Reasoning About Group Polarization: From Semantic Games to Sequent Systems

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Abstract

Group polarization, the phenomenon where individuals become more extreme after interacting, has been gaining attention, especially with the rise of social media shaping people's opinions. Recent interest has emerged in formal reasoning about group polarization using logical systems. In this paper we consider the modal logic PNL that captures the notion of agents agreeing or disagreeing on a given topic. Our contribution involves enhancing PNL with advanced formal reasoning techniques, instead of relying on axiomatic systems for analyzing group polarization. To achieve this, we introduce a semantic game tailored for (hybrid) extensions of PNL. This game fosters dynamic reasoning about concrete network models, aligning with our goal of strengthening PNL's effectiveness in studying group polarization. We show how this semantic game leads to a provability game by systemically exploring the truth in all models. This leads to the first cut-free sequent systems for some variants of PNL. Using polarization of formulas, the proposed calculi can be modularly adapted to consider different frame properties of the underlying model.

1 Introduction

Group polarization – where the opinions or beliefs of individuals within a group become more extreme or polarized after interacting with each other – is rapidly gaining attraction, especially with the advent of social media platforms that have played a key role in the polarization of social, political, and democratic processes. This phenomenon is mainly studied in psychology [21, 15] and political philosophy [28, 29]. More recently, logicians have taken up the challenge of formal reasoning about social networks and changes in agents's beliefs. Take for instance the Facebook logic [26] (an epistemic logic endowed with a symmetric "friendship" relation); the Tweeting logic [33] (formalizing announcements in a network); the logic for reasoning about social belief and change propagation [17], etc.

We focus on the modal logic **PNL** [32], which refers to Kripke frames with two types of disjoint and symmetric reachability relations. The individuals in a social network are identified with worlds of the frame, and they are related either as "friends" (positive) or as "enemies" (negative), but not both at the same time. These relationships can be understood in different ways: Instead of genuine friendship or enduring enmity, they may simply signify agreement or disagreement on a particular issue. Unlike the aforementioned logics, **PNL** was designed to reason about positive and negative relations among agents, a key aspect for defining and measuring polarization [4]. In fact, polarization can actually be studied in a neutral, network theory framework. Under the framework of balance theory, it is possible to investigate the essential conditions required for network stability in a fully polarized social network. For instance, the *local balance* condition that prohibits triangles of nodes with two positive and one negative connection can be associated with the formation of clusters of pairwise positively connected nodes that are negatively connected to all nodes outside the cluster (see Example 1).

We take inspiration from a work by Pedersen, Smets, and Ågnotes [25], where **PNL** is extended in various ways to axiomatically characterize modally undefinable frame properties, including the disjointness of the two relations and collective connectedness. The main challenge is the axiomatization of the balance property, which requires extensions of **PNL** with nominals, dynamic and hybrid operators.

Our approach to logical reasoning about group polarization is also based on **PNL** but focuses on a different aspect of formal reasoning about the corresponding models via games and proof systems. Games are a powerful tool to bridge the gap between intended and formal semantics, often offering a conceptually more natural approach to logic than the common paradigm of model-theoretic semantics. In semantic games [14], every instance of the game is played over a formula ϕ and a model $\mathbb{M}$ by two players, usually called *I* (or *Me*) and *You*. At each point in the game, one of the players acts as the proponent (**P**), while the other acts as the opponent (**O**) of the current formula. The set of actions at each stage is dictated by the main connective of the current formula. In contrast to semantic games, provability games [18] do not refer to truth in a particular model but to *logical validity*. The game is also played by two players, *Me* and *You*, and consists of attacking assertions of formulas made by the other player and defending against these attacks. In this work, we will introduce both a semantic game and a provability game for (hybrid) extensions of **PNL**.

We start by proposing a semantic game that characterizes the truth in a given network model. This provides an alternative to the standard definition of an evaluation function which supports a dynamic form of reasoning about concrete network models (Section 2). We move on by arguing that effective formal reasoning with the relevant logics requires more than (just) Hilbert-style axiom systems. Rather, the automated search for proofs calls for Gentzen-style systems that respect (a restricted form of) the subformula property. In proof-theoretic terms, we are looking for a cut-free sequent system. Hence, our next step is to turn the semantic game over single models into a provability game (Section 3), characterizing logical validity. To this end, we define disjunctive states for a game that is not restricted to a single model, but systematically explores the truth in all models. This method leads to *the first* Gentzen-style systems for variants of **PNL** (Section 4), which modularly adapts to different frame properties by faithfully capturing the rules for *elementary* games.

Models of social learning and opinion dynamics aim to understand the role of certain social factors in the acceptance/rejection of opinions. They can be useful to explain alternative scenarios, such as consensus or polarization. In this context, the positive and negative relationships are not permanent. Instead, they can vary over time when *enemies* reconcile, new *friendships*/agreements emerge, or some actors begin to disagree with others. In Section 5, we show how the *global adding* and *local link change* modalities of [25] (inspired by sabotage modal logic [2, 3, 30]) can be defined in our framework. As a plus, we present in [11] a prototypical implementation of the proposed games using rewriting logic and Maude [20, 6].

2 A Game Semantics for PNL

In this section, we revisit the positive and negative relations logic [32, 25] with nominals (**PNL**) and its standard Kripke semantics, proposing a novel semantic game for **PNL** that we prove to be adequate. Paving the way for the provability game introduced in Section 3, we also propose an alternative presentation of **PNL** that internalizes the nominals.

$\mathbb{M}, \mathsf{a} \Vdash p$	iff $\mathsf{a} \in \mathsf{V}(p)$	$\mathbb{M}, \mathsf{a} \Vdash \neg\phi$	iff $\mathbb{M}, \mathsf{a} \nVdash \phi$
$\mathbb{M}, \mathsf{a} \Vdash R^+(i,j)$	iff $(\mathsf{g}(i), \mathsf{g}(j)) \in \mathsf{R}^+$	$\mathbb{M}, \mathsf{a} \Vdash R^-(i,j)$	iff $(\mathsf{g}(i), \mathsf{g}(j)) \in \mathsf{R}^-$
$\mathbb{M}, \mathsf{a} \Vdash \phi \wedge \psi$	iff $\mathbb{M}, \mathsf{a} \Vdash \phi$ and $\mathbb{M}, \mathsf{a} \Vdash \psi$	$\mathbb{M}, \mathsf{a} \Vdash \phi \vee \psi$	iff $\mathbb{M}, \mathsf{a} \Vdash \phi$ or $\mathbb{M}, \mathsf{a} \Vdash \psi$
$\mathbb{M}, \mathsf{a} \Vdash \diamondplus\, \phi$	iff there is $\mathsf{b} \in \mathsf{A}$ such that $(\mathsf{a}, \mathsf{b}) \in \mathsf{R}^+$ and $\mathbb{M}, \mathsf{b} \Vdash \phi$		
$\mathbb{M}, \mathsf{a} \Vdash \diamondminus\, \phi$	iff there is $\mathsf{b} \in \mathsf{A}$ such that $(\mathsf{a}, \mathsf{b}) \in \mathsf{R}^-$ and $\mathbb{M}, \mathsf{b} \Vdash \phi$		
$\mathbb{M}, \mathsf{a} \Vdash [A]\phi$	iff $\mathbb{M}, \mathsf{b} \Vdash \phi$ for all $\mathsf{b} \in \mathsf{A}$		

Figure 1: Kripke semantics for **PNL** [25]

2.1 Kripke semantics for PNL

Let $\mathsf{A} = \{\mathsf{a}, \mathsf{b}, \ldots\}$ be a non-empty set of agents, $\mathsf{At} = \{p, q, \ldots\}$ be a countable set of propositional variables, and $N = \{i, j, \ldots\}$ be a countable set of *nominals*. The language of **PNL** is generated by the following grammar:

$$\phi ::= p \mid R^+(i,j) \mid R^-(i,j) \mid \neg\phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \diamondplus\, \phi \mid \diamondminus\, \phi \mid [A]\phi$$

where $p \in \mathsf{At}$, and $i, j \in N$. The propositional connectives $\top$, $\bot$, $\to$, and the (dual) modalities $\boxplus$ and $\boxminus$ can be obtained in the usual way.

Formulas of the form p, $R^+(i,j)$, or $R^-(i,j)$ are called *elementary*. The proposition $R^+(i,j)$ states that i is a *friend* of (or, more generally, *agrees* with) j, and proposition $R^-(i,j)$ states that agent i is an *enemy* of (or *disagrees* with) j. The formula $\diamondplus\, \phi$ (resp. $\diamondminus\, \phi$) states that ϕ holds for a friend (resp. an enemy). The global modality $[A]\phi$ states that ϕ holds for all the agents. We use $R^\pm$ to denote either R^+ or R^-, and $\Diamond^\pm$ to denote either $\diamondplus$ or $\diamondminus$.

A model $\mathbb{M}$ is a tuple $\langle \mathsf{A}, \mathsf{R}^+, \mathsf{R}^-, \mathsf{V}, \mathsf{g}\rangle$ where A is a set (of agents), $\mathsf{g} : N \to \mathsf{A}$ is called *denotation function*, $\mathsf{R}^+, \mathsf{R}^- \subseteq \mathsf{A} \times \mathsf{A}$, and $\mathsf{V} : \mathsf{At} \to \mathcal{P}(\mathsf{A})$. A model is a **PNL**-model if R^+ is reflexive, and R^+ and R^- are both symmetric and non-overlapping, i.e., for all $\mathsf{a}, \mathsf{b} \in \mathsf{A}$, $(\mathsf{a}, \mathsf{b}) \notin \mathsf{R}^+$ or $(\mathsf{a}, \mathsf{b}) \notin \mathsf{R}^-$. We say that a model $\mathbb{M}$ is *collectively connected*, or a **cc-PNL**-model, if, additionally, for all $\mathsf{a}, \mathsf{b} \in \mathsf{A}$, $(\mathsf{a}, \mathsf{b}) \in \mathsf{R}^+$ or $(\mathsf{a}, \mathsf{b}) \in \mathsf{R}^-$. The class of all **PNL** models (**cc-PNL**-models) is denoted by $\mathfrak{M}_{PNL}$ ($\mathfrak{M}_{ccPNL}$).

The Kripke semantics of **PNL** is in Figure 1. A formula ϕ is true over $\mathbb{M}$, written $\mathbb{M} \Vdash \phi$ iff $\mathbb{M}, \mathsf{a} \Vdash \phi$, for all agent $\mathsf{a} \in \mathsf{A}$. For a set of formulas Φ, we write $\mathbb{M} \models \Phi$ iff $\mathbb{M} \Vdash \Phi$ for all $\phi \in \Phi$. A formula ϕ is ((**cc-**)**PNL-**) valid iff $\mathbb{M} \Vdash \phi$ for every ((**cc-**)**PNL**)-model $\mathbb{M}$. For a class of models $\mathfrak{M}$, we write $\Phi \models_{\mathfrak{M}} \phi$ iff $\mathbb{M} \Vdash \phi$ for every model $\mathbb{M} \in \mathfrak{M}$ with $\mathbb{M} \models \Phi$.

A model is called *named* if its denotation function is onto, i.e., every agent has a name. Let $\mathfrak{M}_N$ be the class of named models. The following result is immediate.

Lemma 1. *Let $\Phi \cup \{\phi\}$ be a finite set of formulas. Then $\Phi \Vdash_{\mathfrak{M}_{PNL}} \phi \iff \Phi \Vdash_{\mathfrak{M}_{PNL} \cap \mathfrak{M}_N} \phi$ and $\Phi \Vdash_{\mathfrak{M}_{ccPNL}} \phi \iff \Phi \Vdash_{\mathfrak{M}_{ccPNL} \cap \mathfrak{M}_N} \phi$*

Using this lemma, we give an alternative presentation of the semantics where we explicitly test, using elementary formulas, the existence of $R^\pm$-successors. Let $\mathsf{a} = \mathsf{g}(i)$. Then, we define:

$\mathbb{M}, \mathsf{a} \Vdash \Diamond^\pm\phi$	iff there is $j \in N$ such that $\mathbb{M}, \mathsf{g}(j) \Vdash R^\pm(i,j)$ and $\mathbb{M}, \mathsf{g}(j) \Vdash \phi$
$\mathbb{M}, \mathsf{a} \Vdash [A]\phi$	iff $\mathbb{M}, \mathsf{g}(j) \Vdash \phi$, for all $j \in N$,

Remark 1. *At a later point, we need the following observation: if g is surjective, even if restricted to $N' \subseteq N$, then N in the above truth conditions can equivalently be replaced by N'.*

($\mathbf{P}_\wedge$) At $\mathbf{P}, \mathsf{a} : \phi_1 \wedge \phi_2$, *You* choose between $\mathbf{P}, \mathsf{a} : \phi_1$ and $\mathbf{P}, \mathsf{a} : \phi_2$ to continue the game.

($\mathbf{O}_\wedge$) At $\mathbf{O}, \mathsf{a} : \phi_1 \wedge \phi_2$, *I* choose between $\mathbf{O}, \mathsf{a} : \phi_1$ and $\mathbf{O}, \mathsf{a} : \phi_2$ to continue the game.

($\mathbf{P}_\vee$) At $\mathbf{P}, \mathsf{a} : \phi_1 \vee \phi_2$, *I* choose between $\mathbf{P}, \mathsf{a} : \phi_1$ and $\mathbf{P}, \mathsf{a} : \phi_2$ to continue the game

($\mathbf{O}_\vee$) At $\mathbf{O}, \mathsf{a} : \phi_1 \vee \phi_2$, *You* choose between $\mathbf{O}, \mathsf{a} : \phi_1$ and $\mathbf{O}, \mathsf{a} : \phi_2$ to continue the game.

($\mathbf{P}_\neg$) At $\mathbf{P}, \mathsf{a} : \neg\phi$, the game continues with $\mathbf{O}, \mathsf{a} : \phi$.

($\mathbf{O}_\neg$) At $\mathbf{O}, \mathsf{a} : \neg\phi$, the game continues with $\mathbf{P}, \mathsf{a} : \phi$.

($\mathbf{P}_{\Diamond^\pm}$) At $\mathbf{P}, \mathsf{a} : \Diamond^\pm\phi$, *You* win if there are no $\mathsf{R}^\pm$-successors of a. Otherwise, *I* choose an $\mathsf{R}^\pm$-successor b and the game continues with $\mathbf{P}, \mathsf{b} : \phi$.

($\mathbf{O}_{\Diamond^\pm}$) At $\mathbf{O}, \mathsf{a} : \Diamond^\pm\phi$, *I* win if there are no $\mathsf{R}^\pm$-successors of a. Otherwise, *You* choose an $\mathsf{R}^\pm$-successor b and the game continues with $\mathbf{O}, \mathsf{b} : \phi$.

($\mathbf{P}_{[A]}$) At $\mathbf{P}, \mathsf{a} : [A]\phi$, *You* choose an agent b and the game continues with $\mathbf{P}, \mathsf{b} : \phi$.

($\mathbf{O}_{[A]}$) At $\mathbf{O}, \mathsf{a} : [A]\phi$, *I* choose an agent b and the game continues with $\mathbf{O}, \mathsf{b} : \phi$.

($\mathbf{P}_{el}$) Let ϕ_e be an elementary formula. *I* win and *You* lose at $\mathbf{P}, \mathsf{a} : \phi_e$ iff $\mathbb{M}, \mathsf{a} \models \phi_e$. Otherwise, *You* win and *I* lose.

($\mathbf{O}_{el}$) At $\mathbf{O}, \mathsf{a} : \phi_e$, *I* win and *You* lose iff $\mathbb{M}, \mathsf{a} \not\models \phi_e$. Otherwise, *You* win and *I* lose.

Figure 2: Semantic game given a **PNL**-model $\mathbb{M}$.

2.2 Semantic Game

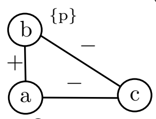
The proposed *semantic game* is played over a **PNL**-model $\mathbb{M} = (\mathsf{A}, \mathsf{R}, \mathsf{V}, \mathsf{g})$ by two players, *Me* and *You*, who argue about the truth of a formula ϕ at an agent a. At each stage of the game, one player acts as *proponent*, while the other acts as *opponent* of the claim that ϕ is true at a. We represent the situation where *I* am the proponent (and *You* are the opponent) by the *game state* $\mathbf{P}, \mathsf{a} : \phi$, and the situation where *I* am the opponent (and *You* are the proponent) by $\mathbf{O}, \mathsf{a} : \phi$. We call a game state *elementary* if its involved formula is elementary. For a game state g, we denote the game starting at g over the model $\mathbb{M}$ by $\mathbf{G}_\mathbb{M}(g)$.

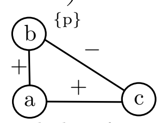
The game proceeds by reducing the involved formula ϕ to an elementary formula. The following rules of the game describe the possible choices of the players depending on the current game state, when playing a game over the model $\mathbb{M}$.

Definition 1. *Let $\mathbb{M}$ be a **PNL**-model. The semantic game is defined by the rules in Figure 2.*

In general, every two-person, zero-sum, win-lose game is usually represented by a *game tree*, i.e., a labeled tree whose nodes are game states. In our case, the root of the game tree representing the game $\mathbf{G}_\mathbb{M}(g)$ is g. The children of each node in the game tree are exactly the possible choices of the corresponding player. For instance, if $h = \mathbf{P}, \mathsf{a} : \phi_1 \wedge \phi_2$ appears in the game tree, then its children are $\mathbf{P}, \mathsf{a} : \phi_1$ and $\mathbf{P}, \mathsf{a} : \phi_2$. Each node in the tree is labeled either "I", or "Y", depending on which player is to move in the corresponding game state, and we label the nodes $\mathbf{P}, \mathsf{a} : \neg\phi$ and $\mathbf{O}, \mathsf{a} : \neg\phi$ with "I" (even though there is no choice involved in these game states). For instance, the node corresponding to the game state h above is "Y", since it is *Your* choice in $\mathbf{P} : \phi_1 \wedge \phi_2$. The leaves of the tree receive the label of the winning player. A *run* of the game is a maximal path through the game tree.

Example 1. *Let* $(\mathbf{4B}) = ((\Diamond^+\Diamond^+ p \vee \Diamond^-\Diamond^- p) \rightarrow \Diamond^+ p) \wedge ((\Diamond^+\Diamond^- p \vee \Diamond^-\Diamond^+ p) \rightarrow \Diamond^- p)$. *This formulas specifies* local balance *[25] and captures the idea that "the enemy of my enemy is my friend",* *"the friend of my enemy is my enemy", and "the friend of my friend is my friend". A collectively connected network where $[A]4B$ holds is a polarized network, where agents can be divided into two opposing groups [13]. Notions such a weak-balance [5] can be also formalized in **PNL** [25].* I *have a winning strategy for the game* $\mathbf{P}, \mathsf{a} : 4B$ *on* $\mathbb{M}_1$ *while* You *have a winning strategy for the same game on* $\mathbb{M}_2$ *where (omitting self-loops for* R^+*):*

$\mathbb{M}_1 =$ 

$\mathbb{M}_2 =$ 

For $\mathbb{M}_1$*, in the first conjunct,* I *pick* $(\mathbf{P}_\vee)$ $\Diamond^+ p$ *and then* b *in* $(\mathbf{P}_{\Diamond^+})$*; for the second conjunct,* I *pick the first disjunction in* $F = (\Diamond^+\Diamond^- p \vee \Diamond^-\Diamond^+ p) \rightarrow \Diamond^- p)$ *where, in any of* Your *choices* ($\mathbf{P}_\neg$ *followed by* $\mathbf{O}_\vee$ *and* $\mathbf{O}_{\Diamond^\pm}$*),* I *win all the elementary states. For* $\mathbb{M}_2$, I *do not have a winning strategy for the second conjunct:* I *can neither win* $\Diamond^- p$ *(no* R^- *successor), nor the first disjunct in* F *above since, after* $\mathbf{P}_\neg$*,* You *choose* $(\mathbf{O}_\vee)$ $\Diamond^+\Diamond^- p$ *and select* c *and then* b $(\mathbf{O}_{\Diamond^\pm})$ *where* p *holds and* You *win. See the complete game in our tool [11] and Appendix B.*

The following proposition follows from the fact that every rule in the game decomposes the involved formula in subformulas.

Proposition 1. *For all models* $\mathbb{M}$ *and game states* g*, the game tree of* $\mathbf{G}_{\mathbb{M}}(g)$ *is of finite height.*

An immediate consequence of this fact is that every semantic game $\mathbf{G}_{\mathbb{M}}(g)$ is *determined*, i.e., exactly one of the two players has a winning strategy.

2.3 Strategies and adequacy

Now we are ready to define winning strategies and prove the main result of this section: the adequacy of the proposed game semantics with respect to the Kripke semantics for **PNL**.

Definition 2. *A* strategy *for* Me *in the game* $\mathbf{G}_{\mathbb{M}}(g)$ *is a subtree* σ *of the associated game tree such that:* ***(1)*** $g \in \sigma$*,* ***(2)*** *if* $h \in \sigma$ *is a node labeled "Y", then all children of* h *are in* σ*,* ***(3)*** *if* $h \in \sigma$ *is a node labeled "I", then exactly one child of* h *is in* σ*. The strategy* σ *is called* winning *if all leaves in the tree* σ *are labeled "I". (Winning) strategies for* You *are defined dually.*

Note that every combination of strategies from *Me* and *You* defines a unique run of the game. Alternatively, we can define a strategy σ for *Me* to be winning iff the run resulting from σ and a strategy for *You* ends in a winning game state for *Me*.

We can now show the adequacy of the semantic game, *i.e.*, that the existence of winning strategies for Me in a game and truth in a model coincide (proof in Appendix A).

Theorem 1. *Let* $\mathbb{M}$ *be a* ***PNL****-model,* a *an agent, and* ϕ *a formula.*
(1) I *have a winning strategy for* $\mathbf{G}_{\mathbb{M}}(\mathbf{P}, \mathsf{a} : \phi)$ *iff* $\mathbb{M}, \mathsf{a} \models \phi$.
(2) You *have a winning strategy for* $\mathbf{G}_{\mathbb{M}}(\mathbf{P}, \mathsf{a} : \phi)$ *iff* $\mathbb{M}, \mathsf{a} \not\models \phi$.

Internalizing nominals. Remember that in a named model $\mathbb{M}$, every agent a has a name i, i.e. there exists $i \in N$ s.t. $\mathsf{g}(i) = \mathsf{a}$. Therefore, for $\mathbf{Q} \in \{\mathbf{P}, \mathbf{O}\}$, it is unambiguous if we write $\mathbf{Q}, i : \phi$ for the game state $\mathbf{Q}, \mathsf{a} : \phi$. The fact that $\mathbb{M} \models R^{\pm}(i, j)$ iff $(\mathsf{g}(i), \mathsf{g}(j)) \in \mathsf{R}^{\pm}$ gives us the following equivalent formulations of the rules for $\Diamond^+$, $\Diamond^-$ and $[A]$[1]:

[1]The outcome of the game state $\mathbf{Q}, k : R^{\pm}(i, j)$ is independent of k (it only depends on the underlying model $\mathbb{M}$). Hence, we write $\mathbf{Q}, \_ : R^{\pm}(i, j)$ instead of $\mathbf{Q}, k : R^{\pm}(i, j)$.

($\mathbf{P}_{\Diamond^\pm}$) At $\mathbf{P}, i : \Diamond^\pm\phi$, *I* choose a nominal j, and *You* decide whether the game ends in the state $\mathbf{P}, \_ : R^\pm(i,j)$ or continues with $\mathbf{P}, j : \phi$.

($\mathbf{O}_{\Diamond^\pm}$) At $\mathbf{O}, i : \Diamond^\pm\phi$, *You* choose j, and *I* choose between $\mathbf{O}, \_ : R^\pm(i,j)$ and $\mathbf{O}, j : \phi$.

($\mathbf{P}_{[A]}$) At $\mathbf{P}, i : [A]\phi$, *You* choose a nominal j and the game continues with $\mathbf{P}, j : \phi$.

($\mathbf{O}_{[A]}$) At $\mathbf{O}, i : [A]\phi$, *I* choose a nominal j, and the game continues with $\mathbf{O}, j : \phi$.

Following Remark 1, we can restrict branching over a subset N' of nominals if g's restriction to N' is surjective. For future reference, let us formulate this observation precisely. We denote the game over $\mathbb{M}$ starting at the game state g, where branching is over $N' \subseteq N$ by $\mathbf{G}_{\mathbb{M}}^{N'}(g)$. We call a model N'-named if $\mathsf{g} : N' \to \mathsf{A}$ is surjective. We say that two games $\mathbf{G}_1$ and $\mathbf{G}_2$ are *strategically equivalent*, notation $\mathbf{G}_1 \cong \mathbf{G}_2$, iff *I* have a winning strategy in both games or in none of the two. We have the following:

Proposition 2. *Let $\mathbb{M}$ be N'-named, let $g = \mathbf{Q}, \mathsf{a} : \phi$ for some $\mathbf{Q} \in \{\mathbf{P}, \mathbf{O}\}$, some agent a and formula ϕ, and let $\mathsf{a} = \mathsf{g}(i)$. Then $\mathbf{G}_{\mathbb{M}}(\mathbf{Q}, \mathsf{a} : \phi) \cong \mathbf{G}_{\mathbb{M}}^{N'}(\mathbf{Q}, i : \phi)$.*

3 The Disjunctive Game

In this section, we lift the semantic game to two different *disjunctive games*, one for **PNL**-validity and one for **cc-PNL**-validity, which differ only in their respective winning conditions. We prove that the obtained games are adequate and can thus be regarded as *provability games* for their corresponding logics. The main crucial fact is that the rules of the semantic game are independent of the underlying model, except for elementary game states. Using this fact, and identifying certain conditions on winning strategies, this disjunctive game will be the foundation for the sequent calculi proposed in Section 4.

3.1 Playing on all models

The disjunctive game $\mathbf{DG}(\mathbf{P}, i : \phi)$ can be thought of as *Me* and *You* playing all semantic games $\mathbf{G}(\mathbf{P}, i : \phi)$ over all **PNL**-models $\mathbb{M}$ simultaneously. We point out that the rules of the semantic game do not depend on the structure of $\mathbb{M}$ but merely on ϕ. Truth degrees are only needed at the atomic level to determine who wins the particular run of the game. This allows us to require players to play "blindly", i.e., without explicitly referencing a model $\mathbb{M}$. Clearly, if *I* have a winning strategy in such a game, then *I* can win in $\mathbf{G}_{\mathbb{M}}(\mathbf{P}, i : \phi)$, for every $\mathbb{M}$, making this strategy an adequate witness of logical validity.

Defining such a validity game is not straightforward, since the simplest case of disjunction is already problematic. Let us consider the game $\mathbf{P}, i : p \vee \neg p$. Clearly, *I* have a winning strategy in the semantic game over every model. However, there is no uniform way of making a good choice in the first turn: No matter whether *I* choose $\mathbf{P}, i : p$ or $\mathbf{P}, i : \neg p$, there are still models where *You* win the game eventually. To compensate for this, we allow *Myself* to create "backup copies" and *duplicate* game states. Formally, disjunctive game states are finite multisets of the game states defined in Section 2.2. We prefer to write $g_1 \bigvee ... \bigvee g_n$ for the disjunctive game state $\{g_1, ..., g_n\}$, but keep the convenient notation $g \in D$ if g is in the multiset D. We write $D_1 \bigvee D_2$ for the multiset sum $D_1 + D_2$ and $D \bigvee g$ for $D + \{g\}$. A disjunctive state is called *elementary* if all its game states are elementary. We use $\mathbf{DG}(D)$ to denote the disjunctive game starting at D, and we define the **cc**-disjunctive game $\mathbf{DG}^{\mathbf{cc}}(D)$ which is played over all **cc-PNL**-models. It differs from $\mathbf{DG}(D)$ only in its winning conditions (see below).

(Dupl) If no state in D is underlined, I can choose a non-elementary $g \in D$ and the game continues with $D \bigvee g$.

(Sched) If no state in $D = D' \bigvee g$ is underlined, and g is non-elementary, I can choose to continue the game with $D' \bigvee \underline{g}$.

(Move) If $D = D' \bigvee \underline{g}$ then the player who is to move in the semantic game $\mathbf{G}(g)$ at g makes a legal move to the game state g' and the game continues with $D' \bigvee g'$. For example, if g is $\mathbf{P}, i : \phi_1 \wedge \phi_2$, then *You* choose a $k \in \{1, 2\}$ and the game continues with $D' \bigvee \mathbf{P}, i : \phi_k$.

(End) The game ends if there are no non-elementary game states left in D, or if no game state is underlined and I win according to Definition 3. Otherwise, I must move according to **(Dupl)** or **(Sched)**.

Figure 3: Rules for the disjunctive game

In our running example, I duplicate the game state in the first round and the game continues with the *disjunctive state* $\mathbf{P}, i : p \vee \neg p \bigvee \mathbf{P}, i : p \vee \neg p$. Now I move to $\mathbf{P}, i : p$ in the first subgame and to $\mathbf{P}, i : \neg p$ in the second. After a role switch in the second subgame, the final state is $\mathbf{P}, i : p \bigvee \mathbf{O}, i : p$, where I win regardless of the underlying model.

The following *winning condition* reflects the fact that *My* strategy for the disjunctive game $\mathbf{DG}(\mathbf{P}, i : p \vee \neg p)$ was successful.

Definition 3. *Let D^{el} denote the disjunctive state consisting of the elementary game states of D.* I *win and* You *lose at D if for every **PNL**-model there is a game state in D^{el} where* I *win the corresponding semantic game. In the **cc**-disjunctive game,* I *win and* You *lose if for every **cc-PNL**-model there is a game state in D^{el} where* I *win the corresponding semantic game.*

In the disjunctive game, I additionally take the role of a *scheduler*, deciding which game is to be played next. We signal the chosen game state by underlining it as in $\underline{g}$.

Definition 4 (Disjunctive game)**.** *The rules of the disjunctive game are in Figure 3. Infinite runs, and runs that end in elementary disjunctive states where* I *do not win according to Definition 3, are winning for* You *and losing for* Me. ***(Dupl)*** *is referred to as the* duplication rule *and* ***(Sched)*** *as the* scheduling*, or* underlining rule*.*

It follows from the Gale-Stewart Theorem [12] that every disjunctive game $\mathbf{DG}(D)$ and every **cc**-disjunctive game $\mathbf{DG}^{\mathbf{cc}}(D)$ is determined.

3.2 Adequacy and best strategies

Now we state the main result of this section, whose proof is split into two propositions: The left-to-right direction in Proposition 3 and the right-to-left direction in Proposition 4 below.

Theorem 2. I *have a winning strategy in $\mathbf{DG}(D)$ (in $\mathbf{DG}^{\mathbf{cc}}(D)$) iff for every (**cc**-) **PNL**-model $\mathbb{M}$, there is some $g \in D$ such that* I *have a winning strategy in $\mathbf{G}_{\mathbb{M}}(g)$.*

Corollary 1. *The formula ϕ is (**cc**-) **PNL**-valid iff* I *have a winning strategy in $\mathbf{DG}(\mathbf{P}, i : [A]\phi)$ (in $\mathbf{DG}^{\mathbf{cc}}(\mathbf{P}, i : [A]\phi)$).*

Proposition 3. *Let $\mathbb{M}$ be a (**cc**-) **PNL**-model. If* I *have a winning strategy in $\mathbf{DG}(D)$ (in $(\mathbf{DG}^{\mathbf{cc}}(D))$, then there is some $g \in D$ such that* I *have a winning strategy in $\mathbf{G}_{\mathbb{M}}(g)$.*

Proof: Let σ be *My* winning strategy in $\mathbf{DG}(D)$. We show the following claim by bottom-up tree induction on σ: For every $H \in \sigma$ there is some $g \in H$ and a winning strategy σ_H for *Me* in $\mathbf{G}_{\mathbb{M}}(g)$. The case for the root, $H = D$, gives the desired result.

If H is a leaf, then, since σ is winning, there is an elementary $h \in H$ such that *I* win the semantic game at h over $\mathbb{M}$. In this case, σ_H consists of the single leaf h.

If *You* move in H, then it must be of the form $H' \bigvee \underline{h}$, where h is labeled "Y" in the semantic game. By definition of strategy, all successors of H must be in σ, and they are of the form $H' \bigvee h'$, where h' ranges over all possible game states immediately after *Your* choice at h. By inductive hypothesis, for every h', $\sigma_{H' \bigvee h'}$ is a winning strategy for some $g \in H' \bigvee h'$. If for some h', $\sigma_{H' \bigvee h'}$ is a winning strategy for some $g \in H'$, then we can set $\sigma_H = \sigma_{H' \bigvee h'}$. Otherwise, all $\sigma_{H' \bigvee h'}$ is a winning strategy for h'. Hence, we can connect the roots of all these trees to the new common root h to obtain a winning strategy for *Me* in $\mathbf{G}_{\mathbb{M}}(h)$.

If *I* move in H according to the rules **(Dupl)** or **(Sched)**, then the resulting disjunctive state is still H (except maybe some game state could be underlined). Hence, the claim follows from the inductive hypothesis. If *I* move in H according to **(Move)**, then $H = H' \bigvee \underline{h}$, and the unique child of H in σ is $H' \bigvee h'$, where h' is a possible game state after *My* move in the semantic game at h. By the inductive hypothesis, $\sigma_{H' \bigvee h'}$ is a winning strategy for *Me* in $\mathbf{G}_M(g)$, for some $g \in H' \bigvee h'$. If $g \in H'$, we proceed as above. If $g = h'$, then we can append the root of $\sigma_{H' \bigvee h'}$ to h to obtain a winning strategy for *Me* in $\mathbf{G}_{\mathbb{M}}(h)$. ■

My best way to play. We will now describe a strategy σ for *Me* for the game $\mathbf{DG}(D_0)$. This strategy is – in a way – the optimal way to play the disjunctive game. Intuitively σ exploits all of *My* possible choices without sacrificing *My* winning chances. Consequently, σ is winning iff *I* can win the game at all. (This claim follows from Proposition 3 by classical reasoning).

Let us fix an enumeration of pairs (g, h) of game states of the semantic game such that every pair appears in this enumeration infinitely often. Let us denote by $\#(g, h)$ the number of the pair (g, h) under this enumeration. Throughout the game, let us keep track of the number of execution steps n of σ. At D_0, $n = 0$. The strategy σ is as follows:

(C1) If in the current disjunctive state D, D^{el} is winning, *I* end the game.

(C2) Otherwise, let $n = \#(g, h)$. If $D = D' \bigvee g$, according to the label of g we have:
(a) "Y" (otherwise, skip), then underline g and *You* make *Your* move.
(b) "I" and h is a child of g in the evaluation game (otherwise, skip), then duplicate g, schedule a copy of g, and go to h in that copy, i.e., the new disjunctive state is $D' \bigvee g \bigvee h$.

(C3) Increase n by 1 and go to (C1).

In words, until the game reaches a winning state, *My* strategy is to play in a way such that *I* always duplicate a state, and then play by exhausting all possible moves in that state.

Let π be the run of the game $\mathbf{DG}(D_0)$ resulting from *You* playing according to *Your* winning strategy and *Me* playing *My* best way σ. We say that a game state g *appears along* π, and write $g \in \pi$, if it occurs in a disjunctive state in π. We say that g *disappears*, if $g \in \pi_n$ and for some $m > n$, $g \notin \pi_m$. The following holds (proof in Appendix A).

Lemma 2. *Let π be as above. Then:*
1) Let $g \in \pi$ be a non-elementary game state labelled "Y" in the semantic game. Then at least one successor of g appears along π.
2) Let $g \in \pi$ be a non-elementary game state labeled "I" in the semantic game. Then all successors of g appear along π.

We can now show that π gives rise to a model $\mathbb{M}_\pi$ with the property that *You* have a winning strategy for every g appearing along π.

Definition 5. *Let $\mathcal{E}$ be a set of elementary game states. Let $\mathbb{M}_\mathcal{E}$ be the following named model:*
*- **Agents** A: an agent a_i for each nominal i appearing in $\mathcal{E}$;*
*- **Accessibility relations**: $\mathsf{R}^-_\mathcal{E}$ is the least symmetric relation such that $\mathsf{a}_i \mathsf{R}^-_\mathcal{E} \mathsf{a}_j$ whenever (i)$^+$ $\mathbf{O}, \_ : R^-(i,j)$ is in $\mathcal{E}$. $\mathsf{R}^+_\mathcal{E}$ is the least reflexive symmetric relation such that (i)$^-$ $\mathsf{a}_i \mathsf{R}^+_\mathcal{E} \mathsf{a}_j$ whenever $\mathbf{O}, \_ : R^+(i,j)$ is in $\mathcal{E}$;*
*- **Valuation function** $\mathsf{V}_\mathcal{E}$: $\mathsf{a}_i \in \mathsf{V}_\mathcal{E}(p)$ iff the state $\mathbf{O}, i : p$ is in $\mathcal{E}$;*
*- **Assignment** $\mathsf{g}_\mathcal{E}$: $\mathsf{g}_\mathcal{E}(i) = \mathsf{a}_i$.*

The model $\mathbb{M}^{\mathbf{cc}}_\mathcal{E}$ is as $\mathbb{M}_\mathcal{E}$, except R^+ also has (ii)$^+$ $\mathsf{a}_i \mathsf{R}^+ \mathsf{a}_j$ if $\mathbf{P}, \_ : R^-(i,j) \in \mathcal{E}$ or if (iii)$^+$ no $\mathbf{Q}, \_ : R^\pm(i,j)$ for $\mathbf{Q} \in \{\mathbf{P}, \mathbf{O}\}$ is in $\mathcal{E}$, and is closed under reflexivity and symmetry, and R^- also has (ii)$^-$ $\mathsf{a}_i R^- \mathsf{a}_j$ if $\mathbf{P}, \_ : R^+(i,j) \in \mathcal{E}$ and is closed under symmetry.

The degree $\delta(\phi)$ of a formula ϕ to be 0 if ϕ is elementary, $\delta(\phi_1 \medwhitestar \phi_2) = \max\{\delta(\phi_1), \delta(\phi_2)\} + 1$ for $\medwhitestar$ a binary and $\delta(\triangle\psi) = \delta(\psi) + 1$ for $\triangle$ a unary operator. We extend δ to game states by setting $\delta(\mathbf{Q}, i : \phi) = \delta(\phi)$ for $\mathbf{Q} \in \{\mathbf{P}, \mathbf{O}\}$.

Lemma 3. *Let $\mathbb{M}_\pi$ be the model $\mathbb{M}_\mathcal{E}$ ($\mathbb{M}^{\mathbf{cc}}_\mathcal{E}$) from the above definition, where $\mathcal{E}$ is the set of all elementary game states appearing along π. This model is a (**cc**-) **PNL** model. Furthermore, if g appears along π, then* You *have a winning strategy for $\mathbf{G}_{\mathbb{M}_\pi}(g)$.*

Proof: To prove that $\mathbb{M}_\mathcal{E}$ is a **PNL**-model it remains to show that it is non-overlapping. Towards a contradiction, assume that $(\mathsf{a}_i, \mathsf{a}_j) \in \mathsf{R}^+_\pi \cap \mathsf{R}^-_\pi$. By definition, both $\mathbf{O}, \_ : R^+(i,j)$ and $\mathbf{O}, \_ : R^-(i,j)$ appear[2] along π. Since the elementary states of π are accumulative, there is a disjunctive state D in π containing both of these states. But then I could have won the game in D, since in every **PNL**-model at least one of $R^+(i,j)$ and $R^-(i,j)$ must be false. This is a contradiction to the assumption that π results from *You* playing *Your* winning strategy.

We show that $\mathbb{M}^{\mathbf{cc}}_\mathcal{E}$ is a **cc-PNL**-model. First, we note that this model is collectively connected, since, $(iii)^+$ is equivalent to $\neg((i)^+ \vee (ii)^+ \vee (i)^- \vee (ii)^-)$. Also, the model remains non-overlapping. Suppose, we had $(\mathsf{a}_i, \mathsf{a}_j) \in \mathsf{R}^+ \cap \mathsf{R}^-$. According to the definition of R^+ and R^- all possible scenarios how this could happen lead to contradiction:
- **(1)** $(i)^+$ and $(i)^-$: At some point, both states appear in a disjunctive state in D. But I would have won the game at D, since there is no non-overlapping model where both $R^+(i,j)$ and $R^-(i,j)$ are true.
- **(2)** $(i)^+$ and $(ii)^-$: I win the game in a disjunctive state, where both $\mathbf{O}, \_ : R^+(i,j), \mathbf{P}, \_ : R^+(i,j) \in \mathcal{E}$ states occur.
- **(3)** $(ii)^-$ and $(i)^-$: Similar to the case above.
- **(4)** $(ii)^-$ and $(ii)^-$: I would have won in a state containing $\mathbf{P}, \_ : R^+(i,j)$ and $\mathbf{P}, \_ : R^-(i,j)$, since there is no collectively connected models where both $R^+(i,j)$ and $R^-(i,j)$ are false.
- **(5)** $(iii)^+$, excludes $(i)^-$ and $(ii)^-$, hence $(\mathsf{a}_i, \mathsf{a}_j) \notin \mathsf{R}^-$.

We prove the second claim by induction on the degree of g. The elementary cases where g is of the form $\mathbf{O}, i : \phi$ follow directly from the definition of $\mathbb{M}_\pi$. Assume $g = \mathbf{P}, i : p$ appears along π, but $\mathbb{M}_\pi, \mathsf{a}_i \nVdash p$. The latter implies that $\mathbf{O}, i : p$ appears along π. Reasoning as above, there is a disjunctive state D in π containing both $\mathbf{P}, i : p$ and $\mathbf{O}, i : p$, which means that D would be winning for *Me*. The case for $\mathbf{P}, \_ : R^\pm(i,j)$ is similar.

For the inductive step, let $g \in \pi$ be non-elementary with the label "Y". By Lemma 2, some child h of g appears along π. By the inductive hypothesis, there is a winning strategy μ_h for

[2]Since the relations are symmetric, we can identify $R^\pm(i,j)$ with $R^\pm(j,i)$.

You for $\mathbf{G}_{\mathbb{M}_\pi}(h)$. Hence, appending the root of μ_h to g gives a winning strategy for *You* in μ_g in $\mathbf{G}_{\mathbb{M}_\pi}(g)$. If g is non-elementary with label "I", then, by Lemma 2, all children h of g appear along π. For each h there is a winning strategy for *You* in $\mathbf{G}_{\mathbb{M}_\pi}(h)$. Thus, appending the roots of all μ_h to the new common root g gives a winning strategy for *You* in $\mathbf{G}_{\mathbb{M}_\pi}(g)$. ■

Proposition 4. *Assume that for every model $\mathbb{M}$, there is some $g \in D$ such that* I *have a winning strategy in $\mathbf{G}_{\mathbb{M}}(g)$. Then,* I *have a winning strategy in $\mathbf{DG}(D)$.*

Proof: Suppose *I* do not have a winning strategy in $\mathbf{DG}(D)$. By the Gale-Stewart Theorem, *You* have a winning strategy in this game. Let *Me* play according to the strategy σ from above, and let π be the run through $\mathbf{DG}(D)$ resulting from *You* playing *Your* winning strategy and *Me* playing σ. Let $\mathbb{M}_\pi$ be the model from Definition 5. By Lemma 2 and Lemma 3, *You* have a winning strategy for $\mathbf{G}_{\mathbb{M}_\pi}(g)$ for all $g \in D$. ■

4 From strategies to Proofs

Theorems 1 and 2 imply that winning strategies for *Me* in the disjunctive game correspond to validity. In this section, we will extend this result to proof systems. This will be done by introducing a sequent calculus **DS**, where proofs correspond to *My* winning strategies in the disjunctive game. Before that, we will demonstrate that winning strategies can be finitized.

4.1 *Your* optimal choices

In this section, we show how to modify the disjunctive game so that it becomes finitely branching in "Y"-nodes. This will help us to conveniently formulate the disjunctive game as a calculus. Infinite branching occurs only in the case of the rules $\mathbf{P}_{[A]}$ and $\mathbf{O}_{\Diamond^\pm}$ where branching is parametrized by the nominals. We will show that in these situations, there is an optimal choice for *You*, so *I* can expect *You* to play according to this choice.

Proposition 5. *Let j be a nominal different from i not occurring in D nor in ϕ. Then:*
(1) You *have a winning strategy in $\mathbf{DG}(D \bigvee \mathbf{P}, i : [A]\phi)$ iff* You *have a winning strategy in $\mathbf{DG}(D \bigvee \mathbf{P}, j : \phi)$, and similarly for $\mathbf{DG}^{\boldsymbol{cc}}$.*
(2) You *have a winning strategy in $\mathbf{DG}(D \bigvee \mathbf{O}, i : \Diamond^\pm\phi)$ iff* You *have winning strategies in both $\mathbf{DG}(D \bigvee \mathbf{P}, \_ : R^\pm(i,j))$ and $\mathbf{DG}(D \bigvee \mathbf{O}, j : \phi)$, and similarly for $\mathbf{DG}^{\boldsymbol{cc}}$.*

This result implies that *My* winning strategies in the disjunctive game can be finitely represented: in every disjunctive state whose children branch over the nominals, it is enough to consider a single child only, given by a *fresh* nominal j not appearing in that disjunctive state. Intuitively, the proof of Proposition 5 runs by transforming a winning strategy for *You* in the semantic game for all game states in $D \bigvee \mathbf{P}, k : \phi$ (where k is arbitrary) over a model $\mathbb{M}$ into a winning strategy for *You* for all game states in $D \bigvee \mathbf{P}, j : \phi$ over a modified model $\mathbb{M}'$, where j is according to the proposition. By the Theorems 1 and 2, this gives the desired result. The actual proof is carried out in the appendix.

4.2 The proof system DS

Now we detail how to formally transform (provability) games into (sequent) proof systems.

Labeled nominal formulas are either *labeled formulas* of the form $i : \phi$ or *relational atoms* of the form $R(i,j)$, where i and j are nominals and ϕ is a **PNL** formula.[3] *Labeled sequents* have

[3] Observe that here we are abusing the notation, identifying $k : R(i,j)$ with $R(i,j)$. Recall from Footnote 1 that the truth value of these atoms depend only on the underlying model.

the form $\Gamma \Rightarrow \Delta$, where Γ, Δ are multisets containing labeled nominal formulas.

Starting with sequents, every disjunctive state of the form

$$\mathbf{O}, i_1 : \phi_1 \bigvee \ldots \bigvee \mathbf{O}, i_n : \phi_n \bigvee \mathbf{P}, j_1 : \psi_1 \bigvee \ldots \bigvee \mathbf{P}, j_m : \psi_m$$

can be rewritten as the labeled sequent $\Gamma \Rightarrow \Delta$ where $\Gamma = \{i_1 : \phi_1, \ldots, i_n : \phi_n\}$ and $\Delta = \{j_1 : \psi_i, \ldots, j_m : \psi_m\}$. In what follows, we will not distinguish between disjunctive states and their corresponding labeled sequent. For example, the disjunctive game state $\mathbf{O}, i : (\Diamond^{+}\Diamond^{+}\, p \vee \Diamond^{-}\Diamond^{-}\, p) \bigvee \mathbf{P}, i : \Diamond^{+}\, p$ will be identified with the sequent $i : (\Diamond^{+}\Diamond^{+}\, p \vee \Diamond^{-}\Diamond^{-}\, p) \Rightarrow i : \Diamond^{+}\, p$.

The inference rules must be tailored in such a way that *proofs* in the sequent system match exactly *My winning strategies* in the disjunctive game. This means that the user of the proof system takes the role of *Me*, scheduling game states and choosing moves in I-states. Moreover, *provability* in the proof system should correspond to *validity* in the game. For that, it is necessary to establish the formal relationship between elementary game states and logical axioms.

Lemma 4. *Let $\Gamma \Rightarrow \Delta$ be composed of elementary game states only.* I *win the disjunctive game in $\Gamma \Rightarrow \Delta$ iff one of the following holds*[4]
i. *$R^-(i,i) \in \Gamma$ or $R^+(i,i) \in \Delta$ for some i;*
ii. *$\{R^+(i,j), R^-(i,j)\} \subseteq \Gamma$ for some $i \neq j$;*
iii. *$\Gamma \cap \Delta \neq \varnothing$.*

In the case of collectively connected models, additionally,
iv. *$\{R^+(i,j), R^-(i,j)\} \subseteq \Delta$ for some $i \neq j$*

Proof: By definition of the disjunctive game, it is immediate that I win the game if (iii) holds. Moreover, I clearly win the game if either (i) or (ii) hold, since only R^+ is reflexive and since the relations are non-overlapping. Finally, if the model is collectively connected, I clearly win the game if (iv) holds.

Suppose that (i), (ii) and (iii) do not hold. Then *You* have a winning strategy in the model $\mathbb{M}_{\Gamma\Rightarrow\Delta}$ from Definition 5. If additionally (iv) does not hold, we choose the model $\mathbb{M}^{\mathbf{cc}}_{\Gamma\Rightarrow\Delta}$.

By proceeding as in Lemma 3, it is easy to see that R^+ is reflexive, $R^\pm$ is symmetric, both models are non-overlapping and $\mathbb{M}^{\mathbf{cc}}_{\Gamma\Rightarrow\Delta}$ is collectively connected. ■

Figure 4 presents the labeled sequent systems **DS** and $\mathbf{DS^{cc}}$, with the standard initial axiom and structural/propositional rules. The modal rules and the relational rules *sym* and $ref\pm$ coincides with the modal rules originally presented by Viganò in [31], adapted to multi-relational modal logics.

It is routine to show that the rules *no* and *cc* in Figure 4 correspond to the non-overlapping and collectively connected axioms, respectively

$$\forall i,j.\neg(R^+(i,j) \wedge R^-(i,j)) \quad \text{and} \quad \forall i,j.R^+(i,j) \vee R^-(i,j)$$

The following result, proved in Appendix A, entails a normal form on proofs in **DS**, since any proof-search procedure can be restricted so to start with applications of logical rules followed by relational rules and the initial axiom.

Lemma 5. *In a bottom-up reading of derivations, the relational rules permutes up w.r.t. any other logical rule in $\mathbf{DS}/\mathbf{DS^{cc}}$. Moreover, the weakening rules below are admissible in $\mathbf{DS}/\mathbf{DS^{cc}}$.*

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, i : \phi \Rightarrow \Delta}\ L_w \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, i : \phi}\ R_w$$

The following result immediately implies that the disjunctive game **DG** is adequate with respect to the calculus **DS**.

[4] Since relations are symmetric, we will identify $R^\pm(i,j)$ with $R^\pm(j,i)$.

Theorem 3. I *have a winning strategy in the disjunctive game* $\mathbf{DG}(\Gamma \Rightarrow \Delta)$ *(in* $\mathbf{DG^{cc}}(\Gamma \Rightarrow \Delta)$*) iff* $\Gamma \Rightarrow \Delta$ *is provable in* $\mathbf{DS}$ *(in* $\mathbf{DS^{cc}}$*).*

Proof: The proof is by case analysis in the rules of the disjunctive game/last rule applied in the proof, and it is based in the following correspondence:
- **(Dupl)** Duplication in the game corresponds to left and right contraction rules.
- **(Sched)** Scheduling game moves over non-elementary formulas corresponds to choosing propositional or modal rules to be applied. Observe that Lemma 5 guarantees that propositional and modal rules in **DS** can always be chosen before dealing with the elementary case.
- **(Move)** Applying the rule chosen in **(Sched)** corresponds to applying the respective sequent rule in **DS**. Note that *I* should be prepared to any movement from *You*. Hence, branching in *Your* possible moves corresponds to branching in a sequent rule. On the other hand, infinite branching is handled as explained in Section 4.1.
- **(End)** Due to Lemma 4, winning states are completely captured by the axioms. ■

Let us write $\models_{\mathbf{PNL}} \Gamma \Rightarrow \Delta$ iff for every **PNL**-model there is some $i : \gamma \in \Gamma$ such that $\mathbb{M}, \mathsf{g}(i) \not\models \gamma$, or there is some $i : \delta \in \Delta$ such that $\mathbb{M}, \mathsf{g}(i) \models \delta$. Similarly, we define $\models_{\mathbf{ccPNL}} \Gamma \Rightarrow \Delta$. We have the following consequence of Theorems 1, 2, and 3:

Corollary 2. *Let* Γ, Δ *be multisets of labeled formulas. Then* $\models_{\mathbf{PNL}} \Gamma \Rightarrow \Delta$ *(*$\models_{\mathbf{ccPNL}} \Gamma \Rightarrow \Delta$*) iff there is a proof of* $\Gamma \Rightarrow \Delta$ *in* **DS** *(in* $\mathbf{DS^{cc}}$*). In particular,* ϕ *is (***cc***-)* ***PNL****-valid iff there is a proof of* $\Rightarrow \phi$ *in* **DS** *(in* $\mathbf{DS^{cc}}$*).*

The next examples shows how $\mathbf{DS^{cc}}$ elegantly captures collectively connectedness.

Example 2 (Connectedness)**.** *The formula* $(\boxplus p \to [A]p) \vee (\boxminus p \to [A]p)$*, characterizing collective connectedness [25], has the following proof in* $\mathbf{DS^{cc}}$*:*

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{cc}
\UnaryInfC{$\Rightarrow j : p, R^{+}(i,j), R^{-}(i,j)$}
\AxiomC{}
\doubleLine
\RightLabel{$R_{\neg}, init$}
\UnaryInfC{$\Rightarrow j : p, j : \neg p, R^{+}(i,j)$}
\RightLabel{$R_{\diamondminus}$}
\BinaryInfC{$\Rightarrow j : p, i : \diamondminus \neg p, R^{+}(i,j)$}
\AxiomC{}
\doubleLine
\RightLabel{$R_{\neg}, init$}
\UnaryInfC{$\Rightarrow j : p, j : \neg p, i : \diamondminus \neg p$}
\RightLabel{$R_{\diamondplus}$}
\BinaryInfC{$\Rightarrow j : p, i : \diamondplus \neg p, i : \diamondminus \neg p$}
\doubleLine
\RightLabel{$R_{[A]}$}
\UnaryInfC{$\Rightarrow i : [A]p, i : \diamondplus p, i : \diamondminus p$}
\doubleLine
\RightLabel{$R_{\vee}, R_{w}, R_{\neg}, L_{\neg}$}
\UnaryInfC{$\Rightarrow i : (\boxplus p \to [A]p) \vee (\boxminus p \to [A]p)$}
\end{prooftree}
$$

Proving cut-admissibility of labeled systems can be cumbersome due to the presence of relational rules. In [19], a systematic procedure for transforming axioms into rules was presented, based on *focusing* and *polarities* [1]. This procedure not only allows for generalizing different approaches for transforming axioms into sequent rules present in the literature [27, 31, 22], but it also provides a uniform way of proving cut-admissibility for the resulting systems.

While it is out of the scope of this paper to introduce all this machinery just to prove cut-admissibility of $\mathbf{DS}/\mathbf{DS^{cc}}$, we note that it is possible to directly transform the semantic description of (**cc**-)**PNL** into a labeled sequent system equivalent to $\mathbf{DS}/\mathbf{DS^{cc}}$, by using the methodology in [19] and adopting the *negative polarity* to atomic formulas. Hence the cut-admissibility result for $\mathbf{DS}/\mathbf{DS^{cc}}$ is a particular instance of the general result in [19].

Theorem 4. *The following cut rule is admissible in* $\mathbf{DS}/\mathbf{DS^{cc}}$

$$
\frac{\Gamma \Rightarrow \Delta, i : \phi \quad i : \phi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ cut
$$

As a consequence, $\mathbf{DS}/\mathbf{DS^{cc}}$ are consistent, since the only rules that can be applied in an empty sequent is *no* and it is routine to show that it does not trivialize derivations. Moreover, cut-admissibility also serves as a tool for proving meta-theoretical properties: $(\boxplus p \to [A]p) \vee (\boxminus p \to [A]p)$ in Example 2 is not provable in **DS**, that is, the rule *cc* is necessary.

Axiom and Structural Rules

$$\frac{}{\Gamma, i : \phi_{el} \Rightarrow \Delta, i : \phi_{el}}\ init \qquad \frac{\Gamma, i : \phi, i : \phi \Rightarrow \Delta}{\Gamma, i : \phi \Rightarrow \Delta}\ (L_c) \qquad \frac{\Gamma \Rightarrow i : \phi, i : \phi, \Delta}{\Gamma \Rightarrow i : \phi, \Delta}\ (R_c)$$

Propositional Rules

$$\frac{\Gamma \Rightarrow i : \phi, \Delta}{\Gamma, i : \neg\phi \Rightarrow \Delta}\ (L_\neg) \qquad \frac{\Gamma, i : \phi \Rightarrow \Delta}{\Gamma \Rightarrow i : \neg\phi, \Delta}\ (R_\neg)$$

$$\frac{\Gamma, i : \phi \Rightarrow \Delta \quad \Gamma, i : \psi \Rightarrow \Delta}{\Gamma, i : \phi \vee \psi \Rightarrow \Delta}\ (L_\vee) \qquad \frac{\Gamma \Rightarrow i : \phi, \Delta}{\Gamma \Rightarrow i : \phi \vee \psi, \Delta}\ (R^1_\vee) \qquad \frac{\Gamma \Rightarrow i : \psi, \Delta}{\Gamma \Rightarrow i : \phi \vee \psi, \Delta}\ (R^2_\vee)$$

$$\frac{\Gamma, i : \phi \Rightarrow \Delta}{\Gamma, i : \phi \wedge \psi \Rightarrow \Delta}\ (L^1_\wedge) \qquad \frac{\Gamma, i : \psi \Rightarrow \Delta}{\Gamma, i : \phi \wedge \psi \Rightarrow \Delta}\ (L^2_\wedge) \qquad \frac{\Gamma \Rightarrow i : \phi, \Delta \quad \Gamma \Rightarrow i : \psi, \Delta}{\Gamma \Rightarrow i : \phi \wedge \psi, \Delta}\ (R_\wedge)$$

Modal Rules

$$\frac{\Gamma, R^\pm(i,j) \Rightarrow \Delta}{\Gamma, i : \Diamond^\pm\phi \Rightarrow \Delta}\ (L_{\Diamond^\pm})_1 \qquad \frac{\Gamma, j : \phi \Rightarrow \Delta}{\Gamma, i : \Diamond^\pm\phi \Rightarrow \Delta}\ (L_{\Diamond^\pm})_2$$

$$\frac{\Gamma \Rightarrow R^\pm(i,j), \Delta \quad \Gamma \Rightarrow j : \phi, \Delta}{\Gamma \Rightarrow i : \Diamond^\pm\phi, \Delta}\ (R_{\Diamond^\pm}) \qquad \frac{\Gamma, j : \phi \Rightarrow \Delta}{\Gamma, i : [A]\phi \Rightarrow \Delta}\ (L_{[A]}) \qquad \frac{\Gamma \Rightarrow j : \phi, \Delta}{\Gamma \Rightarrow i : [A]\phi, \Delta}\ (R_{[A]})$$

Relational Rules

$$\frac{\Gamma \Rightarrow \Delta, R^\pm(j,i)}{\Gamma \Rightarrow \Delta, R^\pm(i,j)}\ sym \qquad \frac{}{\Gamma \Rightarrow \Delta, R^+(i,i)}\ ref+$$

$$\frac{}{\Gamma, R^-(i,i) \Rightarrow \Delta}\ ref- \qquad \frac{\Gamma \Rightarrow \Delta, R^+(i,j) \quad \Gamma \Rightarrow \Delta, R^-(i,j)}{\Gamma \Rightarrow \Delta}\ no$$

Collectively Connected (for $\mathbf{DS^{cc}}$)

$$\frac{}{\Gamma \Rightarrow \Delta, R^+(i,j), R^-(i,j)}\ cc$$

Figure 4: The proof system **DS**. In the rule init, ϕ_{el} denotes an elementary formula. In the rules $(L_{\Diamond^\pm})_1$, $(L_{\Diamond^\pm})_2$, and $(R_{[A]})$, the nominal j is fresh.The rule $R_{\diamondminus}$ has the proviso that $i \neq j$. The system $\mathbf{DS^{cc}}$ also includes the rule *cc* for reasoning about collectively connected systems.

5 Dynamic operators and extensions

In this section we show how the global link-adding and local link change modalities from [25] can be defined in our framework. Adding such modalities require the underlying model $\mathbb{M}$ to be part of the game state, and calls for a different presentation of resulting sequent calculus.

The logic **dPNL** extends the syntax in Section 2 with the following cases:
$\phi ::= \cdots \mid \langle\text{⩕}+\rangle\phi \mid \langle\text{⩕}-\rangle\phi \mid \langle\text{⩕}\pm\rangle\phi \mid \langle\oplus\rangle\phi \mid \langle\ominus\rangle\phi$
The global link-adding modalities, $\langle\text{⩕}+\rangle\phi$ (resp. $\langle\text{⩕}-\rangle\phi$) is forced if, after adding a positive (resp. negative) link *somewhere* in the network, ϕ holds. $\langle\text{⩕}\pm\rangle$ adds either a positive or a negative link. The local change modality $\langle\oplus\rangle\phi$ (resp. $\langle\ominus\rangle\phi$) holds at agent a whenever ϕ holds after changing one of the a's links from negative to positive (resp. from positive to negative).

In the following, if $\mathbb{M} = \langle \mathsf{A}, \mathsf{R}^+, \mathsf{R}^-, \mathsf{V}, \mathsf{g}\rangle$, we denote by with $\mathbb{M} \cup \{\mathsf{a}R^+_\leftrightarrow\mathsf{b}\}$ the model $\langle \mathsf{A}, \mathsf{R}^+ \cup \{\mathsf{a}\mathsf{R}^+\mathsf{b}, \mathsf{b}\mathsf{R}^+\mathsf{a}\}, \mathsf{R}^-, \mathsf{V}, \mathsf{g}\rangle$. Similarly for $\mathbb{M} \cup \{\mathsf{a}R^-_\leftrightarrow\mathsf{b}\}$. We denote by $\mathbb{M} \setminus \{\mathsf{a}R^+_\leftrightarrow\mathsf{b}\}$ the model where both (a, b) and (b, a) are removed from R^+. Similarly for $\mathbb{M} \setminus \{\mathsf{a}R^-_\leftrightarrow\mathsf{b}\}$.

The semantics of the new operators is the following [25]:

($\mathbf{P}_{\langle\bigwedge-\rangle}$) At $\mathbf{P}, \mathbb{M}, \mathsf{a} : \langle\bigwedge-\rangle\phi$, *I* choose $\mathsf{b}, \mathsf{c} \in A$ s.t. $(\mathsf{b}, \mathsf{c}) \notin \mathsf{R}^+$ and the game continues with $\mathbf{P}, (\mathbb{M} \cup \{\mathsf{b}R^-_{\leftrightarrow}\mathsf{c}\}), \mathsf{a} : \phi$. *You* win if there are no such b and c.

($\mathbf{O}_{\langle\bigwedge-\rangle}$) At $\mathbf{O}, \mathbb{M}, \mathsf{a} : \langle\bigwedge-\rangle\phi$, *You* choose $\mathsf{b}, \mathsf{c} \in A$ s.t. $(\mathsf{b}, \mathsf{c}) \notin \mathsf{R}^+$ and the game continues with $\mathbf{O}, (\mathbb{M} \cup \{\mathsf{b}R^-_{\leftrightarrow}\mathsf{c}\}), \mathsf{a} : \phi$. *I* win if there are no such b and c.

($\mathbf{P}_{\langle\bigwedge+\rangle}$) At $\mathbf{P}, \mathbb{M}, \mathsf{a} : \langle\bigwedge+\rangle\phi$, *I* choose $\mathsf{b}, \mathsf{c} \in A$ s.t. $(\mathsf{b}, \mathsf{c}) \notin \mathsf{R}^-$ and the game continues with $\mathbf{P}, (\mathbb{M} \cup \{\mathsf{b}R^+_{\leftrightarrow}\mathsf{c}\}), \mathsf{a} : \phi$.

($\mathbf{O}_{\langle\bigwedge+\rangle}$) At $\mathbf{O}, \mathbb{M}, \mathsf{a} : \langle\bigwedge+\rangle\phi$, *You* choose $\mathsf{b}, \mathsf{c} \in A$ s.t. $(\mathsf{b}, \mathsf{c}) \notin \mathsf{R}^-$ and the game continues with $\mathbf{O}, (\mathbb{M} \cup \{\mathsf{b}R^+_{\leftrightarrow}\mathsf{c}\}), \mathsf{a} : \phi$.

($\mathbf{P}_{\langle\oplus\rangle}$) At $\mathbf{P}, \mathbb{M}, \mathsf{a} : \langle\oplus\rangle\phi$, *I* choose $\mathsf{b} \in A$ s.t. $(\mathsf{a}, \mathsf{b}) \in \mathsf{R}^-$ and the game continues with $\mathbf{P}, (\mathbb{M} \cup \{\mathsf{a}R^+_{\leftrightarrow}\mathsf{b}\} \setminus \{\mathsf{a}R^-_{\leftrightarrow}\mathsf{b}\}), \mathsf{a} : \phi$. *You* win if there is no such a b.

($\mathbf{O}_{\langle\oplus\rangle}$) At $\mathbf{O}, \mathbb{M}, \mathsf{a} : \langle\oplus\rangle\phi$, *You* choose $\mathsf{b} \in A$ s.t. $(\mathsf{a}, \mathsf{b}) \in \mathsf{R}^-$ and the game continues with $\mathbf{O}, (\mathbb{M} \cup \{\mathsf{a}R^+_{\leftrightarrow}\mathsf{b}\} \setminus \{\mathsf{a}R^-_{\leftrightarrow}\mathsf{b}\}), \mathsf{a} : \phi$. *I* win if there is no such a b.

Figure 5: Rules for **dPNL** adding/changing modalities. The rules for $\langle\bigwedge\pm\rangle$, and the rules $\mathbf{P}_{\langle\ominus\rangle}$ and $\mathbf{O}_{\langle\ominus\rangle}$ are similar and omitted. The global addition modalities do not necessarily add a *new* link. In $\mathbf{P}_{\langle\bigwedge+\rangle}$, *I* (and *You* in $\mathbf{O}_{\langle\bigwedge+\rangle}$) always have a choice, say $(\mathsf{a}, \mathsf{a}) \notin \mathsf{R}^-$.

$\mathbb{M}, \mathsf{a} \Vdash \langle\bigwedge+\rangle\phi$	iff	there are $\mathsf{b}, \mathsf{c} \in \mathsf{A}$ s.t. $(\mathsf{b}, \mathsf{c}) \notin \mathsf{R}^-$ and $\mathbb{M} \cup \{\mathsf{b}R^+_{\leftrightarrow}\mathsf{c}\}, \mathsf{a} \Vdash \phi$
$\mathbb{M}, \mathsf{a} \Vdash \langle\bigwedge-\rangle\phi$	iff	there are $\mathsf{b}, \mathsf{c} \in \mathsf{A}$ s.t. $(\mathsf{b}, \mathsf{c}) \notin \mathsf{R}^+$ and $\mathbb{M} \cup \{\mathsf{b}R^-_{\leftrightarrow}\mathsf{c}\}, \mathsf{a} \Vdash \phi$
$\mathbb{M}, \mathsf{a} \Vdash \langle\bigwedge\pm\rangle\phi$	iff	there are $\mathsf{b}, \mathsf{c} \in \mathsf{A}$ s.t. $(\mathsf{b}, \mathsf{c}) \notin \mathsf{R}^-$ and $\mathbb{M} \cup \{\mathsf{b}R^+_{\leftrightarrow}\mathsf{c}\}, \mathsf{a} \Vdash \phi$
		or there are $\mathsf{b}, \mathsf{c} \in \mathsf{A}$ s.t. $(\mathsf{b}, \mathsf{c}) \notin \mathsf{R}^+$ and $\mathbb{M} \cup \{\mathsf{b}R^-_{\leftrightarrow}\mathsf{c}\}, \mathsf{a} \Vdash \phi$
$\mathbb{M}, \mathsf{a} \Vdash \langle\oplus\rangle\phi$	iff	there is $\mathsf{b} \in \mathsf{A}$ s.t. $(\mathsf{a}, \mathsf{b}) \in \mathsf{R}^-$ and $\mathbb{M} \cup \{\mathsf{a}R^+_{\leftrightarrow}\mathsf{b}\} \setminus \{\mathsf{a}R^-_{\leftrightarrow}\mathsf{b}\}, \mathsf{a} \Vdash \phi$
$\mathbb{M}, \mathsf{a} \Vdash \langle\ominus\rangle\phi$	iff	there is $\mathsf{b} \in \mathsf{A}$ s.t. $\mathsf{a} \neq \mathsf{b}, (\mathsf{a}, \mathsf{b}) \in \mathsf{R}^+$ and $\mathbb{M} \cup \{\mathsf{a}R^-_{\leftrightarrow}\mathsf{b}\} \setminus \{\mathsf{a}R^+_{\leftrightarrow}\mathsf{b}\}, \mathsf{a} \Vdash \phi$

We consider game states of the form $\mathbf{P}, \mathbb{M}, \mathsf{a} : \phi$ and $\mathbf{O}, \mathbb{M}, \mathsf{a} : \phi$, where the model $\mathbb{M}$ over which the game is played is explicit. Figure 5 presents the rules for the semantic game **dPNL**. Adequacy is given by the following result, which has a similar proof to Theorem 1.

Theorem 5. *Let $\mathbb{M}$ be a **PNL**-model, a an agent, and ϕ a **dPNL** formula. Then: (1)* I *have a winning strategy for $\mathbf{G}_{\mathbb{M}}(\mathbf{P}, \mathbb{M}, \mathsf{a} : \phi)$ iff $\mathbb{M}, \mathsf{a} \models \phi$; and (2)* You *have a winning strategy for $\mathbf{G}_{\mathbb{M}}(\mathbf{P}, \mathbb{M}, \mathsf{a} : \phi)$ iff $\mathbb{M}, \mathsf{a} \not\models \phi$.*

Example 3. *Let $\mathbb{M}_2$ be as in Example 1.* I *have a winning strategy for the game state $\mathbf{P}, \mathbb{M}_2, \mathsf{a} : \langle\ominus\rangle(4B)$:* I *just need to change the relation $\mathsf{a}\mathsf{R}^+\mathsf{c}$ to obtain the model $\mathbb{M}_1$ (where* I *have a winning strategy for $4B$).* You *do not have a winning strategy for $\mathbf{O}, \mathbb{M}_2, \mathsf{a} : \langle\ominus\rangle\langle\ominus\rangle(4B)$ (and* I *win $\mathbf{P}, \mathbb{M}_2, \mathsf{a} : \neg(\langle\ominus\rangle\langle\ominus\rangle(4B))$). In words,* You *cannot enforce* balance *by making* a *disagree with her two friends. Finally, the formula $[A] \boxminus \bot$ characterizes "reconciliation" in a network [25], where there are no disagreements between agents.* I *have a winning strategy in the game $\mathbf{P}, \mathbb{M}_2, \mathsf{a} :\diamondplus \langle\oplus\rangle[A] \boxminus \bot$. (See the outputs of the tool in Appendix B and [11]).*

Defining a sequent system for **dPNL** would require passing through a provability game **dDG** as done in Section 3. We skip this step since it is similar to the case of **PNL**. Instead, we will go directly to the design of a proof system, which turns out to be a non-trivial task.

The problem is that the new modalities update the relational values and multisets contexts are not adequate for handling this *linear* behavior. Hence relational atoms will be stored in a separate *linear context*, where information can be updated. A *relational context* $\mathcal{R}$ is a *set* containing only relational predicates, and a *relational sequent* has the form $\mathcal{R}; \Gamma \Rightarrow \Delta$, where Γ, Δ are multisets of labeled formulas (hence no relational atoms).

$$\frac{\mathcal{R}, R^+(i,j); \Gamma, k : \phi \Rightarrow \Delta}{\mathcal{R}; \Gamma, k : \langle \wedge\!\!\wedge + \rangle \phi \Rightarrow \Delta}\ (L_{\langle \wedge\!\!\wedge + \rangle}) \qquad \frac{\mathcal{R}, R^+(i,j); \Gamma \Rightarrow \Delta, k : \phi}{\mathcal{R}; \Gamma \Rightarrow \Delta, k : \langle \wedge\!\!\wedge + \rangle \phi}\ (R_{\langle \wedge\!\!\wedge + \rangle}) \qquad \frac{\mathcal{R}, R^-(i,j); \Gamma, k : \phi \Rightarrow \Delta}{\mathcal{R}; \Gamma, k : \langle \wedge\!\!\wedge - \rangle k : \phi \Rightarrow \Delta}\ (L_{\langle \wedge\!\!\wedge - \rangle})$$

$$\frac{\mathcal{R}, R^-(i,j); \Gamma \Rightarrow \Delta, k : \phi}{\mathcal{R}; \Gamma \Rightarrow \Delta, k : \langle \wedge\!\!\wedge - \rangle \phi}\ (R_{\langle \wedge\!\!\wedge - \rangle}) \qquad \frac{\mathcal{R}, R^+(i,j); \Gamma, k : \phi \Rightarrow \Delta}{\mathcal{R}, R^-(i,j); \Gamma, k : \langle \oplus \rangle \phi \Rightarrow \Delta}\ (L_{\langle \oplus \rangle}) \qquad \frac{\mathcal{R}, R^+(i,j); \Gamma \Rightarrow \Delta, k : \phi}{\mathcal{R}, R^-(i,j); \Gamma \Rightarrow \Delta, k : \langle \oplus \rangle \phi}\ (R_{\langle \oplus \rangle})$$

$$\frac{\mathcal{R}, R^-(i,j); \Gamma, k : \phi \Rightarrow \Delta}{\mathcal{R}, R^+(i,j); \Gamma, k : \langle \ominus \rangle \phi \Rightarrow \Delta}\ (L_{\langle \ominus \rangle}) \qquad \frac{\mathcal{R}, R^-(i,j); \Gamma \Rightarrow \Delta, k : \phi}{\mathcal{R}, R^+(i,j); \Gamma \Rightarrow \Delta, k : \langle \ominus \rangle \phi}\ (R_{\langle \ominus \rangle}) \qquad \frac{\mathcal{R}, R^{\pm}(i,j); \Gamma, j : \phi \Rightarrow \Delta}{\mathcal{R}; \Gamma, i : \Diamond^{\pm} \phi \Rightarrow \Delta}\ (L_{\Diamond^{\pm}})$$

Figure 6: System **dDS**. Rules $L/R_{\langle \wedge\!\!\wedge + \rangle}$ (resp. $L/R_{\langle \wedge\!\!\wedge - \rangle}$) have the proviso that $R^-(i,j) \notin \mathcal{R}$ (resp. $R^+(i,j) \notin \mathcal{R}$), modulo symmetry (see Footnote 4). In rules $L/R_{\langle \ominus \rangle}$, $i \neq j$. Rules for $\langle \wedge\!\!\wedge \pm \rangle$ are similar and omitted. In $L_{\Diamond^{\pm}}$, j is fresh.

The label sequent system **dDS** for **dPNL** is depicted in Figure 6. The rules for the other connectives are obtained by adapting those in Figure 4 with relational contexts. Rule $L_{\Diamond^{\pm}}$ introduces the predicate $R(i,j)$, for a fresh j, into the context $\mathcal{R}$. The proviso of the rules for the global adding-link modalities guarantee non-overlapping, and the rules for $\langle \oplus \rangle$ forbid adding into $\mathcal{R}$ the atom $R^-(i,i)$. Moreover, the only way of adding new elements into $\mathcal{R}$ is using the rules $L_{\Diamond^{\pm}}$ and those for $\langle \wedge\!\!\wedge + \rangle$, $\langle \wedge\!\!\wedge - \rangle$ and $\langle \wedge\!\!\wedge \pm \rangle$. This explains the additional hypothesis in the theorem below, which is proved in Appendix A.

Theorem 6. *Let* Γ *and* Δ *be multisets of* ***dPNL*** *formulas not containing relational predicates.* I *have a winning strategy in the disjunctive game* **dDG**($\Gamma \Rightarrow \Delta$) *iff* $\Gamma \Rightarrow \Delta$ *is provable in* **dDS**.

6 Concluding Remarks

We have introduced two new techniques for **PNL** [25], with the aim of formally reasoning about positive and negative relations among agents and group polarization: a satisfiability game that allows for the verification of properties within concrete networks of agents; and a validity game with the corresponding cut-free sequent calculus. Our contributions offer promising avenues for automated reasoning, as demonstrated by our prototypical tool [11]. Furthermore, by showing that reasoning about frame properties of the underlying model can be delayed until reaching elementary games/formulas, we can modularly adapt to different relational properties.

Currently, we are exploring extensions that relax symmetry assumptions, allowing for representing situations where agent a may influence the opinion of b but not the other way around. Additionally, we are investigating the concept of "budget" as in the game proposed in [16] to characterize scenarios where proponents and opponents operate within a limited *political capital*, where adding/changing relations can potentially decrease such a capital. To this end, the preference of spending as little capital as possible could be expressed in a combination of **PNL** with a suitable *choice logic*, i.e., a logic where preferences are definable at the object level. Semantic games for choice logics have been investigated in [10] and the lifting of game-induced choice logic, **GCL**, to a provability game and proof system was demonstrated in [10]. Finally, following the techniques developed in [23] for analyzing sequent systems in rewrite logic, we are extending our tool [11] to also support the sequent calculi proposed here.

This work can be seen as a continuation of a program of lifting semantic games to analytic calculi [7, 24]. Our approach is a refinement of previous work on modal logic [9, 8] as it replaces model checking at the level of axioms with explicit rules for the classes of **PNL** and **cc-PNL** models. We therefore provide hand-tailored systems for reasoning about group polarization and opens up the aforementioned routes to mechanization.

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A Some selected proofs

Theorem 1. *Let $\mathbb{M}$ be a **PNL**-model,* a *an agent, and* ϕ *a formula.*

1. I *have a winning strategy for* $\mathbf{G}_{\mathbb{M}}(\mathbf{P}, \mathsf{a} : \phi)$ *iff* $\mathbb{M}, \mathsf{a} \models \phi$.
2. You *have a winning strategy for* $\mathbf{G}_{\mathbb{M}}(\mathbf{P}, \mathsf{a} : \phi)$ *iff* $\mathbb{M}, \mathsf{a} \not\models \phi$.

Proof: Both directions of (1) and (2) are shown simultaneously by induction on the degree[5] of the game state $g = \mathbf{P}, \mathsf{a} : \phi$.

If ϕ is elementar, then the result trivially follows by the definition.

If $g = \mathbf{P}, \mathsf{a} : \phi_1 \wedge \phi_2$ then I have a winning strategy for $\mathbf{G}_{\mathbb{M}}(\mathbf{P}, \mathsf{a} : \phi_1 \wedge \phi_2)$ iff I have winning strategies for both $\mathbf{G}_{\mathbb{M}}(\mathbf{P}, \mathsf{a} : \phi_1)$ and $\mathbf{G}_{\mathbb{M}}(\mathbf{P}, \mathsf{a} : \phi_2)$. By the inductive hypothesis, this is the case iff $\mathbb{M}, \mathsf{a} \models \phi_1$ and $\mathbb{M}, \mathsf{a} \models \phi_2$, which is equivalent to $\mathbb{M}, \mathsf{a} \models \phi_1 \wedge \phi_2$.

If $g = \mathbf{P}, \mathsf{a} : \Diamond^{\pm}\phi$, I have a winning strategy for $\mathbf{G}_{\mathbb{M}}(\mathbf{P}, \mathsf{a} : \Diamond^{\pm}\phi)$ iff there is a $\mathsf{R}^{\pm}$-successor b of a and I have a winning strategy for $\mathbf{P}, \mathsf{b} : \phi$. By the inductive hypothesis, this occurs iff $\mathbb{M}, \mathsf{b} \models \phi$, but then $\mathbb{M}, \mathsf{a} \models \Diamond^{\pm}\phi$.

The other cases are similar. ■

Lemma 2. *Let π be as above. Then:*
1) Let $g \in \pi$ be a non-elementary game state labelled "Y" in the semantic game. Then at least one successor of g appears along π.
2) Let $g \in \pi$ be a non-elementary game state labeled "I" in the semantic game. Then all successors of g appear along π.
Proof: First, note that since *You* play according to *Your* winning strategy, π does not end in a winning disjunctive state whose elementary party is winning for *Me*. This means that case (C1) in the definition of σ is never reached.

1. Suppose g appeared in π at stage $n \geq 0$ in the above construction. Since every pair appears in the enumeration infinitely often, there is some minimal $m \geq n$ such that $m = \#(g, h)$, for some h. At step m in the execution of σ against *Your* winning strategy, the current disjunctive state is of the form $D' \bigvee g$. According to σ, I underline g and *You* move to some successor h', according to *Your* winning strategy. This means the new game state is of the form $D' \bigvee h'$, hence h' is the successor of g appearing along $\mathfrak{h}$.
2. Suppose g appeared in π at stage $n \geq 0$. Now we additionally fix an arbitrary successor h of g in the evaluation game. By the properties of $\#$, there is a minimal $m \geq n$ such that $m = \#(g, h)$. Since I always first duplicate game states labeled "I", before I make a move into them, g does not disappear. Hence, at step m in the execution of σ, the current disjunctive state is of the form $D' \bigvee g$. According to σ, I duplicate g and go to h in one copy, i.e. the new disjunctive state is $D' \bigvee g \bigvee h$, which shows that h appears along π.

■

Proposition 5 *Let j be a nominal different from i not occurring in D nor in ϕ. Then:*
(1) You *have a winning strategy in* $\mathbf{DG}(D \bigvee \mathbf{P}, i : [A]\phi)$ *iff* You *have a winning strategy in* $\mathbf{DG}(D \bigvee \mathbf{P}, j : \phi)$*, and similarly for* $\mathbf{DG}^{cc}$*.*
(2) You *have a winning strategy in* $\mathbf{DG}(D \bigvee \mathbf{O}, i : \Diamond^{\pm}\phi)$ *iff* You *have winning strategies in both* $\mathbf{DG}(D \bigvee \mathbf{P}, \_ : R^{\pm}(i, j))$ *and* $\mathbf{DG}(D \bigvee \mathbf{O}, j : \phi)$*, and similarly for* $\mathbf{DG}^{cc}$*.*

[5] Observe that, at this point of the text, the degree has not been introduced yet. Its definition can be found just above Lemma 3 on page 8.

The proof is split into a sequence of lemmas. First, we need to define substitutions formally. For a sequence x of finite or infinite length, let x_n denote its n-th element (if defined), and let $\mathrm{range}(x) = \{x_i : i \in \mathbb{N}\}$. Let ϕ be a formula and a and b two sequences of nominals of the same length, where every nominal occurs only once in each sequence. We define $\phi[a/b]$ as the formula obtained by simultaneously substituting for every number n all occurrences of a_n in ϕ with b_n. For example, let $a = \langle i, j\rangle$, $b = \langle k, l\rangle$ and $\phi = R^+(i, j) \vee R^-(j, l)$. Then $\phi[a/b] = R^+(k, l) \vee R^-(l, l)$. As another example let $a = \langle i_1, i_2, ...\rangle$ and $b = \langle i_2, i_3, ...\rangle$. Then $R^+(i_1, i_2)[a/b] = R^+(i_2, i_3)$, since the substitution happens simultaneously. We extend the notion of substitution to game states: for a game state $g = \mathbf{Q}, i : \phi$ of the evaluation game and two sequences of nominals a, b, we define the substitution $g[a/b]$ as $\mathbf{Q}, i[a/b] : \phi[a/b]$. Similarly to histories, strategies, and disjunctive states.

For two games $\mathbf{G}_1$ and $\mathbf{G}_2$, let us write $\mathbf{G}_1 \cong \mathbf{G}_2$ if they are *strategically equivalent*, i.e., *You* have a winning strategy in $\mathbf{G}_1$ iff *You* have a winning strategy in $\mathbf{G}_2$.

Lemma 6. *Let* $\mathbb{M}_1 = (\mathsf{A}, \mathsf{R}^+, \mathsf{R}^-, \mathsf{V}, \mathsf{g}_1)$ *and* $\mathbb{M}_2 = (\mathsf{A}, \mathsf{R}^+, \mathsf{R}^-, \mathsf{V}, \mathsf{g}_2)$ *be named and* $\mathsf{g}_2(i[b/a]) = \mathsf{g}_1(i)$ *for all nominals* i*. Then for all game states* g, $\mathbf{G}_{\mathbf{M}_1}(g) \cong \mathbf{G}_{\mathbb{M}_2}(g[b/a])$.

Proof: By the assumption, g_2 is surjective, even if restricted to $N[b/a] = \{i[b/a] : i \in N\}$. By Proposition 2, it is, therefore, enough to prove $\mathbf{G}_{\mathbb{M}_1}(g) \cong \mathbf{G}^{N[b/a]}_{\mathbb{M}_2}(g[b/a])$. We proceed by induction on the degree of g.

If g is elementary and of the form $\mathbf{P}, i : j$ then it is winning for *Me* over $\mathbb{M}_1$ if and only if $\mathsf{g}_1(i) = \mathsf{g}_1(j)$. By assumption, this is equivalent to $\mathsf{g}_2(i[b/a]) = g_2(i[b/a])$, which means that $\mathbf{O}, i[b/a] : j[b/a]$ is winning for me over $\mathbb{M}_2$. The other elementary cases are similar.

As an example of a simple induction step, we consider $\mathbf{P}, i : \phi_1 \vee \phi_2$. If I have a winning strategy for this game state over $\mathbb{M}_1$, then there is some $k \in \{1, 2\}$ such that I have a winning strategy in $\mathbf{P}, i : \phi_k$. By the inductive hypothesis, I have a winning strategy in $\mathbf{P}, i[b/a] : \phi_k[b/a]$ over $\mathbb{M}_2$. Hence, I have a winning strategy in $(\mathbf{P}, i : \phi_1 \vee \phi_2)[b/a])$ over that model. The other direction is similar.

The most interesting induction step is for the modal rules, so let us consider $\mathbf{O}, i :\oplus\, \psi$. Suppose, I have a winning strategy in $\mathbf{G}_{\mathbb{M}_1}(\mathbf{O}, i :\oplus\, \psi)$. Then, for every nominal j, I have winning strategies in $\mathbf{G}_{\mathbb{M}_1}(\mathbf{P}, j : R(i, j))$ and $\mathbf{G}_{\mathbb{M}_1}(\mathbf{O}, j : \psi)$. By the inductive hypothesis, I have winning strategies in $\mathbf{G}^{N[b/a]}_{\mathbb{M}_2}(\mathbf{P}, j[b/a] : R(i[b/a], j[b/a]))$ and $\mathbf{G}^{N[b/a]}_{\mathbb{M}_2}(\mathbf{O}, j[b/a] : \psi[b/a])$. In other words, I have winning strategies in $\mathbf{G}^{N[b/a]}_{\mathbb{M}_2}(\mathbf{P}, k : R(i[b/a], k)$ and $\mathbf{G}^{N[b/a]}_{\mathbb{M}_2}(\psi[b/a])$, for every $k \in N[b/a]$. Since branching in this game is restricted over $N[b/a]$, we conclude that I have a winning strategy in $\mathbf{G}^{N[b/a]}_{\mathbb{M}_2}((\mathbf{P}, i :\oplus\, \psi)[b/a])$. The other direction, as well as the other cases of induction steps, are similar. ■

Lemma 7. *If* $\mathsf{g}(k) = \mathsf{g}(l)$*, then* $\mathbf{G}_{\mathbb{M}}(g) \cong \mathbf{G}_{\mathbb{M}}(g[k/l])$.

Proof: We show that $\mathsf{g}(i[k/l]) = \mathsf{g}(i)$ for all nominals i. If $i \neq k$, then $\mathsf{g}(i[k/l]) = \mathsf{g}(i)$. If $i = k$, then by the assumption, $\mathsf{g}(i[k/l]) = \mathsf{g}(l) = \mathsf{g}(k) = \mathsf{g}(i)$. The statement of the lemma follows from this fact and Lemma 6. ■

For a model $\mathbb{M}$ and two sequences of nominals a, b, let $\mathbb{M}[a/b]$ be the same as $\mathbb{M}$, except for the denotation function: $\mathsf{g}_{[a/b]}(i) = \mathsf{g}(i[a/b])$.

Lemma 8. *Let* $\mathbb{M}$ *be named and* a, b *two sequences of nominals with* $\mathrm{range}(a) \subseteq \mathrm{range}(b)$*. Then* $\mathbb{M}[a/b]$ *is* $N[b/a]$*-named. Furthermore,* $\mathbf{G}_{\mathbb{M}}(g) \cong \mathbf{G}_{\mathbb{M}[a/b]}(g[b/a])$.

Proof: We have to show that $\mathsf{g}_{[a/b]}$ is surjective when restricted to $N[b/a] = \{i[b/a] : i \in N\}$. Let a be an agent and i its name under g. If $i \notin \mathrm{range}(b)$, then $i \notin \mathrm{range}(b)$ and we have

$i[b/a][a/b] = i[a/b] = i$. If $i \in$ b, then $i = b_m$ for some m. Then $i[b/a][a/b] = b_m[b/a][a/b] = a_m[a/b] = b_m = i$. This shows that $\mathsf{g}_{[a/b]}(i[b/a]) = \mathsf{g}(i[b/a][a/b]) = \mathsf{g}(i)$, i.e. a has a name in $N[b/a]$ under $g_{[a/b]}$. This identity together with Lemma 6 also implies the strategic equivalence of the game $\mathbf{G}_{\mathbb{M}}(g)$ and $\mathbf{G}_{\mathbb{M}[a/b]}(g[b/a])$. ■

We are now ready to prove Proposition 5.

Proof: [Proposition 5] We will show (2), since (1) is similar and simpler. From right to left is trivial. For the other direction, assume that *You* have a winning strategy in $\mathbf{DG}(D \bigvee \mathbf{O}, i :\oplus \phi)$ with j as in the assumption. By Theorem 2, there is a named model $\mathbb{M}$ such that *You* have winning strategies in $\mathbf{G}_{\mathbb{M}}(g)$ for all $g \in D$ and in $\mathbf{O}, i :\oplus \phi$. The latter implies that *You* have winning strategies in $\mathbf{G}_{\mathbb{M}}(\mathbf{O}, \_ : R^{\pm}(i,k))$ and $\mathbf{G}_{\mathbb{M}}(\mathbf{O}, k : \phi)$ for some nominal k.

Let $j_1, j_2, ...$ be a sequence of nominals not occurring in D or ϕ and different from k, j, and i. Let $a = \langle j, j_1, j_2, ...\rangle$ and $b = \langle k, j, j_1, j_2, ...\rangle$. We have that range$(a) \subseteq$ range(b), therefore Lemma 8 applies. We have the following chain of equivalences:

$$
\begin{aligned}
\mathbf{G}_{\mathbb{M}}(\mathbf{O}, k : \phi) &\cong \mathbf{G}_{\mathbb{M}[a/b]}(\mathbf{O}, k[b/a] : \phi[b/a]) && \text{by Lemma 8}\\
&= \mathbf{G}_{\mathbb{M}[a/b]}(\mathbf{O}, j : \phi[k/j]) && \text{by conditions on } i,j,k\\
&= \mathbf{G}_{\mathbb{M}[a/b]}(\mathbf{O}, j[k/j] : \phi[k/j]) && \\
&\cong \mathbf{G}_{\mathbb{M}[a/b]}(\mathbf{O}, j : \phi) && \text{by Lemma 7 and } \mathsf{g}_{[a/b]}(j) = \mathsf{g}_{[a/b]}(k)
\end{aligned}
$$

A similar argument shows that $\mathbf{G}_{\mathbb{M}}(\mathbf{O}, \_ : R^{\pm}(i,k)) \cong \mathbf{G}_{\mathbb{M}[a/b]}(\mathbf{O}, \_ : R^{\pm}(i,j))$. By this equivalence and the assumption, *You* have winning strategies in $\mathbf{O}, \_ : R^{\pm}(i,j)$ and $\mathbf{O}, j : \phi$ over $\mathbb{M}[a/b]$. Moreover, we obtain a winning strategy for *You* for $g \in D$ by using the equivalence $\mathbf{G}_{\mathbb{M}}(g) \cong \mathbf{G}_{\mathbb{M}[a/b]}(g[b/a]) \cong \mathbf{G}_{\mathbb{M}[a/b]}(g[k/j]) \cong \mathbf{G}_{\mathbb{M}[a/b]}(g)$, the same lemmas as before and the fact that no nominals from a appear in g. Since *You* have winning strategies for the semantic games for every $g \in D$ and $\mathbf{O}, \_ : R^{\pm}(i,j)$ over $\mathbb{M}[a/b]$, *You* have a winning strategy in $\mathbf{DG}(D \bigvee \mathbf{O}, \_ : R^{\pm}(i,j))$, by Proposition 2 and the determinacy of the game. Similarly, we conclude that *You* have a winning strategy in $\mathbf{DG}(D \bigvee \mathbf{O}, j : \phi)$. ■

Lemma 4. *Let $\Gamma \Rightarrow \Delta$ be composed of elementary game states only.* I *win the disjunctive game in $\Gamma \Rightarrow \Delta$ iff one of the following holds*

i. $R^-(i,i) \in \Gamma$ *or* $R^+(i,i) \in \Delta$ *for some* i*;*

ii. $\{R^+(i,j), R^-(i,j)\} \subseteq \Gamma$ *for some* $i \neq j$*;*

iii. $\Gamma \cap \Delta \neq \varnothing$*.*

In the case of collectively connected models, additionally,

iv. $\{R^+(i,j), R^-(i,j)\} \subseteq \Delta$ *for some* $i \neq j$

Proof: $\mathbb{M}_{\Gamma\Rightarrow\Delta}$ is non-overlapping: this follows by (ii). $\mathbb{M}^{\mathbf{cc}}_{\Gamma\Rightarrow\Delta}$ is collectively connected, since $(iii)^+$ in the definition of R^+ equivalent to $\neg((i)^+ \vee (ii)^+ \vee (i)^- \vee (ii)^-)$. $\mathbb{M}^{\mathbf{cc}}_{\Gamma\Rightarrow\Delta}$ is non-overlapping: Suppose, $(\mathsf{a}_i, \mathsf{a}_j) \in \mathsf{R}^+ \cap \mathsf{R}^-$. This is impossible since all possible cases in the definitions, in which a_i and a_j are connected by both relations, are excluded by our assumptions:

- $(i)^+$ and $(i)^-$: Excluded by $\neg(ii)$.
- $(i)^+$ and $(ii)^-$: Excluded by $\neg(iii)$.
- $(ii)^-$ and $(i)^-$: Excluded by $\neg(iii)$.

- $(ii)^-$ and $(ii)^-$: Excluded by $\neg(iv)$.
- $(iii)^+$, excludes $(i)^-$ and $(ii)^-$, hence $(\mathsf{a}_i, \mathsf{a}_j) \notin \mathsf{R}^-$.

By definition of the models, $\mathsf{a}_i \in \mathsf{V}(p)$, whenever $i : p \in \Gamma$, $\mathsf{a}_i \notin \mathsf{V}(p)$, whenever $i : p \in \Delta$, $(\mathsf{a}_i, \mathsf{a}_j) \in \mathsf{R}^\pm$, whenever $R^\pm(i, j) \in \Gamma$. If $R^\pm(i, j) \in \Delta$, then $(\mathsf{a}_i, \mathsf{a}_j) \in \mathsf{R}^\mp$. Since both models are non-overlapping, $(\mathsf{a}_i, \mathsf{a}_j) \notin \mathsf{R}^\pm$. Hence, all game states in $\Gamma \Rightarrow \Delta$ are winning for *You*. ■

Lemma 5. *The following weakening rules are admissible in* **DS**

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, i : \phi \Rightarrow \Delta}\ L_w \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, i : \phi}\ R_w$$

Moreover, in a bottom-up reading of derivations, the relational rules permutes up w.r.t. any other logical rule in **DS**.

Proof: The proof of weakening is standard. The proof of permutability is by straightforward case analysis. For example,

$$\cfrac{\cfrac{\overset{\pi_1}{\Gamma, j : \phi \Rightarrow \Delta, R^+(k, l)}}{\Gamma, i :\diamondminus \phi \Rightarrow \Delta, R^+(k, l)}\ (L_{\diamondminus})_2 \qquad \overset{\pi_2}{\Gamma, i :\diamondminus \phi \Rightarrow \Delta, R^-(k, l)}}{\Gamma, i :\diamondminus \phi \Rightarrow \Delta}\ no$$

with j free, can be transformed to

$$\cfrac{\cfrac{\cfrac{\overset{\pi_1^w}{\Gamma, j : \phi, i :\diamondminus \phi \Rightarrow \Delta, R^+(k, l)} \quad \overset{\pi_2^w}{\Gamma, j : \phi, i :\diamondminus \phi \Rightarrow \Delta, R^-(k, l)}}{\Gamma, j : \phi, i :\diamondminus \phi \Rightarrow \Delta}\ no}{\Gamma, i :\diamondminus \phi, i :\diamondminus \phi \Rightarrow \Delta}\ (L_{\diamondminus})_2}{\Gamma, i :\diamondminus \phi \Rightarrow \Delta}\ L_c$$

where π_1^w, π_2^w are the weakened versions of π_1, π_2 respectively. ■

Theorem 6. *Let* Γ *and* Δ *be multisets of* ***dPNL*** *formulas not containing relational predicates.* I *have a winning strategy in the disjunctive game* **dDG**$(\Gamma \Rightarrow \Delta)$ *iff* $\Gamma \Rightarrow \Delta$ *is provable in* **dDS**.

Proof: First of all, we observe that the rule

$$\frac{R^\pm(i, j), \Gamma, j : \phi \Rightarrow \Delta}{\Gamma, i : \diamond^\pm \phi \Rightarrow \Delta}\ (L'_{\diamond^\pm})$$

is admissible in **DS**. In fact, this is an easy consequence of the presence of the contraction rules. Hence, although the rule $(L_{\diamond^\pm})$ in **dDS** is not directly defined via a provability game, it is equivalent to its contracted version. The rest of the proof follows the same lines as the proof in Theorem 3. ■

B Examples

Below we present *My* strategy for winning the game on model $\mathbb{M}_1$ in Example 1 [6].

[6] The notation $[\checkmark, \mathtt{Q}]$ means that I have a winning strategy starting in that state where $Q \in \{I, You\}$ moves. The notation $[\times, \mathtt{Q}]$ means that *You* have a winning strategy where $Q \in \{I, You\}$ moves.

```
python main.py examples/model-M1.maude a "lb(p)"

[✓,Y] : P @ a : (¬ (⊕ ⊕ p ∨ ⊖ ⊖ p) ∨ ⊕ p) ∧ ¬ (⊕ ⊖ p ∨ ⊖ ⊕ p) ∨ ⊖ p
├── [✓,I] : P @ a : ¬ (⊕ ⊕ p ∨ ⊖ ⊖ p) ∨ ⊕ p
│   └── [✓,I] : P @ a : ⊕ p
│       └── [✓,I] : P @ b : p
└── [✓,I] : P @ a : ¬ (⊕ ⊖ p ∨ ⊖ ⊕ p) ∨ ⊖ p
    └── [✓,I] : P @ a : ¬ (⊕ ⊖ p ∨ ⊖ ⊕ p)
        └── [✓,Y] : O @ a : ⊕ ⊖ p ∨ ⊖ ⊕ p
            ├── [✓,Y] : O @ a : ⊕ ⊖ p
            │   ├── [✓,Y] : O @ a : ⊖ p
            │   │   └── [✓,Y] : O @ c : p
            │   └── [✓,Y] : O @ b : ⊖ p
            │       └── [✓,Y] : O @ c : p
            └── [✓,Y] : O @ a : ⊖ ⊕ p
                └── [✓,Y] : O @ c : ⊕ p
                    └── [✓,Y] : O @ c : p
```

I do not have a winning strategy for the game on the model $\mathbb{M}_2$ in Example 1:

```
python main.py examples/model-M2.maude a "lb(p)" --tree

[✗,Y] : P @ a : (¬ (⊕ ⊕ p ∨ ⊖ ⊖ p) ∨ ⊕ p) ∧ ¬ (⊕ ⊖ p ∨ ⊖ ⊕ p) ∨ ⊖ p
├── [✗,I] : P @ a : ¬ (⊕ ⊖ p ∨ ⊖ ⊕ p) ∨ ⊖ p
│   ├── [✗,I] : P @ a : ¬ (⊕ ⊖ p ∨ ⊖ ⊕ p)
│   │   └── [✗,Y] : O @ a : ⊕ ⊖ p ∨ ⊖ ⊕ p
│   │       ├── [✗,Y] : O @ a : ⊕ ⊖ p
│   │       │   ├── [✗,Y] : O @ c : ⊖ p
│   │       │   │   └── [✗,Y] : O @ b : p
│   │       │   ├── [✓,Y] : O @ a : ⊖ p
│   │       │   └── [✓,Y] : O @ b : ⊖ p
│   │       │       └── [✓,Y] : O @ c : p
│   │       └── [✓,Y] : O @ a : ⊖ ⊕ p
│   └── [✗,I] : P @ a : ⊖ p
└── [✓,I] : P @ a : ¬ (⊕ ⊕ p ∨ ⊖ ⊖ p) ∨ ⊕ p
    ├── [✗,I] : P @ a : ¬ (⊕ ⊕ p ∨ ⊖ ⊖ p)
    │   └── [✗,Y] : O @ a : ⊕ ⊕ p ∨ ⊖ ⊖ p
    │       ├── [✗,Y] : O @ a : ⊕ ⊕ p
    │       │   ├── [✗,Y] : O @ a : ⊕ p
    │       │   │   ├── [✗,Y] : O @ b : p
    │       │   │   ├── [✓,Y] : O @ a : p
    │       │   │   └── [✓,Y] : O @ c : p
    │       │   ├── [✗,Y] : O @ b : ⊕ p
    │       │   │   ├── [✗,Y] : O @ b : p
    │       │   │   └── [✓,Y] : O @ a : p
    │       │   └── [✓,Y] : O @ c : ⊕ p
    │       │       ├── [✓,Y] : O @ a : p
    │       │       └── [✓,Y] : O @ c : p
    │       └── [✓,Y] : O @ a : ⊖ ⊖ p
    └── [✓,I] : P @ a : ⊕ p
        ├── [✗,I] : P @ a : p
        ├── [✗,I] : P @ c : p
        └── [✓,I] : P @ b : p
```

And *I* certainly win in the negated formula:

```
python main.py examples/model-M2.maude a "~ lb(p)"
[✓,I] : P @ a : ¬ ((¬ (⊕ ⊕ p ∨ ⊖ ⊖ p) ∨ ⊕ p) ∧ ¬ (⊕ ⊖ p ∨ ⊖ ⊕ p) ∨ ⊖ p)
└── [✓,I] : O @ a : (¬ (⊕ ⊕ p ∨ ⊖ ⊖ p) ∨ ⊕ p) ∧ ¬ (⊕ ⊖ p ∨ ⊖ ⊕ p) ∨ ⊖ p
    └── [✓,Y] : O @ a : ¬ (⊕ ⊖ p ∨ ⊖ ⊕ p) ∨ ⊖ p
        ├── [✓,I] : O @ a : ¬ (⊕ ⊖ p ∨ ⊖ ⊕ p)
        │   └── [✓,I] : P @ a : ⊕ ⊖ p ∨ ⊖ ⊕ p
        │       └── [✓,I] : P @ a : ⊕ ⊖ p
        │           └── [✓,I] : P @ c : ⊖ p
        │               └── [✓,I] : P @ b : p
        └── [✓,Y] : O @ a : ⊖ p
```

The winning strategy for the game $\mathbf{P}, \mathbb{M}_2, \mathsf{a} : \langle \ominus \rangle (4B)$ is the following:

```
python main.py examples/model-M2.maude a " (-) lb(p)"
[✓,I] : P @ a : (-)((¬ (⊕ ⊕ p ∨ ⊖ ⊖ p) ∨ ⊕ p) ∧ ¬ (⊕ ⊖ p ∨ ⊖ ⊕ p) ∨ ⊖ p)
```

```
└── [✓,Y] : P @ a : (¬ (◈ ◈ p ∨ ◇ ◇ p) ∨ ◈ p) ∧ ¬ (◈ ◇ p ∨ ◇ ◈ p) ∨ ◇ p
    ├── [✓,I] : P @ a : ¬ (◈ ◈ p ∨ ◇ ◇ p) ∨ ◈ p
    │   └── [✓,I] : P @ a : ¬ (◈ ◈ p ∨ ◇ ◇ p)
    │       └── [✓,Y] : O @ a : ◈ ◈ p ∨ ◇ ◇ p
    │           ├── [✓,Y] : O @ a : ◈ ◈ p
    │           │   ├── [✓,Y] : O @ a : ◈ p
    │           │   │   ├── [✓,Y] : O @ a : p
    │           │   │   └── [✓,Y] : O @ c : p
    │           │   └── [✓,Y] : O @ c : ◈ p
    │           │       ├── [✓,Y] : O @ a : p
    │           │       └── [✓,Y] : O @ c : p
    │           └── [✓,Y] : O @ a : ◇ ◇ p
    │               └── [✓,Y] : O @ b : ◇ p
    │                   ├── [✓,Y] : O @ a : p
    │                   └── [✓,Y] : O @ c : p
    └── [✓,I] : P @ a : ¬ (◈ ◇ p ∨ ◇ ◈ p) ∨ ◇ p
        └── [✓,I] : P @ a : ◇ p
            └── [✓,I] : P @ b : p
```

and I can win the following game in Example 3.

```
python main.py examples/model-M2.maude a " ~ ( (-) (-) lb(p))"
[✓,I] : P @ a : ¬ ((-)(-)((¬ (◈ ◈ p ∨ ◇ ◇ p) ∨ ◈ p) ∧ ¬ (◈ ◇ p ∨ ◇ ◈ p) ∨ ◇ p))
└── [✓,Y] : O @ a : (-)(-)((¬ (◈ ◈ p ∨ ◇ ◇ p) ∨ ◈ p) ∧ ¬ (◈ ◇ p ∨ ◇ ◈ p) ∨ ◇ p)
    ├── [✓,Y] : O @ a : (-)((¬ (◈ ◈ p ∨ ◇ ◇ p) ∨ ◈ p) ∧ ¬ (◈ ◇ p ∨ ◇ ◈ p) ∨ ◇ p)
    │   └── [✓,I] : O @ a : (¬ (◈ ◈ p ∨ ◇ ◇ p) ∨ ◈ p) ∧ ¬ (◈ ◇ p ∨ ◇ ◈ p) ∨ ◇ p
    │       └── [✓,Y] : O @ a : ¬ (◈ ◈ p ∨ ◇ ◇ p) ∨ ◈ p
    │           ├── [✓,I] : O @ a : ¬ (◈ ◈ p ∨ ◇ ◇ p)
    │           │   └── [✓,I] : P @ a : ◈ ◈ p ∨ ◇ ◇ p
    │           │       └── [✓,I] : P @ a : ◇ ◇ p
    │           │           └── [✓,I] : P @ c : ◇ p
    │           │               └── [✓,I] : P @ b : p
    │           └── [✓,Y] : O @ a : ◈ p
    │               └── [✓,Y] : O @ a : p
    └── [✓,Y] : O @ a : (-)((¬ (◈ ◈ p ∨ ◇ ◇ p) ∨ ◈ p) ∧ ¬ (◈ ◇ p ∨ ◇ ◈ p) ∨ ◇ p)
        └── [✓,I] : O @ a : (¬ (◈ ◈ p ∨ ◇ ◇ p) ∨ ◈ p) ∧ ¬ (◈ ◇ p ∨ ◇ ◈ p) ∨ ◇ p
            └── [✓,Y] : O @ a : ¬ (◈ ◈ p ∨ ◇ ◇ p) ∨ ◈ p
                ├── [✓,I] : O @ a : ¬ (◈ ◈ p ∨ ◇ ◇ p)
                │   └── [✓,I] : P @ a : ◈ ◈ p ∨ ◇ ◇ p
                │       └── [✓,I] : P @ a : ◇ ◇ p
                │           └── [✓,I] : P @ c : ◇ p
                │               └── [✓,I] : P @ b : p
                └── [✓,Y] : O @ a : ◈ p
                    └── [✓,Y] : O @ a : p
```

In the same example, this is *My* winning strategy for $\mathbf{P}, \mathbb{M}_2, \mathsf{a} : ◈ \langle\oplus\rangle[A] \boxminus \bot$:

```
python main.py examples/model-M2.maude a "<+> (+) [A] [-] (p /\ ~ p)"
[✓,I] : P @ a : ◈ (+)[A]¬ (◇ ¬ (p ∧ ¬ p))
└── [✓,I] : P @ b : (+)[A]¬ (◇ ¬ (p ∧ ¬ p))
    └── [✓,Y] : P @ b : [A]¬ (◇ ¬ (p ∧ ¬ p))
        ├── [✓,I] : P @ a : ¬ (◇ ¬ (p ∧ ¬ p))
        │   └── [✓,Y] : O @ a : ◇ ¬ (p ∧ ¬ p)
        ├── [✓,I] : P @ b : ¬ (◇ ¬ (p ∧ ¬ p))
        │   └── [✓,Y] : O @ b : ◇ ¬ (p ∧ ¬ p)
        └── [✓,I] : P @ c : ¬ (◇ ¬ (p ∧ ¬ p))
            └── [✓,Y] : O @ c : ◇ ¬ (p ∧ ¬ p)
```