:

$$\tilde{f}^{(1)}(x, b_T^2) = \frac{2\pi}{M^2} \int k_T \, dk_T \, \frac{k_T}{b_T} J_1(k_T b_T) f(x, k_T^2)$$

for instance Sivers:

$$f(x, k_T^2) = \frac{M^2}{2\pi} \int b_T \, db_T \, \frac{b_T}{k_T} J_1(b_T k_T) \tilde{f}^{(1)}(x, b_T^2)$$

$$\tilde{f}^{(2)}(x, b_T^2) = \frac{4\pi}{M^4} \int k_T \, dk_T \left(\frac{k_T}{b_T}\right)^2 J_2(k_T b_T) f(x, k_T)$$

$$f(x, k_T^2) = \frac{M^4}{4\pi} \int b_T \, db_T \left(\frac{b_T}{k_T}\right)^2 J_2(k_T b_T) \tilde{f}^{(2)}(x, b_T)$$

The property:

$$\tilde{f}^{(1)}(x, 0) = \int d^2 k_T \, \frac{k_T^2}{2M^2} f(x, k_T^2)$$

$$\tilde{f}^{(2)}(x, 0) = \int d^2 k_T \left(\frac{k_T^2}{2M^2}\right)^2 f(x, k_T^2)$$

Test functions

$$f(x, k_T^2) = f_{1T}^{\perp(1)}(x) \, \frac{2M^2}{\pi \langle k_T^2 \rangle^2} e^{-k_T^2/\langle k_T^2 \rangle}$$

$$\tilde{f}^{(1)}(x, b_T) = f_{1T}^{\perp(1)}(x) \, e^{-\frac{\langle k_T^2 \rangle b_T^2}{4}}$$

$$f(x, k_T^2) = h_{1T}^{\perp(2)}(x) \, \frac{2M^4}{\pi \langle k_T^2 \rangle^3} e^{-k_T^2/\langle k_T^2 \rangle}$$

$$\tilde{f}^{(2)}(x, b_T) = h_{1T}^{\perp(2)}(x) \, e^{-\frac{\langle k_T^2 \rangle b_T^2}{4}}$$

3) For fregmentation functions:

$$D(z, p_T^2) = \int \frac{d^2 b_T}{(2\pi)^2}\, e^{i \bar{p}_T \bar{b}_T / z}\, \tilde{D}(z, b_T^2)$$

$$= \int \frac{b_T\, db_T}{2\pi}\, J_0\left(\frac{p_T b_T}{z}\right) \tilde{D}(z, b_T^2)$$

$$\tilde{D}(z, b_T^2) = \int \frac{d^2 p_T}{z^2}\, e^{-i \bar{b}_T \bar{p}_T / z}\, D(z, p_T^2).$$

$$= 2\pi \int \frac{p_T\, dp_T}{z^2}\, J_0(b_T p_T / z)\, D(z, p_T^2)$$

Test functions:

$$D(z, p_T^2) = D_1(z) \frac{1}{\pi \langle p_T^2 \rangle} e^{-p_T^2 / \langle p_T^2 \rangle}$$

$$\tilde{D}(z, b_T^2) = \frac{1}{z^2} D_1(z)\, e^{-\frac{\langle p_T^2 \rangle b_T^2}{4 z^2}}$$

4) For FFs where moments are important

$$\tilde{D}^{(1)}(z, b_T^2) = \frac{2\pi}{z^2 M_h^2} \int \frac{p_T\, dp_T}{z^2} J_1\left(\frac{p_T b_T}{z}\right) \frac{p_T z}{b_T} D(z, p_T^2)$$

For instance Collins FF

$$D(z, p_T^2) = \frac{z^2 M_h^2}{2\pi} \int b_T\, db_T\, J_1\left(\frac{p_T b_T}{z}\right) \frac{b_T}{z p_T} \tilde{D}^{(1)}(z, b_T^2)$$

$$\tilde{D}^{(2)}(z, b_T^2) = \frac{4\pi}{z^4 M_h^4} \int \frac{p_T\, dp_T}{z^2} J_2\left(\frac{p_T b_T}{z}\right) \left(\frac{p_T z}{b_T}\right)^2 D(z, p_T^2)$$

$$D(z, p_T^2) = \frac{z^4 M_h^4}{4\pi} \int b_T\, db_T\, J_2\left(\frac{p_T b_T}{z}\right) \left(\frac{b_T}{p_T z}\right)^2 \tilde{D}^{(2)}(z, b_T^2)$$

Properties:

$$\tilde{D}^{(1)}(z, 0) = \int d^2 p_T\, \frac{p_T^2}{2 z^2 M_h^2} D(z, p_T^2) \equiv D^{(1)}(z)$$

$$\tilde{D}^{(2)}(z, 0) = \int d^2 p_T \left(\frac{p_T^2}{2 z^2 M_h^2}\right)^2 D(z, p_T^2) \equiv D^{(2)}(z)$$

Test functions:

$$D(z, p_T^2) = H_1^{\perp(1)}(z) \frac{2 z^2 M_h^2}{\pi \langle p_T^2 \rangle^2} e^{-\frac{p_T^2}{\langle p_T^2 \rangle}}$$

$$\tilde{D}(z, b_T^2) = H_1^{\perp(1)}(z) \frac{1}{z^2} e^{-\frac{b_T^2 \langle p_T^2 \rangle}{4 z^2}}$$

$$D(z, p_T^2) = D^{(2)}(z) \frac{2 z^4 M_h^4}{\pi \langle p_T^2 \rangle^3} e^{-p_T^2 / \langle p_T^2 \rangle}$$

$$\tilde{D}(z, b_T^2) = D^{(2)}(z) \frac{1}{z^2} e^{-\frac{b_T^2 \langle p_T^2 \rangle}{4 z^2}}$$

Convolutions in momentum space

$$C[\omega f D] \equiv \sum_q e_q^2 \int d^2k_T\, d^2p_T\, \delta^{(2)}(\bar{P}_{hT} - z\bar{k}_T - \bar{p}_T)$$

$$\omega(k_T, p_T)\, f(x, k_T)\, D(z, p_T)$$

We rewrite $\delta^{(2)}(\bar{P}_{hT} - z\bar{k}_T - \bar{p}_T) =$

$$= \delta^{(2)}(-z\bar{q}_T - z\bar{k}_T - \bar{p}_T) = \frac{1}{z^2}\delta^{(2)}\left(\bar{q}_T + \bar{k}_T + \frac{\bar{p}_T}{z}\right)$$

$$= \frac{1}{z^2}\int \frac{d^2b_T}{(2\pi)^2} e^{-i(\bar{q}_T + \bar{k}_T + \bar{p}_T/z)\bar{b}_T}$$

So that

$$F_{UU} = C[1 f_1 D_1] = \sum_q e_q^2 \int \frac{d^2b_T}{(2\pi)^2} e^{-i\bar{b}_T\bar{q}_T}$$

$$\times \int d^2k_T\, e^{-i\bar{b}_T\bar{k}_T} f_1(x, k_T^2)$$

$$\times \frac{1}{z^2}\int d^2p_T\, e^{-i\bar{b}_T\bar{p}_T/z} D_1(z, p_T^2) =$$

$$= \sum_q e_q^2 \int \frac{d^2b_T}{(2\pi)^2} e^{-i\bar{b}_T\bar{q}_T} \tilde{f}_1(x, b_T^2)\tilde{D}_1(z, b_T^2)$$

$$= \sum_q e_q^2 \int \frac{b_T\, db_T}{2\pi} J_0(q_T b_T)\tilde{f}_1(x, b_T^2)\tilde{D}_1(z, b_T^2)$$

$$F_{UU} = \sum_q e_q^2 \int \frac{b_T\, db_T}{2\pi} J_0(q_T b_T)\tilde{f}_1(x, b_T^2)\tilde{D}_1(z, b_T^2)$$

$= B_1(\tilde{f}_1\tilde{D}_1)$, where $B_1(f, D) \equiv \sum_q e_q^2 \int \frac{b_T\, db_T}{2\pi} J_0(b_T q_T) f D$

$$F_{LL} = \mathcal{C}[g_1 D_1] = \sum_q e_q^2 \int d^2k_T\, d^2p_T$$

$$\times\, \delta^{(2)}(\bar{P}_{hT} - z\bar{k}_T - \bar{p}_T)\, g_1(x, k_T^2)\, D_1(z, p_T^2)$$

$$= \sum_q e_q^2 \int \frac{b_T\, db_T}{2\pi} J_0(b_T q_T)\, \tilde{g}_1(x, b_T^2)\, \tilde{D}_1(z, b_T^2)$$

$$= \mathcal{B}_1[\tilde{g}_1 \tilde{D}_1]$$

$$J_0(b_T q_T) = J_0(b_T P_{hT}/z)$$

We will generically define

$$\boxed{\mathcal{B}_n[\tilde{f}\tilde{D}] \equiv \sum_q e_q^2 \int \frac{b_T\, db_T}{2\pi} b_T^n J_n(b_T q_T)\, \tilde{f}\, \tilde{D}}$$

Sivers asymmetry

$$F_{UT}^{\sin(\phi_h-\phi_S)} = \mathcal{C}\left[-\frac{\hat{h}\cdot\bar{k}_T}{M} f_{1T}^{\perp} D_1\right]$$

$$= \sum_q e_q^2 \int d^2k_T\, d^2p_T\, \delta^{(2)}(\bar{P}_{hT} - z\bar{k}_T - \bar{p}_T) \underbrace{\left(-\frac{\hat{h}\cdot\bar{k}_T}{M}\right)}_{-\frac{k_T}{M}\cos(\varphi-\phi_h)}$$

$$\times f_{1T}^{\perp}(x, k_T^2)\, D_1(z, p_T^2)$$

$$= \sum_q e_q^2 \int \frac{d^2b_T}{(2\pi)^2} e^{+i\bar{b}_T\bar{P}_{hT}/z} \int d^2k_T \left(-\frac{k_T}{M}\right)\cos(\varphi-\phi_h)$$

$$e^{-i\bar{k}_T\bar{b}_T} \times f_{1T}^{\perp}(x, k_T^2) \underbrace{\int d^2p_T \frac{1}{z^2} e^{-ip_T b_T/z} D_1(z, p_T^2)}_{\tilde{D}_1(z, b_T^2)}$$

we have $\bar{k}_T\bar{b}_T = b_T k_T \cos(\varphi-\varphi_b)$

$$\int d\varphi\, e^{-ib_Tk_T\cos(\varphi-\varphi_b)} \cos(\varphi-\phi_h) =$$

$$= -2\pi i\, J_1(b_Tk_T)\cos(\varphi_b-\phi_h)$$

$$F_{UT}^{\sin(\phi_h-\phi_S)} = \sum_q e_q^2 \int \frac{d^2b_T}{(2\pi)^2} e^{i\bar{b}_T\bar{P}_{hT}/z}\, k_T\, dk_T$$

$$(2\pi i)\frac{k_T}{M}\cos(\varphi_b-\phi_h) f_{1T}^{\perp}(x, b_T^2)\, \tilde{D}_1(z, b_T^2)\, J_1(b_Tk_T)$$

we have $\int d\varphi_b \cos(\varphi_b-\phi_h)\, e^{ib_T\frac{P_{hT}}{z}\cos(\varphi_b-\phi_h)}$

$$= 2\pi i\, J_1\left(b_T\frac{P_{hT}}{z}\right)$$

so that

$$F_{UT}^{\sin(\phi_h-\phi_S)} = -\sum_q e_q^2 \int \frac{db_T\, b_T}{2\pi} J_1\left(\frac{b_T P_{hT}}{z}\right) J_1(b_T k_T)$$

$$dk_T \frac{k_T^2}{M} 2\pi f_{1T}^{\perp}(x, k_T^2) \tilde{D}_1(z, b_T^2)$$

$$\tilde{f}_{1T}^{\perp(1)}(x, b_T^2) = \frac{2\pi}{M^2} \int k_T\, dk_T \frac{k_T}{b_T} J_1(k_T b_T) f_{1T}^{\perp}(x, k_T^2)$$

so that

$$F_{UT}^{\sin(\phi_h-\phi_S)} = (-M) \sum_q e_q^2 \int \frac{b_T\, db_T}{2\pi} b_T J_1\left(b_T \frac{P_{hT}}{z}\right)$$

$$\times \tilde{f}_{1T}^{\perp(1)}(x, b_T^2) D_1(z, b_T^2)$$

$$F_{UT}^{\sin(\phi_h-\phi_S)} = (-M)\, \mathcal{B}_1[\tilde{f}_{1T}^{\perp(1)} \tilde{D}_1]$$

Collins asymmetry

$$F_{UT}^{\sin(\phi_h+\phi_S)} = C\left[\frac{\bar{h}\cdot \bar{p}_T}{z M_h} h_1 H_1^\perp\right]$$

$$= \sum_q e_q^2 \int \frac{d^2 b_T}{(2\pi)^2} e^{i \bar{b}_T \bar{P}_{hT}/z} \underbrace{d^2 k_T\, e^{-i \bar{b}_T \bar{k}_T} h_1(x, k_T^2)}_{\tilde{h}_1(x, b_T^2)}$$

$$\times \frac{p_T}{z M_h} \cos(\varphi - \phi_h) \frac{1}{z^2} d^2 p_T\, e^{-i \bar{b}_T \bar{p}_T/z} H_1^\perp(z, p_T^2)$$

$$\int d\varphi \cos(\varphi - \phi_h)\, e^{-i \frac{b_T p_T}{z} \cos(\varphi_b - \varphi)} =$$

$$= -2\pi i\, J_1\left(\frac{b_T p_T}{z}\right) \cos(\varphi_b - \phi_h)$$

$$F_{UT}^{\sin(\phi_h+\phi_S)} = \sum_q e_q^2 \int \frac{d^2 b_T}{(2\pi)^2} e^{i b_T P_{hT}/z} \tilde{h}_1(x, b_T^2)$$

$$\frac{1}{z^2}(-2\pi i)\, J_1\left(\frac{b_T p_T}{z}\right) \cos(\varphi_b - \phi_h) \frac{p_T}{z M_h} p_T\, dp_T\, H_1^\perp(z, p_T^2)$$

$$\int d\varphi_b\, e^{i b_T P_{hT}/z \cos(\varphi_b - \phi_h)} \cos(\varphi_b - \phi_h)$$

$$= 2\pi i\, J_1\left(\frac{b_T P_{hT}}{z}\right)$$

$$F_{UT}^{\sin(\phi_h+\phi_S)} = \sum_q e_q^2 \int \frac{b_T\, db_T}{2\pi} J_1\left(\frac{b_T P_{hT}}{z}\right) \tilde{h}_1(x, b_T^2)$$

$$\frac{1}{z^2} 2\pi\, J_1\left(\frac{p_T b_T}{z}\right) \frac{p_T}{z M_h} p_T\, dp_T\, H_1^\perp(z, p_T^2)$$

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$$\tilde{H}_1^{\perp(1)}(z, b_T^2) = \frac{2\pi}{z^2 M_h^2} \int \frac{p_T\, dp_T}{z^2} J_1\left(\frac{p_T b_T}{z}\right) \frac{p_T z}{b_T} H_1^{\perp}(z, p_T^2$$

so that

$$F_{UT}^{\sin(\phi_h+\phi_S)} = \sum_q e_q^2 \int \frac{b_T\, db_T}{2\pi} b_T J_1\left(\frac{b_T P_{hT}}{z}\right)$$

$$M_h \tilde{h}_1(x, b_T^2)\, \tilde{H}_1^{\perp(1)}(z, b_T^2)$$

$$F_{UT}^{\sin(\phi_h+\phi_S)} = M_h\, \mathcal{B}_1[\tilde{h}_1 \tilde{H}_1^{\perp(1)}]$$

$$\begin{cases} \tilde{h}_1(x, b_T^2) = h_1(x)\, e^{-\frac{b_T^2 \langle k_T^2 \rangle}{4}} \\ \tilde{H}_1^{\perp(1)}(z, b_T^2) = \frac{1}{z^2} H_1^{\perp(1)}(z)\, e^{-\frac{b_T^2 \langle p_T^2 \rangle}{4 z^2}} \end{cases}$$

for Jam, otherwise different b_T dependence

and $\times\, e^{-S(b)/2}$ for Sudakov. both

functions, or $e^{-S(b)}$ for SF.

$A_{UT}^{\sin(3\phi_h-\phi_S)}$ pretzelosity asymmetry

$F_{UT}^{\sin(3\phi_h-\phi_S)} =$

$$C\Big[-\underbrace{\frac{2(\hat{h}\cdot\bar{k}_T)(k_T\cdot p_T)+k_T^2(\hat{h}\cdot p_T)-4(\hat{h}\cdot\bar{k}_T)^2(\hat{h}\cdot\bar{p}_T)}{2zM^2M_h}}_{k_T^2 p_T\cos(2\varphi+\phi_{p_T}-3\phi_q)} h_{1T}^{\perp}H_1^{\perp}\Big]$$

$k_T dk_T d\varphi$ $\quad$ $p_T dp_T d\phi_{p_T}$

$$= \sum_q e_q^2 \int d^2p_T\, d^2k_T \frac{d^2b_T}{(2\pi)^2} e^{+i\bar{b}_T\bar{P}_{hT}/z}$$

$$k_T^2 p_T \cos(2\varphi+\phi_{p_T}-3\phi_q)\, e^{-i\bar{k}_T\bar{b}_T} h_{1T}^{\perp}(x,k_T^2)$$

$$\frac{1}{z^2} H_1^{\perp}(z,p_T^2)\, e^{-i\bar{p}_T\bar{b}_T/z}\left(\frac{1}{2zM^2M_h}\right)$$

we have $k_T b_T = b_T k_T \cos(\varphi-\varphi_b)$

$$\int d\varphi\, e^{-ib_Tk_T\cos(\varphi-\varphi_b)}\cos(2\varphi+\phi_{p_T}-3\phi_q) =$$

$$= -2\pi\, J_2(k_Tb_T)\cos(2\phi_b+\phi_{p_T}-3\phi_q)$$

$$F_{UT}^{\sin(3\phi_h-\phi_S)} = \sum_q e_q^2 \int \frac{d^2b_T}{(2\pi)^2} e^{+i\bar{b}_T\bar{P}_{hT}/z}\, k_T dk_T$$

$$k_T^2 p_T(-2\pi)\cos(2\phi_b+\phi_{p_T}-3\phi_q)\, J_2(k_Tb_T)\, h_{1T}^{\perp}(x,k_T^2)$$

$$d^2p_T\, e^{-i\bar{p}_T\bar{b}_T/z} H_1^{\perp}(z,p_T^2)\frac{1}{(2zM^2M_h)}$$

we have

$$\int d\phi_{p_T} e^{-i\bar{b}_T\bar{p}_T/z} \cos(2\phi_b + \phi_{p_T} - 3\phi_q) =$$

$$= -(2\pi i)\, J_1\left(\frac{b_T p_T}{z}\right) \cos(3\phi_b - 3\phi_q)$$

$$F_{UT}^{\sin(3\phi_h - \phi_s)} = \sum_q e_q^2 \int \frac{d^2 b_T}{(2\pi)^2} e^{+i\bar{b}_T \bar{P}_{hT}/z} k_T^3\, dk_T$$

$$(2\pi)^2 i \cos(3\phi_b - 3\phi_q)\, J_2(k_T b_T)\, h_{1T}^{\perp}(x, k_T^2)$$

$$p_T^2\, dp_T\, J_1\left(\frac{b_T p_T}{z}\right) H_1^{\perp}(z, p_T^2)$$

we have

$$\int d\varphi\, e^{-i\bar{b}_T\bar{q}_T} \cos(3\phi_b - 3\phi_q) = 2\pi i\, J_3(b_T q_T)$$

$$\Rightarrow$$

$$F_{UT}^{\sin(3\phi_h - \phi_s)} = \sum_q e_q^2 \int \frac{b_T\, db_T}{(2\pi)^2} J_3(b_T q_T)\, k_T^3\, dk_T$$

$$(2\pi)^3 (+1)\, J_2(k_T b_T)\, h_{1T}^{\perp}(x, k_T^2) \frac{1}{z^2} p_T^2\, dp_T\, J_1\left(\frac{b_T p_T}{z}\right)$$

$$\times H_1^{\perp}(z, p_T^2) = \sum_q e_q^2 \int \frac{b_T\, db_T}{(2\pi)^2} \frac{M^4}{4\pi} b_T^2\, \tilde{h}_{1T}^{\perp(2)}(x, b_T^2)$$

$$J_3(q_T b_T)\, (2\pi)^3 (-1) \cdot \frac{M_h^2}{2\pi} \frac{1}{\cancel{z}} \cdot b_T \frac{1}{\cancel{z}} \tilde{H}_1^{\perp(1)}(z, b_T^2)$$

$$\frac{1}{2 M^2 M_h} = + \sum_q e_q^2 \int \frac{b_T\, db_T}{2\pi} b_T^3\, J_3(b_T q_T) \frac{M^2 M_h}{4} \tilde{h}_{1T}^{\perp(2)}\, \tilde{H}_1^{\perp(1)}$$

$$\boxed{F_{UT}^{\sin(3\phi_h - \phi_S)} = \frac{M^2 M_h}{4} \mathcal{B}_3\left[\tilde{h}_{1T}^{\perp(2)} \tilde{H}_1^{\perp(1)}\right]}$$

$$\tilde{h}_{1T}^{\perp(2)}(x, b_T^2) = h_{1T}^{\perp(2)}(x)\, e^{-\frac{b_T^2 \langle k_T^2 \rangle}{4}}$$

$F_{UU}^{\cos 2\phi}$ (Boer Mulders)

$$F_{UU}^{\cos 2\phi} = \mathcal{C}\left[\frac{2(\hat{h}\cdot\bar{p}_T)(\hat{h}\cdot k_T) - (\bar{p}_T\cdot\bar{k}_T)}{zMM_h} h_1^\perp H_1^\perp \right]$$

$$\hat{h}\cdot\bar{p}_T = p_T \cos(\phi_q - \phi_{p_T})$$

$$\hat{h}\cdot\bar{k}_T = k_T \cos(\phi_q - \varphi)$$

$$\bar{p}_T\cdot\bar{k}_T = p_T k_T \cos(\phi_{p_T} - \varphi) = k_T p_T \cos(\varphi + \phi_{p_T} - 2\phi_q)$$

$$F_{UU}^{\cos 2\phi} = \sum_q e_q^2 \int \frac{d^2 b_T}{(2\pi)^2} e^{i\bar{b}_T \bar{P}_{hT}/z} \frac{1}{z^2} d^2 p_T\, d^2 k_T\, p_T k_T \frac{1}{zMM_h}$$

$$\cos(\varphi + \phi_{p_T} - 2\phi_q)\, e^{-i\bar{k}_T\bar{b}_T} h_1^\perp(x, k_T^2)\, e^{-i\bar{p}_T\bar{b}_T/z} H_1^\perp(z, p_T^2)$$

we have

$$\int d\varphi\, e^{-ik_T b_T \cos(\varphi - \phi_b)} \cos(\varphi + \phi_{p_T} - 2\phi_q) =$$

$$= (-2\pi i)\, J_1(b_T k_T) \cos(\phi_b + \phi_{p_T} - 2\phi_q)$$

$$F_{UU}^{\cos 2\phi} = \sum_q e_q^2 \int \frac{d^2 b_T}{(2\pi)^2} e^{i\bar{b}_T \bar{P}_{hT}/z} \frac{1}{z^2} d^2 p_T\, k_T^2 dk_T \frac{1}{zMM_h}$$

$$\cos(\phi_b + \phi_{p_T} - 2\phi_q)\, J_1(b_T k_T)(-2\pi i)\, h_1^\perp(x, k_T^2)$$

$$\times\, e^{-i\bar{p}_T\bar{b}_T/z} H_1^\perp(z, p_T^2)$$

we have

$$d\phi_{p_T}\, e^{-i b_T p_T/z \cos(\phi_b - \phi_{p_T})} \cos(\phi_b + \phi_{p_T} - 2\phi_q) =$$

$$= (-2\pi i)\, J_1\left[\frac{p_T b_T}{z}\right] \cos(2\phi_b - 2\phi_q)$$

$$F_{UU}^{\cos 2\phi} = \sum_q e_q^2 \int \frac{d^2 b_T}{(2\pi)^2}\, e^{+i \bar{b}_T \bar{P}_{hT}/z}\, \frac{1}{z^2} p_T^2\, dp_T\, k_T^2\, dk_T$$

$$\frac{1}{z M M_h}\left(-(2\pi)^2\right) \cos(2\phi_b - 2\phi_q)\, J_1(p_T k_T)\, h_1^\perp(x, k_T^2)$$

$$J_1\left(\frac{p_T b_T}{z}\right) H_1^\perp(z, p_T^2)$$

we have $\int d\phi_b\, e^{-i b_T q_T \cos(\phi_b - \phi_q)} \cos(2\phi_b - 2\phi_q) =$

$$= (-2\pi)\, J_2(b_T q_T)$$

$$F_{UU}^{\cos 2\phi} = \sum_q e_q^2 \int \frac{b_T\, db_T}{(2\pi)^2} J_2(b_T q_T) \frac{1}{z^2} p_T^2\, dp_T$$

$$k_T^2\, dk_T\, \frac{1}{z M M_h} (2\pi)^3\, J_1(b_T k_T)\, h_1^\perp(x, k_T^2)$$

$$\times J_1\left(\frac{p_T b_T}{z}\right) H_1^\perp(z, p_T^2) =$$

$$= \sum_q e_q^2 \int \frac{b_T\, db_T}{(2\pi)^2} J_2(b_T q_T) \frac{1}{z M M_h} (2\pi)^3 \cdot \frac{M^2}{2\pi} b_T$$

$$\tilde{h}_1^{\perp(1)}(x, b_T^2) \cdot \frac{z^2 M_h^2}{2\pi} \frac{b_T}{z} \tilde{H}_1^{\perp(1)}(z, b_T^2)$$

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$$F_{uu}^{\cos 2\phi} = M M_h \sum_q e_q^2 \int \frac{b_T\, db_T}{2\pi}\, b_T^2\, J_2(b_T q_T)$$
$$\times\, \tilde{h}_1^{\perp (1)}(x, b_T^2)\, \tilde{H}_1^{\perp (1)}(z, b_T^2)$$

$$F_{uu}^{\cos 2\phi} = M M_h\, \mathcal{B}_2\left[\tilde{h}_1^{\perp (1)}\, \tilde{H}_1^{\perp (1)}\right]$$

$F_{UU}^{\cos 2\phi}$ Cahn effect twist-4

$$F_{UU}^{\cos 2\phi} = \frac{2}{Q^2} \mathcal{C}\left[\left(2(\hat{h}\cdot k_T)^2 - k_T^2 \right) f_1 D_1 \right]$$

$$2(\hat{h}\cdot k_T)^2 - k_T^2 = k_T^2 \cos(2\varphi - 2\phi_q)$$

$$F_{UU}^{\cos 2\phi} = \frac{2\sum_q e_q^2}{Q^2} \int \frac{d^2 b_T}{(2\pi)^2} e^{+i \bar{b}_T \bar{P}_{hT}/z} d^2 k_T \, d^2 p_T \frac{1}{z^2}$$

$$k_T^2 \cos(2\varphi - 2\phi_q) e^{-i \bar{b}_T \bar{k}_T} f_1(x, k_T^2) e^{-i \bar{b}_T \bar{p}_T/z} D_1(z, p_T^2)$$

we have:

$$\int d\varphi \, e^{-i k_T b_T \cos(\varphi - \phi_b)} \cos(2\varphi - 2\phi_q) =$$

$$= (-2\pi) J_2(b_T k_T) \cos(2\phi_b - 2\phi_q)$$

$$\int d\phi_{p_T} e^{-i b_T p_T/z \cos(\phi_b - \phi_{p_T})} \cos(2\phi_b - 2\phi_a) =$$

$$= 2\pi J_0\left(\frac{b_T p_T}{z}\right) \cos(2\phi_b - 2\phi_a)$$

$$\int d\phi_b \, e^{-i b_T q_T \cos(\phi_b - \phi_a)} \cos(2\phi_b - 2\phi_a) =$$

$$= (-2\pi) J_2(b_T q_T)$$

$$F_{UU}^{\cos 2\phi} = \frac{2}{Q^2} \sum_q e_q^2 \int \frac{b_T\, db_T}{(2\pi)^2} J_2\left(\frac{b_T P_{hT}}{z}\right)$$

$$k_T^3\, dk_T\, J_2(b_T k_T)\, f_1(x, b_T^2) \frac{1}{z^2} p_T\, dp_T\, J_0\left(\frac{p_T b_T}{z}\right) D_1(z, p_T^2)$$

$$\times (2\pi)^3 = \frac{2}{Q^2} \sum_q e_q^2 \int \frac{b_T\, db_T}{(2\pi)^2} J_2\left(\frac{b_T P_{hT}}{z}\right) \frac{M^4}{4\pi} b_T^2$$

$$\tilde{f}_1^{(2)}(x, b_T^2) \frac{1}{2\pi} \tilde{D}_1(z, b_T^2)\, (2\pi)^3 =$$

$$= \frac{M^4}{Q^2} \sum_q e_q^2 \int \frac{b_T\, db_T}{2\pi} b_T^2\, J_2\left(\frac{b_T P_{hT}}{z}\right) \tilde{f}_1^{(2)}(x, b_T^2) \tilde{D}_1(z, b_T^2)$$

$$F_{UU}^{\cos 2\phi} = \frac{M^4}{Q^2} B_2\left[\tilde{f}_1^{(2)} \tilde{D}_1\right]$$ for Cahn Effect

Notice that

$$\tilde{f}^{(n)}(x, b_T^2) \equiv (-1)^n\, n! \left(\frac{2}{M^2} \frac{\partial}{\partial b_T^2}\right)^n \tilde{f}(x, b_T^2)$$

$$\tilde{f}_1(x, b_T^2) = \underbrace{f(x, Q_0^2)\, e^{-S/2}}_{\text{for fixed } Q_0 \text{ solution}} e^{-\frac{b_T^2 \langle k_T^2 \rangle}{4}}$$

$$\tilde{f}_1^{(2)}(x, b_T^2) = f_1(x, Q_0^2) \frac{\langle k_T^2 \rangle^2 e^{-\frac{b_T^2 \langle k_T^2 \rangle}{4}}}{2 M^4}$$

in Jam 3D

$$F_{LT}^{\cos(\phi_h-\phi_S)} = \mathcal{C}\left[\frac{\bar{h}\cdot k_T}{M} g_{1T} D_1\right]$$

$$= \sum_q e_q^2 \int \frac{d^2 b_T}{(2\pi)^2} e^{+i\bar{b}_T \bar{P}_{hT}/z} d^2 k_T\, k_T \cos(\varphi-\phi_q) \frac{1}{M}$$

$$g_{1T}(x, k_T^2)\, e^{-i\bar{k}_T \bar{b}_T} d^2 p_T \frac{1}{z^2} e^{-i\bar{p}_T \bar{b}_T/z} D_1(z, p_T^2)$$

$$\int d\varphi\, e^{-ik_T b_T \cos(\varphi-\phi_b)} \cos(\varphi-\phi_q) =$$

$$= (-2\pi i) J_1(b_T k_T) \cos(\phi_b-\phi_q)$$

$$\int d\phi_b\, e^{ib_T P_{hT}/z \cos(\phi_b-\phi_q)} \cos(\phi_b-\phi_q) = J_1\left(\frac{b_T P_{hT}}{z}\right)$$

$$\times (2\pi i)$$

$$= \sum_q e_q^2 \int \frac{b_T\, db_T}{(2\pi)^2} J_1\left(\frac{b_T P_{hT}}{z}\right) (2\pi)^2 k_T^2\, dk_T\, J_1(b_T k_T)$$

$$g_1(x, k_T^2) \frac{1}{M} 2\pi\, dp_T\, p_T \frac{1}{z^2} J_0\left(\frac{p_T b_T}{z}\right) D_1(z, p_T^2)$$

$$= \sum_q e_q^2 \int \frac{b_T\, db_T}{(2\pi)^2} J_1\left(\frac{b_T P_{hT}}{z}\right) (2\pi)^2 \frac{M^2 b_T}{2\pi} \tilde{g}_{1T}^{(1)}(x, b_T^2) \frac{1}{M}$$

$$\times \tilde{D}_1(z, b_T^2)$$

$$F_{LT}^{\cos(\phi_h-\phi_S)} = M\, \mathcal{B}_1\left[\tilde{g}_{1T}^{(1)} \tilde{D}_1\right]$$

$$F_{UL}^{\sin(2\phi_h)} = C\left[\frac{2(\hat{h}\cdot\bar{k}_T)(\hat{h}\cdot p_T) - p_T\cdot k_T}{z\,MM_h} h_{1L}^{\perp} H_1^{\perp}\right]$$

$$2(\hat{h}\cdot\bar{k}_T)(\hat{h}\cdot\bar{p}_T) - \bar{p}_T\cdot\bar{k}_T = p_T k_T \cos(\varphi + \phi_{p_T} - 2\phi_a)$$

using the same equations as for $F_{UU\,BM}^{\cos 2\phi}$ we get:

$$F_{UL}^{\sin 2\phi_h} = MM_h\, B_2\left[\tilde{h}_{1L}^{\perp(1)}\tilde{H}_1^{\perp(1)}\right]$$

Details of evolution

Perturbative Sudakov and CS kernel

The renormalization group equation (RG) for $\tilde{K}$ is

$$\frac{d\tilde{K}(b_T;\mu)}{d\ln\mu} = -\gamma_K(\alpha_s(\mu))$$

where

$$\gamma_K = 2C_F\frac{\alpha_s(\mu)}{\pi} + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2 C_F\left(C_A\left(\frac{67}{18}-\frac{\pi^2}{6}\right)\right.$$

$$\left.-\frac{10}{9}T_F n_f\right) + O(\alpha_s^3(\mu))$$

Sometimes it is denoted as "A" $= \frac{\gamma_K}{2}$

$$A \equiv \sum_{n=1}^{\infty} A^{(n)}\left(\frac{\alpha_s(\mu)}{\pi}\right)^n$$

$$A^{(1)} = C_F$$

$$A^{(2)} = \frac{C_F}{2}\left(C_A\left(\frac{67}{18}-\frac{\pi^2}{6}\right) - \frac{10}{9}T_F n_f\right)$$