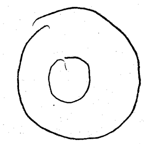
#1 $\{f_n\}_{n\in\mathbb{Z}} = c_n z^n$ where c_n is some constant, is orthonormal on Closed Annulus, $A = \{z \in \mathbb{C} : r \le |z| \le R\}$, $0<r<R<\infty$

$$z = x+iy = ae^{i\phi}$$



$$\langle f_k, f_j \rangle = \int_A c_k z^k \overline{c_j z^j}\, dx\, dy$$

$$= c_k \bar{c}_j \int_A (a^j e^{i\phi_j})(a^k e^{i\phi_k})\, da\, d\phi$$

$$c_k \bar{c}_j \int_r^R a^{j+k+1}\, da \int_0^{2\pi} \underbrace{e^{i(j-k)\phi}}_{\delta_{jk}|_0^{2\pi}}\, d\phi$$

$$= c_k \bar{c}_j \left(\frac{R^{j+k+2} - r^{j+k+2}}{j+k+2} \right) (2\pi \delta_{jk})$$

$$= c_k \bar{c}_j \frac{2\pi (R^{j+k+2} - r^{j+k+2})}{j+k+2} \delta_{jk}, \; j=k$$

$$c_k^2 \cdot \frac{2\pi (R^{2(k+1)} - r^{2(k+1)})}{2(k+1)} = 1$$

$$c_k^2 = \frac{2(k+1)}{2\pi (R^{2(k+1)} - r^{2(k+1)})}$$

$$\boxed{c_k = \sqrt{\frac{2(k+1)}{2\pi (R^{2(k+1)} - r^{2(k+1)})}}}$$

Ryan Richardson | HW 7 | Fourier Analysis

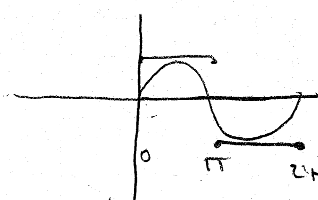
#2

$\{f_n\}_{n\in\mathbb{N}}$ where $f_n(x) = \operatorname{sgn}(\sin(2^n \pi x))$ is orthonormal on $[0,1]$

$$\langle f_j, f_k\rangle = \int_0^1 \operatorname{sgn}(\sin(2^j \pi x))\operatorname{sgn}(\sin(2^k \pi x))\,dx$$

when $j=k$ $\therefore$ $\int_0^1 \operatorname{sgn}^2(\sin 2^j \pi x)\,dx =$ $\qquad \int_0^1 \operatorname{sgn}^2(x) = x$

$$\int_0^1 = x\Big|_0^1 = 1$$



when $j \neq k$ $\quad \operatorname{sgn}(\sin) = \sum 1_{[2l\pi,(2l+1)\pi]}(x) - 1_{[(2l+1)\pi,(2l+2)\pi]}(x)$ $\quad$ from $2\pi \to 2l\pi+1$

$\qquad 0$ from $2l\pi+1 \to 2l\pi+2$

$$1_{\frac{(p\pi,(p+1)\pi)}{2^j\pi}}(2^j \pi x) = 1_{\left[\frac{p}{2^j},\frac{p+1}{2^j}\right]}(x)$$

$$\begin{cases} \frac{1}{2^j} & 2^{j-k}m \le p \le 2^{j-k}_{m+1} \\ \emptyset \end{cases}$$

$$\int_0^1 1_{(p\pi,(p+1)\pi)}(2^j\pi x)\; 1_{[m\pi,(m+1)\pi]}(2^k\pi x)\,dx = \int_0^1 1_{\left[\frac{p}{2^j},\frac{p+1}{2^j}\right]}(x)\; 1_{\left[\frac{2^{j-k}m}{2^j},\frac{2^{j-k}m+1}{2^j}\right]}(x)\,dx$$

$2^{j-k} \le p \le 2^{j-k}m + 2^{j-k}$; $\int_0^{1/2^j} 1 = \frac{1}{2^j}$ $\qquad = \frac{1}{2^j}$ for $j<k$

① else $\qquad$: $\int_0^{1/2^j} 0 = \emptyset$ $\qquad f_n$ is positive @ $\frac{1}{2^j}$

$$\int_0^1 \left(1_{[2p\pi,(2p+1)\pi]}(2^j\pi x) - 1_{[(2p+1)\pi,(2p+2)\pi]}(2^j\pi x)\right) 1_{[m\pi,(m+1)\pi]}(2^k\pi x) =$$

$\frac{1}{2^j} - \frac{1}{2^j} = \emptyset$ when $2^{j-k} \le p \le 2^{j-k}m + 2^{j-k}$

$= \emptyset$ everywhere else $\quad$ So $\operatorname{sgn}(\sin(2^n\pi x))$ is orthonormal on $[0,1]$