# Homological Algebra Notes and Solutions to Exercises from Weibel

zin3724

# Contents

# CHAPTER 1

All categories are assumed to be abelian.

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# Derived Functors

## 1. $\delta$-functors

A functor $T = \{T_i\}_{i \in \mathbb{N}}$ is a *homological $\delta$-functor* if it acts like $H = \{H_i\}_{i \in \mathbb{N}}$, the homology functor in the sense of 2.1 from [Wei95].

**1.1. zin3724.** Let $f$ be a morphism in the category of SESs Follows from condition 2. of definition 2.1.1.

**1.2. zin3724.**

$$
\begin{array}{ccccccccc}
G_1C & \xrightarrow{\delta} & G_0A & \xrightarrow{G_0f} & G_0B & \longrightarrow & G_0C & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \alpha_A} & & \downarrow{\scriptstyle \alpha_B} & & \downarrow{\scriptstyle \alpha_C} & & \downarrow{\scriptstyle 0} \\
0 & \xrightarrow{\delta} & FA & \xrightarrow{Ff} & FB & \longrightarrow & FC & \longrightarrow & 0
\end{array}
$$

Let $G$ be a covariant $\delta$-functor, and suppose there is a natural transformation $\alpha$ from $G_0$ to $F$. Since $F$ is exact, the map $FA \xrightarrow{Ff} FB$ is mono, so

$$Ff \circ \alpha_A \circ \delta = 0 \Leftrightarrow \alpha_A \circ \delta = 0$$

By commutativity of the second square, $Ff \circ \alpha_A \circ \delta = \alpha_B \circ G_0f \circ \delta$, and by exactness of the top row, $\alpha_B \circ G_0f \circ \delta = \alpha_B \circ 0 = 0$, so the first square commutes.

To see that $\alpha$ extends to commute with all $\delta_n$'s,

$$
\begin{array}{ccccccccc}
G_nC & \xrightarrow{\delta} & G_{n-1}A & \xrightarrow{G_{n-1}f} & G_{n-1}B & \longrightarrow & G_{n-1}C & \longrightarrow & G_{n-2}A \\
\downarrow & & \downarrow{\scriptstyle \alpha_A} & & \downarrow{\scriptstyle \alpha_B} & & \downarrow{\scriptstyle \alpha_C} & & \downarrow{\scriptstyle 0} \\
0 & \xrightarrow{\delta} & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}
$$

Note that the first square has a bottom row of 0's.

## 2. projective resolutions

**2.1. zin3724.** Let $P$ be a projective object in **Ch**.

Consider the exact sequence $A \xrightarrow{f} B \to 0 \in \mathbf{Ch}$ such that they're zero everywhere except in the $n$th place. By the projectivity of $P$, we have for $P_n$ and any $P_n \xrightarrow{h} B_n$, that there exists $P_n \xrightarrow{g} A_n$ such that $h = f \circ g$

$$
\begin{array}{ccc}
 & & P_n \\
 & {}^{g}\nearrow & \downarrow{}^{h} \\
A_n & \xrightarrow{f} & B_n \longrightarrow 0
\end{array}
$$

since $A_n$ and $B_n$ are arbitrary, $P_n$ must itself be a projective object. This applies to all $n \in \mathbb{N}$, so $P$ is a complex of projectives.

Now, we know from 5.1 that $\mathrm{cone}(\mathrm{id}_P)$ is split exact, and furthermore, $\mathrm{cone}(\mathrm{id}_P)$ decomposes as $P \oplus P[-1]$ with $i$ the usual inclusion, and $j$ the usual projection (keep in mind that the differential in $P[k]$ is $(-1)^k \partial_P$)

$$
0 \longrightarrow P \xrightarrow{i} \mathrm{cone}(\mathrm{id}_P) \xrightarrow{j} P[-1] \longrightarrow 0
$$
$$
\|
$$
$$
P \oplus P[-1]
$$

By the splitting lemma, there exists $\mathrm{cone}(\mathrm{id}_P) \xrightarrow{p} P$ such that $p \circ i = \mathrm{id}_P$, and combined with the fact that $H_n$ is functorial for all $n$, it follows that $P$ is exact.

Now, note that $i$ and $p$ are morphisms in **Ch**, so they commute with the differentials.

$$
\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial_P} & P_{n+1} & \xrightarrow{\partial_P} & P_n & \xrightarrow{\partial_P} & P_{n-1} & \xrightarrow{\partial_P} & \cdots \\
 & & {}^{p}\!\uparrow\downarrow{}_{i} & & {}^{p}\!\uparrow\downarrow{}_{i} & & {}^{p}\!\uparrow\downarrow{}_{i} & & \\
\cdots & \xrightarrow[s_{P\oplus P[-1]}]{\partial_{P\oplus P[-1]}} & P_{n+1} \oplus P_n & \xrightarrow[s_{P\oplus P[-1]}]{\partial_{P\oplus P[-1]}} & P_n \oplus P_{n-1} & \xrightarrow[s_{P\oplus P[-1]}]{\partial_{P\oplus P[-1]}} & P_{n-1} \oplus P_{n-2} & \xrightarrow[s_{P\oplus P[-1]}]{\partial_{P\oplus P[-1]}} & \cdots
\end{array}
$$

Since $P \oplus P[-1]$ is split with $s_{P\oplus P[-1]}$ the splitting map, we know that

$$\partial_P p s_{P\oplus P[-1]} i \partial_P$$

$$= \partial_P p s_{P\oplus P[-1]} \partial_{P\oplus P[-1]} i$$

$$= p \partial_{P\oplus P[-1]} s_{P\oplus P[-1]} \partial_{P\oplus P[-1]} i$$

$$= p \partial_{P\oplus P[-1]} i = \partial_P$$

so $P$ is split with the splitting map $p s_{P\oplus P[-1]} i$.

For the converse, see [Ral12].

**2.2.** See [ZYX22].

**2.3. zin3724.** The quasi-isomorphism is a thing that makes

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\epsilon} & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & \cdots
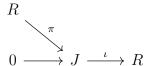\end{array}$$

commute, and induces an isomorphism of the homology groups, which is the same thing as

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \overset{\epsilon}{\longrightarrow} M \longrightarrow 0 \longrightarrow \cdots$$

being exact; the commutativity of the middle square means that the above is a chain complex. The quasi-isomorphism induced by $\epsilon$ at $i = 0$ means that $P_0/\partial(P_1) \cong M$, hence $\mathrm{im}(\partial_1^P) = \ker(\epsilon)$, and the complex above is exact.

## 3. Injective Resolutions

**3.1. zin3724.** Let $J$ be an ideal in $R = \mathbb{Z}/m$, then for some $n$, $J \cong \mathbb{Z}/n$, and the inclusion $\iota : J \hookrightarrow R$ is the map $1 \mapsto \frac{m}{n}$, so there exists a map $\pi$ extending it

$$
\begin{array}{ccc}
R & & \\
& \searrow^{\pi} & \\
0 \longrightarrow J & \xrightarrow{\iota} & R
\end{array}
$$

with $\pi$ the quotient modulo $n$.

If $d \div m$, and there exists a prime $p$ such that $p \div d$ and $p \div \frac{m}{d}$, then consider $\iota_m, \iota_d$ generated by $\iota_m(1) = \frac{m}{p}$ and $\iota_d(1) = \frac{d}{p}$;

$$
\begin{array}{ccc}
0 \longrightarrow \mathbb{Z}/p & \xrightarrow{\iota_m} & \mathbb{Z}/m \\
& \downarrow^{\iota_d} & \\
& \mathbb{Z}/d &
\end{array}
$$

then since $p$ is a prime, and $p \div \frac{m}{d}$, $d \div \frac{m}{p}$, so any map $\mathbb{Z}/m \to \mathbb{Z}/d$ precomposed with $\iota_m$ is 0, and can't be $\iota_d$, which is nonzero. Therefore $\mathbb{Z}/d$ is not injective.

**3.2. zin3724.** If $a$ is in the torsion subgroup and of order $n$, then set $f(a) = \frac{1}{n} \in \mathbb{Q}/\mathbb{Z}$. Since $\mathbb{Q}/\mathbb{Z}$ is injective, $f : a\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ extends to a map $f' : A \to \mathbb{Q}/\mathbb{Z}$. If $a$ is free, then there are several nonzero maps from $a\mathbb{Z}$ to $\mathbb{Q}/\mathbb{Z}$ (e.g. $a \mapsto \frac{1}{2}$).

To prove that $e_A$ is an injection, writing the $f'$ assigned to $a$ as $f'_a$, let $a_1, a_2 \in A$ be distinct. Then $f'_{a_1} - f'_{a_2}$ is nonzero, because we can choose constants $c_1, c_2 \in \mathbb{Z}$ (depending on the orders of $a_1, a_2$ respectively) such that $(f'_{a_1} - f'_{a_2})(c_1 a_1 + c_2 a_2) = \frac{1}{2}$.

**3.3. zin3724.** If there exists $a \in A$ that is nonzero, then it follows from 3.2, that $\mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z}) \neq 0$ since $f'_a(a) = f(a) \neq 0$, so $f'_a \neq 0$.

**3.4. zin3724.** similar to 2.1.
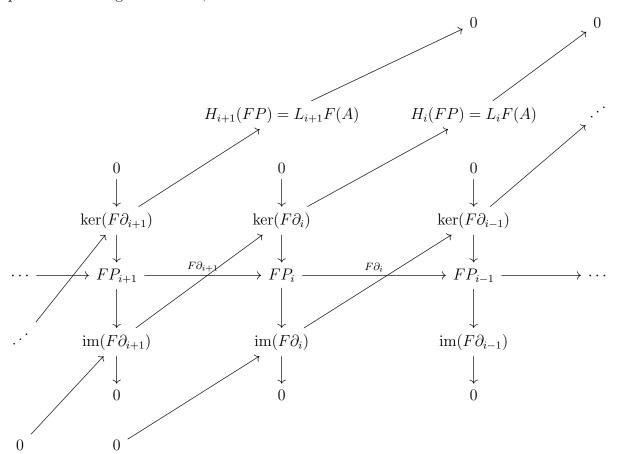
## 4. Left Derived Functors

**4.1.** Too obvious.

**4.2. zin3724.** Let $P$ be a projective resolution of $A$. Since $U$ is exact, it preserves ker and coker (i.e. $U \ker = \ker U$, and $U \operatorname{coker} = \operatorname{coker} U$), because for every $f : X \to Y$, it preserves the exactness of

$$0 \longrightarrow \ker f \longrightarrow A \xrightarrow{f} B \longrightarrow \operatorname{coker} f \longrightarrow 0$$

(note that exact sequence suffice to characterise ker and coker in abelian categories). Since $\operatorname{im}(f) = \ker \operatorname{coker} f$, $U$ preserves them too.

Now, for any functor $F$, $L_i F(A)$ is defined with short exact sequences involving ker and im, like so



Therefore,

$$L_i U F(A) = \ker U F \partial_i / \operatorname{im} U F \partial_{i+1} = U \ker F \partial_i / U \operatorname{im} F \partial_{i+1} = U L_i F(A)$$

and since $i$ is arbitrary, it holds for all $i$. To see that the isomorphism is natural, see [Ped18].

# 5. Right Derived Functors

## 5.1. zin3724.

5.1.1. *1⇔2.* Let $B$ be any object. Then for any exact sequence

$$0 \to W \xrightarrow{i} X \xrightarrow{j} Y \to 0$$

the sequence under $\operatorname{Hom}(-, B)$

$$0 \to \operatorname{Hom}(Y, B) \xrightarrow{j^*} \operatorname{Hom}(X, B) \xrightarrow{i^*} \operatorname{Hom}(W, B)$$

is exact.

PROOF. Suppose $h \in \ker(i^*)$, then $hi = 0$, so $0 \to \operatorname{im}(i) \to \ker(h)$ is exact. But $\operatorname{im}(i) = \ker(j)$ by our assumptions, so by the first isomorphism theorem, $h$ factors through $j$, hence $h \in \operatorname{im}(j^*)$, and $0 \to \ker(i^*) \to \operatorname{im}(j^*)$ is exact. The exactness of $0 \to \operatorname{im}(j^*) \to \ker(i^*)$ follows immediately from $ji = 0$.

By assumptions, $j$ is epic, so $fj = 0 \Rightarrow f = 0$ and it follows that $\ker(j^*) = 0$. □

To prove that $\operatorname{Hom}(-, B)$ is exact when $B$ is injective, note that for any $W \xrightarrow{f} B$, it factors through $W \xrightarrow{i} X$, therefore $\operatorname{im}(i^*) = \operatorname{Hom}(W, B)$ and $\operatorname{Hom}(-, B)$ is right-exact in addition to being left-exact.

Conversely, if $B$ is *not* injective, then there exists some $f \in \operatorname{Hom}(W, B)$ that does not factor through $i$, so $i^*$ would not be surjective.

5.1.2. *1⇒3, note that 3⇒4 is trivial.* $\operatorname{Ext}^i(A, B) = R^i \operatorname{Hom}(A, -)(B)$, and since $B$ is injective, $0 \to B \xrightarrow{\operatorname{id}_B} B \to 0$ is an injective resolution. Functors preserve identity maps by definition, so

$$0 \to \operatorname{Hom}(A, B) \xrightarrow{\operatorname{id}_{\operatorname{Hom}(A,B)}} \operatorname{Hom}(A, B) \to 0$$

is exact, hence $R^i \operatorname{Hom}(A, -)(B)$, the $i$-th cohomology of the above, is 0. $A$ and $i$ were arbitrary, so $\operatorname{Ext}^i(A, B) = 0$ for all $A$ and all $i \neq 0$.

5.1.3. *4⇒2.* Since $\operatorname{Ext}^\bullet$ is a $\delta$-functor, for any exact sequence $0 \to W \to X \to Y \to 0$, we have the long exact sequence
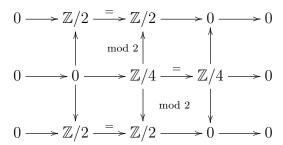
$$
\begin{array}{ccccc}
0 \longrightarrow \operatorname{Hom}(Y, B) \longrightarrow \operatorname{Hom}(X, B) \longrightarrow \operatorname{Hom}(W, B) \\
\\
\operatorname{Ext}^1(Y, B) \longrightarrow \operatorname{Ext}^1(X, B) \longrightarrow \operatorname{Ext}^1(W, B) \\
\\
\cdots
\end{array}
$$

and by assumption, $\operatorname{Ext}^1(Y, B) = \operatorname{Ext}^1(X, B) = \operatorname{Ext}^1(W, B) = 0$, so $\operatorname{Hom}(-, B)$ is exact because it maps the SES to an SES.

# 6. Adjoint Functors and Left/Right Exactness

**6.4. zin3724.** According to Prop 9.4 in [Awo10], the paragraph above (Application 2.6.7 in [Wei95]) is sufficient and necessary for colim to be left adjoint to $\Delta$.

Now, in the category **Ab**, consider

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}/2 & \overset{=}{\longrightarrow} & \mathbb{Z}/2 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow {\scriptstyle\text{mod } 2} & & \uparrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/4 & \overset{=}{\longrightarrow} & \mathbb{Z}/4 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow {\scriptstyle\text{mod } 2} & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}/2 & \overset{=}{\longrightarrow} & \mathbb{Z}/2 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}
$$

under pushout (regarded as a special case of colim with $I = \bullet \leftarrow \bullet \rightarrow \bullet$), which gives

$$0 \longrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2 \longrightarrow 0 \longrightarrow 0$$

but no matter what the arrows are, it can't possibly be left exact.

# 7. Balancing Tor and Ext

Main Result: $\operatorname{Hom}(A, -)(B) \cong \operatorname{Hom}(-, B)(A)$ and $(A \otimes -)(B) \cong (- \otimes B)(A)$ as functors.

CHAPTER 3

# Tor and Ext

1.

# 2. Tor and Flatness

**2.1. zin3724.** $1 \Leftrightarrow 2$ is obvious because $- \otimes B$ being exact means it preserves SES's, which is equivalent to all the $\text{Tor}_i(A, B) = 0$ for $i \geq 1$ because $A$ is a part of the SES

$$0 \longrightarrow A \xrightarrow{\cong} A \longrightarrow 0 \longrightarrow 0$$

$2 \Rightarrow 3$ is trivial, and $3 \Rightarrow 1$ is similar to $\mathbf{4} \Rightarrow \mathbf{2}$ from 5.1.3. In both cases, I can't really see how the first derived functors being zero means all other derived functors are zero.

**3.**

**4.**

**5.**

**6.**

# CHAPTER 4

## 1.

**2.**

**3.**

**4.**

**5.**

**6.**

# Spectral Sequences

## 1. Introduction

**1.1.** Since there are only two nonzero columns,

**2.**

**3.**

**4.**

**5.**

**6.**

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