Homological Algebra Notes and Solutions to Exercises from Weibel

zin3724

CHAPTER 1

All categories are assumed to be abelian.

CHAPTER 2

Derived Functors

1. δ -functors

A functor $T = \{T_i\}_{i \in \mathbb{N}}$ is a homological δ -functor if it acts like $H = \{H_i\}_{i \in \mathbb{N}}$, the homology functor in the sense of 2.1 from [Wei95].

1.1. zin3724. Let f be a morphism in the category of SESs Follows from condition 2. of definition 2.1.1.

1.2. zin3724.

$$G_1C \xrightarrow{\delta} G_0A \xrightarrow{G_0f} G_0B \longrightarrow G_0C \longrightarrow 0$$

$$\downarrow \qquad \qquad \downarrow \alpha_A \qquad \qquad \downarrow \alpha_B \qquad \qquad \downarrow \alpha_C \qquad \downarrow 0$$

$$0 \xrightarrow{\delta} FA \xrightarrow{Ff} FB \longrightarrow FC \longrightarrow 0$$

Let G be a covariant δ -functor, and suppose there is a natural transformation α from G_0 to F. Since F is exact, the map $FA \xrightarrow{Ff} FB$ is mono, so

$$Ff \circ \alpha_A \circ \delta = 0 \Leftrightarrow \alpha_A \circ \delta = 0$$

By commutativity of the second square, $Ff \circ \alpha_A \circ \delta = \alpha_B \circ G_0 f \circ \delta$, and by exactness of the top row, $\alpha_B \circ G_0 f \circ \delta = \alpha_B \circ 0 = 0$, so the first square commutes.

To see that α extends to commute with all δ_n 's,

$$G_n C \xrightarrow{\delta} G_{n-1} A \xrightarrow{G_{n-1} f} G_{n-1} B \xrightarrow{} G_{n-1} C \xrightarrow{} G_{n-2} A$$

$$\downarrow \qquad \qquad \downarrow^{\alpha_A} \qquad \downarrow^{\alpha_B} \qquad \downarrow^{\alpha_C} \qquad \downarrow^{0}$$

$$0 \xrightarrow{\delta} 0 \xrightarrow{} 0 \xrightarrow{} 0 \xrightarrow{} 0 \xrightarrow{} 0$$

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Note that the first square has a bottom row of 0's.

2. projective resolutions

2.1. zin3724. Let P be a projective object in Ch.

Consider the exact sequence $A \xrightarrow{f} B \to 0 \in \mathbf{Ch}$ such that they're zero everywhere except in the *n*th place. By the projectivity of P, we have for P_n and any $P_n \xrightarrow{h} B_n$, that there exists $P_n \xrightarrow{g} A_n$ such that $h = f \circ g$

$$A_n \xrightarrow{g} B_n \longrightarrow 0$$

since A_n and B_n are arbitrary, P_n must itself be a projective object. This applies to all $n \in \mathbb{N}$, so P is a complex of projectives.

Now, we know from 5.1 that cone(id_P) is split exact, and furthermore, cone(id_P) decomposes as $P \oplus P[-1]$ with i the usual inclusion, and j the usual projection (keep in mind that the differential in P[k] is $(-1)^k \partial_P$)

$$0 \longrightarrow P \xrightarrow{i} \operatorname{cone}(\operatorname{id}_{P}) \xrightarrow{j} P[-1] \longrightarrow 0$$

$$\parallel$$

$$P \oplus P[-1]$$

By the splitting lemma, there exists cone(id_P) \xrightarrow{p} P such that $p \circ i = \text{id}_P$, and combined with the fact that H_n is functorial for all n, it follows that P is exact.

Now, note that i and p are morphisms in \mathbf{Ch} , so they commute with the differentials.

$$\cdots \xrightarrow{\partial_{P}} P_{n+1} \xrightarrow{\partial_{P}} P_{n} \xrightarrow{\partial_{P}} P_{n-1} \xrightarrow{\partial_{P}} \cdots$$

$$\downarrow p \uparrow \downarrow i \qquad p \uparrow \downarrow i$$

Since $P \oplus P[-1]$ is split with $s_{P \oplus P[-1]}$ the splitting map, we know that

$$\partial_{P} p s_{P \oplus P[-1]} i \partial_{P}$$

$$= \partial_{P} p s_{P \oplus P[-1]} \partial_{P \oplus P[-1]} i$$

$$= p \partial_{P \oplus P[-1]} s_{P \oplus P[-1]} \partial_{P \oplus P[-1]} i$$

$$= p \partial_{P \oplus P[-1]} i = \partial_{P}$$

so P is split with the splitting map $ps_{P\oplus P[-1]}i$. For the converse, see [Ral12].

2.2. See [ZYX22].

2.3. zin3724. The quasi-isomorphism is a thing that makes

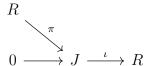
commute, and induces an isomorphism of the homology groups, which is the same thing as

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \stackrel{\epsilon}{\longrightarrow} M \longrightarrow 0 \longrightarrow \cdots$$

being exact; the commutativity of the middle square means that the above is a chain complex. The quasi-isomorphism induced by ϵ at i=0 means that $P_0/\partial(P_1)\cong M$, hence $\operatorname{im}(\partial_1^P)=\ker(\epsilon)$, and the complex above is exact.

3. Injective Resolutions

3.1. zin3724. Let J be an ideal in $R = \mathbb{Z}/m$, then for some n, $J \cong \mathbb{Z}/n$, and the inclusion $\iota : J \hookrightarrow R$ is the map $1 \mapsto \frac{m}{n}$, so there exists a map π extending it



with π the quotient modulo n.

If $d \div m$, and there exists a prime p such that $p \div d$ and $p \div \frac{m}{d}$, then consider ι_m, ι_d generated by $\iota_m(1) = \frac{m}{p}$ and $\iota_d(1) = \frac{d}{p}$;

$$0 \longrightarrow \mathbb{Z}/p \xrightarrow{\iota_m} \mathbb{Z}/m$$

$$\downarrow^{\iota_d}$$

$$\mathbb{Z}/d$$

then since p is a prime, and $p \div \frac{m}{d}$, $d \div \frac{m}{p}$, so any map $\mathbb{Z}/m \to \mathbb{Z}/d$ precomposed with ι_m is 0, and can't be ι_d , which is nonzero. Therefore \mathbb{Z}/d is not injective.

3.2. $\operatorname{zin3724}$. If a is in the torsion subgroup and of order n, then set $f(a) = \frac{1}{n} \in \mathbb{Q}/\mathbb{Z}$. Since \mathbb{Q}/\mathbb{Z} is injective, $f : a\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ extends to a map $f' : A \to \mathbb{Q}/\mathbb{Z}$. If a is free, then there are several nonzero maps from $a\mathbb{Z}$ to \mathbb{Q}/\mathbb{Z} (e.g. $a \mapsto \frac{1}{2}$).

To prove that e_A is an injection, writing the f' assigned to a as f'_a , let $a_1, a_2 \in A$ be distinct. Then $f'_{a_1} - f'_{a_2}$ is nonzero, because we can choose constants $c_1, c_2 \in \mathbb{Z}$ (depending on the orders of a_1, a_2 respectively) such that $(f'_{a_1} - f'_{a_2})(c_1a_1 + c_2a_2) = \frac{1}{2}$.

- **3.3. zin3724.** If there exists $a \in A$ that is nonzero, then it follows from 3.2, that $\operatorname{Hom}(A, \mathbb{Q}/\mathbb{Z}) \neq 0$ since $f'_a(a) = f(a) \neq 0$, so $f'_a \neq 0$.
 - **3.4. zin3724.** similar to 2.1.

4. Left Derived Functors

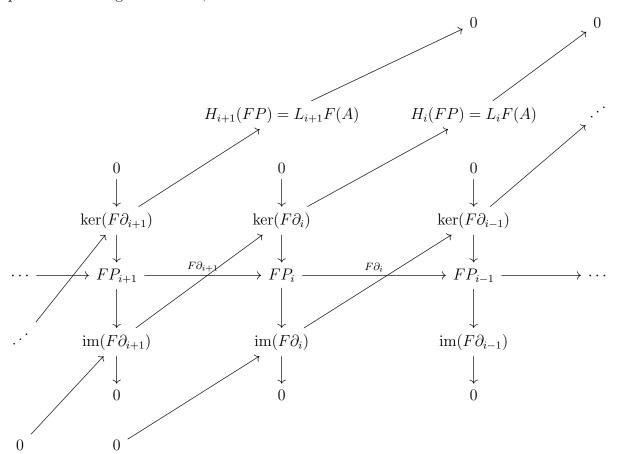
4.1. Too obvious.

4.2. zin3724. Let P be a projective resolution of A. Since U is exact, it preserves ker and coker (i.e. $U \ker = \ker U$, and $U \ker = \operatorname{coker} U$), because for every $f: X \to Y$, it preserves the exactness of

$$0 \longrightarrow \ker f \longrightarrow A \xrightarrow{f} B \longrightarrow \operatorname{coker} f \longrightarrow 0$$

(note that exact sequence suffice to characterise ker and coker in abelian categories). Since $\operatorname{im}(f) = \ker \operatorname{coker} f$, U preserves them too.

Now, for any functor F, $L_iF(A)$ is defined with short exact sequences involving ker and im, like so



Therefore,

$$L_iUF(A) = \ker UF\partial_i/\operatorname{im} UF\partial_{i+1} = U\ker F\partial_i/U\operatorname{im} F\partial_{i+1} = UL_iF(A)$$

and since i is arbitrary, it holds for all i. To see that the isomorphism is natural, see [Ped18].

5. Right Derived Functors

5.1. zin3724.

5.1.1. $1 \Leftrightarrow 2$. Let B be any object. Then for any exact sequence

$$0 \to W \xrightarrow{i} X \xrightarrow{j} Y \to 0$$

the sequence under Hom(-, B)

$$0 \to \operatorname{Hom}(Y, B) \xrightarrow{j^*} \operatorname{Hom}(X, B) \xrightarrow{i^*} \operatorname{Hom}(W, B)$$

is exact.

PROOF. Suppose $h \in \ker(i^*)$, then hi = 0, so $0 \to \operatorname{im}(i) \to \ker(h)$ is exact. But $\operatorname{im}(i) = \ker(j)$ by our assumptions, so by the first isomorphism theorem, h factors through j, hence $h \in \operatorname{im}(j^*)$, and $0 \to \ker(i^*) \to \operatorname{im}(j^*)$ is exact. The exactness of $0 \to \operatorname{im}(j^*) \to \ker(i^*)$ follows immediately from ji = 0.

By assumptions, j is epic, so $fj = 0 \Rightarrow f = 0$ and it follows that $\ker(j^*) = 0$.

To prove that $\operatorname{Hom}(-,B)$ is exact when B is injective, note that for any $W \xrightarrow{f} B$, it factors through $W \xrightarrow{i} X$, therefore $\operatorname{im}(i^*) = \operatorname{Hom}(W,B)$ and $\operatorname{Hom}(-,B)$ is right-exact in addition to being left-exact.

Conversely, if B is not injective, then there exists some $f \in \text{Hom}(W, B)$ that does not factor through i, so i^* would not be surjective.

5.1.2. $1\Rightarrow 3$, note that $3\Rightarrow 4$ is trivial. $\operatorname{Ext}^i(A,B)=R^i\operatorname{Hom}(A,-)(B)$, and since B is injective, $0\to B\xrightarrow{\operatorname{id}_B} B\to 0$ is an injective resolution. Functors preserve identity maps by definition, so

$$0 \to \operatorname{Hom}(A,B) \xrightarrow{\operatorname{id}_{\operatorname{Hom}(A,B)}} \operatorname{Hom}(A,B) \to 0$$

is exact, hence $R^i \operatorname{Hom}(A, -)(B)$, the *i*-th cohomology of the above, is 0. A and *i* were arbitrary, so $\operatorname{Ext}^i(A, B) = 0$ for all A and all $i \neq 0$.

5.1.3. $4\Rightarrow 2$. Since Ext $^{\bullet}$ is a δ -functor, for any exact sequence $0 \rightarrow W \rightarrow X \rightarrow Y \rightarrow 0$, we have the long exact sequence

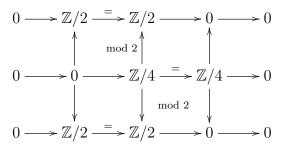
$$0 \longrightarrow \operatorname{Hom}(Y,B) \longrightarrow \operatorname{Hom}(X,B) \longrightarrow \operatorname{Hom}(W,B)$$
$$\operatorname{Ext}^{1}(Y,B) \longrightarrow \operatorname{Ext}^{1}(X,B) \longrightarrow \operatorname{Ext}^{1}(W,B)$$

and by assumption, $\operatorname{Ext}^1(Y,B) = \operatorname{Ext}^1(X,B) = \operatorname{Ext}^1(W,B) = 0$, so $\operatorname{Hom}(-,B)$ is exact because it maps the SES to an SES.

6. Adjoint Functors and Left/Right Exactness

6.4. zin3724. According to Prop 9.4 in [Awo10], the paragraph above (Application 2.6.7 in [Wei95]) is sufficient and necessary for colim to be left adjoint to Δ .

Now, in the category **Ab**, consider



under pushout (regarded as a special case of colim with $I = \bullet \leftarrow \bullet \rightarrow \bullet$), which gives

$$0 \longrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2 \longrightarrow 0 \longrightarrow 0$$

but no matter what the arrows are, it can't possibly be left exact.

7. Balancing Tor and Ext

Main Result: $\text{Hom}(A,-)(B)\cong \text{Hom}(-,B)(A)$ and $(A\otimes -)(B)\cong (-\otimes B)(A)$ as functors.

CHAPTER 3

Tor and Ext

2. Tor and Flatness

2.1. zin3724. $1\Leftrightarrow 2$ is obvious because $-\otimes B$ being exact means it preserves SES's, which is equivalent to all the $Tor_i(A, B) = 0$ for $i \geq 1$ because A is a part of the SES

$$0 \longrightarrow A \xrightarrow{\cong} A \longrightarrow 0 \longrightarrow 0$$

 $2 \Rightarrow 3$ is trivial, and $3 \Rightarrow 1$ is similar to $4 \Rightarrow 2$ from 5.1.3. In both cases, I can't really see how the first derived functors being zero means all other derived functors are zero.

CHAPTER 4

CHAPTER 5

Spectral Sequences

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