LA2 (1)

VP → Množina splňující 7 axiomů:

↳ . umět sečíst dva vektory $\oplus\ V\times V \to V$ (~~x+x~~

↳ → $\odot\ T\times V \to V$

1. sčítání komutativní: $\forall x,y \in V: x\oplus y = y \oplus x$
2. -"- asoc. $\forall x,y,z \in V: (x+y)+z = x+(y+z)$
3. násobení skalárem asoc: $\forall \alpha,\beta \in T\ \forall x\in V: \alpha\cdot(\beta\cdot x) = (\alpha\cdot\beta)\cdot x$
4. násobení distributivní zleva $\forall \alpha \in T\ \forall x,y \in V: \alpha(x+y) = (\alpha x)+(\alpha y)$
5. zprava $(\forall \alpha\in T)(\forall x,y\in V): (x+y)\alpha = \alpha x + \alpha y$
6. neutrální prvek vůči násobení $\forall x \in V: 1\cdot x = x$
7. existence nulového prvku $\forall x\in V: 0\cdot x = \theta$

1) $LN \to (x_1 \dots x_n) \iff \forall \alpha_1 \dots \alpha_n \in T\ ;\ \alpha_1 x_1 \dots \alpha_n x_n = \theta \Rightarrow \alpha_1 \dots \alpha_n = 0$

kolik existuje řešení:

a) $= (\overset{x_1}{(2,1,2)}, \overset{x_2}{(1,0,1)}, \overset{x_3}{(1,1,1)})\ ;\ \alpha\cdot(2,1,2) + \beta\cdot(1,0,1) + \gamma\cdot(1,1,1) = (0,0,0)$

$(2\alpha+\beta+\gamma,\ \alpha+\gamma,\ 2\alpha+\beta+\gamma) = (0,0,0)$

$$\begin{aligned} 2\alpha+\beta+\gamma &= 0 \\ \gamma+\alpha &= 0 \\ 2\alpha+\beta+\gamma &= 0 \end{aligned} \Rightarrow \left(\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right) \sim \dots \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \text{existuje více řešení}$$

⇒ existuje netriviální řešení ⇒ $\underline{LZ}$

b) $\left(\overset{x_1}{\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}}, \overset{x_2}{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}, \overset{x_2}{\begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}}\right)$; kolik existuje řešení?

$$\alpha\cdot\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} + \beta\cdot\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \gamma\cdot\begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2\alpha+\beta & \alpha+\beta+2\gamma \\ 2\alpha+\gamma & \alpha+\beta+2\gamma \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{aligned} 2\alpha+\beta &= 0 \\ \alpha+\beta+2\gamma &= 0 \\ \alpha+\gamma &= 0 \end{aligned} \qquad \alpha+\beta+2\gamma = 0 \qquad \left(\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 \end{array}\right) \sim \dots$$

$$\begin{pmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$ → existuje nekon. řešení → LZ → není báze

$x_1 \quad x_2 \quad x_3$

c) $(x^2+x+1, x+1, x^2+1)$; zkoušíme ekvivalenci.

$$\alpha \cdot (x^2+x+1) + \beta \cdot (x+1) + \gamma \cdot (x^2+1) = 0 \; ?$$

vytknout x(x)

$$(\alpha x^2 + \alpha x + \alpha) + (\beta x + \beta) + (\gamma x^2 + \gamma) = 0$$

$$x^2 \cdot (\alpha+\gamma) + x(\alpha+\beta) + (\alpha+\beta+\gamma) = 0 \quad (0x^2 + 0x + 0)$$

$$(\alpha+\gamma) = 0$$
$$\alpha+\beta = 0$$
$$\alpha+\beta+\gamma = 0$$

$$\begin{pmatrix} 1 & 0 & 1 & | & 0 \\ 1 & 1 & 0 & | & 0 \\ 1 & 1 & 1 & | & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 1 & 0 & | & 0 \end{pmatrix} \sim$$

$$\begin{pmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{pmatrix}$$ → pouze triv. kombinace dává 0

soubor ⇒ LN

je to báze ⇒ není $x^3 \notin \langle x^2+x+1, x+1, x^2+1 \rangle$

2) $(x_1, \dots, x_m)$; $z \in \langle x_1, \dots x_z \rangle$ → tedy zda lze se dá nakombinovat pomocí vektorů z LO; $z = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_m x_z$?

a) $z = (1,2,1)$; $\langle \underset{x_1}{(1,2,3)}, \underset{x_2}{(-1,2,1)}, \underset{x_3}{(1,0,1)} \rangle \subset \mathbb{R}^3$

Jestli je vektor z dá nakombinovat

$$\alpha (1,2,3) + \beta \cdot (-1,2,1) + \gamma \cdot (1,0,1) = (1,2,1)$$

$$(\alpha-\beta+\gamma, 2\alpha+2\beta, 3\alpha+\beta+\gamma) = (1,2,1)$$

$$\alpha - \beta + \gamma = 1$$
$$2\alpha + 2\beta = 2$$
$$3\alpha + \beta + \gamma = 1$$

→ $$\begin{pmatrix} 1 & -1 & 1 & | & 1 \\ 2 & 2 & 0 & | & 2 \\ 3 & 1 & 1 & | & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 & | & 1 \\ 0 & 4 & -2 & | & 0 \\ 0 & 4 & -2 & | & -2 \end{pmatrix} \sim$$

$$\begin{pmatrix} 1 & -1 & 1 & | & 1 \\ 0 & 2 & -1 & | & 0 \\ 0 & 0 & 0 & | & -2 \end{pmatrix}$$ → nemá řešení, vektor z nepatří do LO.

nejsou LN

b) $\mathcal{Z} = \begin{pmatrix}0&4\\-2&-1\end{pmatrix}$; $\langle \overset{x_1}{\begin{pmatrix}2&1\\1&1\end{pmatrix}}, \overset{x_2}{\begin{pmatrix}1&1\\0&1\end{pmatrix}}, \overset{x_3}{\begin{pmatrix}0&2\\1&2\end{pmatrix}} \rangle$

zjisti: $z \in \langle O\rangle \Rightarrow$ pak ma....

$$\begin{pmatrix}0&4\\-2&-1\end{pmatrix} \overset{?}{=} \alpha\cdot\begin{pmatrix}2&1\\1&1\end{pmatrix} + \beta\begin{pmatrix}1&1\\0&1\end{pmatrix} + \gamma\cdot\begin{pmatrix}0&2\\1&2\end{pmatrix}$$

$$\begin{pmatrix}0&4\\-2&-1\end{pmatrix} = \begin{pmatrix}2\alpha+\beta & \alpha+\beta+2\gamma\\ \alpha+\gamma & \alpha+\beta+2\gamma\end{pmatrix}$$

$2\alpha+\beta=0$

$\alpha+\beta+2\gamma=4$

$\alpha+\gamma=-2$

$\alpha+\beta+2\gamma=-1$

3) Nalezněte bázi VP $\langle \begin{pmatrix}1&0\\2&1\end{pmatrix}, \begin{pmatrix}1&2\\2&2\end{pmatrix}, \begin{pmatrix}0&2\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\0&0\end{pmatrix} \rangle \subseteq \mathbb{Z}_3^{2,2}$

souřadnice $\begin{pmatrix}2&2\\2&2\end{pmatrix}$ vůči bázi.

Zjisty LN. $\alpha\begin{pmatrix}1&0\\2&1\end{pmatrix} + \beta\cdot\begin{pmatrix}1&2\\2&2\end{pmatrix} + \gamma\begin{pmatrix}0&2\\0&1\end{pmatrix} + \delta\cdot\begin{pmatrix}1&0\\0&0\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix}$

$$\left(\begin{array}{cc|cc}\alpha+\beta+\delta & 2\beta+2\gamma & 0 & 0\\ 2\alpha+2\beta+\delta & \alpha+2\beta+\gamma & 0 & 0\end{array}\right)$$

$$\left(\begin{array}{cccc|c}1&1&0&1&0\\0&2&\cancel{1}\,2&0&0\\2&2&0&1&0\\1&2&1&0&0\end{array}\right) \sim \left(\begin{array}{cccc|c}1&1&0&1&0\\0&2&1&0&0\\0&0&0&2&0\\0&1&1&2&0\end{array}\right) \sim$$

$$\left(\begin{array}{cccc|c}1&1&0&1&0\\0&1&1&2&0\\0&2&1&0&0\\0&0&0&2&0\end{array}\right) \sim \left(\begin{array}{cccc|c}1&1&0&1&0\\0&1&1&0&0\\0&0&0&1&0\\0&0&0&0&\cancel{0}\end{array}\right)$$

$\rightarrow$ více řeš. $\rightarrow$ soubor je LZ

$\uparrow$ x_3

$x_3 \in \langle x_1, x_2\rangle$

generuj. obal $\leftarrow \langle x_1, x_2, x_4\rangle \rightarrow$ je LN

$\begin{pmatrix}2&2\\2&2\end{pmatrix} = \alpha\cdot x_1 + \beta\cdot x_2 + \gamma\cdot x_3$

!

4) Nalezněte bázi: $\left(\begin{pmatrix}2&1\\1&1\end{pmatrix}, \begin{pmatrix}1&0\\2&2\end{pmatrix}\right)$ i rozš. $\begin{pmatrix}1&1\\1&1\end{pmatrix}$

doplníme standardní bází

$\alpha\cdot\begin{pmatrix}2&1\\1&1\end{pmatrix}+\beta\cdot\begin{pmatrix}1&0\\2&2\end{pmatrix}+\gamma\cdot\begin{pmatrix}1&0\\0&0\end{pmatrix}+\delta\cdot\begin{pmatrix}0&1\\0&0\end{pmatrix}+\varphi\cdot\begin{pmatrix}0&0\\1&0\end{pmatrix}+\varepsilon\cdot\begin{pmatrix}0&0\\0&1\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix}$

$$\left(\begin{array}{cc|cc} 2\alpha+\beta+\gamma & \alpha+\delta & 0 & 0\\ \alpha+2\beta+\varphi & \alpha+2\beta+\varepsilon & 0 & 0\end{array}\right)$$

$$\begin{array}{c} \begin{array}{cccccc} \alpha & \beta & \gamma & \delta & \varphi & \varepsilon \end{array} \\ \left(\begin{array}{cccccc|c} 2&1&1&0&0&0&0\\ 1&0&0&1&0&0&0\\ 1&2&0&0&1&0&0\\ 1&2&0&0&0&1&0 \end{array}\right) \end{array} \sim \left(\begin{array}{cccccc|c} 2&1&1&0&0&0&0\\ 0&1&1&1&0&0&0\\ 0&0&1&0&1&0&0\\ 0&0&1&0&0&1&0 \end{array}\right)$$

$$\sim \left(\begin{array}{cccccc|c} 2&1&1&0&0&0&0\\ 0&1&1&1&0&0&0\\ 0&0&1&0&1&0&0\\ 0&0&0&0&2&1&0 \end{array}\right)$$

$\delta, \varepsilon \in \langle \alpha, \beta, \gamma, \varphi \rangle$

$\left(\begin{pmatrix}2&1\\1&1\end{pmatrix}, \begin{pmatrix}1&0\\2&2\end{pmatrix}, \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right)$ je LN

$\dim \mathbb{Z}_3^{2,2} = 4$

5) (x, y, z) je LN ve VP nad $\mathbb{R}$

a) $(x+y, 2x, y+z)$ je LN/LZ

Kolik existuje řešení?

$\alpha(x+y) + \beta\cdot(2x) + \gamma\cdot(y+z) = \theta$

$\alpha x + \alpha y + 2\beta x + \gamma y + \gamma z = \theta$

$(\alpha+2\beta)\cdot x + (\alpha+\gamma)\, y + (\gamma)\cdot z = \theta$

(x, y, z) je LN $\Rightarrow$ LK. je $\theta \Leftrightarrow (\alpha, \beta, \gamma) = 0$

$$\begin{aligned} \alpha + 2\beta &= 0 \\ \alpha + \gamma &= 0 \\ \gamma &= 0 \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$ ☺ proto L.N. dává 0

b) $(x + 2y + z,\ 7x + y + 5z,\ 3x + y + 8z)$

(6). Máme vektory x, y, z z VP V nad T: $z \in \langle x, y \rangle$

$\langle x, y \rangle = \langle x, y, z \rangle$

Dk. $\langle x, y \rangle \subseteq \langle x, y, z \rangle$

$\forall v \in \langle x, y \rangle \Rightarrow \exists \alpha_1 \alpha_2 \in T\ v = \alpha_1 x + \alpha_2 y \Rightarrow v = \alpha_1 x + \alpha_2 y + 0 \cdot z \in \langle x, y, z \rangle$

$\langle x, y, z \rangle \subseteq \langle x, y \rangle$

$w \in \langle x, y, z \rangle \Rightarrow \exists \alpha_1 \alpha_2 \alpha_3 \in T: z = \alpha_1 x + \alpha_2 y + \alpha_3 z_3$

$z \in \langle x, y \rangle \Rightarrow \exists \beta_1 \beta_2 \in T\ z = \beta_1 x + \beta_2 y$

$w = \alpha_1 x + \alpha_2 y + \alpha_3 \cdot (\beta_1 x + \beta_2 y)$

$= (\alpha_1 + \alpha_3 \cdot \beta_1) x + (\alpha_2 + \alpha_3 \cdot \beta_2) y \in \langle x, y \rangle$ □

(7) $M \ll VP$

$\hookrightarrow$ 1) $M \neq \emptyset$; 2) $\forall x, y \in M$: $x + y \in M$; 3) $\forall x \in M\ \forall \alpha \in T\ \alpha x \in M$

Máme danou $A \in \mathbb{R}^{2,2}$ $M := \{ X \in \mathbb{R}^{2,2} \mid A \cdot X = X \cdot A \}$

$M \ll \mathbb{R}^{2,2}$ 1) je $M \neq \emptyset$? (A, Θ)

$E \in \mathbb{R}^{2,2}$ $A \cdot E = E \cdot A$ (obě $= A$)

$E \in M \Rightarrow M \neq \emptyset$

2) $x \in M$: $Ax = xA$?
$y \in M$ $Ay = yA$ $\Rightarrow x + y \in M$

$A (x + y) = Ax + Ay = xA + yA = (x + y) \cdot A \Rightarrow x + y \in M$

3) $X \in M;\ \alpha \in \mathbb{R} : AX = XA \Rightarrow \alpha \cdot X \in M$

$A \cdot (\alpha X) = \alpha \cdot (AX) = \alpha (XA) = (\alpha X) \cdot A \Rightarrow \alpha X \in M$

$M \subset \mathbb{R}^{2 \times 2}$

$M = \{ X \in \mathbb{R}^{2 \times 2} \mid AX = XA \}$ $A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$

$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 2a+c & 2b+d \\ c & d \end{pmatrix} = \begin{pmatrix} 2a & a+b \\ 2c & c+d \end{pmatrix}$

$2a+c = 2a \Rightarrow c = 0$

$2b$

$a-b-d = 0$

$\left(\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array} \right) =$

$S = \langle (1,1,0,0), (1,0,0,1) \rangle$

$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$M = \langle \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

LA2 - ②

Lineární zobrazení: Nechť P, Q VP nad T, pak zobrazení $A: P \to Q$ nazveme lineární $\Leftrightarrow$ platí:

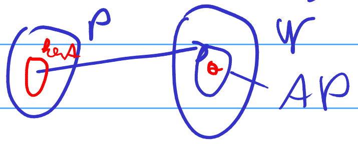
1) aditivita: $(\forall x,y \in P)(A(x+y) = Ax + Ay)$
(obraz součtu je součet obrazů)

2) homogenita $(\forall \alpha \in T)(\forall x \in P)(A(\alpha x) = \alpha Ax)$
(obraz násobku $\Rightarrow$ násobek obrazu)

Množina LZ: $L(P,Q)$

$A(\langle x_1, \ldots x_n \rangle) = \langle Ax_1, \ldots Ax_n \rangle$
obraz LO   lin. obal obrazů

Hodnost zobrazení je $h(A) = \dim P$ (obrazu hodnot)



Jádro zobrazení je množina $= \ker A := \{x \in P \mid Ax = \theta_Q\}$

množina všech vektorů, které se zobrazí na θ vzor θ vektoru $A^{-1}\{\theta\}$

Defekt je dimenze jádra $d(A) = \dim \ker A$

↳ $A \in L(P,Q)$ A je prosté $\Leftrightarrow \ker A = \{\theta_P\}$
↳

Věta o dim
$A \in L(P,Q) \Rightarrow h(A) + d(A) = \dim P$

(1) Jestli zobrazení je lin.?

a) $A(a,b) = (\overset{1.}{a} + \overset{2.}{b}, 2b, a)$; $a, b \in \mathbb{R}$

$A: \mathbb{R}^2 \to \mathbb{R}^3$

↳ 1) $A(x+y) = A(x) + A(y)$ $\forall x, y \in P$ ✓
2) $A(\alpha X) = \alpha A(X)$ $\forall X \in P, \forall \alpha \in T$ ✓ } zobrazení je lin.

1)

Mějme dva lib. vektory $x = (a,b) \in \mathbb{R}^2$; $y = (c,d) \in \mathbb{R}^2$

$A(x+y) = A(\overset{1.\,složka}{a+c}, \overset{2.\,složka}{b+d}) = (a+c+b+d ; 2\cdot(b+d) ; a+c) =$
$= (a+b+c+d, 2b+2d, a+c)$
$=$ ‖ ✓
$A(x) + A(y) = A((a,b)) + A((c,d)) = (a+b, 2b, a) + (c+d, 2d, c)$

2) Mějme lib. $x = (a,b) \in \mathbb{R}^2$ a $\alpha \in \mathbb{R}$ libovolné

$A(\alpha x) = A(\alpha a, \alpha b) = (\alpha a + \alpha b ; 2\cdot(\alpha b) ; \alpha a) =$
$\alpha \cdot (a+b), 2b, a) = \alpha A(x)$ ✓

b) $A: \mathbb{R}^{2,2} \to P$

$A\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b)x^2 + 2c + d$, $a, b, c, d \in \mathbb{R}$

1) aditivita $A(x+y) = A(x) + A(y)$ ✓
$x, y \in \mathbb{R}^{2,2}$

Mějme dvě lib. matice $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2,2}$ a $y = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in \mathbb{R}^{2,2}$

$A(x+y) = A\begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} = \underline{(a+e+b+f)\cdot x^2 + 2\cdot(c+g) + d+h}$

$A(x) + A(y) = (a+b)\cdot x^2 + 2c + d + (e+f)\cdot x^2 + 2g + h$
$= ax^2 + fx^2 + ex^2 + bx^2 + 2c + 2g + d + h$ ‖
$= \underline{(a+f+e+b)\cdot x^2 + (c+g)\cdot 2 + d + h}$

2) homogenita $A(\alpha X) \overset{X \in P}{=} \alpha A(X)$; $\checkmark$

Mějme lib. matici $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2,2}$ a $\alpha \in T$

$A(\alpha X) = (\alpha a + \alpha b) \cdot x^2 + 2\alpha c + \alpha d = \alpha \cdot ((a+b) x^2 + 2c + d) =$
$= \alpha A(X)$

$\alpha A(X)$

c) $A: \mathbb{R}^2 \to \mathbb{R}^2$
$A(a,b) = (|A|, b) \quad a, b \in \mathbb{R}$

1) aditivita $A(x+y) = A(x) + A(y)$
Mějme dva lib. dvojice $x = (a,b)$, $y = (c,d) \in \mathbb{R}^2$

$|a+c| \leq |a| + |c|$

$A(x+y) = A(a+c, b+d) = (|a+c|, b+d) \neq$

$A(x) + A(y) = (|a|, b) + (|c|, d) = (|a| + |c|, b+d)$

2) homogenita $A(\alpha X) = \alpha A(X)$

Mějme lib. dvojici $X = (a,b)$ a $\alpha \in \mathbb{R}$

$A(\alpha X) = A(\alpha a, \alpha b) = (|\alpha a|, \alpha b) = (|\alpha| \cdot |a|, \alpha b) \neq$

$\alpha A(X) = \alpha \cdot (|a|, b) = (\alpha |A|, \alpha b)$

Př.:

1) $A \in \mathcal{L}(P, Q)$ a $M \subset\subset Q \Rightarrow A(M) \subset\subset Q$

obraz podprostoru je podprostorem

↳ dk.:

1) neprázdnost

$A(\Theta_P) = \Theta_Q \qquad \Theta_Q \in A(M)$

$\Theta_P \in M$

$A(\Theta_P) = A(0 \cdot \Theta_P) = 0 \cdot A(\Theta_P) = \Theta_Q$

2) $x, z \in A(M) \overset{?}{\Rightarrow} x + z \in A(M)$

↳ $\exists v \in M \; Av = x$; $\exists w \in M \; A(w) = z$ $\Rightarrow$ obraz součtu se musí rovnat díky line- árite $A(\underbrace{v+w}_{\in M}) = \underbrace{A(v) + A(w)}_{= x+z} \Rightarrow x + z \in M$

$\Downarrow$

$v + w \in M \qquad M \subset\subset$

3) x lin. veličin $A(M)$ $\exists v: A(v) = x$

$\alpha \in T \Rightarrow \alpha x \in A(M)$?

↳ $A(\alpha v) = \alpha \underbrace{A(v)}_{\in M} = \alpha x$

$\Rightarrow \alpha x \in A(M)$

b) obraz lin. ob. je lin. o. obalu

$A(\langle x_1, \dots x_n \rangle) = \langle A(x_1), \dots A(x_n) \rangle$?

$A(\langle x_1, \dots, x_n \rangle) \subseteq \langle A(x_1), \dots, A(x_n) \rangle$

mějme $v \in A(\langle x, y \rangle) \Rightarrow \exists_{\alpha, \beta} \in T: A(\alpha x + \beta y) \Rightarrow$

v se dá díky lin. napsat jako $A(\alpha x) + A(\beta y) = \alpha A(x) + \beta A(y)$

$\Rightarrow$ taková veličina $\in \langle A(x), A(y) \rangle$

$\langle A(x), A(y)\rangle \subseteq A(\langle x, y\rangle)$?

$\hookrightarrow$ koeficienty z těl. K.

využij linearito míčin — můžeme vektor $w \in \langle A(x), A(y)\rangle$; $\exists_{\alpha,\beta}$ $w = \alpha A(x) + \beta A(y)$

$w = A(\alpha x + \beta y) \in \langle x, y\rangle \Rightarrow \langle x, y\rangle \subseteq A(\langle x, y\rangle)$

napsal jako obraz $\square$

Př. ker, d, bázi obou podprostorů, hodnost

jádro $\ker A \overset{\text{ži vzor}}{=} A^{-1}(\Theta_Q)$

a) $A: \mathbb{R}^3 \to \mathbb{R}^2$

$A(a,b,c) = (2a - c, b + c)$, $a,b,c \in \mathbb{R}$

$\hookrightarrow A(a,b,c) = (2a-c, b+c) = (0,0)$

$$\begin{aligned} 2a - c &= 0 \\ b + c &= 0 \end{aligned} \qquad \left(\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \quad S = \langle (1,-2,2) \rangle$$

$d(A) = 1$

$\Rightarrow \dim \ker(A) = \underline{1}$

$$\begin{aligned} \dim \mathbb{R}^3 &= h(A) + d(A) \\ 3 &= h(A) + 1 \\ h(A) &= 2 \end{aligned}$$

$\dim A(\mathbb{R}^3) = 2$. $\Rightarrow A(\mathbb{R}^3) \subset \mathbb{R}^2$

$A(\mathbb{R}^3) = \mathbb{R}^2 \to$ báze obou podprostorů je libovolná b)

$A: \mathbb{R}^2 \to \mathbb{R}^3$

$A(a,b) = (a+b, a-b, 2a) \quad a,b \in \mathbb{R}$

$\ker(A) = A^{-1}(\Theta_Q)$

$A(a,b) = (a+b, a-b, 2a) = (0,0,0)$

$a+b=0 \quad a=0 \Rightarrow \ker A = \Theta = \{(0,0)\}$

$a-b=0 \quad b=0 \quad d(A)=0$

$2a=0$

závěr odtud

$h(A) = 2$

$\dim \mathbb{R}^2 - d(A) = h(A) \qquad \dim A(\mathbb{R}) = 2$

$2 - 0 = 2$

důkaz

chci najít obraz kanonických vektorů

$A(\mathbb{R}^2) = A(\langle(1,0),(0,1)\rangle) = \langle A(1,0), A(0,1)\rangle =$

$= \langle(1,1,2),(1,-1,0)\rangle \Rightarrow LN$

generující obal, generující obal $\Rightarrow$ tvoří bázi

c) $A: \mathbb{R}^{2/2} \to P$

$A\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b)\cdot x^2 + c + d \quad ; a,b,c,d \in \mathbb{R}$

Jádro $\ker A = A^{-1}(\Theta_Q) \qquad A(x) = 0$

vše, které se zobrazí na Θ.

$A\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b)\cdot x^2 + 2c + d - 0x^2 + 0x + 0\cdot 1$

$a+b=0$

$2c+d=0$

$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \end{array}\right) \Rightarrow$ řešení je lin. obal dvou vektorů

$S = \langle(-1,1,0,0),(0,0,1,-2)\rangle = \ker A$

$\subset \mathbb{R}^{2/2}$

$\|$

$x\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\mathrm{Ker}\,A = \left\langle \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -2 \end{pmatrix} \right\rangle$

$d(A) = 2 \qquad \dim \mathbb{R}^{2,2} = d(A) + h(A)$

$4 = 2$

$A(\mathbb{R}^{2,2}) = A\left(\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle \right) =$

$\left\langle A\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, A\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, A\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, A\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle =$

$= \langle x^2, \cancel{x^2}, \cancel{x}, 1 \rangle = \langle x^2, 1 \rangle$ LN

báze $(x^2, 1)$ je báze $A(\mathbb{R}^{2,2})$

$h(A) = 2$

②

$A: \mathbb{Z}_5^2 \to \mathbb{Z}_5^2$ im báz: $A(1,2) = (2,3)$; $A(2,1) = (1,1)$

$B = ((1,2),(2,1))$

→ vyjádřím si to

$(a,b) = \alpha \cdot (1,2) + \beta (2,1)$ $\qquad \alpha = 3a + 4b$, $\beta = 4a + 3b$

$\left(\begin{array}{cc|c} 1 & 2 & a \\ 2 & 1 & b \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & 2 & a \\ 0 & 2 & 3a+b \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & 0 & 3a+4b \\ 0 & 1 & 4a+3b \end{array} \right)$

$A(a,b) = A(\alpha(1,2), \beta(2,1)) = \alpha \cdot A(1,2) + \beta(2,1) =$

$= \alpha \cdot (2,3) + \beta \cdot (1,1) = (3a+4b)(2,3) + (4a+3b)(1,1) =$

$= \underline{(b, 3a)}$ → předpis

$\left({}^{E}A^{E} = \;{}^{X}A^{E}\, E^{E,X} \right)$

$A(n) = A_n$

$A = \begin{pmatrix} c & d \\ e & f \end{pmatrix} \quad A \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = A(1\ 0)$

LA2 (3)

DF: nechť $A \in \mathcal{L}(P,Q)$, $X = (x_1, \dots, x_n)$ báze P a $Y = (y_1, \dots, y_m)$ báze Q, tak matici $^{X}A^{Y} \in T^{m \times n}$, definovanou po sloupcích pro $\forall i \in \hat{n}$: $({}^{X}A^{Y})_{:i} := (A(x_i))_Y$ nazveme matici lin. zobra. A vzhledem k bázím X, Y. $({}^{Y}A^{Y} = {}^{Y}A)$

$${}^{X}A^{Y} = \left((A(x_1))_Y \; A(x_2))_Y \dots A(x_n))_Y \right) \in T^{m \times n}$$

věta: $(\forall r \in P) \left(A(r)_Y = {}^{X}A^{Y} \cdot (r)_X \right)$

↳ vektor r je vzorem vektoru w $(r \in A^{-1}(w) = A(r) = w) \iff ({}^{X}A^{Y} \mid (w)_Y)$ (r je řešením soustavy)

↳ je-li $r = (r_1, \dots, r_n) \in T^n \Rightarrow r = r_1 \cdot e_1 + \dots + r_n \cdot e_n$ $(r)_\varepsilon = r$

$${}^{X}A^{\varepsilon} = \left(A(x_1)_\varepsilon \dots A(x_n)_\varepsilon \right) = \left(A(x_1) \dots A(x_n) \right)$$

věta: $h(A) = h({}^{X}A^{Y})$

↳ $A^{-1}(\{\theta\}) = \ker A$

Př 1

Mějme $A \in \mathcal{L}(P,Q)$

$A: \mathbb{R}^3 \to \mathbb{R}^2$

$A(a,b,c) = (a+2b,\ a+2b+c)$; $a,b,c \in \mathbb{R}$.

Nalézt matici lin. zobr. vůči $\mathcal{E}$

↳ obraz vektoru $(2,1,0)$ ○

↳ vzor vektoru $(1,2)$ □

$$\mathcal{E}_3 = ((1,0,0),(0,1,0),(0,0,1))$$

$$\mathcal{E}_2 = ((1,0),(0,1))$$

obraz

$${}_{\mathcal{E}_3}A^{\mathcal{E}_2} = \left((A(e_1))_{\mathcal{E}_2}\ \ (A(e_2))_{\mathcal{E}_2}\ \ (A(e_3))_{\mathcal{E}_2} \right) = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

$$A(\overset{e_1}{1,0,0}) = (1,1)$$

$$A(\overset{e_2}{0,1,0}) = (2,2)$$

$$A(\overset{e_3}{0,0,1}) = (0,1)$$

vstup

○ $${}_{\mathcal{E}_3}A^{\mathcal{E}_2}\cdot(\mathbf{r})_{\mathcal{E}_3} = (A(\mathbf{r}))_{\mathcal{E}_2} = A(\mathbf{r})$$

výstup

$$A(2,1,0) = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

□ $${}_{\mathcal{E}_3}A^{\mathcal{E}_2}\cdot \mathbf{r} = A(\mathbf{r}) = \underbrace{(1,2)}_{b}$$

$$A \cdot x = b$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 1 & 2 & 1 & 2 \end{array}\right) = \left(\begin{array}{ccc|c|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right) \to \quad \bar{x} = (0,1,0) \quad S_0 = \langle(-2,1,0)\rangle$$

$$\text{vzor} = (0,1,0) + \langle(-2,1,0)\rangle$$

$$A^{-1}(1,2)$$

Př. 2 $A(a,b) = (a+2b, a+2b, b)$; $a,b \in \mathbb{R}$
$A: \mathbb{R}^2 \to \mathbb{R}^3$; matici, obraz $(1,2)$, vzor $(1,1,0)$

$${}^{\varepsilon_2}A^{\varepsilon_3} = \left(A(e_1)_\varepsilon \quad A(e_2)_\varepsilon \right) = \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$A(1,0) = (1,1,0)$
$A(0,1) = (2,2,1)$

$${}^{\varepsilon_2}A^{\varepsilon_3} \cdot (x)_{\varepsilon_2} = (A(x))_{\varepsilon_3} = A(x)$$

$$A(1,2) = \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 2 \end{pmatrix} \leftarrow \text{obraz}$$

$${}^{\varepsilon_2}A^{\varepsilon_3} \cdot x = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \to S = \{(1,0)\}$$

$A \cdot x = b$

$$= A^{-1}(1,1,0)$$

Př. 3 $A: \mathbb{Z}_3^2 \to \mathbb{Z}_3^3$; def. pomocí B obrazů $X = ((1,2),(2,0))$

$A(1,2) = (2,2,2)$, $A(2,0) = (1,1,1)$
${}^{X}A^{\varepsilon_3}$; $h(A)$, $d(A)$, $\mathrm{Ker}(A)$

$${}^{X}A^{\varepsilon_3} = \left(A(x_1)_{\varepsilon_3} \quad A(x_2)_{\varepsilon_3} \right) = \underline{\underline{\begin{pmatrix} 2 & 1 \\ 2 & 1 \\ 2 & 1 \end{pmatrix}}} \qquad \left({}^{\varepsilon_3}A^{\varepsilon_2} \right)$$

- $h(A) = h({}^{X}A^{\varepsilon_3})$

$$\begin{pmatrix} 2 & 1 \\ 2 & 1 \\ 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow h(A) = 1 \Rightarrow \dim \mathbb{Z}_3^2 = d(A) + h(A)$$

2. věta

$2 = d(A) + 1$

$d(A) = 1$ (1 vektor)

vzor Θ vektoru

jádro $A^{-1}(\{\Theta\})$

$${}^{X}A^{\varepsilon_3} (x)_X = (Ax)_{\varepsilon_3} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

obraz vůči ε_3 bude $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Downarrow$

$$A\cdot x = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$r = \alpha x_1 + \beta x_2$$

$$\left(\begin{array}{cc|c}2&1&0\\2&1&0\\2&1&0\end{array}\right) \sim \left(\begin{array}{cc|c}2&1&0\\0&0&0\\0&0&0\end{array}\right) \quad S = \langle(1,1)\rangle \neq \text{jádro } A \qquad \Leftrightarrow (r)_X = (a,b)\ !$$

$$\ker A = \langle 1\cdot x_1 + 1 x_2\rangle = \langle(0,2)\rangle$$

Př 4

$$A: P_2 \to \mathbb{R}^{2,2},\ A(ax^2+bx+c) = \begin{pmatrix}a+b & a+b\\ b+c & a+c\end{pmatrix}$$

$a=1\ b=-2\ c+3$

např. obr $A(x^2 - 2x + 3) = \begin{pmatrix}-1 & -1\\ 1 & 4\end{pmatrix}$

$$B = (1, x, x^2)\ ; \mathcal{E}_{2,2} = \left(\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right)$$

$$\left(\begin{pmatrix}a&b\\c&d\end{pmatrix}\right)_{\mathcal{E}_{2,2}} \to (a,b,c,d)$$

$${}^{B}A^{\mathcal{E}_{2,2}} = \left(A(1)_{\mathcal{E}_{2,2}}\quad A(x)_{\mathcal{E}_{2,2}}\quad A(x^2)_{\mathcal{E}_{2,2}}\right) =$$

$$A(1)_{\mathcal{E}_{2,2}} = \begin{pmatrix}0&0\\1&1\end{pmatrix}_{\mathcal{E}_{2,2}} = (0,0,1,1)$$

$$A(x)_{\mathcal{E}_{2,2}} = \begin{pmatrix}1&1\\1&0\end{pmatrix}_{\mathcal{E}_{2,2}} = (1,1,1,0)$$

$$A(x^2)_{\mathcal{E}_{2,2}} = \begin{pmatrix}1&1\\0&0\end{pmatrix}_{\mathcal{E}_{2,2}} = (1,1,0,1)$$

$${}^{B}A^{\mathcal{E}_{2,2}} = \begin{pmatrix}0&1&1\\0&1&1\\1&1&0\\1&0&1\end{pmatrix}$$

$$h(A) = \begin{pmatrix}0&1&1\\0&1&1\\1&1&0\\1&0&1\end{pmatrix} = \begin{pmatrix}1&0&1\\1&1&0\\0&1&1\\0&1&1\end{pmatrix} \sim \begin{pmatrix}1&0&1\\0&1&-1\\0&0&2\\0&0&0\end{pmatrix}$$

$$h(A) = 3$$

; 2. věta o dimenzi

$${}^{B}A^{\mathcal{E}_{2,2}}\cdot(2)_B = \begin{pmatrix}0&0\\0&0\end{pmatrix}_{\mathcal{E}_{2,2}} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} = b \qquad \text{dim}$$

$$A\cdot x = b \qquad x \Rightarrow (r)_B$$

$$S = \{(\theta)\} = \{0,0,0\} \Rightarrow \ker A = \{0\cdot 1 + 0x + 0x^2\} = \{0\}$$

Pr. 5

$X = (x_1, x_2, x_3)$, $A : P \to P$, VP - P

$Ax_1 = x_1 + x_2 + x_3$, $Ax_2 = x_2 + 2x_3$, $Ax_3 = x_1 - x_3$

$${}_X A^X = \left((Ax_1)_X \; (Ax_2)_X \; (A(x_3))_X \right) = \underline{\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & -1 \end{pmatrix}}$$

$(Ax_1)_X = (\overset{1}{x_1} + \overset{1}{x_2} + \overset{1}{x_3})_X = (1,1,1)$

$(Ax_2)_X = (0,1,2)$

$(Ax_3)_X = (1,0,-1)$

$$h(A) = h({}_X A^X) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{pmatrix} \sim$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \| \quad \underline{2}$$

$\dim P - h(A) = 1 = d(A)$

$${}_X A^X \cdot (n)_X = (An)_X = \Theta_X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad S = \langle (\overset{X = (n)_X}{-1}, 1, 1) \rangle =$$

$\underline{\ker A = \langle -x_1 + x_2 + x_3 \rangle}$

LA2-4

Def: nechť $X = (x_1, x_2, \dots x_n)$ a $Y = (y_1, \dots, y_n)$ jsou báze P
Matici identického operátoru ${}^{X}E^{Y}$ - matice přechodu z X do Y. $Ex = x$ $\in T^{n,n}$

$${}^{X}E^{Y} = \left((x_1)_Y \dots (x_n)_Y \right) \quad {}^{X}A^{Y} = \left((A(x_1))_Y \dots (A(x_n))_Y \right)$$

věta: X, Y a Z jsou báze P, $\dim P < \infty$:

1. matice ${}^{X}E^{Y}$ je regulární a platí
$({}^{X}E^{Y})^{-1} = {}^{Y}E^{X}$ $\qquad ({}^{X}A^{Y})^{-1} = {}^{Y}A^{X}$

2. pro libovolné $x \in P$ platí:
${}^{X}E^{Y} \cdot (x)_X = (x)_Y$ $\qquad (Az)_Y = {}^{X}A^{Y} (z)_X$

$E^{-1} = E$

3. $\to$ ${}^{Y}E^{Z} \cdot {}^{X}E^{Y} = {}^{X}E^{Z}$ $\qquad {}^{X}(AB)^{U} = {}^{Y}A^{U} \cdot {}^{X}B^{Y}$

$${}^{X}E^{\varepsilon_n} = \left(x_1 \quad x_2 \quad x_n \right) \in T^{n,n}$$

$${}^{\varepsilon_n}E^{X} = \left({}^{X}E^{\varepsilon_n} \right)^{-1} = \left((x_1 \quad x_2 \dots x_n) \right)^{-1}$$

$${}^{X}E^{Y} = {}^{\varepsilon_n}E^{Y} \cdot {}^{X}E^{\varepsilon_n} = \left({}^{Y}E^{\varepsilon_n} \right)^{-1} \cdot {}^{X}E^{\varepsilon_n}$$

věta: $\to$ ${}^{\tilde{X}}A^{\tilde{Y}} = {}^{Y}E^{\tilde{Y}} \cdot {}^{X}A^{Y} \cdot {}^{\tilde{X}}E^{X}$

Př. 1 báze $\mathbb{R}^2$, $X = (\overset{x_1}{(1,2)}, \overset{x_2}{(2,2)})$, $Y = ((1,0), (-1,1))$

matice přechodu ${}^{X}E^{\varepsilon_2}$, ${}^{Y}E^{\varepsilon_2}$, ${}^{X}E^{Y}$, $r = 2x_1 - 3x_2$ $(r)_y$

$${}^{X}E^{\varepsilon_2} = \begin{pmatrix} A(x_1)_{\varepsilon_2} & A(x_2)_{\varepsilon_2} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$$

$${}^{Y}E^{\varepsilon_2} = \begin{pmatrix} A(y_1)_{\varepsilon_2} & A(x_2)_{\varepsilon_2} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & +1 \end{pmatrix}$$

$${}^{X}E^{Y} = {}^{\varepsilon_2}E^{Y} \cdot {}^{X}E^{\varepsilon_2} \qquad {}^{\varepsilon_2}E^{Y} = ({}^{Y}E^{\varepsilon_2})^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & +1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{array}\right)$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 2 \end{pmatrix}$$

$${}^{X}E^{Y} = \begin{pmatrix} 3 & 4 \\ 2 & 2 \end{pmatrix}$$

(souřadnice = koeficienty)

$(r)_y$? $\quad$, $(r)_x = (2, -3)$

$${}^{X}E^{Y} \cdot (r)_x = (r)_y$$

$$\begin{pmatrix} 3 & 4 \\ 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -6 \\ -2 \end{pmatrix} \Rightarrow (r)_y \Rightarrow r = -6 \cdot y_1 - 2y_2$$

Př. 2

$A: \mathbb{Z}_5^3 \to \mathbb{Z}_5^2$; $A(a,b,c) = (a+b+c, b+2c)$; $a,b,c \in \mathbb{Z}$

${}^{\varepsilon_3}A^{\varepsilon_2}$, ${}^{X}A^{\varepsilon_2}$, ${}^{X}A^{Y}$; $X = ((1,2,1), (2,0,1), (0,0,1))$

$Y = ((1,1), (3,2))$

$\varepsilon_3 = ((1,0,0), (0,1,0), (0,0,1))$, $\varepsilon_2 = ((1,0), (0,1))$

$${}^{\varepsilon_3}A^{\varepsilon_2} = \begin{pmatrix} A(e_1)_{\varepsilon_2} & A(e_2)_{\varepsilon_2} & A(e_3)_{\varepsilon_2} \end{pmatrix}$$

$${}^{\varepsilon_3}A^{\varepsilon_2} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$${}_X A^{\mathcal{E}_2} = {}_{\mathcal{E}_3}A^{\mathcal{E}_2} \cdot {}_X E^{\mathcal{E}_3} \quad \Big| \quad {}_Y A^{\mathcal{E}_3} = \left((A(x_1))_{\mathcal{E}_3} \; (A(x_2))_{\mathcal{E}_3} \; A(x_3))_{\mathcal{E}_3} \right)$$

$$= \begin{pmatrix} 4 & 3 & 1 \\ 4 & 2 & 2 \end{pmatrix}$$

$${}_X A^Y = {}_{\mathcal{E}_3}E^Y \cdot {}_X A^{\mathcal{E}_2} = \begin{pmatrix} 3 & 3 \\ 1 & 4 \end{pmatrix} \cdot \begin{pmatrix} 4 & 3 & 1 \\ 4 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 4 \\ 0 & 1 & 4 \end{pmatrix}$$

$${}_{\mathcal{E}_3}E^Y = ({}_Y E^{\mathcal{E}_3})^{-1} = \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array} \right) \sim \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 4 \end{array} \right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 3 & 3 \\ 0 & 1 & 1 & 4 \end{array} \right)$$

$${}_{\mathcal{E}_3}E^Y = \begin{pmatrix} 3 & 3 \\ 1 & 4 \end{pmatrix}$$

Pr. 3.

$$A: \mathbb{Z}_5^2 \to \mathbb{Z}_5^3 \qquad X = (\overset{x_1}{(1,1)}, \overset{x_2}{(3,2)}),\; Y = ((1,0,1), (2,1,1), (0,1,1))$$

$${}_X A^Y = \begin{pmatrix} 1 & 2 \\ 4 & 3 \\ 1 & 2 \end{pmatrix} \qquad {}_{\mathcal{E}_2}A^{\mathcal{E}_3};\; A(a,b) = ?,\; \ker, h, d$$

$$h(A) = h({}_X A^Y) \qquad \begin{pmatrix} 1 & 2 \\ 4 & 3 \\ 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow h(A) = 1$$

Z. věta o dimenzi

$$\dim \mathbb{Z}_5^2 = h(A) + d(A) \Rightarrow 2 = 1 + d(A) \Rightarrow d(A) = 1$$

$$\ker A \Rightarrow {}_X A^Y \cdot (\mathcal{Z})_X \overset{\text{dostaneme obraz}}{=} (A\,\mathcal{Z})_Y = \Theta_Y = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$A \cdot x = b$$

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 4 & 3 & 0 \\ 1 & 2 & 0 \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad S = \langle (1,2) \rangle \neq \ker A \; !$$

tohle jsou souřadnice $(\mathcal{Z})_X$

$$\mathcal{Z} = 1 \cdot x_1 + 2 \cdot x_2$$

$$\ker A = \langle (1,1) + 2 \cdot (3,2) \rangle$$

$$= \underline{\langle (2,0) \rangle}$$

musím to otevřít vhodnými maticemi přechodu

$$ {}^{\varepsilon_2}A^{\varepsilon_3} = {}^{Y}E^{\varepsilon_3} \cdot {}^{X}A^{Y} \cdot {}^{\varepsilon_2}E^{X} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 4 & 3 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 3 & 3 \\ 1 & 4 \end{pmatrix} = * $$

$$ {}^{Y}E^{\varepsilon_3} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad {}^{\varepsilon_2}E^{X} = \left({}^{X}E^{\varepsilon_2}\right)^{-1} = \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 4 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 3 & 3 \\ 0 & 1 & 1 & 4 \end{array}\right) $$

$$ {}^{\varepsilon_3}E^{X} = \begin{pmatrix} 3 & 3 \\ 1 & 4 \end{pmatrix} $$

$$ * \begin{pmatrix} 4 & 3 \\ 0 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 3 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} $$

$$ Az = {}^{\varepsilon_2}A^{\varepsilon_3} \cdot z = \begin{pmatrix} 0 & 4 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 4b \\ 0 \\ b \end{pmatrix} \Rightarrow (4b, 0, b) $$

$\underset{\varepsilon_2}{z} = (a, b)$ když tam dáme obecný vektor z, tak získáme jeho obraz

$$ A(a, b) = (4b, 0, b) $$

Př. 4 $A : P_2 \Rightarrow \mathbb{R}^{2,2}$

$$ A(ax^2 + bx + c) = \begin{pmatrix} a & b \\ c & a+b+c \end{pmatrix} \quad ; \ {}^{X}A^{Y} = ? $$

$$ X = (\underset{x_1}{x^2+1}, \underset{x_2}{x+1}, \underset{x_3}{1}) \qquad \varepsilon_{2,2} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) $$

$$ Y = \left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) $$

$$ A(x^2+1) = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \ ; A(x+1) = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \ , A(1) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} $$

$$ {}^{X}A^{\varepsilon_{2,2}} = \left((Ax_1)_{\varepsilon_{2,2}} \quad A(x_2)_{\varepsilon_{2,2}} \quad A(x_3)_{\varepsilon_{2,2}} \right) $$

$$ = \left(\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}\right)_{\varepsilon} = (1, 0, 1, 2) $$

$$ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix} $$

$\otimes \; {}_{A}^{Y} = {}^{\varepsilon_{212}}E^{Y} \; \otimes \; A^{\varepsilon_{212}}$ ← tohoto výsledek

$${}^{\varepsilon_{212}}E^{Y} = \left({}^{Y}E^{\varepsilon_{212}}\right)^{-1} = \begin{pmatrix} 1 & 1 & 0 & 1 & \\ 1 & 0 & 0 & 0 & \\ 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

Př. 5

$$A: \mathbb{Z}_5^3 \to \mathbb{Z}_5^3$$

$$X = ((1,2,1), (2,1,1), (1,0,0))$$

$x_1 = (1,2,1) \to (2,4,2)$, $x_2 = (2,1,1) \to (1,3,3)$, $x_3 = (1,0,0) \to (0,0,0)$

$${}^{X}A^{\varepsilon_3} = \begin{pmatrix} A(x_1) & A(x_2) & A(x_3) \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 4 & 3 & 0 \\ 2 & 3 & 0 \end{pmatrix}$$

$${}^{\varepsilon_3}A^{\varepsilon_3} = {}^{X}A^{\varepsilon_3} \cdot {}^{\varepsilon_3}E^{X} = \begin{array}{c} Ae_1 \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 4 & 2 \end{pmatrix} \end{array}$$

$${}^{\varepsilon_3}E^{X} = \left({}^{X}E^{\varepsilon_3}\right)^{-1} = \begin{pmatrix} 0 & 1 & 4 \\ 0 & 4 & 2 \\ 1 & 1 & 3 \end{pmatrix}$$

$$Az = {}^{\varepsilon_3}A^{\varepsilon} \cdot z = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 4 & 2 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = (b, 2c, 4b+2c) \text{ (špatně)}$$

$(b, 2c+b, 4b+4c)$

$z = (a,b,c)$

$(1,2,1)$ ✓

$(2,4,2)$