Homological Algebra Notes and Solutions to Exercises from Weibel

zin3724

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CHAPTER 1

All categories are assumed to be abelian. Arrows are named after the source.

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CHAPTER 2

Derived Functors

1. δ-functors

A functor $T = \{T_i\}_{i\in\mathbb{N}}$ is a *homological δ-functor* if it acts like $H = \{H_i\}_{i\in\mathbb{N}}$, the homology functor in the sense of 2.1 from [Wei95].

1.1. zin3724. Let f be a morphism in the category of SESs Follows from condition 2. of definition 2.1.1.

1.2. zin3724.

$$\begin{tikzcd}
G_1C \arrow[r, "\delta"] \arrow[d] & G_0A \arrow[r, "G_0f"] \arrow[d, "\alpha_A"] & G_0B \arrow[r] \arrow[d, "\alpha_B"] & G_0C \arrow[r] \arrow[d, "\alpha_C"] & 0 \arrow[d, "0"] \\
0 \arrow[r, "\delta"] & FA \arrow[r, "Ff"] & FB \arrow[r] & FC \arrow[r] & 0
\end{tikzcd}$$

Let G be a covariant δ-functor, and suppose there is a natural transformation α from G_0 to F. Since F is exact, the map $FA \xrightarrow{Ff} FB$ is mono, so

$$Ff \circ \alpha_A \circ \delta = 0 \Leftrightarrow \alpha_A \circ \delta = 0$$

By commutativity of the second square, $Ff \circ \alpha_A \circ \delta = \alpha_B \circ G_0 f \circ \delta$, and by exactness of the top row, $\alpha_B \circ G_0 f \circ \delta = \alpha_B \circ 0 = 0$, so the first square commutes.

To see that α extends to commute with all δ_n's,

$$\begin{tikzcd}
G_nC \arrow[r, "\delta"] \arrow[d] & G_{n-1}A \arrow[r, "G_{n-1}f"] \arrow[d, "\alpha_A"] & G_{n-1}B \arrow[r] \arrow[d, "\alpha_B"] & G_{n-1}C \arrow[r] \arrow[d, "\alpha_C"] & G_{n-2}A \arrow[d, "0"] \\
0 \arrow[r, "\delta"] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Note that the first square has a bottom row of 0's.

2. projective resolutions

2.1. zin3724. Let P be a projective object in **Ch**.

Consider the exact sequence $A \xrightarrow{f} B \to 0 \in \mathbf{Ch}$ such that they're zero everywhere except in the nth place. By the projectivity of P, we have for P_n and any $P_n \xrightarrow{h} B_n$, that there exists $P_n \xrightarrow{g} A_n$ such that $h = f \circ g$

$$\begin{tikzcd} & P_n \arrow[dl, dashed, "g"'] \arrow[d, "h"] & \\ A_n \arrow[r, "f"'] & B_n \arrow[r] & 0 \end{tikzcd}$$

since A_n and B_n are arbitrary, P_n must itself be a projective object. This applies to all $n \in \mathbb{N}$, so P is a complex of projectives.

Now, we know from 5.1 that $\mathrm{cone}(\mathrm{id}_P)$ is split exact, and furthermore, $\mathrm{cone}(\mathrm{id}_P)$ decomposes as $P \oplus P[-1]$ with i the usual inclusion, and j the usual projection (keep in mind that the differential in $P[k]$ is $(-1)^k \partial_P$)

$$\begin{tikzcd} 0 \arrow[r] & P \arrow[r, "i"] & \mathrm{cone}(\mathrm{id}_P) \arrow[r, "j"] \arrow[d, equal] & P[-1] \arrow[r] & 0 \\ & & P \oplus P[-1] & & \end{tikzcd}$$

By the splitting lemma, there exists $\mathrm{cone}(\mathrm{id}_P) \xrightarrow{p} P$ such that $p \circ i = \mathrm{id}_P$, and combined with the fact that H_n is functorial for all n, it follows that P is exact.

Now, note that i and p are morphisms in **Ch**, so they commute with the differentials.

$$\begin{tikzcd} \cdots \arrow[r, "\partial_P"] & P_{n+1} \arrow[r, "\partial_P"] \arrow[d, shift left, "i"] & P_n \arrow[r, "\partial_P"] \arrow[d, shift left, "i"] & P_{n-1} \arrow[r, "\partial_P"] \arrow[d, shift left, "i"] & \cdots \\ \cdots \arrow[r, shift left, "\partial_{P\oplus P[-1]}"] & P_{n+1} \oplus P_n \arrow[l, shift left, "s_{P\oplus P[-1]}"] \arrow[u, shift left, "p"] \arrow[r, shift left, "\partial_{P\oplus P[-1]}"] & P_n \oplus P_{n-1} \arrow[l, shift left, "s_{P\oplus P[-1]}"] \arrow[u, shift left, "p"] \arrow[r, shift left, "\partial_{P\oplus P[-1]}"] & P_{n-1} \oplus P_{n-2} \arrow[l, shift left, "s_{P\oplus P[-1]}"] \arrow[u, shift left, "p"] \arrow[r, shift left, "\partial_{P\oplus P[-1]}"] & \cdots \arrow[l, shift left, "s_{P\oplus P[-1]}"] \end{tikzcd}$$

Since $P \oplus P[-1]$ is split with $s_{P\oplus P[-1]}$ the splitting map, we know that

$$\begin{aligned} &\partial_P p s_{P\oplus P[-1]} i \partial_P \\ &= \partial_P p s_{P\oplus P[-1]} \partial_{P\oplus P[-1]} i \\ &= p \partial_{P\oplus P[-1]} s_{P\oplus P[-1]} \partial_{P\oplus P[-1]} i \\ &= p \partial_{P\oplus P[-1]} i = \partial_P \end{aligned}$$

so P is split with the splitting map $p s_{P\oplus P[-1]} i$.

For the converse, see [Ral12].

2.2. See [ZYX22].

2.3. zin3724. The quasi-isomorphism is a thing that makes

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d, "\epsilon"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

commute, and induces an isomorphism of the homology groups, which is the same thing as

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\epsilon} M \longrightarrow 0 \longrightarrow \cdots$$

being exact; the commutativity of the middle square means that the above is a chain complex. The quasi-isomorphism induced by ϵ at $i = 0$ means that $P_0/\partial(P_1) \cong M$, hence $\mathrm{im}(\partial_1^P) = \ker(\epsilon)$, and the complex above is exact.

3. Injective Resolutions

3.1. zin3724. Let J be an ideal in $R = \mathbb{Z}/m$, then for some n, $J \cong \mathbb{Z}/n$, and the inclusion $\iota : J \hookrightarrow R$ is the map $1 \mapsto \frac{m}{n}$, so there exists a map π extending it

$$\begin{tikzcd} & R \arrow[d, "\pi"] & \\ 0 \arrow[r] & J \arrow[r, "\iota"] & R \end{tikzcd}$$

with π the quotient modulo n.

If $d \div m$, and there exists a prime p such that $p \div d$ and $p \div \frac{m}{d}$, then consider ι_m, ι_d generated by $\iota_m(1) = \frac{m}{p}$ and $\iota_d(1) = \frac{d}{p}$;

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}/p \arrow[r, "\iota_m"] \arrow[d, "\iota_d"] & \mathbb{Z}/m \\ & \mathbb{Z}/d & \end{tikzcd}$$

then since p is a prime, and $p \div \frac{m}{d}$, $d \div \frac{m}{p}$, so any map $\mathbb{Z}/m \to \mathbb{Z}/d$ precomposed with ι_m is 0, and can't be ι_d, which is nonzero. Therefore $\mathbb{Z}/d$ is not injective.

3.2. zin3724. If a is in the torsion subgroup and of order n, then set $f(a) = \frac{1}{n} \in \mathbb{Q}/\mathbb{Z}$. Since $\mathbb{Q}/\mathbb{Z}$ is injective, $f : a\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ extends to a map $f' : A \to \mathbb{Q}/\mathbb{Z}$. If a is free, then there are several nonzero maps from $a\mathbb{Z}$ to $\mathbb{Q}/\mathbb{Z}$ (e.g. $a \mapsto \frac{1}{2}$).

To prove that e_A is an injection, writing the f' assigned to a as f'_a, let $a_1, a_2 \in A$ be distinct. Then $f'_{a_1} - f'_{a_2}$ is nonzero, because we can choose constants $c_1, c_2 \in \mathbb{Z}$ (depending on the orders of a_1, a_2 respectively) such that $(f'_{a_1} - f'_{a_2})(c_1 a_1 + c_2 a_2) = \frac{1}{2}$.

3.3. zin3724. If there exists $a \in A$ that is nonzero, then it follows from 3.2, that $\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}) \neq 0$ since $f'_a(a) = f(a) \neq 0$, so $f'_a \neq 0$.

3.4. zin3724. similar to 2.1.

4. Left Derived Functors

4.1. Too obvious.

4.2. zin3724. Let P be a projective resolution of A. Since U is exact, it preserves ker and coker (i.e. $U \ker = \ker U$, and $U \operatorname{coker} = \operatorname{coker} U$), because for every $f : X \to Y$, it preserves the exactness of

$$0 \longrightarrow \ker f \longrightarrow A \xrightarrow{f} B \longrightarrow \operatorname{coker} f \longrightarrow 0$$

(note that exact sequence suffice to characterise ker and coker in abelian categories). Since $\operatorname{im}(f) = \ker \operatorname{coker} f$, U preserves them too.

Now, for any functor F, $L_iF(A)$ is defined with short exact sequences involving ker and im, like so

$$\begin{tikzcd}
 & & & 0 & 0 \\
 & & H_{i+1}(FP) = L_{i+1}F(A) \arrow[ur] & H_i(FP) = L_iF(A) \arrow[ur] & \iddots \\
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & \ker(F\partial_{i+1}) \arrow[d] \arrow[uur] & \ker(F\partial_i) \arrow[d] \arrow[uur] & \ker(F\partial_{i-1}) \arrow[d] \arrow[uur] & \\
 \cdots \arrow[r] & FP_{i+1} \arrow[r, "F\partial_{i+1}"] \arrow[d] & FP_i \arrow[r, "F\partial_i"] \arrow[d] & FP_{i-1} \arrow[r] \arrow[d] & \cdots \\
 \iddots \arrow[uur] & \operatorname{im}(F\partial_{i+1}) \arrow[d] \arrow[uur] & \operatorname{im}(F\partial_i) \arrow[d] \arrow[uur] & \operatorname{im}(F\partial_{i-1}) \arrow[d] & \\
 & 0 & 0 & 0 & \\
 0 \arrow[uur] & 0 \arrow[uur] & & &
\end{tikzcd}$$

Therefore,

$$L_iUF(A) = \ker UF\partial_i / \operatorname{im} UF\partial_{i+1} = U \ker F\partial_i / U \operatorname{im} F\partial_{i+1} = UL_iF(A)$$

and since i is arbitrary, it holds for all i. To see that the isomorphism is natural, see [Ped18].

5. Right Derived Functors

5.1. zin3724.

5.1.1. *1⇔2.* Let B be any object. Then for any exact sequence

$$0 \to W \xrightarrow{i} X \xrightarrow{j} Y \to 0$$

the sequence under $\operatorname{Hom}(-, B)$

$$0 \to \operatorname{Hom}(Y, B) \xrightarrow{j^*} \operatorname{Hom}(X, B) \xrightarrow{i^*} \operatorname{Hom}(W, B)$$

is exact.

Proof. Suppose $h \in \ker(i^*)$, then $hi = 0$, so $0 \to \operatorname{im}(i) \to \ker(h)$ is exact. But $\operatorname{im}(i) = \ker(j)$ by our assumptions, so by the first isomorphism theorem, h factors through j, hence $h \in \operatorname{im}(j^*)$, and $0 \to \ker(i^*) \to \operatorname{im}(j^*)$ is exact. The exactness of $0 \to \operatorname{im}(j^*) \to \ker(i^*)$ follows immediately from $ji = 0$.

By assumptions, j is epic, so $fj = 0 \Rightarrow f = 0$ and it follows that $\ker(j^*) = 0$. ☐

To prove that $\operatorname{Hom}(-, B)$ is exact when B is injective, note that for any $W \xrightarrow{f} B$, it factors through $W \xrightarrow{i} X$, therefore $\operatorname{im}(i^*) = \operatorname{Hom}(W, B)$ and $\operatorname{Hom}(-, B)$ is right-exact in addition to being left-exact.

Conversely, if B is *not* injective, then there exists some $f \in \operatorname{Hom}(W, B)$ that does not factor through i, so i^* would not be surjective.

5.1.2. *1⇒3, note that 3⇒4 is trivial.* $\operatorname{Ext}^i(A, B) = R^i \operatorname{Hom}(A, -)(B)$, and since B is injective, $0 \to B \xrightarrow{\mathrm{id}_B} B \to 0$ is an injective resolution. Functors preserve identity maps by definition, so

$$0 \to \operatorname{Hom}(A, B) \xrightarrow{\mathrm{id}_{\operatorname{Hom}(A,B)}} \operatorname{Hom}(A, B) \to 0$$

is exact, hence $R^i \operatorname{Hom}(A, -)(B)$, the i-th cohomology of the above, is 0. A and i were arbitrary, so $\operatorname{Ext}^i(A, B) = 0$ for all A and all $i \neq 0$.

5.1.3. *4⇒2.* Since $\operatorname{Ext}^\bullet$ is a δ-functor, for any exact sequence $0 \to W \to X \to Y \to 0$, we have the long exact sequence

$$\begin{array}{ccccccc} 0 \longrightarrow & \operatorname{Hom}(Y, B) & \longrightarrow & \operatorname{Hom}(X, B) & \longrightarrow & \operatorname{Hom}(W, B) & \\ & \swarrow & & & & & \\ & \operatorname{Ext}^1(Y, B) & \longrightarrow & \operatorname{Ext}^1(X, B) & \longrightarrow & \operatorname{Ext}^1(W, B) & \\ & \swarrow & & & & & \\ & \cdots & & & & & \end{array}$$

and by assumption, $\operatorname{Ext}^1(Y, B) = \operatorname{Ext}^1(X, B) = \operatorname{Ext}^1(W, B) = 0$, so $\operatorname{Hom}(-, B)$ is exact because it maps the SES to an SES.

6.4. zin3724. According to Prop 9.4 in [Awo10], the paragraph above (Application 2.6.7 in [Wei95]) is sufficient and necessary for colim to be left adjoint to Δ.

Now, in the category **Ab**, consider

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/2 \arrow[r, "="] & \mathbb{Z}/2 \arrow[r] & 0 \arrow[r] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[u] \arrow[d] & \mathbb{Z}/4 \arrow[r, "="] \arrow[u, "\text{mod } 2"] \arrow[d] & \mathbb{Z}/4 \arrow[r] \arrow[u] \arrow[d, "\text{mod } 2"'] & 0 \\
0 \arrow[r] & \mathbb{Z}/2 \arrow[r, "="] & \mathbb{Z}/2 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

under pushout (regarded as a special case of colim with $I = \bullet \leftarrow \bullet \rightarrow \bullet$), which gives

$$0 \longrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2 \longrightarrow 0 \longrightarrow 0$$

but no matter what the arrows are, it can't possibly be left exact.

7. Balancing Tor and Ext

Main Result: $\mathrm{Hom}(A,-)(B) \cong \mathrm{Hom}(-,B)(A)$ and $(A \otimes -)(B) \cong (- \otimes B)(A)$ as functors.

CHAPTER 3

Tor and Ext

1.

2. Tor and Flatness

2.1. zin3724. 1⇔2 is obvious because $- \otimes B$ being exact means it preserves SES's, which is equivalent to all the $\mathrm{Tor}_i(A, B) = 0$ for $i \geq 1$ because A is a part of the SES

$$0 \longrightarrow A \xrightarrow{\cong} A \longrightarrow 0 \longrightarrow 0$$

$2 \Rightarrow 3$ is trivial, and $3 \Rightarrow 1$ is similar to $\mathbf{4} \Rightarrow \mathbf{2}$ from 5.1.3. In both cases, I can't really see how the first derived functors being zero means all other derived functors are zero.

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CHAPTER 4

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CHAPTER 5

Spectral Sequences

1. Introduction

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots & \bullet \arrow[l] \arrow[d] & \bullet \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\cdots & E_{p-1,q+1} \arrow[l] \arrow[d] & \bullet \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\cdots & \bullet \arrow[l] \arrow[d] & E_{p,q} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\cdots & \bullet \arrow[l] \arrow[d] & \bullet \arrow[l] \arrow[d] & \cdots \arrow[l] \\
 & \vdots & \vdots &
\end{tikzcd}$$

1.1. $\mathrm{Tot}(E)$ refers to the total complex of the zeroth page, with $d = d^h + d^v$ denoting its differential. Consider the linear map

$$\phi : E^2_{p-1,q+1} \to H_{p+q}(T) : a \mapsto (a, b)$$

where $b \in E^0_{p,q}$ is chosen so that $b \in \mathrm{im}(d^v) \cap \ker(d^h)$, and b. Now, suppose that $a \in E^0_{p-1,q+1}$ is in the same class as zero on the second page, i.e. $a \in \ker(d^v)$, $\exists a' \in \mathrm{im}(d^v)$ such that $a + a' \in \mathrm{im}(d^h)$, then for any $c \in (d^h)^{-1}(a + a')$, $d^v \circ d^h(c) = d^h \circ d^v(c) = 0$. So for its image in $H_{p+q}(T)$, we have

$$d(\phi(a)) = (d^h + d^v)(a, b) = d^h(b) + d^v(a) + d^v(b) = 0,$$

implying that $\phi(a) \in \ker(d)$.

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