# Homological Algebra Notes and Solutions to Exercises from Weibel

zin3724

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All categories are assumed to be abelian. Arrows are named after the source.

#### **Derived Functors**

#### 1. $\delta$ -functors

A functor  $T = \{T_i\}_{i \in \mathbb{N}}$  is a homological  $\delta$ -functor if it acts like  $H = \{H_i\}_{i \in \mathbb{N}}$ , the homology functor in the sense of 2.1 from [Wei95].

1.1. zin3724. Let f be a morphism in the category of SESs Follows from condition 2. of definition 2.1.1.

#### 1.2. zin3724.

$$G_1C \xrightarrow{\delta} G_0A \xrightarrow{G_0f} G_0B \longrightarrow G_0C \longrightarrow 0$$

$$\downarrow \qquad \qquad \downarrow \alpha_A \qquad \qquad \downarrow \alpha_B \qquad \qquad \downarrow \alpha_C \qquad \downarrow 0$$

$$0 \xrightarrow{\delta} FA \xrightarrow{Ff} FB \longrightarrow FC \longrightarrow 0$$

Let G be a covariant  $\delta$ -functor, and suppose there is a natural transformation  $\alpha$  from  $G_0$  to F. Since F is exact, the map  $FA \xrightarrow{Ff} FB$  is mono, so

$$Ff \circ \alpha_A \circ \delta = 0 \Leftrightarrow \alpha_A \circ \delta = 0$$

By commutativity of the second square,  $Ff \circ \alpha_A \circ \delta = \alpha_B \circ G_0 f \circ \delta$ , and by exactness of the top row,  $\alpha_B \circ G_0 f \circ \delta = \alpha_B \circ 0 = 0$ , so the first square commutes.

To see that  $\alpha$  extends to commute with all  $\delta_n$ 's,

$$G_n C \xrightarrow{\delta} G_{n-1} A \xrightarrow{G_{n-1} f} G_{n-1} B \xrightarrow{} G_{n-1} C \xrightarrow{} G_{n-2} A$$

$$\downarrow \qquad \qquad \downarrow^{\alpha_A} \qquad \downarrow^{\alpha_B} \qquad \downarrow^{\alpha_C} \qquad \downarrow^{0}$$

$$0 \xrightarrow{\delta} 0 \xrightarrow{} 0 \xrightarrow{} 0 \xrightarrow{} 0 \xrightarrow{} 0$$

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Note that the first square has a bottom row of 0's.

#### 2. projective resolutions

#### **2.1.** zin3724. Let P be a projective object in Ch.

Consider the exact sequence  $A \xrightarrow{f} B \to 0 \in \mathbf{Ch}$  such that they're zero everywhere except in the *n*th place. By the projectivity of P, we have for  $P_n$  and any  $P_n \xrightarrow{h} B_n$ , that there exists  $P_n \xrightarrow{g} A_n$  such that  $h = f \circ g$ 

$$A_n \xrightarrow{g} B_n \longrightarrow 0$$

since  $A_n$  and  $B_n$  are arbitrary,  $P_n$  must itself be a projective object. This applies to all  $n \in \mathbb{N}$ , so P is a complex of projectives.

Now, we know from 5.1 that cone(id<sub>P</sub>) is split exact, and furthermore, cone(id<sub>P</sub>) decomposes as  $P \oplus P[-1]$  with i the usual inclusion, and j the usual projection (keep in mind that the differential in P[k] is  $(-1)^k \partial_P$ )

$$0 \longrightarrow P \xrightarrow{i} \operatorname{cone}(\operatorname{id}_{P}) \xrightarrow{j} P[-1] \longrightarrow 0$$

$$\parallel$$

$$P \oplus P[-1]$$

By the splitting lemma, there exists cone(id<sub>P</sub>)  $\xrightarrow{p}$  P such that  $p \circ i = \text{id}_P$ , and combined with the fact that  $H_n$  is functorial for all n, it follows that P is exact.

Now, note that i and p are morphisms in  $\mathbf{Ch}$ , so they commute with the differentials.

$$\cdots \xrightarrow{\partial_{P}} P_{n+1} \xrightarrow{\partial_{P}} P_{n} \xrightarrow{\partial_{P}} P_{n-1} \xrightarrow{\partial_{P}} \cdots$$

$$\downarrow p \uparrow \downarrow i \qquad p \uparrow \downarrow i$$

Since  $P \oplus P[-1]$  is split with  $s_{P \oplus P[-1]}$  the splitting map, we know that

$$\partial_{P} p s_{P \oplus P[-1]} i \partial_{P}$$

$$= \partial_{P} p s_{P \oplus P[-1]} \partial_{P \oplus P[-1]} i$$

$$= p \partial_{P \oplus P[-1]} s_{P \oplus P[-1]} \partial_{P \oplus P[-1]} i$$

$$= p \partial_{P \oplus P[-1]} i = \partial_{P}$$

so P is split with the splitting map  $ps_{P \oplus P[-1]}i$ . For the converse, see [Ral12].

#### **2.2.** See [ZYX22].

2.3. zin3724. The quasi-isomorphism is a thing that makes

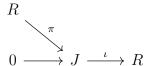
commute, and induces an isomorphism of the homology groups, which is the same thing as

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \stackrel{\epsilon}{\longrightarrow} M \longrightarrow 0 \longrightarrow \cdots$$

being exact; the commutativity of the middle square means that the above is a chain complex. The quasi-isomorphism induced by  $\epsilon$  at i=0 means that  $P_0/\partial(P_1)\cong M$ , hence  $\operatorname{im}(\partial_1^P)=\ker(\epsilon)$ , and the complex above is exact.

#### 3. Injective Resolutions

**3.1.** zin3724. Let J be an ideal in  $R = \mathbb{Z}/m$ , then for some n,  $J \cong \mathbb{Z}/n$ , and the inclusion  $\iota : J \hookrightarrow R$  is the map  $1 \mapsto \frac{m}{n}$ , so there exists a map  $\pi$  extending it



with  $\pi$  the quotient modulo n.

If  $d \div m$ , and there exists a prime p such that  $p \div d$  and  $p \div \frac{m}{d}$ , then consider  $\iota_m, \iota_d$  generated by  $\iota_m(1) = \frac{m}{p}$  and  $\iota_d(1) = \frac{d}{p}$ ;

$$0 \longrightarrow \mathbb{Z}/p \xrightarrow{\iota_m} \mathbb{Z}/m$$

$$\downarrow^{\iota_d}$$

$$\mathbb{Z}/d$$

then since p is a prime, and  $p \div \frac{m}{d}$ ,  $d \div \frac{m}{p}$ , so any map  $\mathbb{Z}/m \to \mathbb{Z}/d$  precomposed with  $\iota_m$  is 0, and can't be  $\iota_d$ , which is nonzero. Therefore  $\mathbb{Z}/d$  is not injective.

**3.2.**  $\operatorname{zin3724}$ . If a is in the torsion subgroup and of order n, then set  $f(a) = \frac{1}{n} \in \mathbb{Q}/\mathbb{Z}$ . Since  $\mathbb{Q}/\mathbb{Z}$  is injective,  $f: a\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$  extends to a map  $f': A \to \mathbb{Q}/\mathbb{Z}$ . If a is free, then there are several nonzero maps from  $a\mathbb{Z}$  to  $\mathbb{Q}/\mathbb{Z}$  (e.g.  $a \mapsto \frac{1}{2}$ ).

To prove that  $e_A$  is an injection, writing the f' assigned to a as  $f'_a$ , let  $a_1, a_2 \in A$  be distinct. Then  $f'_{a_1} - f'_{a_2}$  is nonzero, because we can choose constants  $c_1, c_2 \in \mathbb{Z}$  (depending on the orders of  $a_1, a_2$  respectively) such that  $(f'_{a_1} - f'_{a_2})(c_1a_1 + c_2a_2) = \frac{1}{2}$ .

- **3.3. zin3724.** If there exists  $a \in A$  that is nonzero, then it follows from 3.2, that  $\operatorname{Hom}(A, \mathbb{Q}/\mathbb{Z}) \neq 0$  since  $f'_a(a) = f(a) \neq 0$ , so  $f'_a \neq 0$ .
  - **3.4. zin3724.** similar to 2.1.

#### 4. Left Derived Functors

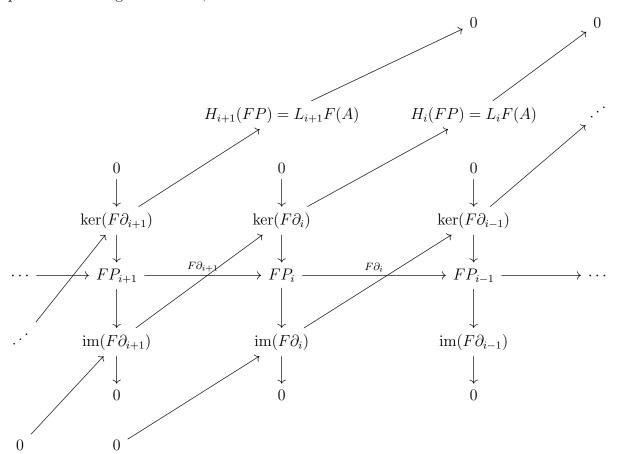
#### **4.1.** Too obvious.

**4.2. zin3724.** Let P be a projective resolution of A. Since U is exact, it preserves ker and coker (i.e.  $U \ker = \ker U$ , and  $U \ker = \operatorname{coker} U$ ), because for every  $f: X \to Y$ , it preserves the exactness of

$$0 \longrightarrow \ker f \longrightarrow A \xrightarrow{f} B \longrightarrow \operatorname{coker} f \longrightarrow 0$$

(note that exact sequence suffice to characterise ker and coker in abelian categories). Since  $\operatorname{im}(f) = \ker \operatorname{coker} f$ , U preserves them too.

Now, for any functor F,  $L_iF(A)$  is defined with short exact sequences involving ker and im, like so



Therefore,

$$L_iUF(A) = \ker UF\partial_i/\operatorname{im} UF\partial_{i+1} = U \ker F\partial_i/U \operatorname{im} F\partial_{i+1} = UL_iF(A)$$

and since i is arbitrary, it holds for all i. To see that the isomorphism is natural, see [Ped18].

#### 5. Right Derived Functors

#### 5.1. zin3724.

5.1.1.  $1 \Leftrightarrow 2$ . Let B be any object. Then for any exact sequence

$$0 \to W \xrightarrow{i} X \xrightarrow{j} Y \to 0$$

the sequence under Hom(-, B)

$$0 \to \operatorname{Hom}(Y, B) \xrightarrow{j^*} \operatorname{Hom}(X, B) \xrightarrow{i^*} \operatorname{Hom}(W, B)$$

is exact.

PROOF. Suppose  $h \in \ker(i^*)$ , then hi = 0, so  $0 \to \operatorname{im}(i) \to \ker(h)$  is exact. But  $\operatorname{im}(i) = \ker(j)$  by our assumptions, so by the first isomorphism theorem, h factors through j, hence  $h \in \operatorname{im}(j^*)$ , and  $0 \to \ker(i^*) \to \operatorname{im}(j^*)$  is exact. The exactness of  $0 \to \operatorname{im}(j^*) \to \ker(i^*)$  follows immediately from ji = 0.

By assumptions, j is epic, so  $fj = 0 \Rightarrow f = 0$  and it follows that  $\ker(j^*) = 0$ .

To prove that  $\operatorname{Hom}(-,B)$  is exact when B is injective, note that for any  $W \xrightarrow{f} B$ , it factors through  $W \xrightarrow{i} X$ , therefore  $\operatorname{im}(i^*) = \operatorname{Hom}(W,B)$  and  $\operatorname{Hom}(-,B)$  is right-exact in addition to being left-exact.

Conversely, if B is not injective, then there exists some  $f \in \text{Hom}(W, B)$  that does not factor through i, so  $i^*$  would not be surjective.

5.1.2.  $1\Rightarrow 3$ , note that  $3\Rightarrow 4$  is trivial.  $\operatorname{Ext}^i(A,B)=R^i\operatorname{Hom}(A,-)(B)$ , and since B is injective,  $0\to B\xrightarrow{\operatorname{id}_B} B\to 0$  is an injective resolution. Functors preserve identity maps by definition, so

$$0 \to \operatorname{Hom}(A, B) \xrightarrow{\operatorname{id}_{\operatorname{Hom}(A, B)}} \operatorname{Hom}(A, B) \to 0$$

is exact, hence  $R^i \operatorname{Hom}(A, -)(B)$ , the *i*-th cohomology of the above, is 0. A and *i* were arbitrary, so  $\operatorname{Ext}^i(A, B) = 0$  for all A and all  $i \neq 0$ .

5.1.3.  $4\Rightarrow 2$ . Since Ext $^{\bullet}$  is a  $\delta$ -functor, for any exact sequence  $0 \rightarrow W \rightarrow X \rightarrow Y \rightarrow 0$ , we have the long exact sequence

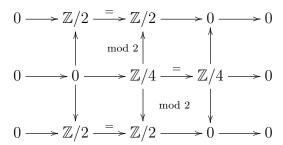
$$0 \longrightarrow \operatorname{Hom}(Y,B) \longrightarrow \operatorname{Hom}(X,B) \longrightarrow \operatorname{Hom}(W,B)$$
$$\operatorname{Ext}^{1}(Y,B) \longrightarrow \operatorname{Ext}^{1}(X,B) \longrightarrow \operatorname{Ext}^{1}(W,B)$$

and by assumption,  $\operatorname{Ext}^1(Y,B) = \operatorname{Ext}^1(X,B) = \operatorname{Ext}^1(W,B) = 0$ , so  $\operatorname{Hom}(-,B)$  is exact because it maps the SES to an SES.

#### 6. Adjoint Functors and Left/Right Exactness

**6.4.** zin3724. According to Prop 9.4 in [Awo10], the paragraph above (Application 2.6.7 in [Wei95]) is sufficient and necessary for colim to be left adjoint to  $\Delta$ .

Now, in the category **Ab**, consider



under pushout (regarded as a special case of colim with  $I = \bullet \leftarrow \bullet \rightarrow \bullet$ ), which gives

$$0 \longrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2 \longrightarrow 0 \longrightarrow 0$$

but no matter what the arrows are, it can't possibly be left exact.

# 7. Balancing Tor and $\operatorname{Ext}$

Main Result:  $\text{Hom}(A,-)(B)\cong \text{Hom}(-,B)(A)$  and  $(A\otimes -)(B)\cong (-\otimes B)(A)$  as functors.

# Tor and Ext

#### 2. Tor and Flatness

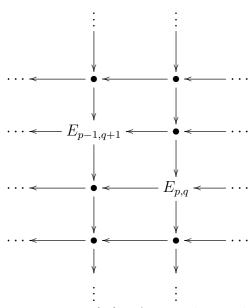
**2.1.** zin3724.  $1\Leftrightarrow 2$  is obvious because  $-\otimes B$  being exact means it preserves SES's, which is equivalent to all the  $Tor_i(A, B) = 0$  for  $i \geq 1$  because A is a part of the SES

$$0 \longrightarrow A \xrightarrow{\cong} A \longrightarrow 0 \longrightarrow 0$$

 $2 \Rightarrow 3$  is trivial, and  $3 \Rightarrow 1$  is similar to  $4 \Rightarrow 2$  from 5.1.3. In both cases, I can't really see how the first derived functors being zero means all other derived functors are zero.

### Spectral Sequences

#### 1. Introduction



1.1. Tot(E) refers to the total complex of the zeroth page, with  $d = d^h + d^v$  denoting its differential. Consider the linear map

$$\phi: E_{p-1,q+1}^2 \to H_{p+q}(T): a \mapsto (a,b)$$

where  $b \in E_{p,q}^0$  is chosen so that  $b \in \operatorname{im}(d^v) \cap \ker(d^h)$ , and b. Now, suppose that  $a \in E_{p-1,q+1}^0$  is in the same class as zero on the second page, i.e.  $a \in \ker(d^v)$ ,  $\exists a' \in \operatorname{im}(d^v)$  such that  $a + a' \in \operatorname{im}(d^h)$ , then for any  $c \in (d^h)^{-1}(a+a')$ ,  $d^v \circ d^h(c) = d^h \circ d^v(c) = 0$ . So for its image in  $H_{p+q}(T)$ , we have

$$d(\phi(a)) = (d^h + d^v)(a, b) = d^h(b) + d^v(a) + d^v(b) = 0,$$

implying that  $\phi(a) \in \ker(d)$ .

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