Formules for e^+e^-

We will use just one method called A_0

$$A_0^{UL(C)}(z_1, z_2, \theta, P_{h\perp}) = \frac{\langle \sin^2\theta \rangle}{\langle 1+\cos^2\theta \rangle} \left(\frac{Z_c^U}{Z_{uu}^U} - \frac{Z_c^{L(C)}}{Z_{uu}^{L(C)}} \right)$$

$$A_0^{UL(C)}(z_1, z_2, \theta) = \frac{\langle \sin^2\theta \rangle}{\langle 1+\cos^2\theta \rangle} \left(\frac{\int dP_\perp P_\perp \ldots}{\int \ldots} - \frac{\int \ldots}{\int \ldots} \right)$$

$Z_{uu}^U \equiv Z_{uu}^{\pi^+\pi^-} + Z_{uu}^{\pi^-\pi^+}$ (unlike sign)

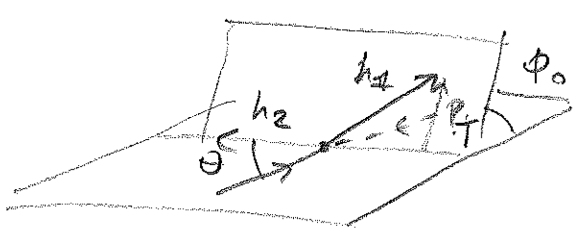
$Z_{uu}^L = Z_{uu}^{\pi^+\pi^+} + Z_{uu}^{\pi^-\pi^-}$, (like sign)

$Z_{uu}^C = Z_{uu}^U + Z_{uu}^L$, (charged hadrons)

Generic cross section

$$\frac{d\sigma}{dz_1\, dz_2\, d^2P_T\, d\cos\theta} = \frac{N_c \pi \alpha^2}{2Q^2} \left[(1+\cos^2\theta)\, Z_{uu} + \sin^2\theta \cos 2\phi_0\, Z_c \right]$$

azimuthal modulation is in $\cos 2\phi_0$

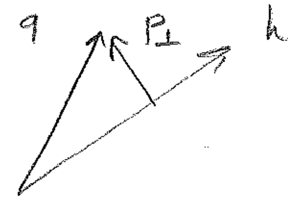


Unpolarized fragmentation function.

We start from definition of Yuan et al

$$D_1(z,p_\perp) = \int_0^\infty \frac{b\,db}{2\pi} \frac{D_1(z)}{z^2} e^{-\frac{b^2 g_h}{z^2}} J_0\left(\frac{p_\perp b}{z}\right) =$$

$$= D_1(z) \frac{e^{-p_\perp^2/4g_h}}{\pi\, 4 g_h} \quad , \text{ we will use } \langle p_\perp^2\rangle \equiv 4 g_h$$



$$D_1(z,p_\perp) = D_1(z) \frac{e^{-p_\perp^2/\langle p_\perp^2\rangle}}{\pi \langle p_\perp^2\rangle}$$

b space fragmentation

$$D_1(z,b) = \frac{D_1(z)}{z^2} e^{-\frac{b^2 g_h}{z^2}} \equiv \frac{D_1(z)}{z^2} e^{-\frac{b^2\langle p_\perp^2\rangle}{4z^2}}$$

Structure function Z_{uu} in b space is:

$$\tilde{Z}_{uu}(b) \equiv \sum e_q^2 \frac{D_1(z_1)\, D_2(z_2)}{z_1^2} e^{-b^2\left(\frac{\langle p_\perp^2\rangle_1}{4z_1^2} + \frac{\langle p_\perp^2\rangle_2}{4z_2^2}\right)}$$

P_T space

$$Z_{uu}(P_T) = \int_0^\infty \frac{b\,db}{2\pi} J_0\left(\frac{b P_T}{z_1}\right) \tilde{Z}_{uu}(b) =$$

$$= \sum e_q^2 z_2^2\, D_1(z_1)\, D_2(z_2) \frac{e^{-\frac{P_T^2 z_2^2}{z_2^2\langle p_\perp^2\rangle_1 + z_1^2\langle p_\perp^2\rangle_2}}}{\pi\left(z_2^2\langle p_\perp^2\rangle_1 + z_1^2\langle p_\perp^2\rangle_2\right)}$$

If we call $\langle P_T^2 \rangle (z_1, z_2) = \dfrac{z_2^2 \langle p_\perp^2 \rangle_1 + z_1^2 \langle p_\perp^2 \rangle_2}{z_2^2}$

$$F_{UU}(P_T) = D_1(z_1)\, D_2(z_2)\, \frac{e^{-P_T^2 / \langle P_T^2 \rangle}}{\pi \langle P_T^2 \rangle}$$

Inverse Fourier definition

$$D(z, b^2) = \frac{1}{z^2} \int d^2 p_\perp\, e^{-\vec{p}_\perp \cdot \vec{b}/z}\, D(z, p_\perp^2) =$$

$$= \frac{2\pi}{z^2} \int d p_\perp\, p_\perp\, J_0(p_\perp b / z)\, D(z, p_\perp) =$$

$$= \frac{1}{z^2}\, D_1(z)\, e^{-\frac{b^2 \langle p_\perp^2 \rangle}{4 z^2}}$$

Contribution from Collins function:

$$Z_C = \frac{1}{z_1^2}\frac{1}{4z_1z_2}\int_0^\infty \frac{db\, b^3}{2\pi} J_2\left(\frac{P_T b}{z_1}\right) e^{-\left(\frac{g_{H_1}-g_{C_1}}{z_1^2}+\frac{g_{H_2}-g_{C_2}}{z_2^2}\right)b^2}$$

$$\times \sum_q e_q^2\, \hat{H}^{(3)}(z_1)\,\hat{H}^{(3)}(z_2) =$$

$$= \sum_q e_q^2\, \hat{H}^{(3)}(z_1)\,\hat{H}^{(3)}(z_2)\, \frac{e^{-\frac{P_T^2 z_2^2}{4(g_{H_2}-g_{C_2})z_1^2+4(g_{H_1}-g_{C_1})z_2^2}}}{64\pi\left((g_{H_2}-g_{C_2})z_1^2+(g_{H_1}-g_{C_1})z_2^2\right)^3}\, P_T^2 z_1 z_2^5$$

Relation of $\hat{H}^{(3)}$ and Collins FF

$$\hat{H}^{(3)}(z) = \int d^2p_\perp \frac{p_\perp^2}{M_h}\left(-\frac{1}{z}\right) H_1^\perp(z, p_\perp^2)\Big|_{\text{Trento}}$$

We use the following parametrisation

$$H_1^\perp(z, p_\perp) = H_1^{\perp(1)}(z)\frac{2z^2M_h^2}{\pi\langle p_\perp^2\rangle^2_{H_1^\perp}}\, e^{-p_\perp^2/\langle p_\perp^2\rangle_{H_1^\perp}}$$

so that:

$$\hat{H}^{(3)}(z) = -2z M_h H_1^{\perp(1)}(z)$$

or

$$\hat{H}^{(3)}(z)\,\frac{e^{-p_\perp^2/4(g_H-g_C)}}{\pi(4(g_H-g_C))^2}\,p_\perp = \frac{p_\perp}{M_h}\left(-\frac{1}{z}\right)H_1^\perp(z,p_\perp^2)\Big|_{\text{Trento}}$$

$$\Rightarrow\quad 4(g_H-g_C) \equiv \langle p_\perp^2\rangle_{H_1^\perp}$$

So that:

$$z_c = \sum_q e_q^2 \; H_1^{\perp(1)}(z_1)\, H_1^{\perp(1)}(z_2) \; \frac{e^{-\frac{P_T^2 z_2^2}{\langle p_{\perp 2}^2\rangle_{H_1^\perp} z_1^2 + \langle p_{\perp 1}^2\rangle_{H_1^\perp} z_2^2}}}{\pi\left(\langle p_{\perp 2}^2\rangle_{H_1^\perp} z_1^2 + \langle p_{\perp 1}^2\rangle_{H_1^\perp} z_2^2\right)^3}$$

$$\times \; P_\perp^2 \, z_1 z_2^5 \; 4 z_1 M_h z_2 M_h$$

Finally

$$z_c = \sum_q e_q^2 \; H_1^{\perp(1)}(z_1)\, H_1^{\perp(1)}(z_2) \; \frac{e^{-\frac{P_T^2 z_2^2}{(\langle p_{\perp 2}^2\rangle_{H_1^\perp} z_1^2 + \langle p_{\perp 1}^2\rangle_{H_1^\perp} z_2^2)}}}{\pi\left(\langle p_{\perp 2}^2\rangle_{H_1^\perp} z_1^2 + \langle p_{\perp 1}^2\rangle_{H_1^\perp} z_2^2\right)^3}$$

$$\times \; P_\perp^2 \; 4 M_h^2 z_1^2 z_2^6$$

If we define $\quad \langle P_T^2\rangle_{H_1^\perp} = \dfrac{\langle p_{\perp 2}^2\rangle_{H_1^\perp} z_1^2 + \langle p_{\perp 1}^2\rangle_{H_1^\perp} z_2^2}{z_2^2}$

then

$$z_c = \sum_q e_q^2 \; H_1^{\perp(1)}(z_1)\, H_1^{\perp(1)}(z_2) \; \frac{e^{-P_T^2/\langle P_T^2\rangle_{H_1^\perp}}}{\pi \langle P_T^2\rangle_{H_1^\perp}^3}$$

$$\times \; P_\perp^2 \; 4 M_h^2 z_1^2$$

Let us derive formulas for integrated SF

$$\int dP_T\, P_T\, Z_{UU}(P_T) = \sum_q e_q^2\, D_1^q(z_1)\, D_1^{\bar q}(z_2)$$

$$\int dP_T\, P_T\, Z_c(P_T) = -\sum_q e_q^2\, \hat{H}_1^{(3)}(z_1)\, \hat{H}^{(3)}(z_2)$$

$$\times \frac{z_1 z_2}{(\langle p_\perp^2\rangle_{1H_1^\perp} z_2^2 + \langle p_\perp^2\rangle_{2H_1^\perp} z_1^2)}$$

$$= -\sum_q e_q^2\, H_1^{\perp(1)}(z_1) H_1^{\perp(1)}(z_2)\, \frac{m_h^2\, z_1^2 z_2^2}{2(\langle p_\perp^2\rangle_{1H_1^\perp} z_2^2 + \langle p_\perp^2\rangle_{2H_1^\perp} z_1^2)}$$

Let us compare to Anselmino:

Eq(31) $$N = \frac{1}{4}\,\frac{z_1 z_2}{z_1^2+z_2^2}\,\sin^2\theta_2 \sum e_q^2\,\tilde{\Delta}^N D(z_1)\,\tilde{\Delta}^N D(z_2)\;\frac{2\,e\,P_{1T}^2}{\tilde{M}_c^2+\langle\tilde{p}_\perp^2\rangle}\;\frac{\exp\left[-\frac{P_{1T}^2}{\tilde{M}_c^2}-\frac{P_{1T}^2}{\langle\tilde{p}_\perp^2\rangle}\right]}{\pi\langle\tilde{p}_\perp^2\rangle}$$

where $$\tilde{M}_c^2 = M_c^2\,\frac{z_1^2+z_2^2}{z_2^2}\,,\quad \langle\tilde{p}_\perp^2\rangle = \langle p_\perp^2\rangle\,\frac{z_1^2+z_2^2}{z_2^2}$$

Eq(30) $$D = (1+\cos^2\theta_2)\,\underbrace{\sum e_q^2\,D(z_1)\,D(z_2)\,\frac{e^{-P_{1T}^2/\langle\tilde{p}_\perp^2\rangle}}{\pi\langle\tilde{p}_\perp^2\rangle}}$$

It corresponds to result of page(3)

Let us check N:

$$\frac{1}{\tilde{M}_c^2}+\frac{1}{\langle\tilde{p}_\perp^2\rangle} = \frac{\langle\tilde{p}_\perp^2\rangle+\tilde{M}_c^2}{\tilde{M}_c^2\langle\tilde{p}_\perp^2\rangle} = 1\Big/\left(\left(\underbrace{\frac{M_c^2\langle p_\perp^2\rangle}{M_c^2+\langle p_\perp^2\rangle}}_{\langle p_\perp^2\rangle_{H_1^\perp}\text{ in my notations}}\right)\frac{z_1^2+z_2^2}{z_2^2}\right)$$

and Anselmino does not distinguish 1 and 2.

$\tilde{\Delta}^N D = 2\, N^c_{(z)} D(z)$ so we have

$$N = \frac{z_1 z_2}{z_1^2 + z_2^2} \left(\frac{z_2^2}{z_1^2 + z_2^2} \right)^2 \sin^2\theta_2 \sum_i e_q\, N^c(z_1)\, N^c(z_2)\, D_1(z_1) D_1(z_2)$$

$$\frac{2 e\, P_{1T}^2}{M_c^2 + \langle p_\perp^2 \rangle} \quad \frac{\exp\left[- P_{1T}^2 / \langle p_\perp^2 \rangle_{H_1^\perp} \right]}{\pi \langle p_\perp^2 \rangle}$$

Now let us calculate $H_1^{\perp(1)}(z)$ from Torinos parametrization

$$\Delta^N D = 2\, N^c(z)\, D_1(z) \sqrt{2e}\, \frac{p_\perp}{M_c}\, e^{-p_\perp^2/M_c^2}\, e^{-p_\perp^2/\langle p_\perp^2\rangle} \frac{1}{\pi \langle p_\perp^2 \rangle}$$

$$\Delta^N D = \frac{2 p_\perp}{z\, m_h} H_1^\perp(z, p_\perp)$$

$$\Rightarrow \quad H_1^\perp(z, p_\perp) = z\, m_h\, N^c(z)\, D_1(z) \sqrt{2e}\, \frac{1}{M_c} \frac{1}{\pi \langle p_\perp^2 \rangle}\, e^{-p_\perp^2 / \underbrace{\langle p_\perp^2 \rangle_{H_1^\perp}}_{\frac{M_c^2 \langle p_\perp^2 \rangle}{M_c^2 + \langle p_\perp^2 \rangle}}}$$

$$H_1^\perp(z, p_\perp) = H_1^{\perp(1)}(z)\, \frac{2 z^2 M_h^2}{\pi \langle p_\perp^2 \rangle_{H_1^\perp}^2}\, e^{-p_\perp^2 / \langle p_\perp^2 \rangle_{H_1^\perp}}$$

(see page 4)

$$\Rightarrow \quad H_1^{\perp(1)}(z) = z\, m_h\, N^c(z)\, D_1(z) \sqrt{2e}\, \frac{1}{M_c} \frac{1}{\pi \langle p_\perp^2 \rangle} \cdot \frac{\pi \langle p_\perp^2 \rangle^2_{H_1^\perp}}{2 z^2\, m_h}$$

$$= \frac{N_c(z)\, D_1(z)}{2 z\, m_h} \sqrt{2e}\, \frac{\langle p_\perp^2 \rangle^2_{H_1^\perp}}{M_c \langle p_\perp^2 \rangle}$$

$$N = \frac{z_1 \, z_2^5}{(z_1^2 + z_2^2)^3} \sin^2\theta_2 \sum_q e_q^2 \, H_1^{\perp(1)}(z_1) \, H_1^{\perp(1)}(z_2)$$

$$\cdot \; 2\frac{z_1 \, m_h}{\sqrt{2e}} \, 2\frac{z_2 \, m_h}{\sqrt{2e}} \; \frac{(M_C \langle p_\perp^2 \rangle)^2}{\langle p_\perp^2 \rangle^4_{H_1^\perp}} \; \frac{2e \, P_{1T}^2}{M_C^2 + \langle p_\perp^2 \rangle} \; \frac{e^{-P_{1T}^2 / \langle p_\perp^2 \rangle_{H_1^\perp} \cdot \left(\frac{z_2^2}{z_1^2 + z_2^2}\right)}}{\pi \langle p_\perp^2 \rangle}$$

$$= \frac{4 \, z_1^2 \, z_2^6}{(z_1^2 + z_2^2)^3} \sin^2\theta_2 \sum_q e_q^2 \, H_1^{\perp(1)}(z_1) \, H_1^{\perp(1)}(z_2)$$

$$M_h^2 \; \frac{(M_C \langle p_\perp^2 \rangle)^2}{\pi \langle p_\perp^2 \rangle (M_C^2 + \langle p_\perp^2 \rangle) \langle p_\perp^2 \rangle^4_{H_1^\perp}} \; P_{1T}^2 \; e^{-P_{1T}^2 / \langle p_\perp^2 \rangle_{H_1^\perp} \cdot \left(\frac{z_2^2}{z_1^2 + z_2^2}\right)}$$

$$\frac{(M_C \langle p_\perp^2 \rangle)^2}{(M_C^2 + \langle p_\perp^2 \rangle) \langle p_\perp^2 \rangle \langle p_\perp^2 \rangle^4_{H_1^\perp}} = \frac{M_C^2 \langle p_\perp^2 \rangle}{(M_C^2 + \langle p_\perp^2 \rangle) \langle p_\perp^2 \rangle^4_{H_1^\perp}} =$$

$$= \frac{1}{\langle p_\perp^2 \rangle^3_{H_1^\perp}}$$

so $$N = \frac{z_1^2 \, z_2^6}{(z_1^2 + z_2^2)^3} \sin^2\theta_2 \sum_q e_q^2 \, H_1^{\perp(1)}(z_1) \, H_1^{\perp(1)}(z_2)$$

$$4 M_h^2 \; \frac{P_{1T}^2}{\pi \langle p_\perp^2 \rangle^3_{H_1^\perp}} \; e^{-\frac{P_{1T}^2}{\langle p_\perp^2 \rangle_{H_1^\perp}} \frac{z^2}{(z_1^2 + z_2^2)}}$$

if $\langle P_T^2 \rangle_{H_1^\perp} = \langle p_\perp^2 \rangle_{H_1^\perp} \frac{z_1^2 + z_2^2}{z_2^2}$

then

$$N = \frac{z_1^2 \, z_2^6}{(z_1^2 + z_2^2)^3} \sin^2\theta_2 \sum_q e_q^2 \, H_1^{\perp(1)}(z_1) \, H_1^{\perp(1)}(z_2)$$

$$M_h^2 \frac{P_{1T}^2}{\pi \langle P_T^2 \rangle_{H_1^\perp}^3} \frac{(z_1^2 + z_2^2)^3}{z_2^6} e^{-P_{1T}^2 / \langle P_T^2 \rangle_{H_1^\perp}}$$

$$= z_1^2 \sin^2\theta_2 \sum_q e_q^2 \, H_1^{\perp(1)}(z_1) \, H_1^{\perp(1)}(z_2)$$

$$4 M_h^2 \frac{P_{1T}^2}{\pi \langle P_T^2 \rangle_{H_1^\perp}^3} e^{-P_{1T}^2 / \langle P_T^2 \rangle_{H_1^\perp}} .$$

$\Rightarrow$ the same result as on page 5 .