QCD evolution is governed by the so-called Collins-Soper equation two Renormalization Group equations

<u>CS equation</u>

b_T space

$$(1) \quad \frac{\partial \ln \tilde{F}(x, b_T, \mu, \zeta)}{\partial \ln \sqrt{\zeta}} = \underset{\substack{\uparrow \\ \text{CS kernel}}}{\tilde{K}}(b_T, \mu)$$

<u>RG equations</u>

$$(2) \quad \frac{d \tilde{K}(b_T, \mu)}{d \ln \mu} = -\gamma_K(g(\mu)) \leftarrow \text{Cusp anomalous dimention of } K$$

Very universal in QCD., γ_K depend only on μ

$$(3) \quad \frac{d \ln \tilde{F}(x, b_T, \mu, \zeta)}{d \ln \mu} = \gamma_F(g(\mu), \zeta/\mu^2)$$

$\uparrow$
anomalous dimention of F.

$$\frac{d}{d\ln\mu}\left(\frac{\partial \ln F(x, b_T, \mu, \zeta)}{\partial \ln\sqrt{\zeta}}\right) = \frac{d}{d\ln\mu}\tilde{K}(b_T;\mu) = -\gamma_K(\mu)$$

$$\frac{\partial}{\partial\ln\sqrt{\zeta}}\left(\frac{d\ln F(x, b_T, \mu, \zeta)}{d\ln\mu}\right) = -\gamma_K(\mu)$$

$$\gamma_F(\mu, \zeta/\mu^2)$$

$$\gamma_F(\mu, \zeta_0/\mu^2) - \gamma_F(\mu, \zeta/\mu^2) = -\gamma_K(\mu)\ln\sqrt{\zeta_0} + \gamma_K\ln\sqrt{\zeta}$$

if $\zeta_0 = \mu^2$ then

$$\underline{\gamma_F(\mu, \zeta/\mu^2) = \gamma_F(\mu, 1) - \frac{1}{2}\gamma_K(\mu)\ln\zeta/\mu^2}$$

Solutions

1) $$\frac{d\tilde{K}(b_T,\mu)}{d\ln\mu} = -\gamma_K(\mu) \Rightarrow$$

$$\int_{\mu_0}^{\mu} d\tilde{K}(b_T,\mu) = -\int_{\mu_0}^{\mu} \gamma_K(\mu')\frac{d\mu'}{\mu'}$$

$$\tilde{K}(b_T,\mu) = -\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\gamma_K(\mu') + \tilde{K}(b_T,\mu_0)$$

2) $$\tilde{F}(x,b_T,\mu,\zeta) = \tilde{F}(x,b_T,\mu,\zeta_0)\exp\left[\tilde{K}(b_T,\mu)\ln\sqrt{\frac{\zeta}{\zeta_0}}\right]$$

3) $$\tilde{F}(x,b_T,\mu_0,\zeta) = \tilde{F}(x,b_T,\mu_0,\zeta)\exp\left[\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\gamma_F(\mu';\zeta/\mu'^2)\right]$$

Implementing evolution

We start with low b_T

$$\tilde{F}_f(x, b_T, \mu, \zeta) = \sum_j \int_x \frac{d\hat{x}}{\hat{x}} \underbrace{\tilde{C}_{j/f}\left(\frac{x}{\hat{x}}, b_T, \mu, \zeta\right)}_{\text{coefficient functions}} \underbrace{f_j(\hat{x}, \mu)}_{\text{collinear pdfs}}$$

at the lowest order

$$\tilde{C}_{j/f} = \delta_{jf}\, \delta\left(\frac{x}{\hat{x}} - 1\right)$$

Next step: combine perturbative & non perturbative $\to b_*$

Problem: $\tilde{K}(b_T)$, $\tilde{F}(b_T)$ are non perturbative at large b_T

We want: write functions such that they are perturbatively calculable <u>with</u> nonperturb. corrections

$$b_*(b_T) = \frac{b_T}{\sqrt{1 + b_T^2/b_{max}^2}}$$

$$\tilde{K}(b_T,\mu) = \tilde{K}(b_*,\mu) + \underbrace{\left[\tilde{K}(b_T,\mu) - \tilde{K}(b_*,\mu)\right]}_{\substack{g_K(b_T) \\ \text{non pert. function}}}$$

$$\underbrace{\tilde{K}(b_T,\mu) = \tilde{K}(b_*,\mu_0) - \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'}\gamma_K(\mu') - g_K(b_T)}$$

study in Mathematica

$$\underbrace{g_K = \tfrac{1}{2} g_2 b_T^2, \quad g_K = g_0 b_T b_*, \quad g_K = g_2 \ln b_T/b_*}_{\text{different groups}}$$

We can choose $\mu_0 \sim 1/b_T$, $\mu_0 = \dfrac{2e^{-\gamma_E}}{b_*}$ is the standard choice

Euler gamma

$$\tilde{F}(x, b_T, \mu, \zeta) = \tilde{F}(x, b_*, \mu, \zeta)\left[\frac{\tilde{F}(x, b_T, \mu, \zeta)}{\tilde{F}(x, b_*, \mu, \zeta)}\right] =$$

$$= \tilde{F}(x, b_*, \mu, \zeta_0)\exp\left[\tilde{K}(b_*, \mu)\ln\sqrt{\frac{\zeta}{\zeta_0}}\right]\underbrace{\left[\frac{\tilde{F}(x, b_T, \mu, \zeta_0)}{\tilde{F}(x, b_*, \mu, \zeta_0)}\right]}_{\exp[-g(x, b_T)]}$$

$$\times \exp\left[\ln\sqrt{\frac{\zeta}{\zeta_0}}\left(\tilde{K}(b_T, \mu) - \tilde{K}(b_*, \mu)\right)\right]$$

$$= \tilde{F}(x, b_*, \mu, \zeta_0)\exp\left[\ln\sqrt{\frac{\zeta}{\zeta_0}}\,\tilde{K}(b_*, \mu)\right]\exp\left[-g(x, b_T) - \ln\sqrt{\frac{\zeta}{\zeta_0}}\,g_K(b_T)\right]$$

$$= \tilde{F}(x, b_*, \mu_0, \zeta_0)\exp\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\left(\gamma_F(\mu'; 1) - \ln\underline{\sqrt{\frac{\zeta_0}{\mu'^2}}}\,\gamma_K(\mu')\right)$$

$$\exp\left[\ln\sqrt{\frac{\zeta}{\zeta_0}}\,\tilde{K}(b_*, \mu_0) - \int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\underline{\ln\sqrt{\frac{\zeta}{\zeta_0}}}\,\gamma_K(\mu')\right]$$

$$\exp\left[-g(x, b_T) - \ln\sqrt{\frac{\zeta}{\zeta_0}}\,g_K(b_T)\right]$$

$$= \tilde{F}(x, b_*, \mu_0, \zeta_0) \exp\Big\{ \ln\sqrt{\frac{\zeta}{\zeta_0}}\, \tilde{K}(b_*, \mu_0)$$

$$+ \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \Big[\gamma_F(\mu', 1) - \ln\sqrt{\frac{\zeta}{\mu'^2}}\, \gamma_K(\mu') \Big] \Big\}$$

$$\times \exp\Big[-g(x, b_T) - \ln\sqrt{\frac{\zeta}{\zeta_0}}\, g_K(b_T) \Big]$$

at small b_T and large μ:

$$\tilde{K}(b_T, \mu) = -\frac{g^2 C_F}{4\pi^2}\Big(\ln(\mu^2 b_T) - \underbrace{\ln 4 + 2\gamma_E}_{} \Big)$$

the reason why

$$\mu_b \sim \frac{2e^{-\gamma_E}}{b_*} \text{ is chosen}$$

$$\bar{K}(b_*, \mu_b) = 0$$

ζ_0 is a scale $\sim 1 - 2\ (\mathrm{GeV}^2)$.

Generalized parton model usually is.

$$\tilde{F}(x, b_T) \sim F(x) \exp\left[-g(x, b_T)\right].$$

Lecture 3

Elements of evolution of TMDs

We have studied so far how structure function can be written in terms of TMDs.

For instance

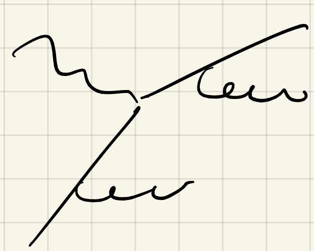
$$F_{UU} = \mathcal{C}\left[1\, f_1\, D_1\right]$$

In the Generalized Parton Model one often uses

$$f_1(x, k_T) = f_1(x)\,\frac{1}{\pi\langle k_T^2\rangle}\, e^{-k_T^2/\langle k_T^2\rangle}$$

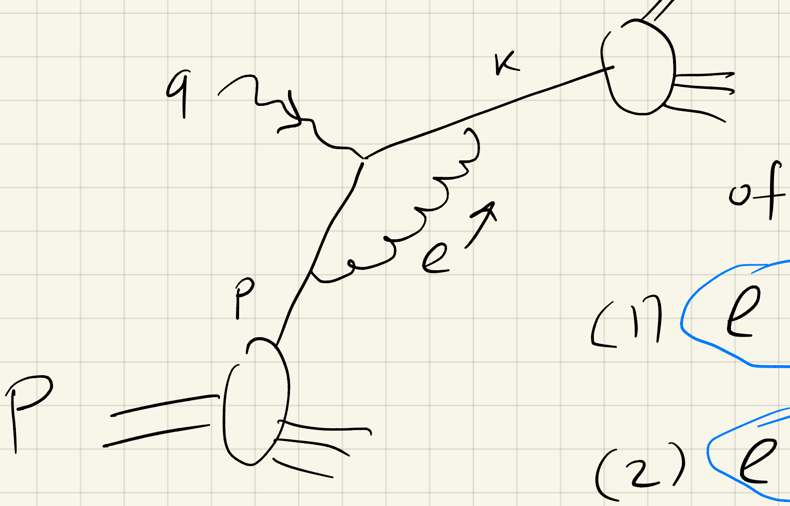
$$D_1(z, p_T) = D_1(z)\,\frac{1}{\pi\langle p_T^2\rangle}\, e^{-p_T^2/\langle p_T^2\rangle}$$

Of course this Gaussian dependence can be a good approximation of intrinsic k_T dependence but what happens if we take into account gluon radiation?



?

Consider 1 gluon



$$\int \frac{d^4 l}{(2\pi)^4} \to \text{factorisation}$$

$\lambda \ll 1$

of regions

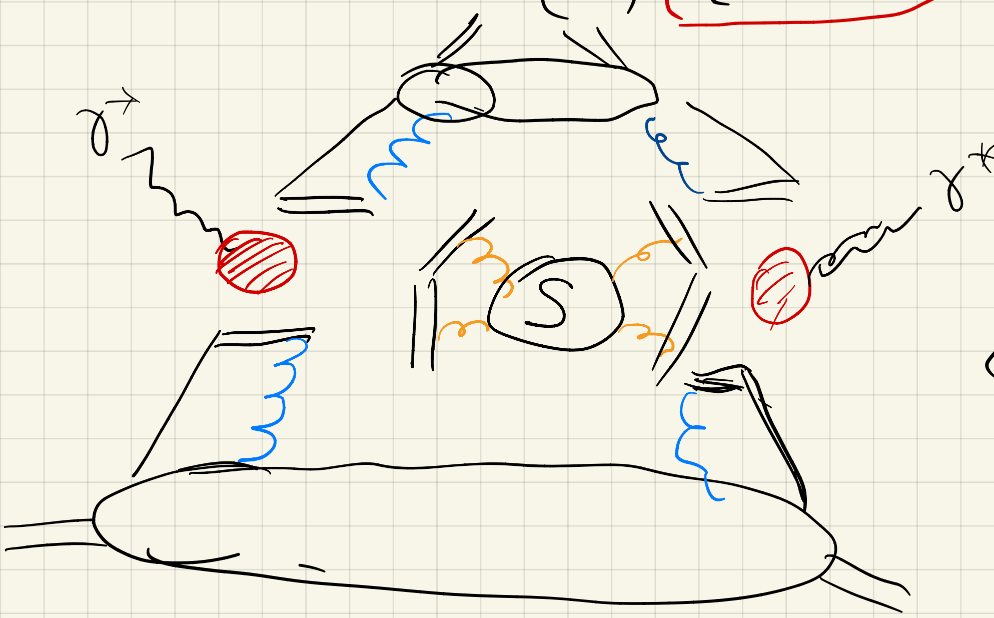
$\qquad\qquad\qquad\qquad\quad + \quad - \quad T$

(1) $l \parallel p$ $\quad l \sim Q(1, \lambda^2, \lambda)$

(2) $l \parallel k$ $\quad l \sim Q(\lambda^2, 1, \lambda)$

(3) l soft $\quad l \sim Q(\lambda, \lambda, \lambda)$

(4) l hard $\quad l \sim Q(1, 1, 1)$



= are Wilson lines (I will not talk about them here)

$$\sigma \sim \int d^2k_T \, d^2p_T \, d^2l_T \, H(Q) \, f(x, k_T) \, D(z, p_T)$$

$$\cdot \, S(l_T) \, \delta^{(2)}(\vec{P}_{hT} - z\vec{k}_T - \vec{p}_T - \underbrace{\vec{l}_T}_{\text{gluon radiation}})$$

$$\underbrace{G \sim \int \frac{d^2 b_T}{(2\pi)^2} e^{i \vec{P}_{hT} \vec{b}_T / z} H(Q) \tilde{f}(x, b_T) \tilde{D}(z, b_T) S(b_T)}$$

See next lecture for the proof

The additional factor $S(b_T)$ is absorbed into $\tilde{f}$ & $\tilde{D}$. It leads to cancelation of divergencies and self consistent definition

$$\tilde{f}(x, b_T, Q, \zeta) \to \tilde{f}(x, b_T) \sqrt{S(b_T)}$$

$$\tilde{D}(z, b_T, \underset{\text{UV scale}}{Q}, \underset{\text{rapidity scale}}{\zeta}) \to \tilde{D}(z, b_T) \underbrace{\sqrt{S(b_T)}}_{\text{effect of radiation}}$$

$$G \sim H(Q) \int \frac{d^2 b_T}{(2\pi)^2} e^{i \vec{P}_{hT} \vec{b}_T / z} \underbrace{\tilde{f}(x, b_T, Q, \zeta) \tilde{D}(z, b_T, Q, \zeta)}_{\text{exactly like in Generalised parton model!}}$$

We would like to write

$\tilde{f}(x, b_T, Q, \zeta)$ starting from some initial scales Q_0, ζ_0

QCD evolution of TMDs is governed by 3 equations

1) Collins-Soper equation (CS)

$$\frac{\partial \ln \tilde{f}(x, b_T, \mu, \zeta)}{\partial \ln \sqrt{\zeta}} = \tilde{K}(b_T, \mu)$$

$\tilde{K}$ is the so-called Collins-Soper kernel

it can be calculated perturbatively for small b_T & large μ (so that $\alpha_s(\mu)$ is small)

$$\tilde{K}(b_T, \mu) = -8 C_F \frac{\alpha_s(\mu)}{4\pi} \ln\left(\frac{b_T \mu}{2e^{-\gamma_E}}\right) + O(\alpha_s^2)$$

$\gamma_E \simeq 0.57$ Euler constant

The problem:

We need to $\int \frac{d^2 b_T}{(2\pi)^2} \to \int_0^\infty b_T \, db_T$

but $\ln\left(\frac{b_T \mu}{2e^{-\gamma_E}}\right)$ will become large for $b_T \to \infty$

it corresponds to non-perturbative regime of $k_T \to 0$.

Solution later

② Renormalisation group equation

$$\frac{d\tilde{K}(b_T,\mu)}{d\ln\mu} = -\gamma_K(\alpha_s(\mu))$$

γ_K is Cusp anomalous dimention. It is present in many areas of physics

$$\gamma_K(\alpha_s) = \sum_{i=1}^{\infty} \gamma_K^i \left(\frac{\alpha_s}{4\pi}\right)^i = 8C_F\left(\frac{\alpha_s}{4\pi}\right) + \mathcal{O}(\alpha_s^2)$$

③ $$\frac{d\ln\tilde{f}(x,b_T,\mu,\zeta)}{d\ln\mu} = \gamma_F(\alpha_s(\mu), \zeta/\mu^2)$$

γ_F is the anomalous dimention of f

$$\gamma_F(\alpha_s(\mu),1) = \sum_{i=1}^{\infty} \gamma_F^i \left(\frac{\alpha_s}{4\pi}\right)^i = 6C_F\left(\frac{\alpha_s(\mu)}{4\pi}\right) + \mathcal{O}(\alpha_s^2)$$

Let us write the solutions:

$$\frac{d \ln \tilde{f}(x, b_T, \mu, \zeta)}{d \ln \mu} = \gamma_F(\mu, \zeta/\mu^2)$$

$$\int_{\mu_0}^{\mu} d \ln \tilde{f}(x, b_T, \mu, \zeta) = \int_{\mu_0}^{\mu} \gamma_F(\mu', \zeta/\mu'^2) \frac{d\mu'}{\mu'}$$

$$\frac{\tilde{f}(x, b_T, \mu, \zeta)}{\tilde{f}(x, b_T, \mu_0, \zeta)} = \exp\left[\int_{\mu_0}^{\mu} \gamma_F(\mu', \zeta/\mu'^2) \frac{d\mu'}{\mu'} \right]$$

$$\tilde{f}(x, b_T, \mu, \zeta) = \tilde{f}(x, b_T, \mu_0, \zeta) \exp\left[\int_{\mu_0}^{\mu} \gamma_F(\mu', \zeta/\mu'^2) \frac{d\mu'}{\mu'} \right]$$

(2) $$\frac{\partial \ln \tilde{f}(x, b_T, \mu, \zeta)}{\partial \ln \sqrt{\zeta}} = \tilde{K}(b_T, \mu)$$

$$\Rightarrow \quad \tilde{f}(x, b_T, \mu, \zeta) = f(x, b_T, \mu, \zeta_0) \exp\left[\tilde{K}(b_T, \mu) \ln \sqrt{\frac{\zeta}{\zeta_0}}\right]$$

(3) $$\frac{d \tilde{K}(b_T, \mu)}{d \ln \mu} = -\gamma_K(\mu)$$

$$\Rightarrow \quad \tilde{K}(b_T, \mu) = \tilde{K}(b_T, \mu_0) - \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \gamma_K(\mu')$$

Let us also combine 2 equations

$$\frac{d}{d\ln\mu}\left(\frac{\partial \ln \tilde{f}(x,b_T,\mu,\zeta)}{\partial \ln\sqrt{\zeta}}\right) = \frac{d}{d\ln\mu}\tilde{K}(b_T,\mu) = -\gamma_K(\mu)$$

those commute!

$$\frac{\partial}{\partial \ln\sqrt{\zeta}}\left(\underbrace{\frac{d\ln \tilde{f}(x,b_T,\mu,\zeta)}{d\ln\mu}}_{\gamma_F(\mu,\,\zeta/\mu^2)}\right) = -\gamma_K(\mu)$$

$$\Rightarrow \quad \gamma_F(\mu,\zeta/\mu^2) - \gamma_F(\mu,\zeta_0/\mu^2) = -\gamma_K(\mu)\ln\sqrt{\zeta/\zeta_0}$$

If we use $\zeta_0 = \mu^2$ then

$$\gamma_F(\mu,\zeta/\mu^2) = \gamma_F(\mu,1) - \tfrac{1}{2}\gamma_K(\mu)\ln(\zeta/\mu^2)$$

Implementing the evolution

1) Operator Product Expantion (OPE)
at low b_T:

$$\tilde{f}_f(x, b_T, \mu, \zeta) = \sum_j \int_x^1 \frac{d\hat{x}}{\hat{x}} \underbrace{\tilde{C}_{j/f}\left(\frac{x}{\hat{x}}, b_T, \mu, \zeta\right)}_{\text{coefficient functions}} \underbrace{f_j(\hat{x}, \mu)}_{\text{collinear pdfs for unpol. f}} + O(b_T^2)$$

$$\tilde{C}_{j/f} = \delta_{jf}\, \delta\left(\frac{x}{\hat{x}} - 1\right) + O(\alpha_s^2)$$

2) Combine perturbative & non perturbative
(solution to problem of $\tilde{K}(b_T)$ nonpert @ large b_T)

$$b_* = \frac{b_T}{\sqrt{1 + b_T^2 / b_{max}^2}}$$

if b_{max} is small ($\sim 1\ \text{GeV}^{-1}$) then
b_* is always perturbative for $\forall\, b_T$

Start from CS kernel

$$\tilde{K}(b_T,\mu) = \tilde{K}(b_*,\mu) + \underbrace{\tilde{K}(b_T,\mu) - \tilde{K}(b_*,\mu)}_{g_K(b_T)}$$

universal non pert. function!

g_K does not depend on μ, in fact

$$\frac{d}{d\ln\mu}\left[\tilde{K}(b_T,\mu) - \tilde{K}(b_*,\mu)\right] = \gamma_K(\mu) - \gamma_K(\mu) = \phi$$

next

$$\tilde{K}(b_T,\mu) = \tilde{K}(b_*,\mu_0) - \int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'}\gamma_F(\mu') - g_K(b_T)$$

For convergence, remember,

$$\tilde{K} = -8 C_F \frac{\alpha_s(\mu)}{4\pi}\ln\left(\frac{b_T\mu}{2e^{-\gamma_E}}\right)$$

$$\mu_0 = \frac{2e^{-\gamma_E}}{b_T} \equiv \mu_b \quad \text{but at large } b_T \to \infty$$

$$\mu_b \to 0 \Rightarrow \alpha_s(0) \to \infty \quad \underline{\text{Landau pole}}$$

Because the function is non perturbative.

To avoid it $\mu_b = \dfrac{2e^{-\gamma_E}}{b_*} \to \dfrac{2e^{-\gamma_E}}{b_{max}} \gg \Lambda_{QCD}$

Study in Mathematica

$$\tilde{K}(b_T, \mu) = \tilde{K}(b_*, \mu_b) - \int_{\mu_b}^{\mu} \frac{d\mu'}{\mu'} \gamma_K(\mu') - g_K(b)$$

for a realistic $g_K(b) = g_0 \cdot \ln\left(b/b_*\right)$

Now let us write $f(x,b_T,\mu,\zeta)$ in terms of $f(x,b_T,\mu_0,\zeta_0)$:

$$\tilde f(x,b_T,\mu,\zeta) = \tilde f(x,b_*,\mu,\zeta)\left[\frac{\tilde f(x,b_T,\mu,\zeta)}{\tilde f(x,b_*,\mu,\zeta}\right] =$$

$$= \tilde f(x,b_*,\mu,\zeta_0)\exp\left[\tilde K(b_*,\mu)\ln\sqrt{\frac{\zeta}{\zeta_0}}\right]\underbrace{\left[\frac{\tilde f(x,b_T,\mu,\zeta_0)}{\tilde f(x,b_*,\mu,\zeta_0)}\right]}_{\exp[-g(x,b_T)]}$$

$$\cdot\exp\left[\ln\sqrt{\frac{\zeta}{\zeta_0}}\underbrace{\left(\tilde K(b_T,\mu)-\tilde K(b_*,\mu)\right)}_{g_K(b_T)}\right]$$

np behav. of TMD

$$= \tilde f(x,b_*,\mu_0,\zeta_0)\exp\left[\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\left(\gamma_F(\mu';1)-\ln\sqrt{\frac{\zeta_0}{\mu'^2}}\gamma_K(\mu')\right)\right]$$

$$\cdot\exp\left[\ln\sqrt{\frac{\zeta}{\zeta_0}}\tilde K(b_*,\mu_0) - \int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\ln\sqrt{\frac{\zeta}{\zeta_0}}\gamma_K(\mu')\right]$$

$$\cdot\exp\left[-g(x,b_T)-\ln\sqrt{\frac{\zeta}{\zeta_0}}g_K(b_T)\right]$$

$$= \tilde f(x,b_*,\mu_0,\zeta_0)\exp\left[\ln\sqrt{\frac{\zeta}{\zeta_0}}\tilde K(b_*,\mu_0)+\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\left[\gamma_F(\mu';1)-\ln\sqrt{\frac{\zeta}{\mu'^2}}\gamma_K(\mu')\right]\right]$$

$$\cdot\exp\left[-g(x,b_T)-\ln\sqrt{\frac{\zeta}{\zeta_0}}g_K(b_T)\right]$$

Let us use $\mu_0 = \mu_b = \frac{2e^{-\gamma_E}}{b_*}$

$$\zeta_0 = Q_0^2 \sim 1-2\ (\mathrm{GeV}^2)$$

Then: , $\zeta = Q^2$ (the scale)

$$\tilde{f}(x, b_T, Q, Q^2) = \tilde{f}(x, b_*, \mu_b, Q_0^2)\left(\frac{Q}{Q_0}\right)^{\tilde{K}(b_*, \mu_b) - g_K(b)}$$

$$\cdot \exp\left[\int_{\mu_b}^{\mu} \frac{d\mu'}{\mu'}\left(\gamma_F(\mu', 1) - \ln\frac{Q}{\mu'}\gamma_K(\mu')\right)\right]$$

$$\cdot \exp\left[-g(x, b_T)\right]$$

$\downarrow$ Sadakov formfactor

$$\exp[S]$$

Contains result of gluon radiation

$$\underbrace{\tilde{f}(x, b_T, Q, Q^2) = \tilde{f}(x, b_*, \mu_b, Q_0^2)\, e^{-g(x, b_T)}\, e^{S}}_{\text{almost like GPM!}}$$

$$\simeq \tilde{f}(x, \mu_b)\, e^{-g(x, b_T)}\, e^{S} \quad \text{if} \begin{cases} g(x, b_T) \simeq \frac{b_T^2 \langle k_T^2 \rangle}{4} \\ S \simeq 0 \end{cases}$$

study in Mathematica.

We will use Mathematica to study how it differs from GPM at higher scales.