

Bootleg Solutions Manual to
*Introduction to An Homological
Algebra* by Weibel

zin3724

Contents

Chapter 1.	5
1.	5
2.	6
3.	7
4.	8
5.	9
6.	10
Chapter 2. Derived Functors	11
1. δ -functors	11
2. projective resolutions	12
3. Injective Resolutions	14
4. Left Derived Functors	15
5. Right Derived Functors	17
6. Adjoint Functors and Left/Right Exactness	19
7. Balancing Tor and Ext	20
Chapter 3. Tor and Ext	21
1.	21
2. Tor and Flatness	22
3.	23
4.	24
5.	25
6.	26
Chapter 4.	27
1.	27
2.	28
3.	29
4.	30
5.	31
6.	32
Chapter 5. Spectral Sequences	33
1. Introduction	33
2.	35
3.	36
4.	37
5.	38

6.	39
Bibliography	41

CHAPTER 1

All categories are assumed to be abelian. Arrows are named after the source.

1.

2.

3.

4.

5.

6.

CHAPTER 2

Derived Functors

1. δ -functors

A functor $T = \{T_i\}_{i \in \mathbb{N}}$ is a *homological δ -functor* if it acts like $H = \{H_i\}_{i \in \mathbb{N}}$, the homology functor in the sense of 2.1 from [Wei95].

1.1. zin3724. Let f be a morphism in the category of SESs. Follows from condition 2. of definition 2.1.1.

1.2. zin3724.

$$\begin{array}{ccccccccc}
 G_1 C & \xrightarrow{\delta} & G_0 A & \xrightarrow{G_0 f} & G_0 B & \longrightarrow & G_0 C & \longrightarrow & 0 \\
 \downarrow & & \downarrow \alpha_A & & \downarrow \alpha_B & & \downarrow \alpha_C & & \downarrow 0 \\
 0 & \xrightarrow{\delta} & F A & \xrightarrow{F f} & F B & \longrightarrow & F C & \longrightarrow & 0
 \end{array}$$

Let G be a covariant δ -functor, and suppose there is a natural transformation α from G_0 to F . Since F is exact, the map $F A \xrightarrow{F f} F B$ is mono, so

$$F f \circ \alpha_A \circ \delta = 0 \Leftrightarrow \alpha_A \circ \delta = 0$$

By commutativity of the second square, $F f \circ \alpha_A \circ \delta = \alpha_B \circ G_0 f \circ \delta$, and by exactness of the top row, $\alpha_B \circ G_0 f \circ \delta = \alpha_B \circ 0 = 0$, so the first square commutes.

To see that α extends to commute with all δ_n 's,

$$\begin{array}{ccccccccc}
 G_n C & \xrightarrow{\delta} & G_{n-1} A & \xrightarrow{G_{n-1} f} & G_{n-1} B & \longrightarrow & G_{n-1} C & \longrightarrow & G_{n-2} A \\
 \downarrow & & \downarrow \alpha_A & & \downarrow \alpha_B & & \downarrow \alpha_C & & \downarrow 0 \\
 0 & \xrightarrow{\delta} & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0
 \end{array}$$

Note that the first square has a bottom row of 0's.

2. projective resolutions

2.1. zin3724. Let P be a projective object in \mathbf{Ch} .

Consider the exact sequence $A \xrightarrow{f} B \rightarrow 0 \in \mathbf{Ch}$ such that they're zero everywhere except in the n th place. By the projectivity of P , we have for P_n and any $P_n \xrightarrow{h} B_n$, that there exists $P_n \xrightarrow{g} A_n$ such that $h = f \circ g$

$$\begin{array}{ccccc} & & P_n & & \\ & \swarrow g & \downarrow h & & \\ A_n & \xrightarrow{f} & B_n & \longrightarrow & 0 \end{array}$$

since A_n and B_n are arbitrary, P_n must itself be a projective object. This applies to all $n \in \mathbb{N}$, so P is a complex of projectives.

Now, we know from 5.1 that $\text{cone}(\text{id}_P)$ is split exact, and furthermore, $\text{cone}(\text{id}_P)$ decomposes as $P \oplus P[-1]$ with i the usual inclusion, and j the usual projection (keep in mind that the differential in $P[k]$ is $(-1)^k \partial_P$)

$$\begin{array}{ccccccc} 0 & \longrightarrow & P & \xrightarrow{i} & \text{cone}(\text{id}_P) & \xrightarrow{j} & P[-1] \longrightarrow 0 \\ & & & & \parallel & & \\ & & & & P \oplus P[-1] & & \end{array}$$

By the splitting lemma, there exists $\text{cone}(\text{id}_P) \xrightarrow{p} P$ such that $p \circ i = \text{id}_P$, and combined with the fact that H_n is functorial for all n , it follows that P is exact.

Now, note that i and p are morphisms in \mathbf{Ch} , so they commute with the differentials.

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\partial_P} & P_{n+1} & \xrightarrow{\partial_P} & P_n & \xrightarrow{\partial_P} & P_{n-1} \xrightarrow{\partial_P} \cdots \\ & & \uparrow p \downarrow i & & \uparrow p \downarrow i & & \uparrow p \downarrow i \\ \cdots & \xrightarrow[\underset{s_{P \oplus P[-1]}}{\partial_{P \oplus P[-1]}}]{} & P_{n+1} \oplus P_n & \xrightarrow[\underset{s_{P \oplus P[-1]}}{\partial_{P \oplus P[-1]}}]{} & P_n \oplus P_{n-1} & \xrightarrow[\underset{s_{P \oplus P[-1]}}{\partial_{P \oplus P[-1]}}]{} & P_{n-1} \oplus P_{n-2} \xrightarrow[\underset{s_{P \oplus P[-1]}}{\partial_{P \oplus P[-1]}}]{} \cdots \end{array}$$

Since $P \oplus P[-1]$ is split with $s_{P \oplus P[-1]}$ the splitting map, we know that

$$\begin{aligned} & \partial_P p s_{P \oplus P[-1]} i \partial_P \\ &= \partial_P p s_{P \oplus P[-1]} \partial_{P \oplus P[-1]} i \\ &= p \partial_{P \oplus P[-1]} s_{P \oplus P[-1]} \partial_{P \oplus P[-1]} i \\ &= p \partial_{P \oplus P[-1]} i = \partial_P \end{aligned}$$

so P is split with the splitting map $p s_{P \oplus P[-1]} i$.

For the converse, see [Ral12].

2.2. See [ZYG22].

2.3. zin3724. The quasi-isomorphism is a thing that makes

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 \longrightarrow 0 \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \epsilon \\
 \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & M \longrightarrow 0 \longrightarrow \cdots
 \end{array}$$

commute, and induces an isomorphism of the homology groups, which is the same thing as

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\epsilon} M \longrightarrow 0 \longrightarrow \cdots$$

being exact; the commutativity of the middle square means that the above is a chain complex. The quasi-isomorphism induced by ϵ at $i = 0$ means that $P_0/\partial(P_1) \cong M$, hence $\text{im}(\partial_1^P) = \ker(\epsilon)$, and the complex above is exact.

3. Injective Resolutions

3.1. zin3724. Let J be an ideal in $R = \mathbb{Z}/m$, then for some n , $J \cong \mathbb{Z}/n$, and the inclusion $\iota : J \hookrightarrow R$ is the map $1 \mapsto \frac{m}{n}$, so there exists a map π extending it

$$\begin{array}{ccc} R & & \\ & \searrow \pi & \\ 0 & \longrightarrow J & \xrightarrow{\iota} R \end{array}$$

with π the quotient modulo n .

If $d \div m$, and there exists a prime p such that $p \div d$ and $p \div \frac{m}{d}$, then consider ι_m, ι_d generated by $\iota_m(1) = \frac{m}{p}$ and $\iota_d(1) = \frac{d}{p}$;

$$\begin{array}{ccc} 0 & \longrightarrow \mathbb{Z}/p & \xrightarrow{\iota_m} \mathbb{Z}/m \\ & \downarrow \iota_d & \\ & \mathbb{Z}/d & \end{array}$$

then since p is a prime, and $p \div \frac{m}{d}$, $d \div \frac{m}{p}$, so any map $\mathbb{Z}/m \rightarrow \mathbb{Z}/d$ precomposed with ι_m is 0, and can't be ι_d , which is nonzero. Therefore \mathbb{Z}/d is not injective.

3.2. zin3724. If a is in the torsion subgroup and of order n , then set $f(a) = \frac{1}{n} \in \mathbb{Q}/\mathbb{Z}$. Since \mathbb{Q}/\mathbb{Z} is injective, $f : a\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z}$ extends to a map $f' : A \rightarrow \mathbb{Q}/\mathbb{Z}$. If a is free, then there are several nonzero maps from $a\mathbb{Z}$ to \mathbb{Q}/\mathbb{Z} (e.g. $a \mapsto \frac{1}{2}$).

To prove that e_A is an injection, writing the f' assigned to a as f'_a , let $a_1, a_2 \in A$ be distinct. Then $f'_{a_1} - f'_{a_2}$ is nonzero, because we can choose constants $c_1, c_2 \in \mathbb{Z}$ (depending on the orders of a_1, a_2 respectively) such that $(f'_{a_1} - f'_{a_2})(c_1 a_1 + c_2 a_2) = \frac{1}{2}$.

3.3. zin3724. If there exists $a \in A$ that is nonzero, then it follows from 3.2, that $\text{Hom}(A, \mathbb{Q}/\mathbb{Z}) \neq 0$ since $f'_a(a) = f(a) \neq 0$, so $f'_a \neq 0$.

3.4. zin3724. similar to 2.1.

4. Left Derived Functors

4.1. Too obvious.

4.2. zin3724. Let P be a projective resolution of A . Since U is exact, it preserves \ker and coker (i.e. $U \ker = \ker U$, and $U \operatorname{coker} = \operatorname{coker} U$), because for every $f : X \rightarrow Y$, it preserves the exactness of

$$0 \longrightarrow \ker f \longrightarrow A \xrightarrow{f} B \longrightarrow \operatorname{coker} f \longrightarrow 0$$

(note that exact sequence suffice to characterise \ker and coker in abelian categories). Since $\operatorname{im}(f) = \ker \operatorname{coker} f$, U preserves them too.

Now, for any functor F , $L_i F(A)$ is defined with short exact sequences involving \ker and im , like so

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \searrow & & \searrow \\
 & & & H_{i+1}(FP) = L_{i+1}F(A) & & H_i(FP) = L_iF(A) & & \cdots \\
 & & 0 & \downarrow & 0 & \downarrow & 0 & \downarrow \\
 & & \ker(F\partial_{i+1}) & & \ker(F\partial_i) & & \ker(F\partial_{i-1}) & \\
 & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow \\
 \cdots & \longrightarrow & FP_{i+1} & \xrightarrow{F\partial_{i+1}} & FP_i & \xrightarrow{F\partial_i} & FP_{i-1} & \longrightarrow \cdots \\
 & \searrow & \downarrow & \searrow & \downarrow & \searrow & \downarrow & \searrow \\
 & \cdots & \operatorname{im}(F\partial_{i+1}) & & \operatorname{im}(F\partial_i) & & \operatorname{im}(F\partial_{i-1}) & \\
 & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow \\
 0 & & 0 & & 0 & & 0 &
 \end{array}$$

Therefore,

$$L_i U F(A) = \ker U F \partial_i / \operatorname{im} U F \partial_{i+1} = U \ker F \partial_i / U \operatorname{im} F \partial_{i+1} = U L_i F(A)$$

and since i is arbitrary, it holds for all i . To see that the isomorphism is natural, see [Ped18].

5. Right Derived Functors

5.1. zin3724.

5.1.1. $1 \Leftrightarrow 2$. Let B be any object. Then for any exact sequence

$$0 \rightarrow W \xrightarrow{i} X \xrightarrow{j} Y \rightarrow 0$$

the sequence under $\text{Hom}(-, B)$

$$0 \rightarrow \text{Hom}(Y, B) \xrightarrow{j^*} \text{Hom}(X, B) \xrightarrow{i^*} \text{Hom}(W, B)$$

is exact.

PROOF. Suppose $h \in \ker(i^*)$, then $hi = 0$, so $0 \rightarrow \text{im}(i) \rightarrow \ker(h)$ is exact. But $\text{im}(i) = \ker(j)$ by our assumptions, so by the first isomorphism theorem, h factors through j , hence $h \in \text{im}(j^*)$, and $0 \rightarrow \ker(i^*) \rightarrow \text{im}(j^*)$ is exact. The exactness of $0 \rightarrow \text{im}(j^*) \rightarrow \ker(i^*)$ follows immediately from $ji = 0$.

By assumptions, j is epic, so $fj = 0 \Rightarrow f = 0$ and it follows that $\ker(j^*) = 0$. \square

To prove that $\text{Hom}(-, B)$ is exact when B is injective, note that for any $W \xrightarrow{f} B$, it factors through $W \xrightarrow{i} X$, therefore $\text{im}(i^*) = \text{Hom}(W, B)$ and $\text{Hom}(-, B)$ is right-exact in addition to being left-exact.

Conversely, if B is *not* injective, then there exists some $f \in \text{Hom}(W, B)$ that does not factor through i , so i^* would not be surjective.

5.1.2. $1 \Rightarrow 3$, note that $3 \Rightarrow 4$ is trivial. $\text{Ext}^i(A, B) = R^i \text{Hom}(A, -)(B)$, and since B is injective, $0 \rightarrow B \xrightarrow{\text{id}_B} B \rightarrow 0$ is an injective resolution. Functors preserve identity maps by definition, so

$$0 \rightarrow \text{Hom}(A, B) \xrightarrow{\text{id}_{\text{Hom}(A, B)}} \text{Hom}(A, B) \rightarrow 0$$

is exact, hence $R^i \text{Hom}(A, -)(B)$, the i -th cohomology of the above, is 0. A and i were arbitrary, so $\text{Ext}^i(A, B) = 0$ for all A and all $i \neq 0$.

5.1.3. $4 \Rightarrow 2$. Since Ext^\bullet is a δ -functor, for any exact sequence $0 \rightarrow W \rightarrow X \rightarrow Y \rightarrow 0$, we have the long exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}(Y, B) & \longrightarrow & \text{Hom}(X, B) & \longrightarrow & \text{Hom}(W, B) \\ & & & & \swarrow & & \\ & & \text{Ext}^1(Y, B) & \longrightarrow & \text{Ext}^1(X, B) & \longrightarrow & \text{Ext}^1(W, B) \\ & & & & \swarrow & & \\ & & \dots & & & & \end{array}$$

and by assumption, $\text{Ext}^1(Y, B) = \text{Ext}^1(X, B) = \text{Ext}^1(W, B) = 0$, so $\text{Hom}(-, B)$ is exact because it maps the SES to an SES.

6. Adjoint Functors and Left/Right Exactness

6.4. zin3724. According to Prop 9.4 in [Awo10], the paragraph above (Application 2.6.7 in [Wei95]) is sufficient and necessary for colim to be left adjoint to Δ .

Now, in the category **Ab**, consider

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{Z}/2 & \xrightarrow{=} & \mathbb{Z}/2 & \longrightarrow & 0 \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & & \text{mod } 2 & & & \\
 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/4 & \xrightarrow{=} & \mathbb{Z}/4 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & & \text{mod } 2 & & & \\
 0 & \longrightarrow & \mathbb{Z}/2 & \xrightarrow{=} & \mathbb{Z}/2 & \longrightarrow & 0 \longrightarrow 0
 \end{array}$$

under pushout (regarded as a special case of colim with $I = \bullet \leftarrow \bullet \rightarrow \bullet$), which gives

$$0 \longrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2 \longrightarrow 0 \longrightarrow 0$$

but no matter what the arrows are, it can't possibly be left exact.

7. Balancing Tor and Ext

Main Result: $\text{Hom}(A, -)(B) \cong \text{Hom}(-, B)(A)$ and $(A \otimes -)(B) \cong (- \otimes B)(A)$ as functors.

CHAPTER 3

Tor and Ext

1.

2. Tor and Flatness

2.1. zin3724. $1 \Leftrightarrow 2$ is obvious because $-\otimes B$ being exact means it preserves SES's, which is equivalent to all the $\text{Tor}_i(A, B) = 0$ for $i \geq 1$ because A is a part of the SES

$$0 \longrightarrow A \xrightarrow{\cong} A \longrightarrow 0 \longrightarrow 0$$

$2 \Rightarrow 3$ is trivial, and $3 \Rightarrow 1$ is similar to $\mathbf{4} \Rightarrow \mathbf{2}$ from 5.1.3. In both cases, I can't really see how the first derived functors being zero means all other derived functors are zero.

3.

4.

5.

6.

CHAPTER 4

1.

2.

3.

4.

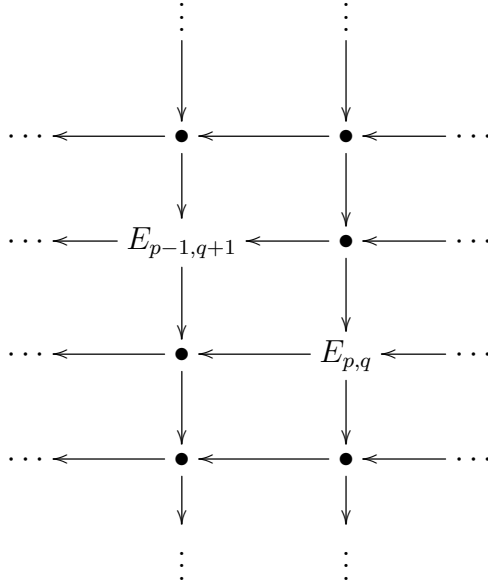
5.

6.

CHAPTER 5

Spectral Sequences

1. Introduction



1.1. $T = \text{Tot}(E)$ refers to the total complex of the zeroth page, with $d = d^h + d^v$ denoting its differential. Consider the linear map

$$\phi : E_{p-1,q+1}^2 \rightarrow H_{p+q}(T) : a \mapsto (a, 0).$$

Suppose that $a \in E_{p-1,q+1}^0$ is in the same class as zero in $E_{p-1,q+1}^2$, i.e. $a \in \ker(d^v)$, $\exists a' \in \text{im}(d^v)$ such that $a + a' \in d^h(\ker(d_{p,q+1}^v))$, because on the page $E_{\bullet,\bullet}^1$, $a = a + a'$ ends up in $\text{im}(d_1^h)$. It now follows that there exists $b' \in E_{p-1,q+2}^0$ and $c \in \ker(d_{p,q+1}^v) \subset E_{p,q+1}^0$ such that

$$\phi(a) = (a, 0) = ((-a') + (a + a'), 0) = d^v(b') + d^h(c) + d^v(c),$$

so $(a, 0) \in \text{im}(d)$, hence $\phi(0) = 0$, so ϕ must be well-defined.

Now, suppose that $\phi(a) = 0$, then $(a, 0) \in \text{im}(d)$, so there exist $a' \in \text{im}(d^v)$ such that $a - a' \in d^h(\ker(d^v))$. Furthermore, for any $c \in \ker(d_{p,q+1}^v)$ such that $d^h(c) = a - a'$, we have

$$0 = d^h \circ d^v(c) = d^v \circ d^h(c) = d^v(a - a'),$$

so it follows that $d^v(a) = 0$. We have established that $a \in \ker(d^v)$ and on the page $E_{\bullet,\bullet}^1$, $a = a - a' \in \text{im}(d^h)$, so on the page $E_{\bullet,\bullet}^2$, $a = 0$, hence ϕ is injective.

For the remaining part of the required short exact sequence, consider

$$\psi : H_{p+q}(T) \rightarrow E_{p,q}^2 : (a, b) \rightarrow b.$$

Let $(a, b) \in H_{p+q}(T)$ be in the same class as 0, then there exists $(r, s) \in E_{p-1,q+2}^0 \oplus E_{p,q+1}^0 = T_{p+q+1}$ such that $d^v(r) + d^h(s) = a$ and $d^v(s) = b$, so it's immediate that $\psi(a, b) = b$ is in $\text{im}(d^v)$, hence $\psi(0) = 0$, and ψ is well-defined.

It follows from previous results that for all $a \in E_{p-1,q+1}^2$, $\psi \circ \phi(a) = \psi(a, 0) = 0$, so $\text{im}(\phi) \subset \ker(\psi)$. Conversely, suppose that $(a, b) \in \ker(\psi)$. Then $\psi(a, b) = b \in \text{im}(d^v)$, so we can choose $c \in (d^v)^{-1}(b)$ such that

$$\begin{aligned} (a, b) &= (a, b) - d(0, c) = (a - d^h(c), b - d^v(c)) = (a - d^h(c), b - b) \\ &= (a - d^h(c), 0) \end{aligned}$$

as a class in $H_{p+q}(T)$, and conclude that $(a, b) \in \text{im}(\phi)$. Therefore, $\text{im}(\phi) = \ker(\psi)$.

Lastly consider some arbitrary $b \in E_{p,q}^2$, then it is represented by some $b \in \ker(d^v) \subset E_{p,q}^0$ such that there exists some $b' \in \text{im}(d^v)$ with $b + b' \in \ker(d^h)$. We now choose some $c' \in (d^v)^{-1}(b') \subset E_{p,q+1}^0$ and see that

$$\begin{aligned} (-d^h(c'), b) &= (-d^h(c'), b) + d(0, c') = (-d^h(c') + d^h(c'), b + d^v(c')) \\ &= (0, b + b') \end{aligned}$$

which satisfies

$$d(-d^h(c'), b) = d(0, b + b') = (d^h(b + b'), d^v(b + b')) = (0, 0),$$

it follows that there exists a class in $H_{p+q}(T) := \ker(d)/\text{im}(d)$ that maps to b under ψ , and ψ must be surjective.

Finally, we have shown that

$$0 \rightarrow E_{p-1,q+1}^2 \xrightarrow{\phi} H_{p+q}(T) \xrightarrow{\psi} E_{p,q}^2 \rightarrow 0$$

is exact.

2.

3.

4.

5.

6.

Bibliography

- [Awo10] Steve Awodey. *Category theory*. en. 2nd ed. Oxford logic guides. Oxford ; New York: Oxford University Press, 2010. ISBN: 978-0-19-958736-0.
- [Ped18] Pedro. *Answer to “Exercise 2.4.2 In Weibel An Introduction to Homological Algebra”*. Sept. 2018. URL: <https://math.stackexchange.com/a/2932030>.
- [Ral12] Ralph. *Answer to “Projective objects in the category of chain complexes”*. Dec. 2012. URL: <https://mathoverflow.net/a/115454>.
- [Wei95] Charles A. Weibel. *An Introduction to Homological Algebra*. en. Cambridge University Press, Oct. 1995. ISBN: 978-0-521-55987-4.
- [ZYX22] ZYX. *Answer to “If A has enough projectives, then so does the category $Ch(A)$ of chain complex over A ”*. Mar. 2022. URL: <https://math.stackexchange.com/a/4415054>.