Unified Opinion Formation Analysis in Rewriting Logic

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Abstract

Social media platforms significantly influence the polarization of social, political, and democratic processes, primarily through their role in opinion formation. Opinion dynamic models are essential for understanding the impact of specific social factors on the acceptance or rejection of opinions. This extended paper builds upon the conference presentation documented in [1], introducing improvements, and new opinion models that explore biases and collective human behaviors. It presents a framework based on concurrent set relations that formalizes, simulates, and analyzes social interaction systems with dynamic opinion models. Within this framework, standard models for social learning are realized as specific instances. Implemented in the Maude system as a fully executable rewrite theory, the framework enables a detailed examination of how agents' opinions can be influenced within a system. The authors report on new formalization of several and existing social learning models, exploring their relationships with different concurrency models. Furthermore, new experimentation involving reachability analysis, probabilistic simulation, and statistical model checking has been conducted. These experiments are crucial for validating significant properties related to dynamic opinion models in Maude, offering new insights into the mechanisms of opinion shaping in social media environments.

Keywords: Concurrent set relations, opinion dynamic models, social interaction systems, belief revision, rewriting logic, formal verification

1. Introduction

Social media platforms have increasingly influenced the polarization of social, political, and democratic processes worldwide. Incidents such as social uprisings in the Middle East, Asia, and Central and South America have catalyzed significant shifts in societal structures and norms over the past decade [2, 3, 4, 5, 6, 7]. This polarization has often led to a shift in political attitudes toward ideological extremes, resulting in unstable institutions, erratic policymaking, diminished political dialogue, and the revival of previously discredited regimes [8, 9, 10, 2, 11, 12]. Such dynamics threaten the foundational principles of democracies, where electoral campaigns, including those for presidential elections, have seen compromises that challenge the essence of democratic processes [13, 14, 15, 16]. Central to these developments is the process of social learning through media interactions, which serves as a powerful medium for *opinion formation* and fuels further polarization. Social learning primarily explores how individuals adapt their opinions in light of new information acquired from their social networks, often influenced by cognitive biases.

To understand how social factors influence the acceptance or rejection of opinions, researchers have developed and improved opinion dynamics models. These models, which are central to this study, enable the analysis of key phenomena such as consensus, polarization, and fragmentation in opinion dynamics [17, 18, 19, 20]. The extended approach presented in this work adopts a micro-level perspective, modeling individuals as agents who exchange views on a given topic. By simulating interactions among these agents —including those influenced by figures such as influencers, family, and friends— the study explores how opinions evolve over discrete time intervals. Depending on whether updates are deterministic (affecting all agents simultaneously) or non-deterministic (affecting agents individually or some groups of agents), the choice of model influences the social phenomena that emerge. This paper builds on previous work by introducing new theoretical developments and experimental studies to deepen the understanding of how societal opinions are shaped over time.

This paper introduces a framework that uses concurrent set relations as the formal basis for specifying, simulating, and analyzing social interaction systems built on dynamic opinion models. At the core of this framework are *influence graphs*, which represent the structure of agent interactions: agents are depicted as vertices, and a directed weighted edge from vertex a to vertex b indicates the influence of agent a's opinion on agent b. Building on the concepts of set relations discussed in [21], the framework incorporates two key mechanisms to describe opinion dynamics across these graphs. The *atomic set relation* defines how to update the opinion of a single vertex based on a set of incoming edges and their associated vertices. Additionally, the *strategy* component selects edges to synchronously update the opinions of the connected vertices, ensuring parallel updates. This approach enables dynamic opinion models to be represented as systems of concurrent set relations, where opinion updates are parameterized by combining an atomic relation and a strategy through closures. A notable feature of this model is its ability to capture deterministic or non-deterministic aspects of opinion dynamics, which align with the deterministic or non-deterministic nature of the concurrent set relations used. Overall, the proposed framework provides a formalized representation of user communication within each opinion dynamic model, characterizing the nature of interaction concurrency from fully synchronous to entirely asynchronous.

Standard models for social learning are seamlessly integrated as specific instances within the proposed framework. For example, the DeGroot model [22], a seminal work for opinion formation in social networks where all agents update their opinion simultaneously by averaging the opinions of their contacts, is derived as the synchronous closure under the maximal redexes strategy applied to a given atomic set relation. Similarly, the gossip-based model [23], another representative framework for opinion formation that uses pairwise interactions to simulate opinion formation processes, is realized through an asynchronous closure where the strategy selectively applies to individual edges of the atomic set relation. This framework also accommodates other opinion dynamics models as variations of these standard ones, positioned as midpoints between the DeGroot and gossip-based models via synchronous closures of atomic set relations. Moreover, by introducing new update functions and combining them with various strategies, the framework can generate novel variants of these models. An example includes adaptations of the spiral of silence theory [24], and other specific biases such as confirmation, backfire, fanaticism, and insular biases [25], further enriching the spectrum of social learning simulations possible within this framework.

The implementation of the proposed framework is carried out in the Maude system [26], leveraging the reflective capabilities of rewriting logic to instantiate various opinion models of interest. The framework is structured as a rewrite theory, where a state is represented as an object-like configuration that encapsulates both the structural and opinion-based elements of the system. Specifically, an agent u with a state o_u is represented as $\langle u : o_u \rangle$, while the influence of agent u on agent v with a weight i_{uv} is denoted as $\langle (u, v) : i_{uv} \rangle$. In most simple cases, o_u is a scalar in the range $[0, 1]$ representing the opinion of agent u about a given proposition or assertion. In other cases, this preference can exhibit certain structure and be composed, e.g., of a tuple of scalars.

The atomic set relation and the strategy components draw inspiration from [27]. Conceptualized as a (non-executable) rewrite rule, the atomic set relation processes an agent $\langle u : o_u \rangle$ and a subset $A \subseteq E$ of edges from the current state, updating the agent's state o_u to a new one o'_u based on the update function μ, which for each specific model is defined equationally. Atomically, this transformation only affects the state of agent u with respect to the influences in A, thus rewriting each involved object $\langle u : o_u \rangle$ to its updated form $\langle u : o'_u \rangle$. At the metalevel, the atomic rewrite rule is applied to agents in the state as dictated by the strategy, which selects edges influencing potential updates. The strategy, defined equationally by the user, determines sets of edge subsets where each subset A leads to a parallel rewrite under the maximal redexes strategy through a serialization process [28]. Given that the atomic rewrite relation is deterministic, the introduction of non-determinism stems solely from the strategy, enabling concurrent updates for each edge subset A identified. This fully executable object-like rewrite theory in Maude provides a powerful tool for exploring how opinions within a system of agents evolve and contribute to phenomena like polarization. The framework supports formal methods techniques, including reachability analysis and statistical model checking, to yield deeper insights into the dynamics of opinion formation and polarization.

This work is part of a broader initiative to harness computational ideas and methods for analyzing social network phenomena such as polarization, consensus, and fragmentation. This encompasses the integration of concurrency models, modal and probabilistic logics, and formal methods —including frameworks, techniques, and tools tailored for these purposes. The research presented in this paper marks the first effort to apply rewriting logic to address these challenges. As discussed in later sections, a major challenge faced by opinion dynamic models is the state space explosion problem. This work begins to explore the use of execution strategies, probabilistic simulation, and statistical analysis as potential solutions to mitigate this issue. However, extending the current framework to a fully probabilistic setting —where, for example, the strategy for selecting edge sets is governed by a probability distribution function— remains beyond the scope of this work. Such an extension warrants further investigation and is identified as a crucial area for future research. It promises the potential for a broader spectrum of statistical model checking, employing novel metrics that could significantly advance the analysis of quantitative attributes beyond the capabilities of existing techniques used in opinion dynamic models.

This paper is an extended and revised version of [1], including:

Cognitive biases. The proposed framework underscores its adaptability by seamlessly incorporating an important extension of the DeGroot model. As explained in Section 3.5, the framework facilitates the natural representation of systems wherein agents' opinion updates are influenced by various *cognitive biases* (i.e., systematic deviations from rational updates [29]). These biases, which are ubiquitous in the opinion formation in social networks and include confirmation bias, authority bias, and the backfire effect, can coexist within the same system.

The spiral of silence. Section 3.6 presents two novel extensions inspired by the Spiral of Silence [30], a prominent social science theory stating that individuals may withhold their opinions when they believe they are in the minority. The first extension is a memoryless model that captures a social scenario in which opinions (expressed as messages, posts, etc.) are removed once they have been accessed. In contrast, the second extension is a history-dependent model where certain opinions are remembered. These models expand the framework's capacity to depict a more diverse spectrum of opinion dynamics where agents may choose to remain silent at some point depending on present or past opinions of their contacts following the Spiral of Silence theory. This enhances the utility of the framework for studying a wider range of systems and behavioral dynamics.

Parametric opinion representation. The initial framework, introduced in [1], has been augmented to incorporate an abstract data type for representing agents' states, including their opinions, both conceptually and operationally. This modification, while seemingly minor, significantly preserves the integrity of the original theory's core structure. Simultaneously, it facilitates the seamless integration of the two new models discussed previously. Such adaptability not only underscores the robustness of the proposed framework, but also demonstrates its capacity to accommodate a wide range of opinion dynamics models effectively.

New experimental insights. The executable rewriting theory proposed in this work has been used to conduct additional analyses beyond those presented in [1]. The experimental results align with those reported in [25], which examines consensus conditions under scenarios where agents follow different cognitive biases and the spiral of silence dynamics, thereby allowing for the empirical testing of different hypotheses concerning agents' behavior under different cognitive influences. The Maude files needed to reproduce these experiments are publicly available in the companion tool for this paper [31].

Organization. After recalling the notion of set relations in Section 2, Section 3 shows how different models for social learning can be seen as particular instances (atomic set relation and strategy) of this framework. The implementation in Maude is described in Section 4, while different analyses performed on the proposed rewrite theory are introduced in Section 5. Section 6 concludes the paper. The Maude specification supporting the set relations framework and the experimental setups are available at [31].

2. Set Relations

This section introduces set relations and their notation, as used in this paper. It defines the asynchronous, parallel, and synchronous set relations as closures of an atomic set relation. This section is based mainly on [21].

Let $\mathcal{U}$ be a set whose elements are denoted $A, B, \ldots$ and let $\rightarrow$ be a binary relation on $\mathcal{U}$. An element A of $\mathcal{U}$ is called a $\rightarrow$-*redex* iff there exists $B \in \mathcal{U}$ such that the *pair* $\langle A; B\rangle \in \rightarrow$. The expressions $A \rightarrow B$ and $A \not\rightarrow B$ denote $\langle A; B\rangle \in \rightarrow$ and $\langle A; B\rangle \notin \rightarrow$, respectively. The *identity* and *reflexive-transitive* closures of $\rightarrow$ are defined as usual and denoted $\xrightarrow{0}$ and $\xrightarrow{*}$, respectively.

It is assumed that $\mathcal{U}$ is the family of all *nonempty* finite subsets of an abstract and possibly infinite set T whose members are called *elements* (i.e., $\mathcal{U} \subseteq \mathcal{P}(T)$, $\emptyset \notin \mathcal{U}$, and if $A \in \mathcal{U}$, then $\operatorname{card}(A) \in \mathbb{N}$). Therefore, $\rightarrow$ is a binary relation on finite subsets of elements in T. When it is clear from the context, curly brackets are omitted from set notation; e.g., $a, b \rightarrow b$ denotes $\{a, b\} \rightarrow \{b\}$. Because this convention, the symbol ‘,’ is overloaded to denote set union. For example, if A denotes the set $\{a, b\}$, B the set $\{c, d\}$, and D the set $\{d, e\}$, the expression $A, B \rightarrow B, D$ denotes $a, b, c, d \rightarrow c, d, e$.

Given a set of elements, in the asynchronous set relation *exactly one* redex is selected to be updated.

Definition 2.1 (Asynchronous Set Relation)**.** *The* asynchronous relation $\xrightarrow{\square}$ *is defined as the asynchronous closure of* $\rightarrow$*, i.e., the set of pairs* $\langle A; B\rangle \in \mathcal{U} \times \mathcal{U}$ *such that* $A \xrightarrow{\square} B$ *iff there exists a* $\rightarrow$*-redex* $A' \subseteq A$ *and an element* $B' \in \mathcal{U}$ *such that* $A' \rightarrow B'$ *and* $B = (A \setminus A') \cup B'$*.*

In the parallel set relation, a nonempty collection of redexes is identified to be updated in parallel (i.e., without interleaving).

Definition 2.2 (Parallel Set Relation)**.** *The* parallel relation $\xrightarrow{||}$ *is defined as the parallel closure of* $\rightarrow$*, i.e., the set of pairs* $\langle A; B\rangle \in \mathcal{U} \times \mathcal{U}$ *such that* $A \xrightarrow{||} B$ *iff there exist (nonempty) pairwise disjoint* $\rightarrow$*-redexes* $A_1, \ldots, A_n \subseteq A$*, and elements* $B_1, \ldots, B_n$ *in* $\mathcal{U}$ *such that* $A_i \rightarrow B_i$*, for* $1 \leq i \leq n$*, and* $B = \left(A \setminus \bigcup_{1 \leq i \leq n} A_i\right) \cup \left(\bigcup_{1 \leq i \leq n} B_i\right)$*.*

The synchronous set relation $\xrightarrow{s}$ applies atomic reductions in parallel. However, in contrast to the previous two closures, the redexes are selected with the help of a *strategy* s, a function that identifies a nonempty subset of redexes. As a consequence, the synchronous set relation is a subset of the parallel set relation [21]. It is important to note that the notion of strategy used for defining the synchronous closure of the atomic set relation is different to the one introduced in Section 1 for the framework; the name used in this section is kept from [21].

Definition 2.3 ($\rightarrow$-strategy)**.** *A* $\rightarrow$-strategy *is a function* s *that maps any element* $A \in \mathcal{U}$ *into a set* $s(A) \subseteq \mathcal{P}(\rightarrow)$ *such that if* $s(A) = \{\langle A_1; B_1\rangle, \ldots \langle A_n; B_n\rangle\}$*, then* $A_i \subseteq A$ *and* $A_i \rightarrow B_i$*, for* $1 \leq i \leq n$*, and* $A_1, \ldots, A_n$ *are pairwise disjoint.*

Definition 2.4 (Synchronous Relation)**.** *Let* s *be a* $\rightarrow$*-strategy. The* synchronous relation $\xrightarrow{s}$ *is defined as the synchronous closure of* $\rightarrow$ *w.r.t.* s*, i.e., the set of pairs* $\langle A; B\rangle \in \mathcal{U} \times \mathcal{U}$ *such that* $A \xrightarrow{s} B$ *iff* $B = \left(A \setminus \bigcup_{1 \leq i \leq n} A_i\right) \cup \left(\bigcup_{1 \leq i \leq n} B_i\right)$ *where* $s(A) = \{\langle A_1; B_1\rangle, \ldots \langle A_n; B_n\rangle\}$*.*

This section is concluded with an example that illustrates the notions introduced so far. Consider the directed weighted graph $G = (V, E, i)$ in Figure 1. It represents a social system with six agents $V = \{a, b, c, d, e, f\}$ and twelve opinion influences. The label $i(u, v)$ associated to each edge (u, v) from agent u to agent v denotes the opinion influence $i_{uv} = i(u, v)$ of agent u over the opinion of agent v (about a given topic): these values are in the real interval $[0, 1]$ (i.e., $i : E \rightarrow [0, 1]$); the higher the value, the stronger the influence. In this example, the influence of f over a is the strongest possible. Notice that agents may also have *self-influence*, representing agents whose opinion need not be completely influenced by the opinion of the others.

The initial opinions (or beliefs) of the agents are depicted within the box below each node. They are specified by a function $o : V \to [0,1]$, which is assumed to represent the opinion value $o_u = o(u)$ of each agent u on the given topic. The greater the value, the stronger (weaker) the agreement (disagreement) with the proposition, and 0 represents total disagreement. In this example such a proposition is *vaccines are safe*. Intuitively, the agents a, b, and c are in strong disagreement with vaccines being safe (the anti-vaxxers) and the rest are in strong agreement (the pro-vaxxers).

Notice that although a is the most extreme anti-vaxxer, the most extreme pro-vaxxer f has a strong influence over a. Hence, it is expected that the evolution of a's opinion will be highly influenced by the opinion of f. In general, an agent's opinion evolution takes into account a subset of its influences, as will be explained shortly.

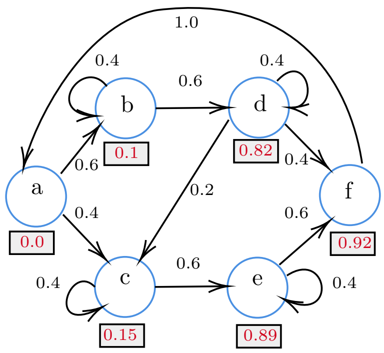

Figure 1: Graph representing opinion and influence interaction in a social system. Initial opinions are given within the box below each node. The labels on each edge (u, v) represent the influence value of agent u over agent v.

Recall the object-like notation in Section 1. The set of elements T is made of pairs of the form $\langle u : r \rangle$ or $\langle (u, v) : r \rangle$, with $u, v \in V$, $(u, v) \in E$, and $r \in [0, 1]$. The graph in Figure 1 can be specified as the set of elements Γ:

$$\begin{aligned}\Gamma = \{ &\langle a : 0.0\rangle, \langle b : 0.1\rangle, \langle c : 0.15\rangle, \langle d : 0.82\rangle, \langle e : 0.89\rangle, \langle f : 0.92\rangle, \\ &\langle (a, b) : 0.6\rangle, \langle (a, c) : 0.4\rangle, \langle (b, d) : 0.6\rangle, \langle (c, e) : 0.6\rangle, \langle (d, c) : 0.2\rangle, \langle (d, f) : 0.4\rangle, \\ &\langle (e, f) : 0.6\rangle, \langle (f, a) : 1.0\rangle, \langle (b, b) : 0.4\rangle, \langle (c, c) : 0.4\rangle, \langle (d, d) : 0.4\rangle, \langle (e, e) : 0.4\rangle\}.\end{aligned}$$

The atomic relation $\to_A$ is defined over elements representing agents and is parametric on a set A of elements representing edges in Γ. In this example, it follows the pattern

$$\langle u : o_u \rangle \to_A \left\langle u : \sum_{\langle (x,u):i_{xu}\rangle \in A} o_x \cdot \frac{i_{xu}}{\sum_{\langle (y,u):i_{yu}\rangle \in A} i_{yu}} \right\rangle, \tag{1}$$

where the summation in the denominator is assumed to be non-zero. The opinion o_u of an agent u w.r.t. to A is updated to be the weighted average of the opinion values of those agents adjacent to u and whose influence is present in A. For instance, let $A = \{\langle (a, b) : 0.6\rangle, \langle (b, b) : 0.4\rangle, \langle (c, e) : 0.6\rangle\}$. Then, the atomic set relation $\to_A$ has the following two pairs:

$$\langle b : 0.1\rangle \to_A \langle b : 0.04\rangle \qquad\qquad \langle e : 0.89\rangle \to_A \langle e : 0.15\rangle .$$

In the case of agent b, its opinion is updated to $0.04 = 0.0 \cdot \frac{0.6}{1.0} + 0.1 \cdot \frac{0.4}{1.0}$ because, w.r.t. A, it is influenced both by itself and by agent a, whose opinion value is 0.0 and influence over b is 0.6. In the case of agent e, its opinion is influenced only by agent c. In this case, the value is updated to $0.15 = 0.15 \cdot \frac{0.6}{0.6}$. It can be said that, w.r.t. A, agent e acts like a *puppet* whose own opinion is not taken into account when it is updated.

The asynchronous closure of $\to_A$ has exactly two pairs, one for each redex determined by $\to_A$ (i.e., one for agent b and another for agent e):

$$\Gamma \xrightarrow{\square}_A (\Gamma \setminus \{\langle b : 0.1\rangle\}) \cup \{\langle b : 0.04\rangle\} \qquad\qquad \Gamma \xrightarrow{\square}_A (\Gamma \setminus \{\langle e : 0.89\rangle\}) \cup \{\langle e : 0.15\rangle\}.$$

The parallel closure $\xrightarrow{||}$ has three pairs: one in which the opinions of both b and e are updated, in addition to the same two pairs present in the asynchronous closure:

$$\begin{aligned}
&\Gamma \xrightarrow{||}_A (\Gamma \setminus \{\langle b : 0.1\rangle\}) \cup \{\langle b : 0.04\rangle\} \\
&\Gamma \xrightarrow{||}_A (\Gamma \setminus \{\langle e : 0.89\rangle\}) \cup \{\langle e : 0.15\rangle\} \\
&\Gamma \xrightarrow{||}_A (\Gamma \setminus \{\langle b : 0.1\rangle, \langle e : 0.89\rangle\}) \cup \{\langle b : 0.04\rangle, \langle e : 0.15\rangle\}.
\end{aligned}$$

Finally, to illustrate the synchronous closure of $\rightarrow$, let $s = A$ be the strategy. That is, all redexes in $\rightarrow_A$ are identified to be reduced. Therefore, this relation has the only pair in which the opinions of both b and e are updated in parallel:

$$\Gamma \xrightarrow{s}_A (\Gamma \setminus \{\langle b : 0.1\rangle, \langle e : 0.89\rangle\}) \cup \{\langle b : 0.04\rangle, \langle e : 0.15\rangle\}.$$

3. Opinion Dynamic Models

This section demonstrates how opinion dynamics models can be specified as set relations (see Section 2). In particular, it introduces the gossip-based (Section 3.2) and classical DeGroot (Section 3.3) opinion models. The proposed framework is flexible enough to accommodate a generalized version of both the DeGroot and gossip models (under certain conditions), here called the *hybrid opinion model* (Section 3.4). The flexibility of the framework allows also for considering extensions of the DeGroot model that incorporate individual cognitive biases (Section 3.5) and the Spiral of Silence theory (Section 3.6).

The above-mentioned models are defined, as stated in Section 2, over a directed weighted graph $G = (V, E, i)$ representing a social system, with agents V, directed opinion influences $E \subseteq V \times V$, and influence values $i : E \to [0, 1]$. A given topic (i.e., proposition) is fixed. The weight $i_{uv} = i(u, v)$ associated to each edge $(u, v) \in E$ from agent u to agent v denotes the opinion influence value of agent u over the opinion value of agent v on the given topic. For a given datatype O, the state $o_u = o(u) \in O$ associated to each agent $u \in V$ is or includes its opinion about the given topic, which is assumed to be known by all agents in the system. In all cases treated in this section, the opinion is a value in $[0, 1]$ equipped with the usual total order on the real numbers. As in Section 2, the "higher" the value of a opinion (resp., influence), the stronger the agreement (resp., influence).

The set of elements T in the set relations framework is made of pairs of the form $\langle u : o\rangle$ or $\langle (u, v) : r\rangle$, with $u, v \in V$, $(u, v) \in E$, $o \in O$, and $r \in [0, 1]$. A *G-configuration* (or *configuration*) is the set of elements in T that exactly represent the structure of G, and the values of opinions and influences. Therefore, in the rest of this section, it is assumed that any configuration Γ can be partitioned in two sets Γ_o and Γ_i, respectively containing elements of the form $\langle u : o_u\rangle$ specifying opinions and $\langle (u, v) : i_{uv}\rangle$ specifying influences.

A model specifies how opinions (associated to agents) can be updated. Each model definition comprises three pieces; namely, an atomic relation, a strategy, and an update function for agents' states. Therefore, a model specifies how a G-configuration $\Gamma = \Gamma_o \cup \Gamma_i$ can change to another G-configuration $\Gamma' = \Gamma_{o'} \cup \Gamma_i$, where only the state of the agents is updated. It is important to note that the notion of strategy introduced in this section generalizes the notion of strategy introduced in Section 2, as will be explained later.

The atomic relation is defined in Section 3.1 for all the models. Each model is introduced by identifying a specific strategy and a specific update function in subsequent sections.

3.1. The Atomic Relation

The atomic relation $\rightarrow_A$ is parametric on a subset $A \subseteq \Gamma_i$ and defines how the state of a single agent may evolve. The set of influences A directly identifies the influences (and indirectly the opinions) to update the state of each agent in the configuration Γ (i.e., in Γ_o). For each one of the three models, the atomic relation $\rightarrow_A$ follows the pattern:

$$\langle u : o_u \rangle \rightarrow_A \langle u : \mu(\Gamma, A, u) \rangle \,, \tag{A-Rel}$$

where μ is the update function specific to each model. This function takes as input a G-configuration (e.g., Γ), a subset of its influences (e.g., A), and the agent whose state is to be updated (e.g., u), and outputs the new state for agent u w.r.t. Γ and A in the corresponding model.

3.2. Gossip-based Models

In a gossip-based model, single peer-to-peer interactions are used to update the opinion of a single user at each time-step. In general, a strategy in the proposed framework identifies a collection of subsets of interactions in Γ_i. In particular, the strategy ρ_{gossip} maps a G-configuration to the collection of singletons made from the influences in Γ_i:

$$\rho_{\text{gossip}}(\Gamma) = \{\{x\} \mid x \in \Gamma_i\}.$$

This means that, at each time-step, the opinion value of agent v can be updated w.r.t. the opinion value of agent u for each singleton $\{\langle (u, v) : i_{uv} \rangle\}$ computed by the strategy $\rho_{\text{gossip}}(\Gamma)$.

Opinions in this model are real numbers in the interval $[0, 1]$ (i.e., $O = [0, 1]$). The update function μ_{gossip} is defined for any $u \in V$ and $A = \{\langle (v, u) : i_{vu} \rangle\} \in \rho_{\text{gossip}}(\Gamma)$ as:

$$\mu_{\text{gossip}}(\Gamma, A, u) = o_u + (o_v - o_u) \cdot i_{vu}.$$

It can be checked that $\mu_{\text{gossip}}(\Gamma, A, u) \in O$ given $o_u, o_v, i_{vu} \in [0, 1]$.

Each singleton $A \in \rho_{\text{gossip}}(\Gamma)$ determines an atomic relation that updates exactly one agent's opinion in the given configuration. Recall, from Section 3.1, that each pair in the atomic set relation $\rightarrow_A$ has the form:

$$\langle u : o_u \rangle \rightarrow_A \langle u : \mu_{\text{gossip}}(\Gamma, A, u) \rangle \,.$$

Hence, in this model, the opinion of an agent u is updated by identifying an edge from an agent v (it may be u itself if it has a self-loop) with influence i_{vu} over u and by adding to its current opinion o_u the weighted difference of opinion $(o_v - o_u) \cdot i_{vu}$ of v over u.

A gossip-based model is identified as a binary set relation on G-configurations in terms of the asynchronous closure of $\rightarrow_A$, for each singleton $A \in \rho_{\text{gossip}}(\Gamma)$.

Definition 3.1. *The $\rightarrow_{gossip}$ set relation is defined as the set of pairs $\langle \Gamma; \Gamma' \rangle$ of G-configurations such that:*

$$\Gamma \rightarrow_{gossip} \Gamma' \quad \textit{iff} \quad (\exists A \in \rho_{gossip}(\Gamma))\; \Gamma \overset{\square}{\rightarrow}_A \Gamma'.$$

From the viewpoint of concurrency, the gossip-based opinion dynamic model realized by $\rightarrow_{\text{gossip}}$ is nondeterministic in the sense that from each state (i.e., G-configuration) exactly $|\Gamma_i|$ transitions are possible, one per edge in E.

3.3. DeGroot Model

In the DeGroot model, the opinion value of every agent in the network is updated at each time-step. All influences are considered at the same time.

The strategy for DeGroot in the proposed framework identifies the whole set of interactions in the network, i.e., Γ_i. In particular, the strategy ρ_{DeGroot} maps a G-configuration to the singleton whose only element is Γ_i:

$$\rho_{\text{DeGroot}}(\Gamma) = \{\Gamma_i\}.$$

Opinions in this model are also real numbers in the interval $[0, 1]$. The update function μ_{DeGroot} is defined for any $u \in V$ and $A \in \rho_{\text{DeGroot}}(\Gamma)$ (i.e., $A = \Gamma_i$) as:

$$\mu_{\text{DeGroot}}(\Gamma, A, u) = o_u + \sum_{\langle (v,u):i_{vu}\rangle \in A} (o_v - o_u) \cdot \frac{i_{vu}}{\sum_{\langle (x,u):i_{xu}\rangle \in A} i_{xu}},$$

where the summation in the denominator is assumed to be non-zero. Otherwise, the value of this function is assumed to be o_u (i.e., the opinion of agent u does not change). In either case, $\mu_{\text{DeGroot}}(\Gamma, A, u) \in [0, 1]$.

The DeGroot model is identified as a binary set relation on G-configurations in terms of the synchronous closure of $\rightarrow_{\Gamma_i}$ under the maximal redexes strategy for $s = \Gamma_i$.

Definition 3.2. *The $\rightarrow_{DeGroot}$ set relation is defined as the set of pairs $\langle \Gamma; \Gamma' \rangle$ of G-configurations such that:*

$$\Gamma \rightarrow_{DeGroot} \Gamma' \quad \textit{iff} \quad \Gamma \xrightarrow{\Gamma_i}_{\Gamma_i} \Gamma'.$$

From the viewpoint of concurrency, the DeGroot opinion dynamic model captured by $\rightarrow_{\text{DeGroot}}$ is deterministic in the sense that, at each state, there is exactly only one possible transition where all influences are taken into account to update each agent's opinion without interleaving.

3.4. The Hybrid Model

The hybrid model considers every possible influence scenario in the network, i.e., any possible combination of influences are used to update the opinion of agents that may be affected by them at each time-step. Therefore, the strategy in the proposed framework identifies all nonempty subsets of interactions in Γ_i. In particular, the strategy ρ_{hybrid} maps a G-configuration to the collection of nonempty subsets made from the influences in Γ_i:

$$\rho_{\text{hybrid}}(\Gamma) = \{A \mid A \subseteq \Gamma_i \text{ and } A \neq \emptyset\}.$$

This means that, at each time-step, the opinion value of an agent v can be updated with a subset of its influencers.

The update function μ_{hybrid} is the same as function ρ_{DeGroot} (hence $O = [0, 1]$). That is, it is defined for any $u \in V$ and $A \in \rho_{\text{hybrid}}(\Gamma)$ as:

$$\mu_{\text{hybrid}}(\Gamma, A, u) = o_u + \sum_{\langle (v,u):i_{vu}\rangle \in A} (o_v - o_u) \cdot \frac{i_{vu}}{\sum_{\langle (x,u):i_{xu}\rangle \in A} i_{xu}},$$

where the summation in the denominator is assumed to be non-zero. Otherwise, the value of this function is assumed to be o_u (i.e., the opinion of agent u does not change). Each subset $A \in \rho_{\text{hybrid}}(\Gamma)$ determines an atomic relation that may update more that one agent's opinion. Hence, in this model, the opinion of an agent is updated by identifying some edges that may have influence over it.

The hybrid model is identified as a binary set relation on G-configurations in terms of the synchronous closure of $\rightarrow_A$, for each subset $A \in \rho_{\text{hybrid}}(\Gamma)$.

Definition 3.3. *The $\rightarrow_{hybrid}$ set relation is defined as the set of pairs $\langle \Gamma; \Gamma' \rangle$ of G-configurations such that:*

$$\Gamma \rightarrow_{hybrid} \Gamma' \quad \textit{iff} \quad (\exists A \in \rho_{hybrid}(\Gamma))\ \Gamma \xrightarrow{A}_A \Gamma'.$$

From the viewpoint of concurrency, the hybrid opinion dynamic model has the maximum degree of non-determinism possible. Moreover, this model is more general than the DeGroot model.

Theorem 3.1. $\rightarrow_{DeGroot} \subseteq \rightarrow_{hybrid}$.

Proof. It follows by noting that $\Gamma_i \in \rho_{\text{hybrid}}(\Gamma)$ and, for each vertex $u \in V$, the equality $\mu_{\text{DeGroot}}(\Gamma, \Gamma_i, u) = \mu_{\text{hybrid}}(\Gamma, \Gamma_i, u)$ holds. □

It is not necessarily the case that $\to_{\text{gossip}} \subseteq \to_{\text{hybrid}}$. This is because the update functions do not always agree when the collection of selected influences A is a singleton. In particular, for each singleton $A = \{\langle (v, u) : i_{vu} \rangle\}$, $\mu_{\text{hybrid}}(\Gamma, A, u) = o_v$, meaning that agent u in the hybrid model behaves always like a *puppet* when $u \neq v$. Note that this is not (necessarily) the case in $\to_{\text{gossip}}$. Nevertheless, there is a class of graphs for which this inclusion holds.

Theorem 3.2. *If G is such that each vertex has a self-loop and is influenced at most by another vertex, and the summation of its incoming influences is 1, then $\to_{gossip} \subseteq \to_{hybrid}$.*

Proof. If $\Gamma \to_{\text{gossip}} \Gamma'$, there is a singleton $A \in \rho_{\text{gossip}}(\Gamma)$ such that $\Gamma \overset{\square}{\to}_A \Gamma'$. Let $A = \{\langle (v, u) : i_{vu} \rangle\}$. If u has exactly one incoming edge, then $v = u$ (by the initial assumption) and $\mu_{\text{gossip}}(\Gamma, A, u) = o_u = \mu_{\text{hybrid}}(\Gamma, A, u)$. Since $A \in \rho_{\text{hybrid}}(\Gamma)$, it follows that $\Gamma \to_{\text{hybrid}} \Gamma'$. If u has two edges, and the self-loop is taken, the case $v = u$ is as above. Otherwise, if $u \neq v$, the same transition is obtained in the hybrid model by taking $A' \in \rho_{\text{hybrid}}(\Gamma)$ where $A' = A \cup \{\langle (u, u) : 1 - i_{vu} \rangle\}$ (the denominator in μ_{hybrid} becomes 1). □

3.5. Cognitive Biases in DeGroot

The DeGroot model precludes the possibility of considering agents who may update their opinions differently depending on their individual cognitive biases. Such biases include, for instance, *confirmation bias* where agents are more receptive to opinions that align closely with their own, the *backfire effect* where agents strengthen their position of disagreement in the presence of opposing views, and *authority bias* where individuals tend to follow authoritative or influential figures, often to an extreme. In [25], a DeGroot-based model featuring the above mentioned biases is proposed.

Being a synchronous extension of the DeGroot model, in [25] *all* the agents update their opinions in each iteration. This means that the strategy $\rho_{\text{DeGrootCB}}$ selects the whole set of interactions:

$$\rho_{\text{DeGrootCB}}(\Gamma) = \{\Gamma_i\}.$$

The set relation framework proposed here is general enough to capture the situation where agents may update their opinions by using different confirmation biases. In particular, a bias can be subsumed as part of the update function in the framework.

Definition 3.4 (Bias Update [25])**.** *Let $G = (V, E, i)$ be a social system, $o : V \longrightarrow [0, 1]$ be the opinion of agents V about a given proposition, and $A \subseteq \Gamma_i$. The function μ_{CB} is a (disagreement) bias update if for every agent $u \in V$,*

$$\mu_{CB}(u, A) = \left[o_u + \sum_{\langle (v,u):i_{vu} \rangle \in A} \beta_{uv}(o_v - o_u) \cdot \frac{i_{vu}}{\sum_{\langle (x,u):i_{xu} \rangle \in A}} \right]_0^1 \tag{2}$$

where each β_{uv}, called the disagreement bias from u towards v, is an endo-function on $[-1, 1]$, and $[\cdot]_0^1$ is the clamp function defined as $[r]_0^1 = min(max(r, 0), 1)$ for any $r \in \mathbb{R}$.

The clamp function $[\cdot]_0^1$ guarantees that the right-hand side of Eq. 2 yields a valid belief value (a value in $[0, 1]$). Intuitively, the function β_{uv} represents the direction and magnitude of how agent u reacts to their disagreement $(o_v - o_u)$ with agent v. If $\beta_{uv}(o_v - o_u)$ is a negative term in the sum of Eq. 2, then the bias of agent u towards v contributes with a magnitude proportional to $|\beta_{uv}(o_v - o_u)|$ to *decreasing* u's belief in the underlying proposition. Conversely, if $\beta_{uv}(o_v - o_u)$ is positive, it contributes to *increasing* i's belief with the same magnitude.

Given a set of bias functions $B = \{\beta_{uv} \mid u, v \in V\}$ and $A \in \rho_{\text{DeGrootCB}}(\Gamma)$ (i.e., $A = \Gamma_i$), the update function is defined as:

$$\mu_{\text{DeGrootCB}}(\Gamma, A, u) = \mu_{CB}(u, A).$$

Similar to Section 3.3, the DeGroot model extended with individual cognitive bias functions is defined as a binary set relation on G-configurations in terms of the synchronous closure of $\rightarrow_{\Gamma_i}$ under the maximal redexes strategy.

Definition 3.5. *The $\rightarrow_{DeGrootCB}$ set relation is defined as the set of pairs $\langle \Gamma; \Gamma' \rangle$ of G-configurations such that:*

$$\Gamma \rightarrow_{DeGrootCB} \Gamma' \quad \textit{iff} \quad \Gamma \xrightarrow{\Gamma_i}_{\Gamma_i} \Gamma'.$$

3.6. The Spiral of Silence in the DeGroot Model

The DeGroot model assumes that each agent is always willing to interact with all other agents in its neighborhood to express its opinion and be influenced. However, this model does not account for some social and psychological factors that may inhibit agents from sharing their opinions. The Spiral of Silence [30] is a social theory that describes how individuals may be unwilling to express their opinions when they perceive themselves to be in the minority. Recently, an extension of the DeGroot model incorporating this theory was proposed in [24]. The extension introduces two versions of the model: the memoryless silence opinion model (SOM$^-$) and the memory-based silence opinion model (SOM$^+$). Both versions are discussed next.

3.6.1. Memoryless Silence Opinion Model

In this model, silent agents are excluded from the opinion updates of the agents they would typically influence. Additionally, agents become silent at a given time (thus withholding their opinions) if their views do not align with the majority of their non-silent contacts.

Each agent $u \in V$ is associated to a state (o_u, s_u, τ_u) where $o_u \in [0, 1]$ is its opinion value, $s_u \in \{0, 1\}$ is its *willingness to speak* (i.e., to not be silent), and $\tau_u \in [0, 1]$ is its *tolerance* to other opinions. The value s_u is 1 when the agent u is willing to express its opinion; otherwise, it is 0. Furthermore, any opinion value within a distance less or equal the tolerance value $\tau_u \in [0, 1]$ is considered to be aligned with the opinion of agent u.

The spiral of silence models are extensions of the synchronous DeGroot model and, consequently, each agent updates its opinion at each time considering the whole set of interactions:

$$\rho_{\text{som}^-}(\Gamma) = \{\Gamma_i\}.$$

The update function μ_{som^-} for the memoryless version is defined for any $u \in V$ and $A \in \rho_{\text{som}^-}(\Gamma)$ as follows:

$$\mu_{\text{som}^-}(\Gamma, A, u) = (\mu_{\text{opinion}^-}(\Gamma, A, u), \mu_{\text{silence}^-}(\Gamma, A, u), \tau_u),$$

where μ_{opinion^-} and μ_{silence^-} stand for the functions for updating the opinion and the silence value for each agent. They are defined as:

$$\mu_{\text{opinion}^-}(\Gamma, A, u) = o_u + \sum_{\langle (v,u):i_{vu} \rangle \in A} i_{vu} \cdot s_v \cdot (o_v - o_u) \text{ and}$$

$$\mu_{\text{silence}^-}(\Gamma, A, u) = \begin{cases} 1 & \text{, if } \left\lceil \frac{|N_u|}{2} \right\rceil \leq |\{v \in N_u \mid |o_u - o_v| \leq \tau_u\}| \\ 0 & \text{, otherwise} \end{cases}$$

where $N_u = \{v \mid \langle (v, u) : i_{vu} \rangle \in A\}$. In this way, the agent u only considers the non-silent neighbors when it updates its opinion and it will be silent if and only if the opinions of the majority of its non-silent neighbors do not align with its opinion value.

The memoryless model for the spiral of silence is defined as a binary set relation on G-configurations in terms of the synchronous closure of $\rightarrow_{\Gamma_i}$ under the maximal redexes strategy.

Definition 3.6. *The $\rightarrow_{som^-}$ set relation is defined as the set of pairs $\langle \Gamma; \Gamma' \rangle$ of G-configurations such that:*

$$\Gamma \rightarrow_{som^-} \Gamma' \quad \textit{iff} \quad \Gamma \xrightarrow{\Gamma_i}_{\Gamma_i} \Gamma'.$$

3.6.2. Memory-based Silence Opinion Model

In this model, agents choose to be silent if their opinion does not align with the *most recent* public opinions of *the majority of* their contacts. In addition, when their current opinion is unknown, their most recent public opinion is taken into account in the update. Furthermore, silent agents are not excluded from the opinion updates of the agents they influence.

Each agent $u \in V$ is associated to a state (o_u, s_u, τ_u, po_u) where o_u, s_u, and τ_u, are defined as in SOM$^-$ model, and $po_u \in [0, 1]$ is the opinion value of agent u the last time it was not silent, i.e., po_i represents the agent's *public opinion value.*

Similar to the memoryless and Degroot models, the strategy ρ_{som^+} identifies the whole set of interactions at once:

$$\rho_{\text{som}^+}(\Gamma) = \{\Gamma_i\}.$$

The update function μ_{som^+} is defined for any $u \in V$ and $A \in \rho_{\text{som}^+}(\Gamma)$ (i.e., $A = \Gamma_i$) as:

$$\mu_{\text{som}^+}(\Gamma, A, u) = (\mu_{\text{opinion}^+}(\Gamma, A, u), \mu_{\text{silence}^+}(\Gamma, A, u), \tau_u, \mu_{\text{public}^+}(\Gamma, A, u)),$$

where μ_{opinion^+}, μ_{silence^+}, and μ_{public^+} update the opinion, silence, and public opinion values for each agent. They are defined as follows:

$$\mu_{\text{opinion}^+}(\Gamma, A, u) = o_u + \sum_{\langle (v,u) : i_{vu} \rangle \in A} i_{vu} \cdot (po_v - o_u),$$

$$\mu_{\text{silence}^+}(\Gamma, A, u) = \begin{cases} 1 & \text{if } \left\lceil \frac{|N_u|}{2} \right\rceil \leq |\{v \in N_u \mid |o_u - po_v| \leq \tau_u\}|, \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_{\text{public}^+}(\Gamma, A, u) = \begin{cases} \mu_{\text{opinion}^+}(\Gamma, A, u) & \text{, if } \left\lceil \frac{|N_u|}{2} \right\rceil \leq |\{v \in N_u \mid |o_u - po_v| \leq \tau_u\}| \\ po_u & \text{, otherwise,} \end{cases}$$

where N_u is as in the memoryless model. Correspondingly, the agent u does not consider explicitly the silence value of its neighbors and, instead, it updates its opinion and silence values by considering, only, their public opinions.

The memory-based silence opinion model is also identified as a binary set relation on G-configurations in terms of the synchronous closure of $\rightarrow_{\Gamma_i}$ under the maximal redexes strategy.

Definition 3.7. *The $\rightarrow_{som^+}$ set relation is defined as the set of pairs $\langle \Gamma; \Gamma' \rangle$ of G-configurations such that:*

$$\Gamma \rightarrow_{som^+} \Gamma' \quad \textit{iff} \quad \Gamma \xrightarrow{\Gamma_i}_{\Gamma_i} \Gamma'.$$

4. The Framework in Rewriting Logic

This section introduces a rewrite theory $\mathcal{R}$ that implements the set relations framework in Section 2. Off-the-shelf definitions are provided to instantiate the framework with opinion dynamic models, such as the ones introduced in Section 3. This section assumes familiarity with rewriting logic [32] and Maude [26]; Section 4.1 presents some preliminaries on these two subjects. The full Maude specification supporting the set relations framework is available at [31].

4.1. Overview of Rewriting Logic and Maude

A *rewrite theory* [32] is a tuple $\mathcal{R} = (\Sigma, E, L, R)$ such that: (Σ, E) is an equational theory where Σ is a signature that declares sorts, subsorts, and function symbols; E is a set of (conditional) equations of the form $t = t'$ **if** ψ, where t and t' are terms of the same sort, and ψ is a conjunction of equations; L is a set of *labels*; and R is a set of labeled (conditional) rewrite rules of the form $l : q \longrightarrow r$ **if** ψ, where $l \in L$ is a label, q and r are terms of the same sort, and ψ is a conjunction of equations. Condition ψ in equations and rewrite rules can be more general than conjunction of equations, but this extra expressiveness is not needed in this paper.

The expression $T_{\Sigma,s}$ denotes the set of ground terms of sort s and $T_\Sigma(X)_s$ denotes the set of terms of sort s over a set of sorted variables X. The expressions $T_\Sigma(X)$ and T_Σ denote all terms and ground terms, respectively. A substitution $\sigma : X \to T_\Sigma(X)$ maps each variable to a term of the same sort and $t\sigma$ denotes the term obtained by simultaneously replacing each variable x in a term t with $\sigma(x)$.

A *one-step rewrite* $t \longrightarrow_{\mathcal{R}} t'$ holds if there is a rule $l : q \longrightarrow r$ **if** ψ, a subterm u of t, and a substitution σ such that $u = q\sigma$ (modulo equations), t' is the term obtained from t by replacing u with $r\sigma$, and $v\sigma = v'\sigma$ holds in (Σ, E) for each $v = v'$ in ψ. The reflexive-transitive closure of $\longrightarrow_{\mathcal{R}}$ is denoted as $\longrightarrow^*_{\mathcal{R}}$.

Maude [26] is a language and tool supporting the specification and analysis of rewrite theories. A Maude module (`mod` M `is` ... `endm`) specifies a rewrite theory $\mathcal{R}$. Sorts and subsort relations are declared by the keywords `sort` and `subsort`; the syntax `op` f `:` s_1 ... s_n `->` s introduces the function symbol (or operator) f where $s_1, \ldots, s_n$ are the sorts of its arguments, and s is its (value) sort. Operators can have user-definable syntax, with underbars '`_`' marking each of the argument positions (e.g., `_+_`). Some operators can have equational attributes, such as `assoc`, `comm`, and `id:` t, stating that the operator is, respectively, associative, commutative, and/or has identity element t. Equations are specified with the syntax `eq` t `=` t' or `ceq` t `=` t' `if` ψ; and rewrite rules as `rl [`l`] :` u `=>` v or `crl [`l`] :` u `=>` t' `if` ψ. The mathematical variables in such statements are declared with the keywords `var` and `vars`.

Maude provides a large set of analysis methods, including computing the canonical form of a term t (command `red` t), simulation by rewriting (`rew` t), reachability analysis (`search` t `=>*` t' `such that` ψ), and rewriting according to a given rewrite strategy (`srew` t `using` *str* and `dsrew` t `using` *str* for breadth- and depth-first search respectively). Basic rewrite strategies include $r[\sigma]$ (apply rule with label r once with the optional ground substitution σ), `idle` (identity), `fail` (empty set), and `match` P `s.t.` C, which checks whether the current term matches the pattern P subject to the constraint C. Compound strategies can be defined using concatenation $(\alpha\,;\,\beta)$, disjunction $(\alpha \,|\, \beta)$, iteration $(\alpha\,*)$, α `or-else` β (execute β if α fails), among other options.

The Unified Maude model-checking tool [33] (`umaudemc`) enables the use of different model checkers to analyze Maude specifications. Besides being an interface for the standard LTL model checker of Maude, it also offers the possibility of interfacing external CTL and probabilistic model checkers. For the purpose of this paper, the command `scheck` [34] is used to assign probabilities to the transition system generated by an initial term t and to perform statistical model checking to estimate quantitive expressions written in the Quantitative Temporal Expressions (QuaTEx) language [35]. QuaTEx supports parameterized recursive temporal operator definitions using primitive non-temporal operators (e.g., conditional statements, values from the current state of the system, etc) and the *next* temporal operator (notation #). The QuaTEx query `eval E[`*expr*`]` returns the expected value of the expression *expr* using the Monte Carlo method.

Meta-programming. Maude supports *meta-programming*, where a Maude module M (resp., a term t) can be (meta-)represented as a Maude *term* $\overline{M}$ of sort `Module` (resp., as a Maude term $\bar{t}$ of sort `Term`) in Maude's `META-LEVEL` module. Given a term t, the Maude function `upTerm(`t`)` returns $\bar{t}$. Maude provides built-in functions such as `metaRewrite` and `metaSearch`, which are the "meta-level" functions corresponding to "user-level" commands to perform rewriting and search, respectively.

4.2. Influences, Opinions, and States

An agent a and its state, and the influence of agent a over agent b with weight i_{ab}, are specified in the rewrite theory $\mathcal{R}$ with the help of the following sorts and function symbols:

```
sorts Agent Opinion AgentState Edge .
op  <_:_>    :     Agent AgentState  -> Opinion [ctor] .
op  <`(_,_`):_> : Agent Agent Float -> Edge    [ctor] .
```

The user is expected to provide appropriate constructors for the sort `Agent` and the sort `AgentState`, e.g., by extending $\mathcal{R}$ with the subsort relation `subsort Nat < Agent` to use natural numbers as identifiers for agents. Some distinguished constructor for `AgentState` include the following for the DeGroot and the spiral of silence models:

```
op [_]    :     Float                   -> AgentState [ctor] .
op [_,_,_] :    Float Float Float       -> AgentState [ctor] .
op [_,_,_,_] : Float Float Float Float -> AgentState [ctor] .
```

Sets of agents, opinions, and edges (sorts `SetAgent`, `SetOpinion`, and `SetEdge` respectively) are defined as "`,`"-separated sets of elements in the usual way. A G-configuration $\Gamma = \Gamma_o \cup \Gamma_i$ is represented by a term of sort `Network`, defining the set of agents' opinions (Γ_o) and influences (Γ_i) with the following sort and function symbol:

```
sort Network .
op < nodes:_ ; edges:_ > : SetOpinion SetEdge -> Network [ctor] .
```

Analyzing opinion dynamics usually requires determining the number of interactions between agents and the time needed to reach a given state. A term of the form "N `in step:`t `comm:` nc" of sort `State` represents the state of a network N at time instant t, where a number of interactions/communications nc have taken place:

```
sort State .
op _in step:_ comm:_ : Network Nat Nat -> State [ctor] .
```

4.3. Strategies and the Atomic Relation

The framework is parametric on a strategy ρ and an update function μ, as explained in Section 3. The atomic relation $\rightarrow_A$ is parametric on a nonempty subset $A \subseteq \Gamma_i$. A strategy identifies each one of such subsets at each time-step. A `SetSetEdge` is a "`;`"-separated set of sets of edges.

```
sort SetSetEdge .
subsort NeSetEdge < SetSetEdge .
op mt : -> SetSetEdge [ctor] .
op _;_ : SetSetEdge SetSetEdge -> SetSetEdge [ctor assoc comm id: mt] .
```

Some distinguished `SetSetEdge`s include the singleton with all the edges in the network (DeGroot model), the set containing only singletons (Gossip model) and the set of nonempty subsets of edges (Hybrid model).

```
var SE : SetEdge .   var E : Edge .
op deGroot   : SetEdge     -> SetSetEdge .
eq deGroot(SE) = SE .

op gossip   : SetEdge       -> SetSetEdge .
eq gossip(empty) = mt .
eq gossip((E, SE)) = E ; gossip(SE) .

op hybrid : SetEdge -> SetSetEdge .
eq hybrid(SE) = power-set(SE) \ empty .

op strategy :  -> SetSetEdge . --- user defined strategy
```

The operator `strategy` must be defined by the user to identify the subsets $A \subseteq \Gamma_i$ available in each transition. This can be done, e.g., by adding the equation

```
eq strategy =  gossip(edges) .
```

where `edges` is the set of edges in the network currently being modeled.

The atomic relation (pattern (A-Rel) on page 7) is defined as a non-executable rewrite rule (`nonexec`) and the set relation framework is implemented using the meta-programming facilities in Maude. In particular, the atomic rewrite relation updates the belief of a given agent (u in pattern (A-Rel)) when a set of edges (A in pattern (A-Rel)) is selected.

```
var   AGENT          : Agent .
vars  BELIEF BELIEF' : AgentState .
var   STATE          : State .
vars  SETEDGE EDGES  : SetEdge .

op update : State SetEdge Agent -> AgentState .  --- user defined μ

crl [atomic] : < AGENT : BELIEF >   =>  < AGENT :  BELIEF' >
    if BELIEF' := update(STATE, SETEDGE, AGENT) [nonexec] .
```

The function `update` (μ in pattern (A-Rel)) must be specified by the user. The framework provides instances of this function for the models presented in Section 3.

An asynchronous, parallel, or synchronous rewrite step, depending on the underlying strategy, is captured by the rewrite rule `step` below:

```
var  SETNODE    : SetNode .
vars STEPS COMM : Nat .

op moduleName : -> Qid . --- Name of the module with the user's network

crl [step] : STATE => STATE'
    if  EDGES ; SSE := strategy /\
        STATE'      := step([moduleName], STATE, EDGES) .
```

In this rule, the current `STATE` is updated to `STATE'` by non-deterministically selecting a set of `EDGES` from the set of set of edges available according to the `strategy`. The function `step` below takes as parameters the meta-representation of the user's module defining the network (`[moduleName]`), the current state, and the selected set of edges.

```
var SETAG : SetAgent .
var SETOP : SetOpinion .
var M     : Module .

op step : Module State SetAgent SetOpinion SetEdge -> State .
op step : Module State                     SetEdge -> State .

eq step(M, STATE, EDGES) =
   step(M, STATE, incidents(EDGES), empty, EDGES) .
eq step(M, STATE, empty, SETOP, EDGES) =
  < nodes: (nodes(STATE) / SETOP) ; edges: edges(STATE) >
  in step: (steps(STATE) + 1) comm: (comm(STATE) + | non-self(EDGES) |) .
eq step(M, STATE, (AGENT, SETAG), SETOP, EDGES) =
   step(M, STATE, SETAG, (SETOP, next(M, AGENT, EDGES, STATE)), EDGES) .
```

The function `step` recursively computes the beliefs of the agents `incident` to `EDGES`. The updated beliefs are accumulated in the set of opinions `SETOP`. The opinions of the other agents remain as in `STATE` (operator for set difference `/`), and the number of steps and the number of communications are updated accordingly. The expression `| non-self(.) |` returns the number of edges that are not self-loops and `nodes(.)` returns the opinions (Γ_o) in a state.

The function `next` computes the outcome of the transition $\langle u : o_u \rangle \to_A \langle u : o'_u \rangle$ by applying, at the meta-level (`metaApply`), the rule `atomic` with the needed substitutions to make this rule executable (and deterministic). Namely, it fixes the opinion to be updated (`AGENT` and `BELIEF`), the current `STATE`, and the set of `EDGES` to be considered during the update.

```
var OP   : Opinion .
var RES? : ResultTriple? .
```

```
var SUBS : Substitution .

op next : Module Agent SetEdge State -> Opinion .
ceq  next(M, AGENT, EDGES, STATE) = OP
 if  SUBS := 'AGENT:Agent   <- upTerm(AGENT) ;
             'STATE:State   <- upTerm(STATE) ;
             'EDGES:SetEdge <- upTerm(EDGES) /\
     RES? := metaApply(M, upTerm(< AGENT :  state(AGENT, STATE) >),
             'atomic, SUBS, 0) /\
     OP   := if RES? == failure then error
             else downTerm(getTerm(RES?), error) fi .
```

5. Experimentation

This section shows how Maude and some of its tools can be used to analyze opinion dynamic models using the framework introduced in Section 4 as the rewrite theory $\mathcal{R}$ to formally understand the evolution of agents' opinions. Of special interest is checking the (im)possibility of reaching a consensus (i.e., agent's opinions converge to a given value) or stability, computing the number of steps to reach consensus, computing an optimal strategy to reach consensus, and measuring the polarization of a group of agents at each time-step, among others. It is noticed that for DeGroot and gossip-like models, there exist theoretical results identifying topological conditions that guarantee consensus. In particular, in these models, the agents reach consensus if the graph is strongly connected and aperiodic (i.e., the greatest common divisor of the lengths of its cycles is one) [17].

5.1. Finding Consensus

Let `Example-DG` be the module/theory instantiating $\mathcal{R}$ with the following operators and equations:

```
op init : -> Network .                  --- Initial state (as in Fig 1)
eq init = < nodes: ... ; edges: ... > in step: 0 comm: 0 .
eq moduleName = 'Example-DG .           --- Name of the theory

--- Predefined μ for DeGroot
eq update(STATE, SETEDGE, AGENT)  = deGrootUpdate(STATE, SETEDGE, AGENT) .
eq strategy    = deGroot(edges(init)) . --- DeGroot strategy
```

The following command answers the question of whether it is possible to reach a consensus from the `initial` state. Function `consensus(.)` checks if all (real-valued) opinions o_i and o_j in a given state satisfy $|o_i - o_j| < \epsilon$, where ϵ is an error bound.

```
Maude> search [1]  init =>* STATE such that consensus(STATE) .

Solution 1 (state 34)
STATE --> < nodes: < 0 : [ 4.80e-1 ]  >, < 1 : [ 4.79e-1 ] >, < 2 : [ 4.79e-1 ]  >, ...
            edges: <(0,1): 5.99e-1 >, <(0,2): 4.00e-1 >, ... >
            in step: 34 comm: 272
```

Consensus is approximately 0.48 and it is reached in 34 steps. Since in the DeGroot model all the 12 edges are considered in each interaction, there is a total of $272 = 34 \times 8$ communications (the interactions on the self-loops are not considered). Note that an application of rule `step` in this case is completely deterministic because the strategy only considers one possible outcome, including all the edges of the network.

Cognitive Biases. Consider the same example, but analyzed under different cognitive biases. The framework provides the following operator that implements the update function $\mu_{\text{DeGrootCB}}(\Gamma, A, u)$ in Section 3.5:

```
op update-beta : State SetEdge Agent -> AgentState .
```

In turns, the definition of `update-beta` considers the function below that specifies the family of bias functions $B = \{\beta_{uv} \ | \ u, v \in V\}$:

```
op beta : Agent Agent Float -> Float . --- Defined by the user
```

The framework provides suitable definitions to specify the following cognitive biases:

- The identity on disagreement, corresponding to the classical DeGroot update function [22]:
```
op id : Float -> Float .
eq id(R) = R .
```
- Confirmation bias [29], where individuals are more inclined to accept opinions that align closely with their own:
```
op conf : Float -> Float .
eq conf(R) = R * (1.0001 - abs(R)) / (1.0001) .
```
- The backfire effect [36], where agents reinforce their position toward another person's viewpoint if their opinions differ significantly:
```
op backf : Float -> Float .
eq backf(R) = -1.0 * R * R * R .
```
- Authority Bias (die-hard fanaticism) [37], where individuals are inclined to blindly follow authoritative or influential figures:
```
op fan : Float -> Float .
eq fan(R) = if R == 0.0 then 0.0 else R / abs(R) fi .
```

By selecting `update-beta` as the update function and defining equationally `beta`, different behaviors can be observed. For instance, by instantiating `beta` with the identity function, the resulting model is equivalent to the classical DeGroot model and the value of the consensus is the same as above (i.e., 0.48).

Consider the following three scenarios where `beta` is instantiated as follows:
```
--- Scenario 1
eq beta(AG, AG', R) = backf(R) .

--- Scenario 2
eq beta(AG, AG', R) = fan(R) .

--- Scenario 3
eq beta(1, 0, R) = fan(R) .
eq beta(2, 0, R) = fan(R) .
eq beta(0, 5, R) = backf(R) .
eq beta(AG, AG', R) = conf(R) [owise] .
```
In the first scenario, where all the agents follow the cognitive bias *backfire*, the consensus is not reached. In fact, the system stabilizes in a configuration where agents a, c, and f adopt the opinion 0.0 and the other agents the opinion 1.0. In the second scenario, the opinion of the agents oscillate between the 0.0 and 1.0 values (agents a and f) and between 0.0 and 0.6 (agents b, c, d and e). In fact, while agent a takes the value 0.0, agent f takes the value 1.0 and vice versa. Moreover, while agents b and c take the higher value of 0.6, agents d and e take the lower value of 0.0. Finally, in the third scenario, agents b and c follow blindly agent a and, hence, adopt the extreme opinion value 0.0. Moreover, agent a does not affect its opinion because β_{05} =`backf`. In the end, the consensus is 0.0 and it is reached in 19 steps (152 communications).

Spiral of Silence. As mentioned in Section 3.6, agents can be prevented from interacting when the climate of opinion in their neighborhood differs from their own personal opinion. This fact can alter the evolution of opinions in the network and induce different outcomes when compared to the evaluation of the system that does not consider a spiral of silence. For instance, the agents may not reach a consensus, reach consensus with a different values, or require a different number of interactions before reaching consensus.

Several scenarios are considered to analyze the example in Section 2 within the memoryless model of spiral of silence (see Section 3.6.1), where each agent u has tolerance radius τ_u equals to 0.05, 0.1, 0.15, and 0.35. In the first scenario ($\tau_u = 0.05$), the consensus is 0.433 instead of 0.48 and is reached after 39 steps (312 communications). In the second scenario ($\tau_u = 0.1$), the value of consensus is 0.536 and is reached after 44 steps (352 communications). The consensus value is 0.472 after 39 steps (312 communications) in the third scenario ($\tau_u = 0.15$). In the last scenario ($\tau_u = 0.35$), the consensus is 0.52 and is reached after 37 steps (296 communications). In an additional scenario, the agents have tolerance radius $\tau_u = 0.1$ and the agent c falls silent. In this case, the value of consensus is 0.618 in 43 steps (344 communications).

Hybrid Model. Let `Example-H` be as `Example-DG`, but considering the strategy and update functions of the hybrid model. As explained in Section 3.4, the hybrid model exhibits the maximum degree of non-determinism possible. Using `search` to check the existence of a reachable state satisfying consensus for the system in Figure 1 (12 edges) becomes unfeasible: a state may have up to 4095 (nonempty subsets of Γ_i) successor states. Certainly, for this network, a solution must exist due to the above output of the `search` command and the fact that $\rightarrow_{\text{DeGroot}} \subseteq \rightarrow_{\text{hybrid}}$.

Consider the following rewrite rule and expression in the Maude's strategy language:

```
crl [step'] :   STATE  => STATE'
    if  STATE'  := step([moduleName], STATE, EDGES) [nonexec] .

var STR : SetSetEdge .

strat round :  SetSetEdge @ State .
sd round(EDGES ; STR) := (match STATE s.t. consensus(STATE))
                          or-else step'[EDGES <- EDGES] ; round(STR) .
```

Unlike `step`, rule `step'` does not use the model strategy to select the set of `EDGES` that will be used to compute the next state (and hence, it is non-executable). The Maude's strategy `round` checks whether the current state satisfies consensus and stops. Otherwise, it non-deterministically chooses a set `EDGES`, applies the rule `step'` instantiating the set of edges with that particular set, and is recursively called without `EDGES`. In other words, `round` starts with a set of possible interactions (a collection of set of edges) and it allows for these interactions to happen only once. This is certainly one of the possible behaviors that can be observed with the hybrid model. Using this strategy, the solutions found by the commands below positively answer the following question for the model in Figure 1: can consensus be reached by making some groups of agents (non necessarily disjoint) interact only once? The expression `filter>=(n,STR)` below returns the sets in `STR` with cardinality at least `n`.

```
Maude> dsrew [1] init using round(hybrid(edges)) .
Solution 1
result State: < nodes: < 0 : [0.0] >, ... edges: ... > in step: 8 comm: 13 .

Maude> dsrew [1] init using round(filter>=(6, hybrid(edges))) .
Solution 1
result State: < nodes: < 0 : [1.50e-1] >, ... > in step: 21 comm: 88 .
```

As expected, the value of consensus (and the number of steps to reach such a state) heavily depend on the choice of edges at each step. In the first output returned by `dsrew` in the first command, all the sets considered by `round` included edges where a acts as an influencer and the edge $f \longrightarrow a$ is never selected. This explains the value of the consensus, where the opinion of a was propagated to the neighbors. In the second command, larger groups are chosen to interact and the edge $f \longrightarrow a$ is selected in 4 out of the 21 interactions. Hence, a eventually changes its opinion.

5.2. Statistical Analysis

An alternative approach to deal with the inherent state space explosion problem is to perform statistical model checking for quantitative properties. In the following, the tool `umaudemc` [33] is used for such a purpose. The `umaudemc` command `scheck` enables Monte-Carlo simulations of a rewrite theory extended with probabilities; it estimates the value of a quantitative temporal expression written in the query language QuaTEx [35].

Consider the following QuaTEx expression that computes the probability of reaching a consensus before N communications:

```
Prob(N) = if (s.rval("consensus(S)")) then 1.0
              else if (s.rval("comm(S)") <= N) then # Prob(N) else 0.0 fi
          fi;
```

The two commands below estimate the probability (output of the tool $\mu = val$) of reaching consensus before 30 (expression `E[Prob(30)]`) and 20 communications, respectively, in the running example when the gossip-based model is considered. The confidence level of these analyses is 95% and the same probability is assigned to every successor state (`-assign uniform`).

```
--- E[Prob(30)]
umaudemc  scheck ex-gossip init formula -a 0.05 -d 0.01 --assign uniform
  (μ = 0.587)

--- E[Prob(20)]
umaudemc  scheck ex-gossip init formula -a 0.05 -d 0.01 --assign uniform
  (μ = 0.348)
```

As expected, reducing the maximum number of communications decreases the changes of reaching a consensus state.

The authors in [38] hypothesize that the less dispersed opinion becomes, the easier it will be to reach consensus. In fact, the variance (a standard measure of dispersion) is used as a measure of opinion polarization in social networks [38]. The following commands aim at testing such a hypothesis in the running example when considering the hybrid model:

```
umaudemc  scheck example-H init ... --assign uniform
  (μ = 0.901)

umaudemc scheck example-H init  ... --assign "term(variance(L,R))"
  (μ = 1.0)

umaudemc scheck example-H init ...  --assign "term(distance(L,R))"
  (μ = 0.987)
```

The above commands estimate the probability of reaching consensus before 300 communications by checking the QuaTEx expression `E[Prob(300)]`. In the first case, all the successor states are assigned the same probability. In the second, successor states whose set of chosen agents has higher `variance` are assigned higher probabilities. In the third command, successor states whose set of chosen agents are more *polarized*, in the sense that the `distance` between the maximal and the minimal opinions is bigger, are assigned higher probabilities. These results confirm the hypothesis that it is more likely (1.0 vs 0.9) to reach consensus sooner when communications of agents with more distant opinions is encouraged to reduce dispersion of opinions.

6. Related Work and Concluding Remarks

The literature on opinion models is extensive; here the focus is specifically on some recent works closely related with the main contribution of the paper. A recent model that generalizes gossip protocols for opinion formation is presented in [39]. In this model, the influence graph is undirected (or equivalently, bidirectional). At each time step, a random agent is selected, and both the agent and all its contacts update their opinions simultaneously. This form of update can be naturally captured within the framework of this paper by randomly selecting the corresponding set of edges at each time step within the hybrid setting. Recent works on gossip-like models [39, 40, 41, 42, 43] study consensus with asynchronous communications for distributed averaging and opinion dynamics. The present hybrid model can capture the asynchronous communications of these works and use Maude to explore consensus as illustrated in Section 5. Nevertheless, the theoretical study of consensus and other reachability properties in the hybrid model is more complex giving its generality and thus deferred to future work.

This paper presented a unified framework for dynamic opinion models. These models are tools to study how opinions on a specific topic evolve within a network of agents, where each agent's opinion can be influenced by others. The framework uses set relations, a formalism for specifying and analyzing concurrent behavior in groups of agents, to unify the specification of these models. It relies on two key mechanisms: (1) an atomic relation that updates the opinion of individual agents based on a set of interactions and (2) a strategy that defines sets of interactions to be considered. The framework is formally specified as a rewrite theory and is designed to be instantiated for specific opinion dynamic models. Several models were demonstrated as instances of this framework, including the DeGroot and gossip-like models. The novel hybrid model is introduced to support interactions that can occur relative to any collection of influences. Moreover, state-of-the-art variations of the DeGroot model incorporating cognitive biases and the spiral of silence were included. Experimental results based on state-space exploration, with and without execution strategies, were reported. Further experimentation suggests that statistical model checking is a promising approach to address the state-space explosion problem, particularly in models with high non-determinism, such as the hybrid model. To the best of the authors' knowledge, this work represents the first documented effort to apply concurrency theory, techniques, and tools to the specification and analysis of opinion dynamics models, including how polarization and consensus can be reached.

The ultimate goal of making available computational ideas and approaches for analyzing phenomena in social networks requires (significant) additional work. First, a more in-depth exploration of interaction properties in social networks is required. This may lead to the proposal of new temporal and probabilistic properties that cannot be handled with current techniques and approaches within the opinion dynamic modeling community, but that may be accessible through developments in the concurrency and computational logic communities. It is worth exploring standard concurrency techniques such as bisimulation and testing equivalences to answer questions such as whether two social systems ought to be equivalent and whether there is a social context, represented as a social system, that can tell the difference between two other social systems. Furthermore, epistemic logics suitable for specifying and analyzing an agent's spatial and real-time behavior in rewriting logic [44] can be useful for understanding structured degrees of opinions. Second, extensions to the current framework in terms of more general dynamic networks (i.e., the value of influences can change), trace-sensitive strategies, temporal networks (i.e., nodes and edges can appear and disappear), and the inclusion of several topics/propositions that may share causal relations are in order. Third, more experimental validation is required, ideally with data gathered from real social networks. Fourth, building on the abstract relations proposed here, new models of interaction such as averaging, flocking, and voter models —and some of their variants— (see [45] for a brief survey) are candidates for specification, simulation, and analysis. In particular, the voter model seems specially interesting because current analysis techniques, based on coalescing random walks, are used to find convergence time bounds and steady state distribution of its dynamics. The simpler statistical model checking approach looks like a promising substitute technique for these analyses.

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