

Homological Algebra Notes and Solutions to Exercises from Weibel

zin3724

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CHAPTER 1

All categories are assumed to be abelian.

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CHAPTER 2

Derived Functors

1. δ -functors

A functor $T = \{T_i\}_{i \in \mathbb{N}}$ is a *homological δ -functor* if it acts like $H = \{H_i\}_{i \in \mathbb{N}}$, the homology functor in the sense of 2.1 from [Wei95].

1.1. zin3724. Let f be a morphism in the category of SESs. Follows from condition 2. of definition 2.1.1.

1.2. zin3724.

$$\begin{array}{ccccccccc}
 G_1 C & \xrightarrow{\delta} & G_0 A & \xrightarrow{G_0 f} & G_0 B & \longrightarrow & G_0 C & \longrightarrow & 0 \\
 \downarrow & & \downarrow \alpha_A & & \downarrow \alpha_B & & \downarrow \alpha_C & & \downarrow 0 \\
 0 & \xrightarrow{\delta} & F A & \xrightarrow{F f} & F B & \longrightarrow & F C & \longrightarrow & 0
 \end{array}$$

Let G be a covariant δ -functor, and suppose there is a natural transformation α from G_0 to F . Since F is exact, the map $F A \xrightarrow{F f} F B$ is mono, so

$$F f \circ \alpha_A \circ \delta = 0 \Leftrightarrow \alpha_A \circ \delta = 0$$

By commutativity of the second square, $F f \circ \alpha_A \circ \delta = \alpha_B \circ G_0 f \circ \delta$, and by exactness of the top row, $\alpha_B \circ G_0 f \circ \delta = \alpha_B \circ 0 = 0$, so the first square commutes.

To see that α extends to commute with all δ_n 's,

$$\begin{array}{ccccccccc}
 G_n C & \xrightarrow{\delta} & G_{n-1} A & \xrightarrow{G_{n-1} f} & G_{n-1} B & \longrightarrow & G_{n-1} C & \longrightarrow & G_{n-2} A \\
 \downarrow & & \downarrow \alpha_A & & \downarrow \alpha_B & & \downarrow \alpha_C & & \downarrow 0 \\
 0 & \xrightarrow{\delta} & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0
 \end{array}$$

Note that the first square has a bottom row of 0's.

2. projective resolutions

2.1. zin3724. Let P be a projective object in \mathbf{Ch} .

Consider the exact sequence $A \xrightarrow{f} B \rightarrow 0 \in \mathbf{Ch}$ such that they're zero everywhere except in the n th place. By the projectivity of P , we have for P_n and any $P_n \xrightarrow{h} B_n$, that there exists $P_n \xrightarrow{g} A_n$ such that $h = f \circ g$

$$\begin{array}{ccccc} & & P_n & & \\ & \swarrow g & \downarrow h & & \\ A_n & \xrightarrow{f} & B_n & \longrightarrow & 0 \end{array}$$

since A_n and B_n are arbitrary, P_n must itself be a projective object. This applies to all $n \in \mathbb{N}$, so P is a complex of projectives.

Now, we know from 5.1 that $\text{cone}(\text{id}_P)$ is split exact, and furthermore, $\text{cone}(\text{id}_P)$ decomposes as $P \oplus P[-1]$ with i the usual inclusion, and j the usual projection (keep in mind that the differential in $P[k]$ is $(-1)^k \partial_P$)

$$\begin{array}{ccccccc} 0 & \longrightarrow & P & \xrightarrow{i} & \text{cone}(\text{id}_P) & \xrightarrow{j} & P[-1] \longrightarrow 0 \\ & & & & \parallel & & \\ & & & & P \oplus P[-1] & & \end{array}$$

By the splitting lemma, there exists $\text{cone}(\text{id}_P) \xrightarrow{p} P$ such that $p \circ i = \text{id}_P$, and combined with the fact that H_n is functorial for all n , it follows that P is exact.

Now, note that i and p are morphisms in \mathbf{Ch} , so they commute with the differentials.

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\partial_P} & P_{n+1} & \xrightarrow{\partial_P} & P_n & \xrightarrow{\partial_P} & P_{n-1} \xrightarrow{\partial_P} \cdots \\ & & \uparrow p \downarrow i & & \uparrow p \downarrow i & & \uparrow p \downarrow i \\ \cdots & \xleftarrow[s_{P \oplus P[-1]}]{\partial_{P \oplus P[-1]}} & P_{n+1} \oplus P_n & \xleftarrow[s_{P \oplus P[-1]}]{\partial_{P \oplus P[-1]}} & P_n \oplus P_{n-1} & \xleftarrow[s_{P \oplus P[-1]}]{\partial_{P \oplus P[-1]}} & P_{n-1} \oplus P_{n-2} \xleftarrow[s_{P \oplus P[-1]}]{\partial_{P \oplus P[-1]}} \cdots \end{array}$$

Since $P \oplus P[-1]$ is split with $s_{P \oplus P[-1]}$ the splitting map, we know that

$$\begin{aligned} & \partial_P p s_{P \oplus P[-1]} i \partial_P \\ &= \partial_P p s_{P \oplus P[-1]} \partial_{P \oplus P[-1]} i \\ &= p \partial_{P \oplus P[-1]} s_{P \oplus P[-1]} \partial_{P \oplus P[-1]} i \\ &= p \partial_{P \oplus P[-1]} i = \partial_P \end{aligned}$$

so P is split with the splitting map $p s_{P \oplus P[-1]} i$.

For the converse, see [Ral12].

2.2. See [ZYZ22].

2.3. zin3724. The quasi-isomorphism is a thing that makes

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 \longrightarrow 0 \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \epsilon \\
 \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & M \longrightarrow 0 \longrightarrow \cdots
 \end{array}$$

commute, and induces an isomorphism of the homology groups, which is the same thing as

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\epsilon} M \longrightarrow 0 \longrightarrow \cdots$$

being exact; the commutativity of the middle square means that the above is a chain complex. The quasi-isomorphism induced by ϵ at $i = 0$ means that $P_0/\partial(P_1) \cong M$, hence $\text{im}(\partial_1^P) = \ker(\epsilon)$, and the complex above is exact.

3. Injective Resolutions

3.1. zin3724. Let J be an ideal in $R = \mathbb{Z}/m$, then for some n , $J \cong \mathbb{Z}/n$, and the inclusion $\iota : J \hookrightarrow R$ is the map $1 \mapsto \frac{m}{n}$, so there exists a map π extending it

$$\begin{array}{ccc} R & & \\ & \searrow \pi & \\ 0 & \longrightarrow & J \xrightarrow{\iota} R \end{array}$$

with π the quotient modulo n .

If $d \div m$, and there exists a prime p such that $p \div d$ and $p \div \frac{m}{d}$, then consider ι_m, ι_d generated by $\iota_m(1) = \frac{m}{p}$ and $\iota_d(1) = \frac{d}{p}$;

$$\begin{array}{ccc} 0 & \longrightarrow & \mathbb{Z}/p \xrightarrow{\iota_m} \mathbb{Z}/m \\ & & \downarrow \iota_d \\ & & \mathbb{Z}/d \end{array}$$

then since p is a prime, and $p \div \frac{m}{d}$, $d \div \frac{m}{p}$, so any map $\mathbb{Z}/m \rightarrow \mathbb{Z}/d$ precomposed with ι_m is 0, and can't be ι_d , which is nonzero. Therefore \mathbb{Z}/d is not injective.

3.2. zin3724. If a is in the torsion subgroup and of order n , then set $f(a) = \frac{1}{n} \in \mathbb{Q}/\mathbb{Z}$. Since \mathbb{Q}/\mathbb{Z} is injective, $f : a\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z}$ extends to a map $f' : A \rightarrow \mathbb{Q}/\mathbb{Z}$. If a is free, then there are several nonzero maps from $a\mathbb{Z}$ to \mathbb{Q}/\mathbb{Z} (e.g. $a \mapsto \frac{1}{2}$).

To prove that e_A is an injection, writing the f' assigned to a as f'_a , let $a_1, a_2 \in A$ be distinct. Then $f'_{a_1} - f'_{a_2}$ is nonzero, because we can choose constants $c_1, c_2 \in \mathbb{Z}$ (depending on the orders of a_1, a_2 respectively) such that $(f'_{a_1} - f'_{a_2})(c_1 a_1 + c_2 a_2) = \frac{1}{2}$.

3.3. zin3724. If there exists $a \in A$ that is nonzero, then it follows from 3.2, that $\text{Hom}(A, \mathbb{Q}/\mathbb{Z}) \neq 0$ since $f'_a(a) = f(a) \neq 0$, so $f'_a \neq 0$.

3.4. zin3724. similar to 2.1.

4. Left Derived Functors

4.1. Too obvious.

4.2. zin3724. Let P be a projective resolution of A . Since U is exact, it preserves \ker and coker (i.e. $U \ker = \ker U$, and $U \operatorname{coker} = \operatorname{coker} U$), because for every $f : X \rightarrow Y$, it preserves the exactness of

$$0 \longrightarrow \ker f \longrightarrow A \xrightarrow{f} B \longrightarrow \operatorname{coker} f \longrightarrow 0$$

(note that exact sequence suffice to characterise \ker and coker in abelian categories). Since $\operatorname{im}(f) = \ker \operatorname{coker} f$, U preserves them too.

Now, for any functor F , $L_i F(A)$ is defined with short exact sequences involving \ker and im , like so

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \searrow & & \searrow \\
 & & & H_{i+1}(FP) = L_{i+1}F(A) & & H_i(FP) = L_iF(A) & & \cdots \\
 & & 0 & \downarrow & 0 & \downarrow & 0 & \downarrow \\
 & & \ker(F\partial_{i+1}) & & \ker(F\partial_i) & & \ker(F\partial_{i-1}) & \\
 & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow \\
 \cdots & \longrightarrow & FP_{i+1} & \xrightarrow{F\partial_{i+1}} & FP_i & \xrightarrow{F\partial_i} & FP_{i-1} & \longrightarrow \cdots \\
 & \searrow & \downarrow & \searrow & \downarrow & \searrow & \downarrow & \searrow \\
 & \cdots & \operatorname{im}(F\partial_{i+1}) & & \operatorname{im}(F\partial_i) & & \operatorname{im}(F\partial_{i-1}) & \\
 & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow \\
 0 & & 0 & & 0 & & 0 &
 \end{array}$$

Therefore,

$$L_i U F(A) = \ker U F \partial_i / \operatorname{im} U F \partial_{i+1} = U \ker F \partial_i / U \operatorname{im} F \partial_{i+1} = U L_i F(A)$$

and since i is arbitrary, it holds for all i . To see that the isomorphism is natural, see [Ped18].

5. Right Derived Functors

5.1. zin3724.

5.1.1. $1 \Leftrightarrow 2$. Let B be any object. Then for any exact sequence

$$0 \rightarrow W \xrightarrow{i} X \xrightarrow{j} Y \rightarrow 0$$

the sequence under $\text{Hom}(-, B)$

$$0 \rightarrow \text{Hom}(Y, B) \xrightarrow{j^*} \text{Hom}(X, B) \xrightarrow{i^*} \text{Hom}(W, B)$$

is exact.

PROOF. Suppose $h \in \ker(i^*)$, then $hi = 0$, so $0 \rightarrow \text{im}(i) \rightarrow \ker(h)$ is exact. But $\text{im}(i) = \ker(j)$ by our assumptions, so by the first isomorphism theorem, h factors through j , hence $h \in \text{im}(j^*)$, and $0 \rightarrow \ker(i^*) \rightarrow \text{im}(j^*)$ is exact. The exactness of $0 \rightarrow \text{im}(j^*) \rightarrow \ker(i^*)$ follows immediately from $ji = 0$.

By assumptions, j is epic, so $fj = 0 \Rightarrow f = 0$ and it follows that $\ker(j^*) = 0$. \square

To prove that $\text{Hom}(-, B)$ is exact when B is injective, note that for any $W \xrightarrow{f} B$, it factors through $W \xrightarrow{i} X$, therefore $\text{im}(i^*) = \text{Hom}(W, B)$ and $\text{Hom}(-, B)$ is right-exact in addition to being left-exact.

Conversely, if B is *not* injective, then there exists some $f \in \text{Hom}(W, B)$ that does not factor through i , so i^* would not be surjective.

5.1.2. $1 \Rightarrow 3$, note that $3 \Rightarrow 4$ is trivial. $\text{Ext}^i(A, B) = R^i \text{Hom}(A, -)(B)$, and since B is injective, $0 \rightarrow B \xrightarrow{\text{id}_B} B \rightarrow 0$ is an injective resolution. Functors preserve identity maps by definition, so

$$0 \rightarrow \text{Hom}(A, B) \xrightarrow{\text{id}_{\text{Hom}(A, B)}} \text{Hom}(A, B) \rightarrow 0$$

is exact, hence $R^i \text{Hom}(A, -)(B)$, the i -th cohomology of the above, is 0. A and i were arbitrary, so $\text{Ext}^i(A, B) = 0$ for all A and all $i \neq 0$.

5.1.3. $4 \Rightarrow 2$. Since Ext^\bullet is a δ -functor, for any exact sequence $0 \rightarrow W \rightarrow X \rightarrow Y \rightarrow 0$, we have the long exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}(Y, B) & \longrightarrow & \text{Hom}(X, B) & \longrightarrow & \text{Hom}(W, B) \\ & & & & \swarrow & & \\ & & \text{Ext}^1(Y, B) & \longrightarrow & \text{Ext}^1(X, B) & \longrightarrow & \text{Ext}^1(W, B) \\ & & & & \swarrow & & \\ & & \dots & & & & \end{array}$$

and by assumption, $\text{Ext}^1(Y, B) = \text{Ext}^1(X, B) = \text{Ext}^1(W, B) = 0$, so $\text{Hom}(-, B)$ is exact because it maps the SES to an SES.

6. Adjoint Functors and Left/Right Exactness

6.4. zin3724. According to Prop 9.4 in [Awo10], the paragraph above (Application 2.6.7 in [Wei95]) is sufficient and necessary for colim to be left adjoint to Δ .

Now, in the category **Ab**, consider

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{Z}/2 & \xrightarrow{=} & \mathbb{Z}/2 & \longrightarrow & 0 \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & & \text{mod } 2 & & & \\
 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/4 & \xrightarrow{=} & \mathbb{Z}/4 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & & \text{mod } 2 & & & \\
 0 & \longrightarrow & \mathbb{Z}/2 & \xrightarrow{=} & \mathbb{Z}/2 & \longrightarrow & 0 \longrightarrow 0
 \end{array}$$

under pushout (regarded as a special case of colim with $I = \bullet \leftarrow \bullet \rightarrow \bullet$), which gives

$$0 \longrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2 \longrightarrow 0 \longrightarrow 0$$

but no matter what the arrows are, it can't possibly be left exact.

7. Balancing Tor and Ext

Main Result: $\text{Hom}(A, -)(B) \cong \text{Hom}(-, B)(A)$ and $(A \otimes -)(B) \cong (- \otimes B)(A)$ as functors.

CHAPTER 3

Tor and Ext

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2. Tor and Flatness

2.1. zin3724. $1 \Leftrightarrow 2$ is obvious because $-\otimes B$ being exact means it preserves SES's, which is equivalent to all the $\text{Tor}_i(A, B) = 0$ for $i \geq 1$ because A is a part of the SES

$$0 \longrightarrow A \xrightarrow{\cong} A \longrightarrow 0 \longrightarrow 0$$

$2 \Rightarrow 3$ is trivial, and $3 \Rightarrow 1$ is similar to $\mathbf{4} \Rightarrow \mathbf{2}$ from 5.1.3. In both cases, I can't really see how the first derived functors being zero means all other derived functors are zero.

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CHAPTER 4

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CHAPTER 5

Spectral Sequences

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