CS 6150: HW4 – Graphs, Randomized algorithms

Submission date: Wednesday, Nov 10, 2021 (11:59 PM)

This assignment has 5 questions, for a total of 50 points. Unless otherwise specified, complete and reasoned arguments will be expected for all answers.

Question	Points	Score
QuickSelect	6	
Sampling from a stream	6	
Walking on a path	12	
Birthdays and applications	12	
Checking matrix multiplication	14	
Total:	50	

Instructions. For all problems in which you are asked to develop an algorithm, write down the pseudocode, along with a rough argument for correctness and an analysis of the running time (unless specified otherwise). Failure to do this may result in a penalty. If you are unsure how much detail to provide, please contact the instructors on Piazza.

Question 1: QuickSelect ..... **[6]**

Recall that given an (unsorted) array of **distinct** integers $A[0, 1, \ldots, n-1]$ and a parameter $1 \leq k \leq n$, the Selection problem asks to find the kth smallest entry of A. In class, we saw an algorithm that used a randomized implementation of ApproximateMedian, and showed that it leads to an $O(n)$ time algorithm. Let us now consider a different procedure, that is similar to QuickSort.

PROCEDURE QUICKSELECT(A, k)

1. If $|A| = 1$, return the only element
2. Select x from A uniformly at random
3. Form arrays B and C, containing the elements of A that are $< x$ and $> x$ respectively
4. If $|B| = (k-1)$, return x, else if $|B| < (k-1)$, return QUICKSELECT$(C, k - |B| - 1)$, else return QUICKSELECT(B, k)

Let $T(n)$ be defined as the **expected running time** of QuickSelect on an array of length n. Using the law of conditional expectation, prove that

$$T(n) \leq n + \sum_{j=1}^{n} \frac{1}{n} \max\{T(j-1), T(n-j)\}.$$

Using this along with $T(1) = 1$, prove that $T(n) \leq 4n$. Write down a description of all the events you use when you use conditional expectation.

(For the purposes of this question, you may ignore the additional $O(1)$ time for steps (1-2) and (4) of the procedure above.) [*Hint:* Follow the analysis for QuickSort seen in class, use induction.]

Side note. It is interesting to see that the constant term (the 4 in $4n$) above is much better than what we had for the deterministic algorithm we saw before. It turns out that there's a way of improving the constant further: instead of choosing x uniformly at random, we pick a small sample from the array and pick the sample median.

Answer: I write down my answer on paper, because it is really hard to write down these using latex.

T_n: the running time of quick select, on array of length n, distinct integers $[0, 1, \ldots n-1]$

→ first step of algorithm, is to choose the pivot

choose first element as pivot	choose second as pivot	choose third as pivot	...	choose n^{th} as pivot
$\mathcal{E}_1$	$\mathcal{E}_2$	$\mathcal{E}_3$		$\mathcal{E}_n$

each of this possbility is $\frac{1}{n}$, we have n choice

$\mathcal{E}_j$: event that algorithm pick jth element as pivot

$\Pr(\mathcal{E}_j) = \frac{1}{n}$

we need to find $E[T_n | \mathcal{E}_j]$

$E[T_n | \mathcal{E}_j] = n + E[T_{j-1}] + E[T_{n-j}]$

↓ time needed to do all the partition

so $E[T] = E[T|\mathcal{E}_1] \cdot \Pr(\mathcal{E}_1) + E(T|\mathcal{E}_2) \cdot \Pr(\mathcal{E}_2) + \ldots + E(T|\mathcal{E}_n) \Pr(\mathcal{E}_n)$

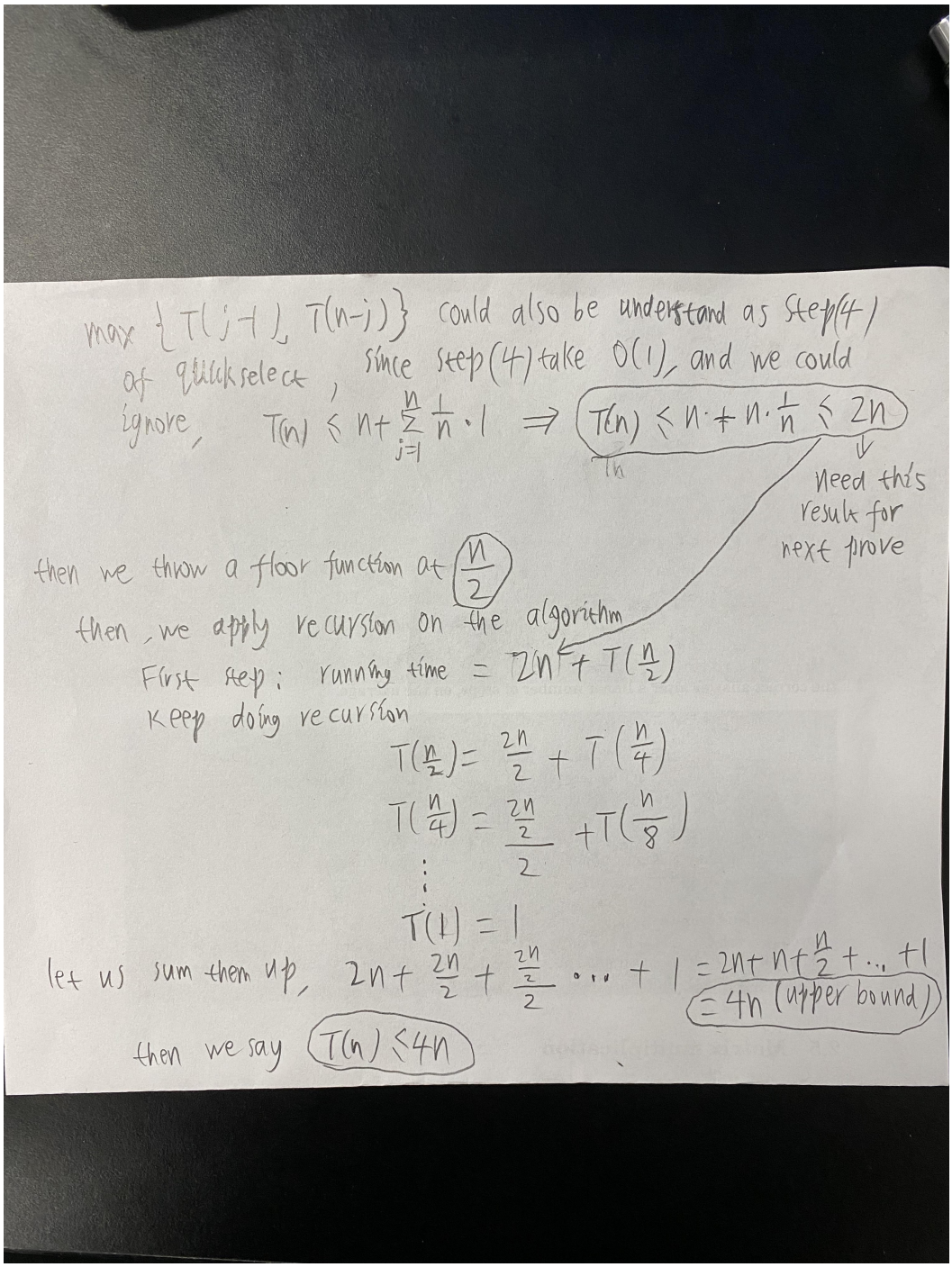
$E[T_n] = \sum_{j=1}^{n} \Pr(\mathcal{E}_j) \cdot E[T_n | \mathcal{E}_j] = \sum_{j=1}^{n} \frac{1}{n} \cdot \max\{T(j-1), T(n-j)\}$

↓ event that jth element was choosen as pivot

This proof is also worst case, because it cover every possibility of every elements, since this is upper bound we could say

Then, we have $T(n) \leq n + \sum_{j=1}^{n} \frac{1}{n} \max\{T(j-1), T(n-j)\}$

↓ n is time needed to do all partition



Question 2: Sampling from a stream ..... **[6]**

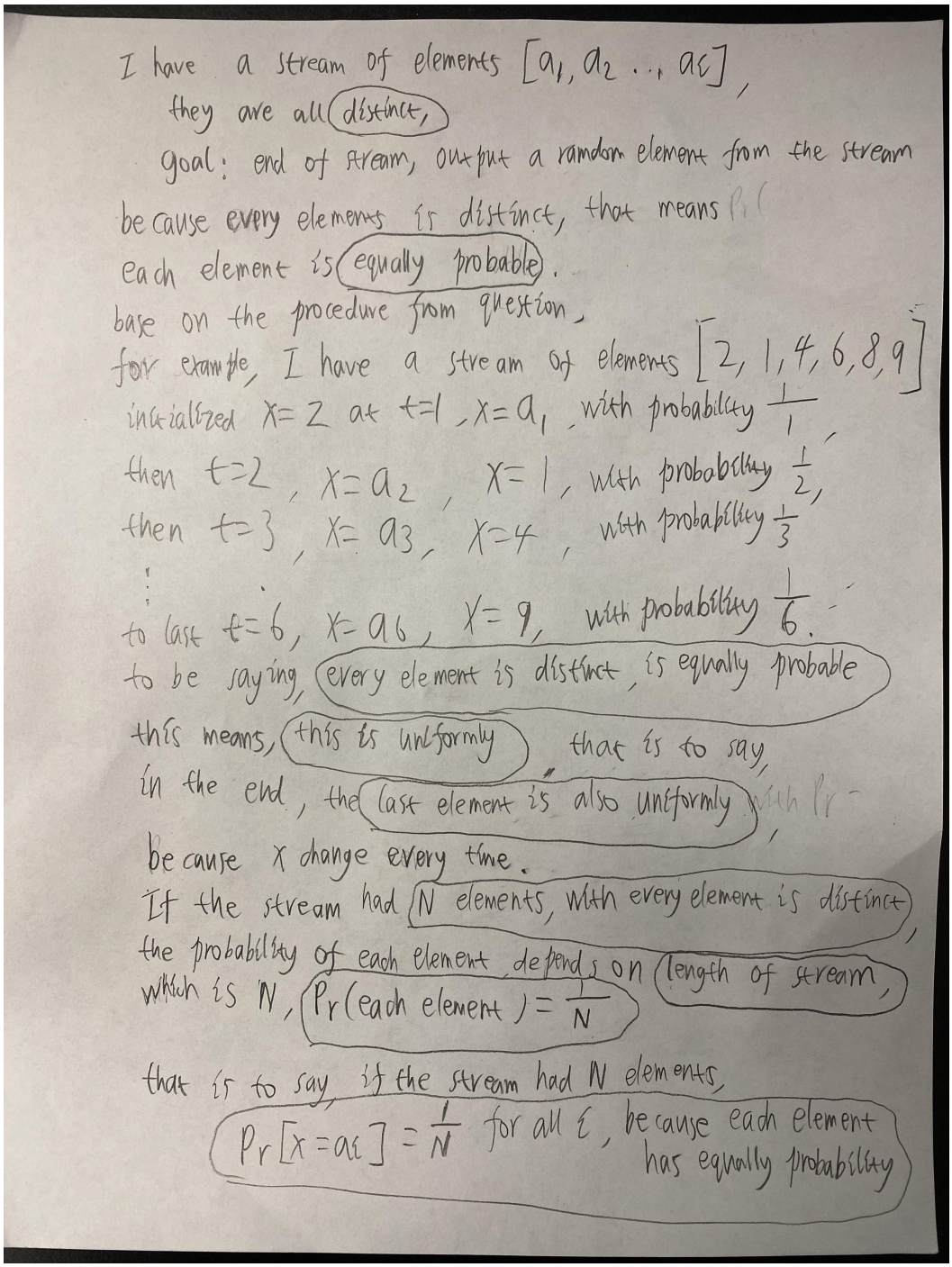
If you have an array of n elements, sampling one at random is easy: you choose an index i at random in $\{0, 1, \ldots, n-1\}$ and return the ith element. Now suppose you have a *stream* of elements $a_1, a_2, \ldots$ (suppose they are all distinct for simplicity), and you don't know how many will arrive beforehand. Your goal is the following: at the end of the stream, you should output a random element from the stream.

The trivial algorithm is to store all the elements in an array (say a dynamic array), and in the end, output a random element. But it turns out that this can be done with very little memory.

Consider the following procedure: we maintain a special variable x, initialized to the first element of the array. At time t, upon seeing a_t, we set $x = a_t$ with probability $1/t$, otherwise we keep x unchanged.

Prove that in the end, the variable x stores a uniformly random sample from the stream. (In other words, if the stream had N elements, $\Pr[x = a_i] = 1/N$ for all i.)

[*Hint:* try doing a direct computation.]
Answer:



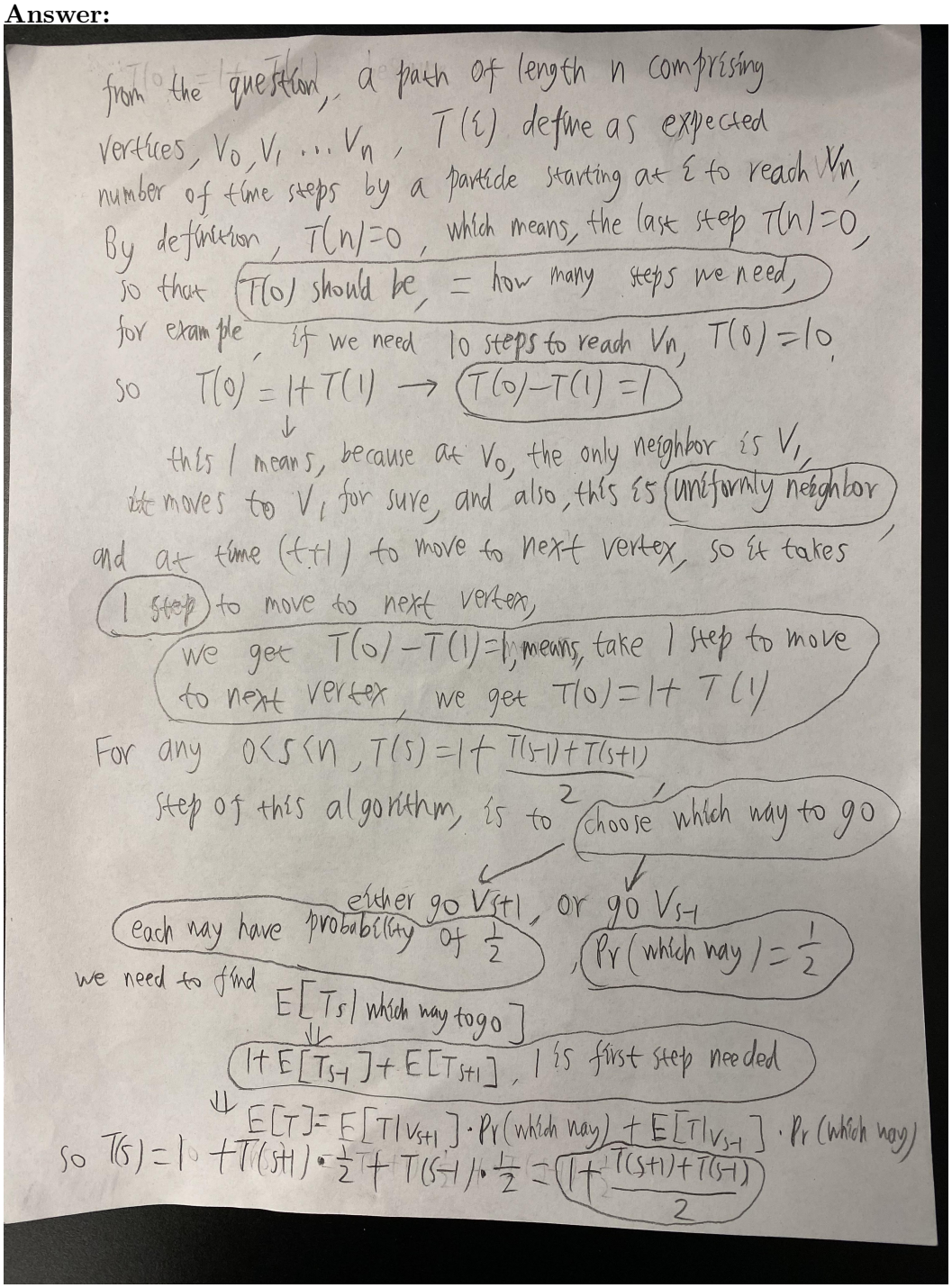
Question 3: Walking on a path ..... **[12]**

Consider a path of length n comprising vertices $v_0, v_1, \ldots, v_n$. A particle starts at v_0 at $t = 0$, and in each time step, it moves to a **uniformly random neighbor** of the current vertex. Thus if it is at v_s at time t for some $s > 0$, then at time $(t + 1)$, it moves to v_{s+1} or v_{s-1} with probability 1/2 each. (If it is at v_0, the only neighbor is v_1 and so it moves there.) The particle gets "absorbed" once it reaches v_n and the walk stops.

Define $T(i)$ as the expected number of time steps taken by a particle *starting at* i to reach v_n. By definition, $T(n) = 0$.

(a) **[5]** Prove that $T(0) = 1 + T(1)$, and further, that for any $0 < s < n$, $T(s) = 1 + \frac{T(s-1)+T(s+1)}{2}$.

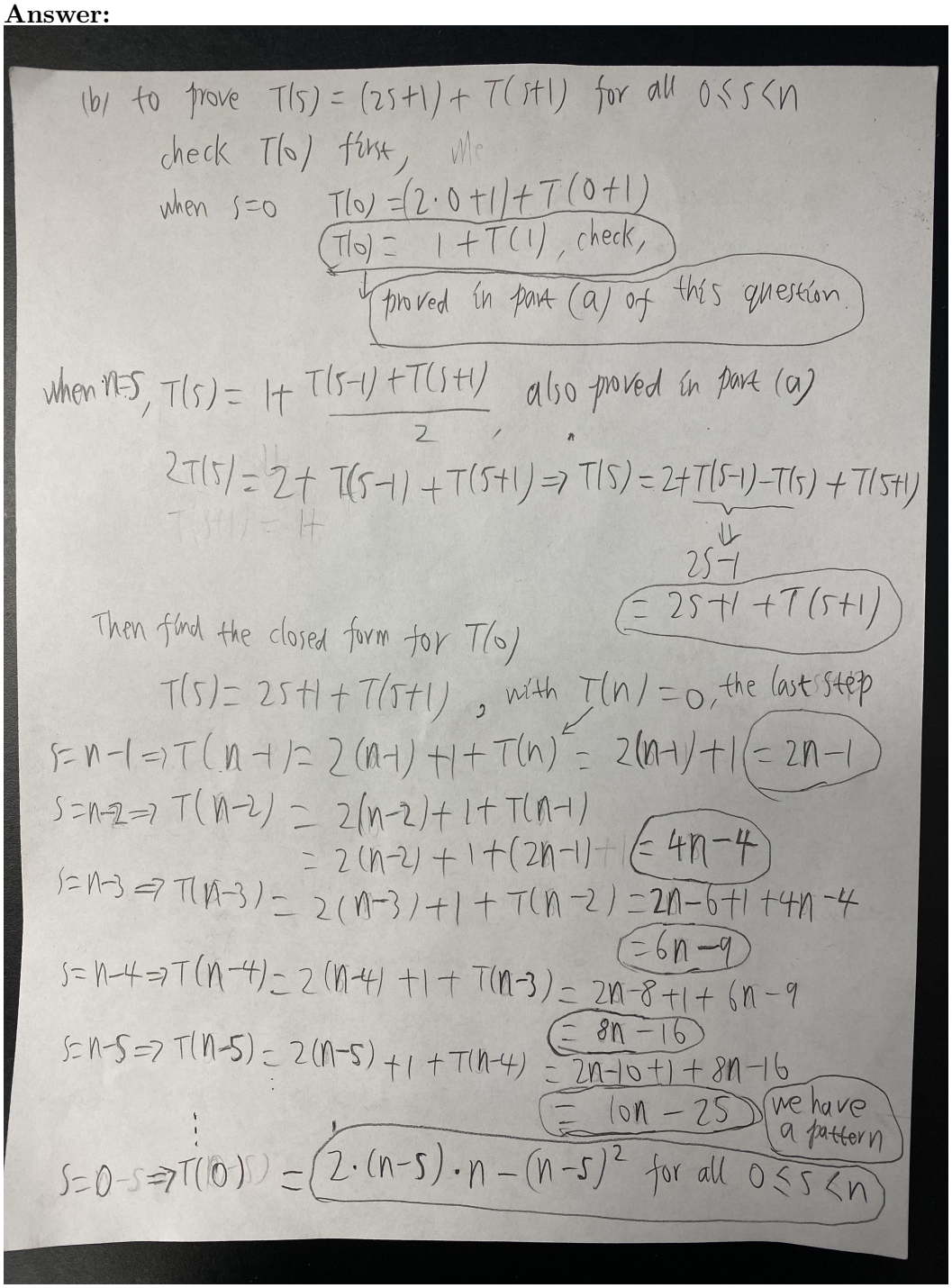
Answer:



(b) **[5]** Use this to prove that $T(s) = (2s + 1) + T(s + 1)$ for all $0 \leq s < n$, and then find a

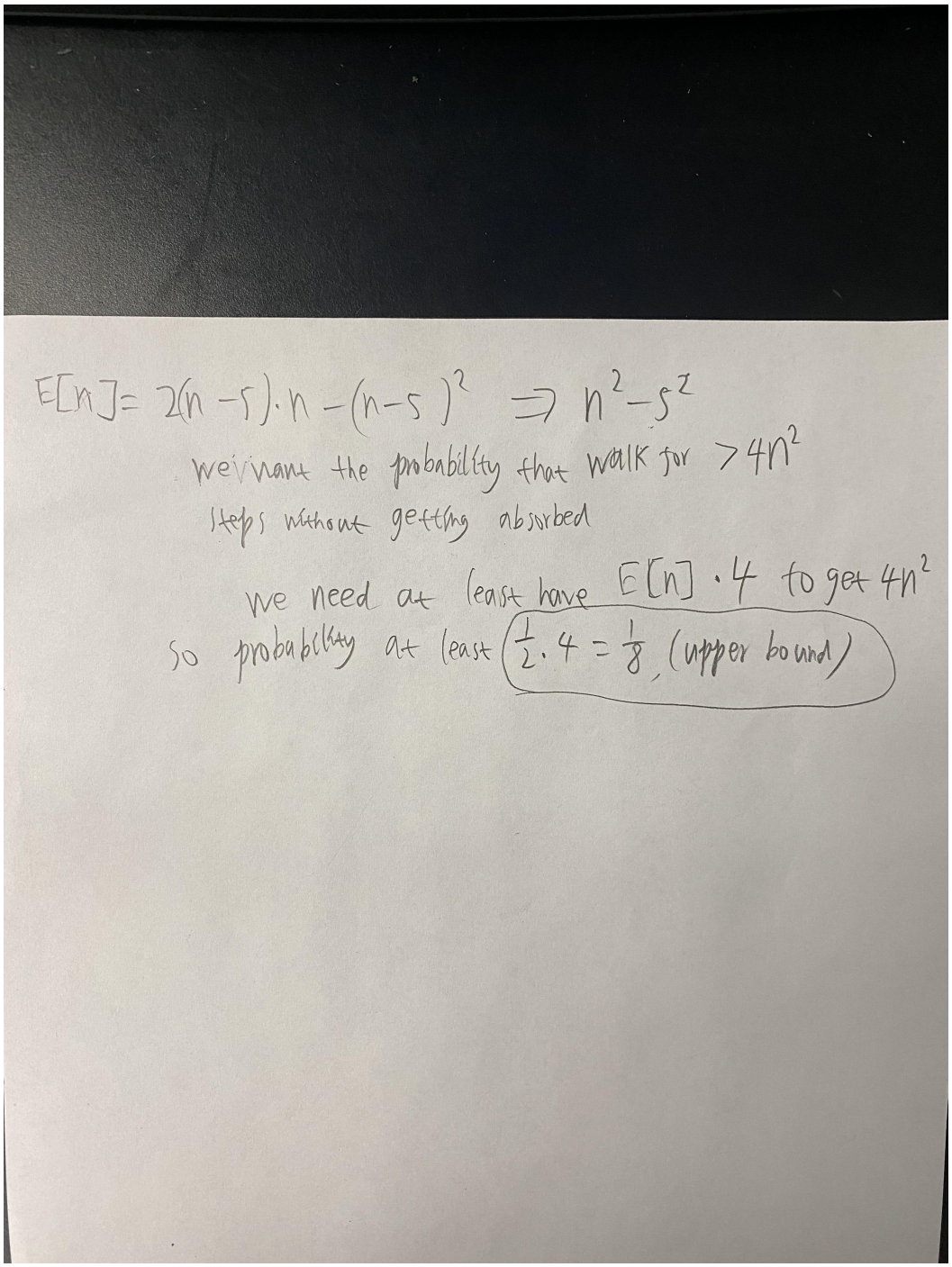
closed form for $T(0)$. [*Hint:* Use induction.]

Answer:



(c) [**2**] Give an upper bound for the probability that the particle walks for $> 4n^2$ steps without getting absorbed.

Answer:

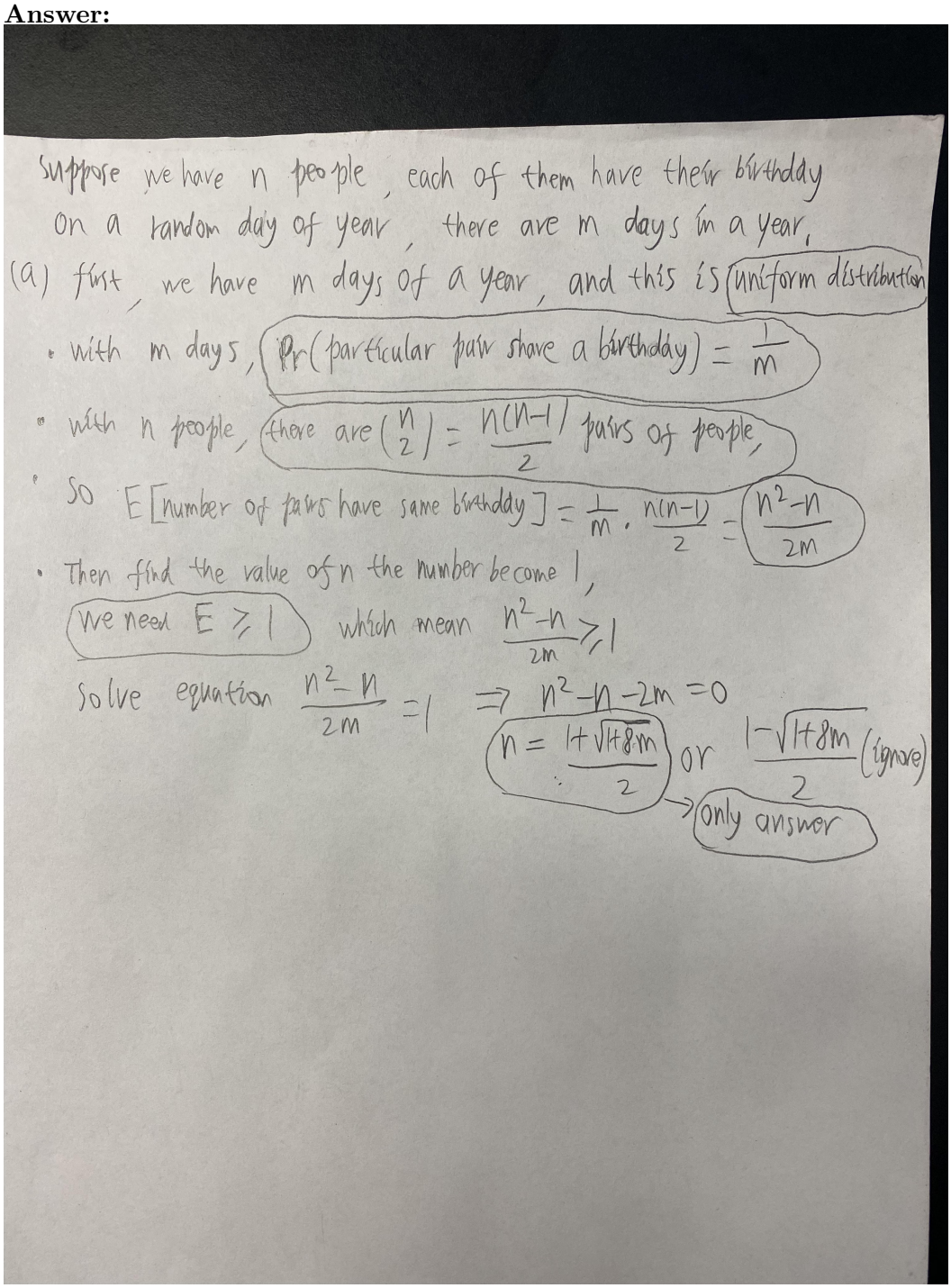


Question 4: Birthdays and applications ..... **[12]**

Suppose we have n people, each of whom has their birthday on a random day of the year. Suppose **there are m days in a year**, and let us pretend that this is some parameter.

(a) **[5]** What is the expected *number of pairs* (i, j) with $i < j$ such that person i and person j have the same birthday? For what value of n (as a function of m) does this number become 1?

Answer:



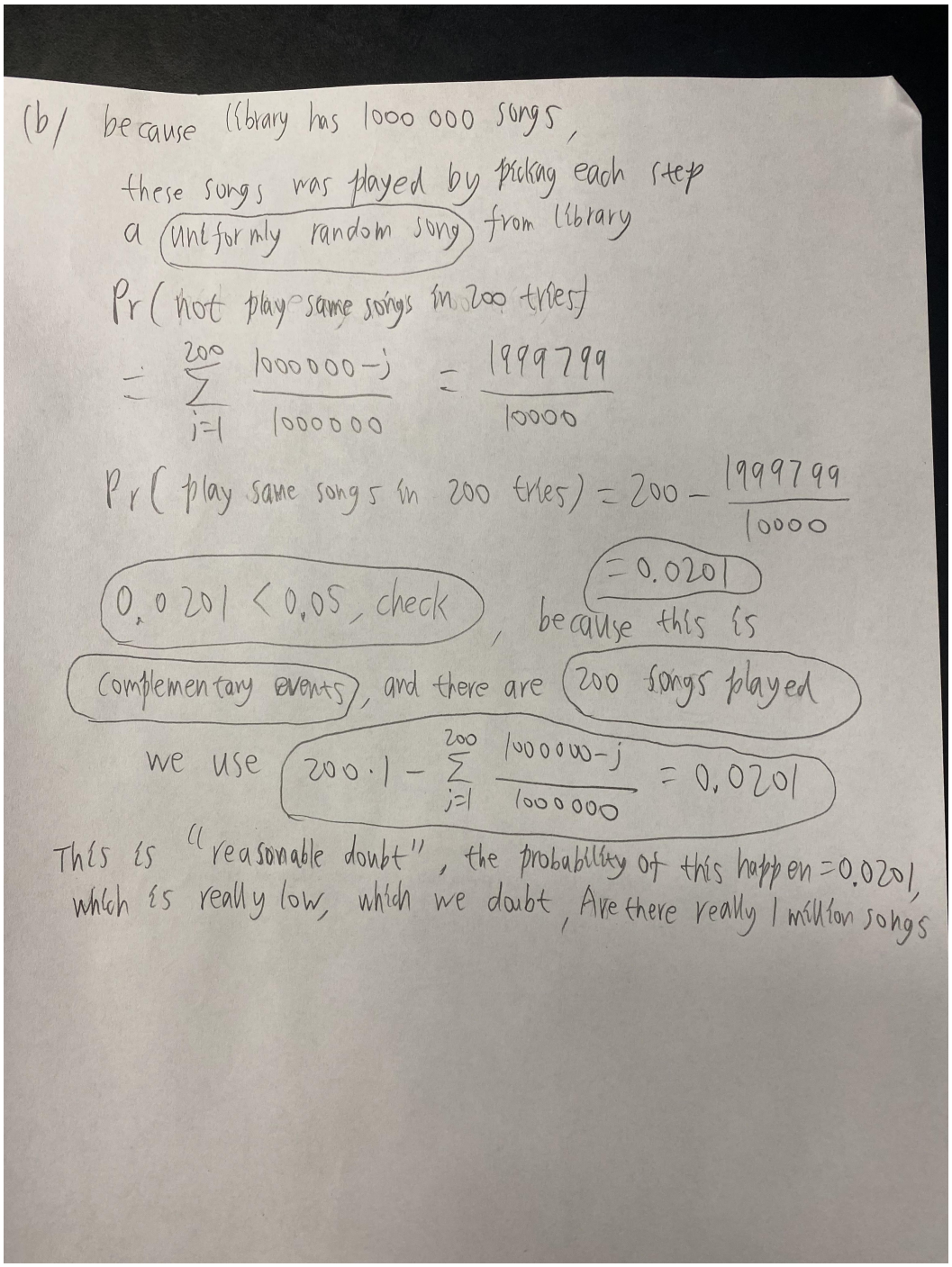
(b) [**7**] This idea has some nice applications in CS, one of which is in estimating the "support" of a distribution. Suppose we have a radio station that claims to have a library of one million songs, and suppose that the radio station plays these songs by picking, at each step a uniformly random song from its library (with replacement), playing it, then picking the next song, and so on.

Suppose we have a listener who started listening when the station began, and noticed that among the first 200 songs, there was a repetition (i.e., a song played twice). Prove that the probability of this happening (conditioned on the library size being a million songs) is < 0.05. Note that this gives us "reasonable doubt" about the station's claim that its library has a million songs.

Hint: Compute the probability of the complementary event —that all songs would be distinct— and prove that it must be large. You may use the inequality $(1-x)^n \geq 1-nx$ (for $x > 0$ and a positive integer n) without proof.

[This idea has many applications in CS, for estimating the size of sets without actually enumerating them.]

Answer:



Question 5: Checking matrix multiplication ..... **[14]**

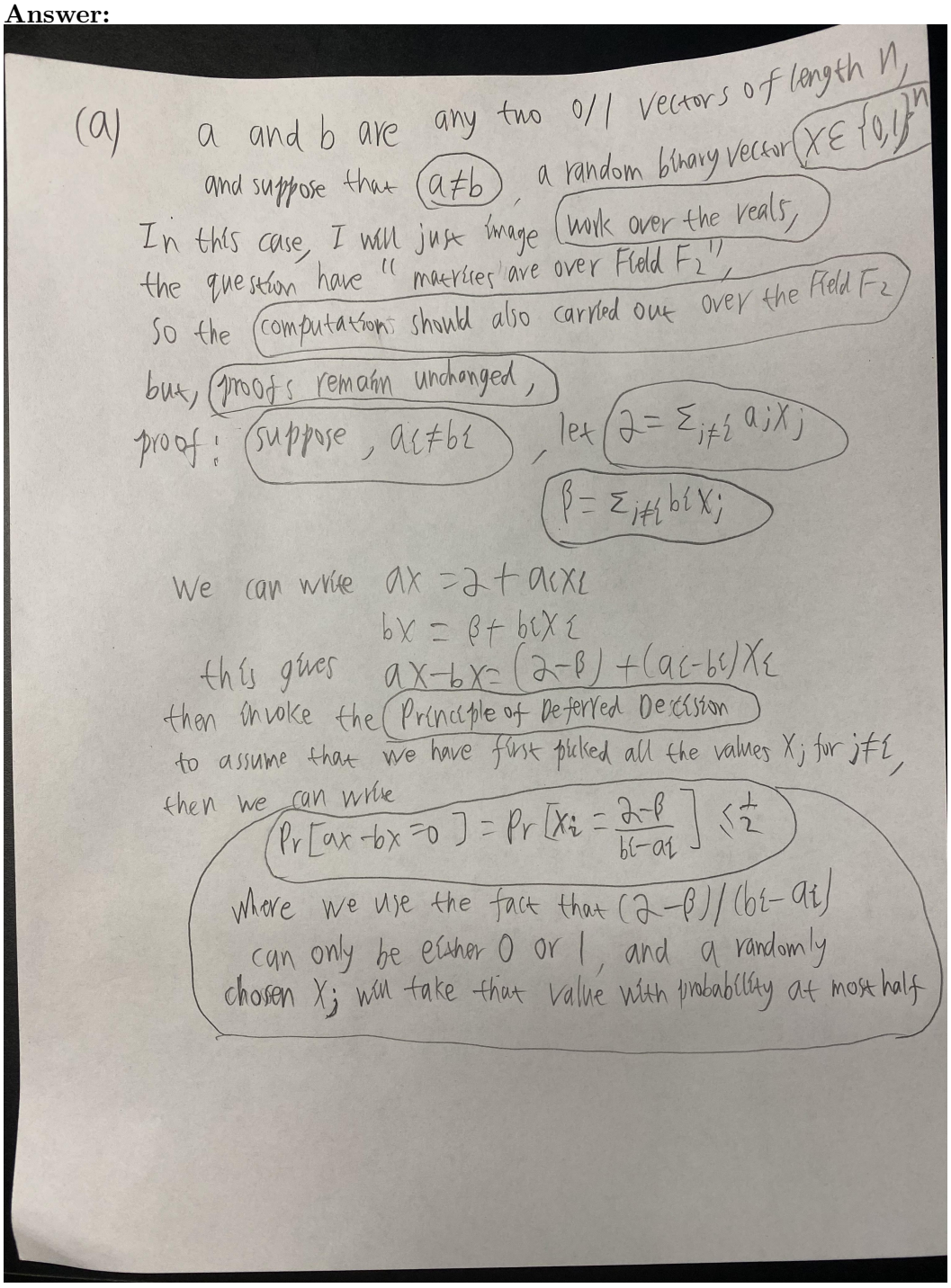
Matrix multiplication is one of the classic algorithmic problems. Consider the problem of multiplying two $n \times n$ matrices over the field $\mathbf{F}_2$ (i.e., we have matrices with entries 0/1, and we perform all computations modulo 2; e.g., 0*0 + 1*1 + 1*1 + 1*0 = 0).

The best known algorithms here are messy and take time $O(n^{2.36\cdots})$. However, the point of this exercise is to prove a simpler statement. Suppose someone gives a matrix C and claims that

$C = AB$, can we *quickly verify* if the claim is true?

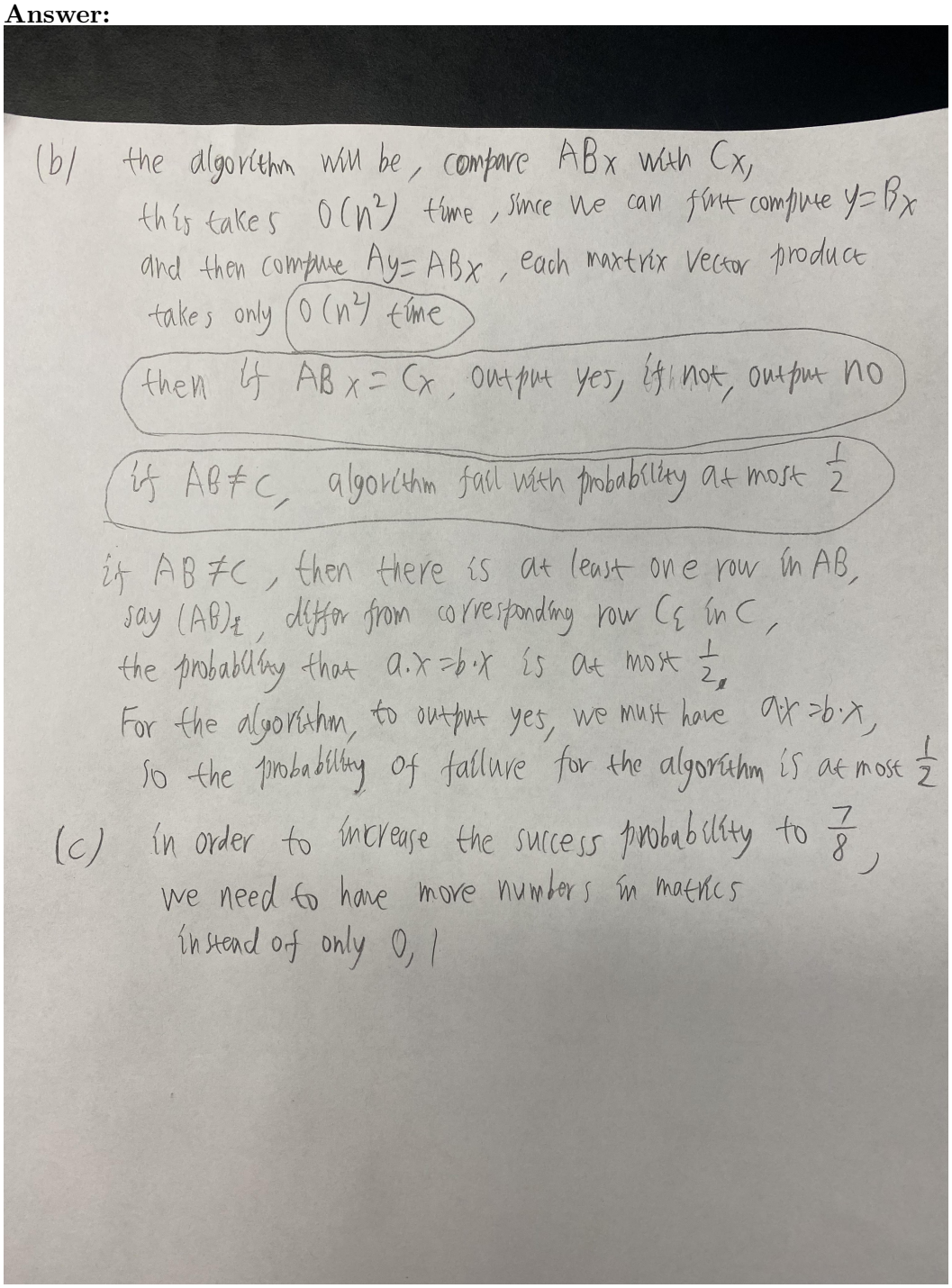
(a) **[5]** First prove a warm-up statement: suppose a and b are any two 0/1 vectors of length n, and suppose that $a \neq b$. Then, for a random binary vector $x \in \{0,1\}^n$ (one in which each coordinate is chosen uniformly at random), prove that $\Pr[\langle a, x\rangle \neq \langle b, x\rangle \pmod 2] = 1/2$. [In other words, with a probability 1/2, we can "catch" the fact that $a \neq b$.]

Answer:



(b) [**6**] Now, design an $O(n^2)$ time algorithm that tests if $C = AB$ and has a success probability $\geq 1/2$. (You need to bound both the running time and probability.)

Answer:



(c) [**3**] Show how to improve the success probability to 7/8 while still having running time $O(n^2)$.

Answer: Answer was included in part (b).