On the Continued Fraction Development of a Quadratic Surd

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March 6, 2012

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Abstract

This paper derives many of the simpler relationships and properties concerning the evaluation of the regular continued fraction representation of a quadratic surd, and does so using very elementary means.

1 Introduction

This note provides some justification of the stages within the Continued Fraction development of a Quadratic Surd

$$\alpha = \alpha_0 = \frac{P_0 + \sqrt{D}}{Q_0} \tag{1.1}$$

subject to the constraints

- $D > 0$ is non-square
- $Q_0 > 0$

- $Q_0 | D - P_0^2$ or $P_0^2 \equiv D \pmod{Q_0}$

defined by the tableau:

$$
\begin{array}{llllllll}
\alpha_0 = & & & & \dfrac{\sqrt{D}+P_0}{Q_0} & = & a_0 + \dfrac{\sqrt{D}-P_1}{Q_0} \\
\alpha_1 = & \dfrac{Q_0}{\sqrt{D}-P_1} & = & \dfrac{Q_0(\sqrt{D}+P_1)}{D-P_1^2} = & \dfrac{\sqrt{D}+P_1}{Q_1} & = & a_1 + \dfrac{\sqrt{D}-P_2}{Q_1} \\
\alpha_k = & \dfrac{Q_{k-1}}{\sqrt{D}-P_k} & = & \dfrac{Q_{k-1}(\sqrt{D}+P_k)}{D-P_k^2} = & \dfrac{\sqrt{D}+P_k}{Q_k} & = & a_k + \dfrac{\sqrt{D}-P_{k+1}}{Q_k} \\
\alpha_{k+1} = & \dfrac{Q_k}{\sqrt{D}-P_{k+1}} & = & \dfrac{Q_k(\sqrt{D}+P_{k+1})}{D-P_{k+1}^2} = & \dfrac{\sqrt{D}+P_{k+1}}{Q_{k+1}} & = & a_{k+1} + \dfrac{\sqrt{D}-P_{k+1}}{Q_{k+1}}
\end{array} \tag{1.2}
$$

The a's arising above are the partial quotients in the continued fraction

$$
\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}} \tag{1.3}
$$

and the tableau provides a convenient method of calculating them exactly using integer arithmetic.

The α_k's (at least for $k \geq 1$) are the complete quotients in the continued fraction (1.3).

We may also use the notation $\alpha = [a_0, a_1, a_2, a_3, \dots]$ for this continued fraction.

The a's and α's above are related by

$$
\begin{aligned}
a_k &\equiv \lfloor \alpha_k \rfloor \\
\alpha_{k+1} &\equiv \frac{1}{\alpha_k - a_k} \qquad k \geq 0
\end{aligned} \tag{1.4}
$$

The scheme can be implemented by [1] [2]

```
dRt  :=  I.Sqrt (d);  (*  Floor of square root  *)
ASSERT (dRt * dRt # d, 20); ASSERT ((d - p * p) MOD q  =  0, 21);
k  :=  0;
a  :=  (dRt + p)  DIV  q; IF  q < 0  THEN  DEC (a)  END;
LOOP
  INC (k);
  p  :=  a * q - p;
  q  :=  (d - p * p)  DIV  q
  a  :=  (dRt + p)  DIV  q; IF  q < 0  THEN  DEC (a)  END
END
```

as will be justified in the next few sections.

Many other papers have addressed this topic over the centuries. Some use rather sophisticated notations and ideas, but are hard to follow if you are not familiar with them. Others use very elementary language, and tend to be both rather verbose and of limited scope. This paper belongs to the second category, but, I hope, makes fewer omissions in the ground it attempts to cover.

If α above is *reduced* the sequence of partial denominators $\{a_k\}$ *ultimately* repeats.

Key ideas omitted from this paper are

- The word *ultimately* above is unnecessary
- Every surd *ultimately* leads to a repeating set of partial denominators $\{a_k\}$
- The repeating cycle of a's is palendromic

I suspect that attempts to prove those facts without using sophisticated ideas and notations would be very tedious.

[1]Some means of terminating the loop is required.

[2]If, as may be the case, q is known to always be positive the final 'IF' statement can be removed.

2 Some definitions

We consider a Quadratic surd of the form:

Definition 2.1 (Quadratic surd)**.**

$$\alpha \equiv \frac{P+\sqrt{D}}{Q}$$

and we always assume D is non-square & $Q > 0$.

Definition 2.2 (Conjugate surd)**.** The conjugate surd to x is defined by

$$\overline{\alpha} \equiv \frac{P-\sqrt{D}}{Q}$$

Definition 2.3 (Weakly reduced)**.** We call α *weakly reduced* if

$$\begin{aligned} \alpha &> 1 \\ \overline{\alpha} &< 0 \end{aligned} \tag{2.1}$$

These conditions are equivalent to

$$\begin{aligned} Q &< P+\sqrt{D} \\ P &< \sqrt{D} \end{aligned} \tag{2.2}$$

Definition 2.4 (Reduced)**.** x is *Reduced* if, in addition to being *weakly reduced*, it also satisfies

$$\overline{\alpha} > -1$$

which is equivalent to

$$Q > \sqrt{D} - P.$$

Note: Reduced x satisfies

$$\begin{aligned} Q < \sqrt{D}+P &\Longleftrightarrow P > Q-\sqrt{D} \\ Q > \sqrt{D}-P &\Longleftrightarrow P > \sqrt{D}-Q \\ &\Longrightarrow 2P > 0 \\ &\Longrightarrow P > 0 \end{aligned} \tag{2.3}$$

3 One iteration

In this section we consider a generic stage from the tableau in section 1. For variables which had the subscript k we shall use no subscript, and for variables which had the subscript $k+1$ we shall use a prime ($'$) mark.

The input surd

$$\alpha = \frac{P+\sqrt{D}}{Q}$$

is replaced by the sequence:

$$\begin{aligned} a &\leftarrow \lfloor \alpha \rfloor \\ \beta &\leftarrow \alpha - a \\ \alpha' &\leftarrow= \frac{1}{\beta}. \end{aligned} \tag{3.1}$$

a is the next partial denominator, and we can write

$$\begin{aligned} \beta &= \frac{\widetilde{P}+\sqrt{D}}{Q} \\ \alpha' &= \frac{P'+\sqrt{D}}{Q'}. \end{aligned} \tag{3.2}$$

Note that it is only possible to write α' in this form if $Q|D-P^2$, and then this invarient is preserved in the form that $Q'|D-P'^2$, as will be shown later. This invarient can always be established for any surd α by multiplying its numerator and denominator by the factor

$$\frac{Q}{\mathrm{Hcf}(Q, D-P^2)}$$

We will also establish that if α is weakly reduced then α' will be reduced.

3.1 Offset surd

3.1.1 Division property

The surd β offset from α by an integer a can clearly be written in the form

$$\beta = \frac{P-aQ+\sqrt{D}}{Q} = \frac{\widetilde{P}+\sqrt{D}}{Q}$$

and we note that $Q|D^2-\widetilde{P}$ since

$$\begin{aligned} \frac{D-\widetilde{P}^2}{Q} &= \frac{D-(P-aQ)^2}{Q} \\ &= \frac{D-P^2-Q(-2aP+a^2Q)}{Q} \\ &= \frac{D-P^2}{Q} - a(aQ-2P) \end{aligned} \tag{3.3}$$

which is an integer provided α satisfies the property $Q|D^2-P$.

3.1.2 Positive quotient

We write

$$D^2 - \widetilde{P} = Q\widetilde{Q} \tag{3.4}$$

and note that if β is positive, which implies that $\widetilde{P} > -\sqrt{D}$, and sufficiently small, specifically if $\widetilde{P} < \sqrt{D}$, then we have $D^2 - \widetilde{P} > 0$ which with $Q > 0$ implies that $\widetilde{Q} > 0$.

In the case that α is *weakly* reduced we already have $P < \sqrt{D}$. Since $\alpha > 1$ we have $a = \lfloor \alpha \rfloor \geq 1$. This means that $\widetilde{P} = P - aQ$ is also $< \sqrt{D}$. Thus and all the conditions above are met, so $\widetilde{Q} > 0$.

3.1.3 Negative conjugate

(This is possibly the trickiest section in this paper!)

We assume that α is *weakly* reduced (see (2.2)), and that β is as defined in equation (3.2).

We write:

$$\beta = \frac{\widetilde{P}+\sqrt{D}}{Q} = \alpha - a \qquad 0 < \beta < 1$$
$$\overline{\beta} = \frac{\widetilde{P}-\sqrt{D}}{Q} = \beta - \frac{2\sqrt{D}}{Q} \tag{3.5}$$

We consider two cases:

1. $Q < \sqrt{D}$.

   In this case $\frac{2\sqrt{D}}{Q}$ is clearly greater than 2, so we have $\overline{\beta} < -1$.

2. $Q > \sqrt{D}$.

$$\begin{aligned} P+\sqrt{D} < 2\sqrt{D} &< 2Q \\ &\Rightarrow 1 < \alpha < 2 \\ &\Rightarrow a = 1 \\ &\Rightarrow \alpha = 1+\beta \end{aligned} \tag{3.6}$$

   so

$$\frac{2\sqrt{D}}{Q} > \frac{P+\sqrt{D}}{Q} = \alpha = 1+\beta$$

   which, with (3.5), again gives

$$\overline{\beta} < \beta - (1+\beta) = -1$$

3.2 Reciprocal surd

We define α' by

$$\begin{aligned} \alpha' \equiv \frac{1}{\beta} &= \frac{Q}{\widetilde{P}+\sqrt{D}} = \frac{Q(\sqrt{D}-\widetilde{P})}{(\sqrt{D}-\widetilde{P})(\sqrt{D}+\widetilde{P})} \\ &= \frac{Q(\sqrt{D}-\widetilde{P})}{D-\widetilde{P}^2} = \frac{\sqrt{D}-\widetilde{P}}{Q'} \\ &= \frac{P'+\sqrt{D}}{Q'} \end{aligned} \tag{3.7}$$

where we have defined

$$\begin{aligned} P' &\equiv -\widetilde{P} \\ Q' &\equiv \frac{D-\widetilde{P}^2}{Q} = \frac{D-P'^2}{Q} \end{aligned} \tag{3.8}$$

and Q' is the integer detailed in equation (3.3).

In the case that α is *weakly* reduced, and $\beta = \alpha - \lfloor\alpha\rfloor$ we have

$$\begin{aligned} P' &> 0 \\ Q' &> 0 \end{aligned} \tag{3.9}$$

We have not (yet) proved that $P' > 0$, however we know $Q' > 0$ because it equals $\widetilde{Q}$ in section 3.1.2.

It is worth noting the relation

$$Q'Q = D - P'^2. \tag{3.10}$$

It is also apparent from (3.5)

$$\alpha' > 1. \tag{3.11}$$

3.2.1 Conjugate reciprocal

The conjugate of α' is given by

$$\overline{\alpha'} = \frac{P'-\sqrt{D}}{Q'}$$

so

$$\begin{aligned} \overline{\alpha'}\,\overline{\beta} &= \frac{P'-\sqrt{D}}{Q'} \times \frac{\widetilde{P}-\sqrt{D}}{Q} \\ &= \frac{P'-\sqrt{D}}{Q'} \times \frac{-P'-\sqrt{D}}{Q} \\ &= \frac{D-P'^2}{Q'Q} \\ &= 1 \end{aligned} \tag{3.12}$$

In section 3.1.3 we show that, provided α is *weakly* reduced, $\overline{\beta} < -1$ which, with the above, implies that $-1 < \overline{\alpha'} < 0$.

3.2.2 Surd is reduced

We have now established, on the assumptions:

- α is *weakly* reduced
- $Q | D - P^2$

that

- $\alpha > 1$
- $-1 < \overline{\alpha'} < 0$
- $Q' > 1$
- $Q' | D - P'^2$.

The first three bullets establish that α' is (fully) reduced, so we have

$$\begin{aligned} 0 < P' &< \sqrt{D} \\ \sqrt{D} - P' < Q' &< \sqrt{D} + P' \end{aligned} \tag{3.13}$$

4 Periodicity

If α_k is reduced the subsequent a's are periodic, and vica versa. If the period length is K the partial denominators

$$a_k, a_{k+1}, \ldots, a_{k+K-1}$$

form the period, and are also palendromic.

These facts are not proven in this paper.

4.1 Surds are *eventually* periodic

This seems to be somewhat involved to prove, so I shall simply refer to [Kin97] page 48, theorem 28.

An alternative derivation of the theory is in [Lac88]. Again the full story is too long for me to reproduce, in my own terms, here. Maybe on another day!

4.2 Reduced Surds are *immediately* periodic

The first observation is that from the point where the α_k values become reduced their P & Q values are restricted by

$$\begin{aligned} 0 &< P_k < \sqrt{D} \\ 0 &< Q_k < 2\sqrt{D} \end{aligned} \tag{4.1}$$

and hence there is only a finite number of possibilities. So, ultimately, there must be a repeat. Call these two equal values α_A and α_B. Then the immediate successor of each must be the same since; they have the common value

$$\frac{1}{\alpha_A - \lfloor \alpha_A \rfloor} = \frac{1}{\alpha_B - \lfloor \alpha_B \rfloor}.$$

This argument can be applied repeatedly, and the conclusion is that the sequence of α's must repeat from the prior of α_A and α_B. The latter marks the start of the second period.

4.3 *Immediately* periodic Surds are reduced

The next observation concerns the immediate predecessors of α_A and α_B. They both (call them α and β) satisfy the relation

$$\alpha - \lfloor \alpha \rfloor = \frac{1}{\alpha_A} = \frac{1}{\alpha_B} = \beta - \lfloor \beta \rfloor$$

so

$$\beta = \alpha + (\lfloor \beta \rfloor - \lfloor \alpha \rfloor)$$

Why is β reduced - just because it ends a period?

4.4 Square root of D

The value

$$\alpha = \alpha_0 = \lfloor \sqrt{D} \rfloor + \sqrt{D}$$

is reduced because $\alpha > 1$ and

$$-1 < \overline{\alpha} \equiv \lfloor \sqrt{D} \rfloor - \sqrt{D} < 0$$

and hence is periodic, say of period K, and can be written

$$\lfloor \sqrt{D} \rfloor + \sqrt{D} = [a_0, a_1, a_2, a_{K-1}, a_0, \ldots].$$

If we write a for $\lfloor \sqrt{D} \rfloor$, it is apparent that $a_0 = 2a$.

We can easily subtract a from both sides of the previous equation to leave

$$\sqrt{D} = [a, a_1, a_2, a_{K-1}, 2a, a_1, \ldots] \tag{4.2}$$

This is also periodic, of period K, but the periodicity does not start until the second denominator a_1.

4.5 Recognising the period

The

$a_k = 2\lfloor \sqrt{D} \rfloor$ means that

$$\alpha_k = \frac{P_k + \sqrt{D}}{Q_k} > 2\lfloor \sqrt{D} \rfloor$$

The palendromic property is that

$$a_k = a_{K-k} \qquad \text{for} \quad 1 \le k < K.$$

So, for $\sqrt{D}$, the period end can be recognised by $a_K = 2\lfloor \sqrt{D} \rfloor$.

(It is not obvious why this value can not appear earlier!).

4.5.1 Example continued fractions

Some example results for the continued fractions of $\sqrt{D}$ are (2 periods are shown)

D	K	a0	a1	a2	...									
2	1	1	2	2										
3	2	1	1	2	1	2								
5	1	2	4	4										
6	2	2	2	4	2	4								
7	4	2	1	1	1	4	1	1	1	4				
8	2	2	1	4	1	4								
10	1	3	6	6										
11	2	3	3	6	3	6								
12	2	3	2	6	2	6								
13	5	3	1	1	1	1	6	1	1	1	1	6		
14	4	3	1	2	1	6	1	2	1	6				
15	2	3	1	6	1	6								
17	1	4	8	8										
18	2	4	4	8	4	8								
19	6	4	2	1	3	1	2	8	2	1	3	1	2	8
20	2	4	2	8	2	8								
21	6	4	1	1	2	1	1	8	1	1	2	1	1	8
22	6	4	1	2	4	2	1	8	1	2	4	2	1	8
23	4	4	1	3	1	8	1	3	1	8				
24	2	4	1	8	1	8								
26	1	5	10	10										
27	2	5	5	10	5	10								
28	4	5	3	2	3	10	3	2	3	10				
29	5	5	2	1	1	2	10	2	1	1	2	10		
30	2	5	2	10	2	10								

References

[Kin97] A.Ya. Kinchin, *Continued Fractions*, Dover Publications, INC, Mineola, New York, 1997, ISBN 0-486-69630-8

[Lac88] Gilles Lachaud, *Continued Fractions, Binary Quadratic Forms, Quadratic Fields, and Zeta Function*, 1988 Proceedings of KIT Mathematics Workshop, http://iml.univ-mrs.fr/editions/biblio/files/lachaud/1988e.pdf