

Bootleg Solutions Manual to  
*Introduction to An Homological  
Algebra* by Weibel

zin3724



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## CHAPTER 1

All categories are assumed to be abelian. Arrows are named after the source.

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## CHAPTER 2

### Derived Functors

#### 1. $\delta$ -functors

A functor  $T = \{T_i\}_{i \in \mathbb{N}}$  is a *homological  $\delta$ -functor* if it acts like  $H = \{H_i\}_{i \in \mathbb{N}}$ , the homology functor in the sense of 2.1 from [Wei95].

**1.1. zin3724.** Let  $f$  be a morphism in the category of SESs. Follows from condition 2. of definition 2.1.1.

**1.2. zin3724.**

$$\begin{array}{ccccccccc}
 G_1 C & \xrightarrow{\delta} & G_0 A & \xrightarrow{G_0 f} & G_0 B & \longrightarrow & G_0 C & \longrightarrow & 0 \\
 \downarrow & & \downarrow \alpha_A & & \downarrow \alpha_B & & \downarrow \alpha_C & & \downarrow 0 \\
 0 & \xrightarrow{\delta} & F A & \xrightarrow{F f} & F B & \longrightarrow & F C & \longrightarrow & 0
 \end{array}$$

Let  $G$  be a covariant  $\delta$ -functor, and suppose there is a natural transformation  $\alpha$  from  $G_0$  to  $F$ . Since  $F$  is exact, the map  $F A \xrightarrow{F f} F B$  is mono, so

$$F f \circ \alpha_A \circ \delta = 0 \Leftrightarrow \alpha_A \circ \delta = 0$$

By commutativity of the second square,  $F f \circ \alpha_A \circ \delta = \alpha_B \circ G_0 f \circ \delta$ , and by exactness of the top row,  $\alpha_B \circ G_0 f \circ \delta = \alpha_B \circ 0 = 0$ , so the first square commutes.

To see that  $\alpha$  extends to commute with all  $\delta_n$ 's,

$$\begin{array}{ccccccccc}
 G_n C & \xrightarrow{\delta} & G_{n-1} A & \xrightarrow{G_{n-1} f} & G_{n-1} B & \longrightarrow & G_{n-1} C & \longrightarrow & G_{n-2} A \\
 \downarrow & & \downarrow \alpha_A & & \downarrow \alpha_B & & \downarrow \alpha_C & & \downarrow 0 \\
 0 & \xrightarrow{\delta} & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0
 \end{array}$$

Note that the first square has a bottom row of 0's.

## 2. projective resolutions

**2.1. zin3724.** Let  $P$  be a projective object in  $\mathbf{Ch}$ .

Consider the exact sequence  $A \xrightarrow{f} B \rightarrow 0 \in \mathbf{Ch}$  such that they're zero everywhere except in the  $n$ th place. By the projectivity of  $P$ , we have for  $P_n$  and any  $P_n \xrightarrow{h} B_n$ , that there exists  $P_n \xrightarrow{g} A_n$  such that  $h = f \circ g$

$$\begin{array}{ccccc} & & P_n & & \\ & \swarrow g & \downarrow h & & \\ A_n & \xrightarrow{f} & B_n & \longrightarrow & 0 \end{array}$$

since  $A_n$  and  $B_n$  are arbitrary,  $P_n$  must itself be a projective object. This applies to all  $n \in \mathbb{N}$ , so  $P$  is a complex of projectives.

Now, we know from 5.1 that  $\text{cone}(\text{id}_P)$  is split exact, and furthermore,  $\text{cone}(\text{id}_P)$  decomposes as  $P \oplus P[-1]$  with  $i$  the usual inclusion, and  $j$  the usual projection (keep in mind that the differential in  $P[k]$  is  $(-1)^k \partial_P$ )

$$\begin{array}{ccccccc} 0 & \longrightarrow & P & \xrightarrow{i} & \text{cone}(\text{id}_P) & \xrightarrow{j} & P[-1] \longrightarrow 0 \\ & & & & \parallel & & \\ & & & & P \oplus P[-1] & & \end{array}$$

By the splitting lemma, there exists  $\text{cone}(\text{id}_P) \xrightarrow{p} P$  such that  $p \circ i = \text{id}_P$ , and combined with the fact that  $H_n$  is functorial for all  $n$ , it follows that  $P$  is exact.

Now, note that  $i$  and  $p$  are morphisms in  $\mathbf{Ch}$ , so they commute with the differentials.

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\partial_P} & P_{n+1} & \xrightarrow{\partial_P} & P_n & \xrightarrow{\partial_P} & P_{n-1} \xrightarrow{\partial_P} \cdots \\ & & \uparrow p \downarrow i & & \uparrow p \downarrow i & & \uparrow p \downarrow i \\ \cdots & \xrightarrow[\underset{s_{P \oplus P[-1]}}{\partial_{P \oplus P[-1]}}]{} & P_{n+1} \oplus P_n & \xrightarrow[\underset{s_{P \oplus P[-1]}}{\partial_{P \oplus P[-1]}}]{} & P_n \oplus P_{n-1} & \xrightarrow[\underset{s_{P \oplus P[-1]}}{\partial_{P \oplus P[-1]}}]{} & P_{n-1} \oplus P_{n-2} \xrightarrow[\underset{s_{P \oplus P[-1]}}{\partial_{P \oplus P[-1]}}]{} \cdots \end{array}$$

Since  $P \oplus P[-1]$  is split with  $s_{P \oplus P[-1]}$  the splitting map, we know that

$$\begin{aligned} & \partial_P p s_{P \oplus P[-1]} i \partial_P \\ &= \partial_P p s_{P \oplus P[-1]} \partial_{P \oplus P[-1]} i \\ &= p \partial_{P \oplus P[-1]} s_{P \oplus P[-1]} \partial_{P \oplus P[-1]} i \\ &= p \partial_{P \oplus P[-1]} i = \partial_P \end{aligned}$$

so  $P$  is split with the splitting map  $p s_{P \oplus P[-1]} i$ .

For the converse, see [Ral12].

**2.2.** See [ZYX22].

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**2.3. zin3724.** The quasi-isomorphism is a thing that makes

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 \longrightarrow 0 \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \epsilon \\
 \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & M \longrightarrow 0 \longrightarrow \cdots
 \end{array}$$

commute, and induces an isomorphism of the homology groups, which is the same thing as

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\epsilon} M \longrightarrow 0 \longrightarrow \cdots$$

being exact; the commutativity of the middle square means that the above is a chain complex. The quasi-isomorphism induced by  $\epsilon$  at  $i = 0$  means that  $P_0/\partial(P_1) \cong M$ , hence  $\text{im}(\partial_1^P) = \ker(\epsilon)$ , and the complex above is exact.

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### 3. Injective Resolutions

**3.1. zin3724.** Let  $J$  be an ideal in  $R = \mathbb{Z}/m$ , then for some  $n$ ,  $J \cong \mathbb{Z}/n$ , and the inclusion  $\iota : J \hookrightarrow R$  is the map  $1 \mapsto \frac{m}{n}$ , so there exists a map  $\pi$  extending it

$$\begin{array}{ccc} R & & \\ & \searrow \pi & \\ 0 & \longrightarrow J & \xrightarrow{\iota} R \end{array}$$

with  $\pi$  the quotient modulo  $n$ .

If  $d \div m$ , and there exists a prime  $p$  such that  $p \div d$  and  $p \div \frac{m}{d}$ , then consider  $\iota_m, \iota_d$  generated by  $\iota_m(1) = \frac{m}{p}$  and  $\iota_d(1) = \frac{d}{p}$ ;

$$\begin{array}{ccc} 0 & \longrightarrow \mathbb{Z}/p & \xrightarrow{\iota_m} \mathbb{Z}/m \\ & \downarrow \iota_d & \\ & \mathbb{Z}/d & \end{array}$$

then since  $p$  is a prime, and  $p \div \frac{m}{d}$ ,  $d \div \frac{m}{p}$ , so any map  $\mathbb{Z}/m \rightarrow \mathbb{Z}/d$  precomposed with  $\iota_m$  is 0, and can't be  $\iota_d$ , which is nonzero. Therefore  $\mathbb{Z}/d$  is not injective.

**3.2. zin3724.** If  $a$  is in the torsion subgroup and of order  $n$ , then set  $f(a) = \frac{1}{n} \in \mathbb{Q}/\mathbb{Z}$ . Since  $\mathbb{Q}/\mathbb{Z}$  is injective,  $f : a\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z}$  extends to a map  $f' : A \rightarrow \mathbb{Q}/\mathbb{Z}$ . If  $a$  is free, then there are several nonzero maps from  $a\mathbb{Z}$  to  $\mathbb{Q}/\mathbb{Z}$  (e.g.  $a \mapsto \frac{1}{2}$ ).

To prove that  $e_A$  is an injection, writing the  $f'$  assigned to  $a$  as  $f'_a$ , let  $a_1, a_2 \in A$  be distinct. Then  $f'_{a_1} - f'_{a_2}$  is nonzero, because we can choose constants  $c_1, c_2 \in \mathbb{Z}$  (depending on the orders of  $a_1, a_2$  respectively) such that  $(f'_{a_1} - f'_{a_2})(c_1 a_1 + c_2 a_2) = \frac{1}{2}$ .

**3.3. zin3724.** If there exists  $a \in A$  that is nonzero, then it follows from 3.2, that  $\text{Hom}(A, \mathbb{Q}/\mathbb{Z}) \neq 0$  since  $f'_a(a) = f(a) \neq 0$ , so  $f'_a \neq 0$ .

**3.4. zin3724.** similar to 2.1.

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## 4. Left Derived Functors

**4.1.** Too obvious.

**4.2. zin3724.** Let  $P$  be a projective resolution of  $A$ . Since  $U$  is exact, it preserves  $\ker$  and  $\operatorname{coker}$  (i.e.  $U \ker = \ker U$ , and  $U \operatorname{coker} = \operatorname{coker} U$ ), because for every  $f : X \rightarrow Y$ , it preserves the exactness of

$$0 \longrightarrow \ker f \longrightarrow A \xrightarrow{f} B \longrightarrow \operatorname{coker} f \longrightarrow 0$$

(note that exact sequence suffice to characterise  $\ker$  and  $\operatorname{coker}$  in abelian categories). Since  $\operatorname{im}(f) = \ker \operatorname{coker} f$ ,  $U$  preserves them too.

Now, for any functor  $F$ ,  $L_i F(A)$  is defined with short exact sequences involving  $\ker$  and  $\operatorname{im}$ , like so

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \searrow & & \searrow \\
 & & & H_{i+1}(FP) = L_{i+1}F(A) & & H_i(FP) = L_i F(A) & & \cdots \\
 & & 0 & \downarrow & 0 & \downarrow & 0 & \downarrow \\
 & & \ker(F\partial_{i+1}) & & \ker(F\partial_i) & & \ker(F\partial_{i-1}) & \\
 & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow \\
 \cdots & \longrightarrow & FP_{i+1} & \xrightarrow{F\partial_{i+1}} & FP_i & \xrightarrow{F\partial_i} & FP_{i-1} & \longrightarrow \cdots \\
 & \searrow & \downarrow & \searrow & \downarrow & \searrow & \downarrow & \searrow \\
 & & \operatorname{im}(F\partial_{i+1}) & & \operatorname{im}(F\partial_i) & & \operatorname{im}(F\partial_{i-1}) & \\
 & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow \\
 0 & & 0 & & 0 & & 0 & 
 \end{array}$$

Therefore,

$$L_i U F(A) = \ker U F \partial_i / \operatorname{im} U F \partial_{i+1} = U \ker F \partial_i / U \operatorname{im} F \partial_{i+1} = U L_i F(A)$$

and since  $i$  is arbitrary, it holds for all  $i$ . To see that the isomorphism is natural, see [Ped18].

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## 5. Right Derived Functors

### 5.1. zin3724.

5.1.1.  $1 \Leftrightarrow 2$ . Let  $B$  be any object. Then for any exact sequence

$$0 \rightarrow W \xrightarrow{i} X \xrightarrow{j} Y \rightarrow 0$$

the sequence under  $\text{Hom}(-, B)$

$$0 \rightarrow \text{Hom}(Y, B) \xrightarrow{j^*} \text{Hom}(X, B) \xrightarrow{i^*} \text{Hom}(W, B)$$

is exact.

PROOF. Suppose  $h \in \ker(i^*)$ , then  $hi = 0$ , so  $0 \rightarrow \text{im}(i) \rightarrow \ker(h)$  is exact. But  $\text{im}(i) = \ker(j)$  by our assumptions, so by the first isomorphism theorem,  $h$  factors through  $j$ , hence  $h \in \text{im}(j^*)$ , and  $0 \rightarrow \ker(i^*) \rightarrow \text{im}(j^*)$  is exact. The exactness of  $0 \rightarrow \text{im}(j^*) \rightarrow \ker(i^*)$  follows immediately from  $ji = 0$ .

By assumptions,  $j$  is epic, so  $fj = 0 \Rightarrow f = 0$  and it follows that  $\ker(j^*) = 0$ .  $\square$

To prove that  $\text{Hom}(-, B)$  is exact when  $B$  is injective, note that for any  $W \xrightarrow{f} B$ , it factors through  $W \xrightarrow{i} X$ , therefore  $\text{im}(i^*) = \text{Hom}(W, B)$  and  $\text{Hom}(-, B)$  is right-exact in addition to being left-exact.

Conversely, if  $B$  is *not* injective, then there exists some  $f \in \text{Hom}(W, B)$  that does not factor through  $i$ , so  $i^*$  would not be surjective.

5.1.2.  $1 \Rightarrow 3$ , note that  $3 \Rightarrow 4$  is trivial.  $\text{Ext}^i(A, B) = R^i \text{Hom}(A, -)(B)$ , and since  $B$  is injective,  $0 \rightarrow B \xrightarrow{\text{id}_B} B \rightarrow 0$  is an injective resolution. Functors preserve identity maps by definition, so

$$0 \rightarrow \text{Hom}(A, B) \xrightarrow{\text{id}_{\text{Hom}(A, B)}} \text{Hom}(A, B) \rightarrow 0$$

is exact, hence  $R^i \text{Hom}(A, -)(B)$ , the  $i$ -th cohomology of the above, is 0.  $A$  and  $i$  were arbitrary, so  $\text{Ext}^i(A, B) = 0$  for all  $A$  and all  $i \neq 0$ .

5.1.3.  $4 \Rightarrow 2$ . Since  $\text{Ext}^\bullet$  is a  $\delta$ -functor, for any exact sequence  $0 \rightarrow W \rightarrow X \rightarrow Y \rightarrow 0$ , we have the long exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}(Y, B) & \longrightarrow & \text{Hom}(X, B) & \longrightarrow & \text{Hom}(W, B) \\ & & & & \swarrow & & \\ & & \text{Ext}^1(Y, B) & \longrightarrow & \text{Ext}^1(X, B) & \longrightarrow & \text{Ext}^1(W, B) \\ & & & & \swarrow & & \\ & & \dots & & & & \end{array}$$

and by assumption,  $\text{Ext}^1(Y, B) = \text{Ext}^1(X, B) = \text{Ext}^1(W, B) = 0$ , so  $\text{Hom}(-, B)$  is exact because it maps the SES to an SES.

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## 6. Adjoint Functors and Left/Right Exactness

**6.4. zin3724.** According to Prop 9.4 in [Awo10], the paragraph above (Application 2.6.7 in [Wei95]) is sufficient and necessary for colim to be left adjoint to  $\Delta$ .

Now, in the category **Ab**, consider

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{Z}/2 & \xrightarrow{=} & \mathbb{Z}/2 & \longrightarrow & 0 \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & & \text{mod } 2 & & & \\
 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/4 & \xrightarrow{=} & \mathbb{Z}/4 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & & \text{mod } 2 & & & \\
 0 & \longrightarrow & \mathbb{Z}/2 & \xrightarrow{=} & \mathbb{Z}/2 & \longrightarrow & 0 \longrightarrow 0
 \end{array}$$

under pushout (regarded as a special case of colim with  $I = \bullet \leftarrow \bullet \rightarrow \bullet$ ), which gives

$$0 \longrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2 \longrightarrow 0 \longrightarrow 0$$

but no matter what the arrows are, it can't possibly be left exact.

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## 7. Balancing Tor and Ext

Main Result:  $\text{Hom}(A, -)(B) \cong \text{Hom}(-, B)(A)$  and  $(A \otimes -)(B) \cong (- \otimes B)(A)$  as functors.

## CHAPTER 3

### **Tor and Ext**

#### **1.**

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## 2. Tor and Flatness

**2.1. zin3724.**  $1 \Leftrightarrow 2$  is obvious because  $-\otimes B$  being exact means it preserves SES's, which is equivalent to all the  $\text{Tor}_i(A, B) = 0$  for  $i \geq 1$  because  $A$  is a part of the SES

$$0 \longrightarrow A \xrightarrow{\cong} A \longrightarrow 0 \longrightarrow 0$$

$2 \Rightarrow 3$  is trivial, and  $3 \Rightarrow 1$  is similar to  $\mathbf{4} \Rightarrow \mathbf{2}$  from 5.1.3. In both cases, I can't really see how the first derived functors being zero means all other derived functors are zero.

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## CHAPTER 4

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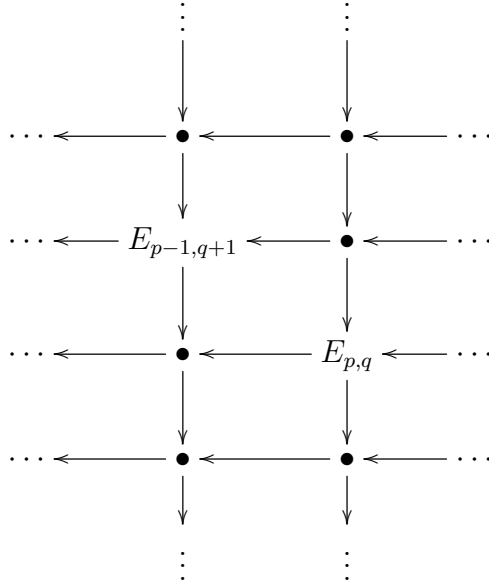
**6.**



## CHAPTER 5

# Spectral Sequences

### 1. Introduction



**1.1.**  $T = \text{Tot}(E)$  refers to the total complex of the zeroth page, with  $d = d^h + d^v$  denoting its differential. Consider the linear map

$$\phi : E_{p-1,q+1}^2 \rightarrow H_{p+q}(T) : a \mapsto (a, 0).$$

Suppose that  $a \in E_{p-1,q+1}^0$  is in the same class as zero in  $E_{p-1,q+1}^2$ , i.e.  $a \in \ker(d^v)$ ,  $\exists a' \in \text{im}(d^v)$  such that  $a + a' \in d^h(\ker(d_{p,q+1}^v))$ , because on the page  $E_{\bullet,\bullet}^1$ ,  $a = a + a'$  ends up in  $\text{im}(d_1^h)$ . It now follows that there exists  $b' \in E_{p-1,q+2}^0$  and  $c \in \ker(d_{p,q+1}^v) \subset E_{p,q+1}^0$  such that

$$\phi(a) = (a, 0) = ((-a') + (a + a'), 0) = d^v(b') + d^h(c) + d^v(c),$$

so  $(a, 0) \in \text{im}(d)$ , hence  $\phi(0) = 0$ , so  $\phi$  must be well-defined.

Now, suppose that  $\phi(a) = 0$ , then  $(a, 0) \in \text{im}(d)$ , so there exist  $a' \in \text{im}(d^v)$  such that  $a - a' \in d^h(\ker(d^v))$ . Furthermore, for any  $c \in \ker(d_{p,q+1}^v)$  such that  $d^h(c) = a - a'$ , we have

$$0 = d^h \circ d^v(c) = d^v \circ d^h(c) = d^v(a - a'),$$

so it follows that  $d^v(a) = 0$ . We have established that  $a \in \ker(d^v)$  and on the page  $E_{\bullet,\bullet}^1$ ,  $a = a - a' \in \text{im}(d^h)$ , so on the page  $E_{\bullet,\bullet}^2$ ,  $a = 0$ , hence  $\phi$  is injective.

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For the remaining part of the required short exact sequence, consider

$$\psi : H_{p+q} \rightarrow E_{p,q}^2 : (a, b) \rightarrow b$$

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