The elements of Quantum Field Theory

The wave function $\psi(x)$ is a coordinate projection of the state vector in the Hilbert space $|\psi\rangle$

$$\int d^3x \, |\psi(x)|^2 < \infty$$

The scalar product

$$\langle \psi | \phi \rangle = \int d^3x \, \psi^*(x) \, \phi(x)$$

We will use $|P;S\rangle$ to denote the proton with momentum P and spin vector S.

Unitarity of S matrix and optical theorem

(Taylor „Scattering theory")

The probability of one state going to the other is described by S matrix

$$w(\chi \leftarrow \phi) = |\langle \chi | S | \phi \rangle|^2$$

One can write

$$S_{ab} = \underbrace{\delta_{ab}}_{\text{no interaction}} + i(2\pi)^4 \underbrace{\delta^{(4)}(p_a - p_b)}_{\text{momentum conservation}} \overbrace{T_{ab}}^{\text{interaction}}$$

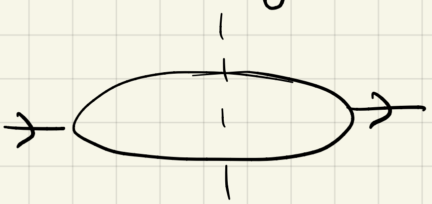
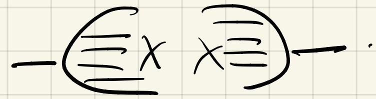
Probability is conserved

$$S S^\dagger = S^\dagger S = 1 \quad \text{Unitarity of S matrix}$$

One can use it to write

$$\mathrm{Im}\, \langle \chi | T | \phi \rangle = \frac{1}{2} \sum_X \langle \chi | T | \overbrace{X \rangle \langle X}^{\text{all possible states}} | T^\dagger | \phi \rangle (2\pi)^4 \delta^{(4)}(p_\phi - p_X)$$

Diagrammatically

2 Im  $= \sum_X$ 

Im part

$$SS^\dagger \simeq (1 + i(2\pi)^4\delta^{(4)}(P_\phi - P_\chi)T)(1 - i(2\pi)^4\delta^4(P_\phi - P_\chi)T^\dagger) = 1$$

$$\cancel{1} - i(2\pi)^4(T^\dagger - T)\delta^{(4)}(P_\phi - P_\chi) + (2\pi)^8\, TT^\dagger(\delta^4(P_\phi - P_\chi))^2 = \cancel{1}$$

$$T = \mathrm{Re}\,T + i\,\mathrm{Im}\,T, \quad T^\dagger = \mathrm{Re}\,T - i\,\mathrm{Im}\,T$$

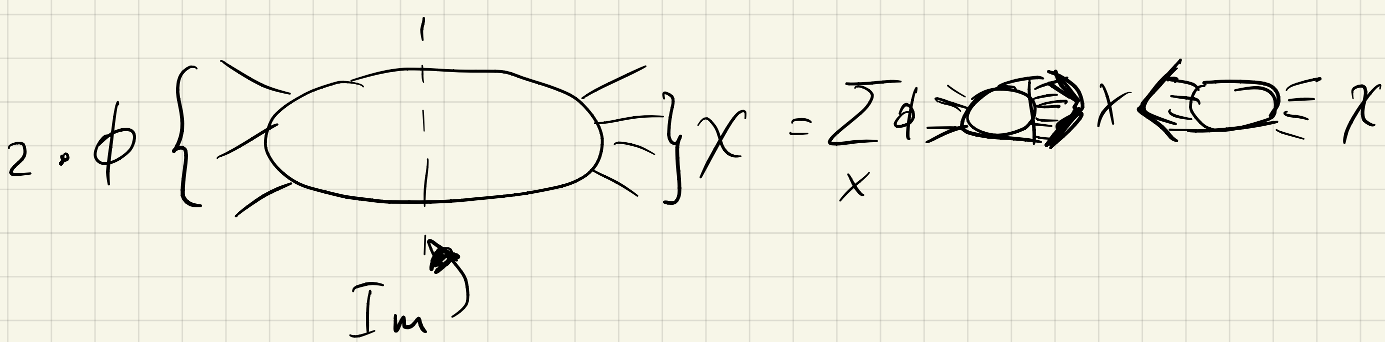
$$T^\dagger - T = -2i\,\mathrm{Im}\,T$$

$$2(2\pi)^4\,\mathrm{Im}\,T = (2\pi)^8\, T T^\dagger\, \delta^{(4)}(P_\phi - P_\chi)$$

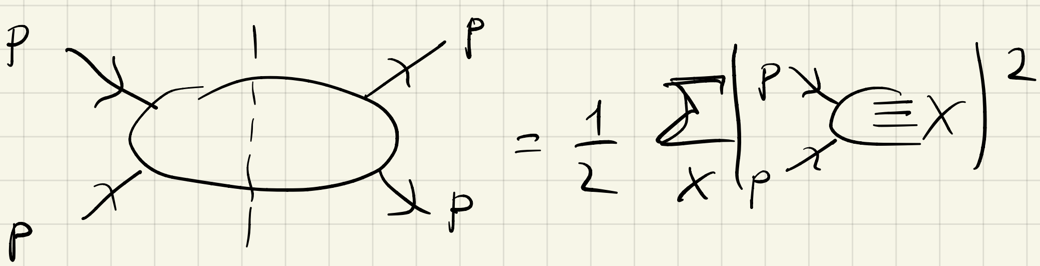
Let us insert $\mathbb{1} = \sum_X |X\rangle\langle X|$

and sandwich this expression with $\langle\chi| \dots |\phi\rangle$

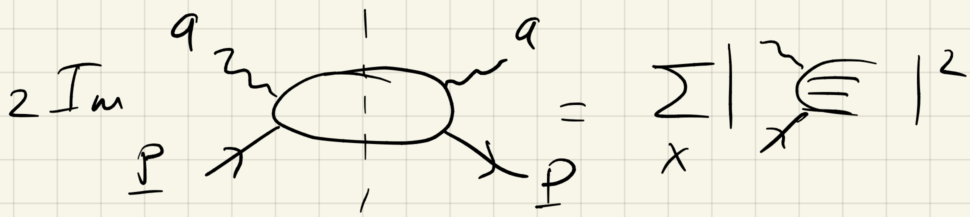
$$\mathrm{Im}\,\langle\chi|T|\phi\rangle = \frac{1}{2}\sum_X \langle\chi|T|X\rangle\langle X|T^\dagger|\phi\rangle(2\pi)^4\delta(P_\phi - P_\chi)$$



If $\phi = \chi$ then, for instance $pp \to pp$



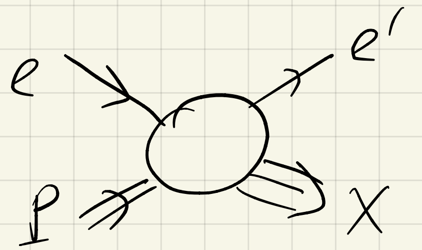
We are interested in photon-proton interactions



Experimentally one measures cross-sections

$$\sigma_{\phi\to\psi} = \underbrace{\frac{1}{F_\phi}}_{\text{Flux of } \phi} \left|A_{\phi\to\psi}\right|^2 \underbrace{\frac{d^3 P_\psi}{(2\pi)^3\, 2E_\psi}}_{\text{Phase space of } \psi}$$

We want to calculate cross-section of this process

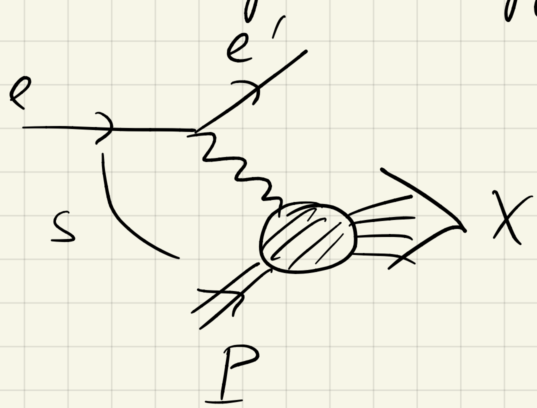


$e + P \to e' + X$

$q^2 = (e-e')^2 \qquad P \to X$

Deep Inelastic Scattering

We use one photon approximation



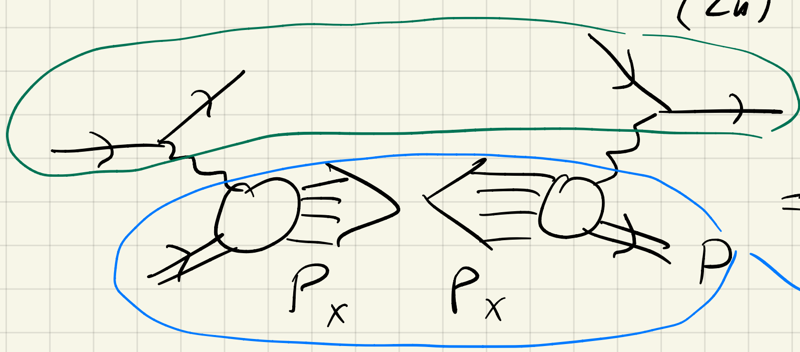
$$\sigma = \frac{1}{\mathcal{F}} |M|^2 \, d\mathcal{PS}$$

$$\mathcal{F} \simeq 2s = 2(e+P)^2 \text{ flux}$$

$$d\mathcal{PS} = \frac{d^3 e'}{(2\pi)^3 2E'}$$

leptonic tensor

$|M|^2 =$



$= \frac{1}{Q^4} L_{\mu\nu} W^{\mu\nu}$

hadronic tensor

$\frac{1}{Q^4} \sim$ product of photon propagators

Before we calculate $L_{\mu\nu}$ and $W^{\mu\nu}$ let us recapitulate some basics of QFT.

Consider the Lagrangean of a spin-½ particle with the mass m:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

γ^μ – gamma matrices

$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$, $\psi(x)$, $\bar{\psi}(x) \equiv \psi^\dagger(x)\gamma^0$ fields

Global gauge transformations

$$\psi'(x) = e^{i\alpha}\psi(x)$$

$$\bar{\psi}'(x) = \bar{\psi}(x)e^{-i\alpha}$$

$\mathcal{L} \to \mathcal{L}'$, current $j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$

Local gauge transformations

$$\begin{cases} \psi'(x) = e^{i\alpha(x)}\psi(x) \\ \bar{\psi}'(x) = \bar{\psi}(x)\, e^{-\alpha(x)} \end{cases}$$

$$\partial_\mu \psi(x) = e^{-i\alpha(x)}(\partial_\mu - i\partial_\mu \alpha(x))\psi'(x)$$

Thus

$$\mathcal{L} = \bar{\psi}'(x)(i\gamma^\mu(\partial_\mu - i\partial_\mu\alpha(x)) - m)\psi'(x)$$

We can restore gauge invariance if we use

$$(\partial_\mu + ieA_\mu(x))\psi(x)$$

$$(\partial_\mu + ieA_\mu(x))\psi(x) = e^{-i\alpha(x)}(\partial_\mu + ieA'_\mu(x))\psi'(x)$$

where

$$A'_\mu(x) = A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x)$$

$\partial_\mu + ieA_\mu(x) \rightarrow$ covariant derivative D_μ

$$\mathcal{L} = \bar{\psi}(x)\left(i\gamma^\mu(\partial_\mu + ieA_\mu(x)) - m\right)\psi(x)$$

or

$$\mathcal{L} = \bar{\psi}(x)\left(i\gamma^\mu D_\mu - m\right)\psi(x)$$

is invariant also under the local gauge transform.

We also have introduced interactions!

$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$, where

$\mathcal{L}_I = -e j^\mu A_\mu$ where $j^\mu = \bar{\psi}(x)\gamma^\mu\psi(x)$

Remember

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \equiv \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

$$\gamma^0 (\gamma^\mu)^\dagger \gamma^0 = \gamma^\mu \;, \Rightarrow (\gamma^0)^\dagger = \gamma^0 \;, (\gamma^k)^\dagger = -\gamma^k$$

Independent fields $\psi(x)$ & $\bar\psi(x) = \psi^\dagger(x)\gamma^0$

Euler-Lagrange equations (for $\mathcal{L} \equiv \mathcal{L}_0$)

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \dfrac{\partial \mathcal{L}}{\partial \partial_\mu \psi} = 0 \\[2ex] \dfrac{\partial \mathcal{L}}{\partial \bar\psi} - \partial_\mu \dfrac{\partial \mathcal{L}}{\partial \partial_\mu \bar\psi} = 0 \end{cases}$$

$$\frac{\partial \mathcal{L}}{\partial \bar\psi} = (i\gamma^\mu\partial_\mu - m)\psi \;, \quad \frac{\partial \mathcal{L}}{\partial \partial_\mu \bar\psi} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \psi} = -m\bar\psi \;, \quad \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi} = i\bar\psi\gamma^\mu$$

$$\Rightarrow \begin{cases} (i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \\ i\partial_\mu \bar\psi(x)\gamma^\mu + m\bar\psi(x) = 0 \end{cases}$$

4 solutions, 2 with $p_0 > 0$, 2 with $p_0 < 0$

Let us consider only positive energy

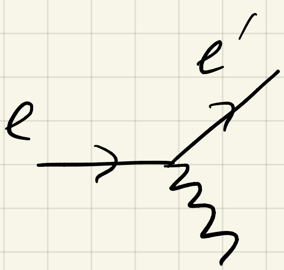
$$\psi(x) = u(p,s)\, e^{-ip\cdot x}, \quad p^2 = m^2, \; p_0 \geq 0$$

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$$

$$\Rightarrow \quad (\gamma^\mu p_\mu - m)\, u(p) = 0, \quad \gamma^\mu p_\mu \equiv \not{p}$$

$(\not{p} - m)\, u(p) = 0$, $u(p)$ is called spinor

$$\bar{u}(p)(\not{p} - m) = 0$$



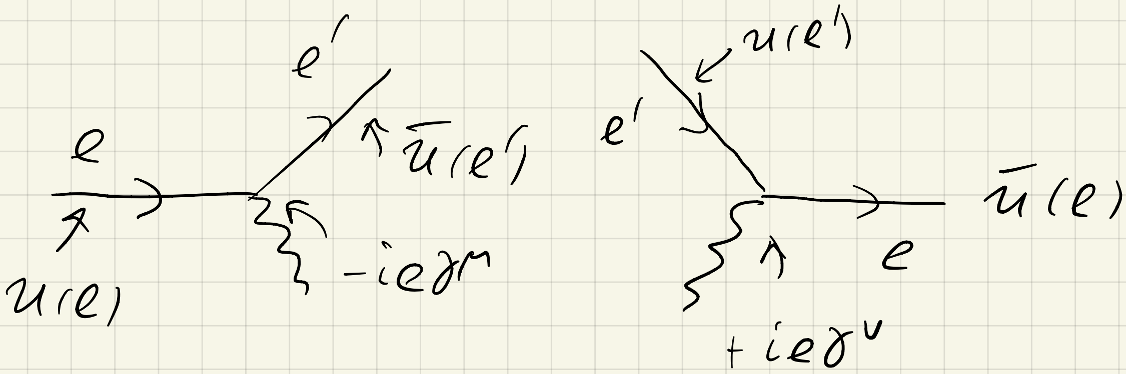
$$q = e - e'$$

Current conservation $\partial_\mu J^\mu(x) = 0$

$$J^\mu(x) = \bar{u}(e')\gamma^\mu u(e)\, e^{-i(e-e')\cdot x}$$

$$\partial_\mu J^\mu(x) = -i q_\mu J^\mu = 0 \Rightarrow q_\mu J^\mu(x) = 0$$

Let us calculate $L^{\mu\nu}$:



$$L^{\mu\nu} = \frac{1}{2s+1}\sum_{s'} \bar{u}_{\alpha}(e,s)(-ie\gamma^{\nu})_{\alpha\beta}\, u(e',s')_{\beta}\, \bar{u}_{a}(e',s')(+ie\gamma^{\mu})_{ab}\, u(e)_{b}$$

Spin products $\quad \sum_{s'} u_{\beta}(e',s')\bar{u}_{a}(e',s') = (\not{e}'+m)_{\beta a}$

$$u_{b}(e,s)\bar{u}_{\alpha}(e,s) = \left[\frac{(\not{e}+m)(1+\gamma_5 \not{s})}{2}\right]_{b\alpha}$$

where $\quad \gamma_5 = +i\gamma^0\gamma^1\gamma^2\gamma^3$, $\gamma_5^{+} = \gamma_5$, $(\gamma_5)^2 = 1$, $\{\gamma_5, \gamma^{\mu}\} = 0$

$$L^{\mu\nu} = \frac{e^2}{2}\underbrace{(\not{e}+m)_{b\alpha}\,\gamma^{\nu}_{\alpha\beta}(\not{e}'+m)_{\beta a}(\gamma^{\mu})_{ab}}_{\text{trace}}$$

neglect m and we have

$$L^{\mu\nu} = \frac{e^2}{2}\,\mathrm{Tr}(\not{e}\gamma^{\mu}\not{e}'\gamma^{\nu})$$

Traces

$$\mathrm{Tr}(\text{odd } \# \gamma) = 0$$

$$\mathrm{Tr}(\not{a}\not{b}) = 4\, a\cdot b$$

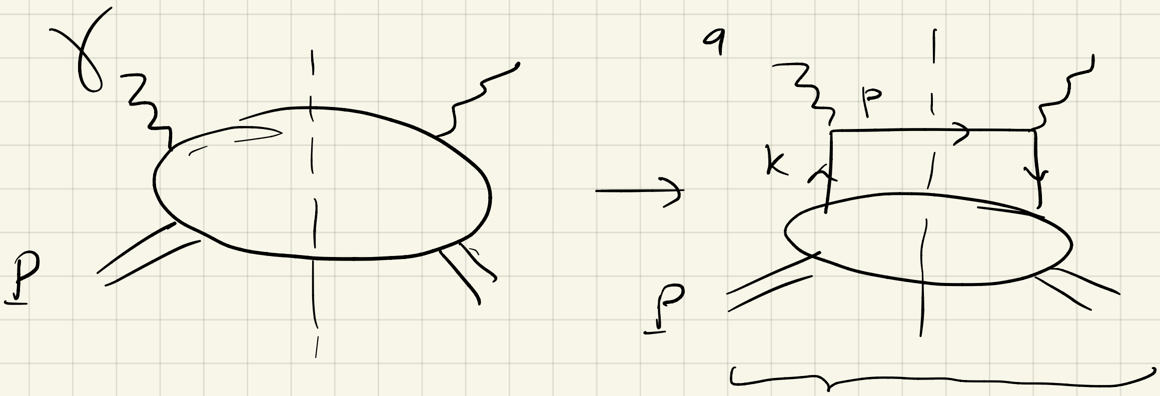
$$\mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) = 4\left[(a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(c\cdot b)\right]$$

$$\mathrm{Tr}(\gamma^a\gamma^b\gamma^c\gamma^d) = 4(g^{ab}g^{cd} - g^{ac}g^{bd} + g^{ad}g^{cb})$$

Thus

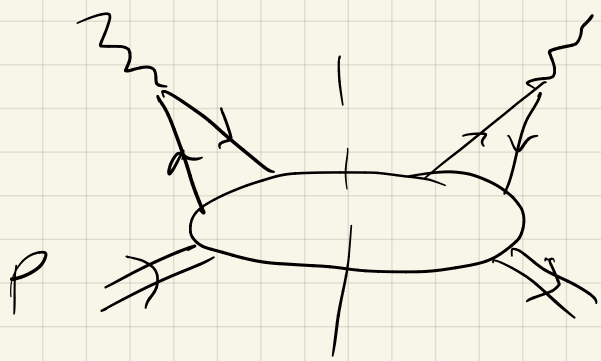
$$L^{\mu\nu} = 2e^2(e^\mu e'^\nu + e^\nu e'^\mu - g^{\mu\nu}(e\cdot e'))$$

Now let us consider the hadronic tensor



hand bag diagram

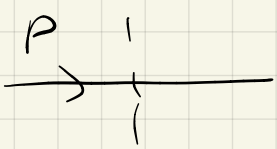
why do we not consider?



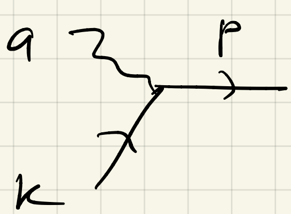
$\propto e_{q_1} e_{q_2}$

suppressed by $\left(\frac{1}{Q^2}\right)^2$

as at least one of the propagators is hard

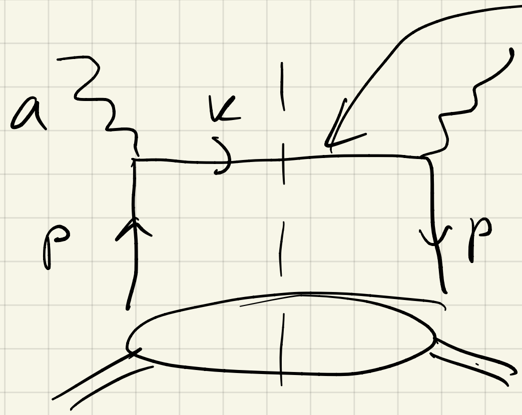


$= \mathrm{Im}\, \frac{1}{p^2 + i\epsilon} = \pi\, \delta(p^2)$



$\rightarrow \delta^{(4)}(q + k - p) \Rightarrow p = k + q$

Why partons are almost on mass shell?

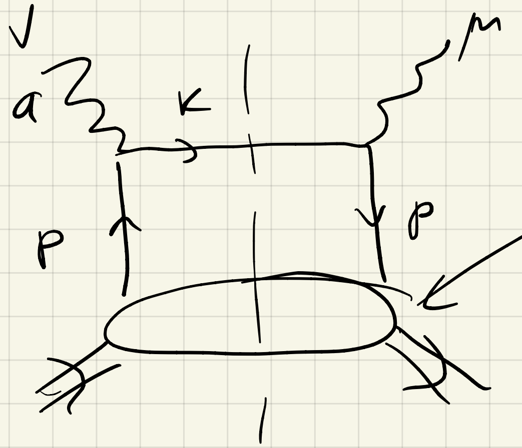


$\delta(k^2) \Rightarrow k^2 = 0$

virtual, but

$$\underbrace{\int d^4p \frac{1}{(p^2+i\epsilon)} \frac{1}{(p^2-i\epsilon)}}_{} \to p^2 \approx 0$$

The contribution from this integral is max then $p^2 \simeq 0$ as well!



let us call this matrix $\phi(p,P)$

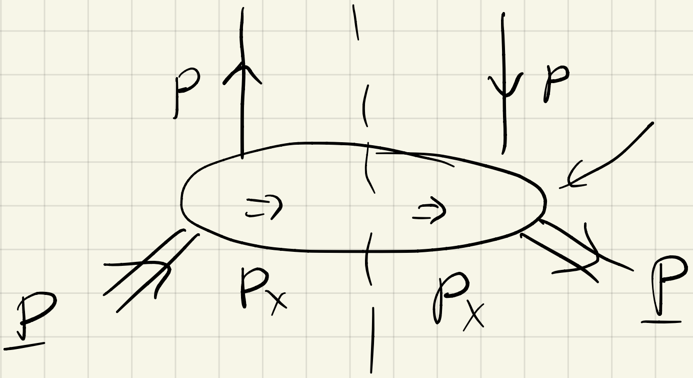
$$W^{\mu\nu} = \sum_q e_q^2 \int \frac{d^4 p}{(2\pi)^4} \mathrm{Tr}\left(\gamma^\mu (\not p + \not q)\gamma^\nu \Phi(p,P)\right) \delta\left((p+q)^2\right)$$

Let us parametrize $p = x P$, $x \in (-\infty, \infty)$

$$\delta((p+q)^2) \simeq \delta(-Q^2 + 2xP\cdot q) = \frac{1}{2P\cdot q}\,\delta(x_{BJ} - x)$$

quarks are probed at $x = x_{BJ}$!

What is Φ?



$\delta^{(4)}(p + P_X - P)$

$$\delta^4(p) = \int \frac{d^4\xi}{(2\pi)^4} e^{-ip\cdot\xi}$$

$$\sum\!\!\!\!\!\!\int_X \equiv \int \frac{d^3P_X}{2E_X(2\pi)^3} = \int \frac{d^4P_X}{(2\pi)^4}\theta(E_X)$$

$$\underline{\Phi} = \sum\!\!\!\!\!\!\int_X \delta^{(4)}(p + P_X - P)\langle P|\bar\Psi(0)|X\rangle\langle X|\Psi(0)|P\rangle$$

$$= \sum\!\!\!\!\!\!\int_X \int \frac{d^4\xi}{(2\pi)^4} e^{-i\xi(p+P_X-P)}\langle \underline{P}|\bar\Psi(0)|X\rangle\langle X|\Psi(0)|P\rangle$$

$$= \sum\!\!\!\!\!\!\int_X \int \frac{d^4\xi}{(2\pi)^4} e^{-i\xi\cdot p}\,\underbrace{\langle P|e^{i\xi \underline{P}}}\bar\Psi(0)\underbrace{e^{-i\xi P_X}|X\rangle}\langle X|\Psi(0)|P\rangle$$

$$\langle P|\underbrace{e^{i\xi\cdot\hat{\underline{P}}}\bar\Psi(0)e^{-i\xi\cdot\hat{P}}}_{\bar\Psi(\xi)}|\cancel{X}\rangle$$

$\bar\Psi(\xi)$ - <u>shift of the field</u>

Thus

$$\underline{\Phi}(p,\underline{P}) = \sum\!\!\!\!\!\!\int_X \int \frac{d^4\xi}{(2\pi)^4} e^{-ip\cdot\xi} \langle P|\bar{\psi}(\xi)|X\rangle\langle X|\psi(0)|P\rangle$$

now we use

$$\sum\!\!\!\!\!\!\int_X |X\rangle\langle X| = \mathbb{1} \text{ completeness}$$

of states and obtain

$$\underline{\Phi}(p,\underline{P}) = \int \frac{d^4\xi}{(2\pi)^4} e^{-ip\cdot\xi} \langle P|\bar{\psi}(\xi)\psi(0)|P\rangle$$

Let us introduce light cone variables $A^{\pm} = \frac{A^0 \pm A^3}{\sqrt{2}}$

$$A\cdot B = A^+B^- + A^-B^+ - \vec{A}_T\cdot\vec{B}_T \quad , \quad A_T \equiv (A^1, A^2)$$

$$\underline{P} \simeq \left(P^+, \frac{M^2}{2P^+}, 0\right) \simeq (P^+, 0, 0)$$

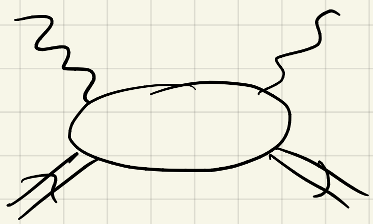
$$p = x\underline{P} \simeq (p^+, 0, p_\perp)$$

important for TMDs

$$p\cdot\xi \to p^+\xi^- - \underbrace{\vec{p}_T\cdot\vec{\xi}_T}_{\text{small}} \simeq p^+\xi^-$$

$$\Rightarrow \xi \simeq (0, \xi^-, 0)$$

Let us see how distributions are introduced in <u>DIS</u>.



$$W^{\mu\nu} = -\left(g^{\mu\nu} + \frac{q^\mu q^\nu}{Q^2}\right)W_1 + \left(P^\mu + \frac{q^\mu}{2x}\right)\left(P^\nu + \frac{q^\nu}{2x}\right)W_2$$

(only $W^{\mu\nu} = W^{\nu\mu}$ & $q_\mu W^{\mu\nu} = 0$)

Remember $P \cdot q = \nu$

One usually uses $\begin{cases} F_1(x,Q^2) = W_1(x,Q) \\ F_2(x,Q) = \nu\, W_2(x,Q) \end{cases}$

$$F_L = F_2 - 2x F_1 \simeq 0$$

$$p^\mu = (P, 0, 0\, P) \qquad p^2 = n^2 = 0$$

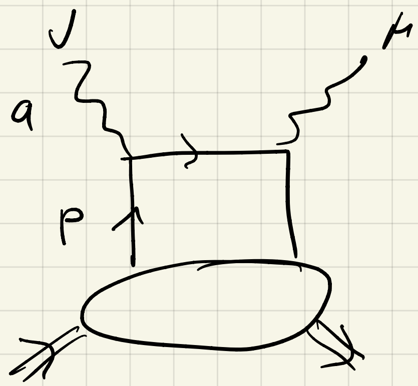
$$n^\mu = \left(\frac{1}{2P}, 0, 0, -\frac{1}{2P}\right) \qquad p \cdot n = 1$$

$$q^\mu = q_\perp^\mu + \nu n^\mu$$

$$q^2 = -\vec{q}_\perp^{\,2} = -Q^2$$

Then

$$n^\mu n^\nu W_{\mu\nu} = W_2 = \frac{1}{\nu} F_2$$



$$W^{\mu\nu} = e_q^4 \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left(\gamma^\mu (\not{q}+\not{p})\gamma^\nu \phi\right) \cdot \underbrace{\delta\left((p+q)^2\right)}_{\frac{1}{2\nu}\delta(x-x_{Bj})}$$

$$p \simeq x P$$

$$F_2 = \nu n^\mu n^\nu W_{\mu\nu} = \frac{1}{2} e_q^2 \int \frac{d^4p}{(2\pi)^4} \underbrace{\mathrm{Tr}(\not{n}\underbrace{\not{p}\not{n}}_{-\not{n}\not{p}+2n\cdot p}\phi)}_{2x\,\mathrm{Tr}(\not{n}\phi)} \delta(x-x_{Bj})$$

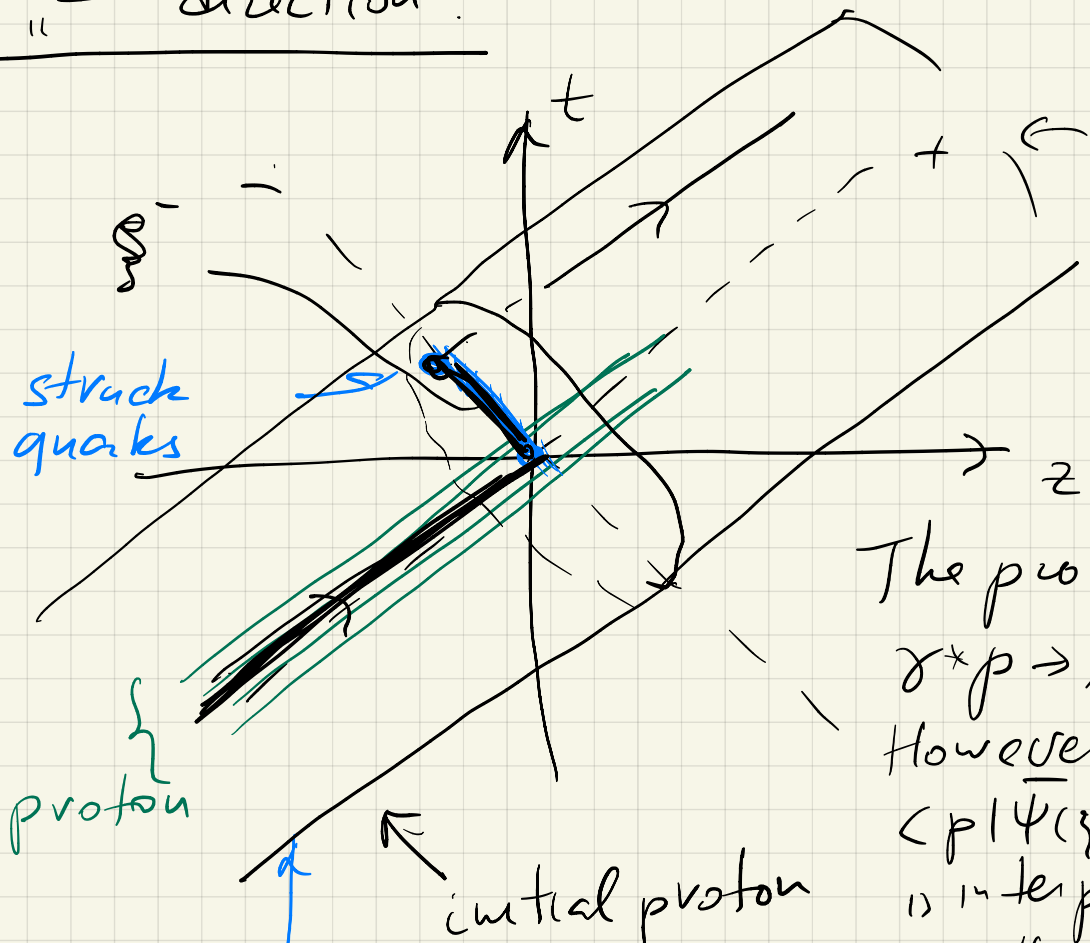
we can define

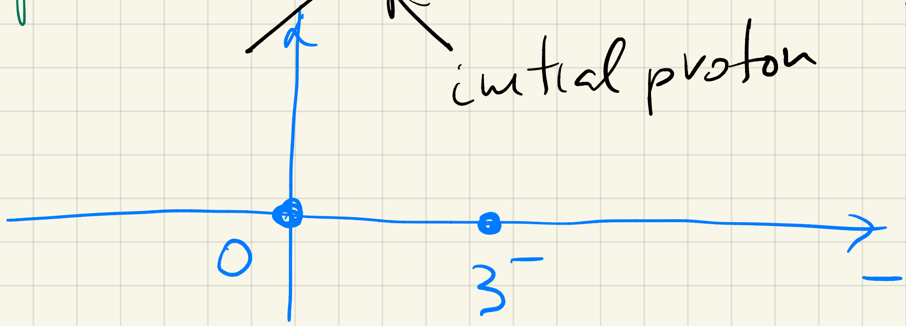
$$\underbrace{f(x_{Bj}) \equiv \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left(\not{n}\,\phi(p,P)\right)\delta(x-x_{Bj})}_{\text{Parton distributions}}$$

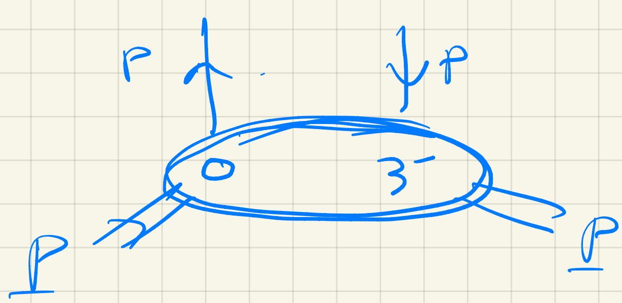
$$\Rightarrow \underbrace{F_2(x,Q^2) = \sum_q e_q^2\, x\, f(x)}_{\text{Bjorken scaling!}}$$

$$\underline{\not{n} = \gamma^+}$$

The fields are separated by some distance z^- in "−" direction!







The process
$\gamma^* p \to X$
However
$\langle p|\bar{\psi}(z)\psi(0)|P\rangle$
is interpreted as the proton at $-\infty$, then at 0 the quark field is shifted to z and then the proton continues to $+\infty$

Analogously to DIS we can define the following projections of the correlator $\Phi(p,P)$ in the case when transverse motion is not ignored.

It is customary to call the parton's momentum k so

$$\Phi(x,k_T)_{ij} = \int \frac{d\xi^- d^2\xi_T}{(2\pi)^3} e^{-ixP^+\xi^- + ik_T\cdot\xi_T}$$

$$\langle P|\,\bar{\Psi}_j(\xi)\,\Psi_i(0)|P\rangle\Big|_{\xi^+=0}$$

$$\frac{1}{2}\,\mathrm{Tr}(\gamma^+\phi) = f_1 - \frac{\epsilon^{jk}k_T^j S_T^k}{M} f_{1T}^{\perp}$$

$$\frac{1}{2}\,\mathrm{Tr}(\gamma^+\gamma_5\phi) = S_L g_1 + \frac{\bar{k}_T\cdot\bar{S}_T}{M} g_{1T}$$

$$\frac{1}{2}\,\mathrm{Tr}(i\sigma^{j+}\gamma_5\phi) = S_T^j h_1 + S_L\frac{k_T^j}{M} h_{1L}^{\perp} + \frac{\kappa^{jk}S_T^k}{M} h_{1T}^{\perp}$$

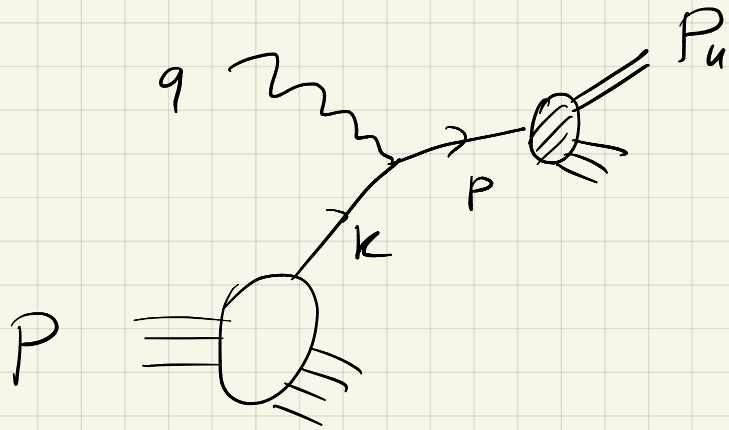
$$+ \frac{\epsilon^{jk}k_T^k}{M} h_1, \quad \text{where } \kappa^{jk} \equiv \left(k_T^j k_T^i - \frac{1}{2}k_T^2\delta^{jk}\right)$$

$$\epsilon^{ij} \equiv \epsilon^{-+ij}, \quad \epsilon^{0123} = +1$$

Let us work out convolutions for SIDIS

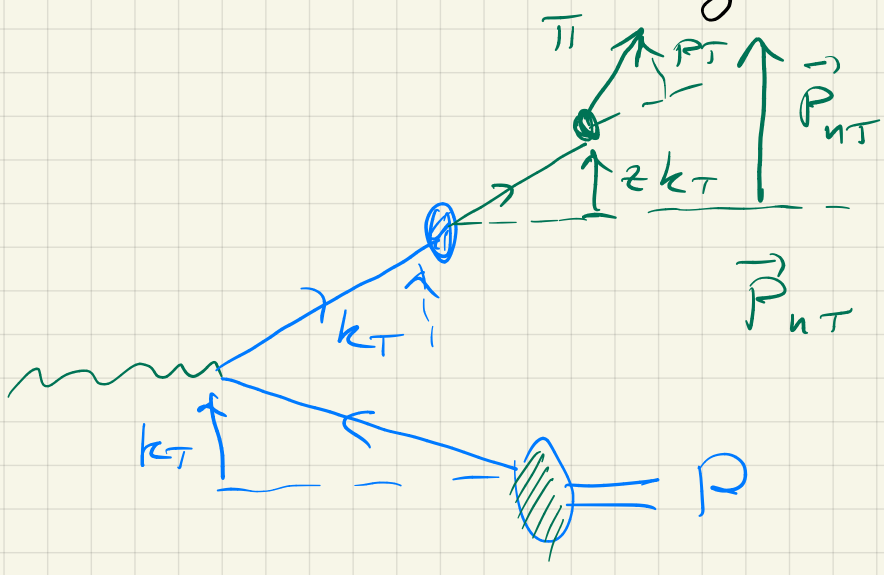
$$C[wfD] = \sum_q e_a^2 \int d^2k_T\, d^2p_T\, \delta^{(2)}(\vec{P}_{hT} - z\vec{k}_T - \vec{p}_T)$$

$$w(k_T, p_T)\, f(x, k_T)\, D(z, p_T)$$



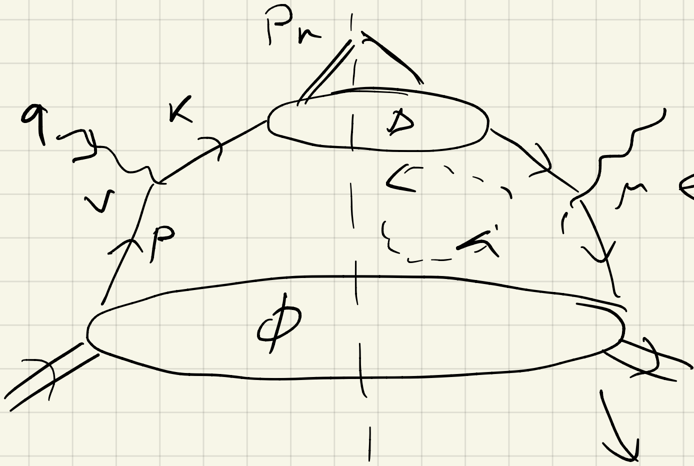
In order to study evolution we need to rewrite the convolutions in the configuration space.

We will see that TMD evolution equations are to be solved in configuration space



$$\vec{P}_{hT} = z\vec{k}_T + \vec{p}_T$$

Let us write the cut amplitude:



Starting from here we „read" the diagram similar to others in counterclockwise

$$W^{\mu\nu} = \sum_q e_q^2 \int \frac{d^4p}{(2\pi)^4} Tr(\gamma^\mu \Delta \gamma^\nu \phi)\, \delta^{(4)}(k-p-q)$$

ϕ and Δ contain TMDs

$$\frac{1}{2} Tr[\gamma^+ \phi] = f_1 - \frac{\epsilon^{jk} p_T^j S_T^k}{M} f_{1T}^\perp$$

$$\frac{1}{2} Tr[\gamma^- \Delta] = D_1$$

$$\sigma \sim \frac{1}{Q^4} L_{\mu\nu} W^{\mu\nu} \sim \sigma_0 (F_{UU} + \dots)$$

structure functions:

$$F_{UU} = C[1 f_1 D_1] \text{ etc}$$

Asymmetries are ratios of polarised over unpolarised structure functions

$$A_{UT}^{\sin(\phi_h - \phi_S)} = \frac{F_{UT}^{\sin(\phi_h - \phi_S)}}{F_{UU}} \text{ etc}$$